Inequalities

In the topic inequalities, we generally come across various sets on real line. So, let us first know about these sets which are generally called intervals.

Closed Interval

Let a and b be two given real numbers such that a < b. Then the set of all real numbers x such that $a \le x \le b$ is a closed interval and is denoted by [a, b].

[a, b] = $\{x \in R; a \le x \le b\}$

[a, b] is the set of all real numbers lying between a and b including the end points.

Open interval

Let a and b two given real numbers such that a < b. Then the set of all real numbers x such that a < x < b is a open interval and is devoted by (a, b).

[a, b] = {x R: a < x < b}

(a, b) is the set of all real numbers lying between and b excluding the end points.

Semi open or Semi closed interval

Let a and b be two real numbers such that a < b. Then the sets (a, b] = {x R: a < x $\le$ b} and [a, b) = {x R: a $\le$ x < b} are known as semi-open or semi closed intervals.

Rules pertaining to operations on Inequalities:

(i) When any number is added or subtracted from both sides of an inequality, the sign of inequality remains same.

If 5x – 4 > 4x –1, we can surely say that 5x – 4x > –1 + 4 i.e. x > 3 because we are essentially adding 4 and subtracting 3x from both sides. Thus one can transpose terms from one side to other side by changing their signs and the inequality sign will remain the same

(ii) if both sides of an inequality are multiplied or divided by the same number, the inequality sign does not always remain the same. If the number multiplied or divided with is positive, the sign remains the same but if the number is negative the inequality sign reverses.

Thus $\frac{x}{y} > \frac{3}{4}$ cannot be restated as 4x > 3y. This is because here we are multiplying both sides by 4y and since we do not know the sign of 4y, we cannot be sure of the inequality remaining the same. For more proof, though $\frac{-2}{-1} > \frac{3}{4}$, yet –8 > –3 is wrong.

Thus when we cross multiply, we have to be careful and should know the sign of the expression we are multiplying with. In the above example if we knew that x is negative, we can infer that y is negative (x/y is positive and if x is negative, y has to be negative) and then we can be sure that 4x < 3y. Note that the inequality sign has changed as we are multiplying with 4y which is negative.

Solved Examples

1. Solve $\frac{1}{x} > \frac{1}{5}$

(a) We cannot cross multiply since we don't know whether x is positive or negative.

Case 1: If x > 0

$\Rightarrow$ 5 > x

$\Rightarrow$ x < 5

So 0 < x < 5

Case 2: If x < 0

$\Rightarrow$ 5 < x

$\Rightarrow$ x > 5

No solution.

So Solution set for given inequation is 0 < x < 5.

2. Solve $\frac{1}{x} < \frac{1}{5}$

We cannot cross multiply since we don't know whether x is positive or negative.

Case 1: If x > 0

5 < x

$\Rightarrow$ x > 5

Case 2: If x < 0

5 > x

$\Rightarrow$ x < 5

But we know that x < 0

Solution set for given inequation is x < 0 or x > 5.

3. Solve: $\frac{x+3}{x+4} \geq 1$

$$\frac{x+3}{x+4} - 1 \geq 0$$
$$\frac{x+3-x-4}{x+4} \geq 0$$
$$\frac{-1}{x+4} > 0$$
$$[\frac{a}{b} > 0 \text{ and } a < 0 \Rightarrow b < 0]$$

x (– ∞ , 4]

4. If x^2 – 7x + 12 < 0. find the range of x.

x^2 – 7x + 12 < 0

(x – 3) (x – 4) < 0

3 < x < 4

This is of type a – b < 0. This is possible when the product of two is negative. Rather than trying possibilities, a simpler way also exists. (x – 3) will be negative when if x < 3 and positive of x > 3.

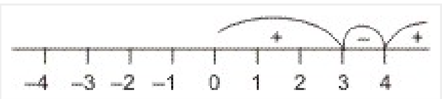
Similarly, (x – 4) will be negative if x < 4 and positive if x > 4

Combining above possibilities, we have-

If x > 4, both terms will be positive and hence the product will be positive.

If 3 < x < 4, (x – 3) remain positive and by (x – 4) will be negative. Hence the product will be negative. since, we need the product to be negative, the solution will be 3 < x < 4.

Rather writing above all, we can simply plot the points where the terms will change the signs, on a number line as follows.



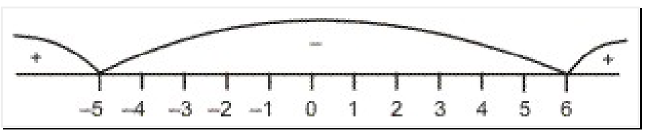
The number line represents all values from – ∞ to + ∞ . However, it is broken in various regions, three regions for this inequality. Now we have to identify those values of x that satisfy the given inequality,

For the right most region, x > 4, all terms will be positive and hence the product will be positive. For the region from the right, one term will turn negative and thus the product will be negative in this range. For the third region from right side, two terms will turn negative making product positive for this range of x.

5. x^2 – x – 30 > 0, find the range of x.

(x + 5) (x – 6) > 0

Drawing and representing this on number line.

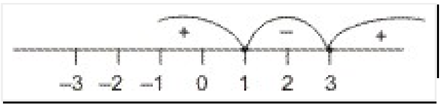


i, e. X doesn't lie between – 5 and 6. So, x > 6 or x < – 5

6. x^2 – 4x + 3 ≤ 0, find the range of x.

$(x - 1)(x - 3) \leq 0$

Representing this on number line-

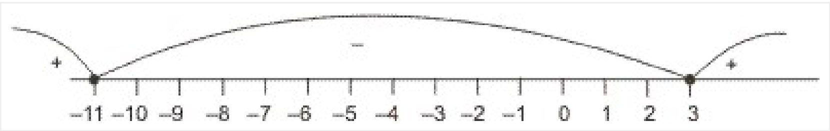


i, e x lies between $1 \leq x \leq 3$.

7. $x^2 + 8x - 33 \geq 0$

$(x + 11)(x - 3) \geq 0$

Representing this on number line-



Hence, $x3 \geq$ or $x \leq -11$.

8. Solve for the following conditions $x^2 + 8x - 33 \geq 0$, $x^2 \geq 36$

Solution: First we have to find the solution set for separate equations.

(a) $x^2 + 8x - 33 \geq 0$

$(x + 11)(x - 3) \geq 0$

Hence, $x \geq 3$ or $x \leq -11$

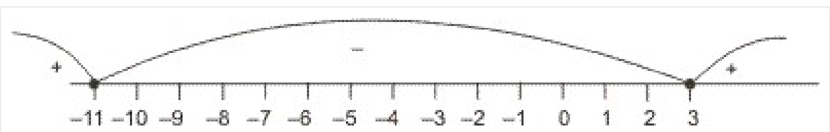
(b) $x^2 \geq 36$

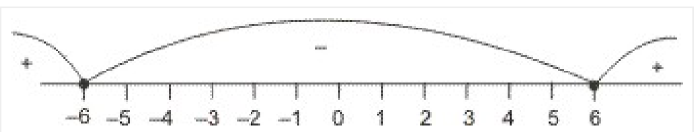
$x^2 - (6)2 \geq 0$

$(x + 6)(x - 6) \geq 0$

Hence, $x \leq -6$ or $x \geq 6$

Now combining the solutions from (a) and (b).





From above two number line we can conclude that the region from $x \le -11$ or $x \ge 6$ is possible for both. Hence, required solution is $x \le -11$ or $x \ge 6$.

9. $(\frac{x+2}{x^2-16}) - (\frac{5}{x+4}) < (\frac{6x}{4x-x^2})$

(1) 4

(2) 3

(3) 2

(4) 1

(5) -1

Ans:

$$(\frac{x+2}{x^2-16}) - (\frac{5}{x+4}) < (\frac{6x}{4x-x^2})$$

Cancelling x, in the right hand term

$$(\frac{x+2}{x^2-16}) - (\frac{5}{x+4}) < (\frac{6}{4-x})$$

Multiplying the numerator and denominator by - 1 in right hand term

$$(\frac{x+2}{x^2-16}) - (\frac{5}{x+4}) < -(\frac{6}{x-4})$$

$$\frac{x+2}{(x-4)(x+4)} - \frac{5}{x+4} + \frac{6}{x-4} < 0$$

$$\frac{2x+46}{(x-4)(x+4)} < 0$$

$$\frac{x+23}{(x-4)(x+4)} < 0$$

So the above inequality is valid for, numerator is negative and denominator is positive. So x < -23 and x < -4, x > 4.

No number satisfy this range

or the above inequality is valid for, numerator is positive and denominator is negative. So x > -23 and -4 < x < 4.

Hence 3 is the largest integer which satisfies the inequality.

Hence option 2

10. $(\frac{x-10}{x-9}) - (\frac{30}{x^2-81}) > (\frac{15x}{x+9})$

(1) 12

(2) 15

(3) 9

(4) 0

(5) 8

Ans:

$$\frac{x-10}{x-9} - \frac{30}{x^2-81} > \frac{15x}{x+9}$$

$$\frac{x-10}{x-9} - \frac{30}{(x-9)(x+9)} > \frac{15x}{x+9}$$

$$\frac{(x-10)(x+9) - 30 - 15x(x-9)}{(x-9)(x+9)} > 0$$

$$\frac{x^2 - x - 90 - 30 - 15x^2 + 135x}{(x-9)(x+9)} > 0$$

$$\frac{-120 - 14x^2 + 134x}{(x-9)(x+9)} > 0$$

$$\frac{-7x^2 + 67x - 60}{(x-9)(x+9)} > 0$$

$$\frac{7x^2 - 67x + 60}{(x+9)(x-9)} < 0$$

$$\frac{(7x-60)(x-1)}{(x+9)(x-9)} < 0$$

$x \in (-9, 1) \cup (60/7, 9)$

So the largest integer that satisfies the inequality is 0.

Hence option 4

11. $(x-3)(6-x)(x-8)^2 > 0$

(1) 3<x>10 and $x \neq 8$

(2) 3<x<9

(3) 6<x<8 and $x \neq 3$

(4) 3<x<6

(5) 6<x<8

Ans:

$(x-3)(6-x)(x-8)^2 > 0$

$(x-3)(x-6)(x-8)^2 < 0$

As $(x-8)^2$ is always positive, the sign of the above inequality depends on $(x-3)(x-6) < 0$

So x should lie in between 3 and 6

Hence option 4

12. $\frac{x^2-7x+12}{x^2+x+1} < 0$

(1) $3<x<4$

(2) $x<3$ and $x>4$

(3) $x<3$

(4) $4<x$

(5) $x<4$

Ans:

$$\frac{x^2-7x+12}{x^2+x+1} < 0$$
$$\frac{(x-3)(x-4)}{x^2+x+1} < 0$$

For denominator $2x^2+2x+1$ we have $D<0$. Hence is positive for all real values of x.

The inequality reduces to $(x+8)(x-2)<0$

So we have , $-8<x<2$

Hence option 3

Inequalities under square roots:

To solve these questions, we have to make sure that the expression within the square root must always be positive.

All the rules related to inequalities apply to these questions. Let us have a look at some solved examples.

1. $(x-3)\sqrt{(x^2)-2x-15} \geq 0$

1. $x \geq 5$

2. $x \leq 5$

3. $x \geq -8$

4. $x \leq 3$

5. $x \geq -5$

Ans:

$$(x-3)\sqrt{(x^2)-2x-15} \geq 0$$
$$(x-3)\sqrt{(x^2)-5x+3x-15} \geq 0$$
$$(x-3)\sqrt{(x-5)(x+3)} \geq 0$$

The above inequality is valid for

$x-5 \geq 0$

That is, $x \geq 5$

Hence option 1

2. $(x-8)\sqrt{x^2-3x+40} \geq 0$

1. $x \geq 5$

2. $x \geq 5$

3. $x \geq -8$

4. $x \geq 13$

5. $x \geq -5$

Ans:

$$(x-8)\sqrt{(x^2)+8x-5x-40} \geq 0$$
$$(x-8)\sqrt{(x+8)(x-5)} \geq 0$$

The above inequality is vallid for

That is, $x \geq 5$

Hence option 2

3. $2\sqrt{\frac{7x-1}{3-x}} > 1$

(1) 0<x<3

(2) 0.5<x<6

(3) 0.5<x<3

(4) 0<x<6

(5) 1<x<3

Ans:

$$2\sqrt{\frac{7x-1}{3-x}} > 1$$
$$\sqrt{\frac{7x-1}{3-x}} > 1$$
$$\frac{7x-1}{3-x} > 1$$
$$\frac{7x-1}{3-x} - 1 > 0$$
$$\frac{8x-4}{3-x} > 0$$
$$\frac{8x-4}{x-3} < 0$$
$$\frac{2x-1}{x-3} < 0$$

The above inequality is negative, when numerator is positive and denominator is negative or numerator is negative and denominator is positive, So 2x - 1 > 0 and x - 3 < 0, ⇒ x > 1/2 and x < 3 ⇒ for 0.5 <x<3 the above inequality is satisfied.

Hence option 2

4. $\sqrt{\frac{6x-8}{5-2x}} > 2$

(1)0.75<x<25

(2) 0<x<5

(3) 2<x<5

(4) 5<X<11

Ans:

$$\frac{6x-8}{5-2x} > 2^2$$

$$\frac{6x-8}{5-2x} > 4$$

$$\frac{6x-8}{5-2x}-4>0$$

$$\frac{14x-28}{5-2x}>0$$

$$\frac{x-2}{2x-5}<0$$

So, we have,

2<x<2.5

Hence option 5

5. $\sqrt{1-\left(\frac{2x-1}{x^2}\right)} < \frac{1}{2}$

1. $\frac{2}{3}<x<2$

2. $x>\frac{2}{3}$

3. $x>2$

4. $x<2$

5. None of these

Ans:

$$\sqrt{1-\left(\frac{2x-1}{x^2}\right)} < \frac{1}{2}$$

$$1-\frac{(2x-1)}{x^2}<\frac{1}{4}$$

$$\frac{x^2-2x+1}{x^2}<\frac{1}{4}$$

As x^2 is positive, we can cross multiply.