B.Sc. (Part—III) Semester—V Examination
MATHEMATICS (New)
(Mathematical Analysis)
Paper—IX

Time : Three Hours]   [Maximum Marks : 60

N.B. :— (1) Question No. **1** is compulsory. Attempt once.

(2) Attempt **ONE** question from each Unit.

1. Choose the correct alternatives :

   (i) Let f: $[0, 1] \to$ R be Riemann integrable. Which of the following is always true :

   (a) f is continuous

   (b) f is monotone

   (c) f has only finite number of discontinuities

   (d) the set of discontinuties of f may be infinite ?   1

   (ii) An improper integral $\int\limits_{a}^{\infty} \dfrac{dx}{x^p}$, $a \in$ R is convergent if :

   (a) $p < 1$ (b) $p > 1$

   (c) $p \geq 1$ (d) $p = 1$ 1

   (iii) $\beta(m, n)$ is :

   (a) $\overline{m}\,\overline{n}$ (b) $\dfrac{\overline{(m + n)}}{\overline{m}\,\overline{n}}$

   (c) $\dfrac{\overline{m}\,\overline{n}}{\overline{(m + n)}}$ (d) $\dfrac{\overline{m}\,\overline{n}}{\overline{(m - n)}}$ 1

   (iv) In the real line R, which of the following is true ?

   (a) Every bounded sequence converges   (b) Every sequence converges

   (c) Every Cauchy sequence converges   (d) None of the above   1

   (v) Every neighbourhood is a/an :

   (a) Closed set   (b) Open set

   (c) Open closed set   (d) None of the above   1

   (vi) A function u(x, y) is harmonic in region D if :

   (a) $u_{xx} - u_{yy} = 0$ (b) $u_{xy} + u_{yx} = 0$

   (c) $u_{xy} - u_{yx} = 0$ (d) $u_{xx} + u_{yy} = 0$ 1

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(vii) The function $f(z) = \sqrt{|xy|}$ at the origin :

(a) Harmonic function

(b) Analytic function

(c) Conjugate function

(d) Not analytic function   1

(viii) If $f(z)$ and $\overline{f(z)}$ are both analytic functions then $f(z)$ is :

(a) Identically zero

(b) Constant

(c) Unbounded

(d) None of the above   1

(ix) The points z where $|e^z| = 10$ form a :

(a) Circle

(b) Straight line

(c) Hyperbola

(d) Parabola   1

(x) A bilinear transformation with two non-infinite fixed points α and β having Normal form

$$\frac{w - \alpha}{w - \beta} = k\left(\frac{z - \alpha}{z - \beta}\right) \text{ is Elliptic if :}$$

(a) $|k| \neq 1$, k is real

(b) $k \neq 1$, k is not real

(c) $|k| = 1$

(d) None of the above   1

UNIT—I

2. (a) Prove that every continuous function is integrable.   4

   (b) Let the function f be defined as :

   $f(x) = 1$,   when x is rational

   $= -1$,   when x is irrational

   Show that f is not R-integrable over [0, 1] but $|f| \in R [0, 1]$.   3

   (c) Show that any constant function defined on a bounded closed interval is integrable.   3

3. (p) If f is a bounded and integrable function over [a, b] and M, m are bounds of f over [a, b], prove that :

   $$m(b - a) \leq \int_a^b f(x)\, dx \leq M(b - a).$$ 4

   (q) Prove that $\dfrac{2}{17} < \displaystyle\int_{-1}^{2} \dfrac{x}{1 + x^4}\, dx < 1/2$.   3

   (r) If f is continuous and non-negative on [a, b], then show that $\displaystyle\int_a^b f(x)\, dx \geq 0$.   3

4. (a) Prove that the integral $\int\limits_{a}^{b} \dfrac{dx}{(x-a)^p}$ converges if $p < 1$ and diverges if $p \geq 1$.   4

(b) Show that $\int\limits_{1}^{\infty} \dfrac{\sin x}{x^2}\, dx$ converges absolutely.   3

(c) Show that $\int\limits_{0}^{\infty} e^{-x^2} dx$ converges.   3

5. (p) Prove that $\beta(m, n) = \dfrac{\overline{m}\,\overline{n}}{\overline{m+n}}$.   4

(q) Prove that $\int\limits_{0}^{\pi/2} \sin^2\theta \cos^4\theta\, d\theta = \dfrac{\pi}{32}$.   3

(r) Prove that $\overline{(n+1)} = n\overline{(n)}$.   3

UNIT—III

6. (a) If $f(z) = u(x, y) + iv(x, y)$ be analytic in a region D, then prove that $u(x, y)$ and $v(x, y)$ satisfy Cauchy-Riemann equations.   4

(b) If $f(z)$ and $f(\bar{z})$ are analytic functions, prove that $f(z)$ is constant.   3

(c) Show that $u = 2x - x^3 + 3xy^2$ is harmonic and find its harmonic conjugate function. Hence find $f(z) = u + iv$.   3

7. (p) If u and v are harmonic in region R, prove that $\left(\dfrac{\partial u}{\partial y} - \dfrac{\partial v}{\partial x}\right) + i\left(\dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y}\right)$ is analytic in R.   4

(q) If the function $f(z) = u + iv$ be analytic in domain D then prove that, the family of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ form an orthogonal system, where c_1 and c_2 are arbitrary constants.   3

(r) Determine a, b, c, d so that the function $f(z) = (x^2 + axy + by^2) + i(cx^2 + dxy + y^2)$ is analytic.   3

UNIT—IV

8. (a) Prove that, every bilinear transformation with two non infinite fixed points α, β is of the form $\dfrac{w-\alpha}{w-\beta} = k\dfrac{z-\alpha}{z-\beta}$, when k is constant.   5

(b) Under the transformation $w = \sqrt{2}\, e^{i\pi/4} z$, find the image of the rectangle bounded by $x = 0$, $y = 0$, $x = 2$ and $y = 3$.   5

(p) Prove that the cross ratio remains invariant under a bilinear transformation. 5

(q) Prove that under the transformation $w = \dfrac{z - i}{iz - 1}$ the region $I_m(z) \geq 0$ is mapped into the region $|w| \leq 1$. 5

UNIT—V

10. (a) Show that $d(x, y) = |x - y|, \forall x, y \in R$ defines a metric on R. 5

    (b) Define :

        (i) Limit point

        (ii) Boundary point. 2

    (c) Prove that every neighbourhood is an open set. 3

11. (p) Define :

        (i) Complete metric space

        (ii) Open set. 2

    (q) Prove that every convergent sequence in a metric space is a Cauchy sequence. 3

    (r) Let X be a metric space. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $x_n \to x$ and $y_n \to y$ then, prove that $d(x_n, y_n) \to d(x, y)$. 5