B.Sc. Part-III (Semester-VI) Examination

MATHEMATICS

(Linear Algebra)

Paper—XI

Time : Three Hours] [Maximum Marks : 60

Note :—(1) Question No. **1** is compulsory and attempt this question once only.

(2) Attempt **ONE** question from each unit.

1. Choose the correct alternative (**1** mark each) :

(i) S is a non-empty subset of vector space V, then the smallest subspace of V containing S is :

(a) S (b) {S}

(c) [S] (d) None

(ii) Let U and V be finite dimensional vector spaces and T : U → V be a linear map one-one and onto, then :

(a) dim U = dim V (b) dim U ≠ dim V

(c) U = V (d) U ≠ d

(iii) Let W is subspace of vector space V. Then $\{f \in \hat{V} / f(w) = 0, \forall w \in W\}$ is called as :

(a) Hilatory of W (b) Annihilator of W

(c) Dual space of W (d) None

(iv) The normalized vector (1, –2, 5) is :

(a) (1, –2, 5) (b) $\left(\frac{1}{\sqrt{30}}, \frac{-2}{\sqrt{30}}, \frac{5}{\sqrt{30}}\right)$

(c) $\left(\frac{1}{2}, -1, \frac{5}{2}\right)$ (d) $\left(\frac{1}{5}, \frac{-2}{5}, 1\right)$

(v) In IPS V(F) the relation $\| u + v \|^2 + \| u - v \|^2 = 2 (\| u \|^2 + \| v \|^2)$ is called as :

(a) Schwartz inequality (b) Triangle law

(c) Parallelogram law (d) Bessels inequality

(vi) For two subspaces U and W of V(F), V = U ⊕ W ⇔ ....

(a) U ∩ W = {0} (b) V = U + W

(c) U ∩ W = {0} and V = U + W (d) None of these

(vii) Let $T : M \to N$ be an R-homomorphism. If B is a submodule of N, then :

(a) $T^{-1}(B)$ is submodule of N
(b) $T^{-1}(B)$ is submodule of M
(c) $T^{-1}(B)$ is kernel of R-homomorphism
(d) $T^{-1}(B) = T(M)$

(viii) If $T : U \to V$ then the set $\{T(u) \mid u \in U\}$ = ...

(a) Ker T
(b) R(u)
(c) R(T)
(d) None of these

(ix) If $\| V \| = 1$, then V is called :

(a) Normalised
(b) Orthonormal
(c) Scalar inner product
(d) Standard inner product

(x) If $\hat{V}$ is n-dimensional, then the dimension of V is :

(a) Less than n
(b) Greater than n
(c) Equal n
(d) Zero 10

UNIT—I

2. (a) Let U and W be two subspaces of a vector space V and $Z = U + W$. Then prove that $Z = U \oplus W$ iff $z \in Z$, $z = u + w$ is unique representation for $u \in U$ and $w \in W$. 5

(b) Extend the linearly independent set $\{(1, 1, 1, 1) (1, 2, 1, 2)\}$ in V_4 to a basis for V_4. 5

3. (p) If U and W are finite dimensional subspaces of vector space V, then prove that :

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$ 5

(q) Let R^+ be the set of all positive real numbers. Define the operations of addition $\oplus$ and scalar multiplication $\otimes$ as follows :

$$u \oplus v = u \cdot v \ \forall\ u, v \in R^+$$

and $\propto \otimes u = u^{\propto}$, $\forall\ u \in R^+$ and $\propto \in R$

Prove that R^+ is a real vector space. 5

UNIT—II

4. (a) If U, V is a vector space over a field F and $T : U \to V$ be a linear, then prove that :

$$T(\alpha_1 u_1 + \alpha_2 u_2 + \ldots\ldots + \alpha_n u_n) = \alpha_1 T(u_1) + \alpha_2 T(u_2) + \ldots\ldots + \alpha_n T(u_n)$$

$\forall\ u_i \in U$, $\alpha_i \in F$, $1 \le i \le n$ and $n \in N$. 2

(b) Let $T : V_4 \to V_3$ be a linear map defined by $T(e_1) = (1, 1, 1)$, $T(e_2) = (1, -1, 1)$, $T(e_3) = (1, 0, 0)$, $T(e_4) = (1, 0, 1)$.

Verify Rank-nullity theorem. 4

(c) Find the matrix of the linear map $T : V_2 \to V_3$ defined by $T(x, y) = (-x + 2y, y, -3x + 3y)$ related to the bases $B_1 = \{(1, 2), (-2, 1)\}$ and $B_2 = \{(-1, 0, 2), (1, 2, 3), (1, -1, 1)\}$. 4

5. (p) Let U and V be vector spaces over the same field F. Then prove that function $T : U \to V$ is linear iff $T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$, $\forall\ \alpha, \beta \in F$ and $u, v \in U$. 5

(q) If matrix of a linear map T with respect to bases B_1 and B_2 is :

$$\begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

where $B_1 = \{(1, 2, 0), (0, -1, 0), (1, -1, 1)\}$ and $B_2 = \{(1, 0), (2, -1)\}$, then find $T(x, y, z)$. 5

UNIT—III

6. (a) Let V be the space of all real valued continuous functions of real variable. Define $T : V \to V$ by

$$(T f)(x) = \int_0^x f(t)\, dt, \ \forall\ f \in V, x \in R.$$

Show that T has no eigen value. 5

(b) Prove that if V be a finite dimensional vector space over F and $v(\neq 0) \in V$, then $\exists\ f \in \hat{V}$ such that $f(v) \neq 0$. 5

7. (p) If W_1 and W_2 are subspaces of a finite dimensional vector space V, show that $A(W_1 + W_2) = A(W_1) \cap A(W_2)$. 5

(q) If K_λ is eigenspace, then prove that K_λ is a subspace of vector space V. 3

(r) Define characteristic root and characteristic vector. 2

UNIT—IV

8. (a) In $F^{(n)}$ define for $u = (\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_n)$ and $v = (\beta_1, \beta_2 \ldots\ldots \beta_n)$

$$(u, v) = \alpha_1 \overline{\beta}_1 + \alpha_2 \overline{\beta}_2 + \ldots\ldots\ldots + \alpha_n \overline{\beta}_n .$$

Show that this defines an inner product. 4

(b) If $\{x_1, x_2, x_3, \ldots\ldots\ x_n\}$ be an orthogonal set, then prove that :

$$\| x_1 + x_2 + x_3 + \ldots\ldots\ldots + x_n \|^2 = \| x_1 \|^2 + \| x_2 \|^2 + \ldots\ldots\ldots + \| x_n \|^2$$ 4

(c) Prove that orthogonal complement i.e. $W^\perp$ is subspace of V. 2

9. (p) If $\{w_1, w_2, \ldots\ldots\ w_m\}$ is an orthonormal set in V, then $\sum_{i=1}^{m} |(w_i, v)|^2 \leq \| v \|^2$ for any $v \in V$. 4

(q) If V is a finite dimensional inner product space and W is a subspace of V then prove that $(W^{\perp})^{\perp} = W$. 4

(r) (i) Define inner product in vector space. 1

(ii) Define orthogonal set. 1

UNIT—V

10. (a) Let A be a submodule of an R-module M and T is a mapping from M into M/A defined by $T_m = A + m, \forall m \in M$. Then prove that T is an R-homomorphism of M into M/A and ker T = A. 5

(b) Let T be a homomorphism of an R-module M to an R-module H. Prove that T is one-one iff ker T = {0}. 3

(c) Define :

(i) Submodule

(ii) Unital R-module. 2

11. (p) If A and B are submodules of M, then prove that $\frac{A+B}{B}$ is isomorphic to $\frac{A}{A \cap B}$. 6

(q) Prove that arbitrary intersection of submodules of a module is a submodule. 4