List of Symbols, Notations and Data

$B(n,p)$: Binomial distribution with n trials and success probability p; $n \in \{1,2,\dots\}$ and $p \in (0,1)$

$U(a,b)$: Uniform distribution on the interval $(a,b), -\infty < a < b < \infty$

$N(\mu,\sigma^2)$: Normal distribution with mean μ and variance σ^2, $\mu \in (-\infty,\infty)$, $\sigma > 0$

$P(A)$: Probability of the event A

Poisson(λ): Poisson distribution with mean $\lambda, \lambda > 0$

$E(X)$: Expected value (mean) of the random variable X

If $Z \sim N(0,1)$, then $P(Z \leq 1.96) = 0.975$ and $P(Z \leq 0.54) = 0.7054$

$\mathbb{Z}$: Set of integers

$\mathbb{Q}$: Set of rational numbers

$\mathbb{R}$: Set of real numbers

$\mathbb{C}$: Set of complex numbers

$\mathbb{Z}_n$: The cyclic group of order n

$\mathbb{F}[x]$: Polynomial ring over the field $\mathbb{F}$

$C[0,1]$: Set of all real valued continuous functions on the interval $[0,1]$

$C^1[0,1]$: Set of all real valued continuously differentiable functions on the interval $[0,1]$

ℓ_2 : Normed space of all square-summable real sequences

$L^2[0,1]$: Space of all square-Lebesgue integrable real valued functions on the interval $[0,1]$

$(C[0,1], \|\ \|_2)$: The space $C[0,1]$ with $\|f\|_2 = \left(\int_0^1 |f(x)|^2\,dx\right)^{\frac{1}{2}}$

$(C[0,1], \|\ \|_\infty)$: The space $C[0,1]$ with $\|f\|_\infty = \sup\{|f(x)| : x \in [0,1]\}$

$V^\perp$: The orthogonal complement of V in an inner product space

$\mathbb{R}^n$: n-dimensional Euclidean space

Usual metric d on $\mathbb{R}^n$ is given by $d\big((x_1,x_2,\dots,x_n),(y_1,y_2,\dots,y_n)\big) = \left(\sum_{i=1}^n (x_i - y_i)^2\right)^{1/2}$

I_n: The $n \times n$ identity matrix (I : the identity matrix when order is NOT specified)

$o(g)$: The order of the element g of a group

Q. 1 – Q. 25 carry one mark each.

Q.1 Let $T:\mathbb{R}^4 \to \mathbb{R}^4$ be a linear map defined by

$$T(x, y, z, w) = (x + z, 2x + y + 3z, 2y + 2z, w).$$

Then the rank of T is equal to _________

Q.2 Let M be a 3×3 matrix and suppose that 1, 2 and 3 are the eigenvalues of M. If

$$M^{-1} = \frac{M^2}{\alpha} - M + \frac{11}{\alpha} I_3$$

for some scalar $\alpha \neq 0$, then α is equal to ___________

Q.3 Let M be a 3×3 singular matrix and suppose that 2 and 3 are eigenvalues of M. Then the number of linearly independent eigenvectors of $M^3 + 2M + I_3$ is equal to __________

Q.4 Let M be a 3×3 matrix such that $M\begin{pmatrix}-2\\1\\0\end{pmatrix} = \begin{pmatrix}6\\-3\\0\end{pmatrix}$ and suppose that $M^3\begin{pmatrix}1\\-1/2\\0\end{pmatrix} = \begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}$ for some $\alpha, \beta, \gamma \in \mathbb{R}$. Then $|\alpha|$ is equal to _______

Q.5 Let $f:[0,\infty) \to \mathbb{R}$ be defined by

$$f(x) = \int_0^x \sin^2(t^2)\, dt.$$

Then the function f is

(A) uniformly continuous on $[0, 1)$ but NOT on $(0, \infty)$
(B) uniformly continuous on $(0, \infty)$ but NOT on $[0, 1)$
(C) uniformly continuous on both $[0, 1)$ and $(0, \infty)$
(D) neither uniformly continuous on $[0, 1)$ nor uniformly continuous on $(0, \infty)$

Q.6 Consider the power series $\sum_{n=0}^{\infty} a_n z^n$, where $a_n = \begin{cases} \frac{1}{3^n} & \text{if } n \text{ is even} \\ \frac{1}{5^n} & \text{if } n \text{ is odd.} \end{cases}$

The radius of convergence of the series is equal to __________

Q.7 Let $C = \{ z \in \mathbb{C} : |z - i| = 2\}$. Then $\frac{1}{2\pi} \oint_C \frac{z^2 - 4}{z^2 + 4}\, dz$ is equal to ____________

Q.8 Let $X \sim B(5, \frac{1}{2})$ and $Y \sim U(0,1)$. Then $\frac{P(X+Y \leq 2)}{P(X+Y \geq 5)}$ is equal to ___________

Q.9 Let the random variable X have the distribution function

$$F(x) = \begin{cases} 0 & \text{if } \quad x < 0 \\ \frac{x}{2} & \text{if } 0 \leq x < 1 \\ \frac{3}{5} & \text{if } 1 \leq x < 2 \\ \frac{1}{2} + \frac{x}{8} & \text{if } 2 \leq x < 3 \\ 1 & \text{if } \quad x \geq 3. \end{cases}$$

Then $P(2 \leq X < 4)$ is equal to ___________

Q.10 Let X be a random variable having the distribution function

$$F(x) = \begin{cases} 0 & \text{if } \quad x < 0 \\ \frac{1}{4} & \text{if } 0 \leq x < 1 \\ \frac{1}{3} & \text{if } 1 \leq x < 2 \\ \frac{1}{2} & \text{if } 2 \leq x < \frac{11}{3} \\ 1 & \text{if } \quad x \geq \frac{11}{3}. \end{cases}$$

Then $E(X)$ is equal to _________

Q.11 In an experiment, a fair die is rolled until two sixes are obtained in succession. The probability that the experiment will end in the fifth trial is equal to

(A) $\frac{125}{6^5}$ (B) $\frac{150}{6^5}$ (C) $\frac{175}{6^5}$ (D) $\frac{200}{6^5}$

Q.12 Let $x_1 = 2.2$, $x_2 = 4.3$, $x_3 = 3.1$, $x_4 = 4.5$, $x_5 = 1.1$ and $x_6 = 5.7$ be the observed values of a random sample of size 6 from a $U(\theta - 1, \ \theta + 4)$ distribution, where $\theta \in (0, \infty)$ is unknown. Then a maximum likelihood estimate of θ is equal to

(A) 1.8 (B) 2.3 (C) 3.1 (D) 3.6

Q.13 Let $\Omega = \{(x, y) \in \mathbb{R}^2 | \, x^2 + y^2 < 1\}$ be the open unit disc in $\mathbb{R}^2$ with boundary $\partial\Omega$. If $u(x, y)$ is the solution of the Dirichlet problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \quad &&\text{in } \Omega \\ u(x, y) &= 1 - 2\,y^2 \quad &&\text{on } \partial\Omega, \end{aligned}$$

then $u\left(\frac{1}{2}, 0\right)$ is equal to

(A) -1 (B) $\frac{-1}{4}$ (C) $\frac{1}{4}$ (D) 1

Q.14 Let $c \in \mathbb{Z}_3$ be such that $\frac{\mathbb{Z}_3[X]}{(X^3+c\,X+1)}$ is a field. Then c is equal to __________

Q.15 Let $V = C^1[0,1]$, $X = (C[0,1], \|\ \ \|_\infty)$ and $Y = (C[0,1], \|\ \ \|_2)$. Then V is

(A) dense in X but NOT in Y
(B) dense in Y but NOT in X
(C) dense in both X and Y
(D) neither dense in X nor dense in Y

Q.16 Let $T : (C[0,1], \|\ \ \|_\infty) \to \mathbb{R}$ be defined by $T(f) = \int_0^1 2xf(x)\,dx$ for all $f \in C[0,1]$. Then $\|T\|$ is equal to __________

Q.17 Let τ_1 be the usual topology on $\mathbb{R}$. Let τ_2 be the topology on $\mathbb{R}$ generated by $\mathcal{B} = \{[a,b) \subset \mathbb{R} : -\infty < a < b < \infty\}$. Then the set $\{x \in \mathbb{R} : 4\sin^2 x \le 1\} \cup \left\{\frac{\pi}{2}\right\}$ is

(A) closed in $(\mathbb{R}, \tau_1)$ but NOT in $(\mathbb{R}, \tau_2)$
(B) closed in $(\mathbb{R}, \tau_2)$ but NOT in $(\mathbb{R}, \tau_1)$
(C) closed in both $(\mathbb{R}, \tau_1)$ and $(\mathbb{R}, \tau_2)$
(D) neither closed in $(\mathbb{R}, \tau_1)$ nor closed in $(\mathbb{R}, \tau_2)$

Q.18 Let X be a connected topological space such that there exists a non-constant continuous function $f : X \to \mathbb{R}$, where $\mathbb{R}$ is equipped with the usual topology. Let $f(X) = \{f(x) : x \in X\}$. Then

(A) X is countable but $f(X)$ is uncountable
(B) $f(X)$ is countable but X is uncountable
(C) both $f(X)$ and X are countable
(D) both $f(X)$ and X are uncountable

Q.19 Let d_1 and d_2 denote the usual metric and the discrete metric on $\mathbb{R}$, respectively. Let $f : (\mathbb{R}, d_1) \to (\mathbb{R}, d_2)$ be defined by $f(x) = x$, $x \in \mathbb{R}$. Then

(A) f is continuous but f^{-1} is NOT continuous
(B) f^{-1} is continuous but f is NOT continuous
(C) both f and f^{-1} are continuous
(D) neither f nor f^{-1} is continuous

Q.20 If the trapezoidal rule with single interval $[0,1]$ is exact for approximating the integral $\int_0^1 (x^3 - c\,x^2)\,dx$, then the value of c is equal to ________

Q.21 Suppose that the Newton-Raphson method is applied to the equation $2x^2 + 1 - e^{x^2} = 0$ with an initial approximation x_0 sufficiently close to zero. Then, for the root $x = 0$, the order of convergence of the method is equal to _________

Q.22 The minimum possible order of a homogeneous linear ordinary differential equation with real constant coefficients having $x^2 \sin(x)$ as a solution is equal to __________

Q.23 The Lagrangian of a system in terms of polar coordinates (r, θ) is given by

$$L = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) - m\, g\, r\, (1 - \cos(\theta)),$$

where m is the mass, g is the acceleration due to gravity and $\dot{s}$ denotes the derivative of s with respect to time. Then the equations of motion are

(A) $2\ddot{r} = r\,\dot{\theta}^2 - g\,(1 - \cos(\theta)),\ \frac{d}{dt}(r^2\dot{\theta}) = -g\,r\,\sin(\theta)$

(B) $2\ddot{r} = r\,\dot{\theta}^2 + g\,(1 - \cos(\theta)),\ \frac{d}{dt}(r^2\dot{\theta}) = -g\,r\,\sin(\theta)$

(C) $2\ddot{r} = r\,\dot{\theta}^2 - g\,(1 - \cos(\theta)),\ \frac{d}{dt}(r^2\dot{\theta}) = g\,r\,\sin(\theta)$

(D) $2\ddot{r} = r\,\dot{\theta}^2 + g\,(1 - \cos(\theta)),\ \frac{d}{dt}(r^2\dot{\theta}) = g\,r\,\sin(\theta)$

Q.24 If $y(x)$ satisfies the initial value problem

$$(x^2 + y)dx = x\,dy, \qquad y(1) = 2,$$

then $y(2)$ is equal to __________

Q.25 It is known that Bessel functions $J_n(x)$, for $n \geq 0$, satisfy the identity

$$e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} = J_0(x) + \sum_{n=1}^{\infty} J_n(x) \left(t^n + \frac{(-1)^n}{t^n} \right)$$

for all $t > 0$ and $x \in \mathbb{R}$. The value of $J_0\left(\frac{\pi}{3}\right) + 2\sum_{n=1}^{\infty} J_{2n}\left(\frac{\pi}{3}\right)$ is equal to __________

Q. 26 – Q. 55 carry two marks each.

Q.26 Let X and Y be two random variables having the joint probability density function

$$f(x,y) = \begin{cases} 2 & \text{if } 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then the conditional probability $P\left(X \leq \frac{2}{3} \mid Y = \frac{3}{4}\right)$ is equal to

(A) $\frac{5}{9}$ (B) $\frac{2}{3}$ (C) $\frac{7}{9}$ (D) $\frac{8}{9}$

Q.27 Let $\Omega = (0,1]$ be the sample space and let $P(\cdot)$ be a probability function defined by

$$P((0,x]) = \begin{cases} \frac{x}{2} & \text{if } 0 \leq x < \frac{1}{2} \\ x & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Then $P\left(\left\{\frac{1}{2}\right\}\right)$ is equal to __________

Q.28 Let X_1, X_2 and X_3 be independent and identically distributed random variables with $E(X_1) = 0$ and $E(X_1^2) = \frac{15}{4}$. If $\psi : (0,\infty) \to (0,\infty)$ is defined through the conditional expectation

$$\psi(t) = E(X_1^2 \mid X_1^2 + X_2^2 + X_3^2 = t), \quad t > 0,$$

then $E(\psi((X_1 + X_2)^2))$ is equal to __________

Q.29 Let $X \sim \text{Poisson}(\lambda)$, where $\lambda > 0$ is unknown. If $\delta(X)$ is the unbiased estimator of $g(\lambda) = e^{-\lambda}(3\lambda^2 + 2\lambda + 1)$, then $\sum_{k=0}^{\infty}\delta(k)$ is equal to ___________

Q.30 Let $X_1, \dots, X_n$ be a random sample from $N(\mu, 1)$ distribution, where $\mu \in \{0, \frac{1}{2}\}$. For testing the null hypothesis $H_0: \mu = 0$ against the alternative hypothesis $H_1: \mu = \frac{1}{2}$, consider the critical region

$$R = \left\{(x_1, x_2, \dots, x_n) : \sum_{i=1}^{n} x_i > c\right\},$$

where c is some real constant. If the critical region R has size 0.025 and power 0.7054, then the value of the sample size n is equal to ___________

Q.31 Let X and Y be independently distributed central chi-squared random variables with degrees of freedom $m\ (\geq 3)$ and $n\ (\geq 3)$, respectively. If $E\left(\frac{X}{Y}\right) = 3$ and $m + n = 14$, then $E\left(\frac{Y}{X}\right)$ is equal to

(A) $\frac{2}{7}$ (B) $\frac{3}{7}$ (C) $\frac{4}{7}$ (D) $\frac{5}{7}$

Q.32 Let $X_1, X_2, \dots$ be a sequence of independent and identically distributed random variables with $P(X_1 = 1) = \frac{1}{4}$ and $P(X_1 = 2) = \frac{3}{4}$. If $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, for $n = 1, 2, \dots$, then $\lim_{n\to\infty} P(\overline{X}_n \leq 1.8)$ is equal to __________

Q.33 Let $u(x,y) = 2f(y)\cos(x - 2y)$, $(x,y) \in \mathbb{R}^2$, be a solution of the initial value problem

$$2u_x + u_y = u$$
$$u(x,0) = \cos(x).$$

Then $f(1)$ is equal to

(A) $\frac{1}{2}$ (B) $\frac{e}{2}$ (C) e (D) $\frac{3e}{2}$

Q.34 Let $u(x,t)$, $x \in \mathbb{R}$, $t \geq 0$, be the solution of the initial value problem

$$u_{tt} = u_{xx}$$
$$u(x,0) = x$$
$$u_t(x,0) = 1.$$

Then $u(2,2)$ is equal to ________

Q.35 Let $W = \text{Span}\left\{\frac{1}{\sqrt{2}}(0,0,1,1),\ \frac{1}{\sqrt{2}}(1,-1,0,0)\right\}$ be a subspace of the Euclidean space $\mathbb{R}^4$. Then the square of the distance from the point (1,1,1,1) to the subspace W is equal to ________

Q.36 Let $T: \mathbb{R}^4 \to \mathbb{R}^4$ be a linear map such that the null space of T is

$$\{(x,y,z,w) \in \mathbb{R}^4 : x + y + z + w = 0\}$$

and the rank of $(T - 4\,I_4)$ is 3. If the minimal polynomial of T is $x(x-4)^{\alpha}$, then α is equal to _______

Q.37 Let M be an invertible Hermitian matrix and let $x, y \in \mathbb{R}$ be such that $x^2 < 4y$. Then

(A) both $M^2 + x\,M + y\,I$ and $M^2 - x\,M + y\,I$ are singular
(B) $M^2 + x\,M + y\,I$ is singular but $M^2 - x\,M + y\,I$ is non-singular
(C) $M^2 + x\,M + y\,I$ is non-singular but $M^2 - x\,M + y\,I$ is singular
(D) both $M^2 + x\,M + y\,I$ and $M^2 - x\,M + y\,I$ are non-singular

Q.38 Let $G = \{e, x, x^2, x^3, y, xy, x^2y, x^3y\}$ with $o(x) = 4, o(y) = 2$ and $xy = yx^3$. Then the number of elements in the center of the group G is equal to

(A) 1 (B) 2 (C) 4 (D) 8

Q.39 The number of ring homomorphisms from $\mathbb{Z}_2 \times \mathbb{Z}_2$ to $\mathbb{Z}_4$ is equal to __________

Q.40 Let $p(x) = 9x^5 + 10x^3 + 5x + 15$ and $q(x) = x^3 - x^2 - x - 2$ be two polynomials in $\mathbb{Q}[x]$. Then, over $\mathbb{Q}$,

(A) $p(x)$ and $q(x)$ are both irreducible
(B) $p(x)$ is reducible but $q(x)$ is irreducible
(C) $p(x)$ is irreducible but $q(x)$ is reducible
(D) $p(x)$ and $q(x)$ are both reducible

Q.41 Consider the linear programming problem

$$\begin{aligned} &\text{Maximize } 3x + 9y, \\ &\text{subject to} \quad 2y - x \le 2 \\ &\qquad\qquad\quad 3y - x \ge 0 \\ &\qquad\qquad\quad 2x + 3y \le 10 \\ &\qquad\qquad\quad x, y \ge 0. \end{aligned}$$

Then the maximum value of the objective function is equal to ______

Q.42 Let $S = \{(x, \sin\frac{1}{x}) : 0 < x \le 1\}$ and $T = S \cup \{(0,0)\}$. Under the usual metric on $\mathbb{R}^2$,

(A) S is closed but T is NOT closed
(B) T is closed but S is NOT closed
(C) both S and T are closed
(D) neither S nor T is closed

Q.43 Let $H = \left\{ (x_n) \in \ \ell_2 : \sum_{n=1}^{\infty} \frac{x_n}{n} = 1 \right\}$. Then H

(A) is bounded
(B) is closed
(C) is a subspace
(D) has an interior point

Q.44 Let V be a closed subspace of $L^2[0,1]$ and let $f, g \in L^2[0,1]$ be given by $f(x) = x$ and $g(x) = x^2$. If $V^{\perp} = \text{Span}\{ f \}$ and Pg is the orthogonal projection of g on V, then $(g - Pg)(x),\ x \in [0,1]$, is

(A) $\frac{3}{4}x$
(B) $\frac{1}{4}x$
(C) $\frac{3}{4}x^2$
(D) $\frac{1}{4}x^2$

Q.45 Let $p(x)$ be the polynomial of degree at most 3 that passes through the points $(-2,12), (-1,1)$, $(0,2)$ and $(2,-8)$. Then the coefficient of x^3 in $p(x)$ is equal to _________

Q.46 If, for some $\alpha, \beta \in \mathbb{R}$, the integration formula

$$\int_0^2 p(x)dx = p(\alpha) + p(\beta)$$

holds for all polynomials $p(x)$ of degree at most 3, then the value of $3(\alpha - \beta)^2$ is equal to _____

Q.47 Let $y(t)$ be a continuous function on $[0, \infty)$ whose Laplace transform exists. If $y(t)$ satisfies

$$\int_0^t (1 - \cos(t - \tau))\, y(\tau)\, d\tau = t^4,$$

then $y(1)$ is equal to _______

Q.48 Consider the initial value problem

$$x^2 y'' - 6y = 0, \qquad y(1) = \alpha,\ y'(1) = 6.$$

If $y(x) \to 0$ as $x \to 0^+$, then α is equal to __________

Q.49 Define $f_1, f_2 : [0,1] \to \mathbb{R}$ by

$$f_1(x) = \sum_{n=1}^{\infty} \frac{x \sin(n^2 x)}{n^2} \quad \text{and} \quad f_2(x) = \sum_{n=1}^{\infty} x^2(1 - x^2)^{n-1}.$$

Then

(A) f_1 is continuous but f_2 is NOT continuous
(B) f_2 is continuous but f_1 is NOT continuous
(C) both f_1 and f_2 are continuous
(D) neither f_1 nor f_2 is continuous

Q.50 Consider the unit sphere $S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ and the unit normal vector $\hat{n} = (x,y,z)$ at each point (x,y,z) on S. The value of the surface integral

$$\iint_S \left\{ \left(\frac{2x}{\pi} + \sin(y^2)\right) x + \left(e^z - \frac{y}{\pi}\right) y + \left(\frac{2z}{\pi} + \sin^2 y\right) z \right\} d\sigma$$

is equal to _______

Q.51 Let $D = \{(x, y) \in \mathbb{R}^2 : 1 \le x \le 1000,\ 1 \le y \le 1000\}$. Define

$$f(x, y) = \frac{x\,y}{2} + \frac{500}{x} + \frac{500}{y}.$$

Then the minimum value of f on D is equal to ________

Q.52 Let $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$. Then there exists a non-constant analytic function f on $\mathbb{D}$ such that for all $n = 2, 3, 4, \ldots$

(A) $f\left(\frac{\sqrt{-1}}{n}\right) = 0$

(B) $f\left(\frac{1}{n}\right) = 0$

(C) $f\left(1 - \frac{1}{n}\right) = 0$

(D) $f\left(\frac{1}{2} - \frac{1}{n}\right) = 0$

Q.53 Let $\sum_{n=-\infty}^{\infty} a_n z^n$ be the Laurent series expansion of $f(z) = \frac{1}{2z^2 - 13z + 15}$ in the annulus $\frac{3}{2} < |z| < 5$. Then $\frac{a_1}{a_2}$ is equal to _________

Q.54 The value of $\frac{i}{4-\pi} \int_{|z|=4} \frac{dz}{z\cos(z)}$ is equal to __________

Q.55 Suppose that among all continuously differentiable functions $y(x)$, $x \in \mathbb{R}$, with $y(0) = 0$ and $y(1) = \frac{1}{2}$, the function $y_0(x)$ minimizes the functional

$$\int_0^1 \left(e^{-(y'-x)} + (1+y)y'\right)dx.$$

Then $y_0\left(\frac{1}{2}\right)$ is equal to

(A) 0 (B) $\frac{1}{8}$ (C) $\frac{1}{4}$ (D) $\frac{1}{2}$

END OF THE QUESTION PAPER