

TN ①

Transverse Nature of electromagnetic waves-

Maxwell's four equations are given as follows

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \text{ --- (1)}, \quad \vec{\nabla} \cdot \vec{B} = 0 \text{ --- (2)} \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \text{ --- (3)} \quad \text{and}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 (\vec{J} + \vec{J}_d) \text{ --- (4)}$$

However $\vec{J} = \sigma \vec{E}$ and $\vec{J}_d = \frac{\partial \vec{D}}{\partial t}$ where $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

$$\therefore \vec{J}_d = \frac{\partial}{\partial t} (\epsilon_0 \vec{E} + \vec{P}) = \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \frac{\partial \vec{P}}{\partial t}$$

Put these values in (4), we get

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\sigma \vec{E} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \frac{\partial \vec{P}}{\partial t} \right) \text{ --- (5)}$$

But in vacuum, we have $\sigma = 0$, $\rho = 0$ and $\vec{P} = 0$ Therefore,

Maxwell's four equations simplify to following form

$$\vec{\nabla} \cdot \vec{E} = 0 \text{ --- (6)}, \quad \vec{\nabla} \cdot \vec{B} = 0 \text{ --- (7)}, \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \text{ --- (8)} \quad \text{and} \quad \vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \text{ --- (9)}$$

We know that solution of Maxwell's electromagnetic wave equation in vacuum for electric and magnetic field components are given as follows:-

$$\vec{E} = \vec{E}_0 e^{i(\omega t - \vec{k} \cdot \vec{r})} \quad \text{and} \quad \vec{B} = \vec{B}_0 e^{i(\omega t - \vec{k} \cdot \vec{r})} \text{ --- (10)}$$

Where $\vec{E}_0, \vec{B}_0$ are amplitudes of electric and magnetic fields
 $\vec{E}, \vec{B}$ are instantaneous values of electric and magnetic fields

$\omega = 2\pi \nu$ = angular frequency

$\vec{k} = k_x \hat{i} + k_y \hat{j} + k_z \hat{k}$ = propagation vector along the direction of propagation of electromagnetic wave.

$\vec{r} = x \hat{i} + y \hat{j} + z \hat{k}$ = position vector of any point where $\vec{E}$ & $\vec{B}$ are defined in equations (10).

$$\text{Thus } \vec{k} \cdot \vec{r} = k_x x + k_y y + k_z z \text{ --- (11)}$$

The solutions given in equation (10) must simultaneously obey equations (6) - (9) i.e. Maxwell's equations. Only then (10) will be true solutions of Maxwell's wave equation.

Let E_x, E_y, E_z are components of electric field. TN ②

From ①, we can write ~~E_x~~ $E_x = E_{0x} e^{i(\omega t - \vec{k} \cdot \vec{r})}$
 $E_y = E_{0y} e^{i(\omega t - \vec{k} \cdot \vec{r})}$
 $E_z = E_{0z} e^{i(\omega t - \vec{k} \cdot \vec{r})}$ } — ②

Similarly components of magnetic field can be expressed as:-

$$\left. \begin{aligned} B_x &= B_{0x} e^{i(\omega t - \vec{k} \cdot \vec{r})} \\ B_y &= B_{0y} e^{i(\omega t - \vec{k} \cdot \vec{r})} \\ \text{and } B_z &= B_{0z} e^{i(\omega t - \vec{k} \cdot \vec{r})} \end{aligned} \right\} \text{--- ③}$$

~~Let us assert that~~ $\vec{E}$ will obey equation ⑥ if

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\text{or } \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \text{ --- ④}$$

From ②, we have

$$\begin{aligned} \frac{\partial E_x}{\partial x} &= \left[E_{0x} e^{i(\omega t - \vec{k} \cdot \vec{r})} \right] \times \frac{d}{dx} [i(\omega t - \vec{k} \cdot \vec{r})] \\ &= E_{0x} e^{i(\omega t - \vec{k} \cdot \vec{r})} \times (\cancel{i\omega t}) (i)(-k_x) \\ &= -i k_x E_{0x} e^{i(\omega t - \vec{k} \cdot \vec{r})} \\ &= -i k_x E_x \text{ --- ⑤} \end{aligned}$$

$$\text{Similarly } \frac{\partial E_y}{\partial y} = -i k_y E_y \text{ --- ⑥}$$

$$\text{and } \frac{\partial E_z}{\partial z} = -i k_z E_z \text{ --- ⑦}$$

Put values from ⑤, ⑥ and ⑦ in ④, we get

$$-i(k_x E_x + k_y E_y + k_z E_z) = 0$$

$$\Rightarrow \nabla \cdot (\vec{k} \cdot \vec{E}) = 0$$

$$\Rightarrow \vec{k} \cdot \vec{E} = 0$$

$$\text{or } \vec{E} \perp \vec{k}$$

Thus electric field is $\perp$ to direction of propagation.

Similarly we can show that $\vec{B}$ can satisfy equation (7) only if $\vec{B} \perp \vec{k}$ (i.e. $\vec{k} \cdot \vec{B} = 0$)

Thus magnetic field is also $\perp$ to direction of propagation.

To find relative orientation between $\vec{E}$ & $\vec{B}$, we must validate equations (8) and (9) by the solutions (10) of wave equation.

Now we can write

$$\vec{\nabla} \times \vec{E} = i \left[(\vec{\nabla} \times \vec{E})_x + j (\vec{\nabla} \times \vec{E})_y + k (\vec{\nabla} \times \vec{E})_z \right] \quad (18)$$

Where $(\vec{\nabla} \times \vec{E})_x$ etc. are various components of $\vec{\nabla} \times \vec{E}$.

$$\text{Now } (\vec{\nabla} \times \vec{E})_x = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \quad (19)$$

From (12) we have

$$\frac{\partial E_z}{\partial y} = \left[E_{0z} e^{i(\omega t - \vec{k} \cdot \vec{r})} \right]_x \frac{\partial}{\partial y} [i(\omega t - \vec{k} \cdot \vec{r})]$$

$$= \left[E_{0z} e^{i(\omega t - \vec{k} \cdot \vec{r})} \right] (i)(-k_y)$$

$$= -i k_y E_z$$

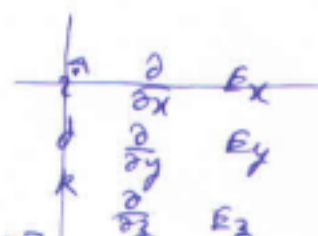
$$\text{Similarly } \frac{\partial E_y}{\partial z} = -i k_z E_y$$

$$\begin{aligned} \text{Put in (19), we get } (\vec{\nabla} \times \vec{E})_x &= -i [k_y E_z - k_z E_y] \\ &= -i (\vec{k} \times \vec{E})_x \quad (20) \end{aligned}$$

Similarly one can prove that

$$(\vec{\nabla} \times \vec{E})_y = -i (\vec{k} \times \vec{E})_y \quad (21)$$

$$\text{and } (\vec{\nabla} \times \vec{E})_z = -i (\vec{k} \times \vec{E})_z \quad (22)$$



Put values from (20), (21) and (22) in (18), we get

$$\vec{\nabla} \times \vec{E} = -i \left[\hat{i} (\vec{k} \times \vec{E})_x + \hat{j} (\vec{k} \times \vec{E})_y + \hat{k} (\vec{k} \times \vec{E})_z \right]$$

$$= -i (\vec{k} \times \vec{E}) \quad \text{--- (23)}$$

Again from (12), we can show that

$$\frac{\partial \vec{B}}{\partial t} = \frac{\partial}{\partial t} \vec{B}_0 e^{i(\omega t - \vec{k} \cdot \vec{r})}$$

$$= \left[\vec{B}_0 e^{i(\omega t - \vec{k} \cdot \vec{r})} \right] \times \frac{\partial}{\partial t} [i(\omega t - \vec{k} \cdot \vec{r})]$$

$$= i\omega \left[\vec{B}_0 e^{i(\omega t - \vec{k} \cdot \vec{r})} \right]$$

$$= i\omega \vec{B} \quad \text{--- (24)}$$

Put values from (23) and (24) in (8), we get

$$-i (\vec{k} \times \vec{E}) = -i\omega \vec{B}$$

$$\text{or } \boxed{\vec{B} = \frac{1}{\omega} (\vec{k} \times \vec{E})} \quad \text{--- (25)}$$

Equation (25) shows that $\vec{B}$ is $\perp$ to $\vec{E}$

Thus from validation of first three equations, we get that $\vec{E}$, $\vec{B}$ & $\vec{k}$ are mutually $\perp$ to each other. Hence electromagnetic waves are transverse in nature.

Note We can also satisfy equation (9) although it is not necessary now. However, it can come as additional problem in exam.

$$\text{We can see that } \vec{\nabla} \times \vec{B} = -i (\vec{k} \times \vec{B}) \quad \text{--- (26) (HW)}$$

$$\text{and } \frac{\partial \vec{E}}{\partial t} = i\omega \vec{E} \quad \text{--- (27) (HW)}$$

Put these values in (9), we get

$$-i (\vec{k} \times \vec{B}) = \mu_0 \epsilon_0 (i\omega \vec{E})$$

$$\therefore \boxed{\vec{E} = \frac{1}{\omega \mu_0 \epsilon_0} (\vec{k} \times \vec{B})} \quad \text{--- (28)}$$

This equation also tells that $\vec{E}$ is $\perp$ to $\vec{B}$.