RW-6386 **571101**

M.Phil. DEGREE EXAMINATION, DECEMBER 2010

Mathematics

COMMUTATIVE ALGEBRA

(CBCS—2008 onwards)

Time : 3 Hours Maximum : 75 Marks

Answer **all** questions. (5 × 15 = 75)

1. (a) Define the term exact sequence of R-modules. Prove that :

(i) for any exact sequence $O - N' \overset{f}{\to} N \overset{g}{\to} N'' \to O$ of R-modules, the induced sequence :

$O \to \mathrm{Hom}_R(M_1N') \overset{f^*}{\to} \mathrm{Hom}_R(M_1N) \overset{g^*}{\to} \mathrm{Hom}_R(M_1N'')$ is exact where M is a -Rmodule.

(ii) An R-module P is projective if and only if for any surjective homomorphism $g : M \to M''$ the induced homomorphism $g^* : \mathrm{Hom}_R(P_1M) \to \mathrm{Hom}_R(P_1M^r)$ is surjective.

(*Or*)

(b) For an R-module M show that the following are equivalent :

(i) A sequence $O \to N' \overset{f}{\to} N \overset{g}{\to} N'' \to O$ of R-modules is exact if and only if the tensored sequence :

$O \to M \otimes N' \overset{f}{\to} M \otimes N \overset{g}{\to} M \otimes N'' \to O$ is exact.

(ii) M is R-flat and for any R-module N, $M \underset{l}{\otimes} N=0$ implies N = 0.

(iii) M is R-flat and for any R-homomorphism $f : N' \to N,$ the induced map $f^* : M \otimes N' \to M \otimes N$ is zero implies that $f = 0$.

Prove that M is faithfully flat if and only if M is flat and for each maximal ideal m of R, $m^{M} \neq M$.

2. (a) Prove :

(i) Let R be a local ring. Any finitely generated projective R-module is free.

(ii) Let R be a local ring with maximal ideal in and M be a finitely presented R-module. If the canonical map $u_{\mathrm{M}} : m \underset{\mathrm{R}}{\otimes} \mathrm{M} \to \mathrm{M}$ given by $u_{\mathrm{M}} : (a \otimes x) = ax, a \in m, x \in \mathrm{M}$ is injective then M is free.

(*Or*)

(b) For a ring R, describe $\mathrm{R_S}$, for a multiplicatively closed subset S of R. If S is a multiplicatively closed set and $f : \mathrm{R} \to \mathrm{R_s}$ be the natural map given by $f(a) = (a/1)$, show that (i) every ideal of $\mathrm{R_s}$ is an extended ideal (ii) the prime ideals of $\mathrm{R_s}$ are in 1 - 1 correspondence with the prime ideals of R not intersecting S ; iii) f preserves the ideal operations of taking finite sums, products, intersections and radical.

3. (a) Let R be an Artinian ring. Show that (i) every prime ideal of R is maximal (ii) there are finitely many maximal ideals of R (iii) the nil radical is nilpotent.

(*Or*)

(b) Define the term "length of a module". State and prove the Jordan-Hölder theorem w.r.t. composition series and length.

4. (a) Prove :

(i) Let RCS be domains and S integral over R. The R is a field if and only if S is a field.

(ii) Let R, S be as in (i) for any prime ideal P of R, there exists a prime ideal P' of S such that $P' \cap R = P$.

(iii) Let S be an integral extension of R, $f : \mathrm{R} \to \Omega$ be a ring homomorphism of R into an algebraically closed field Ω. Then f can be extended to a ring homomorphism $g : \mathrm{S} \to \Omega$.

(*Or*)

(b) Let R be an integrally closed domain with quotent field and S normal extension of R with Galoi's group G = G (L/K). Show that :

(i) G is the group of R-automorphisms of S.

(ii) Two prime ideals P' and Q' of S lie over the same prime ideal of R if and only if there exists some $\sigma \in \mathrm{G}$ with $\sigma(\rho') = \mathrm{Q}'$.

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5. (a) Define the term valuation ring. Let p be a fixed prime and RCQ, where Q is the field of rationals is defined by :

$$R = \left\{ p^r \frac{m}{n} : r \geq 0, (n, p = 1, (n, p) > 1 \right\}$$. Show that R is a valuation ring. Prove (i) the ideals of a valuation ring are totally ordered by inclusion (ii) if the ideals of a domain V in the quotient field K are totally ordered by inclusion then V is a valuation ring of K.

(*Or*)

(b) Let R be a Noetherian local domain, with unique maximal ideal $m \neq 0$, and K be the quotient field of R. Show that the following are equivalent :

(i) R is a discrete valuation ring ;

(ii) R is a principal ideal domain ;

(iii) m is principal.

(iv) R is integrally closed and every non-zero prime ideal of R is maximal.

(v) every non-zero ideal of R is a power of m.

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RW-6387 **571102**

M.Phil. DEGREE EXAMINATION, DECEMBER 2010

Mathematics

FUNCTIONAL ANALYSIS

(CBCS–2008 onwards)

Time : 3 Hours Maximum : 75 Marks

Answer **all** questions. $(5 \times 15 = 75)$

1. (a) Let $\wedge$ be a linear functional on a Topological vector space X. Assume $\wedge x \neq 0$ for some $x \in X$. Prove that the following four properties are equivalent :

   (i) $\wedge$ is continuous.

   (ii) The null space $\mathcal{N}(\wedge)$ is closed.

   (iii) $\mathcal{N}(\wedge)$ is not dense in X.

   (iv) $\wedge$ is bounded in some neighbourhood $\vee$ of 0.

(*Or*)

(b) (i) Suppose K and C are subsets of a topological vector space X, K is compact, C is closed and $K \cap C = \phi$. Prove that 0 has a neighbourhood V such that $(K + V) \cap (C + V) = \phi$.

(ii) Prove that every locally compact topological vector space X has finite dimension.

(8 + 7)

2. (a) (i) State and prove Baire's theorem.

(ii) State and prove Banach-Steinhauss theorem.

(7 + 8)

(*Or*)

(b) State and prove Open mapping theorem.

3. (a) State and prove the Banach-Alaoglu theorem. State and prove any two applications of the above theorem.

(9 + 6)

(*Or*)

(b) Let M be a subspace of a real vector space X. Let $p : X \to R$ satisfy $p(x + y) \le p(x) + p(y)$ and $p(tx) = p(x)$ if $x \in X, y \in X$ $t \ge 0$. Suppose $f : M \to R$ is linear and $f(x) \le p(x)$ on M. Prove that there exists a linear $\wedge : X \to R$ such that $\wedge(x) = f(x), (x \in M)$ and $-p(-x) \le \wedge(x) \le p(x)$, $x \in X$.

4. (a) (i) Suppose X and Y are normed spaces. Prove that to each $T \in \mathcal{B}(X,Y)$ there corresponds a unique $T^* \in \mathcal{B}(Y^*, X^*)$ that satisfies $\langle Tx, y^* \rangle = \langle x, T^* y^* \rangle$ for all $x \in X$ and all $y^* \in Y^*$. Also prove that $\| T^* \| = \| T \|$.

(ii) Suppose X and Y are Banach spaces and $T \in \mathcal{B}(X, Y)$. Prove that T is compact if, and only if, T* is compact.

(*Or*)

(b) Suppose X is a Banach Space, $T \in \mathcal{B}(X)$ and T is compact. Then prove the following :

(i) If $\lambda \neq 0$, then the four numbers $\alpha = \dim \mathcal{N}(T - \lambda I)$, $\beta = \dim (X / \mathbb{R}(T - \lambda))$, $\alpha^* = \dim N(T^* - \lambda I)$ and $\beta^* = \dim \left(X^* / \mathbb{R}(T^* - \lambda I)\right)$ are equal and finite.

(ii) if $\lambda \neq 0$, and $\lambda \in \sigma(T)$ then λ is an eigen value of T and of T*.

(iii) $\sigma(T)$ is compact, atmost countable and has atmost one limit point namely 0.

5. (a) State and prove Bishop's theorem. Give an example to illustrate Bishop's theorem.

(*Or*)

(b) Let G be a compact group. Prove that there exists a unique regular Borel probability measure m which is left invariant, right invariant and satisfies the relation $\int_G f(x)\, dm(x) = \int_G f(x^{-1})\, dm(x)$ where $f \in C(G)$.

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RW-6388 **571203**

M.Phil. DEGREE EXAMINATION, DECEMBER 2010

Mathematics

MEASURE THEORY

(CBCS–2008 onwards)

Time : 3 Hours Maximum : 75 Marks

Answer **all** questions.

$(5 \times 15 = 75)$

1. (a) (i) Define measurable space and measurable sets. (5)

(ii) Suppose m is a σ-algebra in X, and Y is a topological space let f map X into Y

1. If Ω is the collection of all sets $E \subset Y$ such that $f^{-1}(E) \in m$, then Ω is a σ-algebra in Y

2. If f is measurable and E is a Borel set in Y then $f^{-1}(E) \in m$.

(10)

(*Or*)

(b) Suppose f and g $\in L'(\mu)$ and α and β are complex numbers

then prove that $\alpha f + \beta g \in L'(\mu)$ and

$$\int_X (\alpha f + \beta g)\, d\mu = \alpha \int_X f\, d\mu + \beta \int_X g\, d\mu .$$

(15)

2. (a) (i) Explain $C_c(X)$. (5)

(ii) State and prove Urysohn's lemma. (10)

(*Or*)

(b) (i) Define σ–finite measure. (3)

(ii) Let X be a locally compact Hausdorff space in which every open set is σ–compact. Let λ be any positive Borel measure on X such that $\lambda(K) < \infty$, $\forall$ K then prove that λ is regular.

(12)

3. (a) (i) If φ is convex on (a, b) then φ is continuous on (a, b) where φ is a real function on (a, b).

(5)

(ii) State and prove Jensen's inequality.

(10)

(*Or*)

(b) (i) Prove that $L^p(\mu)$ is a complete metric space, for $1 \le p \le \infty$ and for every positive measure μ

(10)

(ii) If $\{f_n\}$ is a Cauchy sequence in L^p (μ) with limit f then prove $\{f_n\}$ has a subsequence which converges pointwise almost everywhere to $f(x)$.

(5)

4\. (a) (i) Let μ be a complex measure on a σ-algebra $\mathcal{m}$ in X, then there is a measurable function h such that $|h(x)| = 1 \ \forall\, x \in X$ such that $d\mu = h\, d|\mu|$

(8)

(ii) State and prove the Hahn-decomposition theorem.

(7)

(*Or*)

(b) State and prove the Riesz representation theorem for a unique regular complex Borel measure.

5\. (a) (i) Prove that weak L^1 contains L^1

(5)

(ii) If $f \in L'(R^K)$, then prove that almost every $x \in R^K$ is a Lebesgue point of f.

(10)

(*Or*)

(b) If T (B (x, r) is Lebesgue measurable, the set V is open in R^K, $T : V \to R^K$ is continuous and T is differentiable at some point $x \in V$ then prove that.

$$\lim_{r \to 0} \frac{m\,(T\,(B\,(x, r)))}{m(B(x, r))} = \Delta\,(T'(x))$$

(15)

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