Algebra - 1 (System of Equations)

Algebra is distinct from Arithmetic because of the presence of variables. Thus while 65 means just that, 12x can take any value depending on the value that x, the variable, takes. A Variable, can assume different values, depending on conditions given, if any. Thus if x belongs to set of Real Numbers (usually written as $x \in R$), x can assume any real value where as if x belongs to set of Integers (usually written as $x \in Z$), x can assume only integral values.

Expressions:

Variables and constants combine to form Terms. Thus 9x or –6xy or x^2 are terms. The constant is called the coefficient of the corresponding variable. One or more terms together form an Algebraic Expression. Thus x + y or 3x – xy + 5y or $x^2 - 8x + 15$ are all examples of expressions. Remember expressions do not have solutions i.e. $x^2 - 8x + 15$ does not boil down to the solution that x = 3 or 5.

Degree of an algebraic expression:

The degree of an expression is same as the maximum value of the sum of the powers of the variables in any term. The degree of the expression has nothing to do with the number of variables in the algebraic expression. Thus all of the following $x^3 - 3x + 5$; $x + y + z + xyz$; $a^2b + b^2a$ has a degree of 3.

Expressions of degree 1 are called Linear expressions, usually called them a straight lines of degree 2 are called Quadratic expressions and of degree 3 are called Cubic expressions
An equation in one variable and with degree n can have at maximum n real roots. The number of real roots can be less than n also because few of them may be Imaginary.

Functions:

Functions are just a language of Algebra. We can refer to the expression $x^2 - 8x + 15$ as y i.e. $y = x^2 - 8x + 15$. Now what will be the numerical value of y? Obviously the value of y also keeps changing as the value of x changes i.e. the numerical value of y depends on the value assumed by x. If x = –2, y = 35, if x = 0, y = 15 and if x = 2, y = 3. This dependency is expressed as y = f(x) and is read as y is a function of x. In $y = f(x) = x^2 - 8x + 15$, the right most side explicitly states how y is related to x.

Thus functions is nothing new and even the quadratic expression that we are so used to can be expressed in the language of functions. In this example, if we need to find the value of y when x = 5, we simply mean to evaluate f(5). To evaluate f(5), we just substitute the value of x as 5 in the expression $x^2 - 8x + 15$ and find that f(5) = 0

Equations:

Only when an expression is equated to another expression (even a constant is an expression), do we get an equation. Thus $x^2 - 8x + 15$ is not an equation but $x^2 - 8x + 15 = 0$ is.

Roots or solution set to an equation are those values of the variables which satisfy the equation.

Thus, as seen from the table and graph of the expression $x^2 - 8x + 15$, the expression takes a value equal to 0 only when x = 3 or when x = 5.

Thus 3, 5 is the solution set or the roots of the equation.

Arithmetic operations on Algebraic expressions:

Addition & Subtraction:

The addition or subtraction of algebraic expression follows the same rules as the arithmetic addition/subtraction. We can add the coefficients of the terms with the same degree and in the same variables only.

For example,

$(2X + 3Y) + (5X + 4Y) = 7X + 7Y$

$= 7(X + Y)$

For example,

$(3X^2 + 4XY + 7Z^2) - (5XY - 4Z^2)$

$= 3X^2 + 4XY + 7Z^2 - 5XY + 4Z^2$

$= 3X^2 - XY + 11Z^2$

Multiplication:

The multiplication of any 2 algebraic expressions follows the distributive property of multiplication and the index rules.

For example,

$(2X + 3Y)(5X + 4Y)$

$= (2X)(5X + 4Y) + (3Y)(5X + 4Y)$

$= 10X^2 + 8XY + 15XY + 12Y^2$

$= 10X^2 + 23XY + 12Y^2$

Division:

Just as in the normal arithmetic, division and multiplication of algebraic expressions follow similar rules.

For example, divide the expression
f(x) = $3x^2 + 4x + 3$ by (x – 2).

Solution:

$$
\begin{array}{r|l}
 & 3x + 10 \\
\hline
x - 2 & 3x^2 + 4x + 3 \\
 & \underline{3x^2 - 6x} \\
 & 10x + 3 \\
 & \underline{10x - 20} \\
 & 23
\end{array}
$$

Just as in normal arithmetic division, the quotient of the division process is 3x + 5 and the remainder is 23.

Factorisation:

The expression x^2 – 5x + 6 can be written as product of (x – 2) and (x – 3). This process of writing an expression as a product of different expressions is called as Factorisation. The expressions (x – 2) and (x – 3) are called factors of x^2 – 5x + 6. Just as in Arithmetic, in Algebra also a factor of an expression can completely divide the expression i.e. the remainder is zero. The process of factorization is useful to find the roots of any equation.

Thus $x^2 - 5x + 6 = 0 \Rightarrow (x - 2)(x - 3) = 0$. The product of two numbers can be 0 only if atleast one of them is zero. Thus the LHS of the above equation will become 0 if x = 2 or 3 and the equation will be satisfied.

Remainder Theorem:

To identify whether a given expression is a factor of another expression, we can take help of Remainder Theorem.

According to the remainder theorem, when any expression f(x) is divided by (x – a), the remainder is f(a). (a is any constant in this example).
Thus when the expression, $x^3 + x^2 + 4$ is divided by x + 1, the remainder is $(-1)^3 + (-1)^2 + 4$ i.e. 4.

Factor Theorem:

We can also use the remainder theorem or more specifically the Factor theorem to identify if an expression is a factor of another expression. As already seen, a expression is said to be a factor of another expression only when the remainder is 0 when the latter is divided by the former.

Thus (x – a) is a factor of f(x) if and only if f(a) = 0.

Factor theorem also helps us in factorising higher degree equations. Consider the equation
$f(x) = x^3 + 6x^2 - 19x - 24 = 0$.

By hit and trial (basically substituting values of x as –2, –1, 1 or 2), we see that $f(-1) = (-1)^3 + 6(-1)^2 - 19(-1) - 24 = -1 + 6 + 19 - 24 = 0$.
Thus, we can deduce that (x + 1) is a factor of f(x). i.e.
$x^3 + 6x^2 - 19x - 24 = (x + 1)$ ´ g(x), where g(x) is another algebraic expression in variable x.
Using common sense we can gather that g(x) is a quadratic expression. Why? Only $(x + 1)(ax^2 + bx + c)$ will have a term in x^3, x^2, x and a constant.

Thus, $x^3 + 6x^2 - 19x - 24 = (x + 1)(ax^2 + bx + c)$.

By visual check we can ascertain that a = 1 and c = –24. How?

Equating coefficient of x^3 on RHS and LHS, we get a = 1 (on the RHS only $x \times ax^2$ will result in term in x^3).
Equating the constant terms of RHS and LHS, we get c = –24 (only 1 × c will result in a constant on RHS).

Thus now we have $x^3 + 6x^2 - 19x - 24$
$= (x + 1)(x^2 + bx - 24)$.
To find b, equate the coefficient of either x^2 or of x of the two sides of the equation. Equating the coefficient of x, we have –19 = b – 24 giving us b = 5 (when the LHS is expanded, the only terms in x are bx – 24x).

Thus finally, $x^3 + 6x^2 - 19x - 24$
$= (x + 1)(x^2 + 5x - 24)$.

Consistency of Equations:

When a system of equations has at least one solution, we say that the system is consistent. When it has no solution we say that the system is inconsistent. Let the system of equation be
$a_1 x + b_1 y + c_1 = 0$

$a_2 x + b_2 y + c_2 = 0$

$$\Rightarrow \frac{x}{b_1 c_2 - b_2 c_1} = \frac{y}{c_1 a_2 - a_1 c_2} = \frac{1}{a_1 b_2 - a_2 b_1}$$

and $a_1 b_2 \neq a_2 b_1$

(i) If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, the equations are consistent with unique solution.

(ii) If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, the equations are consistent with infinite solutions.

(iii) $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, the equations are inconsistent i.e, no solution.

Solved Examples

1. If (x + 2)(x - a) = px^2 + qx + 8, what are the values of the constants a, p and q?

Solution: (x + 2)(x - a) = px^2 + qx + 8

$\Rightarrow x^2 + 2x - ax - 2a = px^2 + qx + 8$

$\Rightarrow x^2 + (2 - a)x - 2a = px^2 + qx + 8$

Equating the coefficients of x^2, x and constant terms on both sides

p = 1; q = 2 - a

-2a = 8

Solving, we get a = -4, p = 1, q = 6

2. (x +1)(2x - 2)(3x + 3) = $ax^3 + bx^2 + cx + d$ What are the values of a, b, c and d?

Solution: (x + 1)(2x - 2)(3x + 3) = $ax^3 + bx^2 + cx + d$

6(x + 1)(x - 1)(x + 1) = $ax^3 + bx^2 + cx + d$

$6(x^2 - 1)(x + 1) = ax^3 + bx^2 + cx + d$

$\Rightarrow 6(x^3 + x^2 - x - 1) = ax^3 + bx^2 + cx + d$

$\Rightarrow 6x^3 + 6x^2 - 6x - 6 = ax^3 + bx^2 + cx + d$

Equating the coefficients of like learns on both the sides, a = 6, b = 6, c = -6 and d = -6.

3. Find for what value(s) of k would there be an unique solution for the given set of questions.

2x - 3y = 1 and kx + 5y = 7

Solution: If two equations ax + by = M and cx + dy = N have a unique solution, then $\frac{a}{c} \neq \frac{b}{d}$

So in the above problem, $k \neq \frac{-10}{3}$

4. Find the value(s) of k for which there is no solution for the given set of equations.

2x - ky = -3 and 3x + 2y = 1

Solution: 3x + 2y = 1

If there are no set of solutions, then $\frac{2}{-k} = \frac{3}{2}$ or $k = -\frac{4}{3}$

For k = $-\frac{4}{3}$, the two lines in the coordinate planes are parallel to each other.

5. Find the value of k for which there are infinite solutions for the given set of equations.

5x + 2y = k and 10x + 4y = 3

Solution: For the two sets of equation to have infinite solutions, we have $\frac{5}{10} = \frac{2}{4} = \frac{k}{3}$

. Hence, k = $\frac{3}{2}$

6. What is the solution of the following simultaneous equations?

x + y + z = 6, x + 2y + 3z = 14 and x + 3y + z = 10

Solution: x + y + z = 6 .... (i)

x + 2y + 3z = 14 .... (ii)

x + 3y + z = 10

From (i), we get z = 6 - x - y

Substitute it in (ii) and (iii)

We have from (ii)

x + 2y + 18 - 3x - 3y = 14 or 2x + y = 4

Similarily, from (iii)

x + 3y + 6 - x - y = 10 or 2y = 4 or y = 2

On solving, we get y = 2, x = 1, z = 3.

7. The number of roots common between the two equations $x^3 + 3x^2 + 4x + 5 = 0$ and $x^3 + 2x^2 + 7x + 3 = 0$ is

Solution: Here f(x) = $x^3 + 3x^2 + 4x + 5 = 0$ and g(x) = $x^3 + 2x^2 + 7x + 3 = 0$

To find the common roots we have to solve the equation f(x) - g(x) = 0

i.e. $(x^3 + 3x^2 + 4x + 5) - (x^3 + 2x^2 + 7x + 3) = 0$

x^2 - 3x + 2 = 0

x = 2, x = 1

These are the points of intersection f(x) and g(x). Whether these points are also the common root will have to be checked by putting these vales in f(x) = 0 and g(x) = 0.

For x = 2;

f(2) = g(2) = 43 $\neq$ 0. Hence 2 is not a common root but only a point of intersection.

For x = 1;

f(1) = g(1) = 13 $\neq$ 0. Again 1 is not a common root but a point of intersection. Hence, we find that the two equations do not have any common root between them.

9. For what values of k is the system of equations independent? The equations are as follows:

kx + 5y = 3; 3x + 4y = 9

Solution: For the equations to be independent, conditions is $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

$\frac{k}{3} \neq \frac{5}{4}$ or k $\neq \frac{15}{4}$

10. Gopi gives Rs. 90 plus one turban as salary to his servant for one year. The servant leaves after 9 months and receives Rs. 65 and the turban. Find the price of the turban.

Solution: Let the price of turban be x.

Thus, for one year the salary = (90 + x)

For 9 months he should earn $\frac{3}{4}$ (90 + x).

Now he gets one turban and Rs. 65.

Thus, $\frac{3}{4}$ (90 + x) = 65 + x or 270 + 3x = 260 + 4x

or x = 10

11. Ranjit went to the market with Rs. 100. If he buys 3 pens and 6 pencils he uses up all his money. On the other hand, if he buys 3 pencils and 6 pens he would fall short by 20%. If he wants to buy equal number of pens and pencils, how many pencils can he buy?

Solution: Let price of a pen = x and pencil = y

3x + 6y = 100 ....(i)

and 6x + 3y = 125 ....(ii)

Adding (i) and (ii)

9(x + y) = 225

x + y = 25

multiply by 4

4x + 4y = 100

He can by 4 pens and 4 pencils in Rs. 100.

12. $x^3 - ax^2 + 3x - b = 0$ has one factor as (x - 2). When the equation is divided by (x + 1), it leaves a remainder -12. What are the values of 'a' and 'b'?

Solution: $x^3 - ax^2 + 3x - b = 0$ has one factor as (x - 2). So for x = 2 the equation will satisfy or we can say if we substitute the value of x in the equation as 2 it will result into 0.

8 - 4a + 6 - b = 0 or 4a + b = 14 ....(i)

Now if we say that by dividing the equation by (x + 1) we get the remainder as -12 then if we put x = -1 in the equation then it will result in -12.

-1 - a - 3 - b = -12 or a + b = 8 ....(ii)

By solving (I) and (II) a = 2 and b = 6.

13. For the given equation $x^9 + 5x^8 - x^3 + 7x + 2 = 0,$ how many maximum real roots are possible?

Solution: f(x) = $x^9 + 5x^8 - x^3 + 7x + 2 = 0$

In f(x), there are 2 changes of sign. So, there are two positive roots.

f(-x) = - $x^9 + 5x^8 - x^3 + 7x + 2 = 0$

In f(-x), there are 3 changes of sign. So, there are three negative roots.

So, in all there are 5 real roots possible (2 positive and 3 negative) and as degree of the given equation is 9, there are total 9 roots. So, remaining 4 roots will be imaginary.

Hence answer option is (b).

14. If $x^3 - 6x^2 + 11x - 6 = 0$ and $x^3 + 3x^2 - 6x - 8 = 0$ have one common root between them. What is the value of that root?

Solution: There are two equations,

$x^3 - 6x^2 + 11x - 6 = 0$ ....(i)

$x^3 + 3x^2 - 6x - 8 = 0$ ....(ii)

So both (i) and (ii) have one root in common and we know that at the root the value of both the equations is 0. Suppose that common root is x then,

$x^3 - 6x^2 + 11x - 6 = x^3 + 3x^2 - 6x - 8 \Rightarrow 9x^2 - 17x - 2 = 0$

(x - 2)(9x + 1) = 0 or x = 2 or -1/9. As by applying the Descartes rule in (i) we can see it cannot have any negative root so x = 2. Hence answer is option (e).

Why are we getting two values x = 2 and x = -1/9?

x = 2 and x = -1/9 are the pint of intersections of these two equations out of which x = 2 is the point of intersection at x axis therefore this point (x = 2) is the common root as well.