Probability

The concept of probability is very important for managers to make judicious decisions. The applications of this concept can be found in numerous areas like insurance, stock markets, medical diagnostics, Quality testing to name a few. Learning this concept in a systematic way improves our decision making abilities apart from scoring good marks in exams. Read every definition and try to understand even though your instinct easily get the answer with out these concepts. This understanding helps you get solutions for some tough problems later.

Important concepts and formulas:

Experiment: Phenomenon where outcomes are un-certain. Single throw of a six-sided die is an experiment
Sample space:** Set of all outcomes of the experiment. {1,2, 3,4, 5, 6} are the total outcomes of rolling a six faced dice.
Event:** A collection of outcomes; a subset of S **Example:** An event of occurrence of even number is {2,4,6}
Note: You must remember these two starred definitions

Equally likely Events: If each outcome of an experiment has equal chance of occurrence then we say, the outcomes are equally likely. Getting heads or tail are equally likely.
Exhaustive Events: A set of events is jointly or collectively exhaustive if at least one of the events must occur. For example, when rolling a six-sided die, the outcomes 1, 2, 3, 4, 5, and 6 are collectively exhaustive, because they encompass the entire range of possible outcomes.

Another way to describe collectively exhaustive events, is that their union must cover all the events within the entire sample space. For example, events A and B are said to be collectively exhaustive if $A \cup B = S$

Probability: The ratio between the possible outcomes of an event to the total possible outcomes of an experiment.

$$\text{Probability} = \frac{\text{Possible outcomes of an event}}{\text{Total possible outcomes of an experiment}}$$

Types of probabilities:

Union probability: The probability of either of the two events is called as union probability.
If A and B are two events, then Union probability is $P(A \cup B)$
Example: What is the probability of getting a king or a diamond from a well shuffled pack of 52 cards.
Say, A is the event of drawing a king and B is the event of drawing a diamond and $A \cap B$ is drawing a diamond king and $A \cup B$ is drawing either diamond or king.
we know that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$

Intersection probability: The probability of the occurrence of both the events is called intersection probability. If A and B are two events, then intersection probability is $P(A \cap B)$

$P(A \cap B) = P(A) \times P(B|A) = P(B) \times P(A|B)$

Example: What is the probability of getting a black king and red card from a well shuffled pack of 52 cards with replacement.

We know that a suit contains four kings and 26 red cards. Say, P(A) is drawing a king and P(B|A) is drawing a red card given we have drawn a king in the first event. So drawing a king from a suit $\Rightarrow P(A)=\frac{4}{52} = \frac{1}{13}$

But this first event does not have any affect on the second event. so P(B|A) = P(B)

$\Rightarrow P(B) = P(B|A)=\frac{26}{52} = \frac{1}{2}$

$\Rightarrow P(A \cap B) = P(A) \times P(B|A)=\frac{1}{13} \times \frac{1}{2} = \frac{1}{26}$

Complement probability: The complement of a sample set A is denoted by A^C. A^C contains all the elements that doesn't contain in set A. So $P(A) + P(A^C) = 1$

Example: What is the probability of getting an even number when rolling a dice.

Let A is the event of getting an odd number. So A = {1,3,5}, then A^c = {2,4,6}

Then $P(A^c)$ = 1/3

Conditional probability: The probability of an event A, given event B has already happened is given by conditional probability.

$P(A|B) = \frac{P(A \cap B)}{P(B)}$ or $P(B|A) = \frac{P(A \cap B)}{P(A)}$

$P(A|B)$ means, The probability of A, if event B has already happend.

$P(B|A)$ means, The probability of B, if event A has already happend.

From above $P(A \cap B) = P(A) \times P(B|A) = P(B) \times P(A|B)$

Example:

On rolling a dice, what is the probability of getting 1, given that the number is odd.

Let the probability of getting 1 is P(A), the probability of getting an odd number is P(B) and the probability of getting 1 given the number is odd $P(A|B)$. We know that A = {1}, B= {1,3,5} So $A \cap B$ = {1} $\Rightarrow P(A \cap B) = \frac{1}{6}$

$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/6}{1/2} = \frac{1}{3}$

Mutually exclusive and Independent Events:

Mutually exclusive events: If two events are mutually exclusive, the probability of occurrence of the two events at the same time is zero. Or the happenning of one event prevents all other events from happenning. For example, the probability of getting both 1 and 6 with a single rolling of dice is impossible. So occurrence of 1 and occurence of 6 both are mutually exclusive events. (Have you seen, Exclusive livecast of the cricket match prevents other channels broadcasting the same cricket match!!)

If two events are mutually exclusive, $P(A \cap B) = 0$; and $P(A \cup B) = P(A) + P(B)$

Independent events: It is a common practice to consider independent experiments and independent events are same. For example, the outcomes are independent when we roll a dice for two times. The result of the first experiment may not effect the outcome of the second experiment.

From the definition, we know that an event is a subset of the sample space S. Say A and B are two events.

If two events are independent then $P(A) = P(A|B)$ or $P(B) = P(B|A)$

From conditional probability formula $P(A|B) = \frac{P(A \cap B)}{P(B)}$

But $P(A) = P(A|B)$

So $P(A \cap B) = P(A) \times P(B)$

$P(A \cup B) = 1 - [(1 - P(A)) \times (1 - P(B))]$

Example: An event A is defined as getting either 1 or 2 on rolling a dice. If An event B is getting an odd number on rolling the dice, Then what is the probaility of event A given B has occurred.

A = {1, 2}

B = {1, 3, 5}

$A \cap B$ = {1}

P(B) = 3/ 6 = 1/2

$P(A \cap B)$ = 1/6

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/6}{3/6} = \frac{1}{3}$$

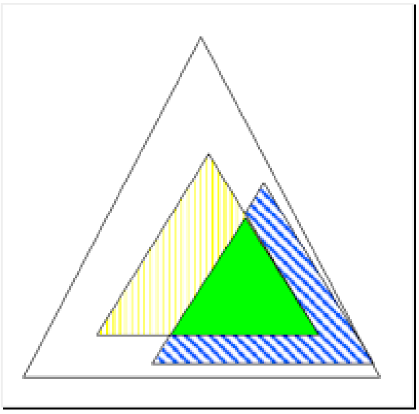
But P(A) = 2/6 = 1/3

This is interesting. Events A and B are independent as the additional information that B is odd does not change the probability of A.

If two events are independent then $P(A \cap B) = P(A) \times P(B)$

$$\Rightarrow P(A \cap B) = \frac{2}{6} \times \frac{3}{6} = \frac{1}{6}$$

What is the probability of yellow triangle, given blue has happend?



Probability of yellow = P(yellow) = $\frac{\text{Yellow}}{\text{white}}$

Probability of Blue = P(Blue) = $\frac{\text{Blue}}{\text{White}}$

Probability of yellow and Blue = $P(\text{Yellow} \cap \text{Blue}) = \frac{\text{Green}}{\text{white}}$

From conditional probability formula

$P(\text{Yellow}|\text{Blue}) = \frac{P(\text{Yellow} \cap \text{Blue})}{P(\text{Blue})} = \frac{\text{Green}}{\text{Blue}}$

If P(yellow) and P(Yellow|blue) are to be same, then the proportion of yellow to white should equal to Green to Blue.

Relation between Mutually exclusive and independent Events:

Firstly, never bother about the two above terms. But remember *if two events are mutually exclusive then they are not independent. and if two events are independent they cannot be mutually exclusive.*

We know that $P(A|B) = \frac{P(A \cap B)}{P(B)}$

If two events mutually exclusive $P(A \cap B) = 0$ so $P(A|B) = \frac{0}{P(B)} = 0$ and For A and B are independent P(A|B) = P(A)

and P(A).P(B) = 0, which is not possible.

If two events are independent $P(A).P(B) = P(A \cap B)$ so there must be common event.

Practice Problems (Level - 1)

1. If a fair dice is thrown twice, what is the probability of getting at least one six?

We can apply complement probability rule to solve this problem easily. Assumed P(A) is the probability of getting atleast one six. then $P(A^C)$ is not getting a single six. A = {6} and $P(A^C)$ = {1, 2, 3, 4, , 5} and S = {1, 2, 3, 4, 5, 6}. On throwing a dice two times the events are independent. So the probability of not getting a six in the first throw is 5/6 and in the second throw is 5/6 as these two events are independent.

So $P(A^C) = \frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$

we know that P(A) + $P(A^C)$ = 1 so P(A) = 1- 25/36 = 11/36

Alternate method:

We can count the points in the event A as {(1,6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5)} = 11

S = 36

P(A) = 11/36

2. There are two positive integers a and b. What is the probability that a + b is odd?

The sample space S for adding two numbers is Even + Even, Even + Odd, Odd + Even, Odd + Odd

If a + b has to be odd, then Even + Odd, Odd + Even are the points in the event. So Probability = 2/4 = 1/2

3. A number is selected at random from the number 1, 2, ...,50. What is the probability that the number is multiple of either 6 or 9?

Say the event A is getting multiples of 6, and the event B is getting multiples of 9.

A = {6, 12, 18, ...48} = 8 $\Rightarrow P(A) = \frac{8}{50}$

B = {9, 18, ...45} = 5 $\Rightarrow P(B) = \frac{5}{50}$

But events A and B have some common points 18, 36. This can be calculated by taking LCM of 6, 9 and finding the multiples of the LCM below 50.

So $A \cap B$ = 2 $\Rightarrow P(A \cap B) = \frac{2}{50}$

From union probability formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow P(A \cup B) = \frac{8}{50} + \frac{5}{50} - \frac{2}{50} = \frac{11}{50}$

4. There are 6 men and 7 women in a committee, 2 people need to be selected as representatives of this committee. What is the probability, that out of these 2 people, 1 is a man and another is a woman?

Number of ways of selecting 2 persons out of 13 persons is $13C_2$

Number of ways of selecting 1 man out of 6 is $6C_1$

Number of ways of selecting 1 woman out of 6 is $7C_1$

As the above two events are independent, the number of ways of selecting one man and one women = $6C_1 \times 7C_1$

So the probability of selecting one man and one women from a group of 13 = $\frac{6C_1 \times 7C_1}{13C_2}$

5. Anil's age is currently between 15-25 years (exclusive of 15, 25). What is the probability that after 15 years, he is 37 years or above in age?

Anil's age after 15 years is in between 30 and 40 exclusive of 30, 40. Number of points in the sample space are {31, 32, ...39}= 9 and number of points in the event are {37, 38, 39} = 3

So required probability = 3/9 = 1/3

6. The probability that A, B and C will live for more than 60 years is $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ respectively. What is the probability that atleast 1 of them will be alive after 60 yrs of age?

We use complement probability to solve this question easily. Let A is the event of atleast one of them will be alive.

Then A^C is the probability that nobody lives.

As the three persons lifes are independent of each other, we use product rule to get total probability

The probability of A dies = $1 - \frac{1}{2} = \frac{1}{2}$

The probability of B dies = $1 - \frac{1}{3} = \frac{2}{3}$

The probability of C dies = $1 - \frac{1}{4} = \frac{3}{4}$

So the probability that all of them die = $\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{4}$

The probability that atleast one person will be alive = 1 - $\frac{1}{4}$ = $\frac{3}{4}$

7. In the last question, what is the probability that atleast 2 are alive?

Any 2 are alive = (A + B) alive and C dead OR (B + C) alive and A dead OR (A + C) alive and B dead. Let $\overline{A}$ is the probability A dies. The the probability of atleast two will be alive will be given as

$A^1 \cap B \cap C + A \cap B^1 \cap C + A \cap B \cap C^1 + A \cap B \cap C$

From above we have already calculated the probabilities of A, B, C not alive and the events are independent.

So by substituting in the above equation and by applying the product rule $\frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} + \frac{1}{2} \times \frac{2}{3} \times$

$\frac{1}{4} + \frac{1}{2} \times \frac{1}{3} \times \frac{3}{4} + \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} = \frac{1}{4}$

8. What is the probability that when 3 cards are pulled from a pack of cards, without replacement, that we get 1 King, 1 Queen and 1 Jack?

Here the order of picking up is not specified and the events are dependent. Total possibility of picking up the given cards are

1st Pick	**2nd Pick**	**3rd Pick**
K	Q	J
K	J	Q
Q	K	J
Q	J	K
J	Q	K
J	K	Q

The probability of Picking up a king in the first attempt = 4/52

The probability of Picking up a queen in the second attempt = 4/51

The probability of Picking up a Jack in the third attempt = 4/50

and total probability = $(\frac{4}{52} \times \frac{4}{51} \times \frac{4}{50}) \times 6 = \frac{16}{5525}$

Conditional Probability:

9. A bag contains 5 black and 3 red balls. A ball is taken out of the bag and is not replaced.

(a) What is the probability of drawing a red ball in the second draw if the ball in the first draw is red?

(b) What is the probability of drawing a black ball in the second draw if the ball in the first draw is red?

Sol:

A: Drawing red in 1st draw

B: Drawing red in 2nd draw

C: Drawing black in 2nd draw

(a) If one ball is drawn already and it is a red ball, the remaining balls are 7 and only 2 red balls left in the bag. So $P(B|A) = \frac{2}{7}$

(b) Similarly $P(C|A) = \frac{5}{7}$

Alternatively:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{n(A \cap B)}{n(A)}$$

n(A) =RR + RB = ${}^3C_1.{}^2C_1 + {}^3C_1.{}^5C_1 = 21$

$n(A \cap B)$ = RR = ${}^3C_1 \times {}^2C_1$ = 6

a. $P(B|A) = \frac{n(A \cap B)}{n(A)} = \frac{6}{21} = \frac{2}{7}$

b. $P(C|A) = \frac{n(A \cap C)}{n(A)} = \frac{{}^3C_1.{}^5C_1}{21} = \frac{15}{21} = \frac{5}{7}$

10. A coin is flipped twice. What is the probability that both flips result in heads given that first flip does?

A = First flip lands heads

B = Second flips is head

$$P(B|A) = \frac{n(A \cap B)}{n(A)} = \frac{P(H,H)}{P[(H,H),(H,T)]} = \frac{1}{2}$$

11. A family has two children. What is the probability that both boys are boys given that one of the children is boy?

Sol: A = One of the children is boy = (BB, BG, GB)

B = Second one is boy = (BB)

Probability = 1/3

12. Ten cards numbered 1 to 10 are placed in a box. And one card is drawn randomly. If is it is known that the drawn card is more than 3, what is the probability that it is an even number?

A = Drawn card is more than 3 = (4, 5, 6, 7, 8, 9, 10) $\Rightarrow$ n(A) = 7

B = It is an even number = (2, 4, 6, 8, 10)

P(B | A) = 4 / 7

Multiplication Theorem:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \Rightarrow P(A \cap B) = P(B|A) \times P(A)$$
$$P(A|B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(A|B) \times P(B)$$
$$P(A \cap B) = P(A|B) \times P(B) = P(B|A) \times P(A)$$

13. Suppose that an Urn contains 8 red balls and 4 white balls. We draw 2 balls without replacement. What is the probability that both balls are red?

Sol: $P(R_1 \cap R_2) = \frac{{}^8C_2}{{}^{12}C_2} = \frac{14}{33}$

Alternatively: $P(R_1 \cap R_2) = P(R_1) \times P(R_2|R_1) = \frac{8}{12} \times \frac{7}{11} = \frac{14}{33}$

14. A bag contains 15 items, of which 4 are defective. The items are selected at random, not put back. What is the probability that 10th one examined is last defective?

In this problem, we should get 3 defects in the first 9 examination and last defect one should be 10th one.

A = {Exactly 3 defects in 9 examined}

B = { 10th one is defective}