Progressions - 1

We have already studied about the progressions in our earlier years. Though, the problems given in the competitive exams are a bit twisted and need careful applications. Study the formulas and solved examples so that you can easily answer the questions given in this chapter.

Arithmetic progression (Key points)

A series in which each term (except first term) differs from its preceeding term by a fixed quantity is called an Arithmetic progression (A.P.) and the fixed quantity is called the common difference. If a is the first term and d is the common difference of an A.P. then that A.P is a + (a+d) + (a+2d)+..... . If the same quantity is added (multiplied) to each term of an A.P. then the resulting series is also an A.P.

n th term of an A.P T_n = a+ (n -1) d.

Sum to first n terms of an A.P. = $S_n = \frac{n}{2}[2a+(n-1)d]$

If we know the last term of a series then = $\frac{n}{2}[a+l]$ where l is the last term.

Arithmetic Mean:

If a,x,b are in A.P. then x is called Arithmetic mean (A.M) between a and b. The arithmetic mean between a and b is $\frac{a+b}{2}$. If a, x_1 , x_2 .... x_n , b are in A.P. then x_1 , x_2 .... x_n are called n-Arithmetic means (A.M's) between a and b.

If x_1 , x_2 .... x_n are n-A.M's between a, b; and d=$\frac{(b-a)}{(n+1)}$, then x_1 =a+d,x_2 =a+2d,......x_n = a+nd and $x_1+x_2+.....x_n$ = .$\frac{n(a+b)}{2}$

If three numbers are in A.P., then they can be taken as a – d, a, a+d.

If four numbers are in A.P., then they can be taken as a – 3d, a – d, a + d, a + 3d.

If five numbers are in A.P., then they can be taken as a – 2d, a – d, a, a + d, a + 2d.

Geometric progression (Key points)

A series in which each term (except first term) is obtained by multiplying the preceeding term by a fixed quantity is called a Geometric progression (G.P) and the fixed quantity is called the common ratio. If a is the first term and r is the common ratio of a G.P. then that G.P is $a + ar + ar^2 + \ldots\ldots$. If every term of a G.P. is multiplied by a fixed real number, then the resulting series is also a G.P. If every term of a G.P is raised to the same power, then the resulting series is also a G.P. The receiprocals of the terms of a G.P. is also a G.P.

nth term of a G.P is ar^{n-1}

The sum of first n terms of a G.P = $\frac{a(r^n - 1)}{r - 1}$ if r>1; = $\frac{a(1 - r^n)}{1 - r}$ if r<1

The sum of infinite G.P. is $\frac{a}{1 - r}$ when |r| <1

Geometric Mean:

If three positive numbers a,x,b are in G.P. then x is called the Geometric Mean (G.M) between a and b. The G.M. between two positive numbers a and b is $\sqrt{a.b}$

If the positive numbers a, x_1 , x_2 .... x_n , b are in G.P., then the numbers x_1 , x_2 .... x_n are called n-Geometric means between a and b.

If x_1 , x_2 .... x_n are n-G.M's between a,b; and r= $\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$ then x_1 = ar, x_2 = $a.r^2$, ... x_n = $a.r^n$ and x_1 x x_2 x....... x_n = $(ab)^{\frac{n}{2}}$

If three numbers are in G.P., then they can be taken $\frac{a}{r}$, a, ar.

If four numbers are in G.P., then they can be taken as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$

If five number are in G.P., then they can be taken as $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$

Harmonic progression (Key points)

If the reciprocals of the terms of a series form An A.P., then the series is called a Harmonic progression (H.P.). If a,x,b are in H.P., then x is called Harmonic Mean between a and b. The harmonic mean between two non zero numbers a and b is $\frac{2ab}{a + b}$.

If a, x_1 , x_2 .... x_n ,b (each is non zero) are in H.P., then x_1 , x_2 .... x_n are called n- Harmonic Means (H.M's) between a and b.

If n-H.M's between a and b, then $x_1 = \frac{ab(n+1)}{b(n+1)+(a-b)}$, $x_2 = \frac{ab(n+1)}{b(n+1)+2(a-b)}$, ...

$x_n = \frac{ab(n+1)}{b(n+1)+n(a-b)}$

Important: If A, G, H are the arithmetic mean, geometric mean, harmonic mean between two positive numbers then $A \geq G \geq H$ and $G^2 = AH$

The sum of first n natural numbers $\sum n = \frac{n(n+1)}{2}$

The sum of first n natural numbers $\sum n^2 = \frac{n(n+1)(2n+1)}{6}$

The sum of cubes of first n natural numbers $\sum n^3 = \frac{n^2(n+1)^2}{4}$

The sum of first n even integers = n(n + 1).

The sum of first n odd integers = n^2.

Arithmetico Geometric progression (Key points)

A series in which each term is the product of two factors so that the first factor is a term of an A.P and the second factor is a corresponding term of a G.P is called Arithmetic Geometric series.

In an AGP series a + (a + d)r + (a + 2d)r^2 +..... nth term = {a + (n - 1) d} r^{n-1} .

Sum of first n terms = $\frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{\{a+(n-1)d\}r^n}{(1-r)}$

Sum of infinite terms = $\frac{a}{1-r} + \frac{dr}{(1-r)^2}$ if |r|<1

Solved Examples (Level - 1)

Arithmetic Progression:

1. 15th term of 3+5+7+ .... is given series is in A.P.

We know that n th term of an A.P T_n = a+ (n -1) d.

First term a = 3; Common difference d = 2, 15th term = a + 14d = 3 + 14(2) = 31

2. How many terms are added in 24+20+16+ .... 10 make the sum 72.

We know that sum to first n terms of an A.P. = S_n = $\frac{n}{2}$ [2a + (n – l) d]

Let the sum of n terms be 72, a = 24, d = -4

$\frac{n}{2}(2a+(n-1)d) = 72 \Rightarrow n(48+(n-1)(-4)) = 144$
$\Rightarrow n(-4n+52) = 144 \Rightarrow n(n-13) = -36$
$\Rightarrow n^2 - 13n + 36 = 0 \Rightarrow n = 4\&9$

n= 4 is ruled out and n=9 is the required answer.

3. If the sum of three numbers in A.P. is 15 and their product is 45, then the numbers are

Let be numbers be a - d, a, a+d

Sum = $a - d + a + a + d = 3a = 15 \Rightarrow a = 5$

Product= $(5-d)5(5+d) \Rightarrow 5(25-d^2) = 45 \Rightarrow d^2 = 16 \Rightarrow d = \pm 4$

Required numbers are 1,5,9 or 9,5,1

4. Divide one hundred loaves of bread among five persons so that the second person receives as much more than the first as the third receives more than the second and the fourth more than the third and the fifth more than the fourth. Also, the ratio of loaves received by the first two to those received by the last three is 4 : 21. How much does the fourth person get?

It is clear from the question that the number of loaves of bread received by the five people will be in A.P.

Let them be (a - 2d), (a - d), a, (a + d), (a + 2d). Their sum is 5a which has to be equal to 100. So, a = 20.

Loaves received by the first two = (a - 2d) + (a - d) = (2a - 3d) and those received by the last three = a + (a + d) + (a + 2d) = (3a + 3d). Their ratio has been given as 4 : 21.

So, (2a - 3d) / (3a + 3d) = 4/21. Substituting a = 20, we get d = 8.

Hence the fourth person gets (a + d) = 28 loaves

Geometric Progression:

5. Which term of the series 1+2+4+8+...................... is 256

If n th term = 256, then $a.r^{n-1}$ = 256, Here a = 1, Common Ratio r = 2

$\Rightarrow 2^{n-1}$ = 256

$\Rightarrow 2^{n-1} = 2^8$

$\Rightarrow$ n= 9

6. If term of a G.P is $-\frac{1}{32}$ and 9th term is $\frac{1}{256}$ then 11th term =

6th term = $a.r^5$ = $-\frac{1}{32}$

9th term = $a.r^8$ = $\frac{1}{256}$

Dividing 9th term with 6th term,

$$r^3 = \frac{ar^8}{ar^5} = \frac{1/256}{-1/32} = \frac{-1}{8} \Rightarrow r = \frac{-1}{2}$$

$$\Rightarrow a\left(\frac{-1}{2}\right)^5 = \frac{-1}{32} \Rightarrow a = 1$$

$$11^{\text{th}}\text{ term} = ar^{10} = \left(\frac{-1}{2}\right)^{10} = \frac{1}{1024}$$

7. If x,y,z are the three geometric means between 6,54 then z =

6, x, y, z, 54 are in G.P. If r is common ratio, then

$$r = \left(\frac{54}{6}\right)^{1/4} = 9^{1/4} = \sqrt{3}$$

$$z = 6(\sqrt{3})^3 = 18\sqrt{3}$$

Harmonic Progression:

8. If the 3rd and 7th terms of a H.P. are $\frac{1}{7}, \frac{1}{15}$ then nth term is 3rd term = $\frac{1}{a+2d} = \frac{1}{7} \Rightarrow a+2d = 7\ldots(1)$

7th term = $\frac{1}{a+6d} = \frac{1}{15} \Rightarrow a+6d = 15\ldots(2)$

Solving (1) and (2) d = 2 and a = 3

9. Two harmonic means between $\frac{1}{2}, \frac{4}{17}$ are

If x,y are the harmonics means, then $\frac{1}{2}$, x,y, $\frac{4}{17}$ are in H.P.

Then $2, \frac{1}{x}, \frac{1}{y}, \frac{17}{4}$ are in A.P.

From above, $d = \dfrac{\frac{17}{4} - 2}{3} = \dfrac{\frac{9}{4}}{3} = \frac{3}{4}$

$$\frac{1}{x} = 2 + \frac{3}{4} = \frac{11}{4}$$

$$\frac{1}{y} = 2 + \frac{6}{4} = \frac{14}{4} = \frac{7}{2}$$

Harmonic means are $\frac{4}{11}, \frac{2}{7}$

A.G.P:

10. The sum of infinite terms of the series 1 + 3/4 + $5/4^2$ + $7/4^3$ .... equals:

Let,

S = 1 + 3/4 + $5/4^2$ + $7/4^3$ + ...upto ∞ ....(a) [divide S by 4]

S/4 = 1/4 + $3/4^2$ + $5/4^3$ $7/4^4$ ....upto ∞ ....(b) [subtract (b) from (a)]

3S/4 = 1 + 2/4 + $2/4^2$ + $2/4^3$ + $2/4^4$ + ...upto ∞

3S/4 = 1 + 2[1/4 + $1/4^2$ + $1/4^3$ ....upto ∞]

3S/4 = 1 + 2[(1/4) / {1 - (1/4)}]

3S/4 = 1 + 2 x 1/3 = 5/3

S = (5 x 4) / (3 x 3) = 20/9