Progressions - 2

(Please read formulas and theory in Progressions - 1 Click here)

Solved Examples (Level - 2)

Arithmetic Progression:

1. If the sum of first p terms of an arithmetic progression is equal to sum of first q terms of the same arithmetic progression, then what is the sum of (p + q) terms? Given that p $\neq$ q.

Compare the values for sum up to p and q terms of the series assuming first term and common difference. Write the equation for sum up to (p + q) terms and fill in the values from the above comparison.

$\Rightarrow \frac{p}{2}[2a+(p-1)d]=\frac{q}{2}[2a+(q-1)d]$

$\Rightarrow$ 2ap + p^2d - pd = 2aq + q^2d - qd

$\Rightarrow$ 2a(p-q) = d(q^2 - p^2 + p - q)

$\Rightarrow$ 2a(p-q) = d(q-p){q+p-1}

$\Rightarrow$ -2a(q-p) = d(q-p)(q+p-1)

$\Rightarrow$ -2a = d(q+p-1)

So sum of (p+q) term is = $\Rightarrow \frac{p+q}{2}[2a+(p+q-1)d]$

Substitute the value of d(q+p+1) in above equation $\frac{p+q}{2}[2a-2a]$ = 0

2. The sum of the 5^{th}, 7^{th} and 9^{th} terms of an arithmetic progression is equal to the sum of the 11^{th}, 13^{th}, 14^{th} and 15^{th} terms of the same arithmetic progression, then which term in the given arithmetic progression is definitely equal to zero?

Let a be the first term and d is the common difference

5^{th} term = a + 4d; 7^{th} term = a + 6d; 9^{th} term = a + 8d; 11^{th} term = a + 10d; 13^{th} term = a + 12d; 14^{th} term = a + 13d; 15^{th} term = a + 14d

Sum of 5^{th}, 7^{th} and 9^{th} term = Sum of 11^{th}, 13^{th} and 15^{th}

$\Rightarrow$ a + 4d + a + 6d + a + 8d = a + 10d + a + 12d + a + 13d + a + 14d

$\Rightarrow$3a + 18d = 4a + 4ad

$\Rightarrow$ a = -31d

Let nth term is zero.

$T_n = 0$

a + (n - 1)d = 0

-31d + (n - 1)d = 0

-31d + nd - d = 0

nd - 32d = 0

n = 32

3. Consider the following two series:

3, 7, 11, 15, ... 120 terms

1, 6, 11, 16, ... 80 terms

Then how many terms are common between the above two series of number?

The first common term is 11. L.C.M. of differences (4 and 5) = 20. So the term will be repeated in an interval of 20. The next common term in the series is 31.

120^{th} term of the first series

T_{120} = a + (n - 1)d

= 3 + (120 - 1) x 4

= 3 + 476

= 479

80^{th} term of the second series

T_{80} = a + (n - 1)d

= 1 + (80 - 1) x 5 = 1 + 395 = 396

Last common term must be less than or equal to 396.

396 $\geq$ 11 + (n - 1) x 20

$(\frac{385}{20}) + 1 \geq n$

n = 20

4. Some stones are placed on a straight road AB. The length of the road is 200 m and the stones are placed 4 m apart beginning at point A. A student was asked to bring these stones to the point B one at a time. He started picking up the stones from point A. If the minimum distance that he has to cover is 3760 m, then how many stones were placed on the road?

As the student starts travelling from A, he will have to travel 200m to reach to B. After that when he will come to pick up the second stone he will travel 196m twice in order to come back to point B. Similarly for the third stone he will have to travel 192m twice and so on for the stone he will have to travel (204 - 4n)m twice. Therefore, Total distance traveled = 200 + 2[196 + 192 + ... + (204 - 4n)]

{Sum of n terms of an arithmetic progression = (n/2)[1st term + last term]}

Here number of terms from 196 to (204 - 4n) is (n - 1). Therefore,

200 + 2[{(n - 1)/2}{196 + 204 - 4n}] = 3760

(n - 1)(400 - 4n) = 3560

n^2 - 101n + 990 = 0

(n - 11)(n - 90) = 0

n = 11 or n = 90 (not possible because the between A and B is only 200m).

5. Consider the following series: 13, 20, 27, 34, ... 370, 377. How many pairs of two distinct numbers can be formed such that the sum of each pair is equal to 390?

13, 20, 27, 34............370, 377

t_m = a + (n - 1)d

377 = 13 + (n - 1) x 7

$$\frac{364}{7} = n - 1$$

n = 53 therefore, no.of such pairs = 26

[Because 377 + 13 = 390, 370 + 20 = 390, and so on]

Geometric Progression:

6. After striking the floor, a ball rebounds to 4/5th of the height from which it fell. Find the total distance it travels before coming to rest if it is dropped from a height of 100 metres.

The ball falls by 100 m, rebounds to a height of 4/5th of 100m and comes down by the same distance, again rebounds to 4/5th of (4/5th of 100m), comes down by the same distance and this goes on and on up to infinity (theoretically).

So, it covers a distance D = 100 + 2(4/5 x 100) + 2(4/5 x 100) + ... up to ∞ .

Or, D = 100 + 2 x 4/5 x 100 (1 + 4/5 + $(4/5)^2$ + ...), the terms inside brackets form a Geometric series whose first term is 1 and common ratio is 4/5.

So, D = 100 + 160(1 - 4/5) = 100 + 160 x 5 = 100 + 800 = 900m.

7. a, b, c are three distinct real numbers in G.P. If a + b + c = xb, then find the range x may take

a + b + c = xb, where a, b and c are in G.P. let the first term of the G.P. is a and the common ratio is r.

So it becomes a + ar + ar^2 = x.ar $\Rightarrow$ r^2 + (1 - x) r + 1 = 0 for the real value of r. (D $\geq$ 0)

$(1-x)^2$ - 4 $\geq$ 0

x^2 - 2x - 3 $\geq$ 0 $\Rightarrow$ (x - 3)(x + 1) $\geq$ 0

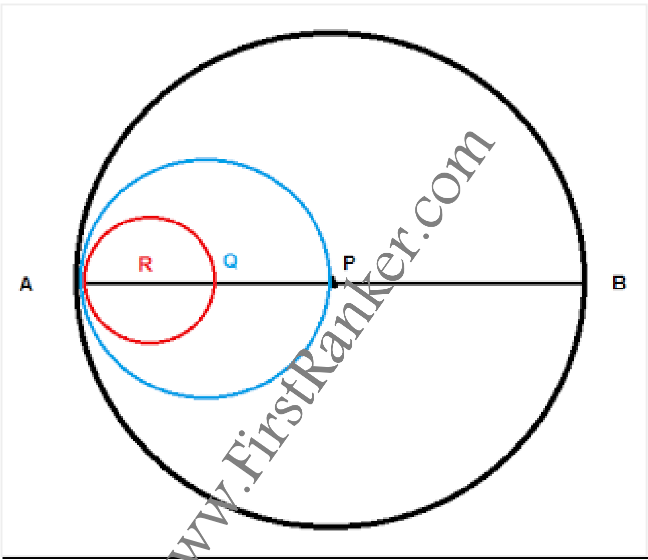
$\Rightarrow$ x $\leq$ -1 or x $\geq$ 3

8. If the third, first and second term of an arithmetic progression forms a geometric progression in the given order,

then what can be the common ratio of the geometric progression?

There are three terms and they are in A.P. so we assume the terms are (a - d), a and (a + d). Now if they are written in the order given in the question i.e. third term then first term and second term then they form a G.P. Hence we can say that (a + d), (a - d) and a are in G.P. or $(a-d)^2$ = a(a + d). Or a^2 - 2ad + d^2 = a^2 + ad or d^2 = 3ad or d = 3a. Substitute this value of d in the G.P. then G.P. will be 4a, -2a and a. Hence we can say the value of common ratio is -1/2

9. A circle is drawn on a plane with center as P and diameter AB. Another is drawn inside this circle by taking AP as the diameter and Q as the center. Another circles is drawn inside the second circle by taking AQ as diameter and R as the center. This process is repeated infinitely. The ratio of the sum of the areas of all the circles to the area of the biggest circle is



Take the diameter of the biggest circle with center P is 16. So its radius is 8. Now the second circle diameter is 8 so its radius is 4 which is half of first circle radius. This pattern continues.

Area of circle with center P = $\pi(\frac{D}{2})^2$

Area of circle with center Q = $\pi(\frac{D}{4})^2$

Area of circle with center R = $\pi(\frac{D}{8})^2$

Sum of areas = $\frac{\pi D^2}{4} + \frac{\pi D^2}{16} + \frac{\pi D^2}{64} \ldots\ldots\ldots\ldots$

= $\frac{\pi D^2}{4}[\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots\ldots]$

$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots\ldots$ is a infinite geometric progression with common difference of 1/4.

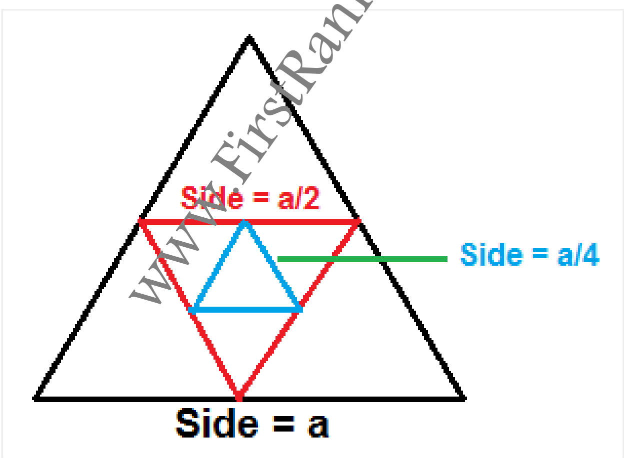
$$\frac{\pi D^2}{4}\left(\frac{\frac{1}{4}}{1-\frac{1}{4}}\right)$$

$$\frac{\pi D^2}{4}\times\frac{4}{3}=\frac{\pi D^2}{3}$$

$$\frac{\text{Sum of area of all circles}}{\text{Area of biggest circle}}=\frac{\frac{\pi D^2}{3}}{\frac{\pi D^2}{4}}=4:3$$

10. An equilateral triangle is drawn on a plane. A new triangle is formed by joining the midpoints of the given equilateral triangle. Third triangle is drawn by joining the midpoints of all previous triangle and the process is repeated indefinitely. What is the ratio of the sum of areas of all the triangles to sum of perimeters of all such triangles, if it is given that the side of the given triangle is ‘a’?

Let ‘a’ be the side of the biggest triangle. We know that the side of the triangle formed by joining the mid points of an equilateral triangle is half of the base (from mid point theorem).. and this pattern continues.



Area of the Biggest triangle = $\frac{\sqrt{3}}{4}a^2$

Area of Second triangle = $\frac{\sqrt{3}}{4}(\frac{a}{2})^2$

Area of third triangle = $\frac{\sqrt{3}}{4}(\frac{a}{4})^2$

Sum of all areas = $\frac{\sqrt{3}}{4}a^2[1+\frac{1}{4}+\frac{1}{16}+\ldots\ldots\ldots]$

$$\frac{\sqrt{3}}{4}a^2\left(\frac{1}{1-\frac{1}{4}}\right)=\frac{\sqrt{3}}{4}a^2\times\frac{4}{3}=a^2/\sqrt{3}$$

Perimeters = 3a + $3\frac{a}{2}+3\frac{a}{4}$ ....

$$= 3a[1+\frac{1}{2}+\frac{1}{4}]=3a\left(\frac{1}{1-\frac{1}{2}}\right)=6a$$

Ratio = $\frac{a^2/\sqrt{3}}{6a} = \frac{a}{6\sqrt{3}}$

11. A certain gentleman has plenty of time at his disposal. He puts his pencil to paper one day and draws a square. He then decides to create a design, so he draws another square inside the earlier one by joining its mid points. He then draws a third square inside the second in a similar fashion. He likes the design and hence goes on making squares inside squares in this manner. The side of the first square drawn is 8 cm. What is the ratio of the sum of area to the sum of perimeter of all the squares?

Area of the 1^{st} square = a^2

Area of the 2^{nd} square = $\left(\frac{a}{\sqrt{2}}\right)^2$

Area of the 3^{rd} square = $\left(\frac{a}{2}\right)^2$

Total area of squares = $a^2 + \left(\frac{a}{\sqrt{2}}\right)^2 + \left(\frac{a}{2}\right)^2$ ....

$$\Rightarrow a^2\left[1 + \frac{1}{2} + \frac{1}{4} + \dots\dots\dots\right] = a^2\left(\frac{1}{1 - \frac{1}{2}}\right) = 2a^2$$

If a square side is a, then the side of the square formed by joining the mid points of the square is $\frac{a}{\sqrt{2}}$.

Perimeters = 4a + 4 x $\frac{a}{\sqrt{2}} + 4 \times \frac{a}{2} + \dots\dots$

$$4a\left[1 + \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{2}{2\sqrt{2}} + \dots\dots\right]$$

$$= 4a\frac{1}{1 - \frac{1}{\sqrt{2}}} = 4a\frac{\sqrt{2}}{\sqrt{2} - 1} = a[8 + 4\sqrt{2}]$$

$$\text{Ratio} = \frac{\text{Sum of Area of square}}{\text{Sum of Perimeter}} = \frac{2a^2}{a[8 + 4\sqrt{2}]}$$

$$= \frac{2a}{8 + 4\sqrt{2}} = \frac{2 \times 8}{8 + 4\sqrt{2}} = \frac{4}{2 + \sqrt{2}}$$

12. For a fibonacci sequence of positive integers, from the third term onwards, each term in the sequence is the sum of the previous two terms in that sequence. If the difference in squares of seventh and sixth terms of this sequence is 517, what is the tenth term of this sequence?

Let sixth and seventh terms are t_6, and t_7, then:

$(t_7)^2-(t_6)^2$ = 517,

$(t_6-t_7)(t_6+t_7)$ = 11 x 47, since both are prime numbers, so we have

t_6-t_7 = 11 and t_6+t_7 = 47, from these two equations

t_6 = 18 and t_7 = 29,

t_8 = 18 + 29 = 47, t_9 = 47 + 29 = 76, t_{10} = 76 + 47 = 123

A.G.M:

13. The infinite sum of 3 + 6/5 + 11/5^2 + 18/5^3 equals:

Observe the coefficients here. The numerators 3, 6, 11 ... are not in arithmetic progression but their differences are in AP. (6-3), (11 - 6), (18 - 11), ... 3, 5, 7 are in AP.

Remember, If we take the differences between the terms of an AP they are equal.

Let Z = 3 + $\frac{6}{5}+\frac{11}{5^2}+\frac{18}{5^3}+\ldots\ldots\ldots\ldots\infty$