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TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS

ANNA UNIVERSITY EXAMINATION, Nov./Dec. 2010

PART - A (10 × 2 = 20 marks)

1. Find the constant term in the expansion of $\cos^2 x$ as a Fourier series in the interval $(-\pi, \pi)$

Solution :

$f(x) = \cos^2 x$ is an even function in the interval $(-\pi, \pi)$

$$\therefore a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2 x \, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} \cos^2 x \, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} \left(\frac{1 + \cos 2x}{2} \right) dx \qquad \because \cos^2 \theta = \frac{1 + \cos 2\theta}{2}$$

$$= \frac{2}{2\pi} \int_{0}^{\pi} (1 + \cos 2x) \, dx$$

$$= \frac{1}{\pi} \left[x + \frac{\sin 2x}{2} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[\pi \right] \qquad [\because \sin 2\pi = 0 ; \sin 0 = 0]$$

$$= 1$$

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2. Find the root mean square value of $f(x) = x^2$ in $(0, l)$

Solution :

$$\text{R.M.S value} = \sqrt{\frac{\int_0^l [f(x)]^2\,dx}{l-0}}$$

$$= \sqrt{\frac{\int_0^l (x^2)^2\,dx}{l-0}}$$

$$= \sqrt{\frac{\int_0^l x^4\,dx}{l}}$$

$$= \sqrt{\frac{\left[\frac{x^5}{5}\right]_0^l}{l}}$$

$$= \sqrt{\frac{l^5/5}{l}}$$

$$= \sqrt{\frac{l^4}{5}}$$

$$= \frac{l^2}{\sqrt{5}}$$

3. Write the Fourier transform pair.

Solution :

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\,dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[f(x)]\, e^{-isx}\,dx$$

4. Find the Fourier sine transform of $f(x) = e^{-ax}$, $a > 0$

Solution :

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2 + s^2} (-a \sin sx - s \cos sx) \right]_0^\infty$$

$$\left[\because \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-1}{a^2 + s^2} \right] \left\{ \underbrace{e^{-ax} \sin sx}_{0} + e^{-ax} s \cos sx \right\}_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-s}{a^2 + s^2} \right] \left[e^{-ax} \cos sx \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-s}{a^2 + s^2} \right] \left[0 - e^0 \cos 0 \right] \qquad \left[e^{-\infty} = 0, \sin 0 = 0 \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-s}{a^2 + s^2} \right] (-1)$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{a^2 + s^2} \right]$$

5. **Form the partial differential equation by eliminating the arbitrary function from $z^2 - xy = f\left(\frac{x}{z}\right)$**

Solution :

$$z^2 - xy = f\left(\frac{x}{z}\right)$$

Let $u = z^2 - xy$ $\qquad v = \frac{x}{z}$

$\therefore$ we get $\phi(u, v) = 0$

$$\phi\left(z^2 - xy, \frac{x}{z}\right) = 0$$

$$\frac{\partial u}{\partial x} = 2z\frac{\partial z}{\partial x} - y$$

$$= 2zp - y$$

$$\frac{\partial u}{\partial y} = 2z\frac{\partial z}{\partial y} - x$$

$$= 2zq - x$$

$$\frac{\partial v}{\partial x} = \frac{z(1) - x\frac{\partial z}{\partial x}}{z^2}$$

$$= \frac{z - xp}{z^2}$$

$$\frac{\partial v}{\partial y} = \frac{z(0) - x\frac{\partial z}{\partial y}}{z^2}$$

$$= \frac{-xq}{z^2}$$

$\therefore$ The PDE is given by

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$

$$\begin{vmatrix} 2zp - y & \frac{z - xp}{z^2} \\ 2zq - x & \frac{-xq}{z^2} \end{vmatrix} = 0$$

$$(2zp - y)\left(\frac{-xq}{z^2}\right) - (2zq - x)\left(\frac{z - xp}{z^2}\right) = 0$$

$$\frac{-2zpqx + xyq - [2z^2 q - xz - 2zpqx + x^2 p]}{z^2} = 0$$

$$-2zpqx + xyq - 2z^2 q + xz + 2zpqx - x^2 p = 0$$

$$xyq - 2z^2 q + xz - x^2 p = 0$$

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6. Find the paticular integral of $(D^2 - 2DD' + D'^2)z = e^{x-y}$

Solution :

$$\text{P.I} = \frac{1}{D^2 - 2DD' + D'^2} e^{x-y}$$

$$= \frac{1}{(1)^2 - 2(1)(-1) + (-1)^2} e^{x-y}$$

$$= \frac{1}{1 - 2 + 1} e^{x-y}$$

$$= \frac{1}{0} e^{x-y} \qquad \text{(case of failure)}$$

$$= \frac{x}{2D - D'} e^{x-y}$$

$$= \frac{x}{2(1) - (-1)} e^{x-y} \qquad [\text{Replacing } D \text{ by } 1,\ D' \text{ by } -1]$$

$$= \frac{x}{2 + 1} e^{x-y}$$

$$= \frac{x}{3} e^{x-y}$$

7. Write down the three possible solutions of one dimensional heat equation.

Solution :

The three possible solutions are

(1) $u = (A e^{px} + Be^{-px})\, Ce^{\alpha^2 p^2 t}$

(2) $u = (A \cos x + B \sin x)\; Ce^{-\alpha^2 p^2 t}$

(3) $u = (Ax + B)\, C$

8. Give three possible solutions of two dimensional steady state heat flow equation.

Solution :

The possible solutions are

(1) $u = (Ae^{px} + Be^{-px})\,(C \cos py + D \sin py)$

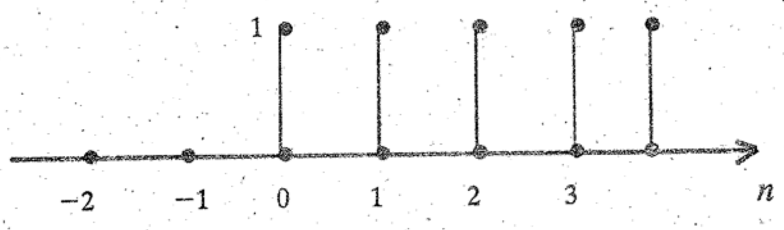
(2) $u = (A \cos px + B \sin px)\,(Ce^{py} + D\, e^{-py})$

(3) $u = (Ax + B)\,(Cy + D)$

9. Define the unit step sequence. Write its Z-transform.

Solution :

The unit step sequence is defined as $u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$



$$Z\{u(n)\} = \frac{z}{z-1} \text{ for } |z| > 1$$

10. Form a difference equation by eliminating the arbitrary constant A from $y_n = A \cdot 3^n$.

Solution :

$$y_n = A\,3^n \qquad \ldots (1)$$

Replacing n by $n+1$ in (1)

$$y_{n+1} = A\,3^{n+1} \qquad \ldots (2)$$

$$\frac{(1)}{(2)} \Rightarrow \quad \frac{y_{n+1}}{y_n} = \frac{A\,3^{n+1}}{A\,3^n}$$

$$= \frac{A\,3^n \cdot 3}{A\,3^n}$$

$$\frac{y_{n+1}}{y_n} = 3$$

$$y_{n+1} = 3y_n$$

$$y_{n+1} - 3y_n = 0$$

PART B (5 × 16 = 80 marks)

11. (a) (i) Find the Fourier series expansion of

$$f(x) = \begin{cases} x & \text{for } 0 \le x \le \pi \\ 2\pi - x & \text{for } \pi \le x \le 2\pi \end{cases}$$

Also, deduce that $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots \infty = \dfrac{\pi^2}{8}$

Solution : $a_0 = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} f(x)\,dx$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} f(x)\,dx + \int_{\pi}^{2\pi} f(x)\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi} (2\pi - x)\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{x^2}{2}\right]_0^{\pi} + \left[2\pi x - \frac{x^2}{2}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\pi^2}{2} + \left[4\pi^2 - \frac{4\pi^2}{2} - \left(2\pi^2 - \frac{\pi^2}{2}\right)\right]\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\pi^2}{2} + 4\pi^2 - 2\pi^2 - 2\pi^2 + \frac{\pi^2}{2}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\pi^2}{2} + \frac{\pi^2}{2}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{2\pi^2}{2}\right\}$$

$$= \pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} f(x)\cos nx\,dx + \int_{\pi}^{2\pi} f(x)\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi} (2\pi - x)\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[\underbrace{x\left(\frac{\sin nx}{n}\right)}_{0} - \left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}\right.$$

$$\left. + \left[\underbrace{(2\pi - x)\left(\frac{\sin x}{n}\right)}_{0} - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{\cos nx}{n^2}\right]_0^{\pi} - \left[\frac{\cos nx}{n^2}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi n^2}\left\{\left[\cos n\pi - \cos 0\right] - \left[\cos 2n\pi - \cos n\pi\right]\right\}$$

$$= \frac{1}{\pi n^2}\left\{(-1)^n - 1 - 1 + (-1)^n\right\} \qquad \because \cos n\pi = (-1)^n \ ; \ \cos 0 = 1$$

$$= \frac{1}{\pi n^2}\left[2(-1)^n - 2\right]$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

Case (i) When n is even number

$$a_n = \frac{2}{\pi n^2}\left[1 - 1\right] = 0$$

Case (ii) When n is odd number

$$a_n = \frac{2}{\pi n^2}\left[-1 - 1\right]$$

$$= \frac{2}{\pi n^2}\left[-2\right]$$

$$= \frac{-4}{\pi n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin x\,dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} f(x)\sin nx\,dx + \int_{\pi}^{2\pi} f(x)\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\sin nx\,dx + \int_{\pi}^{2\pi} (2\pi - x)\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[x\left(\frac{-\cos nx}{n}\right) - \left(\underbrace{\frac{-\sin nx}{n^2}}_{0}\right)\right]_0^{\pi}\right.$$

$$\left. + \left[(2\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\underbrace{\frac{-\sin nx}{n^2}}_{0}\right)\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi n}\left[\left(-x\cos nx\right)_0^{\pi} - \left((2\pi - x)\cos nx\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi n}\left\{-\pi\cos n\pi - \left[0 - (2\pi - \pi)\cos n\pi\right]\right\}$$

$$= \frac{1}{\pi n}\left\{-\pi(-1)^n + \pi(-1)^n\right\}$$

$$= 0$$

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{\pi}{2} + \sum_{n=1,3,5,\ldots} -\frac{4}{\pi n^2}\cos nx$$

11. (a) (ii) Find the Fourier expansion of $f(x) = 1 - x^2$ in the interval $(-1, 1)$

Solution :

$f(x) = 1 - x^2$ is an even function in the interval $(-1, 1)$

$$\therefore b_n = 0$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$2l = \text{U.L} - \text{L.L}$$

$$= 1 - (-1)$$

$$= 2$$

$$\therefore \quad l = 1$$

$$= \frac{1}{1} \int_{-1}^{1} (1 - x^2)\, dx$$

$$= 2 \int_{0}^{1} (1 - x^2)\, dx$$

$$= 2 \left[x - \frac{x^3}{3} \right]_0^1$$

$$= 2 \left[1 - \frac{1}{3} \right]$$

$$= 2 \left(\frac{2}{3} \right)$$

$$= \frac{4}{3}$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l}\, dx$$

$$= \frac{1}{1} \int_{-1}^{1} (1 - x^2) \cos n\pi x\, dx$$

$$= \frac{2}{1} \int_{0}^{1} (1 - x^2) \cos n\pi x\, dx$$

$$= 2\left[(1-x^2)\underbrace{\frac{\sin n\pi x}{n\pi}}_{0} - (-2x)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + (-2\underbrace{\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)}_{0}\right]_0^1$$

$$= 2\left[-2x\frac{\cos n\pi x}{n^2\pi^2}\right]_0^1$$

$$= \frac{-4}{n^2\pi^2}\left[x\cos n\pi x\right]_0^1$$

$$= \frac{-4}{n^2\pi^2}[\cos n\pi - 0]$$

$$= \frac{-4}{n^2\pi^2}(-1)^n \qquad \cos n\pi = (-1)^n$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$= \frac{\frac{4}{3}}{2} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}(-1)^n\cos n\pi x$$

$$= \frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}(-1)^n \cos n\pi x$$

11. (b) (i) Obtain the half range cosine series for $f(x) = x$ in $(0, \pi)$

Solution :

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{2}\right]$$

$$= \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\ \cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \cos n x\, dx$$

$$= \frac{2}{\pi}\left[\underbrace{x\frac{\sin nx}{n}}_{0} - (1)\frac{(-\cos n x)}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}\left[\cos n x\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}[\cos n\pi - \cos 0]$$

$$= \frac{2}{\pi n^2}[(-1)^n - 1]$$

Case (i)

When n is even

$$a_n = \frac{2}{\pi n^2}[1 - 1]$$

$$= 0$$

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Case (ii)

When n is odd

$$a_n = \frac{2}{\pi n^2}[-1-1]$$

$$= \frac{-4}{\pi n^2}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{\pi}{2} + \sum_{n=1,3,5,\ldots}^{\infty} \frac{-4}{\pi n^2} \cos n x$$

11. (b) (ii) Find the Fourier series as far as the second harmonic to represent the function f(x) with period 6, given in the following table.

x	0	1	2	3	4	5
$f(x)$	9	18	24	28	26	20

Solution :

Here the length of the interval is $2l = 6$, i.e., $l = 3$.

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$\therefore$ The Fourier series can be represented by

$$y = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{l} + b_1 \sin\frac{\pi x}{l}\right) + \left(a_2 \cos\frac{2\pi x}{l} + b_2 \sin\frac{2\pi x}{l}\right) + \ldots$$

$$\text{Here } y = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{3} + b_1 \sin\frac{\pi x}{3} \qquad \ldots (1)$$

x	y	$\cos\frac{\pi x}{3}$	$\sin\frac{\pi x}{3}$	$y\cos\frac{\pi x}{3}$	$y\sin\frac{\pi x}{3}$
0	9	1	0	9	0
1	18	0.5	0.866	9	15.588
2	24	–0.5	0.866	–12	20.785
3	28	–1	0	–28	0
4	26	–0.5	–0.866	–13	–22.517
5	20	0.5	–0.866	10	–17.321
	125			–25	–3.465

$$a_0 = \frac{2}{n}\Sigma y = \frac{2}{6}125 = 41.67$$

$$a_1 = \frac{2}{n}\Sigma y\cos\frac{\pi x}{3} = \frac{2}{6}(-25) = -8.33$$

$$b_1 = \frac{2}{n}\Sigma y\sin\frac{\pi x}{3} = \frac{2}{6}(-3.465) = -1.16$$

Substituting the above values in (1) we get

$$y = \frac{41.67}{2} - 8.33\cos x - 1.16\sin x$$

$$= 20.84 - 8.33\cos x - 1.16\sin x$$

12. (a) (i) Derive the Parseval's identity for Fourier Transforms.

Statement :

If $F[f(x)] = F(s)$, then

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(s)|^2\,ds$$

Proof :

By Convolution theorem,

$$F[f(x) * g(x)] = F[f(x)] . F[g(x)]$$

$$= F(s) . G(s) \text{ where}$$

$$F(s) = F[f(x)] \text{ and } G(s) = F[g(x)]$$

$$\therefore f(x) * g(x) = F^{-1}[F(s) G(s)]$$

Using the definition of convolution in LHS and definition of Inverse Fourier Transform in RHS we get

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) g(x-t)\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) . G(s)\; e^{-isx}\, dx$$

Cancelling $\frac{1}{\sqrt{2\pi}}$ on both sides,

$$\int_{-\infty}^{\infty} f(t) g(x-t)\, dt = \int_{-\infty}^{\infty} F(s) G(s)\; e^{-isx}\, ds$$

Putting $x = 0$ on both sides

$$\int_{-\infty}^{\infty} f(t) g(-t)\, dt = \int_{-\infty}^{\infty} F(s) . G(s)\, ds \qquad \text{... (1)}$$

This is true for all $g(t)$

$$\therefore \text{ Let } g(t) = \overline{f(-t)}$$

$$\therefore g(-t) = \overline{f(t)}$$

$$\therefore G(s) = F[g(x)]$$

$$= F[\overline{f(-x)}]$$

$$= \overline{F(s)}$$

Now (1) becomes

$$\int_{-\infty}^{\infty} f(t)\,\overline{f(t)}\,dt = \int_{-\infty}^{\infty} F(s)\,\overline{F(s)}\,ds$$

$$\int_{-\infty}^{\infty} |f(t)|^2\,dt = \int_{-\infty}^{\infty} |F(s)|^2\,ds$$

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(s)|^2\,ds$$

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12. (a) (ii) Find the Fourier integral representation of f(x) defined as

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{2} & \text{for } x = 0 \\ e^{-x} & \text{for } x > 0 \end{cases}$$

Solution :

By Fourier Integral theorem

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cos\lambda(t-x)\,dt\,d\lambda$$

$$\text{Given } f(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{2} & \text{for } x = 0 \\ e^{-x} & \text{for } x > 0 \end{cases}$$

$$\therefore f(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{2} & \text{for } t = 0 \\ e^{-t} & \text{for } t > 0 \end{cases}$$

$$\therefore f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{-t} \cos\lambda(t-x)\,dt\,d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{e^{-t}}{1+\lambda^2} \left(-1 \cos \lambda (t-x) + \lambda \sin \lambda (t-x) \right) \right]_0^{\infty} d\lambda$$

$$\left[\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx) \right]$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[0 - \frac{1}{1+\lambda^2} \left(-\cos \lambda x + \lambda \sin \lambda (-x) \right) \right] d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\cos \lambda x + \lambda \sin \lambda x}{1+\lambda^2} d\lambda \qquad \left[\because \sin(-\theta) = -\sin\theta, \ \cos(-\theta) = \cos\theta \right]$$

i.e., $$f(x) = \frac{2}{2\pi} \int_0^{\infty} \frac{\cos \lambda x + \lambda \sin \lambda x}{1+\lambda^2} d\lambda = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{2} & \text{for } x = 0 \\ e^{-x} & \text{for } x > 0 \end{cases}$$

Note : $\dfrac{\cos \lambda x + \lambda \sin \lambda x}{1+\lambda^2}$ is even function.

At $x = 0$

$$\frac{1}{\pi} \int_0^{\infty} \frac{1}{1+\lambda^2} d\lambda = \frac{1}{\pi} \int_0^{\infty} \frac{\cos 0 + \lambda \sin 0}{1+\lambda^2} d\lambda$$

$$= \frac{1}{\pi} \left[\tan^{-1}(\lambda) \right]_0^{\infty}$$

$$= \frac{1}{\pi} \left[\tan^{-1}(\infty) - \tan^{-1}(0) \right]$$

$$= \frac{1}{\pi} \left(\frac{\pi}{2} \right)$$

$$= \frac{1}{2}$$

12. (b) (i) Find the Fourier sine transform of

$$f(x) = \begin{cases} x\ , & 0 < x < 1 \\ 2 - x\ , & 1 < x < 2 \\ 0\ , & x > 2 \end{cases}$$

Solution :

The Fourier sine transform is defined as

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 f(x) \sin sx\, dx + \int_1^2 f(x) \sin sx \, dx + \int_2^\infty f(x) \sin sx\, dx \right.$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 x \sin sx\, dx + \int_1^2 (2-x) \sin sx\, dx + \int_2^\infty 0 \sin sx\, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 x \sin sx\, dx + \int_1^2 (2-x) \sin sx\, dx \right]$$

Applying Bernoulli's formula

$$= \sqrt{\frac{2}{\pi}} \left[\left[\left\{ x \left(\frac{-\cos sx}{s} \right) - (1) \left(\frac{-\sin sx}{s^2} \right) \right\} \right]_0^1 \right.$$

$$\left. + \quad (2-x)\left(\frac{-\cos sx}{s} \right) - (-1) \left\{ \frac{-\sin sx}{s^2} \right\} \right]_0^2 \Bigg]$$

$$= \sqrt{\frac{2}{\pi}} \left[\left[\left\{ \frac{-x \cos sx}{s} + \frac{\sin sx}{s^2} \right\}_0^1 - \left\{ (2-x) \frac{\cos sx}{s} + \frac{\sin sx}{s^2} \right\}^2 \right] \right.$$

$$= \sqrt{\frac{2}{\pi}} \left[\left[\frac{-\cos s - 0}{s} + \frac{\sin s - 0}{s^2} \right] - \left\{ \frac{0 - \cos s}{s} + \frac{\sin 2s - \sin s}{s^2} \right\} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-\cos s}{s} + \frac{\sin s}{s^2} + \frac{\cos s}{s} - \frac{\sin 2s}{s^2} + \frac{\sin s}{s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2\sin s}{s^2} - \frac{\sin 2s}{s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2\sin s - \sin 2s}{s^2}\right]$$

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12. (b) (ii) Evaluate $\int_0^\infty \frac{dx}{(x^2 + a^2)(x^2 + b^2)}$ using Fourier cosine transforms of e^{-ax} and e^{-bx}

Solution : Let $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$

then $F_c(f(x)) = F_c[e^{-ax}]$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2 + s^2}\right]$$

(i.e.,) $F_c[g(x)] = F_c[e^{-bx}]$

$$G_c(s) = \sqrt{\frac{2}{\pi}} \left[\frac{b}{b^2 + s^2}\right] \quad \text{(by Qn.No.3 Pg.No. 2.37)}$$

By Parseval's theorem

$$\int_0^\infty f(x)\, g(x)\, dx = \int_0^\infty F_c(s)\, G_c(s)\, ds$$

$$\int_0^\infty e^{-ax} e^{-bx} dx = \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2 + s^2}\right) \sqrt{\frac{2}{\pi}} \left(\frac{b}{b^2 + s^2}\right) ds$$

$$\int_0^\infty e^{-(a+b)x} dx = \int_0^\infty \frac{2}{\pi} \frac{ab}{(a^2 + s^2)(b^2 + s^2)} ds$$

$$\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty = \frac{2ab}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)\ (b^2+s^2)}$$

$$-\frac{1}{a+b}\left[e^{-\infty} - e^0\right] = \frac{2ab}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)\,(b^2+s^2)}$$

$$-\frac{1}{a+b}\,(0-1) = \frac{2ab}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)\ (b^2+s^2)}$$

$$-\frac{1}{a+b}\,(-1) = \frac{2ab}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)\,(b^2+s^2)}$$

$$\frac{1}{a+b} = \frac{2ab}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)\,(b^2+s^2)}$$

Cross multiplying

$$\frac{\pi}{2ab\,(a+b)} = \int_0^\infty \frac{ds}{(a^2+s^2)\,(b^2+s^2)}$$

Changing the variable 's' to 'x'

$$\frac{\pi}{2ab\,(a+b)} = \int_0^\infty \frac{dx}{(a^2+x^2)\,(b^2+x^2)}$$

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13. (a) (i) Form the PDE by eliminating the arbitrary function ϕ from $\phi(x^2 + y^2 + z^2, ax + by + cz) = 0$

Solution :

$u = x^2 + y^2 + z^2$	$v = ax + by + cz$
$\frac{\partial u}{\partial x} = 2x + 2z\frac{\partial z}{\partial x}$	$\frac{\partial v}{\partial x} = a + c\frac{\partial z}{\partial x}$
$= 2x + 2zp$	$= a + cp$
$\frac{\partial u}{\partial y} = 2y + 2z\frac{\partial z}{\partial y}$	$\frac{\partial v}{\partial y} = b + c\frac{\partial z}{\partial y}$
$= 2y + 2zq$	$= b + cq$

$\therefore$ PDE is given by $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$

$$\begin{vmatrix} 2(x + zp) & 2(y + zq) \\ a + cp & b + cq \end{vmatrix} = 0$$

$$2(x + zp)(b + cq) - 2(a + cp)(y + zq) = 0$$

$$2[xb + cxq + bzp + czpq - ay - azq - cpy - cpzq] = 0$$

$$\Rightarrow xb + cxq + bzp - ay - azq - cpy = 0$$

13. (a) (ii) Solve the partial differential equation

$$x^2 (y - z) p + y^2 (z - x) q = z^2 (x - y)$$

Solution :

The equation is of the form $Pp + Qq = R$

$$\text{where } \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

The auxillary equations are $\dfrac{dx}{x^2 (y - z)} = \dfrac{dy}{y^2 (z - x)} = \dfrac{dz}{z^2 (x - y)}$

Taking the multipliers $\dfrac{1}{x^2}, \dfrac{1}{y^2}, \dfrac{1}{z^2}$ each term in auxillary equation

$$\frac{\frac{dx}{x^2}}{\frac{x^2 (y - z)}{x^2}} = \frac{\frac{dy}{y^2}}{\frac{y^2 (z - x)}{y^2}} = \frac{\frac{dz}{z^2}}{\frac{z^2 (x - y)}{z^2}}$$

$$\frac{\frac{dx}{x^2}}{y - z} = \frac{\frac{dy}{y^2}}{z - x} = \frac{\frac{dz}{z^2}}{x - y}$$

Summing all ratios we get

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$$\frac{\frac{dx}{x^2}+\frac{dy}{y^2}+\frac{dz}{z^2}}{y-z+z-x+x-y}$$

$$=\frac{\frac{dx}{x^2}+\frac{dy}{y^2}+\frac{dz}{z^2}}{0}$$

$$\Rightarrow \frac{dx}{x^2}+\frac{dy}{y^2}+\frac{dz}{z^2} = 0$$

Integrating

$$\int \frac{dx}{x^2}+\int \frac{dy}{y^2}+\int \frac{dz}{z^2} = 0$$

$$\int x^{-2}\,dx+\int y^{-2}\,dy+\int z^{-2}\,dz = 0$$

$$\frac{x^{-1}}{-1}+\frac{y^{-1}}{-1}+\frac{z^{-1}}{-1} = c_1$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = c_1$$

Taking the multiplier $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ each term in auxillary equation.

$$\frac{\frac{dx}{x}}{\frac{x^2(y-z)}{x}} = \frac{\frac{dy}{y}}{\frac{y^2(z-x)}{y}} = \frac{\frac{dz}{z}}{\frac{z^2(x-y)}{z}}$$

$$\frac{\frac{dx}{x}}{x(y-z)} = \frac{\frac{dy}{y}}{y(z-x)} = \frac{\frac{dz}{z}}{z(x-y)}$$

$$\frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{x(y-z)+y(z-x)+z(x-y)}$$

$$\frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{xy-xz+yz-xy+zx-zy}$$

$$\frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}$$

$$\Rightarrow \frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z} = 0$$

Integrating $\int \frac{dx}{x}+\int \frac{dy}{y}+\int \frac{dz}{z} = 0$

$$\log x+\log y+\log z = \log c_2$$

$$\log xyz = \log c_2 \quad [\because \log a+\log b+\log c = \log abc]$$

$$\therefore xyz = c_2$$

$\therefore$ The solution is $\phi(c_1, c_2) = 0$

$$\phi\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z},\ xyz\right) = 0$$

---

13. (b) (i) Solve the equation

$$[D^3+D^2D'-4DD'^2-4D'^3]\,z = \cos(2x+y)$$

Solution :

The Auxiliary equation is

$$m^3+m^2-4m-4 = 0$$

$$\begin{array}{r|rrrr} -1 & 1 & 1 & -4 & -4 \\ & 0 & -1 & 0 & 4 \\ \hline & 1 & 0 & -4 & \boxed{0} \end{array}$$

$m = -1$ is a root.

Also $m^2 - 4 = 0$

$$m^2 = 4$$

$$m = \pm 2$$

Roots are $-1, 2, -2$

$\therefore$ The complementary function is

$$\text{C.F} = f_1(y-x) + f_2(y-2x) + f_3(y+2x)$$

$$\text{P.I} = \frac{1}{D^3 + D^2 D' - 4DD'^2 - 4D'^3} \cos(2x+y)$$

$$= \frac{1}{DD^2 + D^2 D' - 4DD'^2 - 4D'D'^2} \cos(2x+y)$$

$$= \frac{1}{D(-4) + (-4)D' - 4D(-1) - 4D'(-1)} \cos(2x+y)$$

$$= \frac{1}{-4D - 4D' + 4D + 4D'} \cos(2x+y)$$

$$= \frac{1}{0} \cos(2x+y)$$

$[D^2 = -a^2,\ D'^2 = -b^2$

$DD' = -ab$

$a = 2$

$a^2 = 4,\ -a^2 = -4$

$b = 1$

$b^2 = 1,\ -b^2 = -1$

$ab = 4\ ; \quad -ab = -4]$

Since the denominator becomes zero we apply method 5.

$$\text{P.I} = \frac{1}{D^3 + D^2 D' - 4DD'^2 + 4D'^3} \cos(2x+y)$$

$$= \frac{x}{3D^2 + 2DD' - 4D'^2} \cos(2x + y)$$

$$= \frac{x}{3(-4) + 2(-4) - 4(-1)} \cos(2x + y)$$

$$= \frac{x}{-12 - 8 + 4} \cos(2x + y)$$

[Again replacing D^2 by $-a^2$, D'^2 by $-b^2$, DD' by $-ab$]

$$= \frac{1}{-16} \cos(2x + y)$$

The general solution is z = C.F + P.I.

$$z = f_1(y - x) + f_2(y - 2x) + f_3(y + 2x) - \frac{1}{16} \cos(2x + y)$$

13. (b) (ii) Solve $[2D^2 - DD' - D'^2 + 6D + 3D'] z = x e^y$

Solution :

This is non-homogeneous equation.

Dividing throughout by '2'

$\therefore$ We have to factorize in the form

$$(D - m_1 D' - \alpha_1)(D - m_2 D' - \alpha_2) = 0$$

$$\left[D^2 - \frac{1}{2}DD' - \frac{1}{2}D'^2 + 3D + \frac{3}{2}D'\right] z = \frac{x e^y}{2}$$

Comparing the coefficients of DD'

$$-m_1 - m_2 = -1/2 \qquad \ldots (1)$$

Comparing the coefficient of D'^2

$$m_1 m_2 = -1/2 \qquad \ldots (2)$$

Comparing the coefficient of D

$$-\alpha_1 - \alpha_2 = 3 \qquad \ldots (3)$$

Comparing the co-efficient of D'

$$m_1 \alpha_2 + m_2 \alpha_1 = 3/2 \qquad \ldots (4)$$

Solving (1), (2), (3), (4) we get

$$m_1 = \frac{-1}{2}, \; m_2 = 1; \; \alpha_1 = 0, \; \alpha_2 = -3$$

$$\text{C.F} = e^{\alpha_1 x} f_1 (y + m_1 x) + e^{\alpha_2 x} + f_2 (y + m_2 x)$$

$$= e^{0x} f_1 \left(y - \frac{x}{2}\right) + e^{-3x} f_2 (y + x)$$

$$\text{P.I} = \frac{1}{D^2 - \frac{DD'}{2} - \frac{D'^2}{2} + 3D + \frac{3}{2} D'} \cdot \frac{x e^y}{2}$$

$$= \frac{1}{2} \cdot \frac{1}{\left(D + \frac{D'}{2}\right)\left(D - D' + 3\right)} . x e^y$$

$$= \frac{1}{2} \cdot \frac{1}{\left(D + \frac{D'}{2}\right)} \int x e^{c-x} \, dx$$

$$= \frac{1}{2} \cdot \frac{1}{\left(D + \frac{D'}{2}\right)} \left[x \frac{e^{c-x}}{-1} \cdot (1) \frac{e^{c-x}}{(-1)^2}\right]$$

$$= -\frac{1}{2} \cdot \frac{1}{\left(D + \frac{D'}{2}\right)} \left[x e^{c-x} + e^{c-x}\right]$$

$$= -\frac{1}{2} \cdot \frac{1}{\left(D + \frac{D'}{2}\right)} [x e^y + e^y]$$

$$= -\frac{1}{2}\int \left[x\,e^{c+\frac{x}{2}} + e^{c+\frac{x}{2}}\right] dx$$

$$= -\frac{1}{2}\left[x\,\frac{e^{c+\frac{x}{2}}}{\frac{1}{2}} - (1)\,\frac{e^{c+\frac{x}{2}}}{\left(\frac{1}{2}\right)^2} + \frac{e^{c+\frac{x}{2}}}{\frac{1}{2}}\right]$$

$$= -\frac{1}{2}\left[2x\,e^{c+\frac{x}{2}} - 4\,e^{c+\frac{x}{2}} + 2\,e^{c+\frac{x}{2}}\right]$$

$$= -\frac{1}{2}\left[2x\,e^{c+\frac{x}{2}} - 2\,e^{c+\frac{x}{2}}\right]$$

$$= -\frac{2}{2}\left[x\,e^{c+\frac{x}{2}} - e^{c+\frac{x}{2}}\right]$$

$$= -\left[x\,e^{y} - 3\,e^{y}\right]$$

The General solution is $Z = \text{C.F} + \text{P.I}$

$$= e^{0x} f_1\left(y - \frac{x}{2}\right) + e^{-3x} f_1(y + x) - \left[x\,e^{y} - e^{y}\right]$$

14. (a) **A tightly stretched string of length $2l$ is fastened at both ends. The midpoint of the string is displaced by a distance 'b' transversely and the string is released from rest in this position. Find an expression for the transverse displacement of the string at any time during the subsequent motion.**

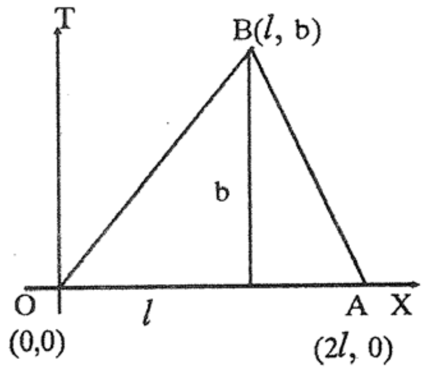
Solution : The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

From the given problem we get the following boundary conditions.

(i) $y(0, t) = 0$ for all $t \geq 0$

(ii) $y(2l, t) = 0$ for all $t \geq 0$

(iii) $\frac{\partial y}{\partial t}(x, 0) = 0, \ 0 < x < 2l$



Equation of OB is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\frac{y - 0}{b - 0} = \frac{x - 0}{l - 0}$$

$$y = \frac{b}{l} x$$

Equation of BA is

$$\frac{y - b}{0 - b} = \frac{x - l}{2l - l}$$

$$y - b = \left(\frac{x - l}{l}\right)(-b)$$

$$y = b + \frac{bl}{l} - \frac{bx}{l}$$

$$= 2b - \frac{bx}{l}$$

$$y = b\left[\frac{2l - x}{l}\right]$$

(iv) $y(x, 0) = \begin{cases} \frac{bx}{l}, & 0 < x < l \\ \frac{b(2l - x)}{l}, & l < x < 2l \end{cases}$

Now the suitable solution which satisfies our boundary conditions is given by

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos p\,at + c_4 \sin p\,at) \quad \dots (1)$$

Applying condition (i) in equation (1) we get

$$y(0,t) = c_1 (c_3 \cos p\,at + c_4 \sin p\,at) = 0$$

Here $c_3 \cos p\,at + c_4 \sin p\,at \neq 0$ [$\because$ it is defined for all t]

Therefore $\boxed{c_1 = 0}$

Substitute $c_1 = 0$ in equation (1) we get

$$y(x,t) = c_2 \sin px\,(c_3 \cos p\,at + c_4 \sin p\,at) \quad \dots (2)$$

Applying condition (ii) in equation (2) we get

$$y(2l, t) = c_2 \sin 2lp\;(c_3 \cos p\,at + c_4 \sin p\,at) = 0$$

Here $c_3 \cos p\,at + c_4 \sin p\,at \neq 0$ [$\because$ it is defined for all t]

Therefore either $c_2 = 0$ or $\sin 2pl = 0$

Suppose if we take $c_2 = 0$ and already we have $c_1 = 0$

then we get a trivial solution,

Therefore $c_2 \neq 0$

The only possibility is $\sin 2pl = 0$

i.e., $2pl = n\pi$ [$\because \sin n\pi = 0$]

$$\boxed{p = \frac{n\pi}{2l}}$$

Substitute $p = \dfrac{n\pi}{2l}$ in equation (2) we get

$$y(x,t) = c_2 \sin\frac{n\pi x}{2l}\left(c_3 \cos\frac{n\pi}{2.l}at + c_4 \sin\frac{n\pi}{2.l}at\right) \quad \dots (3)$$

Before applying condition (iii), diff (3) p.w.r.to t we get

$$\frac{\partial y}{\partial t}(x,t) = c_2 \sin\frac{n\pi x}{2l}\left[-c_3 \sin\frac{n\pi at}{2l}\left(\frac{n\pi a}{2l}\right) + c_4 \cos\frac{n\pi a t}{2l}\left(\frac{n\pi a}{2l}\right)\right]$$

Now we applying condition (iii) we get

$$\frac{\partial y}{\partial t}(x,0) = c_2 \sin\frac{n\pi x}{2l}\left[c_4 \frac{n\pi a}{2l}\right] = 0$$

$$\text{i.e., } c_2 c_4 \sin\frac{n\pi x}{2l}\,\frac{n\pi a}{2l} = 0$$

$$\sin\frac{n\pi x}{2l} \neq 0 \;[\because \text{ It is defined for all } x]$$

$$\frac{n\pi a}{2l} \neq 0 \;[\because \text{ All are constants}]$$

$$c_2 \neq 0 \;[\because \text{ Suppose if } c_2 = 0, \text{ we already explained}]$$

$$\therefore \; c_4 = 0$$

Substitute $c_4 = 0$ in equation (3) we get

$$y(x,t) = c_2 \sin\frac{n\pi x}{2l}\, c_3 \cos\frac{n\pi at}{2l}$$

$$= c_2 c_3 \sin\frac{n\pi x}{2l} \cos\frac{n\pi at}{2l}$$

$$= c_n \sin\frac{n\pi x}{2l} \cos\frac{n\pi a t}{2l} \qquad \ldots (4)$$

Where $c_n = c_2\, c_3$

The most general form of equation (4) can be written as

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{2l} \cos\frac{n\pi a t}{2l} \qquad \ldots (5)$$

Applying condition (iv) in equation (5) we get

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{2l} = \begin{cases} \dfrac{bx}{l}, & 0 < x < l \\ \dfrac{b(2l-x)}{l}, & l < x < 2l \end{cases} \quad \dots (6)$$

To find c_n expand the given value in a half-range Fourier sine series in the interval (0, L)

Here $L = 2l$

$$\left.\begin{matrix} \dfrac{bx}{l}, & 0 < x < l \\ \dfrac{b(2l-x)}{l}, & l < x < 2l \end{matrix}\right\} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \dots (7)$$

where $b_n = \dfrac{2}{(2l)} \displaystyle\int_0^{2l} f(x) \sin\frac{n\pi x}{2l}\, dx$

From (6) and (7) we get $b_n = c_n$

$$\therefore c_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{2l}\,dx$$

$$= \frac{1}{l}\left[\int_0^{l} \frac{bx}{l}\sin\frac{n\pi x}{2l}\,dx + \int_l^{2l}\frac{b(2l-x)}{l}\sin\frac{n\pi x}{2l}\,dx\right]$$

$$= \frac{b}{l^2}\int_0^{l} x\sin\frac{n\pi x}{2l}\,dx + \frac{b}{l^2}\int_l^{2l}(2l-x)\sin\frac{n\pi x}{2l}\,dx$$

$$= \frac{b}{l^2}\left[x\left(\frac{-\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{2l}}{\left(\frac{n\pi}{2l}\right)^2}\right)\right]_0^{l}$$

$$+ \frac{b}{l^2}\left[(2l-x)\left[\frac{-\cos\frac{n\pi x}{2l}}{\left(\frac{n\pi}{2l}\right)}\right] - (-1)\left[\frac{-\sin\frac{n\pi x}{2l}}{\left(\frac{n\pi}{2l}\right)^2}\right]\right]_l^{2l}$$

$$= \frac{b}{l^2}\left[-x\left(\frac{2l}{n\pi}\right)\cos\frac{n\pi x}{2l} + \left(\frac{2l}{n\pi}\right)^2 \sin\frac{n\pi x}{2l}\right]_0^l$$

$$+ \frac{b}{l^2}\left[-(2l-x)\left(\frac{2l}{n\pi}\right)\cos\frac{n\pi x}{2l} - \left(\frac{2l}{n\pi}\right)^2 \sin\frac{n\pi x}{2l}\right]_l^{2l}$$

$$= \frac{b}{l^2}\left[\left(\frac{-2l^2}{n\pi}\cos\frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right) - (-0+0)\right]$$

$$+ \frac{b}{l^2}\left[(0-0) - \left(\frac{-2l^2}{n\pi}\cos\frac{n\pi}{2} - \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right]$$

$$= \frac{b}{l^2}\left[\frac{-2l^2}{n\pi}\cos\frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{2l^2}{n\pi}\cos\frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{b}{l^2}\left[\frac{8l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

i.e., $c_n = \dfrac{8b}{n^2\pi^2}\sin\dfrac{n\pi}{2}$

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$$c_n = \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$= 0 \text{ if } n \text{ is even}$$

$$= \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2} \text{ if } n \text{ is odd}$$

Substitute the value of c_n in equation (5) we get

$$y(x,t) = \sum_{n=\text{odd}}^{\infty} \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2l}\cos\frac{n\pi at}{2l}$$

$$= \sum_{n=1}^{\infty} \frac{8b}{(2n-1)^2\pi^2}\sin\frac{(2n-1)\pi}{2}\sin\frac{(2n-1)\pi x}{2l}\cos\frac{(2n-1)\pi at}{2l}$$

$$= \frac{8b}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}(-1)^{n-1}\sin\frac{(2n-1)\pi x}{2l}\cos\frac{(2n-1)\pi at}{2l}$$

Note : $\sin(2n-1)\frac{\pi}{2} = (-1)^{n-1}$

---

14. (b) A square plate is bounded by the lines $x = 0, y = 0, x = 20$ and $y = 20$. Its faces are insulated. The temperature along the upper horizontal edge is given by

$u(x, 20) = x(20 - x), 0 < x < 20$

while the other two edges are kept at 0°C. Find the steady state temperature distribution in the plate.

Solution :

The two dimensional heat flow equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

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We have the following boundary conditions

(i) $u = 0$ when $x = 0$

(ii) $u = 0$ when $x = 20$

(iii) $u = 0$ when $y = 0$

(iv) $u = x(20 - x)$ when $y = 20$

Consider the first solution

$$u = (A e^{px} + Be^{-px})(C \cos py + D \sin py) \quad \dots (1)$$

Applying condition (i) in (1)

$$0 = (A e^{0} + B e^{0})(C \cos py + D \sin py)$$

$$0 = (A + B)(C \cos py + D \sin py)$$

$$A + B = 0$$

$$\Rightarrow \quad B = -A$$

Substituting $B = -A$ in (1) we get

$$u = A e^{px} - A e^{-px})(C \cos py + D \sin py) \quad \dots (2)$$

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Applying condition (ii) in (2) we get

$$0 = (A\,e^{p20} - A\,e^{P20})\,(C\cos py + D\sin py)$$

$$\Rightarrow A\,e^{p20} - A\,e^{p20} = 0$$

$$A\,(e^{20p} + e^{-20p}) = 0$$

$$A = 0 \qquad\qquad e^{p20} - e^{-20p} \neq 0$$

Put $B = -A$

$$\therefore B = 0$$

But A and B cannot be zero simultaneously.

$\therefore$ First solution fails.

Consider second solution.

$$u = (A\cos px + B\sin px)\,(Ce^{py} + De^{-py}) \qquad \ldots (1)$$

Applying condition (i) in (1)

$$0 = (A\cos 0 + B\sin 0)\,(Ce^{py} + De^{-py})$$

$$0 = (A + 0)\,(Ce^{py} + De^{-py})$$

$$A = 0$$

Substituting $A = 0$ in (1) we get

$$u = B\sin px\,(Ce^{py} + De^{-py}) \qquad \ldots (2)$$

Applying condition (ii) in (2)

$$0 = B\sin 20p\,(Ce^{py} + De^{-py})$$

$$\Rightarrow \qquad B\sin 20p = 0$$

$$\sin 20p = 0$$

$$20p = n\pi$$

$$p = \frac{n\pi}{20}$$

Substitute $p = \dfrac{n\pi}{20}$ in (2)

$$u = B \sin\frac{n\pi x}{20}\left(Ce^{\frac{n\pi y}{20}} + De^{\frac{-n\pi y}{20}}\right) \quad \ldots (3)$$

Applying condition (iii) in (3) we get

$$0 = B \sin\frac{n\pi x}{20}\left(Ce^{0} + De^{0}\right)$$

$$0 = B \sin\frac{n\pi x}{20}(C + D)$$

$$C + D = 0$$

$$D = -C$$

Substituting $D = -C$ in (3) we get

$$u = B \sin\frac{n\pi x}{20}\left(Ce^{\frac{n\pi y}{20}} + De^{\frac{-n\pi y}{20}}\right)$$

$$= B \sin\frac{n\pi x}{20}\, C\left(e^{\frac{n\pi y}{20}} - e^{\frac{-n\pi y}{20}}\right)$$

$$= BC \sin\frac{n\pi x}{20}\left(e^{\frac{n\pi y}{20}} - e^{\frac{-n\pi y}{20}}\right)$$

$$= b \sin\frac{n\pi x}{20}\left(e^{\frac{n\pi y}{20}} - e^{\frac{-n\pi y}{20}}\right)$$

$$u = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{20}\left(e^{\frac{n\pi y}{20}} - e^{\frac{-n\pi y}{20}}\right) \quad \ldots (4)$$

Applying condition (iv) in (4) we get

$$x(20-x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{20}\left(e^{n\pi}-e^{-n\pi}\right)$$

The R.H.S is H.R.S.S. Its co-efficient is given by

$$b_n\left(e^{n\pi}-e^{-n\pi}\right) = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{20}\int_0^{20} x(20-x)\sin\frac{n\pi x}{20}\,dx \qquad [\text{Here } l = 20]$$

$$= \frac{1}{10}\int_0^{20}(20x-x^2)\sin\frac{n\pi x}{20}\,dx$$

$$= \frac{1}{10}\left[\underbrace{(20x-x^2)\left(\frac{-\cos\frac{n\pi x}{20}}{\frac{n\pi}{20}}\right)}_{\downarrow\ 0} - \underbrace{(20-2x)\left(\frac{-\sin\frac{n\pi x}{20}}{\frac{n^2\pi^2}{20^2}}\right)}_{\downarrow\ 0} + (-2)\left(\frac{\cos\frac{n\pi x}{20}}{\frac{n^3\pi^3}{20^3}}\right)\right]_0^{20}$$

$$= \frac{1}{10}\left[\frac{-2\cos\frac{n\pi x}{20}}{\frac{n^3\pi^3}{20^3}}\right]_0^{20}$$

$$= \frac{-2}{10}\times\frac{20^3}{n^3\pi^3}\left(\cos\frac{n\pi x}{20}\right)_0^{20}$$

$$= \frac{-1600}{n^3 \pi^3} \left[\cos n\pi - \cos 0\right]_0^{20}$$

$$= \frac{-1600}{n^3 \pi^3} [(-1)^n - 1]$$

Case : 1.

When 'n' is even number, $(-1)^n = 1$

$$b_n \left(e^{n\pi} - e^{-n\pi}\right) = \frac{-1600}{n^3 \pi^3}(1 - 1)$$

$$= 0$$

$$\therefore b_n = 0$$

Case 2 :

When 'n' is an odd number, $(-1)^n = -1$

$$b_n \left(e^{n\pi} - e^{-n\pi}\right) = \frac{-1600}{n^3 \pi^3}(-1 - 1)$$

$$= \frac{3200}{n^3 \pi^3}$$

$$b_n = \frac{3200}{n^3 \pi^3} \cdot \frac{1}{e^{n\pi} - e^{-n\pi}}$$

Final solution is obtained by substituting b_n in (4)

$$u = \sum_{n=1,3}^{\infty} \frac{3200}{n^3 \pi^3} \frac{1}{e^{n\pi} - e^{-n\pi}} \sin \frac{n\pi x}{20} \left(e^{\frac{n\pi y}{20}} - e^{\frac{-n\pi y}{20}}\right)$$

15. (a) (i) Find the Z-transform of $\cos n\theta$ and $\sin n\theta$. Hence deduce the Z-transforms of $\cos(n+1)\theta$ and $a^n \sin n\theta$

Solution :

Let $a = e^{i\theta}$

$$\therefore Z[a^n] = Z\left[(e^{i\theta})^n\right]$$

$$= \frac{z}{z - e^{i\theta}} \qquad \left[\because Z(a^n) = \frac{z}{z-a}\right]$$

$$= \frac{z}{z - (\cos\theta + i\sin\theta)} \quad [\because e^{i\theta} = \cos\theta + i\sin\theta]$$

$$= \frac{z}{z - \cos\theta - i\sin\theta}$$

$$= \frac{z}{z - \cos\theta - i\sin\theta} \times \frac{z - \cos\theta + i\sin\theta}{z - \cos\theta + i\sin\theta}$$

[Multiplying and dividing by conjugate]

$$= \frac{z(z - \cos\theta) + iz\sin\theta}{(z - \cos\theta)^2 + \sin^2\theta}$$

$$\therefore Z[(e^{i\theta})^n] = \frac{z(z - \cos\theta) + iz\sin\theta}{z^2 + \cos^2\theta - 2z\cos\theta + \sin^2\theta}$$

i.e., $$\left[Z(\cos\theta + i\sin\theta)^n\right] = \frac{z(z-\cos\theta) + iz\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$Z[\cos n\theta + i\sin n\theta] = \frac{z(z-\cos\theta) + iz\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$Z(\cos n\theta) + iZ(\sin n\theta) = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1} + \frac{iz\sin\theta}{z^2 - 2z\cos\theta + 1}$$

Comparing real and imaginary parts

$$Z[\cos n\theta] = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$$

$$Z[\sin n\theta] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

To find $Z[a^n \sin n\theta]$

By change of scale property

$$Z[a^n f_n] = F\left(\frac{z}{a}\right)$$

But, $\quad Z[\sin n\theta] = \dfrac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$

$\therefore$ Replacing $\quad z = \dfrac{z}{a}$ we get

$$Z[a^n \sin n\theta] = \frac{\frac{z}{a}\sin\theta}{\left(\frac{z}{a}\right)^2 - \frac{2z}{a}\cos\theta + 1}$$

To find $Z[\cos(n+1)\theta]$

$$\begin{aligned}
Z[\cos(n+1)\theta] &= Z[\cos(n\theta+\theta)] \\
&= Z[\cos n\theta\cos\theta - \sin n\theta\sin\theta] \\
&= Z[\cos n\theta\cos\theta] - Z[\sin n\theta\sin\theta] \\
&= \cos\theta\, Z[\cos n\theta] - \sin\theta\, Z[\sin n\theta] \\
&= \cos\theta\,\frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1} - \sin\theta\,\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1} \\
&= \frac{z\cos\theta(z-\cos\theta)}{z^2 - 2z\cos\theta + 1} - \frac{z\sin^2\theta}{z^2 - 2z\cos\theta + 1}
\end{aligned}$$

15. (a) (ii) Find the inverse Z-transform of $\frac{z(z+1)}{(z-1)^3}$ by residue method.

Solution :

$$\text{Let } F(z) = \frac{z(z+1)}{(z-3)^3}$$

Multiplying both sides by z^{n-1}

$$\therefore z^{n-1} F(z) = z^{n-1} \frac{z(z+1)}{(z-3)^3}$$

$$= \frac{z^n (z+1)}{(z-3)^3} = g(z)$$

Equating the denominator to zero

$$(z-3)^3 = 0$$

$$z = 3, 3, 3$$

$$\therefore z = 3 \text{ is a pole of order three}$$

The residue for the pole of order three is given by

$$\lim_{z \to a} \frac{1}{2!} \frac{d^2}{dz^2} (z-a)^3 g(z)$$

$\therefore$ Residue for $z = 3$ is given by

$$\lim_{z \to 3} \frac{1}{2!} \frac{d^2}{dz^2} (z-3)^3 . \frac{z^n (z+1)}{(z-3)^3}$$

$$\lim_{z \to 3} \frac{1}{2} \frac{d^2}{dz^2} . z^n (z+1)$$

$$\lim_{z \to 3} \frac{1}{2} \frac{d^2}{dz^2} \left(z^{n+1} + z^n\right)$$

$$\lim_{z \to 3} \frac{1}{2} \frac{d}{dz} \left[(n+1) z^n + n z^{n-1}\right]$$

$$\lim_{z \to 3} \frac{1}{2} \left[(n+1) n z^{n-1} + n (n-1) z^{n-2}\right]$$

$$\frac{1}{2} \left[(n+1) n\, 3^{n-1} + n (n-1)\, 3^{n-2}\right]$$

(OR)

15. (b) (i) Form the difference equation from the relation

$$y_n = a + b \,.\, 3^n$$

Solution :

$$y_n = a + b \,.\, 3^n \qquad \ldots (1)$$

Replacing n by $n+1$

$$y_{n+1} = a + b \,.\, 3^{n+1} \qquad \ldots (2)$$

Replacing n by $n+2$

$$y_{n+2} = a + b \,.\, 3^{n+2} \qquad \ldots (3)$$

Equation (2) can be written as

$$y_{n+1} = a + b \,.\, 3^n \,.\, 3$$

From (1) $b \,.\, 3^n = y_n - a$

Substituting for $b\, 3^n$ we get

$$y_{n+1} = a + (y_n - a)\, 3$$

$$= a + 3 y_n - 3a$$

$$y_{n+1} = 3 y_n - 2a \qquad \ldots (4)$$

Equation (3) can be written as

$$y_{n+2} = a + b \cdot 3^n \cdot 3^2$$

from (1) $\quad b\, 3^n = y_n - a$

Substituting for $b\, 3^n$ we get

$$y_{n+2} = a + (y_n - a)\, 9$$

$$= a + 9y_n - 9a$$

$$y_{n+2} = 9y_n - 8a \qquad \dots (5)$$

(4) $\times$ 4, we get

$$4y_{n+1} = 12y_n - 8a \qquad \dots (6)$$

$$y_{n+2} = 9y_n - 8a \qquad \dots (5)$$

(5) $-$ (6)

$$y_{n+2} - 4y_{n+1} = -3y_n$$

$$y_{n+2} - 4y_{n+1} + 3y_n = 0$$

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15. (b) (ii) Solve $y_{n+2} + 4y_{n+1} + 3y_n = 2^n$ with $y_0 = 0$ and $y_1 = 1$, using Z-transform.

Solution :

Let $\bar{y} = Z[y_n]$

Given $y_{n+2} + 4y_{n+1} + 3y_n = 2^n$

Taking Z-transform on both sides.

$$Z(y_{n+2}) + 4Z(y_{n+1}) + 3Z(y_n) = Z(2^n)$$

$$z^2\left(\bar{y} - y_0 - \frac{y_1}{z}\right) + 4\left[z(\bar{y} - y_0)\right] + 3\bar{y} = \frac{z}{z-2}$$

Using initial conditions

$$z^2\left(\bar{y}-y_0-\frac{1}{z}\right)+4\left[z\left(\bar{y}-0\right]+3\bar{y} = \frac{z}{z-2}$$

$$z^2\left(\bar{y}-\frac{1}{z}\right)+4z\left(\bar{y}\right)+3\bar{y} = \frac{z}{z-2}$$

$$z^2\bar{y}-\frac{z^2}{z}+4z\left(\bar{y}\right)+3\bar{y} = \frac{z}{z-2}$$

$$z^2\bar{y}+4z\bar{y}+3\bar{y}-z = \frac{z}{z-2}$$

$$\bar{y}\left(z^2+4z+3\right)-z = \frac{z}{z-2}$$

$$\bar{y}\left(z^2+4z+3\right) = \frac{z}{z-2}+z$$

$$\bar{y}\left(z^2+4z+3\right) = \frac{z+z\left(z-2\right)}{z-2}$$

$$= \frac{z+z^2-2z}{z-2}$$

$$= \frac{z^2-z}{z-2}$$

$$\bar{y}\left(z^2+4z+3\right) = \frac{z\left(z-1\right)}{z-2}$$

$$\bar{y} = \frac{z\left(z-1\right)}{\left(z-2\right)\left(z^2+4z+3\right)}$$

$$= \frac{z\left(z-1\right)}{\left(z-2\right)\left(z+1\right)\left(z+3\right)}$$

$$y_n = Z^{-1}\left[\frac{z\left(z-1\right)}{\left(z-2\right)\left(z+1\right)\left(z+3\right)}\right]$$

$$\text{Let } F(z) = \frac{z(z-1)}{(z-2)(z+1)(z+3)}$$

$$\frac{F(z)}{z} = \frac{z-1}{(z-2)(z+1)(z+3)}$$

$$\text{Let } \frac{z-1}{(z-2)(z+1)(z+3)} = \frac{A}{z-2} + \frac{B}{z+1} + \frac{C}{z+3}$$

$$= \frac{A(z+1)(z+3) + B(z-2)(z+3) + C(z-2)(z+1)}{(z-2)(z+1)(z+3)}$$

Equating the numerators

$$z-1 = A(z+1)(z+3) + B(z-2)(z+3) + C(z-2)(z+1)$$

Put $z = 2$

$$2-1 = A(2+1)(2+3) + B(0) + C(0)$$

$$1 = A(15)$$

$$A = \frac{1}{15}$$

Put $z = -1$

$$-1-1 = A(0) + B(-1-2) + (-1+3) + C(0)$$

$$-2 = -6B$$

$$B = \frac{1}{3}$$

Put $z = -3$

$$-3-1 = A(0) + B(0) + C(-3-2)(-3+1)$$

$$-4 = 10C$$

$$C = -\frac{2}{5}$$

$$\therefore \frac{F(z)}{z} = \frac{1/15}{z-2} + \frac{1/3}{z+1} + \frac{-2/5}{z+3}$$

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$$= \frac{1}{15} \cdot \frac{1}{z-2} + \frac{1}{3} \frac{1}{z+1} - \frac{2}{5} \frac{1}{z+3}$$

$$F(z) = \frac{1}{15} \frac{z}{z-2} + \frac{1}{3} \frac{z}{z+1} - \frac{2}{5} \frac{z}{z+3}$$

$$y_n = Z^{-1}\left[\frac{1}{15} \frac{z}{z-2} + \frac{1}{3} \frac{z}{z+1} - \frac{2}{5} \frac{z}{z+3}\right]$$

$$= \frac{1}{15} Z^{-1}\left[\frac{z}{z-2}\right] + \frac{1}{3} Z^{-1}\left[\frac{z}{z+1}\right] - \frac{2}{5} Z^{-1}\left[\frac{z}{z+3}\right]$$

$$y_n = \frac{1}{15} 2^n + \frac{1}{3}(-1)^n - \frac{2}{5}(-3)^n$$

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