TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS

ANNA UNIVERSITY EXAMINATION, April/May 2010

PART - A (10 × 2 = 20 marks)

1. Obtain the first order partial differential equation by eliminating the arbitrary constants a and b from the equation $z = (x^2 + a^2)(y^2 + b^2)$

Solution : Let $z = (x^2 + a^2)(y^2 + b^2)$ ... (1)

Differentiating (1) partially with respect to x

$$\frac{\partial z}{\partial x} = 2x(y^2 + b^2)$$

$$p = 2x(y^2 + b^2)$$

$$\therefore y^2 + b^2 = \frac{p}{2x}$$

Differentiating (1) partially with respect to y

$$\frac{\partial z}{\partial y} = (x^2 + a^2)\, 2y$$

$$q = (x^2 + a^2)\, 2y$$

$$\therefore x^2 + a^2 = \frac{q}{2y}$$

Substituting for $x^2 + a^2$ and $y^2 + b^2$ in (1) we get

$$z = \left(\frac{p}{2x}\right)\left(\frac{q}{2y}\right)$$

Corresponding, $4xyz = pq$

2. Form a partial differential equation by eliminating the arbitrary function from the relation $z = xy + f(x^2 + y^2)$

Solution : $$z = xy + f(x^2 + y^2) \quad \ldots (1)$$

Differentiating (1) partially with respect to x

$$\frac{\partial z}{\partial x} = y + f'(x^2 + y^2)\,2x$$

$$p = y + 2xf'(x^2 + y^2)$$

$$p - y = 2xf'(x^2 + y^2) \quad \ldots (2)$$

Differentiating (1) partially with respect to y

$$\frac{\partial z}{\partial y} = x + f'(x^2 + y^2)\,2y$$

$$q = x + 2yf'(x^2 + y^2)$$

$$q - x = 2yf'(x^2 + y^2) \quad \ldots (3)$$

$$\frac{(2)}{(3)} \Rightarrow \quad \frac{p-y}{q-x} = \frac{2xf'(x^2+y^2)}{2yf'(x^2+y^2)}$$

$$\frac{p-y}{q-x} = \frac{x}{y}$$

Cross multiplying,

$$(p - y)\,y = x\,(q - x)$$

$$py - y^2 = xq - x^2$$

$$py - qx = y^2 - x^2$$

3. If $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\pi x + b_n \sin n\pi x)$ is the Fourier series of $f(x) = x$ in $(-1, 1)$, find $a_3^2 + b_3^2$

Solution : Here $f(x) = x$ is an odd function.

$\therefore$ in the interval $(-1, 1)$, $a_0 = 0$, $a_n = 0$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{1}{1} \int_{-1}^{1} x \sin \frac{n\pi x}{1} dx$$

$$= 2 \int_{0}^{1} x \sin n\pi x \, dx \qquad [\because x \sin n\pi x \text{ is even function}]$$

$$= 2\left[x\left(\frac{-\cos n\pi x}{n\pi}\right) - (1)\underbrace{\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)}_{0}\right]_0^1$$

$$= 2\left[x\left(\frac{-\cos n\pi x}{n\pi}\right)\right]_0^1$$

$$= -2\left(\frac{1}{n\pi}\right)\left[x \cos n\pi x\right]_0^1$$

$$= \frac{-2}{n\pi}\left[\cos n\pi - 0\right]$$

$$b_n = \frac{-2}{n\pi}(-1)^n$$

$$\therefore b_3 = -\frac{2}{3\pi}(-1)^3$$

$$= \frac{-2}{3\pi}(-1) = \frac{2}{3\pi}$$

$$b_3^2 = \left(\frac{2}{3\pi}\right)^2 = \frac{4}{9\pi^2}$$

$$a_3 = 0$$

$$a_3^2 = 0$$

$$\therefore\ a_3^2 + b_3^2 = \frac{4}{9\pi^2}$$

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4. If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ is the Fourier cosine series of $f(x)$ in $0 < x < \pi$, state the corresponding Parseval identity.

Solution : The given series is Half Range cosine series.

$\therefore$ The Parseval's identity for half range cosine series in the interval $(0, \pi)$

$$\frac{2}{\pi}\int_0^{\pi} [f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

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5. Write down the one dimensional heat equation.

Solution : The one dimensional heat equation is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$$

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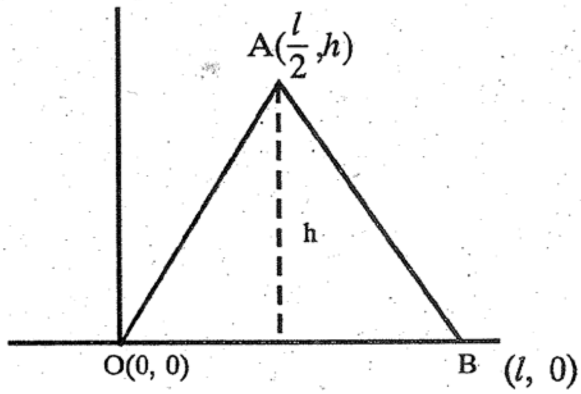
6. If the ends of a string of length l are fixed and the midpoint of the string is drawn aside through a height h and the string is released from rest. Write the boundary and initial conditions.

Solution : The initial position of the string is as follows



The initial position of the string is OAB whose equation can be found by finding the equation of OA and AB separetely.

$\therefore$ Equation of a line joining points (x_1, y_1) and (x_2, y_2) is

$$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$$

Equation of OA is $\frac{y-0}{0-h} = \frac{x-0}{0-\frac{l}{2}}$ $\left[O = (0,0)\,,\ A = \left(\frac{l}{2}, h\right)\right]$

$$\frac{y}{-h} = \frac{x}{\frac{-l}{2}}$$

$$\frac{y}{h} = \frac{x}{\frac{l}{2}}$$

$$\frac{y}{h} = \frac{2x}{l}$$

$$y = \frac{2h}{l}x \text{ between } 0 \text{ and } \frac{l}{2}$$

Equation of BA

$$\frac{y-0}{0-h} = \frac{x-l}{l-\frac{l}{2}} \qquad \left[B = (l,0)\,,\ A = \left(\frac{l}{2}, h\right)\right]$$

$$\frac{y}{-h} = \frac{x-l}{\frac{l}{2}}$$

$$\frac{y}{h} = -\frac{2}{l}(x-l)$$

$$y = \frac{2h}{l}(l-x) \text{ between } \frac{l}{2} \text{ and } l$$

The wave equation is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

We have the following boundary conditions

I. $y = 0$ when $x = 0$

II. $y = 0$ when $x = l$

III. $\frac{\partial y}{\partial t} = 0$ when $t = 0$

IV. $y = f(x)$ when $t = 0$

7. State the inversion formula for Fourier Transform

Solution : The inverse Fourier Transform is

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]\, e^{-isx}\, ds$$

8. State the convolution theorem of Fourier transforms.

Solution : The Fourier Transform of the convolution of $f(x)$ and $g(x)$ is equal to the product of fourier transform is

$$\text{(i.e.,)} \quad F[f*g] = F\left[f(x)\right] F\left[g(x)\right]$$

$$\text{where} \quad f*g = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$$

9. Find z (n a^n)

Solution : We know that $Z(n) = \dfrac{z}{(z-1)^2}$

$\therefore Z(a^n n)$ can be obtained by replacing z by $\dfrac{z}{a}$

$$Z(a^n n) = \frac{\frac{z}{a}}{\left(\frac{z}{a}-1\right)^2}$$

$$= \frac{\frac{z}{a}}{\left(\frac{z-a}{a}\right)^2}$$

$$= \frac{\frac{z}{a}}{\frac{(z-a)^2}{a^2}}$$

$$= \frac{z}{a}\left(\frac{a^2}{(z-a)^2}\right)$$

$$= \frac{az}{(z-a)^2}$$

10. State the convolution theorem of Z-transform.

Solution : $Z^{-1}\left[F(z)\,G(z)\right] = \sum_{m=0}^{n} f(m)\,.\,g(n-m)$

where $f(n) = Z^{-1}\left[F(z)\right]$

$g(n) = Z^{-1}\left[G(z)\right]$

PART-B

11. (a)(i) For the equation $z = px + qy + \left(\frac{q}{p} - p\right)$, find the complete and singular solutions.

Solution : $z = px + qy + \left(\frac{q}{p} - p\right)$ ... (1)

Let $z = ax + by + c$ ... (2) be the solution

Differentiating (2) partially w.r.to x

$$\frac{\partial z}{\partial x} = a$$

$$\therefore p = a$$

Differentiating (2) partially w.r.to y

$$\frac{\partial z}{\partial y} = b$$

$$\therefore q = b$$

Substituting for p and q in (1)

$$z = ax + by + \left(\frac{b}{a} - a\right) \quad \text{... (3)}$$

This itself is the complete integral as there are only two arbitrary constants.

To find the singular integral differentiate (3) partially w.r.to a.

$$\frac{\partial z}{\partial a} = x + \left(\frac{-b}{a^2} - 1\right)$$

$$\frac{\partial z}{\partial a} = 0 \Rightarrow x - \frac{b}{a^2} - 1 = 0$$

$$\therefore b = -a^2(1 - x)$$

Differentiating (3) partially w.r.to 'b'

$$\frac{\partial z}{\partial b} = y + \frac{1}{a}$$

$$\frac{\partial z}{\partial b} = 0 \Rightarrow \quad y + \frac{1}{a} = 0$$

$$a = -\frac{1}{y}$$

Substituting for b and than a in (3) we get the singular integral.

$$z = \left(\frac{-1}{y}\right) x + \left(-a^2 (1-x)\right) y + \left(\frac{-a^2 (1-x)}{-1/y} + \frac{1}{y}\right)$$

$$= \frac{-x}{y} + \left(\frac{-1}{y^2}(1-x)\right) y + \left(\frac{-\frac{1}{y}(1-x)}{-1/y} + \frac{1}{y}\right)$$

$$= \frac{-x}{y} - \frac{1}{y}(1-x) + 1 - x + \frac{1}{y}$$

$$= \frac{-x}{y} - \frac{1}{y} + \frac{x}{y} + 1 - x + \frac{1}{y}$$

$\therefore z = 1 - x$ is the singular integral.

11. (a) (ii) Solve the equation $(D^2 - D'^2) z = e^{x-2y} \sin(2x + y)$

Solution : The A.E is $m^2 - 1 = 0$

$$m^2 = 1$$

$$m = \pm 1$$

$\therefore$ The C.F is $z = f_1(y + x) + f_2(y - x)$

$$\text{P.I} = \frac{1}{D^2 - D'^2} e^{x-2y} \sin(2x + y)$$

$$= e^{x-2y} \frac{1}{(D+1)^2 - (D'-2)^2} \sin(2x + y)$$

$$= e^{x-2y} \frac{1}{(D^2 + 2D + 1) - (D'^2 - 4D' + 4)} \sin(2x + y)$$

$$= e^{x-2y} \frac{1}{D^2 + 2D + 1 - D'^2 + 4D' - 4} \sin(2x + y)$$

$$= e^{x-2y} \frac{1}{D^2 - D'^2 + 2D + 4D' - 3} \sin(2x + y)$$

$$= e^{x-2y} \frac{1}{(-4) - (-1) + 2D + 4D' - 3} \sin(2x + y)$$

$$= e^{x-2y} \frac{1}{2D + 4D' - 6} \sin(2x + y)$$

$$= \frac{e^{x-2y}}{2} \frac{1}{D + 2D' - 3} \sin(2x + y)$$

Multiply and divide by $D + 2D' + 3$

$$= \frac{e^{x-2y}}{2} \frac{1}{D + 2D' - 3} \frac{D + 2D' + 3}{D + 2D' + 3} \sin(2x + y)$$

$$= \frac{e^{x-2y}}{2} \frac{1}{(D + 2D')^2 - 9} (D + 2D' + 3) \sin(2x + y)$$

$$= \frac{e^{x-2y}}{2} \frac{1}{D^2 + 4DD' + 4D'^2 - 9} (D + 2D' + 3) \sin(2x + y)$$

$$= \frac{e^{x-2y}}{2} \frac{1}{(-4) + 4(-2) + 4(-1) - 9} (D+2D' +3) \sin(2x + y)$$

$$= \frac{e^{x-2y}}{2} \frac{1}{-4 - 8 - 4 - 9} (D + 2D' + 3) \sin(2x + y)$$

$$= \frac{e^{x-2y}}{2} \frac{1}{-25} (D + 2D' + 3) \sin(2x + y)$$

$$= \frac{e^{x-2y}}{-50} \left[D [\sin(2x + y)] + 2D' [\sin(2x + y)] + 3 \sin(2x + y)\right]$$

$$= \frac{-e^{x-2y}}{50}\left[2\cos(2x+y)+2\cos(2x+y)+3\sin(2x+y)\right]$$

$$= -\frac{e^{x-2y}}{50}\left[4\cos(2x+y)+3\sin(2x+y)\right]$$

$$\therefore z = \text{C.I} + \text{P.I}$$

$$= f_1(y+x)+f_2(y-x)-\frac{e^{x-2y}}{50}\left[4\cos(2x+y)+3\sin(2x+y)\right]$$

11. (b)(i) Solve the equation $z^2(p^2+q^2) = x+y$

Solution : Given $\quad z^2(p^2+q^2) = x+y$

$$z^2p^2+z^2q^2 = x+y$$

$$(zp)^2+(zq)^2 = x+y \qquad \text{... (1)}$$

This can be reduced to standard type of the following substitution

Put $Z = z^2$

$$P = \frac{\partial Z}{\partial x}$$

$$= \frac{\partial Z}{\partial z}\,\frac{\partial z}{\partial x}$$

$$= 2z\,p$$

$$\therefore P = 2zp \qquad \text{... (2)}$$

$$zp = \frac{P}{2}$$

$$\frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z}\,\frac{\partial z}{\partial y}$$

$$Q = 2z\,q$$

$$zq = \frac{Q}{2} \qquad \text{... (3)}$$

Substituting (2) & (3) in (1)

$$\left(\frac{P}{2}\right)^2 + \left(\frac{Q}{2}\right)^2 = x + y$$

$$\frac{P^2}{4} + \frac{Q^2}{4} = x + y$$

$$P^2 + Q^2 = 4(x + y) \qquad \ldots (4)$$

This is of type 4.

Equation (4) can be written as

$$P^2 + Q^2 = 4x + 4y$$

$$P^2 - 4x = 4y - Q^2 = k \text{ (say)}$$

$$P^2 - 4x = k$$

$$P^2 = k + 4x$$

$$\therefore P = \sqrt{k + 4x}$$

$$4y - Q^2 = k$$

$$Q^2 = 4y - k$$

$$Q = \sqrt{4y - k}$$

We know that

$$dz = P\,dx + Q\,dy$$

$$dz = \sqrt{4 + 4x}\,dx + \sqrt{4y - k}\,dy$$

$$= 2\sqrt{1 + x}\,dx + 2\sqrt{y - \frac{k}{4}}\,dy$$

Integrating on both sides,

$$\int dz = \int 2\sqrt{1 + x}\,dx + \int 2\sqrt{y - \frac{k}{4}}\,dy$$

$$z = 2\int (1+x)^{1/2}\, dx + 2\int \left(y - \frac{k}{4}\right)^{1/2} dy$$

$$= 2\,\frac{(1+x)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + 2\,\frac{\left(y-\frac{k}{4}\right)^{\frac{1}{2}+1}}{\frac{1}{2}+1}$$

$$= 2\,\frac{(1+x)^{3/2}}{\frac{3}{2}} + 2\,\frac{\left(y-\frac{k}{4}\right)^{3/2}}{\frac{3}{2}}$$

$$= \frac{4}{3}\left[(1+x)^{\frac{3}{2}} + \left(y-\frac{k}{4}\right)^{3/2}\right]$$

11.(b)(ii) Solve the equation $(x^2 - y^2 - z^2)\,p + 2xyq = 2xz$

Solution : This is of the form Lagrange's $Pp + Qq = R$

The subsidiary equation is $\dfrac{dx}{x^2 - y^2 - z^2} = \dfrac{dy}{2xy} = \dfrac{dz}{2xz}$

Considering two equations,

$$\frac{dy}{2xy} = \frac{dz}{2xz}$$

Cancelling '2x'

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating both sides,

$$\int \frac{dy}{y} = \int \frac{dz}{z}$$

$$\log y = \log z + \log c_1$$

$$\log y - \log z = \log c_1$$

$$\log\left(\frac{y}{z}\right) = \log c_1$$

$$\frac{y}{z} = c_1$$

Consider the multiplier x, y, z

$$\frac{x\,dx}{x(x^2 - y^2 - z^2)} = \frac{y\,dy}{2xy^2} - \frac{z\,dz}{2xz^2}$$

$$\frac{x\,dx + y\,dy + z\,dz}{x(x^2 - y^2 - z^2) + y(2xy) + z(2zx)}$$

$$\frac{x\,dx + y\,dy + z\,dz}{x^3 - xy^2 - xz^2 + 2xy^2 + 2z^2x}$$

$$\frac{x\,dx + y\,dy + z\,dz}{x(x^2 + y^2 + z^2)}$$

Equating the ratios

$$\frac{x\,dx + y\,dy + z\,dz}{x(x^2 + y^2 + z^2)} = \frac{dz}{2xz}$$

Cancelling 'x'

$$\frac{2(x\,dx + y\,dy + z\,dz)}{x^2 + y^2 + z^2} = \frac{dz}{z}$$

Integrating both sides,

$$\log(x^2 + y^2 + z^2) = \log z + \log c_2$$

$$\log x^2 + y^2 + z^2 - \log z = \log c_2$$

$$\log\frac{x^2 + y^2 + z^2}{z} = \log c_2$$

$$\frac{x^2 + y^2 + z^2}{z} = c_2$$

$\therefore$ The solution is $\phi\,(c_1, c_2) = 0$

$$\phi\left(\frac{y}{z}, \frac{x^2 + y^2 + z^2}{z}\right)$$

12.(a)(i) Find the Fourier series of f(x) of period $2l$ defined as follows : f(x) $= \begin{cases} l - x, & 0 \le x \le l \\ 0, & l \le x \le 2l \end{cases}$

Hence deduce that

(i) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} = \frac{\pi}{4}$

(ii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$

Solution : The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$= \frac{1}{l}\left[\int_0^{l} f(x)\,dx + \int_l^{2l} f(x)\,dx\right]$$

$$= \frac{1}{l}\left[\int_0^{l} (l-x)\,dx + \underbrace{\int_l^{2l} 0\,dx}_{0}\right]$$

$$= \frac{1}{l}\left[\int_0^l (l-x)\,dx\right]$$

$$= \frac{1}{l}\left[lx - \frac{x^2}{2}\right]_0^l$$

$$= \frac{1}{l}\left[\left(l^2 - \frac{l^2}{2}\right) - (0-0)\right]$$

$$= \frac{1}{l}\cdot\frac{l^2}{2}$$

$$a_0 = \frac{l}{2}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx + \int_l^{2l} f(x)\cos\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{1}{l}\left[\int_0^l (l-x)\cos\frac{n\pi x}{l}\,dx + \int_l^{2l} 0\ \cos\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{1}{l}\int_0^l (l-x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[(l-x)\frac{\left(\sin\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}} - (-1)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= -\frac{1}{l}\ \frac{l^2}{n^2\pi^2}\ \left(\cos\frac{n\pi x}{l}\right)_0^l$$

$$= -\frac{l}{n^2\pi^2}\ [\cos n\pi - \cos 0]$$

$$= -\frac{l}{n^2\pi^2}\ [(-1)^n - 1]$$

Case : 1.

When n is even

$$a_n = -\frac{l}{n^2\pi^2}\ [1-1]$$

$$= -\frac{l}{n^2\pi^2}\ [0]$$

$$= 0$$

Case : 2.

When n is odd

$$a_n = -\frac{l}{n^2\pi^2}(-1-1)$$

$$= \frac{2l}{n^2\pi^2}$$

$$b_n = \frac{1}{l} \int_0^{2l} f(x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{1}{l} \left[\int_0^{l} f(x) \sin \frac{n\pi x}{l} dx + \int_l^{2l} f(x) \sin \frac{n\pi x}{l} dx \right]$$

$$= \frac{1}{l} \left[\int_0^{l} (l-x) \sin \frac{n\pi x}{l} dx + \underbrace{\int_0^{2l} 0 \sin \frac{n\pi x}{l} dx}_{\downarrow \; 0} \right]$$

$$= \frac{1}{l} \left[\int_0^{l} (l-x) \sin \frac{n\pi x}{l} dx \right]$$

$$= \frac{1}{l} \left[(l-x) \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - (-1) \underbrace{\left(\frac{\sin \frac{n\pi x}{l}}{\frac{n^2 \pi^2}{l^2}} \right)}_{\downarrow \; 0} \right]_0^l$$

$$= \frac{1}{l} \left[(l-x) \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) \right]_0^l$$

$$= \frac{-1}{l} \cdot \frac{l}{n\pi} \left[(l-x) \cos n\pi x \right]_0^l$$

$$= -\frac{1}{n\pi} \left[0 - l \cos 0 \right]$$

$$= -\frac{1}{n\pi} [-l] \qquad [\because \cos 0 = 1]$$

$$b_n = \frac{l}{n\pi}$$

The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= \frac{\frac{l}{2}}{2} + \sum_{n=1,3,5}^{\infty} \frac{2l}{n^2\pi^2} \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} \frac{l}{n\pi} \sin\frac{n\pi x}{l}$$

$$= \frac{l}{4} + \frac{2l}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{1}{n^2} \cos\frac{n\pi x}{l} + \frac{l}{\pi}\cdot\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi x}{l}}{n}$$

$$= \frac{l}{4} + \frac{2l}{\pi^2}\left[\frac{\cos\frac{\pi x}{l}}{1^2} + \frac{\cos\frac{3\pi x}{l}}{3^2} + \ldots\right]$$

$$+ \frac{l}{\pi}\left[\frac{\sin\frac{\pi x}{l}}{1} + \frac{\sin\frac{2\pi x}{l}}{2} + \ldots\right] \qquad \ldots (1)$$

Put $x = \frac{l}{2}$ in (1)

$$l - \frac{l}{2} = \frac{l}{4} + \frac{2l}{\pi^2}\left[\frac{\cos\frac{\pi}{2}}{1^2} + \frac{\cos\frac{3\pi}{2}}{3^2} + \ldots\right]$$

$$+ \frac{l}{\pi}\left[\frac{\sin\frac{\pi}{2}}{1} + \frac{\sin\frac{2\pi}{2}}{2} + \ldots\right] \qquad [f(x) = l - x \text{ for } x = \tfrac{l}{2}]$$

$$\frac{l}{2} = \frac{l}{4} + \frac{2l}{\pi^2}\left[\frac{\cos\frac{\pi}{2}}{1^2} + \frac{\cos\frac{3\pi}{2}}{3^2} + \ldots\right] + \frac{l}{\pi}\left[\frac{\sin\frac{\pi}{2}}{1} + \frac{\sin\frac{2\pi}{2}}{2} + \ldots\right]$$

$$\frac{l}{2} - \frac{l}{4} = \frac{2l}{\pi^2}[0 + 0 + \ldots] + \frac{l}{\pi}\left[\frac{1}{1} + 0 - \frac{1}{3} + 0 + \frac{1}{5} + \ldots\right]$$

$$\frac{l}{4} = \frac{l}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} + \ldots\right]$$

Cancelling l and Cross multiplying

$$\therefore\ 1 - \frac{1}{3} + \frac{1}{5} + \ldots = \frac{\pi}{4}$$

Put $x = 0$ in (1)

But $x = 0$ is a point of discontinuity of $f(x)$.

$\therefore$ Fourier series conveges to $\dfrac{f(0^+) + f(2l^-)}{2}$

$$\frac{(l-0)+0}{2} = \frac{l}{4} + \frac{2l}{\pi^2}\left[\frac{\cos 0}{1^2} + \frac{\cos 0}{3^2} + \ldots\right] + \frac{l}{\pi}\left[\frac{\sin 0}{1} + \frac{\sin 0}{2} + \ldots\right]$$

$$\frac{l}{2} = \frac{l}{4} + \frac{2l}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \ldots\right] \qquad f(0+) = l-0=l$$

$$\frac{l}{2} - \frac{l}{4} = \frac{2l}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \ldots\right] \qquad f(2l-) = 0$$

$$\frac{l}{4} = \frac{2l}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \ldots\right]$$

$$\therefore\ \frac{1}{1^2} + \frac{1}{3^2} + \ldots. = \frac{l\pi^2}{8l}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \ldots. = \frac{\pi^2}{8}$$

12.(a)(ii) **Find the complex form of the Fourier series of the function $f(x) = e^{-x}$ when $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$**

Solution : $2l = UL - LL$

$$= \pi - (-\pi) = 2\pi$$

$$\therefore l = \pi \qquad C_n = \frac{1}{2l} \int_{-l}^{l} f(x)\, e^{\frac{-in\pi x}{l}}\, dx$$

$$\therefore C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-x}\, e^{\frac{-in\pi x}{\pi}}\, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-x}\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(-1-in)x}\, dx$$

$$= \frac{1}{2\pi} \left[\frac{e^{(-1-in)x}}{-1-in}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi} \times \frac{1}{-1-in} \left[e^{(-1-in)\pi} - e^{-(-1-in)\pi}\right]$$

$$= \frac{1}{2\pi} \times \frac{1}{-1-in} \times \frac{-1+in}{-1-in} \left[e^{(-1-in)\pi} - e^{-(-1-in)\pi}\right]$$

$$= \frac{1}{2\pi}\, \frac{-1+in}{1-i^2n^2} \left[e^{(-1-in)\pi} - e^{-(-1-in)\pi}\right]$$

$$= \frac{1}{2\pi}\, \frac{-1+in}{1+n^2} \left[e^{(-1-in)\pi} - e^{-(-1-in)\pi}\right]$$

$$= \frac{-1+in}{2\pi(1+n^2)}\left[e^{-\pi}e^{-in\pi} - e^{\pi}e^{in\pi}\right]$$

$$= \frac{-1+in}{2\pi(1+n^2)}\left[e^{-\pi}(\cos n\pi - i\sin n\pi) - e^{\pi}(\cos n\pi + i\sin n\pi)\right]$$

$$= \frac{-1+in}{2\pi(1+n^2)}\left[e^{-\pi}\cos n\pi - e^{\pi}\cos n\pi\right] \qquad [\because \sin n\pi = 0]$$

$$= \frac{-1+in}{2\pi(1+n^2)}\left[e^{-\pi}(-1)^n - e^{\pi}(-1)^n\right]$$

$$= (-1)^n \frac{(-1+in)}{2\pi(1+n^2)}[e^{-\pi} - e^{\pi}]$$

12. (b) (i) By using Fourier cosine series for f(x) = x in $0 < x < \pi$.

Show that $\dfrac{\pi^4}{96} = 1 + \dfrac{1}{3^4} + \dfrac{1}{5^4} + \dfrac{1}{7^4} + \ldots$

$$a_0 = \pi \Rightarrow a_0^2 = \pi^2$$

$$a_n = \frac{2}{n^2 \pi} \left[(-1)^n - 1\right]$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{-4}{n^2 \pi} & \text{if } n \text{ is odd} \end{cases}$$

$$a_n^2 = \frac{16}{n^4 \pi^2} \text{ if } n \text{ is odd.}$$

Hence by Parseval's theorem,

$$\frac{1}{\pi} \int_0^{\pi} [f(x)]^2 \, dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} a_n^2 \qquad \dots (1)$$

$$f(x) = x \quad \Rightarrow \left[f(x)\right]^2 = x^2$$

$$\int_0^{\pi} \left[f(x)\right]^2 dx = \int_0^{\pi} x^2 \, dx = \left[\frac{x^3}{3}\right]_0^{\pi} = \frac{\pi^3}{3}$$

$$(1) \Rightarrow \frac{1}{\pi} \left[\frac{\pi^3}{3}\right] = \frac{\pi^2}{4} + \frac{1}{2} \sum_{n=\text{odd}}^{\infty} \frac{16}{n^4 \pi^2}$$

$$\frac{\pi^2}{3} - \frac{\pi^2}{4} = \frac{8}{\pi^2} \sum_{n=\text{odd}}^{\infty} \frac{1}{n^4}$$

$$\frac{\pi^2}{12} = \frac{8}{\pi^2} \sum_{n=\text{odd}}^{\infty} \frac{1}{n^4}$$

$$\Rightarrow \sum_{n=\text{odd}}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{96} \qquad \therefore 1 + \frac{1}{3^4} + \frac{1}{5^4} + \dots = \frac{\pi^4}{96}$$

12. (b) (ii) **Find the Fourier series of period 2π as far as the first harmonic to represent the function $y = f(x)$ defined by the following table :**

x^0	0	30	60	90	120	150	180	210	240	270	300	330	360
y	2.34	3.01	3.69	4.15	3.69	2.20	0.83	0.51	0.88	1.09	1.19	1.64	2.34

Solution :

x	$f(x)$	$\sin x$	$\cos x$	$f(x)\sin x$	$f(x)\cos x$
0	2.34	0	1	0	2.340
30	3.01	0.50	0.87	1.505	2.619
60	3.69	0.87	0.50	3.210	1.845
90	4.15	1	0	4.150	0
120	3.69	0.87	–0.50	3.210	–1.845
150	2.20	0.50	–0.87	1.1000	–1.914
180	0.83	0	–1	0	–0.830
210	0.51	–0.50	–0.87	–0.255	–0.444
240	0.88	–0.87	–0.50	–0.766	–0.440
270	1.09	–1	0	–1.090	0
300	1.19	–0.87	0.5	–1.035	0.595
330	1.64	–0.5	0.87	–0.820	1.427
	$\Sigma y = 25.22$			$\Sigma y(\sin x) = 9.209$	$\Sigma y(\cos x) = 3.353$

$$a_0 = 2\Sigma y$$

$$= \frac{2 \times 25.22}{12}$$

$$= 4.203$$

$$a_1 = 2\,\Sigma\, y\cos x$$

$$a_1 = \frac{2 \times 3.353}{12}$$

$$= 0.559$$

$$b_1 = 2\,\Sigma\, y\sin x$$

$$= \frac{2 \times 9.209}{12}$$

$$= 1.535$$

$$f(x) = \frac{a_0}{2} + a_1\cos x + b_1\sin x$$

$$= 2.105 + 0.559\cos x + 1.535\sin x$$

13. (a) A string is stretched and fastened to two points l apart. Motion is started by displacing the string into the form $y = k\,(lx - x^2)$, from which it is released at time t = 0. Find the displacement of any point on the string at a distance x from one end.

Solution : The wave equation is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

The solution is

$$y = (A\cos px + B\sin px)\,(C\cos p\,ct + D\sin p\,ct) \qquad \ldots (1)$$

We have the following boundary conditions.

(i) $y = 0$ when $x = 0$

(ii) $y = 0$ when $x = l$

(iii) $\frac{\partial y}{\partial t} = 0$ when $t = 0$ [$\because$ initial velocity is zero]

(iv) $y = k(lx - x^2)$ when $t = 0$ ($\because$ initial shape is given)

Applying boundary condition (i) in (1)

$$0 = (A\cos 0 + B\sin 0)(C\cos p\,ct + D\sin p\,ct)$$

$$0 = (A + 0)(C\cos p\,ct + D\sin p\,ct) \qquad [\cos 0 = 1,\ \sin 0 = 0]$$

$$\Rightarrow \boxed{A = 0} \qquad [\because C\cos pct + D\sin pct \neq 0]$$

Substituting $A = 0$ in (1)

Solution (1) becomes

$$y = B\sin px\,(C\cos pct + D\sin pct) \qquad \ldots (2)$$

Applying condition (ii) in (2)

$$0 = B\sin pl\,(C\cos pct + D\sin pct)$$

$$\Rightarrow \sin pl = 0 \qquad [\because B \neq 0]$$

But $\sin n\pi = 0$

$$\therefore pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}}$$

Substitute $p = \frac{n\pi}{l}$ in (2)

(2) becomes

$$y = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi ct}{l} + D\sin\frac{n\pi ct}{l}\right) \qquad \ldots (3)$$

To apply (iii) differentiate (3) partially w.r.to 't'

$$\frac{\partial y}{\partial t} = B \sin\frac{n\pi x}{l}\left(-C\sin\left(\frac{n\pi ct}{l}\right)\frac{n\pi c}{l} + D\cos\left(\frac{n\pi ct}{l}\right)\frac{n\pi c}{l}\right)$$

Now applying condition (iii), we get

$$0 = B\sin\frac{n\pi x}{l}\left(-C\sin 0\frac{n\pi c}{l} + D\cos 0\frac{n\pi c}{l}\right]$$

$$0 = B\sin\frac{n\pi x}{l}\left[0 + D\frac{n\pi c}{l}\right] \qquad [\because \sin 0 = 0;\ \cos 0 = 1]$$

$$\Rightarrow \boxed{D = 0}$$

Substitute $D = 0$ in (3)

$$y = B\sin\frac{n\pi x}{l}C\cos\frac{n\pi ct}{l}$$

$$= BC\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}$$

$$= b\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \qquad [\text{taking } BC = b]$$

Using principle of superposition we can write this as

$$y = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}\cos\frac{n\pi ct}{l}$$

Applying condition (iv) in (4)

$$k(lx - x^2) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos 0$$

$$= \sum_{n=1}^{\infty} b_n \frac{n\pi x}{l} 1 \qquad [\because \cos 0 = 1]$$

R.H.S is half range sine series

$\therefore$ Its co-efficients b_n is given by

$$b_n = \frac{2}{l}\int_0^l f(x)\,\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l k\,(lx - x^2)\,\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2k}{l}\int_0^l (lx - x^2)\,\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2k}{l}\left[\underbrace{(lx - x^2)\frac{\left(-\cos\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}}_{0} - \underbrace{(l-2x)\frac{\left(-\sin\frac{n\pi x}{l}\right)}{\frac{n^2\pi^2}{l^2}}}_{0} + (-2)\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right]_0^l$$

$$= \frac{2k}{l}\left[\frac{-2\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right]_0^l$$

$$= \frac{-4k}{l}\,\frac{l^3}{n^3\pi^3}\left[\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{-4kl^2}{n^3\pi^3}\left[\cos\frac{n\pi l}{l} - \cos 0\right]$$

$$= \frac{-4kl^2}{n^3\pi^3}\left[\cos n\pi - \cos 0\right]$$

$$= \frac{-4kl^2}{n^3\pi^3}\left[(-1)^n - 1\right] \qquad \cos n\pi = (-1)^n$$

Case : 1. When n is even number $(-1)^n = 1$

$$\therefore b_n = \frac{-4kl^2}{n^3\pi^3} [1-1]$$

$$= \frac{-4kl^2}{n^3\pi^3}[0]$$

$$= 0$$

Case : 2. When n is odd number $(-1)^n = -1$

$$\therefore b_n = \frac{-4kl^2}{n^3\pi^3}(-1-1)$$

$$= \frac{-4kl^2}{n^3\pi^3}(-2)$$

$$= \frac{8kl^2}{n^3\pi^3}$$

$\therefore$ the final solution is obtained by substituting for b_n in (4).

$$y = \sum_{n=1,3,5}^{\infty} \frac{8kl^2}{n^3\pi^3} \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l} \quad [\because n \text{ takes only odd values}]$$

13.(a) Find the steady state temperature at any point of a square plate if two adjacent edges are kept at 0°C and others are at 100°C.

Solution :

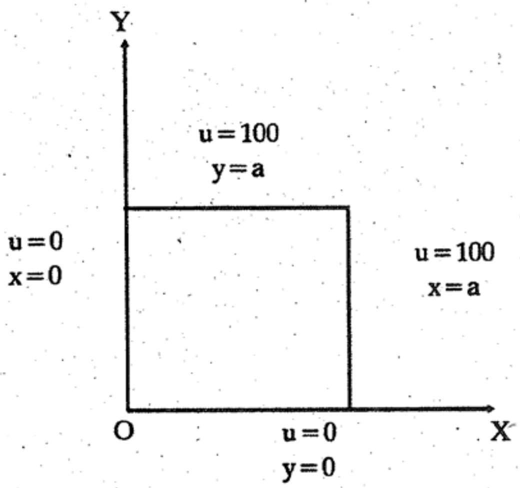
Let a be the side of square.

Let the edges $x = a$, $y = a$ (i.e.,) right vertical edges and upper horizontal edge can be kept at 100°C.

The other two edges can be at 0°C. Hence 2 edges are kept at non-zero temperature. We cannot solve neat flow equation directly.

$\therefore$ we split the solution into two parts as follows

$$u = u_1 + u_2$$



For u_1 we take only one edge as non-zero temperature and other three pages 0^oC temperature. Similarly for u_2 take only one edge as non-zero temperature (other than the above). Then we have to solve for u_1 and u_2 separately. The trial solution is given by

$$u = u_1 + u_2$$

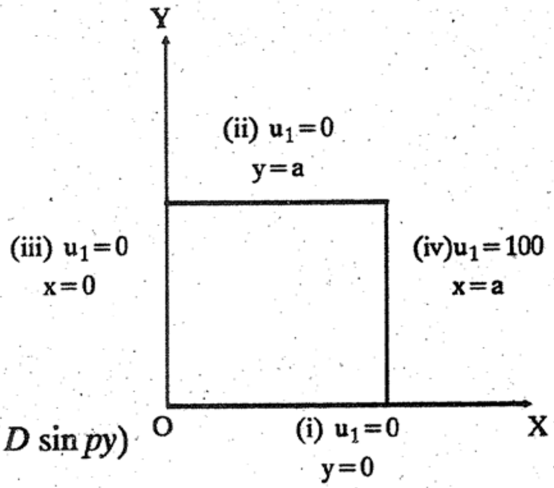
First we will solve for u_1 taking temperature at the right vertical edge as 100^oC. For this we have the following boundary conditions

(i) $u_1 = 0$ when $y = 0$

(ii) $u_1 = 0$ when $y = a$

(iii) $u_1 = 0$ when $x = 0$

(iv) $u_1 = 100$ when $x = a$



Considering the first solution

$$u_1 = (A e^{px} + B e^{-px})(C \cos py + D \sin py)$$

Applying the boundary conditions we get u_1 as

$$u_1 = \sum_{n=1}^{\infty} b_n \left(e^{\frac{n\pi x}{a}} - e^{\frac{-n\pi x}{a}} \right) \sin \frac{n\pi y}{a}$$

Applying condition (iv) we find the value of b_n

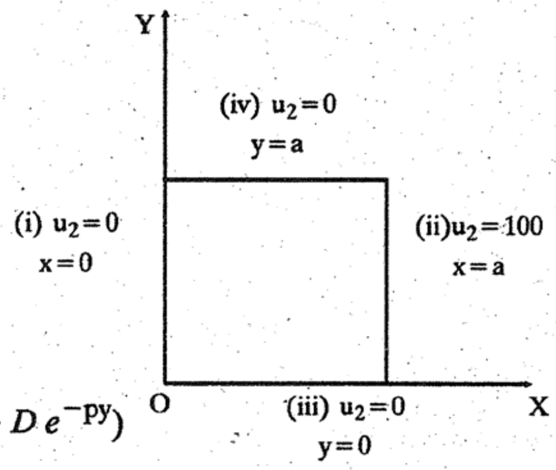
Next we will solve for u_2 taking temperature at upper edge as 100°C. For this we have the following boundary conditions.

(i) $u_2 = 0$ when $x = 0$

(ii) $u_2 = 0$ when $x = a$

(iii) $u_2 = 0$ when $y = 0$

(iv) $u_2 = 100$ when $y = a$



Considering the 2^n solution

$$u_2 = (A \cos px + B \sin px)(Ce^{py} + De^{-py})$$

Applying the boundary conditions,

$$u_2 = \sum_{n=1}^{\infty} b_n \left(e^{\frac{n\pi y}{a}} - e^{\frac{-n\pi y}{a}} \right) \sin \frac{n\pi x}{a}$$

Applying condition (iv) we find the value of b_n

$\therefore$ The final solution $u = u_1 + u_2$

14.(a)(i) Find the Fourier transform of $f(x) = \begin{cases} a - |x|, & \text{if } |x| < a \\ 0, & \text{if } |x| > a > 0 \end{cases}$

(i) Hence show that $\int_0^{\infty} \left(\frac{\sin t}{t} \right)^2 dt = \frac{\pi}{2}$

Solution : $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a - |x|) e^{isx} dx$

Here $a - |x|$ is an even function. But e^{isx} is neither even nor odd function. $(a - |x|) e^{isx}$ is neither even nor odd function.

$\therefore$ as such we cannot integrate this

Hence we write $e^{isx} = \cos sx + i \sin sx$

$$\therefore F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a - |x|)(\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{a} (a - |x|) \cos sx\, dx + i \underbrace{\int_{-a}^{a} (a - |x|) \sin sx\, dx}_{0} \right].$$

The second integral become zero since the function odd and the limits are –1, 1.

Also the function under the first integral is even.

$\therefore$ we get

$$\therefore F[f(x)] = \frac{2}{\sqrt{2\pi}} \int_{0}^{a} (a - |x|) \cos sx\, dx$$

[$\because$ $(a - |x| \cos sx$ is even function]

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{a} (a - x) \cos sx\, dx \quad [\because x \text{ takes only positive values}]$$

$$= \frac{2}{\sqrt{2\pi}} \left[\underbrace{(a - x)\left(\frac{\sin sx}{s}\right)}_{0} - (-1)\left(\frac{-\cos sx}{s^2}\right) \right]_0^a$$

[$\because$ $\sin 0 = 0$]

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{-\cos sx}{s^2} \right]_0^a$$

$$= \frac{-2}{\sqrt{2\pi}} \left[\frac{\cos sx}{s^2} \right]_0^a$$

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$$= \frac{-2}{\sqrt{2\pi}}\left[\frac{\cos sa - \cos 0}{s^2}\right]$$

$$= \frac{-2}{\sqrt{2\pi}}\left[\frac{\cos sa - 1}{s^2}\right]$$

$$= \frac{2}{\sqrt{2\pi}}\left[\frac{1 - \cos sa}{s^2}\right]$$

$$= \frac{2}{\sqrt{2\pi}}\left[\frac{2\sin^2\frac{as}{2}}{s^2}\right] \qquad \left[\because\ 1 - \cos\theta = 2\sin^2\frac{\theta}{2}\right]$$

$$F[f(x)] = \frac{4}{\sqrt{2\pi}}\left(\frac{\sin\frac{as}{2}}{s}\right)^2$$

Now applying inverse Fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]\, e^{-isx}\, ds$$

$$a - |x| = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{4}{\sqrt{2\pi}}\left(\frac{\sin\frac{as}{2}}{s}\right)^2 e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{4}{\sqrt{2\pi}}\right)\int_{-\infty}^{\infty}\left(\frac{\sin\frac{as}{2}}{2}\right)^2(\cos sx - i\sin sx)\, ds$$

$$= \frac{4}{2\pi}\left[\int_{-\infty}^{\infty}\left(\frac{\sin\frac{as}{2}}{s}\right)^2 \cos sx\, ds - \underbrace{\int_{-\infty}^{\infty}\left(\frac{\sin\frac{as}{2}}{s}\right)^2 i\sin sx\, ds}_{0}\right]$$

$\downarrow$ [$\because$ the function is odd]

0

$$a - |x| = \frac{2}{\pi}(2) \int_0^{\infty} \left(\frac{\sin \frac{as}{2}}{s} \right)^2 \cos sx \, ds \quad [\because \text{ the function is even}]$$

$$a - |x| = \frac{4}{\pi} \int_0^{\infty} \left(\frac{\sin \frac{as}{2}}{s} \right)^2 \cos sx \, ds$$

Take $a = 1$ on both sides,

$$1 - |x| = \frac{4}{\pi} \int_0^{\infty} \left(\frac{\sin \frac{s}{2}}{s} \right)^2 \cos sx \, ds$$

Put $x = 0$ on both sides

$$1 = \frac{4}{\pi} \int_0^{\infty} \left(\frac{\sin \frac{s}{2}}{s} \right)^2 (\cos 0) \, ds$$

$$1 = \frac{4}{\pi} \int_0^{\infty} \left(\frac{\sin \frac{s}{2}}{s} \right)^2 ds \qquad [\because \cos 0 = 1]$$

Substitute $\frac{s}{2} = t$

$$s = 2t$$

$$ds = 2\,dt$$

$$\therefore 1 = \frac{4}{\pi} \int_0^{\infty} \left(\frac{\sin t}{2t} \right)^2 2\,dt$$

$$1 = \frac{4}{\pi} \int_0^{\infty} \frac{2}{4} \left(\frac{\sin t}{t} \right)^2 dt$$

$$1 = \frac{4}{\pi} \frac{1}{2} \int_0^{\infty} \left(\frac{\sin t}{t} \right)^2 dt$$

$$1 = \frac{2}{\pi} \int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt$$

$$\therefore \int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$$

14. (a)(ii) Prove that $f(x) = e^{-x^2/2}$ is self-reciprocal under the Fourier cosine transform.

Solution : $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2/2} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2/2} \text{ R.P. } e^{isx} \, dx$$

$$= \text{R.P} \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2/2} e^{isx} \, dx$$

$$= \text{R.P} \sqrt{\frac{2}{\pi}} \int_0^\infty e^{\frac{-x^2}{2} + isx} \, dx$$

$$= \text{R.P} \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{1}{2}(x^2 - 2isx)} \, dx$$

Consider $x^2 - 2isx$

Adding and subtracting $i^2 s^2$

$$x^2 - 2isx + i^2 s^2 - i^2 s^2$$

$$(x - is)^2 - i^2 s^2 \qquad [i^2 = -1]$$

$$(x - is)^2 + s^2$$

$$F_c[f(x)] = \text{R.P} \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-\frac{1}{2}[(x-is)^2+s^2]} dx$$

$$= \text{R.P.} \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-\frac{1}{2}(x-is)^2} e^{\frac{-s^2}{2}} dx$$

$$= \text{R.P} \sqrt{\frac{2}{\pi}} e^{\frac{-s^2}{2}} \int_0^{\infty} e^{-\frac{1}{2}(x-is)^2} dx$$

$$= \text{R.P} \sqrt{\frac{2}{\pi}} e^{\frac{-s^2}{2}} \int_0^{\infty} e^{-\left(\frac{x-is}{\sqrt{2}}\right)^2} dx$$

$$= \text{R.P} \sqrt{\frac{2}{\pi}} \frac{1}{2} e^{\frac{-s^2}{2}} \int_{-\infty}^{\infty} e^{-\left(\frac{x-is}{\sqrt{2}}\right)^2} dx$$

Put $\frac{x-is}{\sqrt{2}} = y$

Differentiating, $\frac{dx}{\sqrt{2}} = dy$

$$\therefore dx = \sqrt{2}\, dy$$

when $x = -\infty$ $\quad$ $y = -\infty$

when $x = \infty$ $\quad$ $y = \infty$

$$\therefore F_c[f(x)] = \text{R.P} \sqrt{\frac{2}{\pi}} \frac{1}{2} e^{\frac{-s^2}{2}} \int_{-\infty}^{\infty} e^{-y^2} \sqrt{2}\, dy$$

$$= \text{R.P} \frac{\sqrt{2}}{\sqrt{\pi}} \frac{1}{2} \sqrt{2}\, e^{-s^2/2} \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \text{R.P} \frac{1}{\sqrt{\pi}} e^{\frac{-s^2}{2}} \sqrt{\pi} \qquad \left[\because \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}\right]$$

[by beta-gamma integral]

$$= \text{R.P } e^{\frac{-s^2}{2}}$$

$$= e^{\frac{-s^2}{2}}$$

Hence $e^{\frac{-x^2}{2}}$ is self reciprocal.

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14. (b)(i) Find the Fourier sine transform of $f(x) = e^{-ax}$ where $a > 0$ and hence deduce that $\int_0^\infty \frac{\sin sx}{a^2+s^2} ds = \frac{\pi}{2} e^{-ax}$

Solution : The Fourier sine transform is defined as

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$\therefore F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2}(-a \sin sx - s \cos sx)\right]_0^\infty$$

$$\because \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-1}{a^2+s^2}\right] \left\{e^{-ax} a \sin sx + e^{-as} s \cos sx\right\}_0^\infty$$

$$\downarrow \quad [\because a = -a, \; b = s]$$
$$0$$

$$= \sqrt{\frac{2}{\pi}} \frac{-s}{a^2+s^2}\left[e^{-ax}\cos sx\right]_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}} \frac{-s}{a^2+s^2}[0 - e^0 \cos 0] \qquad [\because e^{-\infty} = 0]$$

$$= \sqrt{\frac{2}{\pi}} \frac{-s}{a^2+s^2}(-1) \qquad [\because e^0 = 1, \cos 0 = 1]$$

$$= \sqrt{\frac{2}{\pi}} \frac{s}{a^2+s^2}$$

Now we have to apply inverse Fourier sine transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s[f(x)] \sin sx\, ds$$

$$e^{-ax} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} \frac{s}{a^2+s^2} \sin sx\, ds$$

$$e^{-ax} = \sqrt{\frac{2}{\pi}} \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{s}{a^2+s^2} \sin sx\, ds$$

$$e^{-ax} = \frac{2}{\pi} \int_0^{\infty} \frac{s}{a^2+s^2} \sin sx\, ds$$

$$\therefore \int_0^{\infty} \frac{s}{a^2+s^2} \sin sx\, ds = \frac{\pi}{2} e^{-ax}$$

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14. (b) (ii) Find the Fourier transform of $f(x) = \begin{cases} 1, & \text{if } |x| < 1 \\ 0, & \text{if } |x| > 1 \end{cases}$ and hence deduce that $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$

Solution : The Fourier transform of the above function is given by

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} 1\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is}\right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{is} - e^{-isx}}{is}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2\, i \sin s}{is}\right]$$

$$= \frac{2 \sin s}{s\sqrt{2\pi}}$$

$$F\,[f(x)] = \frac{2 \sin s}{s\sqrt{2\pi}}$$

Now applying inverse Fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F\,[f(x)]\, e^{-isx}\, ds$$

Substituting for $F\,[f(x)]$ we get

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{2 \sin s}{s\sqrt{2\pi}}\, e^{-isx}\, ds$$

$$= \frac{2}{2\pi} \int_{-\infty}^{\infty} \frac{\sin s}{s}\, e^{-isx}\, ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin s}{s}\, (\cos sx - i \sin sx)\, ds \quad [\because\ e^{-i\theta} = \cos\theta - i \sin\theta]$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left[\left(\frac{\sin s}{s}\right) \cos sx - \left(\frac{\sin s}{s}\right) i \sin sx\right] ds$$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{\infty}\left(\frac{\sin s}{s}\right)\cos x\,ds - i\int_{-\infty}^{\infty}\left(\frac{\sin s}{s}\right)\sin sx\,ds\right]$$

$\downarrow$ [$\because$ the function is odd]

0

$$\therefore f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s}{s}\right)\cos sx\,ds$$

$$= \frac{2}{\pi}\int_{0}^{\infty}\frac{\sin s}{s}\cos sx\,ds$$

$$\text{(ie.,)}\quad 1 = \frac{2}{\pi}\int_{0}^{\infty}\frac{\sin s}{s}\cos sx\,ds$$

Putting $x = 0$ we get

$$1 = \frac{2}{\pi}\int_{0}^{\infty}\frac{\sin s}{s}\cos 0\,ds$$

$$1 = \frac{2}{\pi}\int_{0}^{\infty}\frac{\sin s}{s}\,ds \qquad [\because \cos 0 = 1]$$

Cross multiplying

$$\frac{\pi}{2} = \int_{0}^{\infty}\frac{\sin s}{s}\,ds$$

Changing dummy variable 's' to 'x' we get

$$\int_{0}^{\infty}\frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

15.(a)(i)(1) Find the Z-transform of $\frac{1}{n(n-1)}$

Solution:

$$x(n) = \frac{1}{n(n-1)}$$

$$= \frac{1}{n-1} - \frac{1}{n}$$

$$Z[x(n)] = Z\left[\frac{1}{n-1}\right] - Z\left[\frac{1}{n}\right]$$

$$= \sum_{n=2}^{\infty} \frac{1}{n-1} z^{-n} - \sum_{n=1}^{\infty} \frac{1}{n} z^{-n}$$

$$= \frac{1}{z} \sum_{n=2}^{\infty} \frac{1}{n-1} z^{-n+1} - \sum_{n=1}^{\infty} \frac{1}{n} z^{-n}$$

$$= \frac{1}{z} \sum_{n=2}^{\infty} \frac{1}{(n-1)} z^{-(n-1)} - \sum_{n=1}^{\infty} \frac{1}{n} z^{-n}$$

$$= \frac{1}{z}\left[\frac{1}{1}\left(\frac{1}{z}\right) + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \cdots\right] - \left[\left(\frac{1}{z}\right) + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \cdots\right]$$

$$= \frac{1}{z}\left[\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \cdots\right] - \left[\left\{\left(\frac{1}{z}\right) + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{3}\right)^2 + \cdots\right\}\right]$$

$$= \frac{1}{z}\left[-\log\left(1 - \frac{1}{z}\right)\right] - \left[-\log\left(1 - \frac{1}{z}\right)\right]$$

$$= \frac{1}{z}\left[\log\frac{z}{z-1}\right] - \left[\log\frac{z}{z-1}\right]$$

$$= \left(\frac{1}{z} - 1\right)\log\left(\frac{z}{z-1}\right)$$

$$= \left(\frac{1-z}{z}\right)\log\left(\frac{z}{z-1}\right)$$

$$= -\left(\frac{z-1}{z}\right)\log\left(\frac{z}{z-1}\right)$$

$$= \left(\frac{z-1}{z}\right)\log\left(\frac{z-1}{z}\right)$$

(2) Find $Z\left(f(n)\right)$, if

$$f(n) = \frac{1}{\sqrt{5}}\left\{\left(\frac{\sqrt{5}+1}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$$

Solution : $Z[f(n)] = Z\left[\frac{1}{\sqrt{5}}\left\{\left(\frac{\sqrt{5}+1}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}\right]$

$$= \frac{1}{\sqrt{5}}\left[Z\left(\frac{\sqrt{5}+1}{2}\right)^n - Z\left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{z}{z-\left(\frac{\sqrt{5}+1}{2}\right)} - \frac{z}{z-\left(\frac{1-\sqrt{5}}{2}\right)}\right]\left[\because z(a^n) = \frac{z}{z-a}\right]$$

15. (a) (ii) Solve the difference equation $y(n+2) + 4y(n+1) + 4y(n) = n$, given $y(0) = 0$ and $y(1) = 1$, by the method of Z-transform.

Solution : Let $\bar{y} = z[y_n]$

Taking z-transform on both sides,

$$Z[y_{n+2}] + 4Z[y_{n+1}] + 4Z[y(n)] = Z[n]$$

$$z^2\left[\bar{y} - y_0 - \frac{y_1}{z}\right] + 4z\left[\bar{y} - y_0\right] + 4\bar{y} = \frac{z}{(z-1)^2}$$

$$z^2\left[\bar{y} - 0 - \frac{1}{z}\right] + 4z[\bar{y} - 0] + 4\bar{y} = \frac{z}{(z-1)^2}$$

$$z^2\bar{y} - z + 4z[\bar{y}] + 4\bar{y} = \frac{z}{(z-1)^2}$$

$$\bar{y}[z^2 + 4z + 4] - z = \frac{z}{(z-1)^2}$$

$$\bar{y}[z^2 + 4z + 4] = z(z-1)^2 + z$$

$$\bar{y}\,[z^2 + 4z + 4] = \frac{z + z\,(z-1)^2}{(z-1)^2}$$

$$\bar{y}\,(z+2)^2 = \frac{z + z\,[z^2 - 2z + 1]}{(z-1)^2}$$

$$\bar{y}\,(z+2)^2 = \frac{z + z^3 - 2z^3 + z}{(z-1)^2}$$

$$\bar{y}\,(z+2)^2 = \frac{z^3 - 2z^2 + 2z}{(z-1)^2}$$

$$\therefore\ \bar{y} = \frac{z^3 - 2z^2 + 2z}{(z-1)^2\,(z+2)^2}$$

$$y = Z^{-1}\left[\frac{z^3 - 2z^2 + 2z}{(z-1)^2\,(z+2)^2}\right]$$

Applying Partial fractions

Let $F(z) = \dfrac{z\,(z^2 - 2z + 2)}{(z-1)^2\,(z+2)^2}$

Consider $\dfrac{F(z)}{z} = \dfrac{z^2 - 2z + 2}{(z-1)^2\,(z-2)^2}$

Let $\dfrac{z^2 - 2z + 2}{(z-1)^2\,(z-2)^2} = \dfrac{A}{(z-1)^2} + \dfrac{B}{z-1} + \dfrac{C}{(z-2)^2} + \dfrac{D}{z-2}$

$$\frac{z^2 - 2z + 2}{(z-1)^2\,(z-2)^2} = \frac{A(z-2)^2 + B(z-1)\,(z-2)^2 + C(z-1)^2 + D(z-1)^2(z-2)}{(z-1)^2(z-2)^2}$$

$$z^2 - 2z + 2 = A(z-2)^2 + B\,(z-1)(z-2)^2 + C(z-1)^2 + D(z-1)^2(z-2)$$

Put $z = 1$

$$1^2 - 2(1) + 2 = A(1-2)^2 + B(1-1)(1-2)^2 + C(1-1)^2 + D(1-1)^2(1-2)$$

$$1-2+2 = A(-1)^2 + B(0) + C(0) + D(0)$$

$$A = 1$$

Put $z = 2$

$$2^2-2(2)+2 = A(2-2)^2+B(2-1)(2-2)^2+C(2-1)^2+D(2-1)^2(2-2)$$

$$4-4+2 = A(0) + B(0) + C(1)^2 + D(0)$$

Removing the brackets in (1)

$$z^2-2z+2 = A(z^2-4z+4) + B(z-1)(z^2-4z+4)+C(z^2-2z+1) + D(z^2-2z+1)(z-2)$$

Comparing the coefficients of z^3

$$B + D = 0 \qquad \dots (2)$$

Comparing the constant terms

$$4A - 4B + C - 2D = 2 \qquad \dots (3)$$

Substituting values of A and C in (3)

$$4-4B+2-2D = 2$$

$$6-4B-2D = 2$$

$$4B+2D = 4$$

$$2B + D = 2 \qquad \dots (4)$$

$$B + D = 0 \qquad \dots (2)$$

$$(4)-(2) \qquad B = 2$$

$$\therefore D = -2$$

$$\therefore \frac{z^2-2z+2}{(z-1)^2(z-2)^2} = \frac{1}{(z-1)^2} + \frac{2}{(z-1)} + \frac{2}{(z-2)^2} + \frac{-2}{(z-2)}$$

$$\therefore F(z) = \frac{z}{(z-1)^2} + \frac{2z}{z-1} + \frac{2z}{(z-2)^2} + \frac{-2z}{z-2}$$

$$\therefore F(z) = Z^{-1}\left[\frac{z}{(z-1)^2} + \frac{2z}{(z-1)} + \frac{2z}{(z-2)^2} + \frac{-2z}{(z-2)}\right]$$

$$= Z^{-1}\left[\frac{z}{(z-1)^2}\right] + Z^{-1}\left[\frac{2z}{(z-1)}\right] + Z^{-1}\left[\frac{2z}{(z-2)^2}\right] + Z^{-1}\left[\frac{-2z}{(z-2)}\right]$$

$$= n + 2(1) + 2(n)\,2^{n-1} - 2(2)^n$$

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15. (b) (i) Find the inverse Z-transform of $\frac{z^2}{(z+2)(z^2+4)}$ by the method of partial fractions.

Solution : Let $F(Z) = \frac{z^2}{(z+2)(z^2+4)}$

$$\frac{F(Z)}{z} = \frac{z}{(z+2)(z^2+4)}$$

Let $\frac{z}{(z+2)(z^2+4)} = \frac{A}{z+2} + \frac{Bz+C}{z^2+4}$

$$= \frac{A(z^2+4) + (Bz+C)(z+2)}{(z+2)(z^2+4)}$$

$$z = A(z^2+4) + (Bz+C)(z+2) \quad \dots (1)$$

Put $z = -2$

$$-2 = A(4+4) + (Bz+C)(0)$$

$$-2 = 8A$$

$$A = \frac{-2}{8}$$

$$A = -\frac{1}{4}$$

Removing the brackets in (1)

$$z = A(z^2 + 4) + (Bz^2 + 2Bz + Cz + 2C)$$

Comparing the coefficient of z^2

$$A + B = 0$$

$$\frac{-1}{4} + B = 0 \Rightarrow B = \frac{1}{4}$$

Comparing the coefficient of constant term

$$4A + 2C = 0$$

$$4\left(\frac{-1}{4}\right) + 2C = 0$$

$$-1 + 2C = 0$$

$$2C = 1$$

$$C = \frac{1}{2}$$

$$\frac{F(z)}{z} = \frac{\frac{-1}{4}}{z+2} + \frac{\left(\frac{1}{4}\right)z + \frac{1}{2}}{z^2+4}$$

$$= \frac{-\frac{1}{4}}{z+2} + \frac{\frac{1}{4}z}{z^2+4} + \frac{\frac{1}{2}}{z^2+4}$$

$$F(z) = \frac{-\frac{1}{4}z}{z+2} + \frac{\frac{1}{4}z^2}{z^2+4} + \frac{\frac{1}{2}z}{z^2+4}$$

$$Z^{-1}[F(z)] = Z^{-1}\left[\frac{-\frac{1}{4}z}{z+2} + \frac{\frac{1}{4}z^2}{z^2+4} + \frac{\frac{1}{2}z}{z^2+4}\right]$$

$$= -\frac{1}{4} Z^{-1}\left[\frac{z}{z+2}\right] + \frac{1}{4} Z^{-1}\left[\frac{z^2}{z^2+4}\right] + \frac{1}{2} Z^{-1}\left[\frac{z}{z^2+4}\right]$$

$$= -\frac{1}{4}(-2)^n + \frac{1}{4} 2^n \cos\frac{n\pi}{2} + \frac{1}{2} 2^n \sin\frac{n\pi}{2}$$

15.(b)(ii) Solve $y(n+3) - 3y(n+1) + 2y(n) = 0$, given $y(0) = 4, y(1) = 0$ and $y(2) = 8$, by using Z-transform.

Solution : Let $\bar{y} = Z[y_n]$

Taking Z-transform on both sides,

$$Z\left[y(n+3) - 3y(n+1) + 2y(n)\right] = Z(0)$$

$$Z[y(n+3)] - 3Z[y(n+1)] + 2Z[y(n)] = 0$$

$$z^3\left[\bar{y} - y_0 - \frac{y_1}{z} - \frac{y_2}{z^2}\right] - 3z[\bar{y} - y_0] + 2z[y(n)] = 0$$

$$z^3\left[\bar{y} - 4 - 0 - \frac{8}{z^2}\right] - 3z[\bar{y} - 4] + 2\bar{y} = 0$$

$$z^3\bar{y} - 4z^3 - 0 - \frac{8z^3}{z^2} - 3z\bar{y} + 12z + 2\bar{y} = 0$$

$$z^3\bar{y} - 4z^3 - 8z - 3z\bar{y} + 12z + 2\bar{y} = 0$$

$$\bar{y}[z^3 - 3z + 2] - 4z^3 + 4z = 0$$

$$\bar{y}[z^3 - 3z + 2] = 4z^3 - 4z$$

$$\bar{y} = \frac{4z^3 - 4z}{z^3 - 3z + 2}$$

$$y = Z^{-1}\left[\frac{4z^3 - 4z}{z^3 - 3z + 2}\right] = Z^{-1}\left[\frac{z(z^3 - z)}{z^3 - 3z + 2}\right]$$

To factorised we apply synthetic division.

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$$\begin{array}{c|cccc} 1 & 1 & 0 & -3 & 2 \\ & 0 & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & \boxed{0} \end{array}$$

$$(z-1)(z^2-z-2) = 0$$

$$z=1, \quad z^2-2z+z-2 = 0$$

$$(z-2)(z+1) = 0$$

$$z-2=0, \; z+1=0$$

$$z=1, \; z=2, \; z=-1$$

Consider $F(z) = \dfrac{4(z^3-z)}{z^3-3z+2}$

We can find the inverse using method of residues.

Cosider $z^{n-1} F(z) = 4z^{n-1} \dfrac{z^3-z}{(z-1)(z-2)(z+1)}$

$$= \frac{4z^{n+2}-z^n}{(z-1)(z-2)(z+1)}$$

Poles are $z=1$, $z=2$, $z=-1$

Residue at $z=1$, $\lim\limits_{z\to 1} (z-1)\dfrac{4z^{n+2}-z^n}{(z-1)(z-2)(z+1)}$

$$= \lim_{z\to 1} \frac{4z^{n+2}-z^n}{(z-2)(z+1)}$$

$$= 4\frac{(1)^{n+2}-(1)^n}{(1-2)(1+1)}$$

$$= 4\frac{(1)^{n+2}-(1)^n}{(-1)(2)}$$

$$= 4\frac{[(1)^{n+2}-(1)^n]}{-2}$$

Residue at $z = 2$, $\lim\limits_{z \to 2} (z-2)\ 4\ \dfrac{z^{n+2} - z^n}{(z-1)(z-2)(z+1)}$

$$= \lim_{z \to 2} 4\ \frac{z^{n+2} - z^n}{(z-1)(z+1)}$$

$$= 4\ \frac{(2)^{n+2} - 2^n}{(2-1)(2+1)}$$

$$= 4\ \frac{(2)^{n+2} - 2^n}{(1)(3)}$$

$$= 4\ \frac{(2)^{n+2} - 2^n}{3}$$

Residue at $z = -1$, $\lim\limits_{z \to -1} (z+1)\ 4\ \dfrac{z^{n+2} - z^n}{(z-1)(z-2)(z+1)}$

$$= \lim_{z \to -1} 4\ \frac{z^{n+2} - z^n}{(z-1)(z-2)}$$

$$= 4\ \frac{(-1)^{n+2} - (-1)^n}{(-1-1)(-1-2)}$$

$$= 4\ \frac{(-1)^{n+2} - (-1)^n}{(-2)(-3)}$$

$$= 4\ \frac{(-1)^{n+2} - (-1)^n}{6}$$

$$\therefore y_n = \frac{4(1)^{n+2} - 4(1)^n}{-2} + \frac{4(2)^{n+2} - 4(2)^n}{3} + \frac{4(-1)^{n+2} - (-1)^n}{6}$$

$$= -2(1)^{n+2} + 2(1)^n + \frac{4}{3}(2)^{n+2} - \frac{4}{3}(2)^n + \frac{2}{3}(-1)^{n+2} - \frac{1}{6}(-1)^n$$

$$= -2(1)^{n+2} + \frac{4}{3}(2)^{n+2} + \frac{2}{3}(-1)^{n+2} + 2(1)^n - \frac{4}{3}(2)^n - \frac{1}{6}(-1)^n$$