TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS

ANNA UNIVERSITY EXAMINATION, Nov/Dec. 2011

PART - A ($10 \times 2 = 20$ marks)

1. State the Dirichlet's conditions for the existence of the Fourier expansion of f(x) in the interval $(0, 2\pi)$

Solution : The function $f(x)$ can be expanded as a Fourier series in the given interval, if the following conditions are satisfied.

(1) $f(x)$ is piecewise continous

(2) $f(x)$ is periodic.

2. Find the root mean square value of the function $f(x) = x$ in $(0, l)$

Solution :

The Root mean square value is given by

$$\sqrt{\frac{\int_a^b [f(x)]^2 dx}{b-a}}$$

Given : $f(x) = x$

$UL = b = l$

$LL = a = 0$

$$= \sqrt{\frac{\int_0^l x^2 dx}{l-0}}$$

$$= \sqrt{\frac{\left[x^3/3\right]_0^l}{l}}$$

$$= \sqrt{\frac{l^3/3}{l}}$$

$$= \sqrt{\frac{l^3}{3} \times \frac{1}{l}}$$

$$= \sqrt{\frac{l^2}{3}}$$

$$= \frac{l}{\sqrt{3}}$$

3. Write the Fourier transform pair.

Solution :

The Fourier transform of $f(x)$ is

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

The inverse Fourier transform is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[f(x)]\, e^{-isx}\, dx$$

4. State Parseval's identity on Fourier transform.

Solution :

Let $F[f(x)] = F(s)$, then

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

5. Find the PDE of the family of spheres having their centers on the Z-axis.

Solution :

The equation of such spheres whose centre lies on Z-axis is

$$x^2 + y^2 + (z-c)^2 = r^2 \qquad \text{... (1)}$$

Differentiating (1) partially w.r.t. 'x'

$$2x + 2(z - c)\frac{\partial z}{\partial x} = 0$$

$$\therefore\ 2x = -2(z-c)\frac{\partial z}{\partial x}$$

$$x = -(z-c)p$$

$$(z-c) = \frac{-x}{p} \qquad \text{... (2)}$$

Differentiating (1) partially w.r.t 'y'

$$2y + 2(z-c)\frac{\partial z}{\partial y} = 0$$

$$2y = -2(z-c)\frac{\partial z}{\partial y}$$

$$y = -(z-c)q \qquad \text{... (3)}$$

Using (2) in (3) [Substituting for $(z-c)$ in (3)]

$$y = -\frac{-x}{p}q$$

$$y = \frac{xq}{p}$$

$py = xq$ is the required p.d.e

6. Solve the equation $(\mathbf{D} - \mathbf{D}')^3 \mathbf{z} = \mathbf{0}$.

Solution :

The auxiliary equation is $(m-1)^3 = 0$

$$m = 1,\ 1,\ 1$$

All the three roots are equal.

$$Z = f_1(y+mx) + xf_2(y+mx) + x^2 f_3(y+mx)$$

$$\therefore\ Z = f_1(y+x) + xf_2(y+x) + x^2 f_3(y+x)$$

7. In the wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ what does c^2 stand for?

Solution :

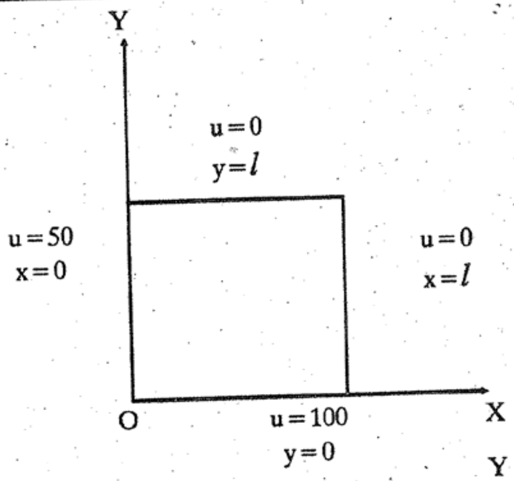
$$C^2 = \frac{T}{m}$$

T = Tension

m = Mass per unit length

8. A plate is bounded by the lines $x = 0$, $y = 0$, $x = l$ and $y = l$. Its faces are insulated. The edge coinciding with x-axis is kept at 100°C. The edge coinciding with y-axis is kept at 50°C. The other two edges are kept at 0°C. Write the boundary conditions that are needed for solving two dimensional heat flow equation.

Solution :



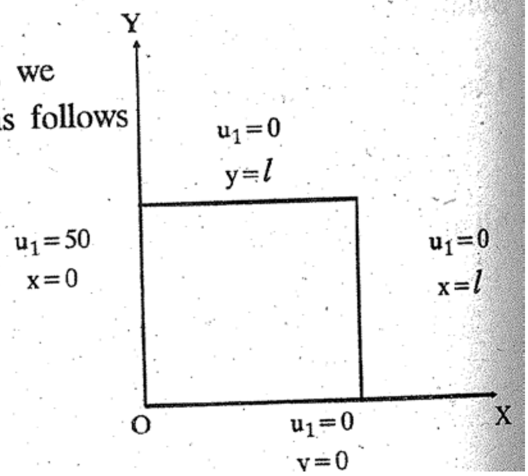
Since there are two non-zero edges, we divide the solution into two parts, as follows

(i) $u_1 = 0$ when $y = 0$

(ii) $u_1 = 0$ when $y = l$

(iii) $u_1 = 0$ when $x = l$

(iv) $u_1 = 50$ when $x = 0$

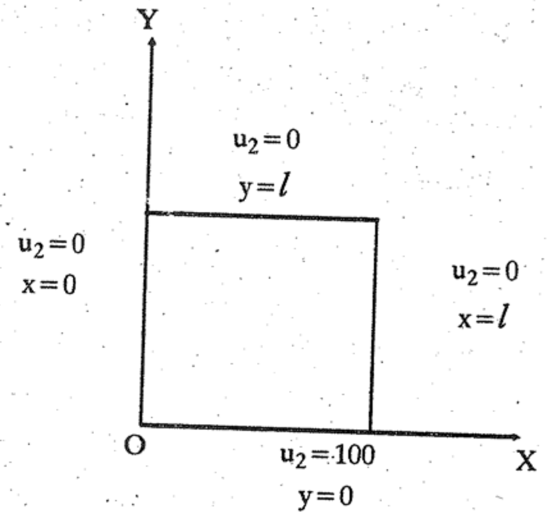


(i) $u_2 = 0$ when $x = 0$

(ii) $u_2 = 0$ when $x = l$

(iii) $u_2 = 0$ when $y = l$

(iv) $u_2 = 100$ when $y = 0$



9. Find the Z-transform of $\frac{1}{n!}$

Solution :

By definition of Z-transform

$$Z[f_n] = \sum_{n=0}^{\infty} f_n z^{-n}$$

$$Z\left[\frac{1}{n!}\right] = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

$$= \frac{1}{0!} + \frac{1}{1!} z^{-1} + \frac{1}{2!} z^{-2} + \ldots$$

$$= 1 + \frac{1}{1!}\frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \ldots$$

$$= 1 + \frac{1/z}{1!} + \frac{(1/z)^2}{2!} + \ldots \qquad [\because 0! = 1]$$

$$= e^{1/z}$$

$$\left[\text{We know that } e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots\right]$$

10. Form a difference equation by eliminating arbitrary constants from $U_n = A\,2^{n+1}$

Solution :

$$U_n = A\,2^{n+1} \quad \dots (1)$$

Replace n by $n+1$

$$U_{n+1} = A\,2^{n+2}$$

$$U_{n+1} = A\,2^{n+1}.2 \quad \dots (2)$$

Substitute (1) in (2)

$$U_{n+1} = U_n.2 \qquad [\because A\,2^{n+1} = U_n]$$

$$U_{n+1} = 2\,U_n$$

PART-B

11. (a) (i) Find the Fourier series of periodicity 3 for $f(x) = 2x - x^2$ in $0 < x < 3$

Solution :

Here $2l = UL - LL$

$$2l = 3 - 0$$

$$2l = 3$$

$$l = \frac{3}{2}$$

The Fourier series in the interval $(0, 2l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \dots (1)$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$= \frac{1}{3/2}\int_0^{2(3/2)} (2x - x^2)\,dx \qquad \left[\text{Here } l = \frac{3}{2}\right]$$

$$= \frac{2}{3}\int_0^{3} (2x - x^2)\,dx$$

$$= \frac{2}{3}\left[2.\frac{x^2}{2} - \frac{x^3}{3}\right]_0^3$$

$$= \frac{2}{3}\left[x^2 - \frac{x^3}{3}\right]$$

$$= \frac{2}{3}\left[9 - \frac{27}{3} - 0\right]$$

$$= \frac{2}{3}[9 - 9]$$

$$= 0$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{3/2}\int_0^{2(3/2)} (2x - x^2)\,.\cos\frac{n\pi x}{(3/2)}\,dx \qquad [\text{Here } l = 3/2]$$

$$= \frac{2}{3}\int_0^{3} (2x - x^2)\cos\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\left[(2x - x^2)\underbrace{\left(\frac{\sin\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}\right)}_{0} - (2-2x)\left(\frac{-\cos\frac{2n\pi x}{3}}{\frac{4n^2\pi^2}{9}}\right) + (-2)\underbrace{\left(\frac{-\sin\frac{2n\pi x}{3}}{\frac{8n^3\pi^3}{27}}\right)}_{0}\right]_0^3$$

$$= \frac{2}{3}\left[(2-2x)\left(\frac{\cos\frac{2n\pi x}{3}}{\frac{4n^2\pi^2}{9}}\right)\right]_0^3$$

$$= \frac{2}{3}\cdot\frac{9}{4n^2\pi^2}\left[(2-2x)\left(\cos\frac{2n\pi x}{3}\right)\right]_0^3$$

$$= \frac{3}{2}\cdot\frac{1}{n^2\pi^2}\left[(2-6)\cos\frac{6n\pi}{3} - (2-0)\cos 0\right]$$

$$= \frac{3}{2}\cdot\frac{1}{n^2\pi^2}\left[-4(1) - 2(1)\right] \qquad [\because \cos 2n\pi = 1;\ \cos 0 = 1]$$

$$= \frac{3}{2}\cdot\frac{1}{n^2\pi^2}[-4-2]$$

$$= \frac{3}{2}\cdot\frac{1}{n^2\pi^2}(-6)$$

$$a_n = \frac{-9}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\,.\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{3/2} \int_0^{2(3/2)} (2x - x^2) \sin \frac{n\pi x}{3/2}\, dx \qquad \{\text{Here } l = \frac{3}{2}\}$$

$$= \frac{2}{3} \int_0^{3} (2x - x^2) \sin \frac{2n\pi x}{3}\, dx$$

$$= \frac{2}{3} \left[(2x - x^2) \left(\frac{-\cos \frac{2n\pi x}{3}}{\frac{2n\pi}{3}} \right) - (2 - 2x) \underbrace{\left(\frac{-\sin \frac{2n\pi x}{3}}{\frac{4n^2\pi^2}{9}} \right)}_{0} + (-2) \left(\frac{\cos \frac{2n\pi x}{3}}{\frac{8n^3\pi^3}{27}} \right) \right]_0^3$$

$$= \frac{2}{3} \left[(6 - 9) \left(\frac{-\cos \frac{6n\pi}{3}}{\frac{2n\pi}{3}} \right) - \frac{2\cos \frac{6n\pi}{3}}{\frac{8n^3\pi^3}{27}} - 0 + \frac{2\cos 0}{\frac{8n^3\pi^3}{27}} \right]$$

$$= \frac{2}{3} \left[(-3) \left(\frac{-1}{\frac{2n\pi}{3}} \right) - 2 \cdot \frac{1}{\frac{8n^3\pi^3}{27}} + 2 \cdot \frac{1}{\frac{8n^3\pi^3}{27}} \right]$$

$$= \frac{2}{3} \left[\frac{9}{2n\pi} - \frac{54}{8n^3\pi^3} + \frac{54}{8n^3\pi^3} \right]$$

$$= \frac{2}{3} \left[\frac{9}{2n\pi} \right]$$

$$b_n = \frac{3}{n\pi}$$

Sub for a_0, a_n and b_n in (1), we get

$$f(x) = \frac{0}{2} + \sum_{n=1}^{\infty} \left(\frac{-9}{n^2 \pi^2}\right) \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} \frac{3}{n\pi} \sin\frac{n\pi x}{l}$$

$$f(x) = -\frac{9}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos\frac{n\pi x}{l} + \frac{3}{\pi} \cdot \sum_{n=1}^{\infty} \frac{1}{n} \cdot \sin\frac{n\pi x}{l}$$

(ii) Obtain the Fourier series of $f(x) = x \sin x$ in $(-\pi, \pi)$

Solution : $f(x) = x \sin x$

x is odd function. $\sin x$ is odd function.

$\therefore$ $x \sin x$ is an even function.

$\therefore$ In the interval $(-\pi, \pi)$, $b_n = 0$

Hence we have to find a_0 and a_n

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin x\, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} x \sin x\, dx \qquad [x \sin x \text{ is an even funtion}]$$

$$= \frac{2}{\pi} \left[x(-\cos x) - (1)\underbrace{(-\sin x)}_{0}\right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[-x \cos x\right]_0^{\pi}$$

$$= -\frac{2}{\pi} \left[x \cos x\right]_0^{\pi}$$

$$= -\frac{2}{\pi} [\pi \cos \pi - 0]$$

$$= -\frac{2}{\pi} [\pi(-1)] \qquad [\because \cos \pi = -1]$$

$$a_0 = 2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\sin x\cos nx\,dx \qquad [x\sin x\cos nx \text{ is an even function}]$$

$$= \frac{2}{2\pi}\int_{0}^{\pi} x(2\sin x\cos nx)\,dx \qquad [\text{multiplying \& divide by } 2]$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x\ \{\sin(x+nx) + \sin(x-nx)\,dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x\ \{\sin(1+n)x + \sin(1-n)x\}\,dx$$

$$= \frac{1}{\pi}\left[x\left\{\frac{-\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n}\right\} - (1)\left\{\underbrace{\frac{-\sin(1+n)x}{(1+n)^2}}_{0} - \underbrace{\frac{\sin(1-n)x}{(1-n)^2}}_{0}\right\}\right]_0^{\pi}$$

Taking '–' sign outside

$$= -\frac{1}{\pi}\left[x\left\{\frac{\cos(1+n)x}{(1+n)} + \frac{\cos(1-n)x}{(1-n)}\right\}\right]_0^{\pi}$$

$$= \frac{-1}{\pi}\left[x\left\{\frac{\cos(1+n)x}{(1+n)} + \frac{\cos(n-1)x}{(1-n)}\right\}\right]_0^{\pi} \qquad [\cos(-\theta) = \cos\theta]$$

$$= \frac{-1}{\pi}\left[\pi\left\{\frac{\cos(1+n)\pi}{(1+n)} + \frac{\cos(n-1)\pi}{(1-n)}\right\} - 0\right]$$

$$= \frac{-1}{\pi}\left[\pi\left\{\frac{(-1)^{1+n}}{1+n}+\frac{(-1)^{n-1}}{1-n}\right\}\right]$$

$$= -\left[\frac{(-1).(-1)^{n}}{1+n}+\frac{(-1)^{n}(-1)^{-1}}{1-n}\right]$$

$$= -\left[\frac{(-1)(-1)^{n}}{1+n}+\frac{(-1)^{n}(-1)}{1-n}\right] \qquad \{(-1)^{-1}=-1$$

$$= -(-1)(-1)^{n}\left[\frac{1}{1+n}+\frac{1}{1-n}\right]$$

$$= (-1)^{n}\left[\frac{1-n+1+n}{(1+n)(1-n)}\right]$$

$$= \frac{2(-1)^{n}}{(1+n)(1-n)}$$

$$a_n = \frac{2(-1)^{n}}{1-n^{2}}$$

For $n = 1$, Denominator beomes 0;

$\therefore$ We have to find a_1 separately.

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos x\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x(2\sin x\cos x)\,dx \qquad \text{[Multiply and divide by 2]}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\sin 2x\,dx \qquad \{\sin 2\theta = 2\sin\theta\cos\theta)$$

$$= \frac{1}{\pi} \int_0^{\pi} x \sin 2x \, dx \qquad \{x \sin 2x \text{ is even function}]$$

$$= \frac{1}{\pi} \left[x \left(\frac{-\cos 2x}{2} \right) - (1) \left(\underbrace{\frac{-\sin 2x}{4}}_{0} \right) \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{-x \cos 2x}{2} \right]_0^{\pi}$$

$$= -\frac{1}{2\pi} \left[x \cos 2x \right]_0^{\pi}$$

$$= -\frac{1}{2\pi} \left[\pi \cos 2\pi - 0 \right]$$

$$= -\frac{1}{2\pi} \left[\pi (1) - 0 \right] \qquad [\cos 2\pi = 1]$$

$$= -\frac{1}{2\pi} (\pi)$$

$$a_1 = -\frac{1}{2}$$

Substitute for a_0, a_n, a_1, we get,

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$= \frac{2}{2} + \left(\frac{-1}{2} \right) \cos x + \sum_{n=2}^{\infty} \frac{2(-1)^n}{1 - n^2} \cos nx$$

$$= 1 - \frac{1}{2} \cos x + 2 \sum_{n=2}^{\infty} \frac{(-1)^n}{1 - n^2} \cos nx$$

11.(b) (i) Obtain the Fourier cosine series expansion of x sin x in $(0, \pi)$ and hence find the value of $1 + \frac{2}{1.3} - \frac{2}{3.5} + \frac{2}{5.7} - \frac{2}{7.9} + \ldots$

Solution :

The half-range cosine series in the interval $(0, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \sin x\, dx$$

$$= \frac{2}{\pi} \left[x(-\cos x) - \underbrace{(1)(-\sin x)}_{0}\right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[-x \cos x\right]_0^{\pi}$$

$$= \frac{-2}{\pi} \left[x \cos x\right]_0^{\pi}$$

$$= \frac{-2}{\pi} [\pi \cos \pi - 0]$$

$$= \frac{-2}{\pi} [\pi(-1)] \qquad \{\cos \pi = -1\}$$

$$a_0 = 2$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \sin x \cos nx\, dx$$

$$= \frac{2}{2\pi} \int_0^{\pi} x \; (2 \sin x \cos nx)\, dx \quad \{\because \text{ Multiply and Divide by } 2\}$$

$$= \frac{1}{\pi} \int_0^{\pi} x \left\{\sin (x + nx) + \sin (x - nx)\right\}$$

$$\{\because \; 2 \sin A \cos B = \sin (A + B) + \sin (A - B)\}$$

$$= \frac{1}{\pi} \int_0^{\pi} x \left\{\sin (1 + n) x + \sin (1 - n) x\right\} dx$$

$$= \frac{1}{\pi} \left[x \left\{\frac{-\cos (1 + n) x}{1 + n} - \frac{\cos (1 - n) x}{1 - n}\right\} - (1) \left\{\underset{\substack{\downarrow \\ 0}}{\frac{-\sin (1 + n) x}{(1 + n)^2}} - \underset{\substack{\downarrow \\ 0}}{\frac{\sin (1 - n) x}{(1 - n)^2}}\right\}\right]_0^{\pi}$$

Taking '–' sign outside

$$= \frac{-1}{\pi} \left[x \left\{\frac{\cos (1 + n) x}{(1 + n)} + \frac{\cos (1 - n) x}{(1 - n)}\right\}\right]_0^{\pi}$$

$$= \frac{-1}{\pi} \left[x \left\{\frac{\cos (1 + n) x}{(1 + n)} + \frac{\cos (n - 1) x}{(1 - n)}\right\}\right]_0^{\pi} \; [\because \cos (-\theta) = \cos \theta]$$

$$= \frac{-1}{\pi} \left[\pi \left\{\frac{\cos (1 + n) \pi}{(1 + n)} + \frac{\cos (n - 1) \pi}{(1 - n)}\right\} - 0\right]$$

$$= \frac{-1}{\pi} \left[\pi \left\{\frac{(-1)^{1+n}}{1 + n} + \frac{(-1)^{n-1}}{1 - n}\right\}\right]$$

$$= - \left[\frac{(-1)\,(-1)^n}{1 + n} + \frac{(-1)^n\,(-1)^{-1}}{1 - n}\right]$$

$$= - \left[\frac{(-1)\,(-1)^n}{1 + n} + \frac{(-1)^n\,(-1)}{1 - n}\right] \qquad \{\because (-1)^{-1} = -1\}$$

$$= -(-1)(-1)^n \left[\frac{1}{1+n} + \frac{1}{1-n}\right]$$

$$= (-1)^n \left[\frac{1-n+1+n}{(1+n)(1-n)}\right]$$

$$= \frac{2(-1)^n}{(1+n)(1-n)}$$

$$a_n = \frac{2(-1)^n}{1-n^2}$$

For $n = 1$, Denominator becomes 0.

$\therefore$ We have to find a_1 separately.

$$a_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\cos x\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,dx$$

$$= \frac{2}{2\pi}\int_0^{\pi} x\,(2\sin x\cos x)\,dx \qquad \{\text{Multiply and divide by } 2\}$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx \qquad \{\because \sin 2\theta = 2\sin\theta\cos\theta\}$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - (1)\underbrace{\left(\frac{-\sin 2x}{4}\right)}_{0}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-x\cos 2x}{2}\right]_0^{\pi}$$

$$= \frac{-1}{2\pi}\left[x\cos 2x\right]_0^{\pi}$$

$$= \frac{-1}{2\pi}\left[\pi \cos 2\pi - 0\right]$$

$$= \frac{-1}{2\pi}\left[\pi (1) - 0\right] \qquad [\because \cos 2\pi = 1\}$$

$$= \frac{-1}{2\pi}(\pi)$$

$$a_1 = \frac{-1}{2}$$

Substituting for a_0, a_1 and a_n we get,

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$= \frac{2}{2} + \left(\frac{-1}{2}\right)\cos x + \sum_{n=2}^{\infty} \frac{2(-1)^n}{1-n^2}\cos nx$$

$$f(x) = 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty} \frac{(-1)^n}{1-n^2}\cos nx$$

$$x \sin x = 1 - \frac{1}{2}\cos x - 2\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-1}\cos nx$$

$$x \sin x = 1 - \frac{1}{2}\cos x - 2\sum_{n=2}^{\infty} \frac{(-1)^n}{(n-1)(n+1)}\cos nx$$

$$x \sin x = 1 - \frac{1}{2}\cos x - 2\left[\frac{1}{1.3}\cos 2x - \frac{1}{2.4}\cos 3x + \frac{1}{3.5}\cos 4x - \ldots\right]$$

[substituting $n = 2, 3, 4, \ldots$]

Put $x = \frac{\pi}{2}$ on both sides.

$$\frac{\pi}{2}\sin\frac{\pi}{2} = 1 - \frac{1}{2}\cos\frac{\pi}{2} - 2\left[\frac{1}{1.3}\cos\frac{2\pi}{2} - \frac{1}{2.4}\cos\frac{3\pi}{2} + \frac{1}{3.5}\cos\frac{4\pi}{2} - \ldots\right]$$

$$\frac{\pi}{2} = 1 - 0 - 2\left[\frac{1}{1.3}\cos\pi - \frac{1}{2.4}\cos\frac{3\pi}{2} + \frac{1}{3.5}\cos 2\pi - \ldots\right]$$

$$\left[\because \cos\frac{\pi}{2} = 0,\ \cos\frac{3\pi}{2} = 0\right]$$

$$\frac{\pi}{2} = 1 - 2\left[\frac{1}{1.3}(-1) - \frac{1}{2.4}(0) + \frac{1}{3.5}(1) - \ldots\right]$$

$$\frac{\pi}{2} = 1 - 2\left[-\frac{1}{1.3} + \frac{1}{3.5} - \ldots\right]$$

$$\therefore \frac{\pi}{2} = 1 + \frac{2}{1.3} - \frac{2}{3.5} + \ldots$$

11. (b) (ii) The following table gives the variations of a periodic current over a period T

x	0	T/6	T/3	T/2	2T/3	5T/6	T
$f(x)$	1.98	1.30	1.05	1.30	−0.88	−0.25	1.98

Find the fundamental and first harmonics of f(x) to express f(x) in a Fourier series in the form $f(x) = a_0/2 + a_1\cos\theta + b_1\sin\theta$, where $\theta = \frac{2\pi x}{T}$

Solution :

Here Period = T

$$\theta = \frac{2\pi x}{T}$$

Substituting x = 0, T/6, T/3, T/2, 2T/3, 5T/6,

we get $\theta = 0$

$$\theta = \frac{2\pi\,(T/6)}{T}$$

$$= \frac{2\pi}{6} = \frac{\pi}{3} = 60^{\circ}$$

Similarly $\theta = 120^\circ$

$\theta = 180^\circ$

$\theta = 240^\circ$

$\theta = 300^\circ$

θ	0°	60°	120°	180°	240°	300°
$f(x)$	1.98	1.30	1.05	1.30	−0.88	−0.25

θ	$y = f(x)$	$\cos\theta$	$y\cos\theta$	$\sin\theta$	$y\sin\theta$
0	1.98	1	1.98	0	0
60	1.30	0.5	0.65	0.866	1.126
120	1.05	−0.5	−0.525	0.866	0.909
180	1.30	−1	−1.30	0	0
240	−0.88	−0.55	0.44	−0.866	0.762
300	−0.25	0.5	−0.125	−0.866	0.216
	$\Sigma y = 4.5$		$\Sigma y\cos\theta = 1.12$		$\Sigma y\sin\theta = 3.013$

$$a_0 = \frac{2}{n}\Sigma y$$

$$= \frac{2}{6}(4.5)$$

$$a_0 = 1.5$$

$$a_1 = \frac{2}{n}\Sigma y\cos\theta$$

$$= \frac{2}{6}(1.12)$$

$$a_1 = 0.373$$

$$b_1 = \frac{2}{n} \Sigma y \sin\theta$$

$$= \frac{2}{6}(3.013)$$

$$b_1 = 1.004$$

$$\therefore f(x) = \frac{a_0}{2} + a_1 \cos\theta + b_1 \sin\theta$$

$$= \frac{1.5}{2} + 0.373 \cos\theta + 1.004 \sin\theta$$

$$f(x) = 0.75 + 0.373 \cos\theta + 1.004 \sin\theta$$

---

12. (a) (i) Show that $e^{-x^2/2}$ is a self reciprocal with respect to Fourier transform.

Solution : $$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\; e^{isx}\, dx$$

$$\therefore F\left[e^{-x^2/2}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2 + isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2\,(x^2 - 2\,isx)}\, dx$$

Consider $x^2 - 2\,i\,sx$

$$x^2 - 2\,isx = x^2 - 2\,isx + (is)^2 - (is)^2$$

[Adding and subtracting $(is)^2$]

$$= \left(x^2 - 2\,isx + (is)^2\right) - (is)^2$$

$$= (x - is)^2 - i^2 s^2$$

$$= (x - is)^2 - (-1) s^2 \qquad [\because i^2 = -1]$$

$$= (x - is)^2 + s^2$$

$$\therefore F\left[e^{-x^2/2}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2\left[(x-is)^2 + s^2\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(x-is)^2 - 1/2 s^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(x-is)^2} . e^{-1/2 s^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-1/2 s^2} \int_{-\infty}^{\infty} e^{-1/2(x-is)^2} dx$$

Put $\dfrac{x - is}{\sqrt{2}} = y$

$$\frac{1}{\sqrt{2}} dx = dy$$

$$\therefore dx = \sqrt{2}\, dy$$

when $x = -\infty, \quad y = -\infty$

when $x = \infty, \quad y = \infty$

$$\therefore F\left[e^{-x^2/2}\right] = \frac{1}{\sqrt{2\pi}} e^{-1/2 s^2} \int_{-\infty}^{\infty} e^{-y^2} \sqrt{2}\, dy$$

$$= \frac{1}{\sqrt{2}\sqrt{\pi}} e^{-1/2 s^2} . 2\sqrt{2} \int_0^{\infty} e^{-y^2} dy$$

$[e^{-y^2}$ is even function$]$

$$= \frac{2}{\sqrt{\pi}} e^{-1/2 s^2} \int_0^{\infty} e^{-y^2} dy$$

But $\int_0^\infty e^{-y^2}\,dy = \frac{\sqrt{\pi}}{2}$ by Beta gamma integrals.

$$\therefore F\left[e^{-x^2/2}\right] = \frac{2}{\sqrt{\pi}}\, e^{-1/2\, s^2}\, \frac{\sqrt{\pi}}{2}$$

$$\therefore F\left[e^{-x^2/2}\right] = e^{-s^2/2}$$

Hence $e^{-s^2/2}$ is self-resiprocal.

---

12. (a) (ii) Find the Fourier transform of the function

$$f(x) = \begin{cases} 1 - |x|, & \text{if } |x| < 1 \\ 0, & \text{if } |x| > 1 \end{cases}$$ **and hence find the value**

of $\int_0^\infty \left(\frac{\sin^4 t}{t^4}\right) dt.$

---

12. (b) (i) Find the Fourier sine transform of e^{-ax} and hence evaluate Fourier cosine transforms of $x e^{-ax}$ and $e^{-ax} \sin ax$

Solution : The Fourier sine transform is defined as,

$$F_s [f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$\therefore \quad F_s \left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$\left[\because \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)\right.$$

$$\left.\text{Here } a = -a, \; b = s\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2 + s^2} (-a \sin sx - s \cos sx)\right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{-1}{a^2 + s^2} \left\{\underbrace{e^{-ax} a \sin sx}_{\downarrow \, 0} + e^{-ax} s \cos sx\right\}_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \frac{-1}{a^2 + s^2} \left[e^{-ax} s \cos x\right]_0^\infty$$

$$[e^{-\infty} = 0 \; ; \; \sin 0 = 0 \; ; \; e^0 = 1]$$

$$= \sqrt{\frac{2}{\pi}} \frac{-s}{a^2 + s^2} \left[e^{-ax} \cos sx\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \frac{-s}{a^2 + s^2} \left[0 - e^0 \cos 0\right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{-s}{a^2 + s^2} [0 - 1]$$

$$= \sqrt{\frac{2}{\pi}} \frac{s}{a^2 + s^2}$$

By definition of Fourier cosine transform and property,

$$F_C\,[x f(x)] = \frac{d}{ds} F_S\,[f(x)]$$

Now $\quad F_S\,[f(x)] = \sqrt{\frac{2}{\pi}}\,\frac{s}{a^2+s^2}$

$$\therefore\ F_C\,[x \, . \, e^{-ax}] = \frac{d}{ds} F_S\,(e^{-ax})$$

$$= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\left(\frac{s}{a^2+s^2}\right)\right]$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{d}{ds}\left[\frac{s}{a^2+s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{(a^2 + s^2) - s\,(2s)}{(a^2 + s^2)^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a^2 + s^2 - 2s^2}{(a + s^2)^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a^2 - s^2}{(a^2 + s^2)}\right]$$

By property,

$$F_C\,[f(x) \sin ax] = \frac{1}{2}\left\{F_S\,(a + s) + F_S\,(a - s)\right\}$$

$$\text{But} \quad F_S = \sqrt{\frac{2}{\pi}}\, \frac{a}{a^2 + s^2}$$

Replacing s by $(a + s)$,

$$F_S\,(a + s) = \sqrt{\frac{2}{\pi}}\, \frac{(a + s)}{a^2 + (a + s)^2}$$

Replacing s by $(a - s)$

$$F_S\,(a - s) = \sqrt{\frac{2}{\pi}}\, \frac{(a - s)}{a^2 + (a - s)^2}$$

$$\therefore\ F_C\,[f(x) \sin ax] = \frac{1}{2}\left[\left\{\sqrt{\frac{2}{\pi}}\, \frac{(a + s)}{a^2 + (a + s)^2} + \sqrt{\frac{2}{\pi}}\, \frac{(a - s)}{a^2 + (a - s)^2}\right\}\right]$$

$$= \frac{1}{2}\cdot\sqrt{\frac{2}{\pi}} \left[\left\{\frac{(a + s)}{a^2 + (a + s)^2} + \frac{(a - s)}{a^2 + (a - s)^2}\right\}\right]$$

(ii) State and prove convolution theorem for Fourier transforms.

Solution :

The Fourier transform of the convolution of $f(x)$ and $g(x)$ is equal to the product of their Fourier transforms

(i.e.,) $F[f(x) * g(x)] = F[f(x)]\ F[g(x)]$

Proof :

$$F[f(x) * g(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[f(x) * g(x)\right] e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt\right\} e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, e^{isx}\, dt\,.\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, e^{isx}\, dx\, dt$$

[on changing the order of integration]

$$= \frac{1}{\sqrt{2\pi}}\, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left[\int_{-\infty}^{\infty} g(x-t)\, e^{isx}\, dx\right] dt$$

Put $x - t = u \qquad x = u + t$

Differentiating $\qquad dx = du$

when $\quad x = -\infty, \quad u = -\infty$

when $\quad x = \infty, \quad u = \infty$

$$\therefore F[f(x) * g(x)] = \frac{1}{\sqrt{2\pi}}\, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left[\int_{-\infty}^{\infty} g(u)\, e^{is(u+t)}\, du\right]$$

$$= \frac{1}{\sqrt{2\pi}}\, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left[\int_{-\infty}^{\infty} g(u)\, e^{isu}\,.\, e^{ist}\, du\right] dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(u)\, e^{isu}\, du \right] e^{ist}\, dt$$

$$= \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt \right] \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(u)\, e^{isu}\, du \right]$$

$$= \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx \right] \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\, e^{isx}\, dx \right]$$

[changing dummy variable]

$$= F[f(x)] \,.\, F[g(x)]$$

13. (a) (i) Find the singular integral if $z = px + qy + \sqrt{1 + p^2 + q^2}$

Solution :

$$z = px + qy + c\sqrt{1 + p^2 + q^2} \quad \text{... (1)}$$

Let $z = ax + by + c$ ... (2) be the solution

Differentiating (2) partially w.r.t x,

$$\frac{\partial z}{\partial x} = a$$

$$\therefore p = a$$

Differentiating (2) partially w.r.t y,

$$\frac{\partial z}{\partial y} = b$$

$$\therefore q = b$$

Substituting for p and q in (1), we get the complete integral as

$$z = ax + by + c\sqrt{1 + a^2 + b^2} \quad \text{... (3)}$$

Differentiating (3) partially w.r.t a

$$\frac{\partial z}{\partial a} = x + \frac{c}{2\sqrt{(1+a^2+b^2)}} \cdot 2a$$

$$\frac{\partial z}{\partial a} = 0 \Rightarrow x + \frac{ac}{\sqrt{1+a^2+b^2}} = 0$$

$$\therefore x = \frac{-ac}{\sqrt{1+a^2+b^2}} \quad \text{... (4)}$$

Differentiating (3) partially w.r.t b,

$$\frac{\partial z}{\partial b} = y + \frac{c}{2\sqrt{1+a^2+b^2}} \cdot 2b$$

$$= y + \frac{bc}{\sqrt{1+a^2+b^2}} = 0$$

$$\therefore y = \frac{-bc}{\sqrt{1+a^2+b^2}} \quad \text{... (5)}$$

$$x^2 = \frac{a^2c^2}{1+a^2+b^2}$$

$$y^2 = \frac{b^2 c^2}{1 + a^2 + b^2}$$

Consider $$x^2 + y^2 = \frac{a^2 c^2}{1 + a^2 + b^2} + \frac{b^2 c^2}{1 + a^2 + b^2}$$

$$= \frac{(a^2 + b^2)\, c^2}{1 + a^2 + b^2}.$$

$$c^2 - (x^2 + y^2) = c^2 - \left(\frac{a^2 + b^2}{1 + a^2 + b^2}\right) c^2$$

$$= c^2 \left(1 - \frac{(a^2 + b^2)}{1 + a^2 + b^2}\right)$$

$$= c^2 \, \frac{[(1 + a^2 + b^2) - (a^2 + b^2)]}{1 + a^2 + b^2}$$

$$c^2 - x^2 - y^2 = \frac{c^2}{1 + a^2 + b^2}$$

$$1 + a^2 + b^2 = \frac{c^2}{c^2 - x^2 - y^2}$$

$$\therefore \sqrt{1 + a^2 + b^2} = \frac{c}{\sqrt{c^2 - x^2 - y^2}} \qquad \text{... (6)}$$

Using (4) and (5) we can find a and b

From (4)

$$a = \frac{-x}{c} \sqrt{1 + a^2 + b^2}$$

$$= \frac{-x}{c} \frac{c}{\sqrt{c^2 - x^2 - y^2}} \qquad \text{[Using (6)]}$$

$$= \frac{-x}{\sqrt{c^2 - x^2 - y^2}}$$

From (5)

$$b = \frac{-y}{c}\sqrt{1+a^2+b^2}$$

$$= \frac{-y}{c}\,\frac{c}{\sqrt{c^2-x^2-y^2}}$$

$$= \frac{-y}{\sqrt{c^2-x^2-y^2}}$$

Now substituting for a, b, $\sqrt{1+a^2+b^2}$ in (3), we get

$$z = \frac{-x}{\sqrt{(c^2-x^2-y^2}}\,x - \frac{y}{\sqrt{c^2-x^2-y^2}}\,y + \frac{c}{\sqrt{c^2-x^2-y^2}}\,c$$

$$= \frac{-x^2}{\sqrt{c^2-x^2-y^2}} - \frac{y^2}{\sqrt{c^2-x^2-y^2}} + \frac{c^2}{\sqrt{c^2-x^2-y^2}}$$

$$= \frac{c^2-x^2-y^2}{\sqrt{c^2-x^2-y^2}}$$

$$= \sqrt{c^2-x^2-y^2}$$

Squaring on both sides,

$$z^2 = c^2-x^2-y^2$$

$$\therefore\; x^2+y^2+z^2 = c^2$$

This is the singular integral.

13. (a) (ii) Solve the partial differential equation

$$x(y-z)p + y(z-x)q = z(x-y)$$

Solution :

The equation is of the form

$$Pp + Qq = R$$

The auxiliary equation is $\dfrac{dx}{x(y-z)} = \dfrac{dy}{y(z-x)} = \dfrac{dz}{z(x-y)}$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Taking the multipliers $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$ each term in auxiliary equation.

$$\frac{\frac{dx}{x}}{\frac{x(y-z)}{x}} = \frac{\frac{dy}{y}}{\frac{y(z-x)}{y}} = \frac{\frac{dz}{z}}{\frac{z(x-y)}{z}}$$

$$\frac{\frac{dx}{x}}{y-z} = \frac{\frac{dy}{y}}{z-x} = \frac{\frac{dz}{z}}{x-y}$$

$$\Rightarrow \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{y-z+z-x+x-y}$$

$$\Rightarrow \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating,

$$\int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = 0$$

$$\log x + \log y + \log z = \log c_1$$

$$\log xyz = \log c_1$$

$$\therefore xyz = c_1$$

Taking the multipliers, 1, 1, 1 each term in auxiliary equation.

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

$$\Rightarrow \quad \frac{dx}{xy - xz} = \frac{dy}{yz - yx} = \frac{dz}{zx - zy}$$

$$\frac{dx + dy + dz}{xy - xz + yz - yx + zx - zy} = \frac{dx + dy + dz}{0}$$

$$\therefore dx + dy + dz = 0$$

Integrating,

$$\int dx + \int dy + \int dz = 0$$

$$\therefore x + y + z = c_2$$

The solution is

$$\phi(c_1, c_2) = 0$$

$$\phi(xyz, x + y + z) = 0$$

13. (b)(i) Solve $(D^3 - 2D^2 D')z = 2e^{2x} + 3x^2 y$

Solution : Replace D by m, and D' by 1.

The auxiliary equation is

$$m^3 - 2m^2 = 0$$

$$m^2(m-2) = 0$$

$$\therefore m^2 = 0 \; ; \; m - 2 = 0$$

$$\therefore m = 0, 0, 2.$$

By case 4, two roots are equal and third is different,

$$z = f_1(y+mx) + xf_2(y+mx) + f_3(y+m_1x)$$

$$= f_1(y+0x) + x.f_2(y+0x) + f_3(y+2x)$$

The C.F is $z = f_1(y) + x.f_2(y) + f_3(y+2x)$

$P.I,$

$$z = \frac{1}{(D^3 - 2D^2D')} \cdot 2e^{2x} + 3x^2 y$$

$$= \frac{1}{D^3 - 2D^2D'} \cdot 2e^{2x} + \frac{1}{D^3 - 2D^2D'} 3x^2 y$$

$$= P.I_1 + P.I_2$$

$$P.I_1 = \frac{1}{D^3 - 2D^2D'} \cdot 2e^{2x}$$

Replace D by 2, D' by 0,

$$P.I_1 = \frac{1}{2^3 - 2\,(2)^2 \,.\, 0} \,.\, 2e^{2x}$$

$$= \frac{1}{8-0}\, 2e^{2x}$$

$$= \frac{1}{8} \,.\, 2\, e^{2x}$$

$$= \frac{1}{4} \,.\, e^{2x}$$

$$P.I_2 = \frac{1}{D^3 - 2D^2 \,.\, D'}\, 3x^2 y$$

$$= \frac{1}{D^3 \left[1 - \frac{2D^2 D'}{D^3}\right]}\, 3x^2 y$$

$$= \frac{1}{D^3} \,.\, \frac{1}{\left[1 - \frac{2\,D'}{D}\right]}\, 3x^2 y$$

$$= \frac{3}{D^3} \left[1 - \frac{2\,D'}{D}\right]^{-1} x^2 y$$

$$= \frac{3}{D^3} \left[1 + \frac{2\,D'}{D} + \frac{4\,D'^2}{D^2} + \ldots\right] x^2 y$$

[Applying binomial series]

$$= \frac{3}{D^3} \left[1 + \frac{2\,D'}{D}\right] x^2 y$$

$$= \frac{3}{D^3} \left[x^2 y + \frac{2}{D}\, D'\,(x^2 y)\right]$$

$$= \frac{3}{D^3} \left[x^2 y + \frac{2}{D}\,(x^2)\right]$$

$$= \frac{3}{D^3}\left[x^2 y + \frac{2x^3}{3}\right]$$

$$= \frac{3}{D^2}\left[\frac{x^3}{3} y + \frac{2x^4}{12}\right]$$

$$= \frac{3}{D^2}\left[\frac{x^3}{3} y + \frac{x^4}{6}\right]$$

$$= \frac{3}{D}\left[\frac{x^4}{12} y + \frac{x^5}{30}\right]$$

$$= 3\left[\frac{x^5}{60} y + \frac{x^6}{180}\right]$$

$$= \frac{x^5}{20} y + \frac{x^6}{60}$$

Solution is $z = C.F + P.I_1 + P.I_2$

$$z = f_1(y+0) + x f_2(y+0) + f_3(y+2x) + \frac{1}{4} e^{2x} + \frac{x^5}{20} y + \frac{x^6}{60}$$

13. (b) (ii) Solve : $(D^2 - 2DD' + D'^2 - 3D + 3D' + 2)z = e^{2x-y}$

Solution :

$$(D^2 - 2DD' + D' - 3D + 3D' + 2)z = e^{2x-y}$$

Let $(D - m_1 D' - \alpha_1)(D - m_2 D' - \alpha_2)$

$$= D^2 - 2DD' + D'^2 - 3D + 3D' + 2$$

Comparing the coefficient of DD',

$$-m_1 - m_2 = -2 \Rightarrow m_1 + m_2 = 2 \quad \text{... (1)}$$

Comparing the coefficient of D'^2,

$$m_1 m_2 = 1 \quad \text{... (2)}$$

Comparing the constant term,

$$\alpha_1 \alpha_2 = 2 \quad \ldots (3)$$

Comparing the coefficient of D

$$-\alpha_1 - \alpha_2 = -3$$

$$\alpha_1 + \alpha_2 = 3 \quad \ldots (4)$$

From (2) $\quad m_2 = \frac{1}{m_1}$

Substituting in equation (1) we get,

$$m_1 + \frac{1}{m_1} = 2$$

$$m_1^2 + 1 = 2m_1$$

$$m_1^2 - 2m_1 + 1 = 0$$

$$(m_1 - 1)^2 = 0$$

$$m_1 = 1, 1$$

From (3) $\quad \alpha_1 = \frac{2}{\alpha_2}$

Substituting in equation (4) we get,

$$\alpha_1 + \alpha_2 = 3$$

$$\frac{2}{\alpha_2} + \alpha_2 = 3$$

$$2 + \alpha_2^2 = 3\alpha_2$$

$$2 + \alpha_2^2 - 3\alpha_2 = 0$$

$$\alpha_2^2 - 3\alpha_2 + 2 = 0$$

$$\alpha_2^2 - 2\alpha_2 - \alpha_2 + 2 = 0$$

$$\alpha_2(\alpha_2 - 2) - 1(\alpha_2 - 2) = 0$$

$$(\alpha_2 - 1)(\alpha_2 - 2) = 0$$

$$\alpha_2 = 1, \alpha_2 = 2$$

$$\therefore \alpha_2 = 1, 2$$

Substituting α_2 values in equation (3) we get,

$$\alpha_1 \alpha_2 = 2$$

$$\alpha_1 = \frac{2}{\alpha_2}$$

$$\alpha_1 = \frac{2}{1} = 2$$

$\therefore$ The complementary function is

$$e^{\alpha_1 x} f_1(y + m_1 x) + e^{\alpha_2 x} f_2(y + m_2 x)$$

Substituting for m_1, m_2, α_1 and α_2 we get

$$e^x f_1(y + x) + e^{2x} f_2(y + x)$$

$$P.I = \frac{1}{D^2 - 2DD' + D'^2 - 3D + 3D' + 2} \cdot e^{2x-y}$$

$$D = 2 \text{ and } D' = -1$$

$$\therefore P.I = \frac{1}{(2)^2 - 2(2)(-1) + (-1)^2 - 3(2) + 3(-1) + 2} \cdot e^{2x-y}$$

$$= \frac{1}{4 + 4 + 1 - 6 - 3 + 2} \cdot e^{2x-y}$$

$$= \frac{1}{2} e^{2x-y}$$

$$z = C.F + P.I$$

$$= e^x f_1(y+x) + e^{2x} f_2(y+x) + \frac{1}{2} e^{2x-y}$$

14. (a) A tightly stretched string of length 'l' is intially at rest in its equilibrium position and each of its points is given the velocity $V_0 \sin^3\left(\frac{\pi x}{l}\right)$. Find the displacement y (x, t)

Solution : The wave equation is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

The solution is

$$y = (A\cos px + B\sin px)(C\cos pct + D\sin pct) \quad \dots (1)$$

We have the following boundary conditions

1. $y = 0$ when $x = 0$
2. $y = 0$ when $x = l$
3. $\frac{\partial y}{\partial t} = 0,$ when $t = 0$
4. $y = y_0 \sin^3 \frac{\pi x}{l}$ when $t = 0$

Applying condition 1. in (1)

$$0 = (A\cos 0 + B\sin 0)(C\cos pct + D\sin pct)$$

$$0 = (A + 0)(C\cos pct + D\sin pct) \quad [\cos 0 = 1, \sin 0 = 0]$$

$$\Rightarrow A = 0$$

Substitute A = 0 in (1)

$$y = B \sin px \,(C \cos pct + D \sin pct) \quad \ldots (2)$$

Applying condition 2. in (2)

$$0 = B \sin pl \,(C \cos pct + D \sin pct)$$

$$\sin pl = 0$$

$$\text{But } \sin n\pi = 0$$

$$\therefore \quad pl = n\pi$$

$$p = \frac{n\pi}{l}$$

Substitute $p = \frac{n\pi}{l}$ in (2)

$$y = B \sin \frac{n\pi x}{l} \left(C \cos \frac{n\pi ct}{l} + D \sin \frac{n\pi ct}{l} \right) \quad \ldots (3)$$

To apply condition 3 differentiate (3) partially w.r.t 't'

$$\frac{\partial y}{\partial t} = B \sin \frac{n\pi x}{l} \left(-C \sin \frac{n\pi ct}{l} \cdot \frac{n\pi c}{l} + D \cos \frac{n\pi ct}{l} \frac{n\pi c}{l} \right)$$

Now applying condition (3)

$$0 = B \sin \frac{n\pi x}{l} \left(-C \sin 0 \frac{n\pi c}{l} + D \cos 0 \frac{n\pi c}{l} \right)$$

$$0 = B \sin \frac{n\pi x}{l} \left(0 + D \frac{n\pi c}{l} \right)$$

$$\Rightarrow \quad D = 0$$

Substitute D = 0 in (3)

$$y = B \sin \frac{n\pi x}{l} \left(C \cos \frac{n\pi ct}{l} + 0 \right)$$

$$= B \sin \frac{n\pi x}{l} \left(C \cos \frac{n\pi ct}{l} \right)$$

$$= BC \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$$

$$= b \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \qquad (BC = b)$$

$$y = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \qquad \dots (4)$$

Applying condition 4 in (4)

$$y_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos 0$$

$$= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \,.\, 1$$

Here we have to apply $\sin 3\theta$ formula and compare the co-efficients of like terms to get the values of b_1, b_2, b_3, ...

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$\therefore \quad 4\sin^3\theta = 3\sin\theta - \sin 3\theta$$

$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$$

$$\therefore \sin^3 \frac{\pi x}{l} = \frac{3}{4}\sin\frac{\pi x}{l} - \frac{1}{4}\sin\frac{3\pi x}{l}$$

Substitute for $\sin^3 \frac{\pi x}{l}$ in L.H.S.

$$y_0 \left[\frac{3}{4}\sin\frac{\pi x}{l} - \frac{1}{4}\sin\frac{3\pi x}{l}\right] = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\frac{3y_0}{4}\sin\frac{\pi x}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l} = b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + b_3 \sin\frac{3\pi x}{l} \dots$$

Comparing the coefficients of $\sin\frac{\pi x}{l}$

$$b_1 = \frac{3y_0}{4}$$

Comparing the coefficients of $\sin \frac{2\pi x}{l}$

$$b_2 = 0$$

Comparing the coefficients of $\sin \frac{3\pi x}{l}$

$$b_3 = \frac{-y_0}{4}$$

$\therefore$ The solution is $y = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot \cos \frac{n\pi ct}{l}$

$$= b_1 \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} + b_2 \sin \frac{2\pi x}{l} \cos \frac{2\pi ct}{l} + b_3 \sin \frac{3\pi x}{l} \cos \frac{3\pi ct}{l} + \ldots$$

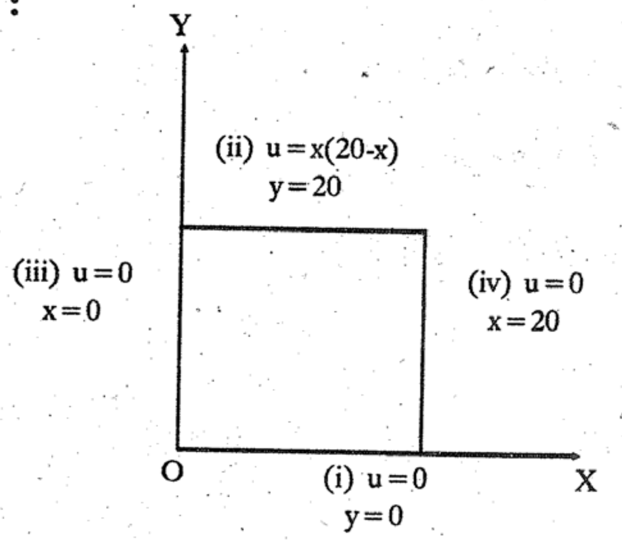
$$y = \frac{3y_0}{4} \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l} \cos \frac{3\pi ct}{l}$$

14. (b) **A square plate is bounded by the lines $x = 0$, $y = 0$, $x = 20$ and $y = 20$. Its faces are insulated. The temperature along the upper horizontal edge is given by**

$u(x, 20) = x(20 - x)$, $0 < x < 20$

while the other three edges are kept at 0°C. Find the steady state temperature distribution in the plate.

Solution :



The two dimensional heat flow equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

∴ We have the following boundary conditions

(i) $u = 0$ when $x = 0$

(ii) $u = 0$ when $x = 20$

(iii) $u = 0$ when $y = 0$

(iv) $u = x(20-x)$ when $y = 20$

Consider the second solution

$$u = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \quad \dots (1)$$

Apply condition (i) in (1) we get

$$0 = (A\cos 0 + B\sin 0)\ (Ce^{py} + De^{-py})$$

$$0 = (A + 0)\ (Ce^{py} + D\,e^{-py})$$

$$0 = A\ (Ce^{py} + De^{-py})$$

$$\Rightarrow \quad A = 0$$

Substituting $A = 0$ in (1) we get

$$u = (B\sin px)\ (Ce^{py} + De^{-py}) \qquad \ldots (2)$$

Applying condition (ii) in (2) we get

$$0 = (B\sin 20\,p)\ (Ce^{py} + De^{-py})$$

$$B\sin P\,(20) = 0$$

$$\sin 20\,p = 0 \qquad [\because B \neq 0]$$

$$20\,p = n\,\pi$$

$$p = \frac{n\,\pi}{20}$$

Substituting $p = \dfrac{n\,\pi}{20}$ in (2) we get

$$u = B \sin \frac{n\pi x}{20}\left(Ce^{\frac{n\pi}{20}y} + De^{\frac{-n\pi}{20}y}\right) \quad \ldots (3)$$

Applying condition (iii) in (3) we get

$$0 = B \sin \frac{n\pi x}{20}\left(Ce^{0} + De^{-0}\right)$$

$$0 = B \sin \frac{n\pi x}{20}(C + D)$$

$$C + D = 0$$

$$D = -C$$

Subtitute $D = -C$ in (3) we get

$$u = B \sin \frac{n\pi x}{20}\left(C \cdot e^{\frac{n\pi y}{20}} - Ce^{\frac{-n\pi y}{20}}\right)$$

$$= B \sin \frac{n\pi x}{20}\ C\left(e^{\frac{n\pi y}{20}} - e^{\frac{-n\pi y}{20}}\right)$$

$$= B\,C \sin \frac{n\pi x}{20}\left(e^{\frac{n\pi y}{20}} - e^{\frac{-n\pi y}{20}}\right)$$

$$= b \sin \frac{n\pi x}{20}\left(e^{\frac{n\pi y}{20}} - e^{\frac{-n\pi y}{20}}\right)$$

$$= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}\left(e^{\frac{n\pi y}{20}} - e^{\frac{-n\pi y}{20}}\right) \quad \ldots (4)$$

Applying condition (iv) in (4) we get

$$x(20 - x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}\left(e^{\frac{n\pi 20}{20}} - e^{\frac{-n\pi 20}{20}}\right)$$

$$x(20 - x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}\left(e^{n\pi} - e^{-n\pi}\right)$$

R.H.S is a half range sine series,

It's coefficient is given by

$$b_n \cdot \left[e^{n\pi} - e^{-n\pi}\right] = \frac{2}{l} \int_0^l f(x) \,.\, \sin \frac{n\pi x}{l}\, dx$$

Where, $l = 20$

$$= \frac{2}{20} \int_0^{20} f(x) \,.\, \sin \frac{n\pi x}{20}\, dx$$

$$= \frac{1}{10} \int_0^{20} x\,(20 - x) \sin \frac{n\pi x}{20}\, dx$$

$$= \frac{1}{10} \int_0^{20} (20x - x^2) \sin \frac{n\pi x}{20}\, dx$$

$$= \frac{1}{10} \int_0^{20} (20x - x^2) \sin \frac{n\pi x}{20}\, dx$$

$$= \frac{1}{10} \left[\underset{\downarrow \atop 0}{(20x - x^2)} \left(\frac{-\cos \frac{n\pi x}{20}}{\frac{n\pi}{20}} \right) - \underset{\downarrow \atop 0}{(20 - 2x)} \cdot \left(\frac{-\sin \frac{n\pi x}{20}}{\frac{n^2\pi^2}{(20)^2}} \right) \right.$$

$$\left. + (-2) \left(\frac{\cos \frac{n\pi x}{20}}{\frac{n^3\pi^3}{(20)^3}} \right) \right]_0^{20}$$

$$= \frac{1}{10} \left[-2 \, \frac{\cos \frac{n\pi x}{20}}{\frac{n^3\pi^3}{8000}} \right]_0^{20}$$

$$= \frac{-2}{10} \cdot \frac{8000}{n^3 \pi^3} \left[\cos \frac{n \pi x}{20} \right]_0^{20}$$

$$= \frac{-1600}{n^3 \pi^3} [\cos n \pi - \cos 0]$$

$$= \frac{-1600}{n^3 \pi^3} [(-1)^n - 1]$$

Case 1 : When n is even

$$(-1)^n = 1$$

$$b_n = \frac{-1600}{n^3 \pi^3} [1 - 1]$$

$$= 0$$

Case 2 : When n is odd

$$b_n \left[e^{n \pi} - e^{-n \pi} \right] = -\frac{1600}{n^3 \pi^3} [-1 - 1]$$

$$= \frac{-1600}{n^3 \pi^3} [-2]$$

$$= \frac{3200}{n^3 \pi^3}$$

$$b_n = \frac{3200}{n^3 \pi^3} \cdot \frac{1}{e^{n \pi} - e^{-n \pi}}$$

The final solution is,

$$u = \sum_{n=1,3}^{\infty} \frac{3200}{n^3 \pi^3} \cdot \frac{1}{e^{n \pi} - e^{-n \pi}} \cdot \sin \frac{n \pi x}{20} \left(e^{\frac{n \pi y}{20}} - e^{\frac{-n \pi y}{20}} \right)$$

15. (a) (i) If Z [f (n)] = F (Z), find Z [f (n − k)] and Z [f (n + k)]

Solution : By definition $Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

(1) $Z[f(n-k)] = \sum_{n=0}^{\infty} f(n-k) z^{-n}$

Put, $n - k = m$

$\therefore \quad n = m + k$

Also when $n = 0$, we get

$m + k = 0 \Rightarrow m = -k$

$$= \sum_{m=-k}^{\infty} f(m) z^{-(m+k)}$$

$$= \sum_{m=-k}^{\infty} f(m)\, z^{-m} z^{-k}$$

$$= z^{-k} \sum_{m=-k}^{\infty} f(m)\, z^{-m}$$

$$= z^{-k} \sum_{m=0}^{\infty} f(m)\, z^{-m}$$

[$f(n)$ is a casual sequence $f(n) = 0$ for $n < 0$]

(2) By definition $Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$Z[f(n+k)] = \sum_{n=0}^{\infty} f(n+k) z^{-n}$$

Putting $n + k = m$

when $n = 0$

$m = k$

$$\therefore Z[f(n+k)] = \sum_{m=k}^{\infty} f(m)\, z^{-(m-k)}$$

$$= \sum_{m=k}^{\infty} f(m)\, z^{-m}\, z^{k}$$

$$= z^{-k} \sum_{m=k}^{\infty} f(m)\, z^{-m}$$

$$= z^{-k} \left[\sum_{m=k}^{\infty} f(m)\, z^{-m} + \sum_{m=0}^{k-1} f(m)\, z^{-m} - \sum_{m=0}^{k-1} f(m)\, z^{-m} \right]$$

[Adding subtracting $\sum_{m=0}^{k-1} f(m)\, z^{-m}$]

$$= z^{-k} \left[\sum_{m=0}^{\infty} f(m)\, z^{-m} - \sum_{m=0}^{k-1} f(m)\, z^{-m} \right]$$

$$= z^{-k} \left[\sum_{m=0}^{\infty} f(m)\, z^{-m} - f(0) - f(1)\, z^{-1} - f(2)\, z^{-2} - f(3)\, z^{-3} - \ldots - f(k-1)\, z^{-k+1} \right]$$

15. (a) (ii) Evaluate $Z^{-1}\left[(z-5)^{-3}\right]$ for $|z| > 5$.

Solution :

$$(z-5)^{-3} = \frac{1}{(z-5)^3}$$

We can find the inverse Z-transform by residue method.

Let $f(z) = \dfrac{1}{(z-5)^3}$

Multiplying both sides by z^{n-1}

$$\therefore z^{n-1} F(z) = z^{n-1} \cdot \frac{1}{(z-5)^3} = g(z)$$

Equating the denominator to zero we get

$$(z-5)^3 = 0$$

$$z = 5, 5, 5$$

Residue for pole of order 3 is

$$\lim_{z\to a} \frac{1}{2!} \frac{d^2}{dz^2} (z-a)^3 f(z)$$

$$\lim_{z\to 5} \frac{1}{2!} \frac{d^2}{dz^2} (z-a)^3 \frac{z^{n-1}}{(z-a)^3}$$

$$= \lim_{z\to 5} \frac{1}{2!} \frac{d^2}{dz^2} z^{n-1}$$

$$= \lim_{z\to 5} \frac{1}{2!} \frac{d}{dz} (n-1) z^{n-2}$$

$$= \lim_{z\to 5} \frac{1}{2} (n-1)(n-2) z^{n-3}$$

$$= \frac{1}{2} (n-1)(n-2)\, 5^{n-3}$$

$$\therefore Z^{-1}\left[(z-5)^{-3}\right] = \frac{1}{2} (n-1)(n-2)\, 5^{n-3}$$

15. (b) (i) Solve $u_{n+2} + 4u_{n+1} + 3u_n = 3^n$ given that $u_0 = 0$, $u_1 = 1$

Solution :

Let $\bar{u} = Z[u_n]$

Given $u_{n+2} + 4u_{n+1} + 3u_n = 3^n$

Taking Z-transform on both sides,

$$Z(u_{n+2}) + 4Z(u_{n+1}) + 3Z(u_n) = Z(3^n)$$

$$z^2\left(\bar{u} - u_0 - \frac{u_1}{z}\right) + 4\left(z\left(\bar{u} - u_0\right)\right) + 3\bar{u} = \frac{z}{z-3}$$

Using initial conditions

$$z^2\left[\bar{u} - 0 - \frac{1}{z}\right] + 4\left[z\left(\bar{u} - 0\right)\right] + 3\bar{u} = \frac{z}{z-3}$$

$$z^2\bar{u} - \frac{z^2}{z} + 4z\bar{u} + 3\bar{u} = \frac{z}{z-3}$$

$$z^2\bar{u} - z + 4z\bar{u} + 3\bar{u} = \frac{z}{z-3}$$

$$\bar{u}\,[z^2 + 4z + 3] - z = \frac{z}{z-3}$$

$$\bar{u}\,[z^2 + 4z + 3] = \frac{z}{z-3} + z$$

$$= \frac{z + z(z-3)}{(z-3)}$$

$$= \frac{z + z^2 - 3z}{z-3}$$

$$\bar{u}\,[z^2 + 4z + z] = \frac{z^2 - 2z}{z-3}$$

$$= \frac{z(z-2)}{z-3}$$

$$\bar{u} = \frac{z(z-2)}{(z-3)(z^2+4z+3)}$$

$$\bar{u} = \frac{z(z-2)}{(z-3)(z+3)(z+1)}$$

$$u = z\left[\frac{z(z-2)}{(z-3)(z+3)(z+1)}\right]$$

$$\text{Let } F[z] = \frac{z(z-2)}{(z-3)(z+3)(z+1)}$$

$$\frac{F[z]}{z} = \frac{(z-2)}{(z-3)(z+3)(z+1)}$$

Applying partial fractions

$$\frac{z-2}{(z-3)(z+3)(z+1)} = \frac{A}{(z-3)} + \frac{B}{(z+3)} + \frac{C}{(z+1)}$$

Let taking LCM and equating numerators

$$(z-2) = A(z+3)(z+1) + B(z-3)(z+1) + C(z-3)(z+3)$$

Put z = 3

$$(3-2) = A(3+3)(3+1)$$

$$1 = A(6)(4)$$

$$24A = 1$$

$$A = \frac{1}{24}$$

Put z = −1

$$(-1-2) = C(-1-3)(-1+3)$$

$$-3 = (-4)(2)C$$

$$-3 = -8C$$

$$C = \frac{3}{8}$$

Put z = −3

$$(-3-2) = B(-3-3)(-3+1)$$

$$-5 = B(-6)(-2)$$

$$-5 = 12B$$

$$B = \frac{-5}{12}$$

$$\therefore \frac{F(z)}{z} = \frac{A}{z-3} + \frac{B}{z+3} + \frac{C}{z+1}$$

$$= \frac{\frac{1}{24}}{z-3} + \frac{\frac{-5}{12}}{z+3} + \frac{\frac{3}{8}}{z+1}$$

$$= \frac{1}{24}\frac{z}{z-3} - \frac{5}{12}\frac{1}{z+3} + \frac{3}{8}\frac{1}{z+1}$$

$$\therefore F(z) = \frac{1}{24}\frac{z}{z-3} - \frac{5}{12}\frac{z}{z+3} + \frac{3}{8}\frac{z}{z+1}$$

$$Z^{-1}[F(z)] = Z^{-1}\left[\frac{1}{24}\frac{z}{z-3} - \frac{5}{12}\frac{z}{z+3} + \frac{3}{8}\frac{z}{z+1}\right]$$

$$= \frac{1}{24}z^n - \frac{5}{12}(-3)^n + \frac{3}{8}(-1)^n$$

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15. (b) (ii) Form the difference equation of second order by eliminating the arbitrary constants A and B from $y_n = A(-2)^n + Bn$

Solution :

Given $y_n = A(-2)^n + Bn$ ... (1)

Replacing n by $n+1$ we get

$$y_{n+1} = A(-2)^{n+1} + B(n+1)$$

$$\therefore y_{n+1} = A(-2)^n(-2) + B(n+1) \quad \text{... (2)}$$

Replacing n by $n+2$ we get

$$y_{n+2} = A(-2)^{n+2} + B(n+2)$$

$$= A(-2)^n (-2)^2 + B(n+2)$$

$$= A(-2)^n 4 + B(n+2)$$

$$\therefore y_{n+2} = 4A(-2)^n + B(n+2) \qquad \dots (3)$$

Divide (2) by -2

$$-\frac{y_{n+1}}{2} = A(-2)^n - \frac{B(n+1)}{2} \qquad \dots (4)$$

Divide (3) by 4

$$\frac{y_{n+2}}{4} = A(-2)^n + \frac{B(n+2)}{4} \qquad \dots (5)$$

Consider the determinant of coefficient of $A(-2)$, B and LHS terms twe get,

$$\begin{vmatrix} y_n & 1 & n \\ \frac{-y_{n+1}}{2} & 1 & \frac{-(n+1)}{2} \\ \frac{y_{n+2}}{2} & 1 & \frac{n+2}{4} \end{vmatrix} = 0$$

$$\{y_n \left[\frac{n+2}{4} + \frac{n+1}{2}\right] - 1\left[-\left(\frac{y_{n+1}}{2}\right)\left(\frac{n+2}{4}\right) + \left(\frac{y_{n+2}}{4}\right)\left(\frac{n+1}{2}\right)\right.$$
$$+ n\left[\frac{-y_{n+1}}{2} - \frac{y_{n+2}}{4}\right] = 0$$

$$y_n \left[\frac{(n+2) + (n+1)\,2}{4}\right] - \left[\frac{-(y_{n-1})(n+2)}{8} + \frac{(y_{n+2})(n+1)}{8}\right]$$
$$+ \left[\frac{-2(y_{n+1}) - y_{n+2}}{4}\right] = 0$$

$$y_n \left[\frac{n+2+2n+2}{4}\right] + \frac{(y_{n+1})(n+2)}{8} - \frac{(y_{n+2})(n+1)}{8}$$

$$-n\left[\frac{2y_{n+1}+y_{n+2}}{4}\right] = 0$$

$$\frac{(3n+4)}{4}y_n + \frac{(n+2)}{8}y_{n+1} - \frac{(n+1)}{8}y_{n+2} - \frac{n}{2}y_{n+1} - \frac{n}{4}y_{n+2} = 0$$

$$y_{n+2}\left(-\frac{n+1}{8} - \frac{n}{4}\right) + y_{n+1}\left(\frac{n+2}{8} - \frac{n}{2}\right) + \frac{3n+4}{4}y_n = 0$$

On simplyfing

$$(3n+1)\,y_{n+2} - (3n-2)\,y_{n+1} - (6n+8)\,y_n = 0$$