TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS

ANNA UNIVERSITY EXAMINATION, April/May 2011

PART - A (10 × 2 = 20 marks)

1. Give the expression for the Fourier Series co-efficient b_n for the function $f(x)$ defined in $(-2, 2)$

Solution :

$$\text{Here } 2l = UL - LL$$
$$= 2-(-2)$$
$$= 4$$
$$\Rightarrow l = 2$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{1}{2}\int_{-2}^{2} f(x) \sin\frac{n\pi x}{2}\, dx$$

when $f(x)$ is even function,

$$b_n = 0$$

when $f(x)$ is odd function,

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x) \sin\frac{n\pi x}{2}\, dx$$

$$= \frac{2}{2}\int_{0}^{2} f(x) \sin\frac{n\pi x}{2}\, dx$$

$$= \int_{0}^{2} f(x) \sin\frac{n\pi x}{2}\, dx$$

2. **Without finding the values of a_0, a_n and b_n, the Fourier coefficients of Fourier series, for the function $f(x) = x^2$ in the interval $(0, \pi)$ find the value of $\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\right]$**

Solution :

The Parseval's identity for the Fourier series in the interval $(0, 2\pi)$ is

$$\frac{1}{\pi} \int_0^{2\pi} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} [a_n^2 + b_n^2]$$

The Parseval's identity for the Fourier series in the interval $(0, \pi)$ is

$$\frac{1}{\pi/2} \int_0^{\pi} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} [a_n^2 + b_n^2]$$

$$\frac{2}{\pi} \int_0^{\pi} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} [a_n^2 + b_n^2]$$

$$\therefore \quad \frac{a_0^2}{2} + \sum_{n=1}^{\infty} [a_n^2 + b_n^2] = \frac{2}{\pi} \int_0^{\pi} [f(x)]^2 dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (x^2)^2 dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x^4 dx$$

$$= \frac{2}{\pi} \left[\frac{x^5}{5}\right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{\pi^5}{5} - 0\right]$$

$$= \frac{2\pi^4}{5}$$

3. State and prove the change of scale property of Fourier Transform.

Solution :

If $F[f(x)] = F(s)$, then

$$F[f(ax)] = \frac{1}{a} F\left(\frac{s}{a}\right)$$

Proof : By definition $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\, e^{isx}\, dx$$

Put $ax = t$

$$\therefore\ x = \frac{t}{a}$$

when $x = -\infty$, $t = -\infty$, when $x = \infty$, $t = \infty$

Differentiating $dx = \frac{dt}{a}$

$$\therefore\ F[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{\frac{ist}{a}}\, \frac{dt}{a}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{a} \int_{-\infty}^{\infty} f(t)\, e^{i\left(\frac{s}{a}\right)t}\, dt$$

Changing the dummy variable 't' to 'x'

$$= \frac{1}{a} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\left(\frac{s}{a}\right)x}\, dx$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right)$$

4. If $F_c(s)$ is the Fourier cosine transform of f(x), prove that the Fourier cosine transform of f (ax) is $\frac{1}{a} F_c\left(\frac{s}{a}\right)$

Solution :

Given, $F_c[f(x)] = F_c(s)$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$\therefore F_c[f(ax)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(ax) \cos sx \, dx$$

Put $ax = y$

Differentiating, $a\,dx = dy$

$$dx = \frac{dy}{a}$$

also $x = \frac{y}{a}$

$$\therefore F_c[f(y)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(y) \cos\left(s\frac{y}{a}\right) \frac{dy}{a}$$

$$= \frac{1}{a}\sqrt{\frac{2}{\pi}} \int_0^\infty f(y) \cos\left(\frac{s}{a}\right) y \, dy$$

$$= \frac{1}{a} F_c\left(\frac{s}{a}\right)$$

5. Form the partial differential equation by eliminating the arbitrary constants a and b from $z = (x^2 + a)(y^2 + b)$.

Solution :

$$z = (x^2 + a)(y^2 + b) \qquad \text{... (1)}$$

Differentiating partially w.r.t 'r'

$$\frac{\partial z}{\partial x} = 2x(y^2+b)$$

$$\therefore \quad p = 2x(y^2+b)$$

$$\frac{p}{2x} = (y^2+b)$$

$$y^2+b = \frac{p}{2x}$$

Differentiating partially w.r.t 'y'

$$\frac{\partial z}{\partial y} = (x^2+a)\,2y$$

$$\therefore \quad q = (x^2+a)\,2y$$

$$\frac{q}{2y} = x^2+a$$

$$\therefore x^2+a = \frac{q}{2y}$$

Substituting for x^2+a and y^2+b in (1) we get,

$$z = \frac{q}{2y}\cdot\frac{p}{2x}$$

$$z = \frac{qp}{4xy}$$

Cross multiplying

$4xyz = qp$ is required PDE.

6. Solve the equation $(D-D')^3 z = 0$

Solution :

This is a Non-homogeneous of the form

$$(D - m\,D' - \alpha)^2 = 0$$

The C.F is given by

$$z = e^{\alpha x} f_1(y + mx) + x e^{\alpha x} f_2(y + mx)$$

Here $\alpha = 0$, $m = 1$

$$\therefore z = e^{0 . x} f_1(y + 1 . x) + x e^{0 . x} f_2(y + 1 . x)$$

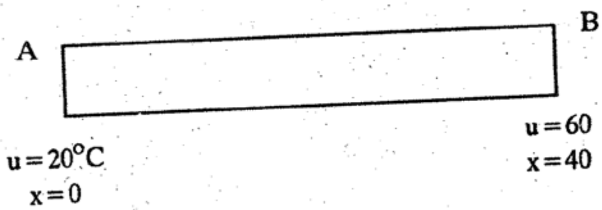
$$z = f_1(y + x) + f_2(y + x)$$

7. A rod 40 cm long with insulated sides has its ends A and B kept at 20°C and 60°C respectively. Find the steady state temperature at a location 15 cm from A.

Solution :

In steady state temperature the heat flow equation is given by

$$u = ax + b \quad \text{... (1)}$$



Applying the condition at the end A in (1)

$$20 = a(0) + b$$

$$\therefore b = 20$$

Applying the condition at the end B in (1)

$$60 = a(40) + b$$

$$60 = 40a + b$$

$$60 = 40a + 20$$

$$60 - 20 = 40a$$

$$40a = 40$$

$$\therefore a = 1$$

Substituting the values a and b in (1) we get

$$u = x + 20$$

At the location 15 cm

$$u = 15 + 20 \qquad [\because x = 15]$$

$$u = 35$$

$\therefore$ The steady state temperature is $35^{o}C$

8. Write down the three possible solutions of Laplace equation in two dimensions.

Solution :

The three possible solutions of Laplace equation are

$$u = (A e^{px} + Be^{-px})(C \cos py + D \sin py)$$

$$u = (A \cos px + B \sin px)(Ce^{py} + D e^{-py})$$

$$u = (Ax + B)(Cy + D)$$

9. Find the Z-transform of a^n

Solution :

$$Z[a^n] = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{a^n}{z^n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

Substituting $n = 0, 1, 2, \ldots$

$$Z[a^n] = 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \ldots$$

$$= \left(1 - \frac{a}{z}\right)^{-1}$$

[Using binomial theorem $(1-x)^{-1} = 1 + x + x^2 + \ldots$

$$= \frac{1}{\pi n^2}\left[(2\pi - 2x)\,.\cos nx\right]_0^{2\pi}$$

$$= \frac{1}{\pi^2 n^2}\left[\left\{(2\pi - 2(2\pi)\,.\cos 2n\pi)\right\} - \left\{\left(2\pi - 0\right)\cos 0\right)\right]$$

$$= \frac{1}{\pi n^2}\left[\left\{(2\pi - 4\pi)\,.\,1\right\} - \left\{(2\pi)\,.\,1\right\}\right]$$

$$= \frac{1}{\pi n^2}[-2\pi - 2\pi]$$

$$= \frac{1}{\pi n^2}[-4\pi]$$

$$= \frac{-4\pi}{\pi n^2}$$

$$= \frac{-4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\,.\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (2\pi x - x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(2\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (2\pi - 2x)\underbrace{\left(\frac{-\sin nx}{n^2}\right)}_{0} + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{-1}{\pi}\left[(2\pi x - x^2)\left(\frac{\cos nx}{n}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi} \quad \text{[Taking } - \text{ sign outside]}$$

$$= \frac{-1}{\pi}\left[\left\{(4\pi^2 - 4\pi^2).\left(\frac{\cos 2n\pi}{n}\right) + 2\left(\frac{\cos 2n\pi}{n^3}\right)\right\} - \left\{0 + \left(\frac{2\cos 0}{n^3}\right)\right\}\right]$$

$$= \frac{-1}{\pi}\left[0 \cdot \frac{1}{n} + \frac{2}{n^3} - \frac{2}{n^3}\right]$$

$$= 0$$

$$\therefore b_n = 0$$

$$\therefore F(x) = \frac{\frac{4\pi^2}{3}}{2} + \sum_{n=1}^{\infty} \frac{-4}{n^2} \cos nx + 0$$

$$2\pi - x^2 = \frac{2\pi^2}{3} - 4 \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

$$= \frac{2\pi^2}{3} - 4\left[\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \dots\right]$$

Putting $x = 0$ on both sides,

$$0 = \frac{2\pi^2}{3} - 4\left[\frac{\cos 0}{1^2} + \frac{\cos 0}{2^2} + \frac{\cos 0}{3^2} + \dots\right]$$

$$0 = \frac{2\pi^2}{3} - 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

$$-\frac{2\pi^2}{3} = -4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

$$\frac{2\pi^2}{12} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

11. (a) (ii) Obtain the Fourier series for the function f(x) given by $f(x) = \begin{cases} 1-x, & -\pi < x < 0 \\ 1+x, & 0 < x < \pi \end{cases}$

Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$

Solution :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} f(x)\, dx + \int_{0}^{\pi} f(x)\, dx \right]$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} (1-x)\, dx + \int_{0}^{\pi} (1+x)\, dx \right]$$

$$= \frac{1}{\pi} \left[\left(x - \frac{x^2}{2} \right)_{-\pi}^{0} + \left(x + \frac{x^2}{2} \right)_{0}^{\pi} \right]$$

$$= \frac{1}{\pi} \left[0 - \left(-\pi - \frac{\pi^2}{2} \right) + \left(\pi + \frac{\pi^2}{2} - 0 \right) \right]$$

$$= \frac{1}{\pi} \left[\pi + \frac{\pi^2}{2} + \pi + \frac{\pi^2}{2} \right]$$

$$= \frac{1}{\pi} \left[2\pi + \pi^2 \right]$$

$$= 2 + \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (1-x)\cos nx\, dx + \int_{0}^{\pi} (1+x)\cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left\{\left[\underbrace{(1-x)\frac{\sin nx}{n}}_{0} - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{-\pi}^{0} + \left[\underbrace{(1+x)\frac{\sin nx}{n}}_{0} - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left[\left(\frac{-\cos nx}{n^2}\right)_{-\pi}^{0} + \left(\frac{\cos nx}{n^2}\right)_{0}^{\pi}\right]$$

$$= \frac{1}{\pi n^2}\left[-\left(\cos nx\right)_{-\pi}^{0} + \left(\cos nx\right)_{0}^{\pi}\right]$$

$$= \frac{1}{\pi n^2}\left[-(\cos 0 - \cos n\pi) + (\cos n\pi - \cos 0)\right]$$

$[\because \cos(-\theta) = \cos\theta]$

$$= \frac{1}{\pi n^2}\left[-\left(1-(-1)^n\right) + \left((-1)^n - 1\right)\right] \quad \{\cos n\pi = (-1)^n$$

$$= \frac{1}{\pi n^2}\left[-1 + (-1)^n + (-1)^n - 1\right]$$

$$= \frac{1}{\pi n^2}\left[2(-1)^n - 2\right]$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

Case (i)

When n is even number, $(-1)^n = 1$

$$a_n = \frac{2}{\pi n^2}[1-1]$$

$$= 0$$

Case (ii)

When n is odd number, $(-1)^n = -1$

$$a_n = \frac{2}{\pi n^2}[-1-1]$$

$$= \frac{2}{\pi n^2}(-2)$$

$$a_n = -\frac{4}{\pi n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(1-x)\sin nx\,dx + \int_{0}^{\pi}(1+x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left\{\left[(1-x)\left(\frac{-\cos nx}{n}\right) - (-1)\underbrace{\left(\frac{-\sin nx}{n^2}\right)}_{0}\right]_{-\pi}^{0} + \left[(1+x)\left(\frac{-\cos nx}{n}\right) - (1)\underbrace{\left(\frac{-\sin nx}{n^2}\right)}_{0}\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[(1-x)\left(\frac{-\cos nx}{n}\right)\right]_{-\pi}^{0} + \left[(1+x)\left(\frac{-\cos nx}{n}\right)\right]_{0}^{\pi}\right\}$$

$$= \frac{-1}{n\pi}\left\{\left[(1-x)\cos nx\right]_{-\pi}^{0} + \left[(1+x)\cos nx\right]_{0}^{\pi}\right\}$$

$$= -\frac{1}{n\pi}\left\{\cos 0 - (1+\pi)\cos n\pi + (1+\pi)\cos n\pi - \cos 0\right\}$$

$$[\because \cos(-\theta) = \cos\theta]$$

$$= -\frac{1}{n\pi}\left[1 - \cos n\pi - \pi\cos n\pi + \cos n\pi + \pi\cos n\pi - 1\right]$$

$$= \frac{-1}{n\pi}[0]$$

$$= 0$$

$$\therefore f(x) = \frac{2+\pi}{2} + \sum_{n=1,3}^{\infty} -\frac{4}{\pi n^2}\cos nx$$

$$f(x) = \frac{2+\pi}{2} - \frac{4}{\pi}\sum_{n=1,3}^{\infty}\frac{\cos nx}{n^2}$$

$$f(x) = \frac{2+\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right]$$

By substituting $x = 0$, In RHS we get the required series. But $x = 0$ is a point of discontinuity in $f(x)$ at $x = 0$.

$\therefore$ The value of $f(x)$ at $x = 0$ is given by

$$\frac{f(x+0) + f(x-0)}{2}$$

$$= \frac{1+x+1-x}{2} \qquad \begin{bmatrix} f(x) = 1+x \text{ for } x > 0 \\ f(x) = 1-x \text{ for } x < 0 \end{bmatrix}$$

$$= \frac{2}{2}$$

$$= 1$$

$$\therefore 1 = 1 + \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos 0}{1^2} + \frac{\cos 0}{3^2} + \frac{\cos 0}{5^2} + \dots\right]$$

$$-\frac{\pi}{2} = -\frac{4}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots$$

11. (b) (i) Obtain the sine series for

$$\mathbf{f(x)} = \begin{cases} x & in \ 0 \le x \le \frac{l}{2} \\ (l-x) & in \ \frac{l}{2} \le x \le l \end{cases}$$

Solution :

The half range sine series is,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\,.\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} f(x)\,.\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} f(x)\,.\sin\frac{n\pi x}{l}\,.dx\right]$$

$$= \frac{2}{l}\left[\int_0^{l/2} x\sin\frac{n\pi x}{l}\,.dx + \int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}\,.dx\right]$$

$$= \frac{2}{l}\left[\left\{x\left[\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right] - (1)\left[\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right]\right\}_0^{l/2}\right.$$

$$\left. + \left\{(l-x)\left[-\cos\frac{n\pi x}{l}\right] - (-1)\left[\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right]\right\}_{l/2}^{l}\ \right]$$

$$= -\frac{2}{l}\left[\left(\frac{-x\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}+\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)_0^{l/2} - \left((l-x)\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}+\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)_{l/2}^{l}\right]$$

$$= \frac{2}{l}\left[\left(-\frac{l}{2}\frac{\cos\frac{n\pi}{2}}{\frac{n\pi}{l}}+\frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{l^2}}-(0+0)\right)-\left(0-\left\{\frac{l}{2}\frac{\cos\frac{n\pi}{2}}{\frac{n\pi}{l}}+\frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{l^2}}\right\}\right)\right]$$

$$= \frac{2}{l}\left[-\frac{l}{2}\frac{\cos\frac{n\pi}{2}}{\frac{n\pi}{l}}+\frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{l^2}}+\frac{l}{2}\frac{\cos\frac{n\pi}{2}}{\frac{n\pi}{2}}+\frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{l^2}}\right]$$

$$= \frac{2}{l}\left[\frac{2\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{l}}\right]$$

$$= \frac{2}{l}\cdot\frac{l^2}{n^2\pi^2}\cdot 2\sin\frac{n\pi}{2}$$

$$= \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$\therefore$ The sine series is

$$f(x) = \sum_{n=1}^{\infty}\frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}$$

11. (b) (ii) Find the Fourier series up to second harmonic for $y = f(x)$ from the following values.

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
y	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Solution :

Here first and last y values are repeated

$\therefore$ we can omit the last value.

$\therefore$ Hence $n = 6$

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x	y	$\cos x$	$y \cos x$	$\cos 2x$	$y \cos 2x$	$\cos 3x$	$y \cos 3x$	$\sin x$	$y \sin x$	$\sin 2x$	$y \sin 2x$	$\sin 3x$	$y \sin 3x$
0	1	1	1	1	1	1	1	0	0	0	0	0	0
60	1.4	0.5	0.7	–0.5	–0.7	–1	–1.4	0.866	1.2124	0.866	1.212	0	0
120	1.9	–0.5	–0.95	–0.5	–0.95	1	1.9	0.866	1.6454	–0.866	–1.6454	0	0
180	1.7	–1	–1.7	1	1.7	–1	–1.7	0	0	0	0	0	
240	1.5	–0.5	–0.75	–0.5	–0.75	1	1.5	–0.866	–1.299	0.866	1.299	0	0
300	1.2	0.5	0.6	–0.5	–0.6	–1	–1.2	–0.866	–1.0392	–0.866	–1.0932	0	0
	$\Sigma y = 8.7$		$\Sigma y \cos x = -1.1$		$\Sigma \cos 2x = -0.3$		$\Sigma \cos 3x = 0.1$		$\Sigma \sin x = 0.5196$		$\Sigma y \sin 2x = -0.1736$		$\Sigma y \sin 3x = 0$

$$a_0 = \frac{2}{n}\Sigma y = \frac{2}{6}\times 8.7 = 2.900$$

$$a_1 = \frac{2}{n}\Sigma y\cos x = \frac{2}{6}\times(-1.1) = -0.366$$

$$a_2 = \frac{2}{n}\Sigma y\cos 2x = \frac{2}{6}\times(-0.3) = -0.1$$

$$a_3 = \frac{2}{n}\Sigma y\cos 3x = \frac{2}{6}\times(0.1) = 0.033$$

$$b_1 = \frac{2}{n}\Sigma y\sin x = \frac{2}{6}\times(0.5196) = 0.1732$$

$$b_2 = \frac{2}{n}\Sigma y\sin 2x = \frac{2}{6}\times(-0.1736) = -0.0578$$

$$b_3 = \frac{2}{n}\Sigma y\sin 3x = \frac{2}{6}\times(0) = 0$$

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + b_1\sin x + b_2\sin 2x + b_3\sin 3x$$

$$= \frac{2.9}{2} - 0.366\cos x - 0.1\cos 2x + 0.033\cos 3x + 0.1732\sin x - 0.0578\sin 2x + 0$$

$$f(x) = 1.45 - 0.366\cos x - 0.1\cos 2x + 0.033\cos 3x + 0.1732\sin x - 0.0578\sin 2x$$

12. (a) (i) Find the Fourier transform of $f(x) = \begin{cases} 1 - x^2 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$

Hence evaluate $\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right) \cos\frac{x}{2}\, dx$

Solution : By definition

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - x^2)\, e^{isx}\, dx \qquad [\because x \text{ lies between } -1, 1]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\underset{\substack{\downarrow \\ 0}}{(1 - x^2) \frac{e^{isx}}{is}} - (-2x) \frac{e^{isx}}{(is)^2} + (-2) \frac{e^{isx}}{(is)^3} \right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left[2x \frac{e^{isx}}{i^2 s^2} - 2 \frac{e^{isx}}{i^3 s^3} \right]_{-1}^{1}$$

$$= \frac{2}{\sqrt{2\pi}} \left[x \frac{e^{isx}}{-s^2} - \frac{e^{isx}}{-is^3} \right]_{-1}^{1} \qquad [\because i^2 = 1,\ i^3 = -i]$$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{1 \cdot e^{is} - (-1)\, e^{is}}{-s^2} + \frac{e^{is} - e^{is}}{is^3} \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[\left(\frac{e^{is} + e^{-is}}{-s^2} \right) + \left(\frac{e^{is} - e^{-is}}{is^3} \right) \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{2\cos s}{-s^2} + \frac{2i\sin s}{is^3} \right] \qquad \left[\because \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}\right]$$

$$= \frac{4}{\sqrt{2\pi}} \left[\frac{-\cos s}{s^2} + \frac{\sin s}{s^3} \right] \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$F[f(x)] = \frac{4}{\sqrt{2\pi}}\left[\frac{-s\cos s + \sin s}{s^3}\right]$$

$$= \frac{4}{\sqrt{2\pi}}\left[\frac{\sin s - s\cos s}{s^3}\right]$$

Now consider inverse Fourier Transform

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]\, e^{-isx}\, ds$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{4}{\sqrt{2\pi}}\left(\frac{\sin s - s\cos s}{s^3}\right) e^{-isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\cdot\frac{4}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)(\cos sx - i\sin sx)\, ds$$

$$[\because e^{-i\theta} = \cos\theta - i\sin\theta]$$

$$= \frac{4}{2\pi}\int_{-\infty}^{\infty}\left[\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sx - \left(\frac{\sin s - s\cos s}{s^3}\right) i\sin sx\right] ds$$

$$= \frac{2}{\pi}\left[\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sx\, ds - i\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\sin sx\, ds\right]$$

The second integral becomes zero since the function is odd and limits are $-\infty, \infty$.

Also the function under the first integral is even.

$\therefore$ we get

$$f(x) = \frac{2}{\pi}\left[2\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sx\, ds\right] \qquad [f(x) = 1 - x^2]$$

$$1 - x^2 = \frac{4}{\pi}\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sx\, ds$$

Substituting $x = \frac{1}{2}$ on both sides,

$$1-\left(\frac{1}{2}\right)^2 = \frac{4}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)\cos s\left(\frac{1}{2}\right) ds$$

$$1-\frac{1}{4} = \frac{4}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)\cos \frac{s}{2}\, ds$$

$$\frac{3}{4} = \frac{4}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)\cos \frac{s}{2}\, ds$$

$$\frac{3\pi}{16} = \int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)\cos \frac{s}{2}\, ds$$

Changing the dummy variable 's' to 'x' we get

$$\therefore \int_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right)\cos \frac{x}{2}\, dx = \frac{3\pi}{16}$$

$$\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos \frac{x}{2}\, dx = \frac{-3\pi}{16}$$

---

12. (a) (ii) Find the Fourier transform of f(x) given by

$$f(x) = \begin{cases} 1 & \text{for } |x| < a \\ 0 & \text{for } |x| > a > 0 \end{cases}$$

and using Parseval's identity prove that $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$

Solution : Fourier transform is defined as

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$|x| \le a$ means x lies between $-a$ and a

$|x| > a$ means x lies between $-\infty$ to $-a$ and a to ∞

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} f(x)\, e^{isx}\, dx + \int_{-a}^{0} f(x)\, e^{isx}\, dx + \int_{0}^{\infty} f(x)\, e^{isx}\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} \underset{0}{\underset{\downarrow}{0}}\, e^{isx}\, dx + \int_{-a}^{a} e^{isx}\, dx + \int_{a}^{\infty} \underset{0}{\underset{\downarrow}{0}}\, e^{isx}\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{isx}}{is}\right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{isa} - e^{-isa}}{is}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2i \sin sa}{is}\right]$$

$$= \frac{2}{\sqrt{2\pi}}\, \frac{\sin sa}{s}$$

By Parseval's identity

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds$$

$$\int_{-a}^{a} (1)^2\, dx = \int_{-\infty}^{\infty}\left[\frac{2}{\sqrt{2\pi}}\left(\frac{\sin sa}{s}\right)\right]^2 ds$$

[$\because f(x)$ is defined between $-a$ and a]

$$\int_{-a}^{a} dx = \int_{-\infty}^{\infty} \frac{4}{2\pi}\left(\frac{\sin sa}{s}\right)^2 ds$$

$$\left[x\right]_{-a}^{a} = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin sa}{s}\right)^2 ds$$

$$a-(-a) = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin sa}{s}\right)^2 ds$$

$$2a = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin sa}{s}\right)^2 ds$$

$$a\pi = \int_{-\infty}^{\infty}\left(\frac{\sin sa}{s}\right)^2 ds$$

Take $a = 1$ on both sides and replace 's' by 't'

$$\pi = \int_{-\infty}^{\infty}\left(\frac{\sin t}{t}\right)^2 dt$$

$$= 2\int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^2 dt$$

$$\therefore \frac{\pi}{2} = \int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^2 dt$$

12. (b) (i) Find the Fourier sine transform of

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$$

Solution :

The Fourier sine transform is defined as

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 f(x) \sin sx \, dx + \int_1^2 f(x) \sin sx \, dx + \int_2^{\infty} f(x) \sin sx \, dx \right.$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 x \sin sx \, dx + \int_1^2 (2-x) \sin sx \, dx + \int_2^{\infty} 0 \sin sx \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 x \sin sx \, dx + \int_1^2 (2-x) \sin sx \, dx \right]$$

Applying Bernoulli's formula

$$= \sqrt{\frac{2}{\pi}} \left[\left[\left\{ x \left(\frac{-\cos sx}{s} \right) - (1) \left(\frac{-\sin sx}{s^2} \right) \right\} \right]_0^1 \right.$$

$$\left. + \quad (2-x) \left(\frac{-\cos sx}{s} \right) - (-1) \left\{ \frac{-\sin sx}{s^2} \right\} \right]_0^2$$

$$= \sqrt{\frac{2}{\pi}} \left[\left[\left\{ \frac{-x \cos sx}{s} + \frac{\sin sx}{s^2} \right\} \right]_0^1 - \left\{ (2-x) \frac{\cos sx}{s} + \frac{\sin sx}{s^2} \right\}_0^2 \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\left[\frac{-\cos s - 0}{s} + \frac{\sin s - 0}{s^2} \right] - \left\{ \frac{0 - \cos s}{s} + \frac{\sin 2s - \sin s}{s^2} \right\} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-\cos s}{s} + \frac{\sin s}{s^2} + \frac{\cos s}{s} - \frac{\sin 2s}{s^2} + \frac{\sin s}{s^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2 \sin s}{s^2} - \frac{\sin 2s}{s^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2 \sin s - \sin 2s}{s^2} \right]$$

---

12. (b) (ii) Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ using Fourier cosine transforms of e^{-ax} and e^{-bx}

Solution :

Solution : Let $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$

then $\quad F_c(f(x)) = F_c[e^{-ax}]$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2+s^2} \right]$$

(i.e.,) $\quad F_c[g(x)] = F_c[e^{-bx}]$

$$G_c(s) = \sqrt{\frac{2}{\pi}} \left[\frac{b}{b^2+s^2} \right] \quad \text{(by Qn.No.3 Pg.No. 2.37)}$$

By Parseval's theorem

$$\int_0^\infty f(x)\, g(x)\, dx = \int_0^\infty F_c(s)\, G_c(s)\, ds$$

$$\int_0^\infty e^{-ax} e^{-bx}\, dx = \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2+s^2} \right) \sqrt{\frac{2}{\pi}} \left(\frac{b}{b^2+s^2} \right) ds$$

$$\int_0^\infty e^{-(a+b)x}\, dx = \int_0^\infty \frac{2}{\pi} \frac{ab}{(a^2+s^2)(b^2+s^2)}\, ds$$

$$\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty = \frac{2ab}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)\,(b^2+s^2)}$$

$$-\frac{1}{a+b}\left[e^{-\infty}-e^0\right] = \frac{2ab}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)\,(b^2+s^2)}$$

$$-\frac{1}{a+b}(0-1) = \frac{2ab}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)\,(b^2+s^2)}$$

$$-\frac{1}{a+b}(-1) = \frac{2ab}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)\,(b^2+s^2)}$$

$$\frac{1}{a+b} = \frac{2ab}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)\,(b^2+s^2)}$$

Cross multiplying

$$\frac{\pi}{2ab\,(a+b)} = \int_0^\infty \frac{ds}{(a^2+s^2)\,(b^2+s^2)}$$

Changing the variable 's' to 'x'

$$\frac{\pi}{2ab\,(a+b)} = \int_0^\infty \frac{dx}{(a^2+x^2)\,(b^2+x^2)}$$

13. (a) (i) Form the partial differential equation by eliminating arbitrary functions f and ϕ from $z = f(x + ct) + \phi(x - ct)$

Solution :

$$Z = f(x+ct) + \phi(x-ct) \quad \dots (1)$$

Here there are 2 arbitrary functions.

$\therefore$ we have to find the second order partial derivatives

$$\frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial^2 z}{\partial y^2}, \quad \frac{\partial^2 z}{\partial x \partial y} \text{ also.}$$

$$\frac{\partial z}{\partial x} = f'(x+ct) + \phi'(x-ct) \quad \dots (2)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ct) + \phi''(x-ct) \quad \dots (3)$$

$$\frac{\partial z}{\partial t} = f'(x+ct)(c) + \phi'(x-ct)(-c) \quad \dots (4)$$

$$\frac{\partial^2 z}{\partial t^2} = f''(x-ct)c^2 + \phi''(x-ct)c^2$$

$$= c^2\left[f''(x+ct) + \phi''(x-ct)\right] \quad \dots (5)$$

$$\frac{\partial^2 z}{\partial x \partial t} = f''(x+ct)(c) + \phi''(x-ct)(-c) \quad \dots (6)$$

From (3) and (5)

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2} \text{ is the required P.D.E.}$$

13. (a) (ii) Solve the partial differential equation $(mz - ny)p + (nx - lz)q = ly - mx$

Solution :

Solution : Given : $(mz - ny)\,p + (nx - lz)\,q = ly - mx$

This equation is of the form $P\,p + Q\,q = R$

where $P = mz - ny$, $Q = nx - lz$, $R = ly - mx$

The Lagrange's subsidiary equations are $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

i.e., $\dfrac{dx}{mz - ny} = \dfrac{dy}{nx - lz} = \dfrac{dz}{ly - mx}$ ... (1)

Using two set of multipliers x, y, z ; l, m, n **each of the ratio in (1)**

$$= \frac{xdx + ydy + zdz}{x\,(mz - ny) + y\,(nx - lz) + z\,(ly - mx)} = \frac{xdx + ydy + zdz}{0}$$

$$= \frac{ldx + mdy + ndz}{l\,(mz - ny) + m\,(nx - lz) + n\,(ly - mx)} = \frac{ldx + mdy + ndz}{0}$$

Hence, $xdx + ydy + zdz = 0$; and $ldx + mdy + ndz = 0$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{a}{2},\quad lx + my + nz = b$$

i.e., $x^2 + y^2 + z^2 = a,\quad lx + my + nz = b$

Hence, the general solution is

$$f(x^2 + y^2 + z^2,\ lx + my + nz) = 0, \text{ where } f \text{ is arbitrary.}$$

13. (b) (i) Solve $(D^2 - D'^2)z = e^{x-y}\sin(2x+3y)$

Solution :

Replacing D by m and D' by 1 we get the AE.

$$m^2 - 1 = 0$$

$$m^2 = 1$$

$$m = 1, 1$$

The C.F is $z = f_1(y+x) + xf_2(y+x)$

P.I

$$z = \frac{1}{D^2 - D'^2} \cdot e^{x-y} + \frac{1}{D^2 - D'^2}\sin(2x+3y)$$

$$= P.I_1 + P.I_2$$

$\mathbf{P.I_1}$

$$z = \frac{1}{D^2 - D'^2} e^{x-y}$$

$$= \frac{1}{(1)^2 - (-1)^2} e^{x-y}$$

$$= \frac{1}{1-1} e^{x-y}$$

$[D = a = 1$

$D' = b = -1]$

Denominator becomes zero.

$$P.I_1 = \frac{x}{2D} e^{x-y} \quad \text{[Applying type IV]}$$

$$= \frac{x}{2.1} e^{x-y}$$

$$= \frac{x}{2} e^{x-y}$$

P.I$_2$ $$z = \frac{1}{D^2 - D'^2} \sin(2x+3y)$$

$$= \frac{1}{-4-(-9)} \sin(2x+3y) \qquad [D^2 - 4,\ D'^2 = -9]$$

$$= \frac{1}{5} \sin(2x+3y)$$

$\therefore$ The solution is $z = \text{C.F} + P.I_1 + P.I_2$

$$z = f_1(y+x) + xf_2(y+x) + \frac{x}{2}e^{x-y} + \frac{1}{5}\sin(2x+3y)$$

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13. (b) (ii) Solve

$$\mathbf{(D^2 - 3DD' + 2D'^2 + 2D - 2D')\,z = x + y + \sin(2x+y)}$$

Solution :

$$D^2 - 3DD' + 2D'^2 + 2D - 2D' = 0$$

$$(D-D')(D-2D') + 2(D-D') = 0$$

$$(D-D')(D-2D'+2) = 0$$

Put $D = m,\ D' = 1$

Auxiliary equation is

$$\therefore \quad (m-1)(m-2+2) = 0$$

$$(m-1)m = 0$$

$$\therefore m_1 = 0,\ m_2 = 1$$

$$\text{The C.F} = f_1(y+x) + f_2(y)$$

$$P.I = \frac{1}{(D^2 - 3DD' + 2D'^2 + 2D - 2D')}(x+y) + \sin(3x+y)$$

$$= \frac{1}{D^2 - 3DD' + 2D'^2 + 2D - 2D'}(x+y)$$

$$+ \frac{1}{D^2 - 3DD' + 2D'^2 + 2D - 2D'}\sin(3x+y)$$

$$= P.I_1 + P.I_2$$

$$P.I_1 = \frac{1}{D^2 - 3DD' + 2D'^2 + 2D - 2D'}(x+y)$$

$$= \frac{1}{D^2\left(1 - \frac{3D'}{D} + \frac{2D'^2}{D^2} + \frac{2}{D} - \frac{2}{D^2}\right)}x+y$$

$$= \frac{1}{D^2}\left[1 - \left(\frac{3D'}{D} - \frac{2D'^2}{D^2} - \frac{2}{D} + \frac{2}{D^2}\right)\right]^{-1}(x+y)$$

$$= \frac{1}{D^2}\left[1 + \frac{3D'}{D} - \frac{2D'^2}{D^2} - \frac{2}{D} + \frac{2}{D^2}\right](x+y)$$

[Applying Binomial series and omitting higher degree terms]

$$= \frac{1}{D^2}\left[x + y + \frac{3}{D}D'(x+y) - \frac{2}{D^2}D'^2(x+y) - \frac{2}{D}(x+y) + \frac{2}{D^2}(x+y)\right]$$

$$= \frac{1}{D^2}\left[x + y + \frac{3}{D}(1) - \frac{2}{D^2}(0) - \frac{2}{D}x - \frac{2}{D}y + \frac{2}{D^2}x + \frac{2}{D^2}y\right]$$

$$= \frac{1}{D^2}\left[x + y + 3x - \frac{2x^2}{2} - 2xy + \frac{x^3}{3} + \frac{2}{2}x^2y\right]$$

$$= \frac{1}{D^2}\left[4x^2 + y - 2x^2 - 2xy + \frac{x^3}{3} + x^2y\right]$$

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$$= \frac{1}{D}\left[\frac{4x^2}{2} + xy - \frac{x^3}{3} - \frac{2x^2}{2}y + \frac{x^4}{12} + \frac{x^3 y}{3}\right]$$

$$= \frac{2x^3}{3} + \frac{x^2 y}{2} - \frac{x^4}{12} - \frac{x^3 y}{3} + \frac{x^5}{60} + \frac{x^4 y}{12}$$

$\therefore$ The solution is $z = f_1(y+x) + f_2(y)$

$$+ \frac{2x^3}{3} + \frac{x^2 y}{2} - \frac{x^4}{12} - \frac{x^3 y}{3} + \frac{x^5}{60} + \frac{x^4 y}{12}$$

$$P.I_2 = \frac{1}{D^2 - 3DD' + 2D'^2 + 2D - 2D'} \sin(2x+y)$$

Replace D^2 by $-a^2$, D'^2 by $-b^2$ and DD' by $-ab$

$$= \frac{1}{-4 - 3(-2) + 2(-1) + 2D - 2D'} \sin(3x+y) \quad [\because a = 2, b = 1]$$

$$= \frac{1}{-4 + 6 - 2 + 2D - 2D'} \sin(3x+y)$$

$$= \frac{1}{2D - 2D'} \sin(3x+y)$$

$$= \frac{1}{2(D - D')} \sin(3x+y)$$

$$= \frac{1}{2(D - D')} \times \frac{D + D'}{D + D'} \sin(3x+y) \quad \text{[Multiply and divide by conjugate]}$$

$$= \frac{D + D'}{2(D^2 - D'^2)} \sin(3x+y)$$

Replace D^2 by $-a^2$ and D'^2 by $-b^2$

$$= \frac{D + D'}{2(-4 + 1)} \sin(3x+y)$$

$$= \frac{D + D'}{-6} \sin(3x + y)$$

$$= \frac{D \sin(3x + y) + D' \sin(3x + y)}{-6}$$

$$= \frac{3 \cos(3x + y) + \cos(3x + y)}{-6}$$

$$= \frac{4 \cos(3x + y)}{-6}$$

$$P.I_2 = \frac{-2}{3} \cos(3x + y)$$

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14. (a) A uniform string is stretched and fastened to two points 'l' apart. Motion is started by displacing the string into the form of the curve $y = kx(l - x)$ and then released from this position at time t = 0. Derive the expression for the displacement of any point of the string at a distance x from one end at time t.

Solution :

Solution : The wave equation is $\dfrac{\partial^2 y}{\partial t^2} = a^2 \dfrac{\partial^2 y}{\partial x^2}$

From the given problem we get the following boundary conditions.

(i) $y(0, t) = 0$ for all $t > 0$

(ii) $y(l, t) = 0$ for all $0 < x < l$

(iii) $y(x, 0) = 0$, $0 < x < l$

(iv) $\dfrac{\partial y}{\partial t}(x, 0) = \lambda x (l - x)$, $0 < x < l$

Now the suitable solution which satisfies our boundary conditions is gien by

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos p\,at + c_4 \sin p\,at) \quad \dots (1)$$

Applying condition (i) in equation (1) we get

$$y(0,t) = c_1(c_3 \cos p\,at + c_4 \sin p\,at) = 0$$

Here $c_3 \cos p\,at + c_4 \sin p\,at \neq 0$ [$\because$ It is defined for all t]

Therefore we get $c_1 = 0$

Substitute $c_1 = 0$ in (1) we get

$$y(x,t) = c_2 \sin p\,x\,(c_3 \cos p\,at + c_4 \sin p\,at) \quad \ldots (2)$$

Applying condition (ii) in equation (2) we get

$$y(l,t) = c_2 \sin p\,l\,(c_3 \cos p\,a\,t + c_4 \sin p\,at) = 0$$

Here $(c_3 \cos p\,at + c_4 \sin p\,at) \neq 0$ [$\because$ It is defined for all t]

Therefore either $c_2 = 0$ or $\sin p\,l = 0$

Suppose if we take $c_2 = 0$ and already we have $c_1 = 0$ then we get a trivial solution.

Therefore we consider $c_2 \neq 0$ and $\sin p\,l = 0$

$$\sin p\,l = 0$$

$$pl = n\pi \qquad [\because \sin n\pi = 0]$$

$$\boxed{p = \frac{n\pi}{l}} \qquad [n \text{ being an integer}]$$

Now substituting $p = \dfrac{n\pi}{l}$ in equation (2) we get

$$y(x,t) = c_2 \sin \frac{n\pi x}{l}\left(c_3 \cos \frac{n\pi at}{l} + c_4 \sin \frac{n\pi at}{l}\right) \quad \ldots (3)$$

Applying condition (iii) in equation (3) we get

$$y(x,0) = c_2 \sin \frac{n\pi x}{l}\, c_3 = 0$$

$$\Rightarrow c_2 c_3 \sin \frac{n\pi x}{l} = 0$$

Here $\sin\frac{n\pi x}{l} \neq 0$ [$\because$ It is a defined for all x]

$c_2 \neq 0$ [$\because$ If $c_2 = 0$ we already explained]

Therefore $c_3 = 0$

Substitute $c_3 = 0$ in equation (3) we get

$$y(x,t) = c_2 c_4 \sin\frac{n\pi x}{l}\sin\frac{n\pi a t}{l}$$

$$= c_n \sin\frac{n\pi x}{l}\sin\frac{n\pi a t}{l} \quad \text{... (4) where } c_n = c_2 c_4$$

The most general solution (4) we get

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l} \quad \text{... (5)}$$

Before applying condition (iv), diff. (5) p.w.r.to 't' we get

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\,\frac{n\pi a}{l}\cos\frac{n\pi at}{l}$$

Now we apply condition (iv) we get

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\,\frac{n\pi a}{l} = \lambda x(l-x)$$

$$= \sum_{n=1}^{\infty}\left[c_n\frac{n\pi a}{l}\right]\sin\frac{n\pi x}{l} = \lambda x(l-x)$$

$$= \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = \lambda x(l-x) \quad \text{... (6)}$$

where $B_n = c_n\frac{n\pi a}{l}$

To find B_n expand $\lambda x(l-x)$ in a half range Fourier sine series in the interval $(0, l)$

$$\lambda x(l-x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{... (7) where}$$

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$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

From (6) and (7) we get $\boxed{b_n = B_n}$

$$\therefore \quad B_n = \frac{2}{l}\int_0^l \lambda x(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$B_n = \frac{2\lambda}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2\lambda}{l}\left[(lx-x^2)\left[\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)}\right]-(l-2x)\left[\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]+(-2)\left[\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}\right]\right]_0^l$$

$$= \frac{2\lambda}{l}\left[-(lx-x^2)\left(\frac{l}{n\pi}\right)\cos\frac{n\pi x}{l}+(l-2x)\left(\frac{l}{n\pi}\right)^2\sin\frac{n\pi x}{l}-2\left(\frac{l}{n\pi}\right)^3\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{2\lambda}{l}\left[\left(-0+0-2\left(\frac{l}{n\pi}\right)^3(-1)^n\right)-\left(-0+0-2\left(\frac{l}{n\pi}\right)^3\right)\right]$$

$$= \frac{2\lambda}{l}\left[-2\left(\frac{l}{n\pi}\right)^3(-1)^n+2\left(\frac{l}{n\pi}\right)^3\right]$$

$$= \frac{2\lambda}{l}\,2\left(\frac{l}{n\pi}\right)^3[1-(-1)^n] = \frac{4\lambda l^2}{n^3\pi^3}[1-(-1)^n]$$

$$B_n = \frac{4\lambda l^2}{n^3\pi^3}[1-(-1)^n]$$

$$C_n\left(\frac{n\pi a}{l}\right) = \frac{4\lambda l^2}{n^3\pi^3}[1-(-1)^n]$$

$$C_n = \left(\frac{l}{n\pi a}\right)\left(\frac{4\lambda l^2}{n^3\pi^3}\right)[1-(-1)^n]$$

$$= \frac{4\lambda l^3}{a n^4 \pi^4}[1-(-1)^n]$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{8\lambda l^3}{a n^4 \pi^4} & \text{if } n \text{ is odd} \end{cases}$$

Substitute the value of c_n in equation (5) we get

$$y(x,t) = \sum_{n=\text{odd}}^{\infty} \frac{8\lambda l^3}{a n^4 \pi^4} \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l}$$

$$= \frac{8\lambda l^3}{a\pi^4} \sum_{n=\text{odd}}^{\infty} \frac{1}{n^4} \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l}$$

$$= \frac{8\lambda l^3}{a\pi^4} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin\frac{(2n-1)\pi x}{l} \sin\frac{(2n-1)\pi at}{l}$$

(OR)

$$y(x,t) = \frac{8\lambda l^3}{a\pi^4} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} \sin\frac{(2n+1)\pi x}{l} \sin\frac{(2n+1)\pi at}{l}$$

14. (b) **A rectangular plate with insulated surfaces is 20 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature of the short edge x = 0 is given by**

$$u = \begin{cases} 10\,y, & \text{for } 0 \le y \le 10 \\ 10\,(20 - y), & \text{for } 10 \le y \le 20 \end{cases}$$

and the two long edges as well as the other short edge are kept at 0°C. Find the steady state temperature distribution in the plate.

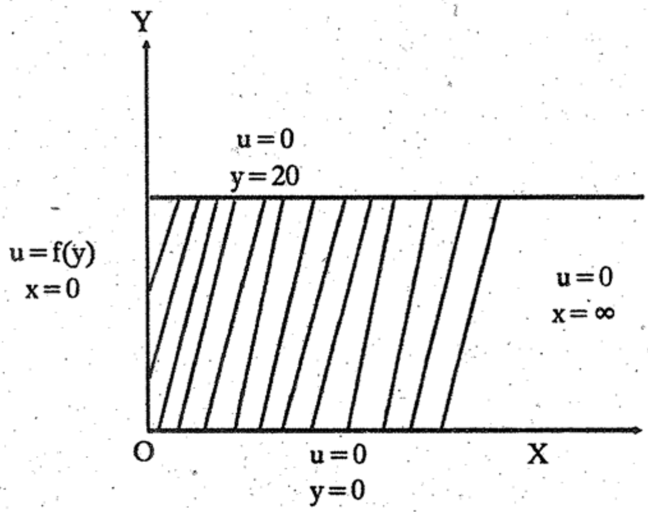
Solution :

The two-dimensional heat equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Consider the first solutions

$$u = (Ae^{px} + B\,e^{-px})\,(C\cos py + D\sin py) \qquad \text{... (1)}$$



We have the following boundary conditions.

(i) $u = 0$, when $y = 0$

(ii) $u = 0$, when $y = 20$

(iii) $u = 0$, when $x = \infty$

(iv) $u = f(y)$, when $x = 0$

Apply condition (i) in (1), we get

$$0 = (A e^{px} + Be^{-px})(C \cos 0 + D \sin 0)$$

$$0 = (A e^{px} + B e^{-px})(C + 0)$$

$$\Rightarrow C = 0$$

Sub $C = 0$ in (1) we get

$$u = (A e^{px} + B e^{-px}) D \sin py \qquad \text{... (2)}$$

Applying (ii) in (2) we get

$$0 = (A e^{px} + B e^{-px}) D \sin p\, 20$$

$$\Rightarrow \sin 20p = 0 \qquad \{D \neq 0\}$$

$$20p = n\pi$$

$$p = \frac{n\pi}{20}$$

Substituting, $p = \frac{n\pi}{20}$ in (2), we get

$$u = \left(A e^{\frac{n\pi a}{20}} + B e^{\frac{-n\pi x}{20}}\right) D \sin \frac{n\pi y}{20} \qquad \text{... (3)}$$

Applying (iii) in (3) we get

$$0 = \left(A e^{\frac{n\pi x}{20}} + 0\right) D \sin \frac{n\pi y}{20} \qquad (e^{-\infty} = 0)$$

$$\text{(i.e.,) } A\ e^{\frac{n\pi x}{20}} \left(D \sin \frac{n\pi y}{20}\right) = 0$$

$$\Rightarrow A = 0$$

Sub $A = 0$ in (3) we get

$$u = B e^{\frac{-n\pi x}{20}} D \sin \frac{n\pi y}{20}$$

$$= BD\ e^{\frac{-n\pi x}{20}} \sin \frac{n\pi y}{20}$$

$$= b\ e^{\frac{-n\pi x}{20}} \sin \frac{n\pi y}{20} \qquad [\text{Taking } BD = b]$$

The most general solution is

$$u = \sum_{n=1}^{\infty} b_n \, e^{\frac{-n\pi x}{20}} \sin \frac{n\pi y}{20}$$

Applying (iv) in (4) we get

$$f(y) = \sum_{n=1}^{\infty} b_n \, e^{-0} \sin \frac{n\pi y}{20}$$

$$f(y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi y}{20}$$

The RHS is half range sine series.

Its coefficient is given by

$$b_n = \frac{2}{l} \int_0^l f(y) \sin \frac{n\pi y}{l} \, dy$$

$$= \frac{2}{20} \int_0^{20} f(y) \sin \frac{n\pi y}{20} \, dy \qquad \{\because l = 20\}$$

$$= \frac{1}{10} \left[\int_0^{10} 10y \sin \frac{n\pi y}{20} \, dy + \int_{10}^{20} 10(20-y) \sin \frac{n\pi y}{20} \, dy \right]$$

$$= \frac{1}{10} \times 10 \left[\int_0^{10} y \sin \frac{n\pi y}{20} \, dy + \int_{10}^{20} (20-y) \sin \frac{n\pi y}{20} \, dy \right]$$

$$= \left[y \left(\frac{-\cos \frac{n\pi y}{20}}{\frac{n\pi}{20}} \right) - (1) \left(\frac{-\sin \frac{n\pi y}{20}}{\frac{n^2\pi^2}{20^2}} \right) \right]_0^{10}$$

$$+ \left[(20-y) \left(\frac{-\cos \frac{n\pi y}{20}}{\frac{n\pi}{20}} \right) - (-1) \left(\frac{-\sin \frac{n\pi y}{20}}{\frac{n^2\pi^2}{20^2}} \right) \right]_{10}^{20}$$

$$= \left[-y \frac{\cos \frac{n\pi y}{20}}{\frac{n\pi}{20}} + \frac{\sin \frac{n\pi y}{20}}{\frac{n^2\pi^2}{20^2}} \right]_0^{10} - \left[(20-y) \frac{\cos \frac{n\pi y}{20}}{\frac{n\pi}{20}} + \frac{\sin \frac{n\pi y}{20}}{\frac{n^2\pi^2}{20^2}} \right]_{10}^{20}$$

$$= \left[\frac{-10\cos\frac{n\pi}{2}}{\frac{n\pi}{20}} + \frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{400}} - (0+0)\right] - \left[(0+0) - \left(\frac{10\cos\frac{n\pi}{2}}{\frac{n\pi}{20}} + \frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{400}}\right)\right]$$

$$= \frac{-10\cos\frac{n\pi}{2}}{\frac{n\pi}{2}} + \frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{400}} + \frac{10\cos\frac{n\pi}{2}}{\frac{n\pi}{20}} + \frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{400}}$$

$$= \frac{2\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{400}}$$

$$= \frac{2\times 400}{n^2\pi^2}\left[\sin\frac{n\pi}{2}\right]$$

$$= \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\therefore u = \sum_{n=1}^{\infty} \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2} . e^{\frac{-n\pi x}{20}} \sin\frac{n\pi y}{20}$$

---

15. (a) (i) Using convolution theorem, find inverse Z-transform of $\frac{z^2}{(z-1)(z-3)}$

Solution :

$$\frac{z^2}{(z-1)(z-3)} = \frac{z}{(z-1)} . \frac{z}{(z-3)}$$

Let $F(z) = \frac{z}{(z-1)}$

$$G(z) = \frac{z}{(z-3)}$$

$$f_n = Z^{-1}\left[\frac{z}{z-1}\right]$$

$$= 1^n$$

$$g_n = Z^{-1}\left[\frac{z}{z-5}\right]$$

$$= 3^n$$

$\therefore$ By convolution theorem $Z^{-1}\left[F(z)\,G(z)\right] = \sum_{m=0}^{n} f_m\, g_{n-m}$

$$Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = \sum_{m=0}^{n} 1^m\, 3^{n-m}$$

$$= 3^n + 3^{n-1} + 3^{n-2} + 3^{n-3} + \ldots + 3 + 1$$

$$= 1 + 3 + 3^2 + 3^3 + \ldots + 3^n$$

$$= \frac{3^{n+1}-1}{3-1}$$

[Geometric series $a + ar + ar^2 + \ldots + ar^n = \frac{a\,(r^{n+1}-1)}{r-1}$]

$$= \frac{3^{n+1}-1}{2}$$

---

15. (a) (ii) Find the Z-transforms of $a^n \cos n\theta$ and $e^{-at} \sin bt$

Solution :

We know that $Z[\cos n\theta] = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$

By change of scale property $Z[a^n f_n] = F\left[\frac{z}{a}\right]$

where $Z[f_n] = F[z]$

Now replacing z by $\frac{z}{a}$

$$Z[a^n \cos n\theta] = \frac{z/a\,(z/a - \cos\theta)}{\left(\frac{z}{a}\right)^2 - 2\left(\frac{z}{a}\right)\cos\theta + 1}$$

$$= \frac{z^2 - za\cos\theta}{z^2 - 2za\cos\theta + a^2} \qquad \text{[on simplyfying]}$$

(ii) $Z[e^{-at}\sin bt]$

Solution : By shifting theorem

$$Z[e^{-at}\sin bt] = \{Z(\sin bt)\}_{z \to ze^{-aT}}$$

we know,

$$Z(\sin n\theta) = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$Z(\sin bt) = \frac{z\sin bt}{z^2 - 2z\cos bt + 1}$$

$$Z[e^{-at}\sin bt] = \left[\frac{z\sin bT}{z^2 - 2z\cos bT + 1}\right]_{z \to ze^{-aT}}$$

$$Z[e^{-at}\sin bt] = \frac{ze^{-aT}\sin bT}{(ze^{-aT})^2 - 2ze^{-aT}\cos bT + 1}$$

---

15. (b) (i) Solve the difference equation
$y(n+3) - 3y(n+1) + 2y(n) = 0$, given that
$y(0) = 4$, $y(1) = 0$ and $y(2) = 8$.

Solution :

Let $\bar{y} = Z[y_n]$

Taking Z-transform on both sides.

$$Z(y_{n+3}) - 3Z(y_{n+1}) + 2Z(y_n) = 0$$

$$z^3\left[\bar{y} - y_0 - \frac{y_1}{z} - \frac{y_2}{z^2}\right] - 3z(\bar{y} - y_0) + 2\bar{y} = 0$$

Apply the Given conditions $y(0) = 4$, $y(1) = 0$ and $y(2) = 8$

$$z^3\left[\bar{y} - 4 - \frac{0}{z} - \frac{8}{z^2}\right] - 3z(\bar{y} - 4) + 2\bar{y} = 0$$

$$z^3\bar{y} - 4z^3 - \frac{8z^3}{z^2} - 3z\bar{y} + 12z + 2\bar{y} = 0$$

$$z^3\bar{y} - 4z^3 - 8z - 3z\bar{y} + 12z + 2\bar{y} = 0$$

$$z^3\bar{y} - 3z\bar{y} + 2\bar{y} = 4z^3 + 8z - 12z$$

$$\bar{y}[z^3 - 3z + 2] = 4z^3 - 4z$$

$$\bar{y} = \left[\frac{4z^3 - 4z}{z^3 - 3z + 2}\right]$$

$$y = Z^{-1}\left[\frac{4z^3 - 4z}{z^3 - 3z + 2}\right]$$

Let $F(Z) = \dfrac{4z^3 - 4z}{z^3 - 3z + 2} = \dfrac{4z(z^2 - 1)}{z^3 - 3z + 2}$

$$\therefore \frac{F(Z)}{z} = \frac{4(z^2 - 1)}{z^3 - 3z + 2}$$

$$\begin{array}{c|cccc} 1 & 1 & 0 & -3 & 2 \\ & 0 & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & \boxed{0} \end{array}$$

$$= \frac{4(z^2 - 1)}{z^3 - 3z + 2}$$

$$(z - 1)(z^2 + z - 2) = 0$$

$$= \frac{4(z^2 - 1)}{(z - 1)^2(z + 2)}$$

$$(z - 1)(z - 1)(z + 2) = 0$$

$$\frac{F(Z)}{z} = \frac{4(z^2-1)}{(z+2)(z-1)^2}$$

$$= \frac{4(z+1)(z-1)}{(z+2)(z-1)^2}$$

$$\frac{F(Z)}{z} = \frac{4(z+1)}{(z+2)(z-1)}$$

Let $$\frac{4(z+1)}{(z+2)(z-1)} = \frac{A}{(z+2)} + \frac{B}{(z-1)}$$

$$4(z+1) = A(z-1) + B(z+2)$$

Putting $z = 1$

$$8 = 3B$$

$$\therefore \quad B = \frac{8}{3}$$

Putting $z = -2$

$$-4 = -3A$$

$$\therefore A = \frac{4}{3}$$

$$\frac{F(Z)}{z} = \frac{\frac{4}{3}}{z+2} + \frac{\frac{8}{3}}{z-1} + \frac{0}{(z-1)^2}$$

$$F(Z) = \frac{4z}{3(z+2)} + \frac{8z}{3(z-1)}$$

$$y(n) = Z^{-1}\left[\frac{4z}{3(z+2)} + \frac{8z}{3(z-1)}\right]$$

$$= \frac{4}{3} Z^{-1}\left[\frac{z}{z+2}\right] + \frac{8}{3} Z^{-1}\left[\frac{z}{z-1}\right]$$

$$= \frac{4}{3}[-2]^n + \frac{8}{3}(1)^n$$

$$= \frac{4}{3}(-2)^n + \frac{8}{3}(1)^n$$

15. (b) (ii) Derive the difference equation from

$$\mathbf{y_n = (A + Bn)(-3)^n}$$

Solution :

$$\text{Given } y_n = (A + Bn)(-3)^n$$

$$y_n = A(-3)^n + Bn(-3)^n \quad \ldots (1)$$

Replacing n by $n + 1$ we get

$$y_{n+1} = A(-3)^{n+1} + B(n+1)(-3)^{n+1}$$

$$y_{n+1} = A(-3)^n(-3) + B(n+1)(-3)^n(-3)$$

Dividing by -3

$$-\frac{y_{n+1}}{3} = A(-3)^n + B(n+1)(-3)^n \quad \ldots (2)$$

Replacing n by $n + 2$ we get

$$y_{n+2} = A(-3)^{n+2} + B(n+2)(-3)^{n+2}$$

$$y_{n+2} = A(-3)^n(-3)^2 + B(n+2)(-3)^n(-3)^2$$

$$y_{n+2} = A(-3)^n 9 + B(n+2)(-3)^n(9)$$

$$\frac{y_{n+2}}{9} = A(-3)^n + B(n+2)(-3)^n \quad \ldots (3)$$

Considering the determinant of coefficient of $A(-3)^n$ and $B(-3)^n$

$$\begin{vmatrix} y_n & 1 & n \\ \dfrac{-y_{n+1}}{3} & 1 & n+1 \\ \dfrac{y_{n+2}}{9} & 1 & n+2 \end{vmatrix} = 0$$

$$y_n\left[(n+2)-(n+1)\right] - 1\left[-\left(\frac{y_{n+1}}{3}\right)(n+2) - \left(\frac{y_{n+2}}{9}\right)(n+1)\right] + n\left[\frac{-y_{n+1}}{3} - \frac{y_{n+2}}{9}\right] = 0$$

$$y_n + \left(\frac{y_{n+1}}{3}\right)(n+2) + \left(\frac{y_{n+2}}{9}\right)(n+1) - \frac{n\,(y_{n+1})}{3} - \frac{n\,(y_{n+2})}{9} = 0$$

$$\frac{9\,y_n + 3\,y_{n+1}\,(n+2) + y_{n+2}\,(n+1) - 3\,n\,y_{n+1} - n\,y_{n+2}}{9} = 0$$

$$9\,y_n + 3\,y_{n+1}\,[n+2-n] + y_{n+2}\,[n+1-n] = 0$$

$$9\,y_n + 6y_{n+1} + y_{n+2} = 0$$