TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS

ANNA UNIVERSITY EXAMINATION, NOVEMBER/DECEMBER 2012

PART - A ($10 \times 2 = 20$ marks)

1. Find the co-efficient b_n of the Fourier series for the function $f(x) = x \sin x$ in $(-2, 2)$.

Solution : Here, the limits are $(-2, 2)$

$$f(x) = x \sin x$$

x is odd function and $\sin x$ is also odd function.

$\therefore$ $x \sin x$ is an even function.

$\therefore$ In the interval $(-2, 2)$, $b_n = 0$

2. Define Root Mean Square value of a function $f(x)$ over the interval (a, b).

Solution :

The Root Mean Square (RMS) value of the function $f(x)$ in the interval (a, b) is defined by

$$\sqrt{\frac{\int_a^b [f(x)]^2 \, dx}{b-a}}$$

3. Find the Fourier transform of $e^{-\alpha|x|}$, $\alpha > 0$

Solution :

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{-\alpha |x|} \cdot e^{isx} \cdot dx + \int_{0}^{\infty} e^{-\alpha |x|} \cdot e^{isx} \, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{-\alpha(-x)} \cdot e^{isx} \, dx + \int_{0}^{\infty} e^{-\alpha (x)} \cdot e^{isx} \, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{\alpha x} \cdot e^{isx} \, dx + \int_{0}^{\infty} e^{-\alpha x} \cdot e^{isx} \cdot dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{\alpha x + isx} \cdot dx + \int_{0}^{\infty} e^{-\alpha x + isx} \cdot dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{(\alpha + is) x} \cdot dx + \int_{0}^{\infty} e^{-(\alpha - is) x} \cdot dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(\frac{e^{(\alpha + is) x}}{\alpha + is} \right)_{-\infty}^{0} + \left(\frac{e^{-(\alpha - is) x}}{-(\alpha - is)} \right)_{0}^{\infty} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{0} - e^{-\infty}}{\alpha + is} + \frac{e^{-\infty} - e^{0}}{-(\alpha - is)} \right] \qquad [e^{-\infty} = 0 \, ; \; e^{0} = 1]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1 - 0}{\alpha + is} - \frac{(0 - 1)}{\alpha - is} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{\alpha + is} + \frac{1}{\alpha - is} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\alpha - is + \alpha + is}{(\alpha + is)(\alpha - is)} \right] = \frac{1}{\sqrt{2\pi}} \left[\frac{2\alpha}{\alpha^2 + s^2} \right]$$

4. State convolution theorem in Fourier transform.

Statement :

The Fourier Transform of the convolution of $f(x)$ and $g(x)$ is equal to the product of their Fourier transforms.

(i.e.) $\quad F[f(x) * g(x)] = F[f(x)] \, F[g(x)]$

5. Eliminate the arbitrary function 'f' from $z = f\left(\frac{y}{x}\right)$ and form the p.d.e.

Solution : Let $z = f\left(\frac{y}{x}\right)$ ... (1)

Differentiating (1) partially w.r.t x

$$\frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

$$p = \left(\frac{-y}{x^2}\right) f'\left(\frac{y}{x}\right) \quad \text{... (2)}$$

Differentiating (1) partially w.r.t y

$$\frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$q = \left(\frac{1}{x}\right) f'\left(\frac{y}{x}\right) \quad \text{... (3)}$$

$$\frac{(2)}{(3)} \Rightarrow \quad \frac{p}{q} = \frac{\frac{-y}{x^2} f'\left(\frac{y}{x}\right)}{\frac{1}{x} f'\left(\frac{y}{x}\right)}$$

$$\frac{p}{q} = -\frac{y}{x^2}x = -\frac{y}{x}$$

$$px = -qy$$

$$\Rightarrow px + qy = 0$$

6. Solve : $(D-1)(D-D'+1)z = 0$

Solution : $(D-1)(D-D'+1)z = 0$ ... (1)

This is non homogeneous equation

The standard form of non homogeneous equation is

$$(D - m_1 D' - \alpha_1)(D - m_2 D' - \alpha_2) = F(x, y)$$

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Hence by comparing with (1)

$$m_1 = 0 \; ; \quad m_2 = 1$$

$$\alpha_1 = 1 \; ; \quad \alpha_2 = -1$$

$$\therefore \text{ C.F is } \quad Z = e^{\alpha_1 x} f_1 (y + m_1 x) + e^{\alpha_2 x} f_2 (y + m_2 x).$$

$$Z = e^{x} f_1 (y) + e^{-x} f_2 (y + x)$$

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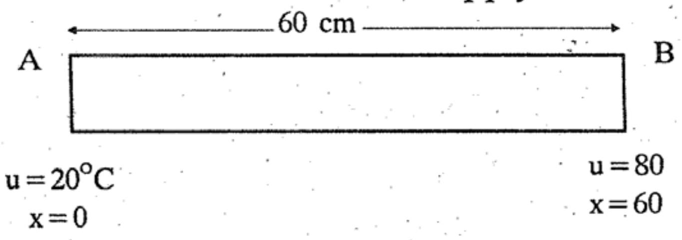
7. An insulated rod of length 60 cm has its ends at A and B maintained at 20°C and 80°C respectively. Find the steady state solution of the rod.

Solution : In steady state condition, the temperature function is

$$u = ax + b$$

Now to find the values of a and b apply the conditions from figure.



Apply the conditions at the end A we get

$$20 = a(0) + b$$

$$\therefore \quad b = 20$$

Apply the conditions at the end B we get

$$80 = a(60) + 20$$

$$60\,a = 80 - 20$$

$$60\,a = 60$$

$$\therefore a = 1$$

Substituting for a and b we get

$$u = x + 20$$

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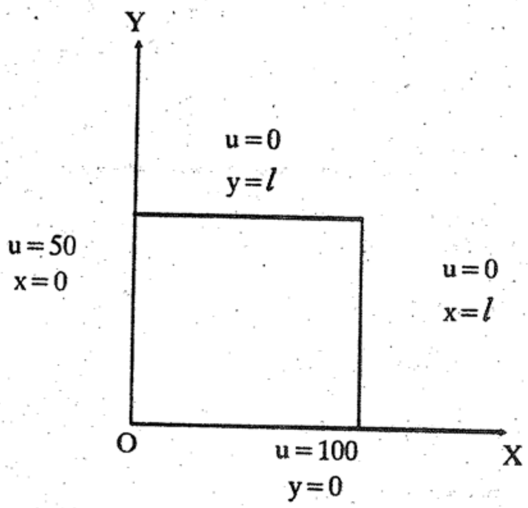
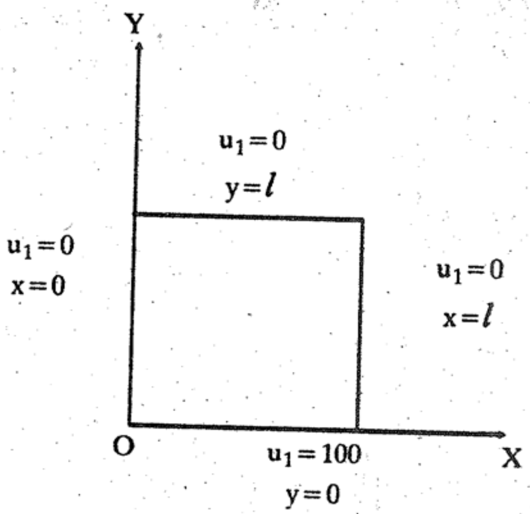
8. A plate is bounded by the lines $x = 0$, $y = 0$, $x = l$ and $y = l$. Its faces are insulated. The edge coinciding with x-axis is kept at 100°C. The edge coinciding with y-axis is kept at 50°C. The other two edges are kept at 0°C. Write the boundary conditions that are needed for solving two dimensional heat flow equation.

Solution :



First we will solve for u_1 taking temperature at horizontal edge as 100°C.



∴ The boundary conditions.

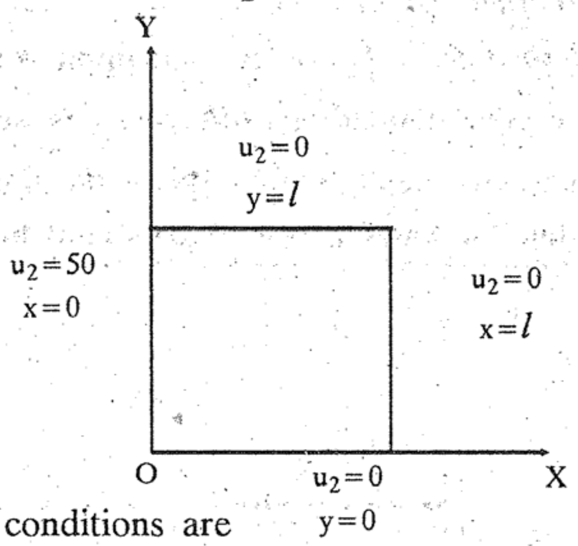
(i) $u_1 = 0$ when $x = 0$

(ii) $u_1 = 0$ when $x = l$

(iii) $u_1 = 0$ when $y = l$

(iv) $u_1 = 100$ when $y = 0$

Second we will solve for u_2 taking temperature at vertical edge as 50°C.



∴ The boundary conditions are

(i) $u_2 = 0$ when $y = 0$

(ii) $u_2 = 0$ when $y = l$

(iii) $u_2 = 0$ when $x = l$

(iv) $u_2 = 50$ when $x = 0$

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9. Find the Z-transform of a^n.

Solution : $Z[a^n] = \sum_{n=0}^{\infty} a^n z^{-n}$

$$= \sum_{n=0}^{\infty} \frac{a^n}{z^n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$= \left(\frac{a}{z}\right)^0 + \left(\frac{a}{z}\right)^1 + \left(\frac{a}{z}\right)^2 + \ldots$$

$$= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \ldots$$

$$= \left(1 - \frac{a}{z}\right)^{-1}$$ [By Binomial theorem

$(1-x)^{-1} = 1 + x + x^2 + \ldots$]

$$= \frac{1}{1 - \frac{a}{z}}$$

$$= \frac{1}{\frac{z-a}{z}}$$

$$= \frac{z}{z-a}$$

$$\therefore Z(a^n) = \frac{z}{z-a}$$

10. Solve $y_{n+1} - 2y_n = 0$, given that $y(0) = 2$

Solution : Let $\bar{y} = Z[y_n]$

$$Z\{y_{n+1}\} - 2Z\{y_n\} = 0$$

$$z(\bar{y} - y_0) - 2(\bar{y}) = 0$$

Applying condition $y(0) = 2$

$$z(\bar{y} - 2) - 2\bar{y} = 0$$

$$z\bar{y} - 2z - 2\bar{y} = 0$$

$$z\bar{y} - 2\bar{y} = 2z$$

$$\bar{y}(z-2) = 2z$$

$$\bar{y} = \frac{2z}{z-2}$$

$$\therefore y = Z^{-1}\left[\frac{2z}{z-2}\right]$$

$$= 2Z^{-1}\left[\frac{z}{z-2}\right]$$

$$= 2\,2^n = 2^{n+1}$$

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PART-B (5 × 16 = 80 marks)

11. (a) (i) Find the Fourier series expansion of $f(x) = x + x^2$ in $(-\pi, \pi)$

Solution : The Fourier series $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

Since $f(x)$ is neither odd function nor even function we have to find a_0, a_n, b_n

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)\,dx$$

$$= \frac{1}{\pi}\left[\underbrace{\int_{-\pi}^{\pi} x\,dx}_{0} + \int_{-\pi}^{\pi} x^2\,dx\right] \qquad [\because x \text{ is an odd function}]$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2\,dx \qquad [\because x^2 \text{ is an even function}]$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{3\pi}\left[x^3\right]_0^{\pi}$$

$$= \frac{2}{3\pi}\left[\pi^3 - 0\right]$$

$$= \frac{2}{3\pi}\left[\pi^3\right] = \frac{2\pi^2}{3}$$

$$a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\underbrace{\int_{-\pi}^{\pi} x\cos nx}_{0} + \int_{-\pi}^{\pi} x^2 \cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x^2 \cos nx\,dx\right]$$

[$\because$ $x\cos nx$ is an odd function]

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx\,dx$$

[$\because$ $x^2 \cos nx$ is an even function]

$$= \frac{2}{\pi}\left[\underbrace{x^2\left(\frac{\sin nx}{n}\right)}_{0} - 2x\left(\frac{-\cos nx}{n^2}\right) + \underbrace{2\left(\frac{-\sin nx}{n^3}\right)}_{0}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[2x\frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{4}{\pi n^2}\left[x\cos nx\right]_0^{\pi}$$

$$= \frac{4}{\pi n^2}\left[\pi\cos n\pi - 0\right]$$

$$= \frac{4}{\pi n^2}\left[\pi(-1)^n\right] \qquad [\because \cos n\pi = (-1)^n]$$

$$a_n = \frac{4}{n^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2) \sin x \, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x \sin nx \, dx + \underbrace{\int_{-\pi}^{\pi} x^2 \sin nx \, dx}_{0}\right]$$

[∵ $x^2 \sin nx$ is an odd function]

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x \sin nx \, dx\right]$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x \sin nx \, dx$$ [∵ $x \sin nx$ is an even function]

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - (1)\underbrace{\left(\frac{-\sin nx}{n^2}\right)}_{0}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-x\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{-2}{n\pi}\left[x \cos nx\right]_0^{\pi}$$

$$= \frac{-2}{n\pi}\left[\pi \cos n\pi - 0\right]$$

$$= \frac{-2}{n\pi}\left[\pi (-1)^n\right]$$

$$b_n = \frac{-2(-1)^n}{n}$$

∴ Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{\frac{2\pi^2}{3}}{2} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx + \sum_{n=1}^{\infty} \frac{-2}{n} (-1)^n \sin nx$$

$$= \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2} - 2 \sum_{n=1}^{\infty} \frac{(-1)^n \sin nx}{n}$$

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11. (a) (ii) Find the Fourier series expansion of

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 \le x < 2 \end{cases}$$

Also deduce $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$ **to** $\infty = \frac{\pi^2}{8}$

Solution :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{2} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

Here $2l = 2 - 0 = 2$

$\therefore \quad l = 1$

$$a_0 = \frac{1}{l} \int_0^{2l} f(x)\, dx$$

$$= \frac{1}{1} \int_0^{2} f(x)\, dx \qquad (l = 1)$$

$$= \int_0^1 f(x)\, dx + \int_1^2 f(x)\, dx$$

$$= \int_0^1 x\, dx + \int_1^2 (2 - x)\, dx$$

$$= \left(\frac{x^2}{2}\right)_0^1 + \left(2x - \frac{x^2}{2}\right)_1^2$$

$$= \left(\frac{1}{2} - 0\right) + 4 - \frac{2^2}{2} - \left(2 - \frac{1}{2}\right)$$

$$= \frac{1}{2} + 4 - \frac{4}{2} - \left(\frac{3}{2}\right)$$

$$= \frac{1}{2} + 4 - 2 - \frac{3}{2}$$

$$= \frac{1}{2} + 2 - \frac{3}{2}$$

$$= 1$$

$$a_0 = 1$$

$$a_n = \frac{1}{l} \int_0^{2l} f(x) \cos \frac{n\pi x}{l} dx$$

$$= \frac{1}{1} \int_0^2 f(x) \cos n\pi x \, dx$$

$$= \int_0^1 f(x) \cos n\pi x \, dx + \int_1^2 f(x) \cos n\pi x \, dx$$

$$= \int_0^1 x \cos n\pi x \, dx + \int_1^2 (2-x) \cos n\pi x \, dx$$

$$= \left[\underbrace{\frac{x \sin n\pi x}{n\pi}}_{0} - (1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$+ (2-x)\underbrace{\left(\frac{\sin n\pi x}{n\pi}\right)}_{0} - (-1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\Bigg]_1^2$$

$$= \left(\frac{\cos n\pi x}{n^2\pi^2}\right)_0^1 - \left(\frac{\cos n\pi x}{n^2\pi^2}\right)_1^2$$

$$= \frac{1}{n^2 \pi^2} \left[\left(\cos n \pi x \right)_0^1 - \left(\cos n \pi x \right)_1^2 \right]$$

$$= \frac{1}{n^2 \pi^2} \left[(\cos n \pi - \cos 0) - (\cos 2 n \pi - \cos n \pi) \right]$$

$$= \frac{1}{n^2 \pi^2} \left[\cos n \pi - 1 - 1 + \cos n \pi \right]$$

$$= \frac{1}{n^2 \pi^2} \left[2 \cos n \pi - 2 \right]$$

$$= \frac{2}{n^2 \pi^2} \left[\cos n \pi - 1 \right]$$

$$= \frac{2}{n^2 \pi^2} \left[(-1)^n - 1 \right]$$

Case 1 :

When n is even number.

$$a_n = \frac{2}{n^2 \pi^2} [1 - 1]$$

$$= \frac{2}{n^2 \pi^2} [0]$$

$$= 0$$

Case 2 :

When n is odd number.

$$a_n = \frac{2}{n^2 \pi^2} [-1 - 1]$$

$$= \frac{2}{n^2 \pi^2} (-2)$$

$$= \frac{-4}{n^2 \pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\,\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{1}\int_0^{2} f(x)\,\sin\frac{n\pi x}{1}\,dx$$

$$= \int_0^1 f(x)\sin n\pi x\,dx + \int_1^2 f(x)\sin n\pi x\,dx$$

$$= \int_0^1 x\sin n\pi x\,dx + \int_1^2 (2-x)\sin n\pi x\,dx$$

$$= \left[x\left(\frac{-\cos n\pi x}{n\pi}\right) - (-1)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[(2-x)\left(\frac{-\cos n\pi x}{n\pi}\right) - (-1)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \left[\frac{-x\cos n\pi x}{n\pi}\right]_0^1 - \left[(2-x)\frac{\cos n\pi x}{n\pi}\right]_1^2$$

$$= -\frac{1}{n\pi}\left[\left(x\cos n\pi x\right)_0^1 + \left((2-x)\cos n\pi x\right)_1^2\right]$$

$$= -\frac{1}{n\pi}\left[\left(\cos n\pi - 0\right) + \left(0 - \cos n\pi\right)\right]$$

$$= -\frac{1}{n\pi}\left[\cos n\pi - \cos n\pi\right]$$

$$= -\frac{1}{n\pi}(0)$$

$$b_n = 0$$

$\therefore$ Fourier series is

$$f(x) = \frac{1}{2} + \sum_{n=1,3,\ldots}^{\infty} \frac{-4}{n^2 \pi^2} \cos n\pi x$$

$$= \frac{1}{2} - \frac{4}{\pi^2} \sum_{n=1,3,\ldots}^{\infty} \frac{\cos n\pi x}{n^2}$$

Put $x = 0$

$$0 = \frac{1}{2} - \frac{4}{\pi^2}\left[\frac{\cos 0}{1^2} + \frac{\cos 0}{3^2} + \frac{\cos 0}{5^2} + \ldots\right]$$

$$[\because f(x) = x = 0 \text{ for } x = 0]$$

$$0 = \frac{1}{2} - \frac{1}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right]$$

$$= \frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right] = \frac{1}{2}$$

Cross multiplying,

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$$

11. (b) (i) Obtain the half range cosine series for $f(x) = x$ in $(0, \pi)$

Solution :

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\, dx$$

$$= \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{\pi^2}{2} \right]$$

$$= \pi$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \cos n x \, dx$$

$$= \frac{2}{\pi} \left[\underbrace{x \frac{\sin nx}{n}}_{0} - (1) \frac{(-\cos n x)}{n^2} \right]_0^{\pi}$$

$$= \frac{2}{\pi n^2} \left[\cos n x \right]_0^{\pi}$$

$$= \frac{2}{\pi n^2} [\cos n \pi - \cos 0]$$

$$= \frac{2}{\pi n^2} [(-1)^n - 1]$$

Case (i)

When n is even

$$a_n = \frac{2}{\pi n^2} [1 - 1]$$

$$= 0$$

Case (ii)

When n is odd

$$a_n = \frac{2}{\pi n^2}[-1-1]$$

$$= \frac{-4}{\pi n^2}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{\pi}{2} + \sum_{n=1,3,5,\ldots}^{\infty} \frac{-4}{\pi n^2} \cos n x$$

11. (b) (ii) Find the Fourier series as far as the second harmonic to represent the function f(x) with period 6, given in the following table.

x	0	1	2	3	4	5
$f(x)$	9	18	24	28	26	20

Solution :

Here the length of the interval is $2l = 6$, i.e., $l = 3$.

$\therefore$ The Fourier series can be represented by

$$y = \frac{a_0}{2} + \left(a_1 \cos \frac{\pi x}{l} + b_1 \sin \frac{\pi x}{l}\right) + \left(a_2 \cos \frac{2\pi x}{l} + b_2 \sin \frac{2\pi x}{l}\right) + \ldots$$

$$\text{Here } y = \frac{a_0}{2} + a_1 \cos \frac{\pi x}{3} + b_1 \sin \frac{\pi x}{3} \quad \ldots (1)$$

x	y	$\cos \frac{\pi x}{3}$	$\sin \frac{\pi x}{3}$	$y \cos \frac{\pi x}{3}$	$y \sin \frac{\pi x}{3}$
0	9	1	0	9	0
1	18	0.5	0.866	9	15.588
2	24	–0.5	0.866	–12	20.785
3	28	–1	0	–28	0
4	26	–0.5	–0.866	–13	–22.517
5	20	0.5	–0.866	10	–17.321
	125			–25	–3.465

$$a_0 = \frac{2}{n} \Sigma y = \frac{2}{6} 125 = 41.67$$

$$a_1 = \frac{2}{n} \Sigma y \cos \frac{\pi x}{3} = \frac{2}{6}(-25) = -8.33$$

$$b_1 = \frac{2}{n} \Sigma y \sin \frac{\pi x}{3} = \frac{2}{6}(-3.465) = -1.16$$

Substituting the above values in (1) we get

$$y = \frac{41.67}{2} - 8.33 \cos x - 1.16 \sin x$$

$$= 20.84 - 8.33 \cos x - 1.16 \sin x$$

12. (a) (i) Find the Fourier transform of

$$f(x) = \begin{cases} 1 - |x| & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases} \text{ and hence evaluate } \int_0^\infty \frac{\sin^4 t}{t^4} dt$$

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12. (a) (ii) Find the Fourier transform of

$$f(x) = \begin{cases} a^2 - x^2, & |x| \le a \\ 0, & |x| > a > 0 \end{cases}.$$ **Hence deduce that**

$$\int_0^\infty \frac{\sin t - t\cos t}{t^3}\, dt = \frac{\pi}{4}$$

Solution : By definition

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$|x| \le a$ means limits for x are $-a$ to a

$$\therefore\ F[f(x)] = \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{a} (a^2 - x^2)\, e^{isx}\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\underbrace{(a^2 - x^2)\left(\frac{e^{isx}}{is}\right)}_{\downarrow\ 0} - (-2x)\left(\frac{e^{isx}}{i^2 s^2}\right) + (-2)\left(\frac{e^{isx}}{i^3 s^3}\right)\right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2x\, e^{isx}}{i^2 s^2} - \frac{2\, e^{isx}}{i^3 s^3}\right]_{-a}^{a}$$

Taking '2' outside

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{x\, e^{isx}}{-s^2} - \frac{e^{isx}}{-is^3}\right]_{-a}^{a} \qquad [i^2 = -1\,;\ i^3 = -i]$$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{x\,e^{isx}}{-s^2} + \frac{e^{isx}}{is^3} \right]_{-a}^{a}$$

$$= \frac{2}{\sqrt{2\pi}} \left[\left\{ \frac{ae^{isa} - (-a)\,e^{-isa}}{-s^2} \right\} + \left\{ \frac{e^{isa} - e^{-isa}}{i\,s^3} \right\} \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[\left\{ \frac{ae^{isa} + ae^{-isa}}{-s^2} \right\} + \left\{ \frac{e^{isa} - e^{-isa}}{i\,s^3} \right\} \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[a \left\{ \frac{e^{isa} + e^{-isa}}{-s^2} \right\} + \left\{ \frac{e^{isa} - e^{-isa}}{i\,s^3} \right\} \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{a\,(2\cos sa)}{-s^2} + \frac{2\,i\sin sa}{i\,s^3} \right] \qquad \left[\begin{aligned} e^{i\theta} + e^{-i\theta} &= 2\cos\theta \\ e^{i\theta} - e^{-i\theta} &= 2\,i\sin\theta \end{aligned} \right]$$

$$= \frac{4}{\sqrt{2\pi}} \left[\frac{-a\cos sa}{s^2} + \frac{\sin sa}{s^3} \right]$$

$$= \frac{4}{\sqrt{2\pi}} \left[\frac{-a\,s\cos sa + \sin sa}{s^3} \right]$$

$$F\,[f\,(x)] = \frac{4}{\sqrt{2\pi}} \left[\frac{\sin sa - a\,s\cos sa}{s^3} \right]$$

Now consider inverse Fourier transform

$$f\,(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F\,[f\,(x)]\,e^{-isx}\,ds$$

$$f\,(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{4}{\sqrt{2\pi}} \left[\frac{\sin sa - as\cos sa}{s^3} \right] .\, e^{-isx}\,ds$$

$$a^2 - x^2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{4}{\sqrt{2\pi}} \left(\frac{\sin sa - as\cos sa}{s^3} \right) (\cos sx - i\sin sx)\,ds$$

$$= \frac{4}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{\sin sa - as\cos sa}{s^3} \right) \cos sx - \left(\frac{\sin sa - as\cos sa}{s^3} \right) .\, i\sin sx\,ds$$

$$= \frac{2}{\pi}\left[\int_{-\infty}^{\infty}\left(\frac{\sin sa - a\, s\cos sa}{s^3}\right)\cos sx\, ds - i\int_{-\infty}^{\infty}\underbrace{\left(\frac{\sin sa - a\, s\cos a}{s^3}\right)\sin sx\, ds}_{\downarrow\ 0}\right.$$

$$a^2 - x^2 = \frac{2}{\pi}\cdot 2\int_0^{\infty}\left(\frac{\sin sx - as\cos sa}{s^3}\right)\cos sx\, ds$$

Take $a = 1$

$$1 - x^2 = \frac{4}{\pi}\int_0^{\infty}\frac{\sin s - s\cos s}{s^3}\cos sx\, ds$$

Putting $x = 0$ on both sides,

$$1 = \frac{4}{\pi}\int_0^{\infty}\frac{\sin s - s\cos s}{s^3}\cos . 0\, ds$$

$$1 = \frac{4}{\pi}\int_0^{\infty}\frac{\sin s - s\cos s}{s^3}\, ds \qquad [\cos 0 = 1]$$

$$\therefore \int_0^{\infty}\frac{\sin b - s\cos s}{s^3}\, ds = \frac{\pi}{4}$$

Changing the dummy variable from 's' to 't'

$$\int_0^{\infty}\frac{\sin t - t\cos t}{t^3}\, dt = \frac{\pi}{4}$$

(OR)

12. (b) (i) Find the Fourier cosine and sine transforms of $f(x) = e^{-ax}$, $a > 0$ and hence deduce the inversion formula.

Solution :

The Fourier cosine transform is defined as

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx\, dx$$

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$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$\left[\because \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}[a \cos bx + b \sin bx]\right.$$

Here $a = -a,\ b = s$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2}(-a \cos sx + s \sin sx)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{a^2+s^2}\left[-e^{-ax} a \cos sx + \underbrace{e^{-ax} s \sin sx}_{\downarrow\ 0}\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{a^2+s^2}\left[-e^{-ax} . a \cos sx\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \frac{-a}{a^2+s^2}\left[e^{-ax} \cos sx\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \frac{-a}{a^2+s^2}\left[0 - e^0 \cos 0\right] \qquad [\because e^{-\infty} = 0]$$

$$= \sqrt{\frac{2}{\pi}} \frac{-a}{a^2+s^2}[0-1]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2+s^2}\right]$$

Now, using Inversion formula for Fourier cosine trasform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c[f(x)] \cos sx \, ds$$

Here $f(x) = e^{-ax}$; $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2+s^2}\right]$

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$$\therefore \quad e^{-ax} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2+s^2}\right] \cos sx \, ds$$

$$e^{-ax} = \frac{2a}{\pi} \int_0^\infty \frac{1}{a^2+s^2} \cos sx \, ds$$

$$\therefore \quad \int_0^\infty \frac{1}{a^2+s^2} \cos sx \, ds = \frac{\pi}{2a} e^{-ax}$$

The Fourier sine transform is defined as

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$\therefore \quad F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$\left[\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}[a \sin bx - b \cos bx]\right]$$

Here $a = -a$, $b = s$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2}(-a \sin sx - s \cos sx)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{-1}{a^2+s^2} \left[\underbrace{e^{-ax} a \sin sx}_{\downarrow \atop 0} + e^{-ax} s \cos sx\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{-1}{a^2+s^2} \left[e^{-ax} s \cos sx\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{-s}{a^2+s^2} \left[0 - e^0 \cos 0\right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{-s}{a^2+s^2} [0 - 1]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{s}{a^2 + s^2}$$

Now, using Inversion formula for Fourier sine transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s[f(x)] \sin sx \, ds$$

Here $f(x) = e^{-ax}$; $F_s[f(x)] = \sqrt{\frac{2}{\pi}} \frac{s}{a^2 + s^2}$

$$\therefore \quad e^{-ax} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{s}{a^2 + s^2} \sin sx \, ds$$

$$e^{-ax} = \frac{2}{\pi} \int_0^\infty \frac{s}{a^2 + s^2} \sin sx \, ds$$

$$\therefore \quad \int_0^\infty \frac{s}{a^2 + s^2} \sin sx \, ds = \frac{\pi}{2} e^{-ax}$$

12. (b)(ii) Find the Fourier cosine transform of $e^{-a^2x^2}$, $a > 0$. Hence, show that the function $e^{-x^2/2}$ is self reciprocal.

Solution :

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$F_c\left[e^{-a^2x^2}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-a^2x^2} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{2} \int_{-\infty}^\infty e^{-a^2x^2} \cos sx \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-a^2x^2} \cos sx \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} \text{ Real part of } e^{isx}\, dx$$

$$= \frac{R.P}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx}\, dx$$

$$= \frac{R.P}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2 + isx}\, dx$$

Consider

$$-a^2x^2 + i\,sx = -(a^2x^2 - i\,s\,x)$$

$$= -\left[a^2x^2 - i\,sx + \frac{i^2 s^2}{4a^2} - \frac{i^2 s^2}{4\,a^2}\right]$$

$$\left[\text{Adding and subtracting } \frac{i^2 s^2}{4\,a^2}\right]$$

$$= -\left[a^2x^2 - i\,s\,x + \frac{i^2 s^2}{4\,a^2} + \frac{s^2}{4\,a^2}\right] \qquad [i^2 = -1]$$

$$= -\left[\left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4\,a^2}\right]$$

$$= -\left(ax - \frac{is}{2a}\right)^2 - \frac{s^2}{4a^2}$$

$$F_c\left[e^{-a^2x^2}\right] = \text{R.P}\, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2 - \frac{s^2}{4a^2}}\, dx$$

$$= \text{R.P}\, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} . e^{\frac{-s^2}{4a^2}}\, dx$$

$$= \text{R.P}\, \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2}\, dx$$

Put $ax - \frac{is}{2a} = y$

Differentiating, $a\,dx = dy$

$$\therefore \quad dx = \frac{dy}{a}$$

When $x = -\infty, \quad y = -\infty$
$x = \infty, \quad y = \infty$

$$F_c\left[e^{-a^2x^2}\right] = \text{R.P}\ \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2}\frac{dy}{a}$$

$$= \text{R.P}\ \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2}\,dy$$

$$= \text{R.P}\ \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \cdot \sqrt{\pi}$$

$\left[\because \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}\right.$ by gamma integrals]

$$= \text{R.P}\ \frac{e^{-s^2/4a^2}}{a\sqrt{2}\sqrt{\pi}} \cdot \sqrt{\pi}$$

$$= \text{R.P}\ \frac{e^{-s^2/4a^2}}{a\sqrt{2}}$$

$$\therefore F_c = \left[e^{-a^2x^2}\right] = \frac{e^{-s^2/4a^2}}{a\sqrt{2}}$$

13. (a)(i) Find the singular integral of $z = px + qy + p^2 + pq + q^2$

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13. (a) (ii) Solve the partial differential equation
$$(x-2z)\,p + (2z-y)\,q = y - x$$

Solution :

The equation of the form $P\,p + Q\,q = R$

where $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

The auxilary equation is

$$\frac{dx}{x-2z} = \frac{dy}{2z-y} = \frac{dz}{y-x}$$

Taking the multipliers as 1, 1, 1 and summing up all the ratios we get

$$\frac{dx+dy+dz}{x-2x+2z-y+y-x} = 0$$

$$\therefore \frac{dx+dy+dz}{0} = 0$$

$$dx+dy+dz = 0$$

Integrating

$$\int dx + \int dy + \int dz = 0$$

$$x+y+z = c_1$$

(OR)

13. (b) (i) Solve : $(D^2 + 3DD' - 4D'^2)\,z = \cos(2x+y) + xy$

Solution :

The auxilary equation is

$$m^2 + 3m - 4 = 0$$

$$(m+4)(m-1) = 0$$

$$\therefore m = -4 \text{ and } m = 1$$

$$\therefore \quad C.F = f_1(y-4x) + f_2(y+x)$$

$$P.I \Rightarrow Z = \frac{1}{D^2+3DD'-4D'^2}[\cos(2x+y)+xy]$$

$$= \frac{1}{D^2+3DD'-4D'^2}\cos(2x+y) + \frac{1}{D^2+3DD'-4D'^2}xy$$

$$\therefore \; P.I_1 = \frac{1}{D^2+3DD'-4D'^2}\cos(2x+y)$$

$$= \frac{1}{-4+3(-2)-4(-1)}\cos(2x+y)$$

$$[D^2=-a^2\;;\; DD'=-ab;\; D'^2=-b^2]$$

$$= \frac{1}{-4-6+4}\cos(2x+y)$$

$$P.I_1 = -\frac{1}{6}\cos(2x+y)$$

$$P.I_2 = \frac{1}{D^2+3DD'-4D'^2}xy$$

$$= \frac{1}{D^2\left[1+\dfrac{3D'D}{D^2}-\dfrac{4D'^2}{D^2}\right]}xy$$

$$= \frac{1}{D^2}\cdot\frac{1}{\left[1+\left(\dfrac{3D'}{D}-\dfrac{4D'^2}{D^2}\right)\right]}xy$$

$$= \frac{1}{D^2}\cdot\left[1+\left(\frac{3D'}{D}-\frac{4D'^2}{D^2}\right)\right]^{-1}xy$$

$$= \frac{1}{D^2}\cdot\left[1-\left(\frac{3D'}{D}-\frac{4D'^2}{D^2}\right)+\ldots\right]xy$$

$$= \frac{1}{D^2}\left[1 - \frac{3D'}{D}\right] xy$$

$$= \frac{1}{D^2}\left[xy - \frac{3D'}{D}(xy)\right]$$

$$= \frac{1}{D^2}\left[xy - \frac{3}{\frac{D}{2}}(x)\right] \qquad [\because D'(xy) = x]$$

$$= \frac{1}{D^2}\left[xy - \frac{3x^2}{2}\right] \qquad [\because \frac{1}{D}(x) = \frac{x^2}{2}]$$

$$= \frac{1}{D}\left[\frac{x^2y}{2} - \frac{3}{2}\cdot\frac{x^3}{3}\right]$$

$$= \frac{1}{2}\,\frac{1}{D}\left[x^2y - x^3\right]$$

$$= \frac{1}{2}\left[\frac{x^3y}{3} - \frac{x^4}{4}\right]$$

$\therefore$ The solution is $Z = \text{C.F} + \text{P.I}_1 + \text{P.I}_2$

$$\therefore Z = f_1(y-4x) + f_2(y+x) - \frac{1}{6}\cos(2x+y) + \frac{1}{2}\left[\frac{x^3y}{3} - \frac{x^4}{4}\right]$$

13. (b) (ii) Solve : $(D^2 - DD' + 2D)z = e^{2x+y} + 4$

Solution :

$D^2 - DD' + 2D$ is non homogeneous type.

Hence factorising we get,

$$D(D - D' + 2)Z = e^{2x+y} + 4$$

The standard form of non homogeneous equation is

$$(D - m_1D' - \alpha_1)\,.\,(D - m_2D' - \alpha_2) = F(x,y)$$

$$\therefore\ (D - 0D' + 0)(D - D' + 2)z = e^{2x+y} + 4$$

Here, $m_1 = 0\ ; \quad \alpha_1 = 0$

$m_2 = 1\ ; \quad \alpha_2 = -2$

$$\therefore \text{ C.F} \Rightarrow Z = e^{\alpha_1 x} f_1(y + m_1 x) + e^{\alpha_2 x} f_2(y + m_2 x)$$

$$Z = e^0 f_1(y + 0) + e^{-2x} f_2(y + x)$$

$$Z = f_1(y) + e^{-2x} f_2(y + x)$$

$$\text{P.I} \Rightarrow Z = \frac{1}{D^2 - DD' + 2D}\left[e^{2x+y} + 4\right]$$

$$Z = \frac{1}{D^2 - DD' + 2D} e^{2x+y} + \frac{1}{D^2 - DD' + 2D} 4 \,.\, e^{0x+0y}$$

$$\therefore\ \text{P.I}_1 = \frac{1}{4 - 2 \times 1 + 2 \times 1} e^{2x+y}$$

$$= \frac{1}{4 - 2 + 2} e^{2x+y} \qquad [\because D = 2;\ D' = 1]$$

$$= \frac{1}{4} e^{2x+y}$$

$$\text{P.I}_2 = 4 \,.\, \frac{1}{D^2 - DD' + 2D} e^{0x+0y}$$

$$= 4 \,.\, \frac{1}{0} \qquad [\because D = 0;\ D' = 0]$$

Since the denominator becomes zero applying type 4.

$$\therefore\ \text{P.I}_2 = 4 \,.\, \frac{x}{2D - D' + 2} e^{0x+0y}$$

$$= 4 \,.\, \frac{x}{2} e^{0x+0y}$$

$$= 2x$$

$\therefore$ The solution is

$$Z = C.F + P.I_1 + P.I_2$$

$$Z = f_1(y) + e^{-2x} f_2(y+x) + \frac{1}{4} e^{2x+y} + 2x$$

14. (a) A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y(x, 0) = y_0 \sin^3 \left(\frac{\pi x}{l}\right)$. It is released from rest from this position. Find the expression for the displacement at any time t.

Solution : The wave equation is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2}$$

The solution is

$$y = (A \cos px + B \sin px)(C \cos pct + D \sin pct) \qquad \dots (1)$$

We have the following boundary conditions

1. $y = 0$ when $x = 0$
2. $y = 0$ when $x = l$
3. $\frac{\partial y}{\partial t} = 0,$ when $t = 0$
4. $y = y_0 \sin^3 \frac{\pi x}{l}$ when $t = 0$

Applying condition 1. in (1)

$$0 = (A \cos 0 + B \sin 0)(C \cos pct + D \sin pct)$$

$$0 = (A + 0)(C \cos pct + D \sin pct) \qquad [\cos 0 = 1, \sin 0 = 0]$$

$$\Rightarrow A = 0$$

Substitute $A = 0$ in (1)

$$y = B \sin px\,(C \cos pct + D \sin pct) \qquad \ldots (2)$$

Applying condition 2. in (2)

$$0 = B \sin pl\,(C \cos pct + D \sin pct)$$

$$\sin pl = 0$$

$$\text{But } \sin n\pi = 0$$

$$\therefore\ pl = n\pi$$

$$p = \frac{n\pi}{l}$$

Substitute $p = \frac{n\pi}{l}$ in (2)

$$y = B \sin \frac{n\pi x}{l}\left(C \cos \frac{n\pi ct}{l} + D \sin \frac{n\pi ct}{l}\right) \qquad \ldots (3)$$

To apply condition 3 differentiate (3) partially w.r.t 't'

$$\frac{\partial y}{\partial t} = B \sin \frac{n\pi x}{l}\left(-C \sin \frac{n\pi ct}{l} \cdot \frac{n\pi c}{l} + D \cos \frac{n\pi ct}{l}\frac{n\pi c}{l}\right)$$

Now applying condition (3)

$$0 = B \sin \frac{n\pi x}{l}\left(-C \sin 0 \frac{n\pi c}{l} + D \cos 0 \frac{n\pi c}{l}\right)$$

$$0 = B \sin \frac{n\pi x}{l}\left(0 + D\frac{n\pi c}{l}\right)$$

$$\Rightarrow \quad D = 0$$

Substitute D = 0 in (3)

$$y = B \sin \frac{n\pi x}{l}\left(C \cos \frac{n\pi ct}{l} + 0\right)$$

$$= B \sin \frac{n\pi x}{l}\left(C \cos \frac{n\pi ct}{l}\right)$$

$$= BC \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}$$

$$= b\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \qquad (BC = b)$$

$$y = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \qquad \ldots (4)$$

Applying condition 4 in (4)

$$y_0 \sin^3\frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos 0$$

$$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \, . \, 1$$

Here we have to apply sin 3θ formula and compare the co-efficients of like terms to get the values of b_1, b_2, b_3, ...

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$\therefore \quad 4\sin^3\theta = 3\sin\theta - \sin 3\theta$$

$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$$

$$\therefore \quad \sin^3\frac{\pi x}{l} = \frac{3}{4}\sin\frac{\pi x}{l} - \frac{1}{4}\sin\frac{3\pi x}{l}$$

Substitute for $\sin^3\frac{\pi x}{l}$ in L.H.S.

$$y_0\left[\frac{3}{4}\sin\frac{\pi x}{l} - \frac{1}{4}\sin\frac{3\pi x}{l}\right] = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\frac{3y_0}{4}\sin\frac{\pi x}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l} = b_1\sin\frac{\pi x}{l} + b_2\sin\frac{2\pi x}{l} + b_3\sin\frac{3\pi x}{l} \ldots$$

Comparing the coefficients of $\sin\frac{\pi x}{l}$

$$b_1 = \frac{3y_0}{4}$$

Comparing the coefficients of $\sin \dfrac{2\pi x}{l}$

$$b_2 = 0$$

Comparing the coefficients of $\sin \dfrac{3\pi x}{l}$

$$b_3 = \frac{-y_0}{4}$$

$\therefore$ The solution is $y = \sum_{n=1}^{\infty} b_n \sin \dfrac{n\pi x}{l} \cdot \cos \dfrac{n\pi ct}{l}$

$$= b_1 \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} + b_2 \sin \frac{2\pi x}{l} \cos \frac{2\pi ct}{l} + b_3 \sin \frac{3\pi x}{l} \cos \frac{3\pi ct}{l} + \ldots$$

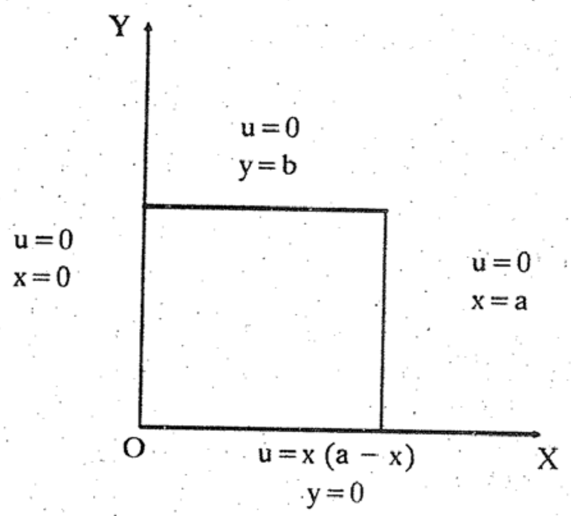
$$y = \frac{3y_0}{4} \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l} \cos \frac{3\pi ct}{l}$$

(OR)

14. (b) Find the steady state temperature distribution in a rectangular plate of sides a and b insulated at the lateral surfaces and satisfying the boundary conditions :
$u(0,y) = u(a,y) = 0$, for $0 \le y \le b$;
$u(x,b) = 0$ and $u(x,0) = x(a-x)$, for $0 \le x \le a$

Solution :

$$u(0,y) = 0 \Rightarrow u = 0 \text{ when } x = 0$$
$$u(a,y) = 0 \Rightarrow u = 0 \text{ when } x = a$$
$$u(x,b) = 0 \Rightarrow u = 0 \text{ when } y = b$$
$$u(x,0) = x(a-x) \Rightarrow u = x(a-x) \text{ when } y = 0$$



The boundary conditions

(i) $u = 0$ when $x = 0$

(ii) $u = 0$ when $x = a$

(iii) $u = 0$ when $y = b$

(iv) $u = x(a-x)$ when $y = 0$

Consider the solution

$$u = (A \cos px \ B \sin px)(Ce^{py} + De^{-py}) \quad \text{... (1)}$$

Apply condition (i) in (1) we get

$$0 = (A \cos 0 + B \sin 0)(C \,.\, e^{py} + D \,.\, e^{-py})$$

$$0 = A(Ce^{py} + De^{-py})$$

$$\therefore A = 0$$

Substitute $A = 0$ in (1)

$$u = (B \sin px)(Ce^{py} + De^{-py}) \quad \text{... (2)}$$

Substitute (ii) in (2) we get

$$0 = B \sin pa (Ce^{py} + De^{-py})$$

$$\therefore B \sin pa = 0$$

$$\sin pa = 0$$

$$pa = n\pi$$

$$\therefore p = \frac{n\pi}{a}$$

Substitute p value in (2)

$$u = B \sin \frac{n\pi x}{a}\left(Ce^{\frac{n\pi y}{a}} + De^{\frac{-n\pi y}{a}}\right) \quad \text{... (3)}$$

Apply (iii) in (3) we get

$$0 = B\sin\frac{n\pi x}{a}\left(Ce^{\frac{n\pi b}{a}} + De^{\frac{-n\pi b}{a}}\right)$$

$$Ce^{\frac{n\pi b}{a}} + De^{\frac{-n\pi b}{a}} = 0$$

$$De^{\frac{-n\pi b}{a}} = -Ce^{\frac{n\pi b}{a}}$$

$$D = \frac{-Ce^{\frac{n\pi b}{a}}}{e^{\frac{-n\pi b}{a}}}$$

$$\therefore D = -Ce^{\frac{n\pi b}{a}} . e^{\frac{n\pi b}{a}}$$

$$D = -Ce^{\frac{2n\pi b}{b}}$$

Substitute D value in (3) we get

$$u = B\sin\frac{n\pi x}{a}\left[Ce^{\frac{n\pi y}{a}} - Ce^{\frac{2n\pi b}{a}} . e^{\frac{n\pi y}{a}}\right]$$

$$\therefore u = BC\sin\frac{n\pi x}{a}\left[e^{\frac{n\pi y}{a}} - e^{\frac{2n\pi b}{a}} . e^{\frac{n\pi y}{a}}\right]$$

[Take $BC = b$]

$$\therefore u = b\sin\frac{n\pi x}{a}\left[e^{\frac{n\pi y}{a}} - e^{\frac{2n\pi b}{a}} . e^{\frac{n\pi y}{a}}\right]$$

$\therefore$ The most general solution is

$$u = \sum_{n=1}^{\infty} b_n\left(e^{\frac{n\pi y}{a}} - e^{\frac{2n\pi b}{a}} . e^{\frac{n\pi y}{a}}\right)\sin\frac{n\pi x}{a} \quad \text{... (1)}$$

Apply (iv) in (4) we get

$$x(a-x) = \sum_{n=1}^{\infty} b_n \left(e^0 - e^{\frac{2n\pi b}{a}} e^0\right) \sin \frac{n\pi x}{a}$$

$$x(a-x) = \sum_{n=1}^{\infty} b_n \left(1 - e^{\frac{2n\pi b}{a}}\right) \sin \frac{n\pi x}{a}$$

The R.H.S is Half range sine series.

It's coefficient is given by

$$b_n \left(1 - e^{\frac{2n\pi b}{a}}\right) = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{a} \, dx$$

[∵ R.H.S contains function of x]

$$= \frac{2}{a} \int_0^a x(a-x) \sin \frac{n\pi x}{a} dx$$

$$= \frac{2}{a} \int_0^a (ax - x^2) \sin \frac{n\pi x}{a} . dx$$

$$= \frac{2}{a} \left[\underbrace{(ax-x^2) \left(\frac{-\cos \frac{n\pi x}{a}}{\frac{n\pi}{a}} \right)}_{0} - \underbrace{(a-2x) \left(\frac{-\sin \frac{np\, x}{a}}{\frac{n^2\pi^2}{a^2}} \right)}_{0} + (-2) \left(\frac{\cos \frac{n\pi x}{a}}{\frac{n^3\pi^3}{a^3}} \right) \right]_0^a$$

$$= \frac{2}{a} \left[-2 \frac{\cos \frac{n\pi x}{a}}{\frac{n^3\pi^3}{a^3}} \right]_0^a$$

$$= -\frac{4}{a} \frac{a^3}{n^3\pi^3} \left[\cos \frac{n\pi x}{a} \right]_0^a$$

$$= \frac{-4a^2}{n^3 \pi^3} [\cos n\pi - \cos 0]$$

$$= \frac{-4a^2}{n^3 \pi^3} [(-1)^n - 1]$$

Case 1 :

When n is even number.

$$b_n = 0$$

Case 2 :

When n is odd number.

$$b_n \left[1 - e^{\frac{2n\pi b}{a}}\right] = \frac{-4a^2}{n^3 \pi^3} [-1 - 1]$$

$$b_n \left[1 - e^{\frac{2n\pi b}{a}}\right] = \frac{8a^2}{n^3 \pi^3}$$

$$\therefore\ b_n = \frac{8a^2}{n^3 \pi^3} \cdot \frac{1}{\left[1 - e^{\frac{2n\pi b}{a}}\right]}$$

$\therefore$ The solution is

$$u = \sum_{n=1,3}^{\infty} \frac{8a^2}{n^3 \pi^3} \cdot \frac{1}{\left[1 - e^{\frac{2n\pi b}{a}}\right]} \sin \frac{n\pi x}{a} \left[e^{\frac{n\pi y}{a}} - e^{\frac{2n\pi}{b}} \cdot e^{\frac{n\pi y}{a}}\right]$$

15. (a) (i) Find the Z-transforms of $\sin^2\left(\frac{n\pi}{4}\right)$ and $\cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)$

Solution : $\sin^2\theta = \frac{1 - \cos 2\theta}{2}$

$$\therefore\ \sin^2 \frac{n\pi}{4} = \frac{1 - \cos \frac{2n\pi}{4}}{2}$$

$$= \frac{1 - \cos \frac{n\pi}{2}}{2}$$

$$Z\left[\sin^2 \frac{n\pi}{4}\right] = Z\left[\frac{1 - \cos \frac{n\pi}{2}}{2}\right]$$

$$= \frac{1}{2} Z\left[1 - \cos \frac{n\pi}{2}\right]$$

$$= \frac{1}{2}\left[z(1) - z\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-1} - \frac{z^2}{z^2+1}\right]$$

Find the Z-transform of $\cos\left(\frac{n\pi}{4} + \frac{n\pi}{4}\right)$

Solution : $z\left[\cos\left(\frac{n\pi}{2} + \frac{n\pi}{2}\right)\right] = z\left[\cos \frac{3n\pi}{4}\right]$

By formula :

$$Z[\cos n\theta] = \frac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1}$$

Here $\theta = \frac{3\pi}{4}$

$$\therefore \quad Z\left[\cos \frac{3n\pi}{4}\right] = \frac{z\left(z - \cos\frac{3\pi}{4}\right)}{z^2 - 2z\cos\frac{3\pi}{4} + 1}$$

$$= \frac{z\left[z - \left(\frac{-1}{\sqrt{2}}\right)\right]}{z^2 - 2z\left(\frac{-1}{\sqrt{2}}\right) + 1}$$

$$= \frac{z\left[z + \frac{1}{\sqrt{2}}\right]}{z^2 + \sqrt{2}z + 1}$$

Note :

It looks there is some mistake in the question. The question may be read as $Z\left[\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right]$

$$\therefore Z\left[\cos\left(\frac{n\pi}{2}+\frac{\pi}{2}\right)\right] = z\left[\cos\frac{n\pi}{2}\cos\frac{\pi}{4} - \sin\frac{n\pi}{2}\sin\frac{\pi}{4}\right]$$

$$= z\left[\cos\frac{n\pi}{2}\cos\frac{\pi}{4}\right] - z\left[\sin\frac{n\pi}{2}\sin\frac{\pi}{4}\right]$$

$$= \cos\frac{\pi}{4} z\left[\cos\frac{n\pi}{2}\right] - \sin\frac{\pi}{4} z\left[\sin\frac{n\pi}{2}\right]$$

$$= \frac{1}{\sqrt{2}}\frac{z^2}{z^2+1} - \frac{1}{\sqrt{2}}\frac{z}{z^2+1}$$

$$= \frac{1}{\sqrt{2}}\left[\frac{z^2}{z^2+1} - \frac{z}{z^2+1}\right]$$

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15. (a)(ii) Using convolution theorem, find the inverse Z-transform of $\frac{z^2}{(z+a)^2}$

Solution : Let $\frac{z^2}{(z+a)^2} = \frac{z}{z+a}\cdot\frac{z}{z+a}$

Let $F(z) = \frac{z}{z+a}$

$$\therefore f_n = Z^{-1}[F(z)]$$

$$= Z^{-1}\left[\frac{z}{z+a}\right] = (-a)^n$$

Replace n by m

$$\therefore f_m = (-a)^m$$

$$G(z) = \frac{z}{z+a}$$

$$\therefore \quad g_n = Z^{-1}\left[G(z)\right]$$

$$= Z^{-1}\left[\frac{z}{z+a}\right]$$

$$= (-a)^n$$

Replace n by $n-m$

$$g_{n-m} = (-a)^{n.m}$$

$\therefore$ By convolution theorem

$$Z^{-1}\left[\frac{z^2}{(z+a)^2}\right] = \sum_{m=0}^{n} f_m \; g_{n-m}$$

$$= \sum_{m=0}^{n} (-a)^m (-a)^{n-m}$$

$$= 1.(-a)^n + (-a)(-a)^{n-1} + (-a)^2(-a)^{n-2} + \ldots + (-a)^r$$

$$= (-a)^n + (-a)^n + (-a)^n + \ldots + (-a)^n$$

$$= (-a)^n + (-a)^n + (-a)^n + \ldots + (-a)^n \; (n+1) \text{ times}$$

$$= (-a)^n [1 + 1 + 1 + \ldots + (n+1) \text{ times}]$$

$$= (-a)^n (n+1)$$

15.(b) (i) Solve the difference equation using Z-transform $y_{(n+3)} - 3y_{(n+1)} + 2y_{(n)} = 0$ given that $y_0 = 4,\ y_1 = 0,\ y_2 = 8$

Solution :

Let $\bar{y} = Z[y_n]$

Taking Z-transform on both side

$$Z(y_{n+3}) - 3Z(y_{n+1}) + 2Z(y_n) = 0$$

$$z^3\left[\bar{y} - y_0 - \frac{y_1}{z} - \frac{y_2}{z^2}\right] - 3z[\bar{y} - y_0] + 2\bar{y} = 0$$

$$z^3\left[\bar{y} - 4 - \frac{8}{z^2}\right] - 3z[\bar{y} - 4] + 2\bar{y} = 0$$

$$z^3\bar{y} - 4z^3 - 8z - 3z\bar{y} + 12z + 2\bar{y} = 0$$

$$z^3\bar{y} - 3z\bar{y} + 2\bar{y} - 4z^3 + 4z = 0$$

$$\bar{y}[z^3 - 3z + 2] = 4z^3 - 4z$$

$$\therefore \bar{y} = \frac{4z^3 - 4z}{z^3 - 3z + 2}$$

To factorising denominator we use synthetic division

1	1	0	−3	2
	0	1	1	−2
	1	1	−2	0

$$\therefore z^3 - 3z + 2 = (z-1)(z^2 + z - 2)$$

$$= (z-1)(z-1)(z+2)$$

$$= (z-1)^2(z+2)$$

$$\bar{y} = \frac{4z^3 - 4z}{(z-1)^2 (z+2)}$$

$$\therefore y_n = Z^{-1}\left[\frac{4z^3 - 4z}{(z-1)^2 (z+2)}\right]$$

$$\therefore F(z) = \frac{4z^3 - 4z}{(z-1)^3 (z+2)}$$

To find inverse Z-transform we apply residue method

Multiply both side by z^{n-1}

$$z^{n-1} F(z) = \frac{z^{n-1}(4z^3 - 4z)}{(z-1)^2 (z+2)}$$

$$= \frac{4z^{n+2} - 4z^n}{(z-1)^2 (z+2)}$$

Equating the denominator to zero

$$(z-1)^2 (z+2) = 0$$

$$(z-1)^2 = 0 \; ; \; (z+2) = 0$$

$$\therefore z = 1, 1 \; ; \; z = -2$$

The residue at $z = -2$

$$\lim_{z \to -2} (z+2) \frac{(4z^{n+2} - 4z^n)}{(z-1)^2 (z+2)}$$

$$= \frac{4(-2)^{n+2} - 4(-2)^n}{(-2-1)^2}$$

$$= \frac{4(-2)^{n+2} - 4(-2)^n}{9}$$

The residue at $z = 1, 1$

$$\lim_{z\to 1} \frac{d}{dz} (z-1)^2 \cdot \frac{4z^{n+2} - 4z^n}{(z+2)(z-1)^2}$$

$$= \lim_{z\to 1} \frac{d}{dz} \frac{4z^{n+2} - 4z^n}{(z+2)}$$

$$= \lim_{z\to 1} \frac{(z+2)\left[(n+2)\,4z^{n+1} - 4nz^{n+1}\right] - \left[4z^{n+2} - 4z^n\right](1)}{(z+2)^2}$$

$$= \frac{(1+2)\left[(n+2)\,4\,(1)^{n+1} - 4n\,(1)^{n-1}\right] - 4\left[(1)^{n+2} - 4\,(1)^n\right]}{(3)^2}$$

$$= \frac{z\left[(n+2)\,4 - 4n\right] - 4\left[1 - 4\,(1)\right]}{9}$$

$$= \frac{z\left[4n + 8 - 4n\right] - 4 + 4}{9}$$

$$= \frac{8}{3}$$

∴ The solution is

$$Z^{-1}\left[\frac{4z^3 - 4z}{(z-1)^2 (z+2)}\right] = \text{Sum of residues.}$$

$$= \frac{4(-2)^{n+2} - 4(-2)^n}{9} + \frac{8}{3}$$

---

15. (b) (ii) Solve $y_{(n+2)} + 6\,y_{(n+1)} + 9\,y_{(n)} = 2^n$ given that $y_0 = y_1 = 0$

Solution :

Take Z-transform on both side

$$Z[y_{n+2}] + 6Z[y_{n+1}] + 9Z[y_n] = Z[2^n]$$

$$z^2\left[\bar{y} - y_0 - \frac{y_1}{z}\right] + 6\left[z(\bar{y} - y_0)\right] + 9\bar{y} = \frac{Z}{Z-2}$$

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$$z^2 [\bar{y}] + 6z\bar{y} + 9\bar{y} = \frac{z}{z-2}$$

$$\bar{y}\,[z^2 + 6z + 9] = \frac{z}{z-2}$$

$$\bar{y} = \frac{z}{(z-2)(z^2+6z+9)}$$

$$y = Z^{-1}\left[\frac{z}{(z-2)(z^2+6z+9)}\right]$$

$$\therefore F(Z) = \left[\frac{z}{(z-2)(z^2+6z+9)}\right]$$

Multiply both side by z^{n-1}

$$z^{n-1} F(Z) = z^{n-1}\frac{z}{(z-2)(z+3)^2}$$

$$= \frac{z^n}{(z-2)(z+3)^2}$$

$$\therefore g(z) = \frac{z^n}{(z-2)(z+3)^2}$$

Equating denominator to zero.

$$(z-2)(z+3)^2 = 0$$

$$\therefore z - 2 = 0 \quad ; \quad (z+3)^2 = 0$$

$$\therefore z = 2 \quad ; \quad z = -3, -3$$

The residue at $z = 2$

$$\lim_{z\to 2} (z-2)\,.\,\frac{z^n}{(z-2)(z+3)^2}$$

$$= \frac{2^n}{25}$$

The residue for $z = -3, -3$

$$\lim_{z \to -3} \frac{d}{dz} (z+3)^2 \cdot \frac{z^n}{(z-2)(z+3)^2}$$

$$= \lim_{z \to -3} \frac{d}{dz} \frac{z^n}{(z-2)}$$

$$= \lim_{z \to -3} \frac{(z-2)\, n\, z^{n-1} - z^n}{(z-2)^2}$$

$$= \frac{-5n(-3)^{n-1} - (-3)^n}{25}$$

$$\therefore Z^{-1}\left[\frac{z}{(z-2)(z+3)^2}\right] = \text{sum of residues}$$

$$= \frac{2^n}{25} - \frac{5n(-3)^{n-1}}{25} - \frac{(-3)^n}{25}$$

$$= \frac{2^n - 5n(-3)^{n-1} - (-3)^n}{25}$$