B.E. /B.Tech. DEGREE EXAMINATION, MAY/JUNE 2012

Third Semester

MA 2211 /18130/MA 31 / 10177 MA 301 / MA 1201 / 080100008 - TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS

(Common to All Branches)

(Regulation 2008)

[Time : Three hours] [Maximum : 100 marks]

Answer ALL questions.

PART-A (10×2=20 marks)

1. **Find the constant term in the expansion of $\cos^2 x$ as a Fourier series in the interval $(-\pi, \pi)$.**

Solution : Given : $f(x) = \cos^2 x = \dfrac{1+\cos 2x}{2}$ is an even function in the interval $(-\pi, \pi)$

$$\text{W.K.T., } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \cos^2 x\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{1+\cos 2x}{2}\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} (1+\cos 2x)\, dx$$

$$= \frac{1}{\pi}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\,[(\pi + 0) - (0 + 0)]$$

$$a_0 = 1$$

Constant term, $\boxed{\frac{a_0}{2} = \frac{1}{2}}$

2. Define Root Mean square value of a function f (x) over the interval (a, b)

Solution : Let $f(x)$ be a function defined in an interval (a, b) then

$$\sqrt{\frac{\int_a^b [f(x)]^2\,dx}{b - a}}$$

is called the R.M.S value or effective value of $f(x)$ and is denoted by $\bar{y}$.

$$\text{Hence, } [\bar{y}]^2 = \frac{1}{b-a}\int_a^b [f(x)]^2\,dx$$

3. What is the Fourier transform of f (x − a), if the Fourier transform of f (x) is F (s) ?

Proof : We know that,

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx$$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)\,e^{isx}\,dx$$

put $t = x - a$ | $x \to -\infty \Rightarrow t \to -\infty$

$dt = dx$ | $x \to \infty \Rightarrow t \to \infty$

$$\text{i.e., } F[f(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{is(t+a)}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{ist}\,e^{isa}\,dt$$

$$= e^{isa} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt$$

$$= e^{isa}\ F\,[s]$$

4. Find the Fourier sine transform of $f(x) = e^{-ax}$, $a > 0$.

Solution : We know that, $F_s\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx\, dx$

$$F_s\,[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2 + a^2}\right] \qquad \boxed{\because \int_0^{\infty} e^{-ax} \sin bx\, dx = \frac{b}{a^2 + b^2}}$$

5. Form the partial differential equation by eliminating the arbitrary function from $z^2 - xy = f\left(\frac{x}{z}\right)$

Solution : Given : $z^2 - xy = f\left(\frac{x}{z}\right)$ ... (1)

Diff. (1) $p.$ w.r.t. x we get,

$$2z \frac{\partial z}{\partial x} - y = f'\left(\frac{x}{z}\right) \left[\frac{z\,(1) - x \frac{\partial z}{\partial x}}{z^2}\right]$$

$$2z\ p - y = f'\left(\frac{x}{z}\right) \left[\frac{z - xp}{z^2}\right] \qquad \text{... (2)}$$

Diff. (1) p w.r.t. y we get,

$$2z \frac{\partial z}{\partial y} - x = f'\left(\frac{x}{z}\right) \left[\frac{z\,(0) - x \frac{\partial z}{\partial y}}{z^2}\right]$$

$$2zq - x = f'\left(\frac{x}{z}\right)\left[\frac{-xq}{z^2}\right] \quad \dots (3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{2zp - y}{2zq - x} = \frac{z - xp}{-xq}$$

$$-2xzpq + xyq = 2z^2 q - 2xzpq - xz + x^2 p$$

$$xyq = 2z^2 q - xz + x^2 p$$

$$x^2 p + 2z^2 q - xyq = xz$$

$$x^2 p - (xy - 2z^2) q = xz$$

Second Method :

Let $u = z^2 - xy, \quad v = \frac{x}{z}$

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 2zp - y & \frac{z - px}{z^2} \\ 2zq - x & \frac{-xq}{z^2} \end{vmatrix} = 0$$

$$\Rightarrow px^2 - q(xy - 2z^2) = xz$$

6. **Solve $(D^2 - 7DD' + 6D'^2)z = 0$**

Solution : Given : $(D^2 - 7DD' + 6D'^2)z = 0$

The auxiliary equation is

$$m^2 - 7m + 6 = 0 \text{ [Replace } D \text{ by } m \text{ and } D' \text{ by } 1]$$

$$(m - 6)(m - 1) = 0$$

$$m = 6, m = 1$$

$$z = \phi_1(y + 6x) + \phi_2(y + x)$$

7. **What is the basic difference between the solutions of one dimensional wave equation and one dimensional heat equation with respect to the time?**

Solution :

	One dimensional wave equation	One dimensional heat equation
	$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$	$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$
1.	It is classified as hyperbolic p.d.e	It is classified as parabolic p.d.e
2.	Solution of one dimensional wave equation $y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos p\,at + c_4 \sin p\,at)$ which is **periodic** w.r.to time 't'	Solution of one dimensional heat equation $u(x,t) = (A \cos px + B \sin px)\, e^{-p^2 \alpha^2 t}$ which is **non-periodic** w.r.to time 't'.

8. Write down the partial differential equation that represents steady state heat flow in two dimensions and name the variables involved.

Solution : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, where $u(x,y)$ is the steady state temp. at the point P (x,y).

9. Find the Z-transform of $x(n) = \begin{cases} \frac{a^n}{n!} & \text{for } n \ge 0 \\ 0, & \text{otherwise} \end{cases}$

Solution : We know that, $Z[a^n f(n)] = F\left[\frac{z}{a}\right]$

$$Z\left[a^n \frac{1}{n!}\right] = \left[Z\left[\frac{1}{n!}\right]\right]_{z \to z/a}$$

$$= \left[e^{1/z}\right]_{z \to z/a} \quad \because Z\left[\frac{1}{n!}\right] = e^{1/z}$$

$$= e^{\frac{1}{(z/a)}}$$

$$= e^{\frac{a}{z}}$$

10. Solve $y_{n+1} - 2y_n = 0$ given $y_0 = 3$

Solution : Given : $y_{n+1} - 2y_n = 0$

Taking Z-transform on both sides of the difference equation, we get

$$Z[y_{n+1}] - 2Z[y_n] = Z(0)$$

$$\left[z\,Y(z) - zy(0)\right] - 2\,Y(z) = 0$$

$$\left[z\,Y(z) - (z)(3)\right] - 2\,Y(z) = 0 \qquad [\because y(0) = 3]$$

$$(z-2)\,Y(z) - 3z = 0$$

$$Y(z) = \frac{3z}{z-2}$$

$$\Rightarrow Z[y_n] = \frac{3z}{z-2}$$

$$y_n = Z^{-1}\left[\frac{3z}{z-2}\right]$$

$$= 3Z^{-1}\left[\frac{z}{z-2}\right]$$

$$= 3\left(2^n\right)$$

$$\left[\because Z^{-1}\left(\frac{z}{z-a}\right) = a^n\right]$$

PART-B (5 × 16 = 80 marks)

11. (a) (i) Find the Fourier series of $f(x) = (\pi - x)^2$ in $(0, 2\pi)$ of periodicity 2π.

Solution : Let $f(x) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx + \sum\limits_{n=1}^{\infty} b_n \sin nx$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2 \, dx$$

$$= \frac{1}{\pi}\left[\frac{(\pi - x)^3}{3}\right]_0^{2\pi}$$

$$= \frac{-1}{3\pi}\left[(\pi - x)^3\right]_0^{2\pi} = -\frac{1}{3\pi}\left[(-\pi)^3 - \pi^3\right]$$

$$= -\frac{1}{3\pi}\left[-\pi^3 - \pi^3\right]$$

$$= \frac{2\pi^3}{3\pi}$$

$$\boxed{a_0 = \frac{2}{3}\pi^2}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2 \cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2 \cos nx \, dx = \frac{1}{\pi}\int_0^{2\pi} (x - \pi)^2 \cos n\, x \, dx$$

$$= \frac{1}{\pi}\left[(x - \pi)^2\left(\frac{\sin nx}{n}\right) - 2(x - \pi)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$[\because (\pi - x)^2 = (x - \pi)^2]$$

$$= \frac{1}{\pi}\left[(x - \pi)^2\left(\frac{\sin nx}{n}\right) + 2(x - \pi)\left(\frac{\cos nx}{n^2}\right) - 2\left(\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(0 + \frac{2\pi}{n^2} - 0\right) - \left(0 + \frac{-2\pi}{n^2} - 0\right)\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{2\pi}{n^2}\right) + \left(\frac{2\pi}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{4\pi}{n^2}\right)\right]$$

$$\boxed{a_n = \frac{4}{n^2}},$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2 \sin nx\, dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} (x - \pi)^2 \sin nx\, dx$$

$$= \frac{1}{\pi}\left[(x-\pi)^2\left(\frac{-\cos nx}{n}\right) - 2(x-\pi)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-(x-\pi)^2\left(\frac{\cos nx}{n}\right) + 2(x-\pi)\left(\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{-\pi^2}{n} + 0 + \frac{2}{n^3}\right) - \left(\frac{-\pi^2}{n} + 0 + \frac{2}{n^3}\right)\right] = 0$$

$$\boxed{b_n = 0}$$

$$f(x) = \frac{(2/3)\pi^2}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos nx = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$$

Note : In $(0, 2\pi)$,

(i) If $f(2\pi - x) = f(x)$, then $b_n = 0$

(ii) If $f(2\pi - x) = -f(x)$, then $a_n = 0$ & $a_n = 0$

11. (a) (ii) Obtain the Fourier series to represent the function

$$\mathbf{f(x) = |x|,\ -\pi < x < \pi \text{ and deduce } \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}}$$

Solution : Given : $f(x) = |x|$ i.e., $f(x) = \begin{cases} -x, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

$$f(-x) = |-x| = x = f(x)$$

Therefore, $f(x)$ is an even function. Hence, $b_n = 0$

Let the required Fourier series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \dots (1)$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^2}{2} - 0\right] = \pi$

$\therefore$ $\boxed{a_0 = \pi}$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) + \left(\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(0 + \frac{(-1)^n}{n^2}\right) - \left(0 + \frac{1}{n^2}\right)\right] = \frac{2}{\pi}\left[\left(\frac{(-1)^n}{n^2}\right) - \left(\frac{1}{n^2}\right)\right]$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$\therefore \quad \boxed{a_n = \begin{cases} \frac{-4}{\pi n^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}}$$

$$\therefore (1) \Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=odd}^{\infty} \frac{-4}{\pi n^2}\cos nx$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=odd}^{\infty} \frac{1}{n^2}\cos nx \quad \dots (2)$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos(2n-1)x$$

Put $x = 0$, Here, $x = 0$ is a point of continuity. $\left[\therefore \left.\begin{matrix} f(x) \\ \text{at } x=0 \end{matrix}\right\} = 0\right]$

$$\therefore (2) \Rightarrow \quad 0 = \frac{\pi}{2} - \frac{\pi}{4}\sum_{n=odd}^{\infty}\frac{1}{n^2} \qquad [\because \cos 0 = 1]$$

$$-\frac{\pi}{2} = -\frac{4}{\pi}\sum_{n=odd}^{\infty}\frac{1}{n^2}$$

$$\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$$

$$\Rightarrow \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots = \frac{\pi^2}{8} \quad \Rightarrow \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

11. (b) (i) Find the half-range Fourier cosine series of $f(x) = (\pi - x)^2$ in the interval $(0, \pi)$. Hence find the sum of the series $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots+\infty$.

Solution : Given : $f(x) = (\pi - x)^2$ in $(0, \pi)$

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \ldots (1)$$

$$\text{when } a_0 = \frac{2}{\pi}\int_0^{\pi}(\pi - x)^2 x = \frac{2}{\pi}\left[\frac{(\pi-x)^3}{-3}\right]_0^{\pi}$$

$$= -\frac{2}{3\pi}\left[(\pi - x)^3\right]_0^{\pi} = -\frac{2}{3\pi}\left[0-\pi^3\right] = \frac{2}{3}\pi^2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(\pi - x)^2 \cos n x\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(x-\pi)^2 \cos nx\, dx \qquad [\because (\pi - x)^2 = (x-\pi)^2]$$

$$= \frac{2}{\pi}\left[(x-\pi)^2\left[\frac{\sin nx}{n}\right] - 2(x-\pi)\left[\frac{-\cos n x}{n^2}\right] + 2\left[\frac{-\sin nx}{n^3}\right]\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[(x-\pi)^2\frac{\sin nx}{n} + \frac{2(x-\pi)}{n^2}\cos nx - \frac{2\sin nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[(0+0-0) - \left(\frac{-2\pi}{n^2}\right)\right] = \frac{2}{\pi}\left[\frac{2\pi}{n^2}\right] = \frac{4}{n^2}$$

$$\therefore (1) \Rightarrow f(x) = \frac{\frac{2}{3}\pi^2}{2} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx$$

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$$

By Parseval's identity,

$$\frac{1}{\pi}\int_0^{\pi}[f(x)]^2\,dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}a_n^2$$

$$\frac{1}{\pi}\int_0^{\pi}(\pi-x)^4\,dx = \frac{\left(\frac{4\pi^4}{9}\right)}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4}$$

$$\frac{1}{\pi}\left[\frac{(\pi-x)^5}{-5}\right]_0^{\pi} = \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{1}{\pi}\left[0-\left(\frac{\pi^5}{-5}\right)\right] = \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{1}{\pi}\left[\frac{\pi^5}{5}\right] = \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{\pi^4}{5} = \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$8\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{5} - \frac{\pi^4}{9}$$

$$= \frac{4}{45}\pi^4$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{90}\pi^4$$

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots + \infty = \frac{\pi^4}{90}$$

11. (b) (ii) Find the Fourier series upto second harmonic for the following data for y with period 6.

x :	0	1	2	3	4	5
y :	9	18	24	28	26	20

Solution : Here, the length of the interval is $2l = 6$, i.e., $l = 3$.

$\therefore$ The Fourier series can be represented by

$$y = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{l} + b_1 \sin\frac{\pi x}{l}\right) + \left(a_2 \cos\frac{2\pi x}{l} + b_2 \sin\frac{2\pi x}{l}\right) + \ldots$$

Here, $y = \dfrac{a_0}{2} + \left(a_1 \cos\dfrac{\pi x}{3} + b_1 \sin\dfrac{\pi x}{3}\right) + \left(a_2 \cos\dfrac{2\pi x}{3} + b_2 \sin\dfrac{2\pi x}{3}\right) \ldots$ (1)

x	y	$\cos\frac{\pi x}{3}$	$\sin\frac{\pi x}{3}$	$y\cos\frac{\pi x}{3}$	$y\sin\frac{\pi x}{3}$	$y\cos\frac{2\pi x}{3}$	$y\sin\frac{2\pi x}{3}$
0	9	1	0	9	0	9	0
1	18	0.5	0.866	9	15.588	−9	15.5885
2	24	−0.5	0.866	−12	20.785	−12	−20.7846
3	28	−1	0	−28	0	28	0
4	26	−0.5	−0.866	−13	−22.517	−13	22.517
5	20	0.5	−0.866	10	−17.321	−10	−17.3205
	125			−25	−3.465	−7	0.0004

$$a_0 = \frac{2}{n}\Sigma y = \frac{2}{6}125 = 41.67$$

$$a_1 = \frac{2}{n}\Sigma y\cos\frac{\pi x}{3} = \frac{2}{6}(-25) = -8.33$$

$$a_2 = \frac{2}{n} \Sigma y \cos\frac{2\pi x}{3} = \frac{2}{6}(-7) = -2.333$$

$$b_1 = \frac{2}{n} \Sigma y \sin\frac{\pi x}{3} = \frac{2}{6}(-3.465) = -1.16$$

$$b_2 = \frac{2}{n} \Sigma y \sin\frac{2\pi x}{3} = \frac{2}{6}(0.0004) = 0.00013$$

Substituting the above values in (1), we get

$$y = 20.84 - 8.33\cos\frac{\pi x}{3} - 1.16\sin\frac{\pi x}{3} - 2.333\cos\frac{2\pi x}{3} + 0.00013\sin\frac{2\pi x}{3}$$

12. (a) (i) Derive the Parseval's identity for Fourier Transforms

Solution : By convolution theorem,

$$F[f(x) * g(x)] = F(s).\ G(s)$$

$$f(x) * g(x) = F^{-1}[F(s).\ G(s)]$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\ g(x-t)\,dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\ G(s)\ e^{-isx}\,ds \quad \dots (1)$$

put $x = 0$ and $g(-t) = \overline{f(t)}$ there it follows that

$$G(s) = \overline{F(s)}$$

$$(1) \Rightarrow \int_{-\infty}^{\infty} f(t)\overline{f(t)}\,dt = \int_{-\infty}^{\infty} F(s)\ \overline{F(s)}\,ds$$

$$\text{(i.e.,)} \quad \int_{-\infty}^{\infty} |f(t)|^2\,dt = \int_{-\infty}^{\infty} |F(s)|^2\,ds$$

$$\text{(or)} \quad \int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(s)|^2\,ds$$

12. (a) (ii) Find the Fourier integral representation of f(x)

defined as $f(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{2} & \text{for } x = 0 \\ e^{-x} & \text{for } x > 0 \end{cases}$

Solution :

We know that, $F(s) = F[f(x)]$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\; e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x}\; e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(1-is)x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-(1-is)x}}{-(1-is)}\right]_{0}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{-1}{1-is}\right] \left[e^{-(1-is)x}\right]_{0}^{\infty}$$

$$= -\frac{1}{\sqrt{2\pi}} \left[\frac{1}{1-is}\right] [0-1]$$

$$= \frac{1}{\sqrt{2\pi}}\; \frac{1}{1-is}$$

$$= \frac{1}{\sqrt{2\pi}}\; \frac{1}{1-is}\; \frac{1+is}{1+is}$$

$$= \frac{1}{\sqrt{2\pi}}\; \frac{1+is}{1+s^2}$$

By inversion formula,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\; e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\; \frac{1+is}{1+s^2}\; e^{-isx}\, ds$$

$$= \frac{1}{2\pi} \left[\int_{-\infty}^{\infty} \left(\frac{1}{1+s^2} + \frac{is}{1+s^2}\right) (\cos sx - i \sin sx)\, ds\right]$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{1}{1+s^2} \cos sx - i\, \frac{1}{1+s^2} \sin sx + i\, \frac{s}{1+s^2} \cos sx + \frac{s}{1+s^2} \sin sx\right] ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\cos sx + s \sin sx}{1+s^2} ds + \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{s \cos sx - \sin sx}{1+s^2} ds$$

$$= \frac{1}{2\pi} \left[2 \int_{0}^{\infty} \frac{\cos sx + s \sin sx}{1+s^2} ds + 0 \right]$$

$\left[\because \frac{\cos x + s \sin sx}{1+s^2}\right.$ is an even function,

$\frac{s \cos sx - \sin sx}{1+s^2}$ is an odd function]

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \frac{\cos sx + s \sin sx}{1+s^2} ds \quad \text{... (1)}$$

which is the required Fourier integral representation of $f(x)$.

Putting $x = 0$ in (1), we get

$$f(0) = \frac{1}{\pi} \int_{0}^{\infty} \frac{1}{1+s^2} ds$$

$$= \frac{1}{\pi} \left[\tan^{-1} s\right]_0^{\infty}$$

$$= \frac{1}{\pi} \left[\frac{\pi}{2} - 0\right] \qquad \left[\because \tan^{-1} \infty = \frac{\pi}{2}\right]$$

$$= \frac{1}{2}$$

Thus the integral representation of (1) holds good for $x = 0$ also.

12. (b) (i) State and prove convolution theorem on Fourier transform.

The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms.

(i.e.,) $F[f(x) * g(x)] = F(s)\, G(s) = F[f(x)]\, F[g(x)]$

Proof : We know that, $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} dx$

$$F[f(x) * g(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [f(x) * g(x)]\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt\right] e^{isx}\, dx$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{1}{\sqrt{2\pi}}\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, e^{isx}\, dt\, dx$$

by changing the order of integration, we get

$$= \left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{1}{\sqrt{2\pi}}\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, e^{isx}\, dx\, dt$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left[\int_{-\infty}^{\infty} g(x-t)\, e^{isx}\, dx\right] dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-t)\, e^{isx}\, dx\right] dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\; F[g(x-t)]\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, G(s)\, dt$$

by shifting property $F[f(x-a)] = e^{-ias} F(s)$

$$= G(s) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt\right]$$

$$= G(s)\, F(s) = F(s)\, G(s)$$

Note : $F^{-1}[F(s)\, G(s)] = f(x) * g(x)$

$$= F^{-1}[F(s)] * F^{-1}[G(s)]$$

12. (b) (ii) Find Fourier sine and cosine transform of x^{n-1} and hence prove $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier sine and cosine transforms.

Solution : (i) $F_c[x^{n-1}] = \sqrt{\frac{2}{\pi}} \int_0^\infty x^{n-1} \cos sx \, dx$

We know that, $\Gamma(n) = \int_0^\infty e^{-y} y^{n-1} dy, \quad n > 0$

put $y = ax$, we get $\int_0^\infty e^{-ax} x^{n-1} dx = \frac{\Gamma(n)}{a^n}, \quad a > 0$

Let $a = is$

$$\therefore \int_0^\infty e^{-isx} x^{n-1} dx = \frac{\Gamma(n)}{(is)^n}$$

$$\int_0^\infty (\cos sx - i \sin sx) x^{n-1} dx = \frac{\Gamma(n)\, i^{-n}}{s^n}$$

$$= \frac{\Gamma(n)}{s^n}\left[\cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right)\right]^{-n}$$

$$= \frac{\Gamma(n)}{s^n}\left[\cos\frac{n\pi}{2} - i \sin\frac{n\pi}{2}\right]$$

Equating real parts, we get

$$\int_0^\infty x^{n-1} \cos sx \, dx = \frac{\Gamma(n)}{s^n} \cos\left(\frac{n\pi}{2}\right)$$

Using this in (1), we get

$$F_c[x^{n-1}] = \sqrt{\frac{2}{\pi}} \frac{\Gamma(n)}{s^n} \cos\left(\frac{n\pi}{2}\right)$$

put $n = \frac{1}{2}$, we get

$$F_c\left[\frac{1}{\sqrt{x}}\right] = \sqrt{\frac{2}{\pi}} \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}} \cos\left(\frac{\pi}{4}\right)$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sqrt{\pi}}{\sqrt{s}} \frac{1}{\sqrt{2}} \qquad [\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}]$$

$$= \frac{1}{\sqrt{s}}$$

Hence, $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier cosine transform.

(ii) $F_s[x^{n-1}] = \sqrt{\frac{2}{\pi}} \int_0^\infty x^{n-1} \sin sx\, dx$

$$= \sqrt{\frac{2}{\pi}} \frac{\Gamma(n)}{s^n} \sin\frac{n\pi}{2} \qquad [\because \int_0^\infty x^{n-1} \sin sx\, dx = \frac{\Gamma(n)}{s^n} \sin\frac{n\pi}{2}]$$

Taking $n = \frac{1}{2}$, we get

$$F_s[x^{\frac{1}{2}-1}] = \sqrt{\frac{2}{\pi}} \frac{\Gamma\frac{1}{2}}{s^{1/2}} \sin\frac{\pi}{4}$$

$$F_s\left[\frac{1}{\sqrt{x}}\right] = \sqrt{\frac{2}{\pi}} \frac{\sqrt{\pi}}{\sqrt{s}} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{s}}$$

Hence, $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier sine transform.

13. (a) (i) Form the p.d.e by eliminating the arbitrary function ϕ from $\phi(x^2 + y^2 + z^2, ax + by + cz) = 0$

Solution :

Given : $u = x^2 + y^2 + z^2$

$$\frac{\partial u}{\partial x} = 2x + 2z\frac{\partial z}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2x + 2pz$$

$$\frac{\partial u}{\partial y} = 2y + 2z\frac{\partial z}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial y} = 2y + 2zq$$

$v = ax + by + cz$

$$\frac{\partial v}{\partial x} = a + c\frac{\partial z}{\partial x}$$

$$\Rightarrow \frac{\partial v}{\partial x} = a + cp$$

$$\frac{\partial v}{\partial y} = b + c\frac{\partial z}{\partial y}$$

$$\Rightarrow \frac{\partial v}{\partial y} = b + cq$$

Required p.d.e is given by

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2x + 2pz & a + cp \\ 2y + 2qz & b + cq \end{vmatrix} = 0$$

$$\Rightarrow (2x + 2pz)(b + cq) - (a + cp)(2y + 2qz) = 0$$

$$\Rightarrow 2bx + 2cqx + 2pbz + 2pczq - 2ay - 2aqz - 2cpy - 2cpqz = 0$$

$$(2bz - 2cy)p + (2cx - 2az)q = 2ay - 2bx$$

$$(bz - cy)p + (cx - az)q = ay - bx$$

(or) $(cy - bz)p + (az - cx)q = bx - ay$

13. (a) (ii) Solve the partial differential equation

$$\mathbf{x^2 (y - z) p + y^2 (z - x) q = z^2 (x - y)}$$

Solution :

Given : $x^2 (y - z) p + y^2 (z - x) q = z^2 (x - y)$

This equation is of the form $Pp + Qq = R$

where $P = x^2 (y - z)$, $Q = y^2 (z - x)$, $R = z^2 (x - y)$

Lagrange's subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

i.e., $$\frac{dx}{x^2 (y - z)} = \frac{dy}{y^2 (z - x)} = \frac{dz}{z^2 (x - y)} \quad \text{... (1)}$$

Taking the Lagrangian multipliers are $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$

we get each ratio in (1),

$$= \frac{\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz}{x(y - z) + y(z - x) + z(x - y)} = \frac{\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz}{0}$$

Hence, $\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

Integrating, we get

$$\log x + \log y + \log z = \log a$$

$$\log(xyz) = \log a$$

$$xyz = a$$

Taking the Lagrangian multiples are $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$

we get each ratio in (1),

$$= \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{(y-z)+(z-x)+(x-y)} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

Hence, $\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$

Integrating, we get $\int x^{-2}dx + \int y^{-2}dy + \int z^{-2}dz = b_1$

$$\frac{x^{-1}}{-1} + \frac{y^{-1}}{-1} + \frac{z^{-1}}{-1} = b_1$$

$$x^{-1} + y^{-1} + z^{-1} = -b_1$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = b$$

> **Aliter :**
>
> Taking the Lagrangian multipliers
>
> $y+z,\ z+x,\ x+y$
>
> we get
>
> $b = xy + yz + zx$

Hence, the general solution is $f(a, b) = 0$

$$\text{i.e., } f\left(xyz, \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 0$$

where f is arbitrary.

13. (b) (i) **Solve the equation**

$$(D^3 + D^2 D' - 4DD'^2 - 4D'^3)z = \cos(2x + y)$$

Solution :

Given : $[D^3 + D^2 D' - 4DD'^2 - 4D'^3]z = \cos(2x + y)$

The auxiliary equation is $m^3 + m^2 - 4m - 4 = 0$

[Replace D by m and D' by 1]

$$m^2(m+1) - 4(m+1) = 0$$

$$\Rightarrow \quad (m+1)(m^2-4) = 0$$

$$\Rightarrow \quad (m+1)(m+2)(m-2) = 0$$

$$\Rightarrow \quad m = -1,\ m = -2,\ m = 2$$

The roots are distinct

$\therefore$ C.F $= \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+2x)$

$$\text{P.I} = \frac{1}{D^3 + D^2D' - 4DD'^2 - 4D'^3}\cos(2x+y) \qquad [\text{Replace } D^2 \text{ by } -2^2$$

$$= \frac{1}{-4D - 4D' + 4D + 4D'}\cos(2x+y) \qquad DD' \text{ by } -2$$

$$D'^2 \text{ by } -1]$$

$$= \frac{1}{0}\cos(2x+y) \qquad [\text{Ordinary rule fails}]$$

$$= x\,\frac{1}{3D^2 + 2DD' - 4D'^2}\cos(2x+y)$$

$$= x\,\frac{1}{-12-4+4}\cos(2x+y) \qquad [\text{Replace } D^2 \text{ by } -2^2$$

$$DD' \text{ by } -2\ ;\ D'^2 \text{ by } -1]$$

$$= \frac{-x}{12}\cos(2x+y)$$

$\therefore$ General solution is

$$z = \text{C.F} + \text{P.I}$$

$$= \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+2x) - \frac{x}{12}\cos(2x+y)$$

13. (b) (ii) Solve $[2D^2 - DD' - D'^2 + 6D + 3D']\,z = x\,e^y$

Solution :

Given : $[2D^2 - DD' - D'^2 + 6D + 3D']\,z = x\,e^y$

$$\Rightarrow [(2D+D')(D-D') + 3(2D+D')]\,z = x\,e^y$$

$$\Rightarrow (2D+D')(D-D'+3)\,z = x\,e^y$$

To find C.F : $(2D + D')(D - D' + 3)z = 0$

$$\Rightarrow \left[D - \left(\frac{-1}{2}\right)D'\right][D - D' - (-3)]z = 0$$

Here, $m_1 = 1,\ c_1 = -3,\ m_2 = -\frac{1}{2},\ c_2 = 0$

$$\text{C.F} = e^{-3x} f_1(y + x) + e^{0x} f_2\left(y - \frac{1}{2}x\right)$$

$$= e^{-3x} f_1(y + x) + f_2\left(y - \frac{1}{2}x\right)$$

$$\text{P.I} = \frac{1}{2D^2 - DD' - D'^2 + 6D + 3D'} x e^y$$

$$= e^y \frac{1}{2D^2 - D(D' + 1) - (D' + 1)^2 + 6D + 3(D' + 1)} x$$

Replace D' by $D' + 1$

$$= e^y \frac{1}{2D^2 - DD' - D - D'^2 - 2D' - 1 + 6D + 3D' + 3} x$$

$$= e^y \frac{1}{2 + 5D + D' + 2D^2 - DD' - D'^2} x$$

$$= \frac{e^y}{2} \frac{1}{1 + \frac{1}{2}(5D + D' + 2D^2 - DD' - D'^2)} x$$

$$= \frac{e^y}{2}\left[1 + \frac{1}{2}(5D + D' + 2D^2 - DD' - D'^2)\right]^{-1} x$$

$$= \frac{e^y}{2}\left[1 - \frac{1}{2}[5D + D' + 2D^2 - DD' - D'^2]\right] x$$

$$= \frac{e^y}{2}\left[x - \frac{1}{2}[5(1) + 0 + 0]\right] = \frac{e^y}{2}\left[x - \frac{5}{2}\right]$$

$$= \frac{e^y}{4}[2x - 5]$$

$\therefore$ The General solution is $z = \text{C.F} + \text{P.I}$

$$= e^{-3x} f_1(y+x) + f_2\left(y - \frac{1}{2}x\right) + \frac{e^y}{4}(2x-5)$$

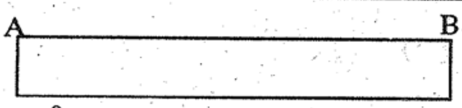
14. (a) The ends A and B of a rod 40 cm long have their temperatures kept at 0° C and 80°C respectively, until steady state condition prevails. The temperature of the end B is then suddenly reduced to 40°C and kept so, while that of the end A is kept is 0°C. Find the subsequent temperature distribution u (x, t) in the rod.

Solution : The equation to be solved in

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{... (A)}$$

Here, there are two steady states

$\therefore$ The solution may be $u(x,t) = u_s(x) + u_T(x,t)$ ... (B)

Steady state 1	Steady state 2
A B  x=0 u=0 x=40 u=80	A B  x=0 u=0 x=40 u=40
The steady state solution is $u = ax + b$ $x = 0 \Rightarrow 0 = 0 + b \Rightarrow \boxed{b = 0}$ $x = 40 \Rightarrow 80 = (a)\,40 + b$ $= 40\,(a)$ $\boxed{a = 2}$ $\therefore \; u = 2x + 0$ i.e., $u = 2x$ $u(x,0) = 2x$	The steady state solution is $u = ax + b$ $x = 0 \Rightarrow 0 = 0 + b \Rightarrow \boxed{b = 0}$ $x = 40 \Rightarrow 40 = (a)\,40 + b$ $= 40\,(a) + 0$ $\boxed{a = 1}$ $\therefore \; u = x + 0$ i.e., $u = x$ $u_s(x) = x$

$$\therefore \text{(B)} \Rightarrow u(x,t) = x + u_T(x,t) \quad \text{... (C)}$$

The boundary conditions are

(i) $u(0,t) = 0$, for all $t \geq 0$

(ii) $u(40,t) = 40$, for all $t \geq 0$

(iii) $u(x,0) = 2x$, for all x

Now, the suitable solution which satisfies our boundary conditions is given by

$$u(x,t) = x + (A\cos px + B\sin px)\, e^{-c^2p^2t} \quad \text{... (1)}$$

Applying condition (i) in (1), we get

$$0 = 0 + A\,e^{-c^2p^2t}$$

$$\Rightarrow \quad A\,e^{-c^2p^2t} = 0$$

$$e^{-c^2p^2t} \neq 0 \;[\because \text{ it is defined for all } t]$$

$$\Rightarrow \quad \boxed{A = 0}$$

Substitute A = 0 in (1), we get

$$u(x,t) = x + B\sin px\; e^{-c^2p^2t} \quad \text{... (2)}$$

Applying condition (ii) in (2), we get

$$40 = 40 + B\sin(40p)\; e^{-c^2p^2t}$$

$$\Rightarrow \quad B\sin(40p)\; e^{-c^2p^2t} = 0$$

$$e^{-c^2p^2t} \neq 0 \quad [\because \text{ it is defined for all } t]$$

$B \neq 0$ [$\because$ suppose $B = 0$, already $A = 0$ then we get a trivial solution]

$$\therefore \quad \sin 40p = 0$$

$$\Rightarrow \quad \sin 40p = \sin n\pi$$

$$40p = n\pi$$

$$\boxed{p = \frac{n\pi}{40}}$$

Substitute $p = \frac{n\pi}{40}$ in (2), we get

$$u(x,t) = x + B \sin\frac{n\pi x}{40}\, e^{\frac{-c^2 n^2 \pi^2 t}{1600}} \quad \ldots (3)$$

The most general solution is given by

$$u(x,t) = \sum_{n=1}^{\infty}\left[x + B_n \sin\frac{n\pi x}{40}\, e^{\frac{-c^2 n^2 \pi^2 t}{1600}}\right]$$

$$= x + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{40}\, e^{\frac{-c^2 n^2 \pi^2 t}{1600}} \quad \ldots (4)$$

Applying condition (iii) in (4), we get

$$2x = x + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{40}$$

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{40} = x = f(x)$$

To find B_n :

Expand $f(x)$ in a Half-range Fourier sine series in the interval $(0, l$

$$B_n = b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$B_n = \frac{2}{40}\int_0^{40} x \sin\frac{n\pi x}{40}\,dx$$

$$= \frac{1}{20}\left[(x)\left[\frac{-\cos\frac{n\pi x}{40}}{\left(\frac{n\pi}{40}\right)}\right] - (1)\left[\frac{-\sin\frac{n\pi x}{40}}{\left(\frac{n\pi}{40}\right)^2}\right]\right]_0^{40}$$

$$= \frac{1}{20}\left[-(x)\left(\frac{40}{n\pi}\right)\cos\frac{n\pi x}{40} + \left(\frac{40}{n\pi}\right)^2 \sin\frac{n\pi x}{40}\right]_0^{40}$$

$$= \frac{1}{20}\left[\left(-\frac{1600}{n\pi}(-1)^n + 0\right) - (-0+0)\right]$$

$$= -\frac{80}{n\pi}(-1)^n$$

$$\therefore (4) \Rightarrow u(x,t) = x + \sum_{n=1}^{\infty} \frac{-80}{\pi}(-1)^n \sin\frac{n\pi x}{40}\, e^{\frac{-c^2 n^2 \pi^2 t}{1600}}$$

$$= x - \frac{80}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\frac{n\pi x}{40}\, e^{\frac{-c^2 n^2 \pi^2 t}{1600}}$$

$$\text{(or)} \quad u(x,t) = x + \frac{80}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi x}{40}\, e^{\frac{-c^2 n^2 \pi^2 t}{1600}}$$

14. (b) A long rectangular plate with insulated surface is l cm wide. If the temperature along one short edge $(y = 0)$ is $u(x, 0) = k\,(lx - x^2)$ degrees, for $0 < x < l$, while the other two long edges $x = 0$ and $x = l$ as well as the other short edge are kept at 0°C, find the steady state temperature function $u(x, y)$.

Solution : The equation to be solved is $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

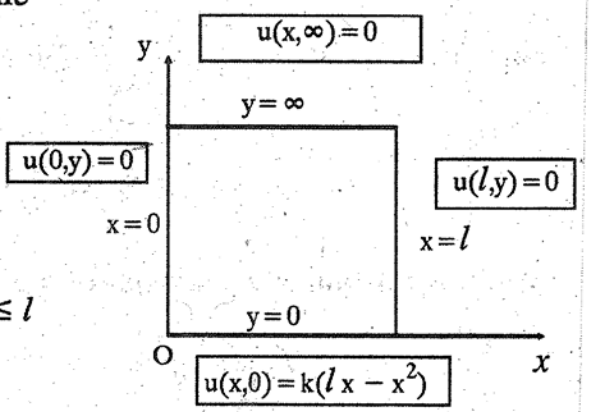
From the given problem, we get the following boundary conditions :

(i) $u(0, y) = 0, \; 0 \le y \le l$

(ii) $u(l, y) = 0, \; 0 \le y \le l$

(iii) $u(x, \infty) = 0, \; 0 \le x \le l$

(iv) $u(x, 0) = k\,(lx - x^2), \; 0 \le x \le l$



Now, the suitable solution which satisfies our boundary conditions is given by

$$u(x, y) = (A \cos px + B \sin px)(C e^{py} + D e^{-py}) \quad \dots (1)$$

Apply condition (i) in equation (1), we get

$$u(0, y) = A(C e^{py} + D e^{-py}) = 0$$

Here, $C e^{py} + D e^{-py} \neq 0$ [$\because$ it is defined for all y]

$$\therefore \boxed{A = 0}$$

Substitute $A = 0$ in (1), we get

$$u(x, y) = B \sin px (C e^{py} + D e^{-py}) \quad \dots (2)$$

Apply condition (ii) in equation (2), we get

$$u(l, y) = B \sin lp (C e^{py} + D e^{-py}) = 0$$

Here, $C e^{py} + D e^{-py} \neq 0$ [$\because$ it is defined for all y]

$B \neq 0$ [$\because$ If $B=0$ we already have $A=0$ then we get a trivial solution]

$$\therefore \sin lp = 0$$

$$\sin lp = \sin n\pi \quad [\because \sin n\pi = 0]$$

$$lp = n\pi$$

$$\boxed{p = \frac{n\pi}{l}}$$

Substitute $p = \frac{n\pi}{l}$ in equation (2), we get

$$u(x, y) = B \sin \frac{n\pi}{l} x \left(C e^{\frac{n\pi}{l} y} + D e^{\frac{-n\pi}{l} y}\right) \quad \dots (3)$$

Apply condition (iii) in equation (3), we get

$$u(x, \infty) = B \sin \frac{n\pi}{l} x [C e^{\infty} + D e^{-\infty}] = 0$$

$$B \sin \frac{n\pi}{l} x \, [C e^{\infty}] = 0 \qquad [\because e^{-\infty} = 0]$$

Here, $e^{\infty} \neq 0$

$$\sin \frac{n\pi x}{l} \neq 0 \qquad [\because \text{ it is defined for all } x]$$

$$B \neq 0 \qquad [\because \text{ we already explained}]$$

$$\Rightarrow \boxed{C = 0}$$

Substitute $C = 0$ in equation (3), we get

$$u(x, y) = B D \sin \frac{n\pi x}{l} e^{\frac{-n\pi}{l} y} \qquad \dots (4)$$

The most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} e^{\frac{-n\pi}{l} y} \qquad \dots (5)$$

Applying condition (iv) in equation (5), we get

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = k(lx - x^2) \qquad \dots (6)$$

To find B_n , expand $f(x)$ in a Fourier half range sine series

$$\therefore k(lx - x^2) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \qquad \dots (7)$$

$$\text{where } B_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

From (6) and (7) we get, $B_n = b_n$

$$\therefore B_n = \frac{2}{l} \int_0^l k(lx - x^2) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2k}{l} \int_0^l (lx - x^2) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2k}{l}\left[(lx-x^2)\left[\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)}\right]-(l-2x)\left[\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]+(-2)\left[\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}\right]\right]_0^l$$

$$= \frac{2k}{l}\left[-(lx-x^2)\left(\frac{l}{n\pi}\right)\cos\frac{n\pi x}{l}+(l-2x)\left(\frac{l}{n\pi}\right)^2\sin\frac{n\pi x}{l}-2\left(\frac{l}{n\pi}\right)^3\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{2k}{l}\left[\left(-0+0-2\left(\frac{l}{n\pi}\right)^3\cos n\pi\right)-\left(-0+0-2\left(\frac{l}{n\pi}\right)^3\right)\right]$$

$$= \frac{2k}{l}\left[-2\left(\frac{l}{n\pi}\right)^3(-1)^n+2\left(\frac{l}{n\pi}\right)^3\right]$$

$$= \frac{2k}{l}\,2\left(\frac{l}{n\pi}\right)^3[1-(-1)^n]$$

$$= \frac{2k}{l}\,\frac{2l^3}{n^3\pi^3}[1-(-1)^n]$$

$$= \frac{4kl^2}{n^3\pi^3}[1-(-1)^n]$$

$$= \begin{cases}\frac{8kl^2}{n^3\pi^3} & \text{if } n \text{ is odd}\\ 0 & \text{if } n \text{ is even}\end{cases}$$

Substitute the value of B_n in equation (5), we get

$$u(x, y) = \sum_{n=\text{odd}}^{\infty}\frac{8kl^2}{n^3\pi^3}\sin\frac{n\pi x}{l}\,e^{\frac{-n\pi}{l}y}$$

$$= \frac{8kl^2}{\pi^3}\sum_{n=\text{odd}}^{\infty}\frac{1}{n^3}\sin\frac{n\pi x}{l}\,e^{\frac{-n\pi}{l}y}$$

$$= \frac{8kl^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{l}\,e^{\frac{-(2n-1)\pi}{l}y}$$

15. (a) (i) Find Z [n (n − 1) (n − 2)]

Solution : $n\,(n-1)\,(n-2) = n\,[n^2 - 2n - n + 2]$

$$= n\,[n^2 - 3n + 2]$$

$$= n^3 - 3n^2 + 2n$$

$$Z\,[n\,(n-1)\,(n-2)] = Z\,[n^3 - 3n^2 + 2n]$$

$$= Z\,[n^3] - 3\,Z\,[n^2] + 2\,Z\,[n]$$

$$= \frac{z\,[z^2 + 4z + 1]}{(z-1)^4} - 3\,\frac{z\,(z+1)}{(z-1)^3} + 2\,\frac{z}{(z-1)^2}$$

$$= \frac{z^3 + 4z^2 + z - 3z\,(z+1)\,(z-1) + 2z\,(z-1)^2}{(z-1)^4}$$

$$= \frac{z^3 + 4z^2 + z - 3z\,(z^2-1) + 2z\,(z^2 - 2z + 1)}{(z-1)^4}$$

$$= \frac{z^3 + 4z^2 + z - 3z^3 + 3z + 2z^3 - 4z^2 + 2z}{(z-1)^4}$$

$$= \frac{6z}{(z-1)^4}$$

15. (a) (ii) Using Convolution theorem, find the inverse Z-transform of $\dfrac{8\,z^2}{(2z-1)\,(4z-1)}$

Solution : Given : $\dfrac{8z^2}{(2z-1)\,(4z-1)} = \dfrac{8z^2}{2\left(z-\frac{1}{2}\right) 4\left(z-\frac{1}{4}\right)}$

$$= \frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$$

$$Z^{-1}\left[\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\right] = Z^{-1}\left[\frac{z}{z-\frac{1}{2}}\cdot\frac{z}{z-\frac{1}{4}}\right]$$

$$= Z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right] * Z^{-1}\left[\frac{z}{z-\frac{1}{4}}\right]$$

$$= \left(\frac{1}{2}\right)^{n} * \left(\frac{1}{4}\right)^{n}$$

$$= \sum_{K=0}^{n}\left(\frac{1}{2}\right)^{n-K}\left(\frac{1}{4}\right)^{K}$$

$$= \left(\frac{1}{2}\right)^{n}\sum_{K=0}^{n}\left(\frac{1}{2}\right)^{-K}\left(\frac{1}{4}\right)^{K}$$

$$= \left(\frac{1}{2}\right)^{n}\sum_{K=0}^{n}\left(\frac{1}{2}\right)^{-K}\left(\frac{1}{2}\right)^{2K}$$

$$= \left(\frac{1}{2}\right)^{n}\sum_{K=0}^{n}\left(\frac{1}{2}\right)^{K}$$

$$= \left(\frac{1}{2}\right)^{n}\left[1+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^{2}+\ldots+\left(\frac{1}{2}\right)^{n}\right]$$

$$= \left(\frac{1}{2}\right)^{n}\left[\frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}}\right] \text{ by G.P.}$$

$$= \left(\frac{1}{2}\right)^{n}\left[\frac{1-\left(\frac{1}{2}\right)^{n+1}}{\frac{1}{2}}\right]$$

$$= \left(\frac{1}{2}\right)^{n-1} \left[1 - \left(\frac{1}{2}\right)^{n+1}\right]$$

$$= \left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{2}\right)^{2n}$$

15. (b) (i) Solve the difference equation y (k + 2) + y (k) = 1, y (0) = y (1) = 0, using Z-transform.

Solution : Given : $y(k+2) + y(k) = 1, y(0) = y(1) = 0$

$$\text{i.e., } y(n+2) + y(n) = 1$$

Taking Z-transform on both sides, we get

$$Z[y(n+2)] + Z[y(n)] = Z[1]$$

$$[z^2 Y(z) - z^2 y(0) - zy(1)] + Y(z) = \frac{z}{z-1}$$

$$z^2 Y(z) + Y(z) = \frac{z}{z-1} \qquad [\because y(0) = y(1) = 0]$$

$$(z^2+1) Y(z) = \frac{z}{z-1}$$

$$Y(z) = \frac{z}{(z-1)(z^2+1)}$$

$$Y(z)\, z^{n-1} = \frac{z}{(z-1)(z+i)(z-i)} z^{n-1}$$

$$= \frac{z^n}{(z-1)(z+i)(z-i)}$$

$z = 1$ is a simple pole.

$z = i$ is a simple pole.

$z = -i$ is a simple pole.

$$\operatorname*{Res}_{z=1} Y(z)\, z^{n-1} = \lim_{z\to 1} (z-1)\frac{z^n}{(z-1)(z+i)(z-i)}$$

$$= \lim_{z\to 1} \frac{z^n}{(z+i)(z-i)}$$

$$= \frac{1}{(1+i)(1-i)} = \frac{1}{2}$$

$$\operatorname*{Res}_{z=i} Y(z)\, z^{n-1} = \lim_{z\to i} (z-i)\frac{z^n}{(z-1)(z+i)(z-i)}$$

$$= \lim_{z\to i} \frac{z^n}{(z-1)(z+i)}$$

$$= \frac{(i)^n}{(i-1)(i+i)} = \frac{i^n}{2i(i-1)}$$

$$\operatorname*{Res}_{z=-i} Y(z)\, z^{n-1} = \lim_{z\to -i} (z+i)\frac{z^n}{(z-1)(z+i)(z-i)}$$

$$= \lim_{z\to -i} \frac{z^n}{(z-1)(z-i)}$$

$$= \frac{(-i)^n}{(-i-1)(-i-i)}$$

$$= \frac{(-i)^n}{(-i-1)(-2i)} = \frac{(-i)^n}{(2i)(1+i)}$$

$$\therefore\ y(n) = \text{sum of the residue}$$

$$= \frac{1}{2} + \frac{i^n}{2i(i-1)} + \frac{(-i)^n}{(2i)(1+i)}$$

$$= \frac{1}{2} + \frac{i^n}{2i}\left[\frac{1}{i-1} + \frac{(-1)^n}{1+i}\right]$$

$$= \frac{1}{2} + \frac{i^n}{2i}\left[\frac{(i+1)+(i-1)(-1)^n}{-1-1}\right]$$

$$= \frac{1}{2} - \frac{i^n}{4i}\left[(i+1) + (-1)^n (i-1)\right]$$

$$= \frac{1}{2} - \frac{1}{4} i^{n-1}\left[(i+1) + (-1)^n (i-1)\right]$$

15. (b) (ii) Solve $y_{n+2} + y_n = n \, . \, 2^n$, using Z-transform.

Solution : Given : $y_{n+2} + y_n = n2^n$

$$Z[y_{n+2}] + Z[y_n] = Z[2^n n]$$

$$\left[z^2 Y(z) - z^2 y(0) - zy(1)\right] + Y(z) = \frac{2z}{(z-2)^2} \qquad \left[\because Z[na^n] = \frac{az}{(z-a)^2}\right]$$

$$(z^2+1)\,Y(z) - z^2 S - zT = \frac{2z}{(z-2)^2}$$

when $S = y(0)$, $T = y(1)$

$$(z^2+1)\,Y(z) = \frac{2z}{(z-2)^2} + z^2 S + zT$$

$$(z^2+1)\,Y(z) = \frac{2z}{(z-2)^2} + Sz^2 + Tz$$

$$Y(z) = \frac{2z}{(z-2)^2(z^2+1)} + \frac{Sz^2}{z^2+1} + \frac{Tz}{z^2+1}$$

$$\frac{Y(z)}{z} = \frac{2}{(z-2)^2(z^2+1)} + \frac{Sz}{z^2+1} + \frac{T}{z^2+1} \qquad \dots (1)$$

Take
$$\frac{2}{(z-2)^2(z^2+1)} = \frac{A}{z-2} + \frac{B}{(z-2)^2} + \frac{Cz+D}{z^2+1} \qquad \dots (2)$$

$$2 = A(z-2)(z^2+1) + B(z^2+1) + (Cz+D)(z-2)^2$$

Put $z = 2$, we get equating z^3

$2 = 5B$

$$\boxed{B = \frac{2}{5}}$$

on bothsides, we get

$0 = A + C$

$A = -C$

(or) $C = -A$ ... (3)

put $z = 0$, we get

$2 = -2A + B + 4D$

$2 = -2A + \frac{2}{5} + 4D$

$2 - \frac{2}{5} = -2A + 4D$

$\frac{8}{5} = -2A + 4D$ ... (4)

equating z^2 on both sides, we get

$$0 = -2A + B - 4C + D$$

$$0 = -2A + \frac{2}{5} + 4A + D$$

$$-\frac{2}{5} = 2A + D \quad \text{... (5)}$$

$$(4) + (5) \Rightarrow \quad 5D = \frac{6}{5}$$

$$\boxed{D = \frac{6}{25}}$$

$$(5) \Rightarrow \quad 2A = \frac{-2}{5} - D$$

$$2A = -\frac{2}{5} - \frac{6}{25}$$

$$2A = -\frac{16}{25}$$

$$\boxed{A = -\frac{8}{25}}$$

$$(3) \Rightarrow \quad C = -A = \frac{8}{25}$$

$$\boxed{C = \frac{8}{25}}$$

$$\therefore (2) \Rightarrow \frac{2}{(z-2)^2(z^2+1)} = -\frac{8}{25}\left(\frac{1}{z-2}\right) + \frac{2}{5}\frac{1}{(z-2)^2} + \frac{\frac{8}{25}z + \frac{6}{25}}{z^2+1}$$

$$(1) \Rightarrow \frac{Y(z)}{z} = -\frac{8}{25}\left(\frac{1}{z-2}\right)+\frac{2}{5}\frac{1}{(z-2)^2}+\frac{8}{25}\left(\frac{z}{z^2+1}\right)+\frac{6}{25}\left(\frac{1}{z^2+1}\right)$$
$$+ S\left(\frac{z}{z^2+1}\right)+T\left(\frac{1}{z^2+1}\right)$$

$$\frac{Y(z)}{z} = -\frac{8}{25}\left(\frac{1}{z-2}\right)+\frac{2}{5}\frac{1}{(z-2)^2}+\left(\frac{8}{25}+S\right)\left(\frac{z}{z^2+1}\right)$$
$$+\left(\frac{6}{25}+T\right)\left(\frac{1}{z^2+1}\right)$$

$$Y(z) = -\frac{8}{25}\left(\frac{z}{z-2}\right)+\frac{2}{5}\left[\frac{z}{(z-2)^2}\right]+\left(\frac{8}{25}+S\right)\left(\frac{z^2}{z^2+1}\right)$$
$$+\left(\frac{6}{25}+T\right)\left(\frac{z}{z^2+1}\right)$$

$$Z[y(n)] = -\frac{8}{25}\left(\frac{z}{z-2}\right)+\frac{2}{5}\left[\frac{z}{(z-2)^2}\right]+\left(\frac{8}{25}+S\right)\left(\frac{z^2}{z^2+1}\right)$$
$$+\left(\frac{6}{25}+T\right)\left(\frac{z}{z^2+1}\right)$$

$$y(n) = -\frac{8}{25}Z^{-1}\left[\frac{z}{z-2}\right]+\frac{1}{5}Z^{-1}\left[\frac{2z}{(z-2)^2}\right]$$
$$+\left(\frac{8}{25}+S\right)Z^{-1}\left(\frac{z^2}{z^2+1}\right)+\left(\frac{6}{25}+T\right)Z^{-1}\left(\frac{z}{z^2+1}\right)$$
$$= -\frac{8}{25}(2)^n+\frac{1}{5}2^n n+E\cos\frac{n\pi}{2}+F\sin\frac{n\pi}{2}$$

Formula : (1) $Z[a^n] = \frac{z}{z-a}$ (2) $Z[a^n n] = \frac{az}{(z-a)^2}$

(3) $Z\left[\cos\frac{n\pi}{2}\right] = \frac{z^2}{z^2+1}$ (4) $Z\left[\sin\frac{n\pi}{2}\right] = \frac{z}{z^2+1}$

Note : In this problem $y(0)$ & $y(1)$ are not given.