ANNA UNIVERSITY EXAMINATION, NOVEMBER/DECEMBER 2013

PART - A (10 × 2 = 20 marks)

1. Find the value of a_0 in the Fourier Series expansion of $f(x) = e^x$ in $(0, 2\pi)$.

Solution : The fourier series in the interval $(0, 2\pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} e^x\, dx$$

$$= \frac{1}{\pi} \left[\frac{e^x}{1}\right]_0^{2\pi}$$

$$= \frac{1}{\pi} [e^{2\pi} - e^0]$$

$$a_0 = \frac{e^{2\pi} - 1}{\pi} \qquad [\because e^0 = 1]$$

2. Find the half range sine series expansion of $f(x) = 1$ in $(0, 2)$.

Solution : The half range sine series in the interval $(0, l)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$b_n = \int_0^l f(x) \quad \sin \frac{n\pi x}{l} dx$$

$$l = \text{U.L} - \text{L.L}$$

$$= 2 - 0 = 2$$

$$b_n = \int_0^2 1 . \sin \frac{n\pi x}{2} dx$$

$$= \int_0^2 \sin \frac{n\pi x}{2} dx$$

$$= \left[\frac{-\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}} \right]_0^2$$

$$= -\frac{2}{n\pi} \left[\cos \frac{n\pi x}{2} \right]_0^2$$

$$b_n = -\frac{2}{n\pi} \left[\cos \frac{2n\pi}{2} - \cos 0 \right]$$

$$= -\frac{2}{n\pi} [(-1)^n - 1]$$

Case (i) :

When n is odd number.

$$b_n = -\frac{2}{n\pi} [-1 - 1]$$

$$= -\frac{2}{n\pi} (-2)$$

$$= \frac{4}{n\pi}$$

Case (ii) :

When n is even number

$$b_n = -\frac{2}{n\pi}[1-1]$$

$$= 0$$

$$\therefore f(x) = \sum_{n=1,3,5}^{\infty} \frac{4}{n\pi} \sin\frac{n\pi x}{2}$$

$$= \frac{4}{\pi} \sum_{n=1,3\ldots}^{\infty} \frac{1}{n} \sin\frac{n\pi x}{2}$$

3. Define self reciprocal with respect to Fourier Transform.

Solution :

A function $f(x)$ is said to be self reciprocal if the Fourier Transform F (S) is obtained by replacing x by s.

4. Prove that $F[f(x-a)] = e^{ias} F[f(x)]$.

Proof : By definition,

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)\, e^{isx}\, dx$$

Putting $x - a = t$ then

Differentiating $dx = dt$

Also $x = t + a$

$$\therefore F[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is(t+a)}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty} f(t)\, e^{ist}\, e^{isa}\, dt$$

$$= e^{isa} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt$$

$$= e^{isa} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

[Changing the dummy variable 't' to 'x']

$$F\,[f(t)] = e^{isa}\, F\,[f(x)]$$

Hence proved.

5. Form a PDE by eliminating the arbitrary constants 'a' and 'b' from $z = ax^2 + by^2$.

Solution : Let $z^2 = ax^2 + by^2$ ... (1)

Differentiating (1) partially w.r.t x

$$2z \frac{\partial z}{\partial x} = 2ax$$

$$\therefore zp = ax$$

$$a = \frac{zp}{x}$$

Differentiating (1) partially w.r.t y

$$2z \frac{\partial z}{\partial y} = 2by$$

$$\therefore zq = by$$

$$b = \frac{zq}{y}$$

Substituting for a and b in (1)

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$$z^2 = \frac{zp}{x}x^2 + \frac{zq}{y}y^2$$

Cacnelling z on both sides

$$z = px + qy$$

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6. Solve : $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} = 0.$

Solution : $D^2 z - DD' z + D z = 0$

$$(D^2 - DD' + D) z = 0$$

$$D (D - D' + 1) z = 0$$

C.F is given by $e^{\alpha_1 x} f_1 (y + m_1 x) + e^{\alpha_2 x} f_2 (y + m_2 x)$

Here $m_1 = 0, \quad \alpha_1 = 0$ and $m_2 = 1, \quad \alpha_2 = -1$

$$\therefore z = e^{0x} f_1 (y + 0x) + e^{-x} f_2 (y + x)$$

$$= f_1 (y) + e^{-x} f_2 (y + x)$$

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7. Define steady state condition on heat flow.

Solution :

The condition at which the temperature remains constant when the time increases is called steady state condition on heat flow.

The temperature function $u (x, t)$ will be a function of x alone. In the steady state condition. $\frac{\partial u}{\partial t} = 0$ under steady state condition.

$\therefore$ The heat flow equation becomes,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial u}{\partial t}$$

$$= \frac{1}{\alpha^2}(0) = 0$$

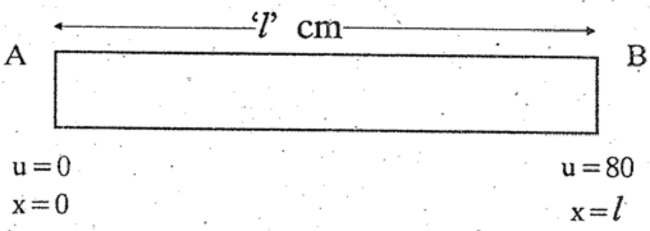
8. An insulated rod of length l cm has its ends A and B maintained at 0° C and 80° C respectively. Find the steady state condition of the rod.

Solution :

In steady state condition, the temperature is given by

$$u = ax + b \quad \text{... (1)}$$

Now to find the values of a and b apply the conditions from figure.



Applying the conditions at the end A in (1)

$$u = ax + b$$

we get,

$$0 = a(0) + b$$

$$b = 0$$

Applying the conditions at the end B we get,

$$80 = a(l) + 0$$

$$80 = al$$

$$a = \frac{80}{l}$$

Substituting for a and b we get

$u = \frac{80}{l} x$ is the steady solution of the rod.

9. $Z\left[\frac{1}{n}\right]$

Solution :

$$Z\left[\frac{1}{n}\right] = \sum_{n=0}^{\infty} \frac{1}{n} z^{-n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} z^{-n} \qquad [n \text{ cannot be } 0]$$

$$= z^{-1} + \frac{1}{2} z^{-2} + \frac{1}{3} z^{-3} + \ldots.$$

$$= \frac{1}{z} + \frac{1}{2}\frac{1}{z^2} + \frac{1}{3}\frac{1}{z^3} + \ldots$$

$$= \frac{1}{z} + \frac{\left(\frac{1}{z}\right)^2}{2} + \frac{\left(\frac{1}{z}\right)^3}{3} + \ldots$$

$$= -\log\left(1 - \frac{1}{z}\right)$$

$$\left[-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right]$$

$$= -\log\left(\frac{z-1}{z}\right)$$

$$= \log\frac{z}{z-1}$$

10. **Find the inverse Z transform of** $\frac{z}{(z+1)^2}$.

Solution :

$$Z^{-1}\left[\frac{z}{(z+1)^2}\right] = n(-1)^{n-1}$$

by formula, $Z^{-1}\left[\frac{z}{(z+a)^2}\right] = n(-a)^{n-1}$

PART - B (5 × 16 = 80 marks)

11. (a) (i) Find the Fourier series of period 2π for the function $f(x) = \begin{cases} 1 & \text{in } (0, \pi) \\ 2 & \text{in } (\pi, 2\pi) \end{cases}$ and hence find the sum of series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \infty$.

Solution :

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \dots (1)$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) \, . \, dx$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} f(x)\, dx + \int_{\pi}^{2\pi} f(x)\, dx \right]$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} 1\, dx + \int_{\pi}^{2\pi} 2 \, . \, dx \right]$$

$$= \frac{1}{\pi} \left[(x)_0^{\pi} + (2x)_{\pi}^{2\pi} \right]$$

$$= \frac{1}{\pi} \left[(\pi - 0) + 2(2\pi - \pi) \right]$$

$$= \frac{1}{\pi} [\pi + 2\pi]$$

$$= \frac{3\pi}{\pi} = 3$$

$$\therefore a_0 = 3$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \, . \cos nx \, . \, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x)\cos nx\,.\,dx + \int_{\pi}^{2\pi} f(x)\cos nx\,.\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} 1\,.\cos nx\,.\,dx + \int_{\pi}^{2\pi} 2\,.\cos nx\,.\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\sin nx}{n}\right)_0^{\pi} + \left(\frac{2\,.\sin nx}{n}\right)_{\pi}^{2\pi}\right]$$

$$[\because \sin n\pi = 0,\ \sin 2n\pi = 0\ \ \sin 0 = 0]$$

$$= \frac{1}{\pi}[0 + 0] = 0$$

$$\therefore a_n = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\,.\sin nx\,.\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x)\sin nx\,.\,dx + \int_{\pi}^{2\pi} f(x)\sin nx\,.\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} 1\,.\sin nx\,.\,dx + \int_{\pi}^{2\pi} 2\,.\sin nx\,.\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{-\cos nx}{n}\right)_0^{\pi} + \left(\frac{-2\,.\cos nx}{n}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{-1}{n\pi}\left[(\cos nx)_0^{\pi} + (2\cos nx)_{\pi}^{2\pi}\right]$$

$$= \frac{-1}{n\pi}\left[(\cos n\pi - \cos 0) + 2(\cos 2n\pi - \cos n\pi)\right]$$

$$= \frac{-1}{n\pi}\left[1(-1)^n - 1) + 2(1 - (-1)^n\right]$$

$$= \frac{-1}{n\pi}\left[(-1)^n - 1 + 2 - 2(-1)^n\right]$$

$$= \frac{-1}{n\pi}\left[1 - (-1)^n\right]$$

Case 1 : When n is odd

$$b_n = \frac{-1}{n\pi}[1+1] = \frac{-2}{n\pi}$$

Case 2 : When n is even

$$b_n = \frac{-1}{\pi}[1-1] = 0$$

$$\therefore\ b_n = \begin{cases} \dfrac{-2}{n\pi}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

Substitute a_0, a_n, b_n in (1) we get

$$f(x) = \frac{3}{2} + \sum_{n=1,3,\ldots}^{\infty} \frac{-2}{n\pi}\sin nx$$

$$= \frac{3}{2} - \frac{2}{\pi}\sum_{n=1,3}^{\infty}\frac{\sin nx}{n}$$

$$= \frac{3}{2} - \frac{2}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right] \quad \ldots (2)$$

By substituting $x = \dfrac{\pi}{2}$ we can get the required series,

When $x = \dfrac{\pi}{2}$, $f(x) = 1$

$$\therefore\ 1 = \frac{3}{2} - \frac{2}{\pi}\left[\frac{\sin\frac{\pi}{2}}{1} + \frac{\sin\frac{3\pi}{2}}{3} + \frac{\sin\frac{5\pi}{2}}{5} + \ldots\right]$$

$$1 = \frac{3}{2} - \frac{2}{\pi}\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{5} + \ldots\right]$$

$$1 - \frac{3}{2} = \frac{-2}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} + \ldots\right] \qquad \sin\frac{\pi}{2} = 1;\ \sin\frac{3\pi}{2} = -1$$

$$\frac{-1}{2} = \frac{-2}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} + \ldots\right] \qquad \sin\frac{5\pi}{2} = 1$$

Cross multiplying,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + \ldots$$

11. (a) (ii) Find the Fourier series expansion of

$$f(x) = \begin{cases} -x+1, & -\pi < x < 0 \\ x+1, & 0 < x < \pi \end{cases}$$

11. (b) (i) Find the half range sine series of $f(x) = lx - x^2$ in $(0, l)$.

Solution : The half range sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l} \int_0^l (lx - x^2) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[\underbrace{(lx-x^2)\left[\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]}_{0} - \underbrace{(l-2x)\left[\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right]}_{0} + (-2)\left[\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right]\right]_0^l$$

$$= \frac{2}{l}\left[\frac{-2}{\frac{n^3\pi^3}{l^3}}\left(\cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{-4l^3}{n^3\pi^3 l}\left[\cos n\pi - \cos 0\right]$$

$$= -\frac{4l^2}{n^3\pi^3}\left[(-1)^n - 1\right]$$

Case (i) :

When n is even number

$$(-1)^n = 1$$

$$\therefore b_n = -\frac{4l^2}{n^3 \pi^3}[1-1]$$

$$= 0$$

Case (ii) :

When n is odd number

$$(-1)^n = -1$$

$$b_n = \frac{-4l^2}{n^3 \pi^3}(-1-1)$$

$$b_n = \frac{8l^2}{n^3 \pi^3}$$

$$f(x) = \sum_{n=1,3,5\ldots}^{\infty} \frac{8l^2}{n^3 \pi^3} \sin \frac{n \pi x}{l}$$

$$f(x) = \frac{8l^2}{\pi^3} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^3} \sin \frac{n \pi x}{l}$$

11. (b) (ii) Find the first three harmonic of function $f(x)$ of period 2π defined by means of the following table

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
y	1	1.4	1.9	1.7	1.5	1.2	1

Solution :

Here first and last y values are repeated

x	y	$\cos x$	$y \cos x$	$\cos 2x$	$y \cos 2x$	$\cos 3x$	$y \cos 3x$	$\sin x$	$y \sin x$	$\sin 2x$	$y \sin 2x$	$\sin 3x$	$y \sin 3x$
0	1	1	1	1	1	1	1	0	0	0	0	0	0
60	1.4	0.5	0.7	–0.5	–0.7	–1	–1.4	0.866	1.212	0.866	1.212	0	0
120	1.9	–0.5	–0.95	–0.5	–0.95	1	1.9	0.866	1.645	–0.866	–1.645	0	0
180	1.7	–1	–1.7	1	1.7	–1	–1.7	0	0	0	0	0	
240	1.5	–0.5	–0.75	–0.5	–0.75	1	1.5	–0.866	–1.299	0.866	1.299	0	0
300	1.2	0.5	0.6	–0.5	–0.6	–1	–1.2	–0.866	–1.039	–0.866	–1.093	0	0
	$\Sigma y = 8.7$		$\Sigma y \cos x = -1.1$		$\Sigma \cos 2x = -0.3$		$\Sigma \cos 3x = 0.1$		$\Sigma \sin x = 0.519$		$\Sigma y \sin 2x = -0.173$		$\Sigma y \sin 3x = 0$

∴ we can omit the last value.

∴ Hence $n = 6$

$$a_0 = \frac{2}{n}\Sigma y = \frac{2}{6} \times 8.7 = 2.900$$

$$a_1 = \frac{2}{n}\Sigma y \cos x = \frac{2}{6} \times (-1.1) = -0.366$$

$$a_2 = \frac{2}{n}\Sigma y \cos 2x = \frac{2}{6} \times (-0.3) = -0.1$$

$$a_3 = \frac{2}{n}\Sigma y \cos 3x = \frac{2}{6} \times (0.1) = 0.033$$

$$b_1 = \frac{2}{n}\Sigma y \sin x = \frac{2}{6} \times (0.5196) = 0.1732$$

$$b_2 = \frac{2}{n}\Sigma y \sin 2x = \frac{2}{6} \times (-0.1736) = -0.0578$$

$$b_3 = \frac{2}{n}\Sigma y \sin 3x = \frac{2}{6} \times (0) = 0$$

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x$$

$$= \frac{2.9}{2} - 0.366 \cos x - 0.1 \cos 2x + 0.033 \cos 3x + 0.1732 \sin x - 0.0578 \sin 2x + 0$$

$$f(x) = 1.45 - 0.366 \cos x - 0.1 \cos 2x + 0.033 \cos 3x + 0.1732 \sin x - 0.0578 \sin 2x$$

12. (a) Find the Fourier transform of $f(x) = \begin{cases} 1 - x^2, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$

(i) $\int_0^{\infty} \frac{\sin s - s\cos s}{s^3} \cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{16}$.

(ii) $\int_{0}^{\infty} \frac{(x \cos x - \sin x)^2}{x^6} dx = \frac{\pi}{15}$

Solution :

$$F[f(x)] = \frac{4}{\sqrt{2\pi}} \left[\frac{\sin s - s \cos s}{s^3}\right]$$

Now applying perseval's identity

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

$$\int_{-1}^{1} (1 - x^2)^2 dx = \int_{-\infty}^{\infty} \left[\frac{4}{\sqrt{2\pi}} \frac{\sin s - s \cos s}{s^3}\right]^2 ds$$

$$2\int_{0}^{1} (1 - x^2)^2 dx = \frac{16}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s \cos s}{s^3}\right)^2 ds$$

$$2\int_{0}^{1} (1 - 2x^2 + x^4) dx = \frac{8}{\pi} \int_{-\infty}^{\infty} \frac{(\sin s - s \cos s)^2}{s^6} ds$$

$$2\left[x - 2\frac{x^3}{3} + \frac{x^5}{5}\right]_0^1 = \frac{8}{\pi} \int_{-\infty}^{\infty} \frac{(s \cos s - \sin s)^2}{s^6} ds$$

$$2\left[1 - \frac{2}{3} + \frac{1}{5}\right] = \frac{8}{\pi} \int_{-\infty}^{\infty} \frac{(s \cos s - \sin s)^2}{s^6} ds$$

$$2\left[\frac{15-10+3}{15}\right] = \frac{8}{\pi}\int_{-\infty}^{\infty}\frac{(s\cos s-\sin s)^2}{s^6}\,ds$$

$$2\left(\frac{8}{15}\right) = \frac{8}{\pi}\int_{-\infty}^{\infty}\frac{(s\cos s-\sin s)^2}{s^6}\,ds$$

$$\frac{2\pi}{15} = \int_{-\infty}^{\infty}\frac{(s\cos s-\sin s)^2}{s^6}\,ds$$

$$\frac{2\pi}{15} = 2\int_{0}^{\infty}\frac{(s\cos s-\sin s)^2}{s^6}\,ds$$

$$\frac{\pi}{15} = \int_{0}^{\infty}\frac{(s\cos s-\sin s)^2}{s^6}\,ds$$

Changing the variable 's' to 'x' we get

$$\int_{0}^{\infty}\frac{(x\cos x-\sin x)^2}{x^6}\,dx = \frac{\pi}{15}$$

(b) (i) Using Fourier cosine transform, evaluate $\int_{0}^{\infty}\frac{dx}{(x^2+a^2)^2}$

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13. (a) (i) Form the PDE by eliminating the arbitrary function 'f' and 'g' from $z = x^2 f(y) + y^2 g(x)$.

Solution :

Given : $z = x^2 f(y) + y^2 g(x)$ ... (1)

Here, there are two arbitrary functions.

We have to find the second order partial derivatives also.

Differentiating (1) partially w.r.t. 'x'

$$\frac{\partial z}{\partial x} = 2xf(y) + y^2 g'(x) \quad \text{... (2)}$$

Differentiating (2) partially w.r.t. 'x'

$$\frac{\partial^2 z}{\partial x^2} = 2f(y) + y^2 g''(x) \quad \text{... (3)}$$

Differentiating (1) partially w.r.t. 'y'

$$\frac{\partial z}{\partial y} = x^2 f'(y) + 2yg(x) \quad \text{... (4)}$$

Differentiating (4) partially w.r.t. 'y'

$$\frac{\partial^2 z}{\partial y^2} = x^2 f''(y) + 2g(x) \quad \text{... (5)}$$

Differentiating (2) partially w.r.t. 'y'

$$\frac{\partial^2 z}{\partial x \partial y} = 2xf'(y) + 2yg'(x) \quad \text{... (6)}$$

From (4)

$$x^2 f'(y) = \frac{\partial z}{\partial y} - 2yg(x)$$

$$f'(y) = \frac{\frac{\partial z}{\partial y} - 2yg(x)}{x^2} \quad \text{... (7)}$$

From (2),

$$y^2 g'(x) = \frac{\partial z}{\partial x} - 2x f(y)$$

$$g'(x) = \frac{\frac{\partial z}{\partial x} - 2x f(y)}{y^2} \qquad \dots (8)$$

Substituting $f'(y)$ and $g'(x)$ in (6)

$$\frac{\partial^2 z}{\partial x \partial y} = 2x \left[\frac{\frac{\partial z}{\partial y} - 2y g(x)}{x^2}\right] + 2y \left[\frac{\frac{\partial z}{\partial x} - 2x f(y)}{y^2}\right]$$

$$= 2 \left[\frac{\frac{\partial z}{\partial y} - 2y g(x)}{x}\right] + 2 \left[\frac{\frac{\partial z}{\partial x} - 2x f(y)}{y}\right]$$

$$= 2 \left[\frac{\left(\frac{\partial z}{\partial y} - 2y g(x)\right)}{x} + \frac{\left(\frac{\partial z}{\partial x} - 2x f(y)\right)}{y}\right]$$

$$= 2 \left[\frac{y \left(\frac{\partial z}{\partial y} - 2y g(x)\right) + x \left(\frac{\partial z}{\partial x} - 2x f(y)\right)}{xy}\right]$$

$$= 2 \left[\frac{y \frac{\partial z}{\partial y} - 2y^2 g(x) + x \frac{\partial z}{\partial x} - 2x^2 f(y)}{xy}\right]$$

$$\frac{\partial^2 z}{\partial x \partial y} = 2 \left[\frac{y \frac{\partial z}{\partial y} + x \frac{\partial z}{\partial x} - 2 [y^2 g(x) + x^2 f(y)]}{xy}\right]$$

Cross multiplying,

$$xy \frac{\partial^2 z}{\partial x \partial y} = 2 \left[y \frac{\partial z}{\partial y} + x \frac{\partial z}{\partial x} - 2z\right]$$

[since $z = y^2 g(x) + x^2 f(y)$]

13. (a) (ii) Solve $[D^2 - DD' - 2D'^2]\, z = 2x + 3y + e^{2x + 4y}$

Refer to Page No. 3.36

$\therefore$ C.F is $z = f_1 (y + 2x) + f_2 (y - x)$

$$P.I_1 = \frac{5x^3}{6} + \frac{3yx^2}{2}$$

$$P.I_2 = \frac{1}{D^2 - DD' - 2D'^2} e^{2x + 4y}$$

$$= \frac{1}{(2)^2 - (2)(4) - 2(4)^2} e^{2x + 4y}$$

[Replacing D by 2, D′ by 4]

$$= \frac{1}{4 - 8 - 32} e^{2x + 4y}$$

$$P.I_2 = \frac{1}{-36} e^{2x + 4y}$$

$\therefore$ The solution is $Z = C.F + P.I_1 + P.I_2$

$$Z = f_1 (y + 2x) + f_2 (y - x) + \frac{5x^3}{6} + \frac{3yx^2}{2} - \frac{1}{36} e^{2x + 4y}$$

13. (b) (i) $(y^2 + z^2)\, p - xyq + xz = 0$

Solution :

$$(y^2 + z^2)\, p - xyq = -xz$$

$$Pp + Q\, q = R$$

$$\frac{dx}{(y^2 + z^2)} = \frac{dy}{-xy} = \frac{dz}{-xz}$$

Taking the multipliers as x, y, z

$$\frac{x\, dx}{x (y^2 + z^2)} = \frac{y\, dy}{y (-xy)} = \frac{z\, dz}{z (-xz)}$$

$$\frac{x\, dx}{xy^2 + xz^2} = \frac{y\, dy}{-xy^2} - \frac{z\, dz}{-z^2 x}$$

Each ratio equal to,

$$\frac{x\, dx + y\, dy + z\, dz}{xy^2 + xz^2 - xy^2 - xz^2}$$

$$\frac{x\,dx + y\,dy + z\,dz}{0}$$

$$\Rightarrow \quad x\,dx + y\,dy + z\,dz = 0$$

Integrating : $\int x\,dx + \int y\,dy + \int z\,dz = 0$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_1$$

Consider, $\dfrac{dy}{-xy} = \dfrac{dz}{-xz}$

Cancelling $-x$ on both sides

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating both sides

$$\log y = \log z + \log c_2$$

$$\log y - \log z = \log c_2$$

$$\log\left(\frac{y}{z}\right) = \log c_2$$

$$\therefore \frac{y}{z} = c_2$$

Hence the solution is $\phi\ (c_1, c_2) = 0$

$$\text{(i.e.,)}\ \phi\left(\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}, \frac{y}{z}\right) = 0$$

(b) (ii) Find the singular interval of $z = px + qy + \sqrt{1 + p^2 + q^2}$.

Solution :

$$z = px + qy + c\sqrt{1 + p^2 + q^2} \quad \ldots (1)$$

Let $z = ax + by + c$ ... (2) be the solution

Differentiating (2) partially w.r.t x,

$$\frac{\partial z}{\partial x} = a$$

$$\therefore\ p = a$$

Differentiating (2) partially w.r.t y,

$$\frac{\partial z}{\partial y} = b$$

$$\therefore\ q = b$$

Substituting for p and q in (1), we get the complete integral as

$$z = ax + by + c\sqrt{1 + a^2 + b^2} \qquad \dots (3)$$

Differentiating (3) partially w.r.t a

$$\frac{\partial z}{\partial a} = x + \frac{c}{2\sqrt{(1 + a^2 + b^2)}} \cdot 2a$$

$$\frac{\partial z}{\partial a} = 0 \Rightarrow x + \frac{ac}{\sqrt{1 + a^2 + b^2}} = 0$$

$$\therefore\ x = \frac{-ac}{\sqrt{1 + a^2 + b^2}} \qquad \dots (4)$$

Differentiating (3) partially w.r.t b,

$$\frac{\partial z}{\partial b} = y + \frac{c}{2\sqrt{1 + a^2 + b^2}} \cdot 2b$$

$$= y + \frac{bc}{\sqrt{1 + a^2 + b^2}} = 0$$

$$\therefore\ y = \frac{-bc}{\sqrt{1 + a^2 + b^2}} \qquad \dots (5)$$

$$x^2 = \frac{a^2 c^2}{1 + a^2 + b^2}$$

$$y^2 = \frac{b^2 c^2}{1 + a^2 + b^2}$$

Consider $$x^2 + y^2 = \frac{a^2 c^2}{1 + a^2 + b^2} + \frac{b^2 c^2}{1 + a^2 + b^2}$$

$$= \frac{(a^2 + b^2) c^2}{1 + a^2 + b^2}.$$

$$c^2 - (x^2 + y^2) = c^2 - \left(\frac{a^2 + b^2}{1 + a^2 + b^2}\right) c^2$$

$$= c^2 \left(1 - \frac{(a^2 + b^2)}{1 + a^2 + b^2}\right)$$

$$= c^2 \frac{[(1 + a^2 + b^2) - (a^2 + b^2)]}{1 + a^2 + b^2}$$

$$c^2 - x^2 - y^2 = \frac{c^2}{1 + a^2 + b^2}$$

$$1 + a^2 + b^2 = \frac{c^2}{c^2 - x^2 - y^2}$$

$$\therefore \sqrt{1 + a^2 + b^2} = \frac{c}{\sqrt{c^2 - x^2 - y^2}} \quad \ldots (6)$$

Using (4) and (5) we can find a and b

From (4)

$$a = \frac{-x}{c} \sqrt{1 + a^2 + b^2}$$

$$= \frac{-x}{c} \cdot \frac{c}{\sqrt{c^2 - x^2 - y^2}} \quad \text{[Using (6)]}$$

$$= \frac{-x}{\sqrt{c^2 - x^2 - y^2}}$$

From (5)

$$b = \frac{-y}{c}\sqrt{1+a^2+b^2}$$

$$= \frac{-y}{c}\,\frac{c}{\sqrt{c^2-x^2-y^2}}$$

$$= \frac{-y}{\sqrt{c^2-x^2-y^2}}$$

Now substituting for a, b, $\sqrt{1+a^2+b^2}$ in (3), we get

$$z = \frac{-x}{\sqrt{(c^2-x^2-y^2}}\,x - \frac{y}{\sqrt{c^2-x^2-y^2}}\,y + \frac{c}{\sqrt{c^2-x^2-y^2}}\,c$$

$$= \frac{-x^2}{\sqrt{c^2-x^2-y^2}} - \frac{y^2}{\sqrt{c^2-x^2-y^2}} + \frac{c^2}{\sqrt{c^2-x^2-y^2}}$$

$$= \frac{c^2-x^2-y^2}{\sqrt{c^2-x^2-y^2}}$$

$$= \sqrt{c^2-x^2-y^2}$$

Squaring on both sides,

$$z^2 = c^2-x^2-y^2$$

$$\therefore\ x^2+y^2+z^2 = c^2$$

This is the singular integral.

14. (a) A tightly stretched string with fixed end points x = 0 and x = l is initially in a position given by y (x, 0) = $k\,(lx - x^2)$. It is released from rest from this position. Find the expression for the displacement at any time 't'.

Solution : The wave equation is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\,\frac{\partial^2 y}{\partial t^2}$$

The solution is

$$y = (A \cos px + B \sin px)(C \cos pct + D \sin pct) \quad \dots (1)$$

We have the following boundary conditions.

1. $y = 0$ when $x = 0$
2. $y = 0$ when $x = l$
3. $\frac{\partial y}{\partial t} = 0$ when $t = 0$ ($\because$ initial velocity is zero)
4. $y = \lambda x(l - x)$ when $t = 0$ (initial shape is given)

Applying boundary conditions 1 in (1)

$$0 = (A \cos 0 + B \sin 0)(C \cos pct + D \sin pct)$$

$$0 = (A + 0)(C \cos pct + D \sin pct) \qquad [\cos 0 = 1, \sin 0 = 0]$$

$$\Rightarrow \quad A = 0 \qquad (\because C \cos pct + D \sin pct \neq 0$$

otherwise y becomes 0)

Substituting $A = 0$ in (1)

Solution (1) becomes

$$y = B \sin px (C \cos pct + D \sin pct) \quad \dots (2)$$

Applying condition 2 in (2)

$$0 = B \sin pl (C \cos pct + D \sin pct)$$

$$\Rightarrow \sin pl = 0 \qquad (\because B \neq 0)$$

But $\quad \sin n\pi = 0$

$$\therefore pl = n\pi$$

$$p = \frac{n\pi}{l}$$

Substitute $p = \frac{n\pi}{l}$ in (2)

(2) becomes

$$y = B \sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi ct}{l} + D\sin\frac{n\pi ct}{l}\right) \quad \ldots (3)$$

To apply condition 3. differentiate (3) partially w.r.t 't'

$$\frac{\partial y}{\partial t} = B\sin\frac{n\pi x}{l}\left(-C\sin\left(\frac{n\pi ct}{l}\right)\frac{n\pi c}{l} + D\cos\left(\frac{n\pi ct}{l}\right)\frac{n\pi c}{l}\right)$$

Now applying condition 3. we get

$$0 = B\sin\frac{n\pi x}{l}\left(-C\sin 0\frac{n\pi c}{l} + D\cos 0\frac{n\pi c}{l}\right)$$

$$0 = B\sin\frac{n\pi x}{l}\left(0 + D\frac{n\pi c}{l}\right) \qquad [\sin 0 = 0, \cos 0 = 1]$$

$$\Rightarrow \quad D = 0$$

Substitute D = 0 in (3)

$$y = B\sin\frac{n\pi x}{l}\,C\cos\frac{n\pi ct}{l}$$

$$= BC\sin\frac{n\pi x}{l}\,.\,\cos\frac{n\pi ct}{l}$$

$$= b\sin\frac{n\pi x}{l}\,.\,\cos\frac{n\pi ct}{l} \qquad (\text{taking } BC = b)$$

Using principle of superposition the general solution is written as

$$y = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \quad \ldots (4)$$

Applying condition 4. in (4)

$$\lambda x(l-x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos 0$$

$$=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\, 1 \qquad (\because \cos 0=1)$$

R.H.S is half range sine series

$\therefore$ Its co-efficient b_n is given by

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$=\frac{2}{l}\int_0^l \lambda x(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$=\frac{2\lambda}{l}\int_0^l (lx-x^2)\sin\frac{n\pi x}{l}\,dx$$

$$=\frac{2\lambda}{l}\left[\underset{\downarrow \atop 0}{(lx-x^2)}\frac{\left(-\cos\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}-\underset{\downarrow \atop 0}{(l-2x)}\frac{\left(-\sin\frac{n\pi x}{l}\right)}{\frac{n^2\pi^2}{l^2}}+(-2)\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right]_0^l$$

[Applying Bernoulli's formula]

$$=\frac{2\lambda}{l}\left[\frac{-2\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right]_0^l$$

$$=\frac{-4\lambda}{l}\,\frac{l^3}{n^3\pi^3}\left[\cos\frac{n\pi x}{l}\right]_0^l$$

$$=\frac{-4\lambda l^2}{n^3\pi^3}\left[\cos\frac{n\pi l}{l}-\cos 0\right]$$

$$= \frac{-4\lambda l^2}{n^3 \pi^3} [\cos n\pi - \cos 0]$$

$$= \frac{-4\lambda l^2}{n^3 \pi^3} [(-1)^n - 1] \qquad \cos n\pi = (-1)^n$$

Case 1 : When n is even $(-1)^n = 1$

$$\therefore b_n = \frac{-4\lambda l^2}{n^3 \pi^3} [1 - 1]$$

$$= \frac{-4\lambda l^2}{n^3 \pi^3} [0] = 0$$

Case 2 : When n is odd $(-1)^n = -1$

$$\therefore b_n = \frac{-4\lambda l^2}{n^3 \pi^3} (-1 - 1)$$

$$= \frac{-4\lambda l^2}{n^3 \pi^3} (-2)$$

$$= \frac{8\lambda l^2}{n^3 \pi^3}$$

$\therefore$ The final solution is obtained by substituting for b_n in (4)

$$y = \sum_{n=1,3,5}^{\infty} \frac{8\lambda l^2}{n^3 \pi^3} \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \quad [\because n \text{ takes only odd values}]$$

---

14. (b) Find the solution to the equation $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$ that satisfies the conditions $u(0, t) = 0$, $u(l, t) = 0$, for $t > 0$ and $u(x, 0) = \begin{cases} x & , 0 \le x \le l/2 \\ l - x, & l/2 < x < l \end{cases}$

Solution :

The heat flow equation is

$$u = (A \cos px + B \sin px) \cdot C e^{-\alpha^2 p^2 t} \qquad \dots (1)$$

The boundary conditions are

I. $u = 0$ when $x = 0$

II. $u = 0$ when $x = l$

III. $u = f(x)$ when $t = 0$

Applying boundary condition (i) we get

$$0 = (A \cos 0 + B \sin 0)\, C\, e^{-\alpha^2 p^2 t}$$

$$0 = (A + 0)\, C\, e^{-\alpha^2 p^2 t}$$

$$\Rightarrow A = 0$$

$$\therefore u = B \sin px\, C\, e^{-\alpha^2 p^2 t} \quad \dots (2)$$

Applying II in (2) we get

$$0 = B \sin pl\, (C\, e^{-\alpha^2 p^2 t})$$

$$\sin pl = 0$$

$$pl = n\pi$$

$$p = \frac{n\pi}{l}$$

Substitute $p = \frac{n\pi}{l}$ in (2)

$$u = B \sin \frac{n\pi}{l}\, C\, e^{\frac{-\alpha^2 n^2 \pi^2}{l^2} t}$$

$$= BC \sin \frac{n\pi x}{l}\, e^{\frac{-\alpha^2 n^2 \pi^2}{l^2} t}$$

$$= b \sin \frac{n\pi x}{l} e^{\frac{-\alpha^2 n^2 \pi^2}{l^2} t} \qquad (BC = b)$$

$$u = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}} \quad \ldots (3)$$

Applying condition (iii) in (3)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^0$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} . dx$$

$$= \frac{2}{l} \left[\int_0^{l/2} f(x) . \sin \frac{n\pi x}{l} . dx + \int_{l/2}^{l} f(x) . \sin \frac{n\pi x}{l} . dx \right]$$

$$= \frac{2}{l} [\int_0^{l/2} x \sin \frac{n\pi x}{l} . dx + \int_{l/2}^{l} (l-x) \sin \frac{n\pi x}{l} . dx$$

$$= \frac{2}{l} \left[x \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - (1) \left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2 \pi^2}{l^2}} \right) \right\}_0^{l/2}$$

$$+ \left\{ (l-x) \left(-\cos \frac{n\pi x}{l} \right) - (-1) \left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right) \right\}_{l/2}^{l} \Bigg]$$

$$= \frac{-2}{l}\left[\left(\frac{-x\dfrac{\cos n\pi x}{l}}{\dfrac{n\pi}{l}}+\frac{\sin\dfrac{n\pi x}{l}}{\dfrac{n^2\pi^2}{l^2}}\right)_0^{l/2}-\left\{(l-x)\frac{\cos\dfrac{n\pi x}{l}}{\dfrac{n\pi}{l}}+\frac{\sin\dfrac{n\pi x}{l}}{\dfrac{n^2\pi^2}{l^2}}\right\}_{l/2}^{l}\right]$$

$$= \frac{2}{l}\left[-\frac{l}{2}\frac{\cos\dfrac{n\pi}{2}}{\dfrac{n\pi}{l}}+\frac{\sin\dfrac{n\pi}{2}}{\dfrac{n^2\pi^2}{l^2}}-(0+0)-\left\{0-\left(\frac{l}{2}\frac{\cos\dfrac{n\pi}{2}}{\dfrac{n\pi}{l}}+\frac{\sin\dfrac{n\pi}{2}}{\dfrac{n^2\pi^2}{l^2}}\right)\right\}\right]$$

$$= \frac{2}{l}\left[-\frac{l}{2}\frac{\cos\dfrac{n\pi}{2}}{\dfrac{n\pi}{l}}+\frac{\sin\dfrac{n\pi}{2}}{\dfrac{n^2\pi^2}{l^2}}+\frac{l}{2}\frac{\cos\dfrac{n\pi}{2}}{\dfrac{n\pi}{l}}+\frac{\sin\dfrac{n\pi}{2}}{\dfrac{n^2\pi^2}{l^2}}\right]$$

$$= \frac{2}{l}\left[\frac{2\sin\dfrac{n\pi}{2}}{\dfrac{n^2\pi^2}{l^2}}\right]$$

$$= \frac{2}{l}\frac{l^2}{n^2\pi^2}\cdot 2\sin\frac{n\pi}{2}$$

$$= \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$\therefore$ The solution is

$$u = \sum_{n=1}^{\infty}\frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}e^{-\frac{\alpha^2 n^2\pi^2}{l^2}t}$$

15. (a) (i) Find the z-transform of $\dfrac{1}{(n+1)(n+2)}$.

Solution :

Apply partial fractions,

Let $\dfrac{1}{(n+2)(n+1)} = \dfrac{A}{n+2} + \dfrac{B}{n+1}$

$$\frac{1}{(n+2)(n+1)} = \frac{A(n+1)+B(n+2)}{(n+2)(n+1)}$$

$$1 = A(n+1) + B(n+2)$$

Put $n = -1$, we get

$$1 = A(0) + B(-1+2)$$

$$\therefore B = 1$$

Put $n = -2$, we get

$$1 = A(-2+1) + B(0)$$

$$1 = A(-1)$$

$$\therefore A = -1$$

$$\frac{1}{(n+2)(n+1)} = \frac{-1}{n+2} + \frac{1}{n+1}$$

$$Z\left[\frac{1}{(n+2)(n+1)}\right] = Z\left[\frac{-1}{n+2} + \frac{1}{n+1}\right]$$

$$= -Z\left[\frac{1}{n+2}\right] + Z\left[\frac{1}{n+1}\right]$$

$$= -\sum_{n=0}^{\infty} \frac{1}{n+2} z^{-n} + z\left[\log \frac{z}{z-1}\right]$$

$$\left[\text{By formula } Z\left[\frac{1}{n+1}\right] = z\left[\log \frac{z}{z-1}\right]\right]$$

15. (a) (ii) Using Z-transform solve the difference equation $y_{n+2} + 2y_{n+1} + y_n = n$ given $y_0 = 0 = y_1$.

Solution :

Let $\bar{y} = Z[y_n]$

Taking Z-transform on both sides,

$$Z[y_{n+2}] + 2Z[y_{n+1}] + Z[y_n] = Z[n]$$

$$z^2\left[\bar{y} - y_0 - \frac{y_1}{z}\right] + 2[z(\bar{y} - y_0)] + \bar{y} = \frac{z}{(z-1)^2}$$

Applying $y_0 = 0, y_1 = 0$ $\left[Z(n) = \frac{Z}{(Z-1)^2}\right]$

$$z^2\bar{y} + 2z\bar{y} + \bar{y} = \frac{z}{(z-1)^2}$$

$$\bar{y}(z^2 + 2z + 1) = \frac{z}{(z-1)^2}$$

$$\bar{y}(z+1)^2 = \frac{z}{(z-1)^2}$$

$$\bar{y} = \frac{z}{(z-1)^2(z+1)^2}$$

$$y_n = z^{-1}\left[\frac{z}{(z-1)^2 + (z+1)^2}\right]$$

$$\text{Let } F(z) = \frac{z}{(z-1)^2(z+1)^2}$$

We can find the inverse Z-transform using residue method.

$$\text{Consider } z^{n-1}F(Z) = z^{n-1}\frac{z}{(z-1)^2(z+1)^2}$$

$$= \frac{z^n}{(z-1)^2(z+1)^2}$$

$$= g(z)$$

Equating the denominator to zero

$$(z-1)^2 = 0 \quad ; \quad (z+1)^2 = 0$$

$$z = 1, 1 \quad ; \quad z = -1, -1$$

Residue at $z = 1, 1$ is given by

$$\underset{z \to 1}{Lt} \frac{d}{dz} (z-1)^2 \cdot \frac{z^n}{(z-1)^2 (z+1)^2} \qquad \left\{ \underset{z \to a}{Lt} \frac{d}{dz} (z-a)^2 g(z) \right\}$$

$$\underset{z \to 1}{Lt} \frac{d}{dz} \frac{z^n}{(z+1)^2}$$

$$\underset{z \to 1}{Lt} \frac{(z+1)^2 nz^{n-1} - z^n 2(z+1)}{(z+1)^4}$$

$$= \frac{(1+1)^2 n (1)^{n-1} - (1)^n 2(1+1)}{(1+1)^4}$$

$$= \frac{4n (1)^{n-1} - (1)^n 4}{2^4}$$

$$= \frac{4[n (1)^{n-1} - (1)^n]}{16}$$

$$= \frac{n (1)^{n-1} - (1)^n}{4} \qquad \{(1)^{n-1} = 1 \; ; \; (1)^n = 1\}$$

$$= \frac{n-1}{4}$$

Residue at $z = -1, -1$ is given by,

$$\underset{z \to -1}{dt} \frac{d}{dz} (z+1)^2 \cdot \frac{z^n}{(z-1)(z+1)^2}$$

$$\underset{z \to -1}{dt} \frac{d}{dz} \frac{z^n}{(z-1)^2}$$

$$\underset{z \to -1}{Lt} \frac{(z-1)^2 nz^{n-1} - z^n 2(z-1)}{(z-1)^4}$$

$$= \frac{(-1-1)^2 n (-1)^{n-1} - (-1)^n 2(-1-1)}{(-1-1)^4}$$

$$= \frac{(-2)^2 n (-1)^{n-1} - (-1)^n 2 (-2)}{(-2)^4}$$

$$= \frac{4n (-1)^{n-1} - (-1)^n (-4)}{16}$$

$$= \frac{4n (-1)^{n-1} + 4 (-1)^n}{16}$$

$$= \frac{4 [-n (-1)^n + (-1)^n]}{16} \qquad [(-1)^{n-1} = (-1)^n (-1) = -(-1)^n]$$

$$= \frac{-n (-1)^n + (-1)^n}{4}$$

$$= \frac{(-1)^n [1 - n]}{4}$$

$$\therefore y_n = z^{-1} \left[\frac{z}{(z+1)^2 (z-1)^2} \right]$$

$$= \frac{(n-1)}{4} + (-1)^n \frac{(1-n)}{4}$$

$$y_n = \frac{n-1}{4} [1 - (-1)^n]$$

15. (b) (i) From the difference equation from $y(n) = (A + Bn) 2^n$.

Solution :

Given $y_n = (A + Bn) 2^n$

$$y_n = A (2^n) + B n (2^n) \qquad \ldots (1)$$

Replacing n by $n + 1$ we get

$$y_{n+1} = A (2)^{n+1} + B (n+1) (2)^{n+1}$$

$$y_{n+1} = A\ (2)^n (2) + B\ (n+1)\ (2)^n\ (2)$$

Dividing by 2,

$$\frac{y_{n+1}}{2} = A\ (2)^n + B\ (n+1)\ (2)^n \qquad \dots (2)$$

Replacing n by $n + 2$ in (1) we get

$$y_{n+2} = A\ (2)^{n+2} + B\ (n+2)\ (2)^{n+2}$$

$$y_{n+2} = A\ (2)^n (2)^2 + B\ (n+2)\ (2)^n\ (2)^2$$

$$y_{n+2} = A\ (2)^n\ (4) + B\ (n+2)\ (2)^n\ (4)$$

Dividing by 4,

$$\frac{y_{n+2}}{4} = A\ (2)^n + B\ (n+2)\ (2)^n \qquad \dots (3)$$

Considering the determinant of coefficient of A $(2)^n$ and B $(2)^n$.

$$\begin{bmatrix} y_n & 1 & n \\ \frac{y_{n+1}}{2} & 1 & n+1 \\ \frac{y_{n+2}}{4} & 1 & n+2 \end{bmatrix} = 0$$

$$y_n\ [\ (n+2) - (n+1)\ ]\ -1\left[\left(\frac{y_{n+1}}{2}\right)(n+2) - \left(\frac{y_{n+2}}{4}\right)(n+1)\right] + n\left[\frac{y_{n+1}}{2} - \frac{y_{n+2}}{4}\right] = 0$$

$$y_n\ [n+2-n-1] - 1\left[\left(\frac{y_{n+1}}{2}\right)(n+2) - \left(\frac{y_{n+2}}{4}\right)(n+1)\right] + n\left[\frac{y_{n+1}}{2} - \frac{y_{n+2}}{4}\right] = 0$$

$$y_n - \frac{y_{n+1}}{2}(n+2) + \frac{y_{n+2}}{4}(n+1) + \frac{n\,y_{n+1}}{2} - \frac{n\,y_{n+2}}{4} = 0$$

$$\frac{4y_n - 2y_{n+1}(n+2) + (y_{n+2})(n+1) + 2n\,y_{n+1} - ny_{n+2}}{4} = 0$$

$$4y_n - 2y_{n+1}(n+2) + (y_{n+2})(n+1) + 2n\,y_{n+1} - ny_{n+2} = 0$$

$$4y_n - 2ny_{n+1} - 4y_{n+1} + ny_{n+2} + y_{n+2} + 2n\,y_{n+1} - ny_{n+2} = 0$$

$$y_{n+2} - 4y_{n+1} + 4y_n = 0$$

15. (b) (ii) Using convolution theorem find $z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right]$.