B.E. /B.Tech. DEGREE EXAMINATION, MAY / JUNE 2013

Third Semester

MA 2211 /MA 31 /MA 1201 A/CK 201/ 10177 MA 301 / 080100008/080210001 -

TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS

(Common to All Branches)

(Regulation 2008 / 2010)

[Time : Three hours] [Maximum : 100 marks]

Answer ALL questions.

PART-A (10×2=20 marks)

1. State the Dirichlet's conditions for Fourier series.

Solution :

Any function $f(x)$ can be developed as a Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ where a_0, a_n & b_n are constants, provided that it satisfies the following Dirichlet's conditions

(i) $f(x)$ is periodic, single valued and finite.

(ii) $f(x)$ has a finite number of finite discontinuities in any one period and has no infinite discontinuity.

(iii) $f(x)$ has at the most a finite number of maxima and minima.

2. What is meant by Harmonic Analysis?

Solution : The process of finding Euler constant for a tabular function is known as Harmonic Analysis.

3. Find the Fourier Sine Transform of e^{-3x}

Solution : $F_s[e^{-3x}] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-3x} \sin sx \, dx$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2 + 3^2}\right]$$

$$\boxed{\text{Formula : } F_s\,[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx\, dx}$$

4. If $F\{f(x)\} = F(s)$, prove that $F\{f(ax)\} = \frac{1}{a} . F\left(\frac{s}{a}\right)$

Proof : We know that, $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\, e^{isx}\, dx$$

put $t = ax$, $x \to -\infty \Rightarrow t \to -\infty$, if $a > 0$

$dt = a\,dx$, $x \to \infty \Rightarrow t \to \infty$, if $a > 0$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist/a} \frac{dt}{a} = \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is(t/a)}\, dt$$

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i(s/a)x}\, dx \qquad [\because t \text{ is a dummy variable}]$$

$$= \frac{1}{a} F\left[\frac{s}{a}\right]$$

5. Form the PDE from $(x-a)^2 + (y-b)^2 + z^2 = r^2$

Sol. Given : $(x-a)^2 + (y-b)^2 + z^2 = r^2$ ... (1)

Here, a and b are the two arbitrary constants.

diff. (1) p.w.r.to x, we get

$$2(x-a) + 2z\frac{\partial z}{\partial x} = 0$$

$$(x-a) + zp = 0 \qquad \text{... (2)} \qquad \left[\because p = \frac{\partial z}{\partial x}\right]$$

diff. (1) p.w.r.to y, we get

$$2(y-b) + 2z\frac{\partial z}{\partial y} = 0$$

$$(y-b) + zq = 0 \qquad \text{... (3)} \qquad [\because q = \frac{\partial z}{\partial y}]$$

Eliminate a and b from (1), (2) and (3)

$$(2) \Rightarrow x - a = -zp$$

$$(3) \Rightarrow y - b = -zq$$

$$\therefore \quad (1) \Rightarrow (-zp)^2 + (-zq)^2 + z^2 = r^2$$

$$z^2p^2 + z^2q^2 + z^2 = r^2$$

$$z^2[p^2 + q^2 + 1] = r^2$$

which is the required p.d.e.

a.c.	I.V.
a, b	x, y
2	2
no. of a.c. = no. of I.V. ∴ we use p & q only, r is a given constant no need to eliminate	

6. Find the complete integral of p + q = pq

Solution :

$$\text{Given : } p + q = pq \qquad \text{... (1)}$$

This equation is of the form $F(p, q) = 0$ ... (2)

Hence, the trial solution is $z = ax + by + c$ ... (3)

To get the complete integral (solution) of (3) we have to eliminate any one of the arbitrary constants. Since in a complete integral

number of a.c. = number of I.V.

$$\therefore (3) \Rightarrow \quad z = ax + by + c$$

$$p = \frac{\partial z}{\partial x} = a$$

$$q = \frac{\partial z}{\partial y} = b$$

$$\therefore (1) \Rightarrow a + b = ab$$

$$b - ab = -a$$

$$b(1-a) = -a$$

$$b = \frac{-a}{1-a}$$

i.e., $b = \frac{a}{a-1}$.

Hence, the complete solution is

$$\therefore (3) \Rightarrow z = ax + \frac{a}{a-1}y + c \qquad \dots (4)$$

Since number of a.c. = number of I.V

7. In the one dimensional heat equation $u_t = c^2 . u_{xx}$, what is c^2 ?

Solution : $c^2 = \frac{k}{s\rho}$,

$c^2 \rightarrow$ differsivity of the material of the body through which heat flows.

$\rho \rightarrow$ density

$s \rightarrow$ specific heat

$k \rightarrow$ thermal conductivity

8. Write down the two dimensional heat equation both in transient and steady states.

Solution : Transient state : $\frac{\partial u}{\partial t} = \alpha^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

Steady state : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

9. Find Z (n)

Solution: We know that, $Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$

$$Z[n] = \sum_{n=0}^{\infty} nz^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{n}{z^n} = \sum_{n=0}^{\infty} n\left(\frac{1}{z}\right)^n$$

$$= 0 + \left[\frac{1}{z}\right] + 2\left[\frac{1}{z}\right]^2 + \ldots$$

$$= \frac{1}{z}\left[1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + \ldots\right]$$

$$= \frac{1}{z}\left[\left(1 - \frac{1}{z}\right)^{-2}\right]$$

$$= \frac{1}{z}\left[\left(\frac{z-1}{z}\right)^{-2}\right]$$

$$= \frac{1}{z}\left[\frac{z}{z-1}\right]^2 = \frac{1}{z}\,\frac{z^2}{(z-1)^2}$$

$$= \frac{z}{(z-1)^2}, \quad |z| > 1$$

Formula :

$(1-x)^{-2} = 1 + 2x + 3x^2 + \ldots$

Here, $x = \frac{1}{z}$, $\left|\frac{1}{z}\right| < 1$

i.e., $|1| < |z|$

i.e., $|z| > 1$

10. Obtain $Z^{-1}\left[\frac{z}{(z+1)(z+2)}\right]$

Solution : Let $X(z) = \frac{z}{(z+1)(z+2)}$

$$\frac{X(z)}{z} = \frac{1}{(z+1)(z+2)}$$

$$= \frac{A}{z+1} + \frac{B}{z+2} \quad \ldots (1)$$

$$1 = A(z+2) + B(z+1)$$

Put $z = -1$, we get

$1 = A$

i.e., $\boxed{A = 1}$

Put $x = -2$, we get

$1 = -B$

i.e., $\boxed{B = -1}$

$$\therefore (1) \Rightarrow \frac{X(z)}{z} = \frac{1}{z+1} - \frac{1}{z+2}$$

i.e., $$X(z) = \frac{z}{z+1} - \frac{z}{z+2}$$

i.e., $$Z[x(n)] = \frac{z}{z+1} - \frac{z}{z+2}$$

$$x(n) = Z^{-1}\left[\frac{z}{z+1}\right] - Z^{-1}\left[\frac{z}{z+2}\right]$$

$$= (-1)^n - (-2)^n$$

i.e., $$Z^{-1}\left[\frac{z}{(z+1)(z+2)}\right] = (-1)^n - (-2)^n$$

$$\boxed{\text{Formula : } Z[a^n] = \frac{z}{z-a} \Rightarrow Z^{-1}\left[\frac{z}{z-a}\right] = a^n}$$

PART-B (5 × 16 = 80 marks)

11. (a) (i) Find the Fourier series of x^2 in $(-\pi, \pi)$ and hence deduce that $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots = \frac{\pi^4}{90}$

Solution : Given : $f(x) = x^2, f(-x) = (-x)^2 = x^2 = f(x)$

Therefore $f(x)$ is an even function in $(-\pi, \pi)$. Hence, $\boxed{b_n = 0}$

$\therefore$ The Fourier series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^3}{3} - 0\right] = \frac{2}{\pi}\left[\frac{\pi^3}{3}\right] = \frac{2\pi^2}{3}$$

$$\therefore \boxed{a_0 = \frac{2\pi^2}{3}}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right) + 2x\left(\frac{\cos nx}{n^2}\right) - 2\left(\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(0+\frac{2\pi(-1)^n}{n^2} - 0\right) - (0+0-0)\right]$$

[$\because$ $\sin n\pi = 0$; $\cos n\pi = (-1)^n$

$\sin 0 = 0$; $\cos 0 = 1$]

$$= \frac{2}{\pi}\left(\frac{2\pi(-1)^n}{n^2}\right)$$

$$\therefore \quad \boxed{a_n = \frac{4}{n^2}(-1)^n}$$

$$\therefore \quad \boxed{a_0 = \frac{2\pi^2}{3}} \quad \boxed{a_n = \frac{4}{n^2}(-1)^n} \quad \boxed{b_n = 0}$$

Using Parseval's theorem,

$$2\pi\left[\frac{a_0^2}{4} + \frac{1}{2}\sum_1^{\infty}[a_n^2 + b_n^2]\right] = \int_{-\pi}^{\pi}[f(x)]^2\,dx = \int_{-\pi}^{\pi} x^4\,dx$$

$$= 2\int_0^{\pi} x^4\,dx = 2\left[\frac{x^5}{5}\right]_0^{\pi}$$

$$= \frac{2}{5}\left[x^5\right]_0^{\pi} = \frac{2}{5}\left[\pi^5 - 0\right]$$

$$= \frac{2}{5}\left[\pi^5 - 0\right] = \frac{2}{5}\pi^5$$

$$2\pi\left[\frac{\pi^4}{9} + \frac{1}{2}\sum_1^{\infty}\frac{16}{n^4}\right] = \left[\frac{2\pi^5}{5}\right]$$

$$\left[\frac{\pi^4}{9} + \frac{1}{2}\sum_1^{\infty}\frac{16}{n^4}\right] = \left[\frac{\pi^4}{5}\right] \Rightarrow \left[\frac{\pi^4}{9} + 8\sum_1^{\infty}\frac{1}{n^4}\right] = \left[\frac{\pi^4}{5}\right]$$

$$8\sum_{1}^{\infty}\frac{1}{n^4} = \left[\frac{\pi^4}{5}-\frac{\pi^4}{9}\right] \Rightarrow 8\sum_{1}^{\infty}\frac{1}{n^4} = \left[\frac{4\pi^4}{45}\right]$$

$$\sum_{1}^{\infty}\frac{1}{n^4} = \left[\frac{\pi^4}{90}\right]$$

$$\frac{\pi^4}{90} = 1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+ \dots$$

11. (a) (ii) Obtain the Fourier cosine series of

$$f(x) = \begin{cases} kx\ , & 0 < x < \frac{l}{2} \\ k(l-x)\ , & \frac{l}{2} < x < l \end{cases}$$

Solution : $\therefore$ The required Fourier cosine series be

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \qquad \dots (1) \text{ where}$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx \qquad \dots (2)$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx \qquad \dots (3)$$

$$(2) \Rightarrow \quad a_0 = \frac{2}{l}\left[\int_0^{l/2} kx\,dx + \int_{l/2}^{l} k(l-x)\,dx\right]$$

$$= \frac{2k}{l}\left[\int_0^{l/2} x\,dx + \int_{l/2}^{l}(l-x)\,dx\right]$$

$$= \frac{2k}{l}\left[\left[\frac{x^2}{2}\right]_0^{l/2} + \left[lx-\frac{x^2}{2}\right]_{l/2}^{l}\right]$$

$$= \frac{2k}{l}\left[\left(\frac{l^2/4}{2}-0\right)+\left(l^2-\frac{l^2}{2}\right)-\left(\frac{l^2}{2}-\frac{l^2/4}{2}\right)\right]$$

$$= \frac{2k}{l}\left[\frac{l^2}{8} + \left(\frac{l^2}{2} - \frac{l^2}{2} + \frac{l^2}{8}\right)\right]$$

$$= \frac{2k}{l}\left[\frac{l^2}{8} + \frac{l^2}{8}\right]$$

$$= \frac{2k}{l}\left[2\frac{l^2}{8}\right]$$

$$= \frac{k}{2}l$$

$$(3) \Rightarrow a_n = \frac{2}{l}\left[\int_0^{l/2} kx\cos\frac{n\pi x}{l}dx + \int_{l/2}^{l} k(l-x)\cos\frac{n\pi x}{l}dx\right]$$

$$= \frac{2k}{l}\left[\int_0^{l/2} x\cos\frac{n\pi x}{l}dx + \int_{l/2}^{l}(l-x)\cos\frac{n\pi x}{l}dx\right] \quad \dots (4)$$

Take

$$\int_0^{l/2} x\cos\frac{n\pi x}{l}dx = \left[x\left[\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right] - (1)\left[\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]\right]_0^{l/2}$$

$$= \left[x\frac{l}{n\pi}\sin\frac{n\pi x}{l} + \left(\frac{l}{n\pi}\right)^2\cos\frac{n\pi x}{l}\right]_0^{l/2}$$

$$= \left(\frac{l}{2}\frac{l}{n\pi}\sin\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\cos\frac{n\pi}{2}\right) - \left(0 + \frac{l^2}{n^2\pi^2}\right)$$

$$= \frac{l^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{l^2}{n^2\pi^2}$$

$$\int_{l/2}^{l} (l-x)\cos\frac{n\pi x}{l}\,dx = \left[(l-x)\left[\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right] - (-1)\left[\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]\right]_{l/2}^{l}$$

$$= \left[(l-x)\frac{l}{n\pi}\sin\frac{n\pi x}{l} - \left(\frac{l}{n\pi}\right)^2 \cos\frac{n\pi x}{l}\right]_{l/2}^{l}$$

$$= \left[0 - \frac{l^2}{n^2\pi^2}(-1)^n\right] - \left[\frac{l}{2}\frac{l}{n\pi}\sin\frac{n\pi}{2} - \frac{l^2}{n^2\pi^2}\cos\frac{n\pi}{2}\right]$$

$$= -\frac{l^2}{n^2\pi^2}(-1)^n - \frac{l^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\cos\frac{n\pi}{2}$$

$$\therefore (4) \Rightarrow a_n = \frac{2k}{l}\left[\frac{2l^2}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{l^2}{n^2\pi^2}[1+(-1)^n]\right]$$

$$= \frac{2k}{l}\frac{l^2}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - (1+(-1)^n)\right]$$

$$= \frac{2kl}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - [1+(-1)^n]\right]$$

$$a_n = 0 \text{ if } n \text{ is odd.}$$

$$a_n = \frac{2kl}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 2\right] \text{ if } n \text{ is even}$$

$$= \frac{4kl}{n^2\pi^2}\left[\cos\frac{n\pi}{2} - 1\right]$$

$$\therefore (1) \Rightarrow f(x) = \frac{[kl/2]}{2} + \sum_{n=\text{even}}^{\infty}\frac{4kl}{n^2\pi^2}\left[\cos\frac{n\pi}{2} - 1\right]\cos\frac{n\pi x}{l}$$

$$= \frac{kl}{4} + \frac{4kl}{\pi^2}\sum_{n=\text{even}}^{\infty}\frac{1}{n^2}\left[\cos\frac{n\pi}{2} - 1\right]\cos\frac{n\pi x}{l}$$

$$= \frac{kl}{4} + \frac{4kl}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \left[\cos\frac{2n\pi}{2} - 1\right] \cos\frac{2n\pi x}{l}$$

[Replace n by $2n$]

$$= \frac{kl}{4} + \frac{4kl}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{4n^2} [\cos n\pi - 1] \cos\frac{2n\pi x}{l}$$

$$= \frac{kl}{4} + \frac{kl}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} [(-1)^n - 1] \cos\frac{2n\pi x}{l}$$

(OR)

11. (b) (i) Find the complex form of Fourier series of cos ax in $(-\pi, \pi)$, where "a" is not an integer.

Solution : We know that, $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$ ... (1)

where $C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$ ... (2)

Here, $f(x) = \cos ax$

(2) $\Rightarrow$ $C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos ax\, e^{-inx}\, dx$

i.e., $C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} \cos ax\, dx$

> Formula : $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx)$
>
> Here, $a = -in,\ b = a$

$$\therefore C_n = \frac{1}{2\pi} \left[\frac{e^{-inx}}{(-in)^2 + a^2} [-in \cos ax + a \sin ax] \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi} \left[\frac{e^{-inx}}{-n^2 + a^2} (-in \cos ax + a \sin ax) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-in\pi}}{a^2-n^2}(-in\cos a\pi + a\sin a\pi) - \frac{e^{in\pi}}{a^2-n^2}(-in\cos a\pi - a\sin a\pi)\right]$$

$$= \frac{(-1)^n}{2\pi(a^2-n^2)}[-in\cos a\pi + a\sin a\pi + in\cos a\pi + a\sin a\pi]$$

$$= \frac{(-1)^n}{2\pi(a^2-n^2)}[2a\sin a\pi] \qquad [\because e^{-in\pi} = (-1)^n = e^{in\pi}]$$

$$= \frac{(-1)^n}{\pi(a^2-n^2)}a\sin a\pi$$

$$\therefore (1) \Rightarrow f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\pi(a^2-n^2)}a\sin a\pi\, e^{inx}$$

$$= \frac{a\sin a\pi}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n e^{inx}}{a^2-n^2} \text{ in the interval } (-\pi,\pi)$$

11. (b) (ii) Obtain the Fourier cosine series of $(x-1)^2$, $0 < x < 1$ and hence show that $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots = \frac{\pi^2}{6}$

Solution : Here, $l = 1$

$\therefore$ The required Fourier cosine series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x \quad \ldots (1) \text{ where}$$

$$a_0 = 2\int_0^1 (x-1)^2\,dx = 2\left[\frac{(x-1)^3}{3}\right]_0^1 \quad \text{(i.e.,)} \quad \boxed{a_0 = \left[\frac{2}{3}\right]}$$

$$a_n = 2\int_0^1 (x-1)^2 \cos n\pi x\,dx$$

$$a_n = 2\left[(x-1)^2\left(\frac{\sin n\pi x}{n\pi}\right) - 2(x-1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + 2\left(\frac{-\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= 2\left[(0+0-0) - \left(0 + \frac{-2}{n^2\pi^2} - 0\right)\right] \quad \text{(i.e.,)} \quad \boxed{a_n = \frac{4}{n^2\pi^2}}$$

$$(1) \Rightarrow \quad f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} \cos n\pi x$$

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi x \qquad \dots (2)$$

At $x=0$, Here $x=0$ is a point of finite continuity in the middle.

[Since $f(x) = (x-1)^2$ in $(0, 1)$

$$f(x) = (-x-1)^2 \text{ in } (-1, 0)$$

$$f(0^-) = f(0^+) = 1 \text{]}$$

Put $x = 0$ in (2) we get,

$$(2) \Rightarrow \quad 1 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$1 - \frac{1}{3} = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{2}{3} = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$

12. (a) (i) Find the Fourier transform of $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$

and hence, find $\displaystyle\int_0^{\infty} \frac{\sin x}{x}\, dx$

Solution : The given function can be written as

$$f(x) = \begin{cases} 1 \text{ if } -a < x < a \\ 0 \text{ otherwise} \end{cases}$$

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (\cos sx + i \sin x\, dx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \cos sx\, dx + \frac{i}{\sqrt{2\pi}} \int_{-a}^{a} \sin sx\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[2\int_{0}^{a} \cos sx\, dx \right] + \frac{i}{\sqrt{2\pi}} [0]$$

$$\boxed{\because \cos sx \text{ is an even function in } (-a, a) \text{ \& } \sin sx \text{ is an odd function in } (-a, a)}$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{a} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s} \right]_{x=0}^{x=a}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sa}{s} - 0 \right]$$

$$F(s) = \sqrt{\frac{2}{\pi}} \left(\frac{\sin as}{s} \right) \qquad \dots (1)$$

Now, by Fourier inversion formula, we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left(\frac{\sin as}{s}\right) (\cos sx - i \sin sx)\, ds \text{ [using (1)]}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right) \cos sx\, ds - \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} \sin sx\, ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right) \cos sx\, ds - \frac{i}{\pi}(0)$$

$\left[\because \left(\frac{\sin as}{s}\right) \sin sx \text{ is an odd function in } (-\infty, \infty)\right]$

$$= \frac{2}{\pi} \int_{0}^{\infty} \left(\frac{\sin as}{s}\right) \cos sx\, ds$$

$\left[\because \left(\frac{\sin as}{s}\right) \cos sx \text{ is an even function in } (-\infty, \infty)\right]$

$$\int_{0}^{\infty} \left(\frac{\sin as}{s}\right) \cos sx\, ds = \frac{\pi}{2} f(x) \quad \ldots (2)$$

put $x = 0$, we get $\int_{0}^{\infty} \frac{\sin as}{s}\, ds = \frac{\pi}{2}$

Now, put $t = as$. Then we have $s = \frac{t}{a}$ and $ds = \frac{dt}{a}$

Further $t \to 0$ as $s \to 0$ and $t \to \infty$ as $s \to \infty$

Hence, $\int_{0}^{\infty} \left(\frac{\sin t}{t}\right) dt = \frac{\pi}{2}$, i.e., $\int_{0}^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$

[$\because$ 't' is a dummy variable]

12. (a) (ii) Verify the convolution theorem under Fourier Transform, for $f(x) = g(x) = e^{-x^2}$

Definition : Convolution theorem for Fourier transforms. The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms.

$$F[f(x) * g(x)] = F\{f(x)\} \cdot F\{g(x)\}$$

Given : $f(x) = g(x) = e^{-x^2}$

We know that, $F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} f(x)\, e^{-isx}\, dx$

$$F[e^{-x^2}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} e^{-isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2 + isx)}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[(x - \frac{is}{2})^2 + \frac{s^2}{4}\right]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(x - \frac{is}{2}\right)^2} e^{-s^2/4}\, dx$$

$$= e^{-s^2/4} \left(\frac{1}{\sqrt{2\pi}}\right) \int_{-\infty}^{\infty} e^{-\left(x - \frac{is}{2}\right)^2}\, dx$$

put $t = x - \frac{is}{2}$, $dt = dx$ | $x \to \infty \Rightarrow t \to -\infty$, $x \to \infty \Rightarrow t \to \infty$

$$= e^{-s^2/4} \left(\frac{1}{\sqrt{2\pi}}\right) \int_{-\infty}^{\infty} e^{-t^2}\, dt$$

$$= e^{-s^2/4} \left(\frac{1}{\sqrt{2\pi}}\right) 2 \int_{0}^{\infty} e^{-t^2}\, dt$$

put $t^2 = u$, $2t\,dt = du$ | $t \to 0 \Rightarrow u \to 0$, $t \to \infty \Rightarrow u \to \infty$

$$dt = \frac{1}{2t} du$$

$$= \frac{1}{2\sqrt{u}} du$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right) 2 e^{-s^2/4} \int_{0}^{\infty} e^{-u} \frac{1}{2\sqrt{u}}\, du$$

$$= e^{-s^2/4}\left(\frac{1}{\sqrt{2\pi}}\right)\int_0^{\infty} e^{-u}\,u^{-1/2}\,du$$

$$= e^{-s^2/4}\left(\frac{1}{\sqrt{2\pi}}\right)\sqrt{\pi} \qquad \left[\because \int_0^{\infty} e^{-t}\,t^{-1/2}\,dt = \sqrt{\pi}\right]$$

$$= \frac{1}{\sqrt{2}}\,e^{-s^2/4}$$

$$F\,[f(x)] = \frac{1}{\sqrt{2}}\,e^{-s^2/4}$$

$$F\,[g(x)] = F\,[f(x)] = \frac{1}{\sqrt{2}}\,e^{-s^2/4}$$

$$\therefore\ F\,[g(x)]\,.\,F\,[g(x)] = \left(\frac{1}{\sqrt{2}}e^{-s^2/4}\right)\left(\frac{1}{\sqrt{2}}e^{-s^2/4}\right) = \frac{1}{2}e^{-s^2/2} \quad \dots (1)$$

$$f(x) * g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,g(x-u)\,du \qquad \text{by convolution definition}$$

$$e^{-x^2} * e^{-x^2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-u^2}\,e^{-(x-u)^2}\,du$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x-u)^2-u^2}\,du$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-2\left[\left(u-\frac{x}{2}\right)^2+\frac{x^2}{4}\right]}\,du$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-2\left(u-\frac{x}{2}\right)^2}\,e^{\frac{-x^2}{2}}\,du$$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-2\left(u-\frac{x}{2}\right)^2}\,du$$

put $t = u - \dfrac{x}{2}$ $\qquad$ $u \to -\infty \Rightarrow t \to -\infty$

$dt = du$ $\qquad$ $u \to \infty \Rightarrow t \to \infty$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2t^2} dt$$

$$= \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} 2\int_{0}^{\infty} e^{-2t^2} dt$$

put $t^2 = y$ | $t \to 0 \Rightarrow y \to 0$

$2t\,dt = dy$ | $t \to \infty \Rightarrow y \to \infty$

$$dt = \frac{1}{2t} dy$$

$$= \frac{1}{2\sqrt{y}} dy$$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}} 2\int_{0}^{\infty} e^{-2y} \frac{1}{2\sqrt{y}} dy$$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-2y} y^{-1/2} dy$$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}} \frac{\sqrt{\pi}}{\sqrt{2}} \qquad [\because \int_{0}^{\infty} e^{-2t} t^{-1/2} dt = \frac{\sqrt{\pi}}{\sqrt{2}}]$$

$$= \frac{1}{2} e^{-x^2/2}$$

$$F[f(x) * g(x)] = F\left[\frac{1}{2} e^{-x^2/2}\right]$$

$$= \frac{1}{2} F[e^{-x^2/2}]$$

We know that, $e^{-x^2/2}$ is self reciprocal under Fourier transform.

$$\therefore \quad F[e^{-x^2/2}] = e^{-s^2/2}$$

$$\therefore \quad F[f(x) * g(x)] = \frac{1}{2} e^{-s^2/2} \qquad \ldots (2)$$

$\therefore$ (1) = (2)

Hence, convolution theorem is verified.

(OR)

12. (b) (i) Obtain the Fourier Transform of $e^{-x^2/2}$

Solution : Given : $f(x) = e^{\frac{-x^2}{2}}$

We know that, $F[s] = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2} + isx}\, dx$$

We know that,

$(a-b)^2 = a^2 - 2ab + b^2$

$a^2 - 2ab = (a-b)^2 - b^2$

Here, $a = x$, $b = is$

$\therefore x^2 - 2\,isx = (x-is)^2 - (is)^2$

$= (x-is)^2 + s^2$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[x^2 - 2i\,sx]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[(x-is)^2 + s^2]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-is)^2 - \frac{1}{2}s^2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-is)^2}\, e^{-\frac{s^2}{2}}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{\frac{-s^2}{2}} \int_{-\infty}^{\infty} e^{-\left(\frac{x-is}{\sqrt{2}}\right)^2}\, dx$$

put $y = \frac{x - is}{\sqrt{2}}$ | $x \to -\infty \Rightarrow y \to -\infty$

$dy = \frac{1}{\sqrt{2}} dx$ | $x \to \infty \Rightarrow y \to \infty$

$\Rightarrow dx = \sqrt{2}\, dy$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{-s^2}{2}} \int_{-\infty}^{\infty} e^{-y^2} \sqrt{2}\, dy$$

$$= \frac{1}{\sqrt{\pi}} e^{\frac{-s^2}{2}} \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \frac{1}{\sqrt{\pi}} e^{-s^2/2}\, 2\int_{0}^{\infty} e^{-y^2} dy \quad [\because e^{-y^2} \text{ is even in } (-\infty, \infty)]$$

$$= \frac{2}{\sqrt{\pi}} e^{-s^2/2} \left(\frac{\sqrt{\pi}}{2}\right) \quad [\because \int_{0}^{\infty} e^{-y^2} dy = \frac{\sqrt{\pi}}{2}]$$

$$= e^{-s^2/2}$$

Hence, $(x) = e^{-x^2/2}$ is self reciprocal with respect to Fourier transform.

12. (b) (ii) Evaluate $\int_{0}^{\infty} \frac{dx}{(x^2 + a^2)^2}$ using Parseval's identity.

Solution : Let $f(x) = e^{-ax}$

We know that, $F_c(s) = F_c[f(x)] = F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2 + a^2}\right]$

By using Parseval's identity,

$$\int_{0}^{\infty} \left|F_c(s)\right|^2 ds = \int_{0}^{\infty} |f(x)|^2 dx$$

$$\int_{0}^{\infty} \left[\sqrt{\frac{2}{\pi}} \frac{a}{s^2 + a^2}\right]^2 ds = \int_{0}^{\infty} (e^{-ax})^2 dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{a^2}{(s^2+a^2)^2}\,ds = \int_0^\infty e^{-2ax}\,dx$$

$$\frac{2a^2}{\pi}\int_0^\infty \frac{1}{(s^2+a^2)^2}\,ds = \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty$$

$$\frac{2a^2}{\pi}\int_0^\infty \frac{ds}{(s^2+a^2)^2} = 0-\left(\frac{1}{-2a}\right) \quad [\because e^{-\infty}=0,\ e^{-0}=1]$$

$$\frac{2a^2}{\pi}\int_0^\pi \frac{ds}{(s^2+a^2)^2} = \frac{1}{2a}$$

$$\int_0^\infty \frac{ds}{(s^2+a^2)^2} = \frac{\pi}{4a^3}$$

(i.e.,) $$\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3} \quad [\because s \text{ is a dummy variable}]$$

13. (a) (i) Solve : $x(y^2-z^2)p+y(z^2-x^2)q=z(x^2-y^2)$

Solution : Given : $x(y^2-z^2)p+y(z^2-x^2)q = z(x^2-y^2)$ ... (1)

This equation is of the form $Pp + Qq = R$

where $P = x(y^2-z^2)$, $Q = y(z^2-x^2)$, $R = z(x^2-y^2)$

The Lagrange's subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

i.e., $$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)} \quad \text{... (2)}$$

Use Lagrange multipliers x, y, z, we get each ratio in (2)

$$= \frac{x\,dx+y\,dy+z\,dz}{x^2(y^2-z^2)+y^2(z^2-x^2)+z^2(x^2-y^2)} = \frac{x\,dx+y\,dy+z\,dz}{0}$$

i.e., $x\,dx+y\,dy+z\,dz = 0$

Integrating, we get

$$\int x\,dx+\int y\,dy+\int z\,dz = 0$$

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2} = \frac{a}{2}$$

i.e., $x^2+y^2+z^2 = a$

Use Lagrange multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, we get

each ratio in (2)

$$= \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{y^2-z^2+z^2-x^2+x^2-y^2} = \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$$

i.e., $\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz = 0$

Integrating, we get

$$\int\frac{1}{x}dx+\int\frac{1}{y}dy+\int\frac{1}{z}dz = 0$$

$$\log x+\log y+\log z = \log b$$

$$\log(xyz) = \log b$$

i.e., $xyz = b$

Hence, the general solution is $f(a,b)=0$ i.e., $f(x^2+y^2+z^2, xyz)=0$, where f is arbitrary.

13. (a) (ii) Solve : $(D^2+DD'-6D'^2)z = y\cos x$

Solution : Given $[D^2+DD'-6D'^2]z = y\cos x$

The auxiliary equation is $m^2+m-6 = 0$

$$m = 2, -3$$

$$C.F = \phi_1(y+2x)+\phi_2(y-3x)$$

$$P.I = \frac{1}{D^2+DD'-6D'^2}\,y\cos x$$

$$= \frac{1}{(D-2D')(D+3D')}\,y\cos x$$

$$= \frac{1}{D-2D'} \int (c+3x)\cos x\,dx \quad \text{when } y = c+3x$$

$$= \frac{1}{D-2D'} [(c+3x)\sin x - 3\int \sin x\,dx] \quad \text{when } c = y-3x$$

$$= \frac{1}{D-2D'} [y\sin x + 3\cos x]$$

$$= \int [(c-2x)\sin x + 3\cos x]\,dx \quad \text{when } y = c-2x$$

$$= [(c-2x)(-\cos x) - (-2)(-\sin x) + 3\sin x) \quad \text{when } c = y+2x$$

$$= -y\cos x + \sin x$$

Hence, the general solution is

$\therefore\ z = \text{C.F} + \text{P.I}$

$$= \phi_1(y+2x) + \phi_2(y-3x) - y\cos x + \sin x$$

Formulae :

$$\frac{1}{D-mD'} f(x,y) = \int F(x, c-mx)\,dx \quad \text{where } y = c-mx$$

$$\frac{1}{D+mD'} f(x,y) = \int F(x, c+mx)\,dx \quad \text{where } y = c+mx$$

(OR)

13. (b) (i) **Solve : $z = px + qy + \sqrt{p^2+q^2+1}$**

Solution : Given $z = px + qy + \sqrt{1+p^2+q^2}$

This is of the form $z = px + qy + f(p,q)$
[Clairaut's form]

Hence, the **complete integral** is $z = ax + by + \sqrt{1+a^2+b^2}$
where a and b are arbitrary constants.

Since the number of a.c. = number of I.V

Singular solution is found as follows.

$$z = ax + by + \sqrt{1+a^2+b^2} \qquad \ldots (1)$$

diff. (1) p.w.r.to a, we get

$$0 = x + \frac{a}{\sqrt{1+a^2+b^2}}$$

diff. (1) p.w.r.to b, we get

$$0 = y + \frac{b}{\sqrt{1+a^2+b^2}}$$

$$\text{i.e.,} \quad x = -\frac{a}{\sqrt{1+a^2+b^2}} \quad \ldots (2)$$

$$\text{i.e.,} \quad y = -\frac{b}{\sqrt{1+a^2+b^2}} \quad \ldots (3)$$

$$x^2+y^2 = \frac{a^2+b^2}{1+a^2+b^2}$$

$$1-(x^2+y^2) = 1 - \frac{a^2+b^2}{1+a^2+b^2}$$

$$1-x^2-y^2 = \frac{1+a^2+b^2-a^2-b^2}{1+a^2+b^2}$$

$$1-x^2-y^2 = \frac{1}{1+a^2+b^2}$$

$$\sqrt{1-x^2-y^2} = \frac{1}{\sqrt{1+a^2+b^2}} \quad \ldots (i)$$

$$\sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1-x^2-y^2}} \quad \ldots (ii)$$

$$(2) \Rightarrow \quad x = -a\sqrt{1-x^2-y^2} \quad \text{by (i)}$$

$$(3) \Rightarrow \quad y = -b\sqrt{1-x^2-y^2} \quad \text{by (i)}$$

$$a = \frac{-x}{\sqrt{1-x^2-y^2}}, \quad b = \frac{-y}{\sqrt{1-x^2-y^2}}$$

Sub in (1), we get

$$z = \frac{-x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}} \quad \text{by (ii)}$$

$$= \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}}$$

$$z = \sqrt{1-x^2-y^2}$$

$$z^2 = 1-x^2-y^2$$

$x^2+y^2+z^2 = 1$ is the singular solution.

To get the general integral,

put $b = \phi(a)$ in (1), we get

$$z = ax + \phi(a)\, y + \sqrt{1+a^2+[\phi(a)]^2} \quad \text{... (4)}$$

diff. (4) p.w.r. to 'a', we get

$$0 = x + \phi'(a)\, y + \frac{[2a + 2\phi(a)\,\phi'(a)]}{2\sqrt{1+a^2+[\phi(a)]^2}} \quad \text{... (5)}$$

eliminate a between (4) & (5), we get the general solution.

13. (b) (ii) Solve : $(D^3 - 7DD'^2 - 6D'^2)z = \sin(2x+y)$

Solution : Given : $(D^3 - 7DD'^2 - 6D'^3)\,z = \sin(2x+y)$

The auxiliary equation is $m^3 - 7m - 6 = 0$

$$(m+1)(m+2)(m-3) = 0$$

i.e., $m = -1$, $m = -2$, $m = 3$

$$\text{C.F} = \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x)$$

$$\text{P.I} = \frac{1}{D^3 - 7\,DD'^2 - 6\,D'^3}\sin(2x+y)$$

$$= \frac{1}{-4D-7(-2)D'-6(-1)D'} \sin(2x+y)$$

$$\boxed{\begin{array}{r} \text{Replace } D^2 \text{ by } -2^2 = -4 \\ D'^2 \text{ by } -1^2 = -1 \\ DD' \text{ by } -(2)(1) = -2 \end{array}}$$

$$= \frac{1}{-4D + 14D' + 6D'} \sin(2x+y)$$

$$= \frac{1}{-4D + 20D'} \sin(2x+y)$$

$$= \frac{1}{-4} \left[\frac{1}{D - 5D'}\right] \sin(2x+y)$$

$$= -\frac{1}{4} \frac{D + 5D'}{D^2 - 25D'^2} \sin(2x+y)$$

$$= -\frac{1}{4} \frac{D + 5D'}{-4 - 25(-1)} \sin(2x+y)$$

$$\boxed{\begin{array}{r} \text{Replace } D^2 \text{ by } -2^2 = -4 \\ D'^2 \text{ by } -1^2 = -1 \end{array}}$$

$$= -\frac{1}{4} \frac{D + 5D'}{21} \sin(2x+y)$$

$$= \frac{-1}{84} (D + 5D') \sin(2x+y)$$

$$= \frac{-1}{84} [2\cos(2x+y) + 5\cos(2x+y)]$$

$$= \frac{-1}{84} [7\cos(2x+y)] = -\frac{1}{12} \cos(2x+y)$$

$$\therefore z = \text{C.F} + \text{P.I}$$

$$= \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x) - \frac{1}{12}\cos(2x+y)$$

14. (a) A tightly stretched string between the fixed end points x = 0 and x = l is initially at rest in its equilibrium position. If each of its points is given a velocity $kx(l-x)$, find the displacement y(x, t) of the string.

Solution : The wave equation is $\dfrac{\partial^2 y}{\partial t^2} = a^2 \dfrac{\partial^2 y}{\partial x^2}$

From the given problem we get the following boundary and initial conditions,

(i) $y(0,t) = 0$ for all $t > 0$

(ii) $y(l, t) = 0$ for all $t > 0$

(iii) $y(x,0) = 0$, $0 < x < l$

(iv) $\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = kx(l-x)$, $0 < x < l$

Now, the suitable solution which satisfies our boundary conditions is gien by

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos p\,at + c_4 \sin p\,at) \quad \dots (1)$$

Applying condition (i) in equation (1), we get

$$y(0,t) = c_1(c_3 \cos p\,at + c_4 \sin p\,at) = 0$$

Here, $c_3 \cos p\,at + c_4 \sin p\,at \neq 0$ [$\because$ It is defined for all t]

Therefore, we get $c_1 = 0$

Substitute $c_1 = 0$ in (1), we get

$$y(x,t) = c_2 \sin px\,(c_3 \cos p\,at + c_4 \sin p\,at) \quad \dots (2)$$

Applying condition (ii) in equation (2), we get

$$y(l, t) = c_2 \sin pl\,(c_3 \cos p\,at + c_4 \sin p\,at) = 0$$

Here, $(c_3 \cos p\,at + c_4 \sin p\,at) \neq 0$ [$\because$ It is defined for all t]

Therefore, either $c_2 = 0$ or $\sin pl = 0$

Suppose, we take $c_2 = 0$ and already we have $c_1 = 0$ then we get a trivial solution.

Therefore, we consider $c_2 \neq 0$ and $\sin pl = 0$

$$\sin pl = 0$$

$$pl = n\pi \qquad [\because \sin n\pi = 0]$$

$$p = \frac{n\pi}{l} \qquad [n \text{ being an integer}]$$

Now, substituting $p = \frac{n\pi}{l}$ in equation (2), we get

$$y(x,t) = c_2 \sin\frac{n\pi x}{l}\left(c_3 \cos\frac{n\pi at}{l} + c_4 \sin\frac{n\pi at}{l}\right) \quad \ldots (3)$$

Applying condition (iii) in equation (3), we get

$$y(x,0) = c_2 \sin\frac{n\pi x}{l}\, c_3 = 0$$

$$\Rightarrow c_2 c_3 \sin\frac{n\pi x}{l} = 0$$

Here, $\sin\frac{n\pi x}{l} \neq 0$ [$\because$ It is defined for all x]

$c_2 \neq 0$ [$\because$ If $c_2 = 0$ we already explained]

Therefore, $c_3 = 0$

Substitute $c_3 = 0$ in equation (3), we get

$$y(x,t) = c_2 c_4 \sin\frac{n\pi x}{l} \sin\frac{n\pi a t}{l}$$

$$= C_n \sin\frac{n\pi x}{l} \sin\frac{n\pi a t}{l} \quad \ldots (4) \text{ where } C_n = c_2 c_4$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l} \quad \ldots (5)$$

Before applying condition (iv), diff. (5) p.w.r.to 't', we get

$$\left(\frac{\partial y}{\partial t}\right)_{(x,t)} = \sum_{n=1}^{\infty} C_n \left(\sin\frac{n\pi x}{l}\right)\left(\frac{n\pi a}{l}\right)\left(\cos\frac{n\pi at}{l}\right)$$

Now, we apply condition (iv), we get

$$\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l}\,\frac{n\pi a}{l} = kx(l-x)$$

$$= \sum_{n=1}^{\infty} \left[C_n \frac{n\pi a}{l}\right] \sin\frac{n\pi x}{l} = kx\,(l-x)$$

$$= \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = kx\,(l-x) \quad \ldots (6)$$

where $B_n = C_n \dfrac{n\pi a}{l}$

To find B_n : Expand $kx\,(l-x)$ in a half range Fourier sine series in the interval $(0, l)$

$$kx\,(l-x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \ldots (7) \text{ where}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

From (6) and (7), we get $b_n = B_n$

$$\therefore \quad B_n = \frac{2}{l}\int_0^l kx\,(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$B_n = \frac{2k}{l}\int_0^l (lx-x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2k}{l}\left[(lx-x^2)\left[\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)}\right] - (l-2x)\left[\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right] + (-2)\left[\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}\right]\right]_0^l$$

$$= \frac{2k}{l}\left[-(lx-x^2)\left(\frac{l}{n\pi}\right)\cos\frac{n\pi x}{l} + (l-2x)\left(\frac{l}{n\pi}\right)^2 \sin\frac{n\pi x}{l} - 2\left(\frac{l}{n\pi}\right)^3 \cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{2k}{l}\left[\left(-0+0-2\left(\frac{l}{n\pi}\right)^3 (-1)^n\right) - \left(-0+0-2\left(\frac{l}{n\pi}\right)^3\right)\right]$$

$$= \frac{2k}{l}\left[-2\left(\frac{l}{n\pi}\right)^3 (-1)^n + 2\left(\frac{l}{n\pi}\right)^3\right]$$

$$= \frac{2k}{l}\, 2\left(\frac{l}{n\pi}\right)^3 [1-(-1)^n] = \frac{4kl^2}{n^3\pi^3}[1-(-1)^n]$$

$$B_n = \frac{4kl^2}{n^3\pi^3}[1-(-1)^n]$$

$$C_n\left(\frac{n\pi a}{l}\right) = \frac{4kl^2}{n^3\pi^3}[1-(-1)^n]$$

$$C_n = \left(\frac{l}{n\pi a}\right)\left(\frac{4kl^2}{n^3\pi^3}\right)[1-(-1)^n]$$

$$= \frac{4kl^3}{a n^4\pi^4}[1-(-1)^n]$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{8kl^3}{a n^4 \pi^4} & \text{if } n \text{ is odd} \end{cases}$$

Substitute the value of C_n in equation (5) we get

$$y(x,t) = \sum_{n=\text{odd}}^{\infty} \frac{8kl^3}{a n^4\pi^4} \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l}$$

$$= \frac{8kl^3}{a\pi^4} \sum_{n=\text{odd}}^{\infty} \frac{1}{n^4} \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l}$$

$$= \frac{8kl^3}{a\pi^4} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin\frac{(2n-1)\pi x}{l} \sin\frac{(2n-1)\pi at}{l}$$

(OR)

$$y(x,t) = \frac{8kl^3}{a\pi^4} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} \sin\frac{(2n+1)\pi x}{l} \sin\frac{(2n+1)\pi at}{l}$$

(OR)

14. (b) **An infinitely long rectangular plate is of width 10 cm. The temperature along the short edge $y = 0$ is given by**

$$u = \begin{cases} 20x & , 0 < x < 5 \\ 20(10 - x) , & 5 < x < 10 \end{cases}$$

If all the other edges are kept at zero temperature, find the steady state temperature at any point on it.

Proof : Let $u(x,y)$ be the temperature at any point (x,y) in the steady state. Then u satisfies the differential equation $\nabla^2 u = 0$

$$\text{i.e., } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{... (1)}$$

The boundary conditions are

(i) $u(0,y) = 0, \ \forall y$

(ii) $u(10,y) = 0, \ \forall y$

(iii) $u(x,\infty) = 0, \ 0 < x < 10$

(iv) $u(x,0) = \begin{cases} 20x & , 0 \le x \le 5 \\ 20(10-x), & 5 < x \le 10 \end{cases}$

Solving (1) by choosing the suitable solution, we have

$$u(x,y) = (A\cos px + B\sin px)\ (Ce^{py} + De^{-py}) \quad \text{... (2)}$$

By (i), $u(0,y) = A\ (Ce^{py} + De^{-py}) = 0$

$$Ce^{py} + De^{-py} \ne 0$$

$$\therefore A = 0$$

$$\therefore (2) \Rightarrow u(x,y) = B\sin px\ (Ce^{py} + De^{-py}) \quad \text{... (3)}$$

Apply condition (ii), we get

$$u(10,y) = B\sin 10p\ (Ce^{py} + De^{-py}) = 0$$

If B = 0, already A = 0 so we get a trivial solution.

$$\therefore B \ne 0$$

$$Ce^{py} + De^{-py} \ne 0 \quad \text{[since defined for all } y]$$

$$\sin 10p = 0$$

$$10p = n\pi$$

$$p = \frac{n\pi}{10}$$

$$(3) \Rightarrow u(x,y) = B \sin \frac{n\pi x}{10}\left[Ce^{\frac{n\pi y}{10}} + De^{\frac{-n\pi y}{10}}\right] \quad \ldots (4)$$

By condition (iii), we get

$$u(x,\infty) = B \sin \frac{n\pi x}{10}\left[Ce^{\infty} + De^{-\infty}\right] = 0 \quad \text{i.e., } C = 0$$

$$(4) \Rightarrow u(x,y) = B \sin \frac{n\pi x}{10} De^{\frac{-n\pi y}{10}}$$

$$= BD \sin \frac{n\pi x}{10} e^{\frac{-n\pi y}{10}} \quad \ldots (5)$$

The most general solution is

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{10} e^{\frac{-n\pi y}{10}} \quad \ldots (6)$$

Apply condition (iv), we get

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{10} = \begin{cases} 20x & , 0 \le x \le 5 \\ 20(10-x), & 5 < x \le 10 \end{cases}$$

$$\therefore B_n = \frac{2}{10}\left[\int_0^5 20x \sin \frac{n\pi x}{10} dx + \int_5^{10} 20(10-x) \sin \frac{n\pi x}{10} dx\right]$$

$$= 4\int_0^5 x \sin \frac{n\pi x}{10} dx + 4\int_5^{10} (10-x) \sin \frac{n\pi x}{10} dx$$

$$= 4\left[x\left[\frac{-\cos \frac{n\pi x}{10}}{\left(\frac{n\pi}{10}\right)}\right] - (1)\left[\frac{-\sin \frac{n\pi x}{10}}{\left(\frac{n\pi}{10}\right)^2}\right]\right]_0^5$$

$$+ 4\left[(10-x)\left[\frac{-\cos \frac{n\pi x}{10}}{\left(\frac{n\pi}{10}\right)}\right] - (-1)\left[\frac{-\sin \frac{n\pi x}{10}}{\left(\frac{n\pi}{10}\right)^2}\right]\right]_5^{10}$$

$$= 4\left[-x\left(\frac{10}{n\pi}\right)\cos\frac{n\pi x}{10} + \left(\frac{10}{n\pi}\right)^2 \sin\frac{n\pi x}{10}\right]_0^5$$

$$+ 4\left[-(10-x)\left(\frac{10}{n\pi}\right)\cos\frac{n\pi x}{10} - \left(\frac{10}{n\pi}\right)^2 \sin\frac{n\pi x}{10}\right]_5^{10}$$

$$= 4\left[-\frac{50}{n\pi}\cos\frac{n\pi}{2} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{50}{n\pi}\cos\frac{n\pi}{2} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$B_n = \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$B_n = 0$ for n is even

$$u(x,y) = \sum_{n=odd}^{\infty} \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{10} e^{\frac{-n\pi y}{10}}$$

$$= \frac{800}{\pi^2}\sum_{n=odd}^{\infty} \frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{10} e^{\frac{-n\pi y}{10}}$$

$$= \frac{800}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\sin\frac{(2n-1)\pi}{2}\sin\frac{(2n-1)\pi x}{10} e^{\frac{-(2n-1)\pi y}{10}}$$

15. (a) (i) Find $Z(\cos n\theta)$ and hence deduce $Z\left(\cos\frac{n\pi}{2}\right)$

Solution : Let $a = e^{i\theta}$

$$a^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta$$

We know that, $Z[a^n] = \frac{z}{z-a}, \quad |z| > |a|$

$$Z[(e^{i\theta})^n] = \frac{z}{z-e^{i\theta}} \qquad \text{Here, } a = e^{i\theta}$$

$$Z[e^{in\theta}] = \frac{z}{z-(\cos\theta + i\sin\theta)} \qquad [\because e^{i\theta} = \cos\theta + i\sin\theta]$$

$$Z[\cos n\theta + i\sin n\theta] = \frac{z}{(z-\cos\theta) - i\sin\theta}$$

$$Z\,[\cos n\theta] + i\,Z\,[\sin n\theta]$$

$$= \left[\frac{z}{(z-\cos\theta) - i\sin\theta}\right]\left[\frac{(z-\cos\theta)+i\sin\theta}{(z-\cos\theta)+i\sin\theta}\right]$$

$$= \frac{z\,(z-\cos\theta) + iz\sin\theta}{(z-\cos\theta)^2 + \sin^2\theta}$$

$$= \frac{z\,(z-\cos\theta) + iz\sin\theta}{z^2 - 2z\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$= \frac{z\,(z-\cos\theta) + iz\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$= \frac{z\,(z-\cos\theta)}{z^2 - 2z\cos\theta + 1} + i\,\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

Equating the real and imaginary parts, we get

$$Z\,\{\cos n\theta\} = \frac{z\,(z-\cos\theta)}{z^2 - 2z\cos\theta + 1},\ |z| > 1 \qquad \dots (1)$$

$$Z\,\{\sin n\theta\} = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1},\ |z| > 1 \qquad \dots (2)$$

Put $\theta = \frac{\pi}{2}$ in (1), we get

$$Z\left[\cos n\frac{\pi}{2}\right] = \frac{z\left[z - \cos\frac{\pi}{2}\right]}{z^2 - 2z\cos\frac{\pi}{2} + 1}$$

$$= \frac{z\,[z-0]}{z^2 - 2z\,(0) + 1}$$

$$= \frac{z^2}{z^2+1},\ |z| > 1$$

15. (a) (ii) Using Z-transform solve : $y_{n+2} - 3y_{n+1} - 10y_n = 0$, $y_0 = 1$ and $y_1 = 0$.

Solution : Given $y_{n+2} - 3y_{n+1} - 10y_n = 0$

$$Z[y_{n+2}] - 3Z[y_{n+1}] - 10Z[y_n] = Z(0)$$

$$[z^2 Y(z) - z^2 y(0) - zy(1)] - 3[zY(z) - zy(0)] - 10Y(z) = 0 \quad [\because Z(0) = 0]$$

$$[z^2 Y(z) - z^2 - z(0)] - 3[zY(z) - z] - 10Y(z) = 0$$

$$z^2 Y(z) - z^2 - 3zY(z) + 3z - 10Y(z) = 0$$

$$[z^2 - 3z - 10]Y(z) = z^2 - 3z$$

$$Y(z) = \frac{z^2 - 3z}{z^2 - 3z - 10}$$

$$Y(z) = \frac{z^2 - 3z}{(z-5)(z+2)}$$

$$\frac{Y(z)}{z} = \frac{z-3}{(z-5)(z+2)} = \frac{A}{z-5} + \frac{B}{z+2} \qquad \dots (1)$$

i.e., $z - 3 = A(z+2) + B(z-5)$

put $z = 5$, we get	put $z = -2$, we get
$2 = 7A$	$-5 = -7B$
$A = \frac{2}{7}$	$B = \frac{5}{7}$

$$\therefore (1) \Rightarrow \frac{Y(z)}{z} = \frac{2}{7}\frac{1}{z-5} + \frac{5}{7}\frac{1}{z+2}$$

$$Y(z) = \frac{2}{7}\frac{z}{z-5} + \frac{5}{7}\frac{z}{z+2}$$

$$Z[y(n)] = \frac{2}{7}\frac{z}{z-5} + \frac{5}{7}\frac{z}{z+2}$$

$$y(n) = \frac{2}{7}Z^{-1}\left[\frac{z}{z-5}\right] + \frac{5}{7}Z^{-1}\left[\frac{z}{z-5}\right]$$

$$= \frac{2}{7}(5)^n + \frac{5}{7}(-2)^n$$

15. (b) (i) State and prove the second shifting property of Z-transform.

Statement : $Z\,[f(t+T)] = zF(z) - zf(0)$

Proof :
$$Z\,[f(t+T)] = \sum_{n=0}^{\infty} f(nT+T)\, z^{-n}$$
$$= \sum_{n=0}^{\infty} f[(n+1)T]\, z^{-n}$$
$$= z \sum_{n=0}^{\infty} f[(n+1)T]\, z^{-(n+1)}$$
$$= z \sum_{m=1}^{\infty} f(mT)\, z^{-m} \quad \text{where } m = n+1$$
$$= z \left[\sum_{m=0}^{\infty} f(mT)\, z^{-m} - f(0) \right]$$
$$= zF(z) - zf(0)$$

15. (b) (ii) Using convolution theorem, find $Z^{-1}\left[\dfrac{z^2}{(z-a)(z-b)}\right]$

Solution :

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = Z^{-1}\left[\frac{z}{z-a}\cdot\frac{z}{z-b}\right]$$
$$= Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-b}\right]$$
$$= a^n * b^n = \sum_{m=0}^{n} a^m b^{n-m} = b^n \sum_{m=0}^{n} \left(\frac{a}{b}\right)^m$$

$$= b^n \frac{1-\left(\frac{a}{b}\right)^{n+1}}{1-\frac{a}{b}} \quad \text{being a G.P}$$
$$= \frac{b^{n+1}-a^{n+1}}{b-a}$$

Formula : $a + ar + ar^2 + \dots + ar^n = \dfrac{a[1-r^n]}{1-r}$, if $r < 1$