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B.E./B.Tech. DEGREE EXAMINATION, MAY/JUNE 2014.

Third Semester

Civil Engineering

MA 2211/MA 31/MA 1201 A/CK 201/080100008/080210001/10177 MA 301 — TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS/ MATHEMATICS — III

(Common to all branches)

(Regulation 2008/2010)

Time : Three hours Maximum : 100 marks

Answer ALL questions.

PART A — (10 × 2 = 20 marks)

1. State the conditions for a function $f(x)$ to be expanded as a Fourier series in a given interval.

Solution :

Any function $f(x)$ can be developed as a Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ where a_0, a_n & b_n are constants, provided that it satisfies the following Dirichlet's conditions

(i) $f(x)$ is periodic, single valued and finite.

(ii) $f(x)$ has a finite number of finite discontinuities in any one period and has no infinite discontinuity.

(iii) $f(x)$ has at the most a finite number of maxima and minima.

2. Expand $f(x) = 1$ as a half range sine series in the interval $(0, \pi)$.

Solution:

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin nx dx = \frac{2}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi} = \frac{2}{n\pi}\left[1-(-1)^n\right]$$

∴ The Fourier sine series of $f(x) = \sum_1^{\infty} b_n \sin nx$.

$$f(x) = \frac{4}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right]$$

3. Find the Fourier sine transform of $f(x) = \frac{1}{x}$.

Solution: By definition

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$\therefore F_s\left(\frac{1}{x}\right) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin sx}{x} dx \qquad (\text{Put } sx = \theta,\ s\,dx = d\theta)$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin \theta}{\theta} d\theta$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left(\frac{\pi}{2}\right) \qquad \left(\because \int_0^\infty \frac{\sin\theta}{\theta} d\theta = \pi/2\right)$$

$$= \sqrt{\pi/2}$$

4. State the Fourier integral theorem.

If f(x) is piecewise continusouly differentiable and absolutely integrable in $(-\infty, \infty)$, then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{i(x-t)s} dt\, ds \quad \text{(OR)}$$

$$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^{\infty} f(t) \cos\lambda(t-x)\ dt\ d\lambda$$

5. Form the PDE by eliminating the arbitrary constants a, b from the relation $z = ax^3 + by^3$.

Form the PDE by eliminating the constants a & b from $z = ax^n + by^n$.

Differentiate the given equation w.r.t. a and b, we get

$p = nax^{n-1}$ and $q = nby^{n-1}$

$a = \frac{p}{nx^{n-1}}$ and $b = \frac{q}{ny^{n-1}}$

using this in the given equation, we get $z = \frac{p}{n}x + \frac{q}{n}y$

Put n = 3, we get , $z = \frac{p}{3}x + \frac{q}{3}y$

6. Solve : $(D^4 - D'^4)z = 0$.

Solution : Given $(D^4 - D'^4)z = 0$

The auxiliary equation is $m^4 - 1 = 0$

[Replace D by m and D' by 1]

Solving $(m^2 - 1)(m^2 + 1) = 0$

$$m^2 - 1 = 0, \; m^2 + 1 = 0$$

$$m^2 = 1, \; m^2 = -1$$

$$m = \pm 1, \; m = \pm\sqrt{-1}$$

i.e., $m = 1, -1, i, -i$

Here the roots are distinct

$\therefore$ C.F $= \phi_1(y + x) + \phi_2(y - ix) + \phi_3(y + ix) + \phi_4(y - ix)$

Since R.H.S. is zero, there is no particular integral

$\therefore$ Hence, the general solution is

$$z = \phi_1(y + x) + \phi_2(y - x) + \phi_3(y + ix) + \phi_4(y - ix)$$

EXERCISES

7. Write all the solutions of the one-dimensional wave equation $y_{tt} = a^2 y_{xx}$.

$$y(x,t) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)\left(Ce^{\lambda at} + De^{-\lambda at}\right)$$

$$y(x,t) = \left(A\cos\lambda x + B\sin\lambda x\right)\left(C\cos\lambda at + D\sin\lambda at\right)$$

$$y(x,t) = (Ax + B)(Cx + D)$$

7. Write all the ...

8. State the assumptions in deriving the one-dimensions heat flow equat (unsteady state).

Heat flows from a higher temperature to lower temperature. The amount of heat required to produce a given temperature change in a body is proportional to the Mass of the body and the temperature change.

9. Find the Z-transform of n^2.

$$\mathbf{Z}[n^2] = \mathbf{Z}[n \, . \, n]$$

$$= -z \frac{d}{dz} \mathbf{Z}[n]$$

$$= -z \frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$= \frac{z^2+z}{(z-1)^3}$$

9. Find the Z-tran...

10. State the convolution theorem on Z-transforms.

If W(n) is the convolution of two sequences x(n) and y(n), then

$$\mathbf{Z}[\,W(n)\,] = \mathbf{Z}[\,x(n)\,] \, . \, \mathbf{Z}[\,y(n)\,]$$

PART B — (5 × 16 = 80 marks)

11. (a) (i) Expand $f(x) = \begin{cases} 1+\frac{2x}{\pi}, & -\pi < x < 0 \\ 1-\frac{2x}{\pi}, & 0 < x < \pi \end{cases}$ as a full range Fourier series in the interval $(-\pi, \pi)$. Hence deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\infty = \frac{\pi^2}{8}$. (8)

Soln/. $f(x) = \begin{cases} 1+2x/\pi, & -\pi < x < 0 \\ 1-2x/\pi, & 0 < x < \pi \end{cases}$

$f(-x) = \begin{cases} 1-2x/\pi & 0 < x < \pi \\ 1+2x/\pi & -\pi < x < 0 \end{cases}$

$= f(x)$

Hence $f(x)$ is an even function. $\therefore b_n = 0$

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot dx = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cdot dx$

$= \frac{2}{\pi}\left[x - \frac{\cancel{2}x^2}{\cancel{2}\pi}\right]_0^{\pi}$

$= \frac{2}{\pi}\left[\pi - \frac{\pi^{\cancel{2}}}{\cancel{\pi}}\right] = 0$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\cdot dx$

$= \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\cdot dx$

$= \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right) - \left(0-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$

$= \frac{4}{\pi^2}\left[-\frac{\cos n\pi}{n^2} + 1/n^2\right]$

$= \frac{4}{\pi^2 n^2}\left[1-\cos n\pi\right] = \frac{4}{\pi^2 n^2}\left[1-(-1)^n\right]$

$$a_n = \frac{8}{\pi n^2} \quad \text{if } n \text{ is odd.}$$

$$\therefore f(x) = 0 + \frac{8}{\pi^2} \sum_{n=1,3,\dots}^{\infty} \frac{1}{n^2} \cos nx.$$

$$= \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \cdots\right]$$

Put $x = 0$ is a disContinuity point at the middle.

$$\therefore \text{At L.H.S,} \quad \frac{(1+0)+(1-0)}{2} = \frac{2}{2} = 1.$$

$$\text{R.H.S} = \frac{8}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \cdots\right]$$

$$\text{L.H.S} = \text{R.H.S}$$

$$1 = \frac{8}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \cdots = \pi^2/8.$$

(ii) Find the half-range sine series of $f(x) = 4x - x^2$ in the interval $(0, 4)$. Hence deduce the value of the series $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots\infty$. (8)

Solution : The half range sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l} \int_0^l (lx - x^2) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l} \left[(lx - x^2) \underbrace{\left[\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right]}_{\downarrow \; 0} - (l-2x) \underbrace{\left[\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right]}_{\downarrow \; 0} + (-2) \left[\frac{\cos \frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}} \right] \right]_0^l$$

$$= \frac{2}{l} \left[\frac{-2}{\frac{n^3\pi^3}{l^3}} \left(\cos \frac{n\pi x}{l} \right) \right]_0^l$$

$$= \frac{-4l^3}{n^3\pi^3 l} [\cos n\pi - \cos 0]$$

$$= -\frac{4l^2}{n^3\pi^3} [(-1)^n - 1]$$

Case (i) :

When n is even number

$$(-1)^n = 1$$

$$\therefore b_n = -\frac{4l^2}{n^3\pi^3}[1-1]$$

$$= 0$$

Case (ii) :

When n is odd number

$$(-1)^n = -1$$

$$b_n = \frac{-4l^2}{n^3\pi^3}(-1-1)$$

$$b_n = \frac{8l^2}{n^3\pi^3}$$

$$f(x) = \sum_{n=1,3,5\ldots}^{\infty} \frac{8l^2}{n^3\pi^3} \sin\frac{n\pi x}{l}$$

$$f(x) = \frac{8l^2}{\pi^3} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^3} \sin\frac{n\pi x}{l}$$

Put l = 4, we get, $f(x) = \frac{128}{\pi^3}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^3}\sin\left(\frac{n\pi x}{l}\right)$.

Or

(b) (i) Expand $f(x) = \sin x$ as a complex form Fourier series in $(-\pi, \pi)$. (8)

Soln:

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$$

where $C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx} \cdot dx$

$$\therefore C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin x\, e^{-inx} \cdot dx$$

$$= \frac{1}{2\pi} \left[\frac{e^{-inx}}{a^2 - n^2} \left[-in \sin x - \cos x \right] \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi} \left\{ \left[\frac{e^{-in\pi}}{a^2 - n^2} \left(-in \sin \pi - \cos \pi \right) \right] - \left[\frac{e^{in\pi}}{a^2 - n^2} \left(-in \sin(-\pi) - \cos(-\pi) \right) \right] \right\}$$

$$= \frac{1}{2\pi} \left[\frac{e^{-in\pi}}{a^2 - n^2} (1) - \frac{e^{in\pi}}{a^2 - n^2} (-1) \right]$$

$$= \frac{1}{2\pi} (-1)^n \left[\frac{1}{a^2 - n^2} + \frac{1}{a^2 - n^2} \right] = \frac{\cancel{2}}{\cancel{2}\pi} \frac{(-1)^n}{a^2 - n^2}$$

$$C_n = \frac{1}{\pi} \frac{(-1)^n}{a^2 - n^2}$$

$$\therefore f(x) = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2 - n^2} e^{inx}$$

(ii) Compute the first three harmonics of the Fourier series for $f(x)$ from the following data : (8)

x:	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
$f(x)$:	1.0	1.4	1.9	1.7	1.5	1.2	1.0

x	y	$\cos x$	$y\cos x$	$\cos 2x$	$y\cos 2x$	$\cos 3x$	$y\cos 3x$	$\sin x$	$y\sin x$	$\sin 2x$	$y\sin 2x$	$\sin 3x$	$y\sin 3x$
0	1	1	1	1	1	1	1	0	0	0	0	0	0
60	1.4	0.5	0.7	–0.5	–0.7	–1	–1.4	0.866	1.212	0.866	1.212	0	0
120	1.9	–0.5	–0.95	–0.5	–0.95	1	1.9	0.866	1.645	–0.866	–1.645	0	0
180	1.7	–1	–1.7	1	1.7	–1	–1.7	0	0	0	0	0	
240	1.5	–0.5	–0.75	–0.5	–0.75	1	1.5	–0.866	–1.299	0.866	1.299	0	0
300	1.2	0.5	0.6	–0.5	–0.6	–1	–1.2	–0.866	–1.039	–0.866	–1.093	0	0
	Σy $=8.7$		$\Sigma y\cos x$ $=-1.1$		$\Sigma\cos 2x$ $=-0.3$		$\Sigma\cos 3x$ $=0.1$		$\Sigma\sin x$ $=0.519$		$\Sigma y\sin 2x$ $=-0.173$		$\Sigma y\sin 3x$ $=0$

$\therefore$ we can omit the last value.

$\therefore$ Hence $n = 6$

$$a_0 = \frac{2}{n}\Sigma y = \frac{2}{6}\times 8.7 = 2.900$$

$$a_1 = \frac{2}{n}\Sigma y\cos x = \frac{2}{6}\times(-1.1) = -0.366$$

$$a_2 = \frac{2}{n}\Sigma y\cos 2x = \frac{2}{6}\times(-0.3) = -0.1$$

$$a_3 = \frac{2}{n}\Sigma y\cos 3x = \frac{2}{6}\times(0.1) = 0.033$$

$$b_1 = \frac{2}{n}\Sigma y\sin x = \frac{2}{6}\times(0.5196) = 0.1732$$

$$b_2 = \frac{2}{n}\Sigma y\sin 2x = \frac{2}{6}\times(-0.1736) = -0.0578$$

$$b_3 = \frac{2}{n}\Sigma y\sin 3x = \frac{2}{6}\times(0) = 0$$

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + b_1\sin x + b_2\sin 2x + b_3\sin 3x$$

$$= \frac{2.9}{2} - 0.366\cos x - 0.1\cos 2x + 0.033\cos 3x + 0.1732\sin x - 0.0578\sin 2x + 0$$

$$f(x) = 1.45 - 0.366\cos x - 0.1\cos 2x + 0.033\cos 3x + 0.1732\sin x - 0.0578\sin 2x$$

12. (a) (i) Find the Fourier transform of $e^{-a|x|}$, $a > 0$ and hence deduce that

(1) $\int_0^\infty \frac{\cos xt}{a^2+t^2} dt = \frac{\pi}{2a} e^{-a|x|}$

(2) $F\{xe^{-a|x|}\} = i\sqrt{\frac{2}{\pi}} \frac{2as}{(s^2+a^2)^2}$, here F stands for Fourier transform. (8)

Solution : We know that,

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$F[e^{-a|x|}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} (\cos sx + i \sin sx) dx$$

$$= \frac{1}{\sqrt{2\pi}} 2\int_0^\infty e^{-ax} \cos sx \, dx \qquad [\because |x| = x \text{ in } (0, \infty)]$$

Since, $e^{-a\|x\|} \cos sx$ is an even function in $(-\infty, \infty)$ $\therefore \int_{-\infty}^{\infty} e^{-a\|x\|} \cos sx\, dx = 2\int_0^{\infty} e^{-ax} \cos sx\, dx$
$e^{-a\|x\|} \sin sx$ is an odd function, $(-\infty, \infty)$ $\int_{-\infty}^{\infty} e^{-a\|x\|} \sin sx\, dx = 0$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2 + a^2}\right] \quad \boxed{\because \text{Formula: } \int_0^{\infty} e^{-ax} \cos bx\, dx = \frac{a}{a^2 + b^2}}$$

Using inversion formula, we get

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[f(x)]\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{a}{s^2 + a^2} e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}}\, a \int_{-\infty}^{\infty} \frac{e^{-isx}}{s^2 + a^2} ds$$

$$= \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{\cos sx - i \sin sx}{s^2 + a^2} ds$$

$$= \frac{a}{\pi} \left[2\int_0^{\infty} \frac{\cos sx}{s^2 + a^2} ds\right]$$

Since, $\frac{\cos sx}{s^2 + a^2}$ is an even function in $(-\infty, \infty)$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos sx}{s^2 + a^2}\, ds = 2 \int_{0}^{\infty} \frac{\cos sx}{s^2 + a^2}\, ds$$

$\frac{\sin sx}{s^2 + a^2}$ is an odd function in $(-\infty, \infty)$

$$\therefore \int_{-\infty}^{\infty} \frac{\sin sx}{s^2 + a^2}\, ds = 0$$

$$f(x) = \frac{2a}{\pi} \int_{0}^{\infty} \frac{\cos sx}{s^2 + a^2}\, ds$$

$$\int_{0}^{\infty} \frac{\cos sx}{s^2 + a^2}\, ds = \frac{\pi}{2a} f(x)$$

$$\int_{0}^{\infty} \frac{\cos xt}{a^2 + t^2}\, dt = \frac{\pi}{2a} e^{-a|x|} \quad [\because s \text{ is a dummy variable}]$$

$$F[xe^{-a|x|}] = -i \frac{d}{ds} F(s)$$

$$= -i \frac{d}{ds} F[e^{-a|x|}]$$

$$= -i \frac{d}{ds} \left[\sqrt{\frac{2}{\pi}} \frac{a}{s^2 + a^2} \right]$$

$$= -i \sqrt{\frac{2}{\pi}}\, a \frac{d}{ds} \left[\frac{1}{s^2 + a^2} \right]$$

(ii) Solve for $f(x)$ from the integral equation (8)

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$$\int_0^\infty f(x)\sin sx\,dx = \begin{cases} 1, & 0 \le s < 1 \\ 2, & 1 \le s < 2 \\ 0, & s \ge 2. \end{cases}$$

By inversion formula, we get

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s\,[f(x)]\,\sin sx\,ds$$

$$= \frac{2}{\pi}\left[\int_0^1 \sin sx\,ds + \int_1^2 2\sin sx\,ds\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{-\cos sx}{x}\right)_0^1 + 2\left(\frac{-\cos sx}{x}\right)_1^2\right]$$

$$= \frac{2}{\pi}\left[\left[\left(-\frac{\cos x}{x}\right) - \left(-\frac{1}{x}\right)\right] + 2\left[\left(\frac{-\cos 2x}{x}\right) - \left(\frac{-\cos x}{x}\right)\right]\right]$$

$$= \frac{2}{\pi}\left[\frac{-\cos x}{x} + \frac{1}{x} - \frac{2\cos 2x}{x} + \frac{2\cos x}{x}\right]$$

$$= \frac{2}{\pi}\left[\frac{1 - 2\cos 2x + \cos x}{x}\right]$$

$$= \frac{2 + 2\cos x - 4\cos 2x}{\pi x},\ x > 0$$

(b) (i) Find the Fourier transform of $f(x) = \frac{1}{\sqrt{|x|}}$ (8)

(ii) Using P...

Solution : We know that,

$$F[s] = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{1}{|x|}}\, e^{isx}\, dx$$

$$\text{Let} \quad I = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} \frac{e^{isx}}{\sqrt{-x}}\, dx + \int_{0}^{\infty} \frac{e^{isx}}{\sqrt{x}}\, dx \right] \quad \ldots (i)$$

$$\text{Since,}\ |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \frac{e^{isx}}{\sqrt{-x}}\, dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{e^{isx}}{\sqrt{x}}\, dx$$

$$= I_1 + I_2$$

In I_1, put $-x = y$; $-dx = dy$

when $x \to -\infty \Rightarrow y \to \infty$

$x \to 0 \quad \Rightarrow y \to 0$

$$\therefore I_1 = \int_{0}^{\infty} \frac{e^{-isy}}{\sqrt{y}}\, dy$$

$$= \int_{0}^{\infty} \frac{e^{-isx}}{\sqrt{x}}\, dx \qquad [\because y \text{ is a dummy variable}]$$

There is no change in I_2

$$\therefore I = I_1 + I_2$$

$$= \left[\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{e^{-isx}}{\sqrt{x}}\, dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{e^{isx}}{\sqrt{x}}\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{e^{isx} + e^{-isx}}{\sqrt{x}}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{2\cos sx}{\sqrt{x}}\, dx \quad \left[\because \cos sx = \frac{e^{isx}+e^{-isx}}{2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \text{R.P. } \frac{e^{-isx}}{\sqrt{x}}\, dx$$

$$= \sqrt{\frac{2}{\pi}}\ \text{R.P.} \int_0^\infty \frac{e^{-isx}}{\sqrt{isx}\,\sqrt{is}}\, d(isx)$$

$$= \sqrt{\frac{2}{\pi}}\ \text{R.P.} \frac{1}{\sqrt{is}} \int_0^\infty \frac{e^{-T}\, dT}{\sqrt{T}} \text{ where } T = isx$$

$$= \sqrt{\frac{2}{\pi}}\ \text{R.P.} \frac{(i)^{-1/2}}{\sqrt{s}} \int_0^\infty e^{-T}\, T^{-1/2}\, dT$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{s}}\ \text{R.P.} \left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]^{-1/2} \Gamma_{1/2}$$

$$\left[\because i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$$

$$= \frac{1}{\sqrt{s}} \sqrt{\frac{2}{\pi}}\ \text{R.P.} \left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right] \sqrt{\pi} \qquad \left[\because \Gamma_{1/2} = \sqrt{\pi}\right]$$

$$= \frac{1}{\sqrt{s}} \sqrt{\frac{2}{\pi}} \cos\frac{\pi}{4} \sqrt{\pi}$$

$$= \sqrt{\frac{2}{s}} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{s}}$$

$$\therefore F\left[\frac{1}{\sqrt{|x|}}\right] = \frac{1}{\sqrt{s}}$$

(iii) Using Parseval's identity evaluate the following integrals

(1) $\int_0^{\infty} \frac{dx}{(a^2+x^2)^2}$

(2) $\int_0^{\infty} \frac{x^2}{(a^2+x^2)^2} dx$ where $a > 0$. (8)

[A.U Tvli M/J 2011][A.U M/J 2013]

Solution : Let $f(x) = e^{-ax}$

We know that, $F_c(s) = F_c[f(x)] = F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}}\left[\frac{a}{s^2+a^2}\right]$

By using Parseval's identity,

$$\int_0^{\infty} \left|F_c(s)\right|^2 ds = \int_0^{\infty} |f(x)|^2 dx$$

$$\int_0^{\infty}\left[\sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}\right]^2 ds = \int_0^{\infty}(e^{-ax})^2 dx$$

$$\frac{2}{\pi}\int_0^{\infty}\frac{a^2}{(s^2+a^2)^2} ds = \int_0^{\infty} e^{-2ax} dx$$

$$\frac{2a^2}{\pi}\int_0^{\infty}\frac{1}{(s^2+a^2)^2} ds = \left[\frac{e^{-2ax}}{-2a}\right]_0^{\infty}$$

$$\frac{2a^2}{\pi}\int_0^{\infty}\frac{ds}{(s^2+a^2)^2} = 0-\left(\frac{1}{-2a}\right) \quad [\because e^{-\infty}=0, e^{-0}=1]$$

$$\frac{2a^2}{\pi}\int_0^{\pi}\frac{ds}{(s^2+a^2)^2} = \frac{1}{2a}$$

$$\int_0^{\infty}\frac{ds}{(s^2+a^2)^2} = \frac{\pi}{4a^3}$$

(i.e.,) $$\int_0^{\infty}\frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3} \quad [\because s \text{ is a dummy variable}]$$

Example 2.3.e(8): Using transform methods, evaluate $\int_0^{\infty}\frac{x^2\,dx}{(x^2+a^2)^2}$ where a > 0. **[A.U CBT Dec. 2009]**

Solution : We know that, $F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}$

By using Parseval's identity,

$$\int_0^\infty \left|F_s[f(x)]\right|^2 ds = \int_0^\infty |f(x)|^2 dx$$

$$\int_0^\infty \left(\sqrt{\frac{2}{\pi}}\ \frac{s}{s^2+a^2}\right)^2 ds = \int_0^\infty (e^{-ax})^2 dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)^2} ds = \int_0^\infty e^{-2ax} dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{x^2}{(x^2+a^2)^2} dx = \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty \quad [\because s \text{ is a dummy variable}]$$

$$= 0 - \left(\frac{1}{-2a}\right) = \frac{1}{2a}$$

$$\therefore \int_0^\infty \frac{x^2}{(x^2+a^2)^2} dx = \frac{\pi}{4a}$$

13. (a) (i) Form the PDE by eliminating the arbitrary function from the relation $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$. (8)

Solution : Given $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

$$p = \frac{\partial z}{\partial x} = 2f'\left(\frac{1}{x} + \log y\right)\left[\frac{-1}{x^2}\right]$$

$$= \frac{-2}{x^2} f'\left(\frac{1}{x} + \log y\right) \quad \ldots (1)$$

a.f
f
1
$\therefore$ **we use p & q only**

$$q = \frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right)\left[\frac{1}{y}\right]$$

$$= 2y + \frac{2}{y}f'\left(\frac{1}{x} + \log y\right) \quad \ldots (2)$$

$$(1) \Rightarrow \frac{-px^2}{2} = f'\left(\frac{1}{x} + \log y\right) \quad \ldots (3)$$

$$(2) \Rightarrow \frac{(q-2y)\,y}{2} = f'\left(\frac{1}{x} + \log y\right) \quad \ldots (4)$$

$$\frac{(3)}{(4)} \Rightarrow \frac{\left(\frac{-px^2}{2}\right)}{\frac{(q-2y)\,y}{2}} = 1$$

$$\Rightarrow \frac{-px^2}{(q-2y)\,y} = 1$$

$$-px^2 = qy - 2y^2$$

$px^2 + qy = 2y^2$ is the required p.d.e.

Or

(b) (i) Solve : $x^2p^2 + y^2q^2 = z^2$. (8)

Solution : Given $z = px + qy + p^2 + pq + q^2$

i.e., $z = px + qy + f(p, q)$

This equation is of the Clairaut's form.

Hence, the complete solution is

$$z = ax + by + a^2 + ab + b^2 \quad \ldots (1)$$

Since the number of a.c. = number of I.V

To find the singular integral.

Differentiation (1) p.w.r. to a and b, we get

$$0 = x + 2a + b \quad \ldots (2)$$

$$0 = y + 2b + a \quad \ldots (3)$$

$$(2) \times 1 \Rightarrow 2a + b + x = 0 \quad \ldots (4)$$

$$(3) \times 2 \Rightarrow 2a + 4b + 2y = 0 \quad \ldots (5)$$

$$(4)-(5) \Rightarrow -3b + x - 2y = 0$$

$$3b = x - 2y$$

$$b = \frac{x-2y}{3}$$

$$(3) \Rightarrow \quad 0 = y + 2\left(x - \frac{2y}{3}\right) + a$$

$$= \frac{3y + 2x - 4y}{3} + a$$

$$= \frac{2x - y}{3} + a$$

$$a = -\left(\frac{2x - y}{3}\right)$$

$$a = \frac{y - 2x}{3}$$

Substitute in (1), we get

$$z = \left(\frac{y-2x}{3}\right)x + \left(\frac{x-2y}{3}\right)y + \left(\frac{y-2x}{3}\right)^2 + \left(\frac{y-2x}{3}\right)\left(\frac{x-2y}{3}\right) + \left(\frac{x-2y}{3}\right)^2$$

$$z = \frac{xy - 2x^2}{3} + \frac{xy - 2y^2}{3} + \frac{y^2 + 4x^2 - 4xy}{9} + \frac{xy - 2y^2 - 2x^2 + 4xy}{9} + \frac{x^2 + 4y^2 - 4xy}{9}$$

$$= \frac{1}{9}[3xy - 6x^2 + 3xy - 6y^2 + y^2 + 4x^2 - 4xy + xy - 2y^2 - 2x^2 + 4xy + x^2 + 4y^2 - 4xy]$$

$$= \frac{1}{9}[-3x^2 - 3y^2 + 3xy]$$

$$z = \frac{3}{9}[-x^2 - y^2 + xy]$$

$$z = \frac{1}{3}[-x^2 - y^2 + xy]$$

$$3z = -x^2 - y^2 + xy$$

$3z + x^2 + y^2 - xy = 0$ is the singular integral.

(ii) Solve : $\left(D^2 + DD' - 6D'^2\right)z = y\cos x$. (8)

Solution : Given $[D^2 + DD' - 6D'^2]z = y\cos x$

The auxiliary equation is $m^2 + m - 6 = 0$

$$m = 2, -3$$

$$C.F = \phi_1(y+2x) + \phi_2(y-3x)$$

$$P.I = \frac{1}{D^2 + DD' - 6D'^2}\, y\cos x$$

$$= \frac{1}{(D-2D')(D+3D')}\, y\cos x$$

$$= \frac{1}{D-2D'} \int (c+3x)\cos x\, dx \quad \text{when } y = c+3x$$

$$= \frac{1}{D-2D'} \left[(c+3x)\sin x - 3\int \sin x\, dx\right] \quad \text{when } c = y-3x$$

$$= \frac{1}{D-2D'} [y\sin x + 3\cos x]$$

$$= \int [(c-2x)\sin x + 3\cos x]\, dx \quad \text{when } y = c-2x$$

$$= [(c-2x)(-\cos x) - (-2)(-\sin x) + 3\sin x) \quad \text{when } c = y+2x$$

$$= -y\cos x + \sin x$$

Hence, the general solution is

$$\therefore z = C.F + P.I$$

$$= \phi_1(y+2x) + \phi_2(y-3x) - y\cos x + \sin x$$

14. (a) A string is stretched and fastened to points at a distance 'l' apart. Motion is started by displacing the string in the form $y = a\sin\left(\frac{\pi x}{l}\right)$, $0 < x < l$, from which it is released at time $t = 0$. Find the displacement at any time t. (16)

Solution: Since the string is fixed at the end points, we have the Boundary Conditions

(i) $y(0, t) = 0$

(ii) $y(l, t) = 0$ for all t

As the string is displaced in the form $y = a\sin\frac{\pi x}{l}$ at $t = 0$, we have

(iii) $y(x, 0) = a\sin\frac{\pi x}{l}$.

Since the string is released from rest, the initial velocity is zero.

(iv) $\frac{\partial y}{\partial t}(x, 0) = 0$

The equation giving the **displacement** $y(x, t)$ is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$$

Its solution is

$$y(x, t) = (A_1\cos px + A_2\sin px)(A_3\cos cpt + A_4\sin cpt) \quad \text{... (1)}$$

By the condition (i), we have

$$A_1(A_3\cos cpt + A_4\sin cpt) = 0 \Rightarrow A_1 = 0$$

[If $A_3\cos cpt + A_4\sin cpt = 0$, then $y(x, t) = 0$ which is the trivial solution]

$\therefore$ (1) reduces to

$$y(x,t) = A_2 \sin px\,[A_3 \cos cpt + A_4 \sin cpt]$$
$$= \sin px\,[b_1 \cos cpt + b_2 \sin cpt]$$

where $\quad b_1 = A_2 A_3;\ b_2 = A_2 A_4$ ... (2)

By the condition (ii), we have

$$\sin pl\,[A_3 \cos cpt + A_4 \sin cpt] = 0 \Rightarrow \sin pl = 0$$
$$\therefore pl = n\pi,\ n = 1, 2, 3, \ldots$$
$$p = \frac{n\pi}{l}$$

$\therefore$ (2) reduces to

$$y(x,t) = \sin\frac{n\pi x}{l}\left[b_1 \cos\frac{n\pi ct}{l} + b_2 \sin\frac{n\pi ct}{l}\right] \quad \ldots (3)$$

Now $\dfrac{\partial y}{\partial t} = \sin\dfrac{n\pi x}{l}\left[-b_1\dfrac{n\pi c}{l}\sin\dfrac{n\pi ct}{l} + b_2\dfrac{n\pi c}{l}\cos\dfrac{n\pi ct}{l}\right]$

By the condition (iv), we have

$$b_2\frac{n\pi c}{l}\sin\frac{n\pi x}{l} = 0 \Rightarrow b_2 = 0$$

$\therefore$ (3) reduces to

$y(x,t) = b_1 \sin\dfrac{n\pi x}{l}\cos\dfrac{n\pi ct}{l}$, which is true for all $n = 1, 2, 3 \ldots$

$\therefore$ The general form of the solution is

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \quad \ldots (4)$$

To determine A_n, apply the condition (iii). By the condition (iii), we have

$$a \sin\frac{\pi x}{l} = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}$$
$$= A_1 \sin\frac{\pi x}{l} + A_2 \sin\frac{2\pi x}{l} + \ldots$$

$\therefore A_1 = a,\ A_2 = A_3 = \ldots = 0$

$\therefore$ (4) reduces to

$$y(x, t) = a \sin \frac{\pi x}{l} \cos \frac{n\pi ct}{l}$$

Hence proved.

Or

(b) An infinitely long rectangular plate with insulated surfaces is 10 cm wide. The two long edges and one short edge are kept at 0°C, while the other short edge $x = 0$ is kept at temperature

$u = 20y, \qquad 0 \le y \le 5$

$u = 20(10 - y), \quad 5 < y \le 10.$

Find the steady state temperature distribution in the plate. (16)

Solution : The equation to be solved is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

From the given problem, we get the following boundary conditions

(i) $u(x, 0) = 0$ for all x

(ii) $u(x, 10) = 0$ for all x

(iii) $u(\infty, y) = 0$ (i.e.,) when $x \to \infty$, $u \to 0$

(iv) $u(0, y) = \begin{cases} f_1(y) = 20y & , 0 \le y \le 5 \\ f_2(y) = 20(10 - y), & 5 \le y \le 10 \end{cases}$

Now, the suitable solution which satisfies our boundary conditions is given by

$$u(x, y) = (A e^{px} + B e^{-px})(C \cos py + D \sin py) \quad \text{... (1)}$$

Applying condition (i) in equation (1), we get

$$u(x, 0) = (A e^{px} + B e^{-px}) C = 0$$

Here, $A e^{px} + B e^{-px} \neq 0 [\because$ it is defined for all $x]$

$$\therefore \quad \boxed{C = 0}$$

Substitute, $C = 0$ in equation (1), we get

$$u(x, y) = (A e^{px} + B e^{-px}) D \sin py \quad \text{... (2)}$$

Applying condition (ii) in equation (2), we get

$$u(x, 10) = (A e^{px} + B e^{-px}) D \sin(10p) = 0$$

Here, $A e^{px} + B e^{-px} \neq 0$ [$\because$ it is defined for all x]

$D \neq 0$ [$\because$ If $D = 0$, already $C = 0$ then we get a trivial solution]

$$\therefore \sin 10p = 0$$

$$10p = n\pi \quad [\because \sin n\pi = 0]$$

$$\boxed{p = \frac{n\pi}{10}}$$

Substitute, $p = \frac{n\pi}{10}$ in equation (2), we get

$$u(x, y) = (A e^{\frac{n\pi x}{10}} + B e^{\frac{-n\pi x}{10}}) D \sin \frac{n\pi y}{10} \quad \text{... (3)}$$

Applying condition (iii) in equation (3), we get

$$u(\infty, y) = (A e^{\infty} + B e^{-\infty}) D \sin \frac{n\pi y}{10} = 0$$

(i.e.,) $A e^{\infty} D \sin\left(\frac{n\pi y}{10}\right) = 0$ [$\because e^{-\infty} = 0$]

Here, $\sin \frac{n\pi y}{10} \neq 0$ [$\because$ it is defined for all y]

$D \neq 0$ [$\because$ If $D = 0$ already explained]

$e^{\infty} \neq 0$

$$\therefore \boxed{A = 0}$$

Substitute, A = 0 in equation (3), we get

$$u(x, y) = B\, e^{\frac{-n\pi x}{10}}\, D \sin\frac{n\pi y}{10}$$

$$= BD\, e^{\frac{-n\pi x}{10}} \sin\frac{n\pi y}{10} \qquad \ldots (4)$$

The most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} B_n\, e^{\frac{-n\pi x}{10}} \sin\frac{n\pi y}{10} \qquad \ldots (5)$$

Applying condition (iv) in equation (5), we get

$$u(0, y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi y}{10} = \begin{cases} 20y & , 0 \le y \le 5 \\ 20(10-y), & 5 \le y \le 10 \end{cases} \qquad \ldots (6)$$

To find B_n, expand $f(y)$ in a half range Fourier sine series in [0,10].

$$f(y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{l} \qquad \ldots (7)$$

where $b_n = \frac{2}{l}\int_0^l f(y) \sin\frac{n\pi y}{l}\, dy$

From equation (6) and (7), we get $\boxed{B_n = b_n}$

$$\therefore B_n = \frac{2}{10}\left[\int_0^5 20y \sin\frac{n\pi y}{10}\, dy + \int_5^{10} 20(10-y) \sin\frac{n\pi y}{10}\, dy\right]$$

$$= \frac{1}{5}\left\{\left[(20y)\left[\frac{-\cos\frac{n\pi y}{10}}{\left(\frac{n\pi}{10}\right)}\right] - (20)\left[\frac{-\sin\frac{n\pi y}{10}}{\left(\frac{n\pi}{10}\right)^2}\right]\right]_0^5\right.$$

$$\left.+\left[20(10-y)\left[\frac{-\cos\frac{n\pi y}{10}}{\left(\frac{n\pi}{10}\right)}\right](-20)\left[\frac{-\sin\frac{n\pi y}{10}}{\left(\frac{n\pi}{10}\right)^2}\right]\right]_5^{10}\right\}$$

$$= \frac{1}{5}\left\{\left[-20y\left(\frac{10}{n\pi}\right)\cos\frac{n\pi y}{10} + 20\left(\frac{10}{n\pi}\right)^2 \sin\frac{n\pi y}{10}\right]_0^5\right.$$

$$\left.+\left[-20(10-y)\left(\frac{10}{n\pi}\right)\cos n\frac{\pi y}{10} - 20\left(\frac{10}{n\pi}\right)^2 \sin\frac{n\pi y}{10}\right]_5^{10}\right\}$$

$$= \frac{1}{5}\left[\frac{-1000}{n\pi}\cos\frac{n\pi}{2} + \frac{2000}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{1000}{n\pi}\cos\frac{n\pi}{2} + \frac{2000}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{1}{5}\left[\frac{4000}{n^2\pi^2}\sin\frac{n\pi}{2}\right] = \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$= 0$ if n is even.

Substitute, B_n in equation (5), we get

$$u(x, y) = \sum_{n=0}^{\infty} \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}\, e^{\frac{-n\pi x}{10}} \sin\frac{n\pi y}{10}$$

$$= \frac{800}{\pi^2}\sum_{n=\text{odd}}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\, e^{\frac{-n\pi x}{10}}\sin\frac{n\pi y}{10}$$

$$= \frac{800}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\sin\left[(2n-1)\frac{\pi}{2}\right] e^{\frac{-(2n-1)\pi x}{10}} \sin\frac{(2n-1)\pi y}{10}$$

15. (a) (i) Find the Z-transforms of $r^n \cos n\theta$ and $e^{-at} \cos bt$. (8)

Sol. Let

$$a = re^{i\theta}$$

$$a^n = (re^{i\theta})^n = r^n e^{in\theta}$$

$$= r^n [\cos n\theta + i \sin n\theta]$$

$$= r^n \cos n\theta + i r^n \sin n\theta$$

We know that $Z[a^n] = \dfrac{z}{z-a}$

$$Z[(re^{i\theta})^n] = \frac{z}{z - re^{i\theta}}$$

$$Z[r^n e^{in\theta}] = \frac{z}{z - r(\cos\theta + i\sin\theta)}$$

$$Z[r^n \cos n\theta + i r^n \sin n\theta] = \frac{z}{z - r\cos\theta - ir\sin\theta}$$

$$= \frac{z[(z - r\cos\theta) + ir\sin\theta]}{[(z - r\cos\theta) - ir\sin\theta][(z - r\cos\theta) + ir\sin\theta)]}$$

$$= \frac{z(z - r\cos\theta) + irz\sin\theta}{(z - r\cos\theta)^2 + r^2\sin^2\theta}$$

$$= \frac{z(z - r\cos\theta) + izr\sin\theta}{z^2 - 2zr\cos\theta + r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= \frac{z(z - r\cos\theta) + izr\sin\theta}{z^2 - 2zr\cos\theta + r^2}$$

$$= \frac{z(z - r\cos\theta)}{z^2 - 2zr\cos\theta + r^2} + i\frac{zr\sin\theta}{z^2 - 2zr\cos\theta + r^2}$$

Equating the real and imaginary parts we get

$$Z\{r^n \cos n\theta\} = \frac{z(z - r\cos\theta)}{z^2 - 2zr\cos\theta + r^2}, \quad |z| > r$$

$$Z\{r^n \sin n\theta\} = \frac{zr\sin\theta}{z^2 - 2zr\cos\theta + r^2}, \quad |z| > r$$

Find Z [$e^{-at}\cos bt$]

tion : We know that, $Z\left[e^{-at}f(t)\right] = \left[Z[f(t)]\right]_{z \to ze^{aT}}$

$$Z\left[e^{-at}\cos bt\right] = \left[Z[\cos bt]\right]_{z \to ze^{aT}}$$

$$= \left[\frac{z(z-\cos bT)}{z^2 - 2z\cos bT + 1}\right]_{z \to ze^{aT}}$$

$$= \frac{z e^{aT}[ze^{aT} - \cos bT]}{z^2 e^{2aT} - 2z e^{aT}\cos bT + 1}$$

(ii) Solve $u_{n+2} - 3u_{n+1} + 2u_n = 4^n$, given that $u_0 = 0$, $u_1 = 1$. (8)

Solution:

Taking Z-Transform on both sides, we get

$$Z\left[y_{n+2} - 3y_{n+1} + 2y_n\right] = Z\left[4^n\right]$$

$$\therefore Z\left[y_{n+2}\right] - 3Z\left[y_{n+1}\right] + 2Z\left[y_n\right] = Z\left[4^n\right]$$

$$\therefore z^2\left[\bar{y}(z) - y_0 - y_1 z^{-1}\right] - 3z\left[\bar{y}(z) - y_0\right] + 2\bar{y}(z) = \frac{z}{z-4}$$

Since, $Z\left[4^n\right] = \dfrac{z}{z-4}$ Also, $y_0 = 0$ and $y_1 = 1$ given

$$\therefore \bar{y}(z)\left[z^2 - 3z + 2\right] = \frac{z}{z-4} + z$$

$$(ie)\quad (z-1)(z-2)\,\bar{y}(z) = \frac{z}{z-4} + z$$

$$\therefore \bar{y}(z) = \frac{z}{(z-1)(z-2)(z-4)} + \frac{z}{(z-1)(z-2)} \quad [(ie)\text{Divide by } z\,]$$

$$\therefore \frac{\bar{y}(z)}{z} = \frac{1}{(z-1)(z-2)(z-4)} + \frac{1}{(z-1)(z-2)} \longrightarrow (1)$$

$$\frac{1}{(z-1)(z-2)(z-4)} = \frac{A}{(z-1)} + \frac{B}{(z-2)} + \frac{C}{(z-4)}$$

$\therefore A=\frac{1}{3};\quad B=\frac{-1}{2};\quad C=\frac{1}{6}$

$$\therefore \frac{1}{(z-1)(z-2)(z-4)}=\frac{1}{3}\left(\frac{1}{(z-1)}\right)-\frac{1}{2}\left(\frac{1}{(z-2)}\right)+\frac{1}{6}\left(\frac{1}{(z-4)}\right) \rightarrow (2)$$

Similarly $\frac{1}{(z-1)(z-2)}=\frac{1}{(z-2)}-\frac{1}{(z-1)}$ $\rightarrow (3)$

Substitute (2) and (3) in (1), we get

$$\therefore \frac{\bar{y}(z)}{z}=\frac{1}{3}\cdot\frac{1}{(z-1)}-\frac{1}{2}\cdot\frac{1}{(z-2)}+\frac{1}{6}\cdot\frac{1}{(z-4)}+\frac{1}{(z-2)}-\frac{1}{(z-1)}$$

$$(ie)\ \frac{\bar{y}(z)}{z}=-\frac{2}{3}\cdot\frac{1}{(z-1)}+\frac{1}{2}\cdot\frac{1}{(z-2)}+\frac{1}{6}\cdot\frac{1}{(z-4)}$$

Multiply by z, we get

$$\bar{y}(z)=-\frac{2}{3}\cdot\frac{z}{(z-1)}+\frac{1}{2}\cdot\frac{z}{(z-2)}+\frac{1}{6}\cdot\frac{z}{(z-4)}$$

Taking inverse z-Transforms, we get

$$Z^{-1}\left[\bar{y}(z)\right]=-\frac{2}{3}Z^{-1}\left(\frac{z}{z-1}\right)+\frac{1}{2}Z^{-1}\left(\frac{z}{z-2}\right)+\frac{1}{6}Z^{-1}\left(\frac{z}{z-4}\right)$$

$$y_n=-\frac{2}{3}(1)+\frac{1}{2}2^n+\frac{1}{6}4^n$$ ANS

which is the Required solution of the given difference equation.

Hence, the solution.

Or

(b) (i) Using convolution theorem find inverse Z-transform of $\frac{z^2}{(z-a)(z-b)}$. (8)

Solution :

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = Z^{-1}\left[\frac{z}{z-a}\cdot\frac{z}{z-b}\right]$$

$$= Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-b}\right]$$

$$= a^n * b^n = \sum_{m=0}^{n} a^m b^{n-m} = b^n \sum_{m=0}^{n}\left(\frac{a}{b}\right)^m$$

> Formula :
>
> $a + ar + ar^2 + \dots + ar^n$
>
> $= \dfrac{a[1-r^n]}{1-r}$, if $r < 1$

$$= b^n \frac{1-\left(\frac{a}{b}\right)^{n+1}}{1-\frac{a}{b}} \text{ being a G.P}$$

$$= \frac{b^{n+1}-a^{n+1}}{b-a}$$

(ii) Solve $y_{n+2} - 3y_{n+1} - 10y_n = 0$, given $y_0 = 1$, $y_1 = 0$. (8)

Solution : *[A.U. Nov./Dec. 2009 Trichy]*

Taking Z – transforms on both sides, we get

$$Z[y(n+2)] - 3Z[y(n+1)] - 10Z[y(n)] = 0$$

$$z^2\bar{y} - z^2 y(0) - z\,y(1) - 3z[\bar{y} - y(0)] - 10\bar{y} = 0 \quad \dots (1)$$

where $\bar{y} = Z[y(n)]$

Given $y(0) = 1, y(1) = 0$ ... (2)

Substituting (2) in (1), we get

$$z^2[\bar{y} - 1] - 3z[\bar{y} - 1] - 10\bar{y} = 0$$

$$\bar{y}[z^2 - 3z - 10] - z^2 + 3z = 0$$

$$\bar{y}(z^2 - 3z - 10) = z^2 - 3z$$

$$\bar{y} = \frac{z(z-3)}{z^2 - 3z - 10}$$

$$= \frac{z(z-3)}{(z-5)(z+2)}$$

$$\frac{\bar{y}}{z} = \frac{z-3}{(z-5)(z+2)} \quad \dots (3)$$

$$\frac{z-3}{(z-5)(z+2)} = \frac{A}{z-5} + \frac{B}{z+2}$$

$$z - 3 = A(z+2) + B(z-5)$$

Put $z = -2$,	Put $z = 5$,
$-5 = -7B$	$2 = 7A$
$B = \frac{5}{7}$	$A = \frac{2}{7}$

$$\therefore \quad \frac{z-3}{(z-5)(z+2)} = \frac{\frac{2}{7}}{z-5} + \frac{\frac{5}{7}}{z+2}$$

$$\frac{\bar{y}}{z} = \frac{\frac{2}{7}}{z-5} + \frac{\frac{5}{7}}{z+2} \qquad \text{[Using (3)]}$$

$$\bar{y} = \frac{2}{7}\,\frac{z}{z-5} + \frac{5}{7}\,\frac{z}{z+2}$$

$$Z[y(n)] = \frac{2}{7}\,\frac{z}{z-5} + \frac{5}{7}\,\frac{z}{z+2}$$

$$y(n) = \frac{2}{7}\,Z^{-1}\left[\frac{z}{z-5}\right] + \frac{5}{7}\,Z^{-1}\left[\frac{z}{z+2}\right]$$

$$= \frac{2}{7}\,(5)^n + \frac{5}{7}\,(-2)^n$$

$$\therefore \quad y(n) = \frac{2}{7}\,(5)^n + \frac{5}{7}\,(-2)^n, \quad n \geq 0$$