**Question Paper Code : 97107**

B.E./B.Tech. DEGREE EXAMINATION, NOVEMBER/DECEMBER 2014.

Third Semester

Civil Engineering

MA 6351 — TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS

(Common to all branches except Environmental Engineering, Textile Chemistry, Textile Technology, Fashion Technology and Pharmaceutical Technology)

(Regulation 2013)

Time : Three hours Maximum : 100 marks

Answer ALL questions.

PART A — (10 × 2 = 20 marks)

1. Form the partial differential equation by eliminating the arbitrary function $f$ from $z = f\left(\frac{y}{x}\right)$.

**Sol. :** Differentiate $z = f\left(\frac{x}{y}\right)$ partially w.r.t. x and y.

$$p = \frac{\partial z}{\partial x} = f'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right) \quad \ldots (1)$$

$$q = \frac{\partial z}{\partial y} = f'\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right) \quad \ldots (2)$$

From (1), $py = f'\left(\frac{x}{y}\right)$ ... (3)

Putting for $f'\left(\frac{x}{y}\right)$ from (3) in (2),

$$q = py\left(-\frac{x}{y^2}\right) = \frac{-px}{y}$$

or $px + qy = 0$

which is the required p.d.e.

2. Find the complete solution of $p + q = 1$.

Let $z = ax + by + c$ be the solution where $a + b = 1$

$$a = (1 - b)$$

$\therefore$ $z = ax + (1 - b)y + c$ is the complete solution

3. State the sufficient conditions for existence of Fourier series.

i. f(x) is bounded function of period $2\pi$

ii. f(x) has finite number of maxima or minima

iii. f(x) has finite number of points of discontinuity

then the Fourier series of f(x) converges to f(x) at all points where f(x) is continuous.

Also, it converges to the average value of the right and left hand limits of f(x) at each point where f(x) is discontinuous.

4. If $(\pi - x)^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$ in $0 < x < 2\pi$, then deduce that value of $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

Given f(x) = $x^2$, $a_0 = \frac{2\pi^2}{3}$, $a_n = \frac{4(-1)^n}{n^2}$ and $b_n = 0$

By Parseval's Identity, we have

$$2\pi\left\{\frac{a_0^2}{4} + \frac{1}{2}\sum_{1}^{\infty}\left(a_n^2 + b_n^2\right)\right\} = \int_{-\pi}^{\pi}[f(x)]^2\,dx$$

$$\left\{\frac{a_0^2}{4} + \frac{1}{2}\sum_{1}^{\infty}\left(a_n^2\right)\right\} = \frac{2}{2\pi}\int_{0}^{\pi}x^4\,dx$$

$$\frac{\pi^4}{9} + 8\sum_{1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{5}$$

$$\sum_{1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

5. Classify the partial differential equation $\left(1-x^2\right)z_{xx} - 2xy\,z_{xy} + \left(1-y^2\right)z_{yy} + xz_x + 3x^2y\,z_y - 2z = 0$.

Here $A = 1-x^2$, $B = -2xy$, $C = 1-y^2$

$\Delta = B^2 - 4AC = 4x^2y^2 - 4(1-x^2)(1-y^2) = 4x^2y^2 - 4(1 - x^2 - y^2 + x^2y^2)$

$\Delta = 4x^2y^2 - 4 + 4x^2 + 4y^2 - 4x^2y^2 = -4 + 4x^2 + 4y^2$

If $x^2 + y^2 - 1 > 0$ ➔ when $x, y \in (-\infty, -1) \cup (1, \infty)$ and hence the pde is hyperbolic

If $x^2 + y^2 - 1 < 0$ ➔ when $x \in (-1, 1)$ and $y \in (-1, 1)$ hence the pde is elliptic.

If $x^2 + y^2 - 1 = 0$ ➔ when $(x, y) \in (1,0)$ , $(x, y) \in (-1,0)$, $(x, y) \in (0,1)$, and

$(x, y) \in (0,-1)$, hence the pde is parabolic.

6. Write down the various possible solutions of one dimensional heat flow equation.

$$u(x,y) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)\left(C\cos\lambda y + D\sin\lambda y\right)$$

$$u(x,y) = \left(A\cos\lambda x + B\sin\lambda x\right)\left(Ce^{\lambda y} + De^{-\lambda y}\right)$$

$$u(x,y) = (Ax+B)(Cy+D)$$

7. State and prove modulation theorem on Fourier transforms.

**If $F_c(s)$ is the Fourier Cosine Transform of f(x),**

**show that $F_c\{ f(x) \cos ax \} = ½ \{ F_c(s+a) + F_c(s-a) \}$**

$$F_C\left[f(x)\cos ax\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \quad \cos ax \quad \cos sx dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \, \left(\cos(s+a)x + \cos(s-a)x\right) \, dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \, \left(\cos(s+a)x\right) \, dx \, +$$

$$\frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \, \left(\cos(s-a)x\right) \, dx$$

8. If $F\{f(x)\} = F(s)$, then find $F\{e^{iax} f(x)\}$.

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$F\left[e^{iax} f(x)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s+a)x} dx$$

= F(s+a)

9. Find the $Z$ transform of $n$.

$$Z\,[f(n)] = \sum_{n=0}^{\infty} f(n) \left(\frac{1}{z}\right)^n$$

$$= \sum_{n=0}^{\infty} n \left(\frac{1}{z}\right)^n$$

$$= \left(\frac{1}{z}\right) + \left(\frac{2}{z^2}\right)^2 + \left(\frac{3}{z^3}\right) + \ldots..$$

$$= \frac{1}{z}\left\{1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + \ldots.\right\}$$

$$= \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-2}$$

$$= \frac{z}{(z-1)^2}$$

10. State initial value theorem on $Z$ transforms.

Initial Value Theorem in **Z** transform is f(0) = $\underset{z \to \infty}{Lt} F(z)$ where **Z** [ f(n) ] = F(z)

PART B — (5 × 16 = 80 marks)

11. (a) (i) Find the singular solution of $z = px + qy + p^2 - q^2$. (8)

This is of the type $z = px + qy + f(p, q)$

The complete integral is

$$z = ax + by + a^2 - b^2 \quad \dots (1)$$

Now, $\frac{\partial z}{\partial a} = x + 2a = 0 \Rightarrow a = \frac{-x}{2}$

$\frac{\partial z}{\partial b} = y - 2b = 0 \Rightarrow b = \frac{y}{2}$ $\quad \dots (2)$

Substituting (2) in (1), we get

$$z = \frac{-x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4} = \frac{-x^2}{4} + \frac{y^2}{4}$$

i.e., $y^2 - x^2 = 4z$ which is the S.I.

(ii) Solve $(D^2 - 2DD')z = x^3 y + e^{2x-y}$. (8)

**Solution:**

Put $D = m$; $D' = 1$

AE is $m^2 - 2m = 0$

$m(m-2) = 0$

$\therefore \quad m = 0, \; m = 2$

$$CF = \phi_1(y) + \phi_2(y + 2x)$$

$$PI_1 = \frac{1}{D^2 - 2D'}(e^{2x})$$

Here $a = 2,\ b = \cancel{0}\ -1$

Put $D = a = 2;\ D' = b = \cancel{0}\ -1$

$$P.I_1 = \frac{e^{2x-y}}{4-2(-1)} = \frac{e^{2x-y}}{6}$$

$$\therefore \quad \mathrm{PI}_1 = \frac{e^{2x}}{4-2(0)} = \frac{e^{2x}}{4}$$

$$\mathrm{PI}_2 = \frac{1}{D^2 - 2DD'}(x^3 y)$$

$$= \frac{1}{D^2\left(1 - \frac{2DD'}{D^2}\right)} x^3 y$$

$$= \frac{1}{D^2}\left(1 - \frac{2D'}{D}\right)^{-1} x^3 y$$

$$= \frac{1}{D^2}\left(1 + \frac{2D'}{D} + \left(\frac{2D'}{D}\right)^2 + \cdots\right) x^3 y \qquad \left[\begin{array}{l}\text{Put } \theta = \frac{2D'}{D} \text{ and} \\ \text{use } (1-\theta)^{-1} = 1 + \theta + \theta^2 + \cdots\end{array}\right.$$

$$= \frac{1}{D^2}\left(1 + \frac{2D'}{D}\right) x^3 y \qquad \text{Omitting the terms with } D'^2$$

$$= \frac{1}{D^2}\left[x^3 y + \frac{2D'}{D}(x^3 y)\right]$$

$$= \frac{1}{D^2}\left[x^3 y + \frac{2x^3}{D}\right] \qquad \left[\because\ D'(x^3 y) = \frac{\partial}{\partial y}(x^3 y) = x^3\right]$$

$$= \frac{1}{D^2}\left[x^3 y + \frac{2x^4}{4}\right] \qquad \left[\because\ \frac{1}{D}(x^3) = \int x^3\,dx = \frac{x^4}{4}\right]$$

$$= \frac{1}{D}\left[\frac{1}{D}\left(x^3 y + \frac{x^4}{2}\right)\right]$$

$$= \frac{1}{D}\left[\frac{x^4 y}{4} + \frac{x^5}{10}\right] \qquad \left[\begin{array}{l}\because\ \frac{1}{D}(x^3 y) = \int x^3 y\,dx = \frac{x^4 y}{4} \\ \text{treating } y \text{ as a constant}\end{array}\right]$$

$$= \frac{x^5 y}{20} + \frac{x^6}{60}$$

Solution is $\qquad Z = \mathrm{CF} + \mathrm{PI}_1 + \mathrm{PI}_2$

i.e., $\qquad Z = \dfrac{e^{2x-y}}{6} + \dfrac{x^5 y}{20} + \dfrac{x^6}{60}$

(b) (i) Solve $x(y-z)p+y(z-x)q=z(x-y)$. (8)

The subsidiary equations are

$$\frac{dx}{x(y-z)} = \frac{dy}{y(y-x)} = \frac{dz}{z(x-y)}$$

Choosing 1, 1, 1 as multipliers, we get each of the above ratios

$$= \frac{dx+dy+dz}{0}$$

i.e., $d(x+y+z) = 0$.

Integrating, we get $u = x+y+z = C_1$

Choosing $x$, $y$, $z$ as multipliers we get each ratios

$$= \frac{xdx+ydy+zdz}{0}.$$

Integrating, we get $v = x^2+y^2+z^2 = C_2$

The solution of the given PDE is $\phi\left[x+y+z,\ x^2+y^2+z^2\right]=0$

(ii) Solve $(D^3-7DD'^2-6D'^3)z=\sin(x+2y)$. (8)

The auxiliary equation is

$$m^3 - 7m - 6 = 0$$

The roots are $m = -1, -2, 3$

$$\therefore \text{C.F.} = f_1(y-x)+f_2(y-2x)+f_3(y+3x)$$

$$\text{P.I.}_2 = \frac{1}{D^3 - 7D D'^2 - 6 D'^3} \sin(x+2y)$$

$$= \frac{1}{-D + 28D + 24D'} \sin(x+2y)$$

[Replace $D^2$ by $-1$ and $D'^2$ by $-4$]

$$= \frac{1}{3(9D + 8D')} \sin(x+2y)$$

$$= \frac{1}{3} \frac{9D - 8D'}{81D^2 - 64 D'^2} \sin(x+2y)$$

$$= \frac{1}{3} \frac{9D - 8D'}{-81 + 256} \sin(x+2y)$$

$$= \frac{1}{3 \times 175} [9D \sin(x+2y) - 8D' \sin(x+2y)]$$

$$= \frac{1}{3 \times 175} [9 \cos(x+2y) - 16 \cos(x+2y)]$$

$$= -\frac{1}{75} \cos(x+2y)$$

Hence the complete solution is $z = \text{C.F.} + \text{P.I.}_1 + \text{P.I.}_2$

i.e.,
$$z = f_1(y-x) + f_2(y-2x) + f_3(y+3x) + \frac{x^5 y}{60} - \frac{1}{75}\cos(x+2y)$$

12. (a) (i) Find the Fourier series of $f(x) = x^2$ in $-\pi < x < \pi$. Hence deduce the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$. (8)

**Solution :** Given : $f(x) = x^2, f(-x) = (-x)^2 = x^2 = f(x)$

Therefore $f(x)$ is an even function in $(-\pi, \pi)$. Hence, $\boxed{b_n = 0}$

$\therefore$ The Fourier series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx = \frac{2}{\pi}\int_0^{\pi} x^2\, dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^3}{3} - 0\right] = \frac{2}{\pi}\left[\frac{\pi^3}{3}\right] = \frac{2\pi^2}{3}$$

$$\therefore \boxed{a_0 = \frac{2\pi^2}{3}}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx$$

$$= \frac{2}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right) + 2x\left(\frac{\cos nx}{n^2}\right) - 2\left(\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(0 + \frac{2\pi(-1)^n}{n^2} - 0\right) - (0+0-0)\right]$$

$[\because \sin n\pi = 0;\ \cos n\pi = (-1)^n$

$\sin 0 = 0;\ \cos 0 = 1\ ]$

$$= \frac{2}{\pi}\left(\frac{2\pi(-1)^n}{n^2}\right)$$

$$\therefore \boxed{a_n = \frac{4}{n^2}(-1)^n}$$

$$\therefore \boxed{a_0 = \frac{2\pi^2}{3}} \quad \boxed{a_n = \frac{4}{n^2}(-1)^n} \quad \boxed{b_n = 0}$$

**Using Parseval's theorem,**

$$2\pi\left[\frac{a_0^2}{4} + \frac{1}{2}\sum_1^{\infty}[a_n^2 + b_n^2]\right] = \int_{-\pi}^{\pi}[f(x)]^2\,dx = \int_{-\pi}^{\pi} x^4\,dx$$

$$= 2\int_0^{\pi} x^4\,dx = 2\left[\frac{x^5}{5}\right]_0^{\pi}$$

$$= \frac{2}{5}\left[x^5\right]_0^{\pi} = \frac{2}{5}\left[\pi^5 - 0\right]$$

$$= \frac{2}{5}\left[\pi^5 - 0\right] = \frac{2}{5}\pi^5$$

$$2\pi\left[\frac{\pi^4}{9} + \frac{1}{2}\sum_1^{\infty}\frac{16}{n^4}\right] = \left[\frac{2\pi^5}{5}\right]$$

$$\left[\frac{\pi^4}{9} + \frac{1}{2}\sum_1^{\infty}\frac{16}{n^4}\right] = \left[\frac{\pi^4}{5}\right] \Rightarrow \left[\frac{\pi^4}{9} + 8\sum_1^{\infty}\frac{1}{n^4}\right] = \left[\frac{\pi^4}{5}\right]$$

(ii) Find the half range cosine series expansion of $(x-1)^2$ in $0<x<1$. (8)

**Solution :** Here, $l = 1$

$\therefore$ The required Fourier cosine series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x \quad \dots (1) \text{ where}$$

$$a_0 = 2\int_0^1 (x-1)^2\,dx = 2\left[\frac{(x-1)^3}{3}\right]_0^1 \quad \text{(i.e.,)} \quad \boxed{a_0 = \left[\frac{2}{3}\right]}$$

$$a_n = 2\int_0^1 (x-1)^2 \cos n\pi x\,dx$$

$$a_n = 2\left[(x-1)^2\left(\frac{\sin n\pi x}{n\pi}\right) - 2(x-1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + 2\left(\frac{-\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= 2\left[(0+0-0) - \left(0 + \frac{-2}{n^2\pi^2} - 0\right)\right] \quad \text{(i.e.,)} \quad \boxed{a_n = \frac{4}{n^2\pi^2}}$$

$$(1) \Rightarrow \quad f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} \cos n\pi x$$

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi x \quad \dots (2)$$

(b) (i) Compute the first two harmonics of the Fourier series of $f(x)$ from the table given. (8)

| $x$ | $0$ | $\frac{\pi}{3}$ | $\frac{2\pi}{3}$ | $\pi$ | $\frac{4\pi}{3}$ | $\frac{5\pi}{3}$ | $2\pi$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 1.0 | 1.4 | 1.9 | 1.7 | 1.5 | 1.2 | 1.0 |

**Solution :**

Here first and last $y$ values are repeated

$\therefore$ we can omit the last value.

$\therefore$ Hence $n = 6$

| $x$ | $y$ | $\cos x$ | $y \cos x$ | $\cos 2x$ | $y \cos 2x$ | $\cos 3x$ | $y \cos 3x$ | $\sin x$ | $y \sin x$ | $\sin 2x$ | $y \sin 2x$ | $\sin 3x$ | $y \sin 3x$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 60 | 1.4 | 0.5 | 0.7 | –0.5 | –0.7 | –1 | –1.4 | 0.866 | 1.2124 | 0.866 | 1.212 | 0 | 0 |
| 120 | 1.9 | –0.5 | –0.95 | –0.5 | –0.95 | 1 | 1.9 | 0.866 | 1.6454 | –0.866 | –1.6454 | 0 | 0 |
| 180 | 1.7 | –1 | –1.7 | 1 | 1.7 | –1 | –1.7 | 0 | 0 | 0 | 0 | 0 | |
| 240 | 1.5 | –0.5 | –0.75 | –0.5 | –0.75 | 1 | 1.5 | –0.866 | –1.299 | 0.866 | 1.299 | 0 | 0 |
| 300 | 1.2 | 0.5 | 0.6 | –0.5 | –0.6 | –1 | –1.2 | –0.866 | –1.0392 | –0.866 | –1.0932 | 0 | 0 |
| | $\Sigma y = 8.7$ | | $\Sigma y \cos x = -1.1$ | | $\Sigma \cos 2x = -0.3$ | | $\Sigma \cos 3x = 0.1$ | | $\Sigma \sin x = 0.5196$ | | $\Sigma y \sin 2x = -0.1736$ | | $\Sigma y \sin 3x = 0$ |

$$a_0 = \frac{2}{n}\Sigma y = \frac{2}{6}\times 8.7 = 2.900$$

$$a_1 = \frac{2}{n}\Sigma y\cos x = \frac{2}{6}\times(-1.1) = -0.366$$

$$a_2 = \frac{2}{n}\Sigma y\cos 2x = \frac{2}{6}\times(-0.3) = -0.1$$

$$a_3 = \frac{2}{n}\Sigma y\cos 3x = \frac{2}{6}\times(0.1) = 0.033$$

$$b_1 = \frac{2}{n}\Sigma y\sin x = \frac{2}{6}\times(0.5196) = 0.1732$$

$$b_2 = \frac{2}{n}\Sigma y\sin 2x = \frac{2}{6}\times(-0.1736) = -0.0578$$

$$b_3 = \frac{2}{n}\Sigma y\sin 3x = \frac{2}{6}\times(0) = 0$$

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + b_1\sin x + b_2\sin 2x + b_3\sin 3x$$

$$= \frac{2.9}{2} - 0.366\cos x - 0.1\cos 2x + 0.033\cos 3x + 0.1732\sin x - 0.0578\sin 2x + 0$$

$$f(x) = 1.45 - 0.366\cos x - 0.1\cos 2x + 0.033\cos 3x + 0.1732\sin x - 0.0578\sin 2x$$

(ii) Obtain the Fourier cosine series expansion of $f(x) = x$ in $0 < x < 4$. Hence deduce the value of $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}$..... to $\infty$. (8)

Solution: Here l = 4

**Solution:**

We have to find the half range cosine series for the function $f(x) = x$ in $(0,\ell)$

$$\text{Let} \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{\ell}x$$

$$\text{where} \quad a_0 = \frac{2}{\ell}\int_0^{\ell} f(x)dx = \frac{2}{\ell}\int_0^{\ell} x dx = \frac{2}{\ell}\left[\frac{x^2}{2}\right]_0^{\ell}$$

$$= \ell$$

$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\frac{n\pi}{\ell}x\,dx$$

$$= \frac{2}{\ell}\int_0^{\ell} x\cos\frac{n\pi}{\ell}x\,dx$$

$$= \frac{2}{\ell}\left\{\left[(x)\frac{\overset{0}{\cancel{(\sin\frac{n\pi}{\ell}x)}}}{(\frac{n\pi}{\ell})}\right]_0^{\ell} - \left[(1)\frac{(-\cos\frac{n\pi}{\ell}x)}{(\frac{n\pi}{\ell})}\right]_0^{\ell}\right\}$$

$$= \frac{2}{\ell}\left\{\frac{\ell^2}{n^2\pi^2}(\cos n\pi - \cos 0)\right\}$$

$$= \frac{2\ell}{n^2\pi^2}[(-1)^n - 1]$$

$$a_n = \begin{cases} \frac{-4\ell}{n^2} & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$$

$$\text{Hence} \quad f(x) = \frac{1}{2}(\ell) + \sum_{n=\text{odd}}^{\infty} \frac{-4\ell}{n^2\pi^2}\cos\frac{n\pi}{\ell}x$$

$$\text{or} \quad f(x) = \frac{\ell}{2} - \frac{4\ell}{\pi^2}\sum_{n=\text{odd}}^{\infty}\frac{1}{n^2}\cos\frac{n\pi}{\ell}x$$

Using Parseval's theorem, to find the sum of the series, applying the formula

$$\int_0^{\ell}(f(x))^2 dx = \ell\left[\frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2\right]$$

$$\text{LHS} = \int_0^{\ell} x^2\,dx = \left[\frac{x^3}{3}\right]_0^{\ell} = \frac{\ell^3}{3}$$

Hence $a_0 = \ell;\ \ a_n \frac{-4\ell}{n^2\ell^2}$ ($n$ is odd)

$$(1) \Rightarrow \quad \therefore \quad \frac{\ell^3}{3} = \left[\frac{1}{4}(\ell)^2 + \frac{1}{2}\sum_{n=\text{odd}}^{\infty}\left[\left(\frac{-4\ell}{n^2\pi^2}\right)^2 + 0\right]\right]\ell$$

$$\div\ell \quad \text{or} \quad \frac{\ell^2}{3} = \frac{\ell^2}{4} + \frac{1}{2}\left(\frac{16\ell^2}{\pi^4}\right)\sum_{n=\text{odd}}^{\infty}\frac{1}{n^4}$$

$$\text{or} \quad \frac{\ell^2}{3} - \frac{\ell^2}{4} = \frac{8\ell^2}{\pi^4}\sum_{n=\text{odd}}^{\infty}\frac{1}{n^4}$$

$$\text{or} \quad \frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$$

13. (a) If a tightly stretched string of length $l$ is initially at rest in equilibrium position and each point of it is given the velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin^3 \frac{\pi x}{l}$, $0 < x < l$, determine the transverse displacement $y(x,t)$. (16)

The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

The boundary conditions are

(i) $y(0, t) = 0$ for every $t > 0$

(ii) $y(l, t) = 0$ for every $t > 0$

(iii) $y(x, 0) = 0$ for all $x$ in $(0, l)$ since the string has no displacement in its equilibrium position.

(iv) $\frac{\partial y(x, 0)}{\partial t} = V_0 \sin^3 \frac{\pi x}{l}$ for all $x$ in $(0, l)$

Solving equation (1) and applying the boundary conditions (i) and (ii) we get the correct solution

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pat + c_4 \sin pat) \quad \ldots (2)$$

Now applying condition (i) in (2) we get

$$y(0, t) = c_1 (c_3 \cos pat + c_4 \sin pat) = 0$$

$$\therefore c_1 = 0 \quad \ldots (3)$$

Substituting (3) in (2) we get

$$y(x, t) = c_2 \sin px\,(c_3 \cos pat + c_4 \sin pat) \quad \ldots (4)$$

Applying condition (ii) in (4) we get

$$y(l, t) = c_2 \sin pl\,(c_3 \cos pat + c_4 \sin pat) = 0$$

Here $c_2 \neq 0$ because if $c_2 = 0$ we get only trivial solution.

Also $c_3 \cos pat + c_4 \sin pat \neq 0$ because it is true for all $t > 0$.

$\therefore \sin pl = 0$ i.e., $pl = n\pi$ $\quad [\because \sin n\pi = 0]$

i.e., $$\boxed{p = \frac{n\pi}{l}} \quad \ldots (5)$$

Substituting (5) in (4) we get

$$y(x, t) = c_2 \sin \frac{n\pi x}{l}\left(c_3 \cos \frac{n\pi at}{l} + c_4 \sin \frac{n\pi at}{l}\right) \quad \ldots (6)$$

Applying condition (iii) in (6) we get

$$y(x, 0) = c_2 \sin \frac{n\pi x}{l} \cdot c_3 = 0$$

Here as usual $c_2 \neq 0$, $\sin\frac{n\pi x}{l} \neq 0$ [$\because$ it is defined for all $x$]

$$\therefore c_3 = 0 \qquad \dots (7)$$

Substituting (7) in (6) we get

$$y(x, t) = c_2 \sin\frac{n\pi x}{l} \cdot c_4 \sin\frac{n\pi at}{l}$$

$$y(x, t) = c_n \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l} \text{ where } c_n = c_2 c_4.$$

By superposition principle, we can write the most general solution as

$$\boxed{y(x, t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l}} \qquad \dots (8)$$

Before applying the condition *(iv)*, find $\frac{\partial y(x, t)}{\partial t}$. Partially differentiating (8) w.r.t. '$t$' we get

$$\frac{\partial y(x, t)}{\partial t} = \sum_{n=1}^{\infty} c_n \frac{n\pi a}{l} \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} \qquad \dots (9)$$

Putting $t = 0$ in (9) we get

$$\frac{\partial y(x, 0)}{\partial t} = \sum_{n=1}^{\infty} c_n \frac{n\pi a}{l} \sin\frac{n\pi x}{l}$$

$$= v_0 \sin^3\frac{\pi x}{l} \qquad \text{[By condition } (iv)\text{]}$$

*i.e.,* $$\sum_{n=1}^{\infty} c_n \frac{n\pi a}{l} \sin\frac{n\pi x}{l} = \frac{v_0}{4}\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right) \qquad \dots (10)$$

$$\left[\sin^3 x = \frac{3\sin x - \sin 3x}{4}\right]$$

Here to find $c_n$, we need not expand the R.H.S. of (10) in a half-range Fourier sine series because it contains only two terms.

From (10) we get

$$c_1\frac{\pi a}{l}\sin\frac{\pi x}{l}+c_2\frac{2\pi a}{l}\cdot\sin\frac{2\pi x}{l}+c_3\frac{3\pi a}{l}\sin\frac{3\pi x}{l}+\ldots\ldots$$

$$=\frac{3v_0}{4}\sin\frac{\pi x}{l}-\frac{v_0}{4}\sin\frac{3\pi x}{l}$$

By equating like coefficients we get

$$c_1\frac{\pi a}{l}=\frac{3v_0}{4},\ c_2=0,\ c_3\frac{3\pi a}{l}=-\frac{v_0}{4},c_4=0,\ c_5=0,\ldots\ldots$$

$\therefore c_1=\dfrac{3v_0}{4}\dfrac{l}{\pi a}$ and $c_3=-\dfrac{v_0 l}{12\pi a}$, the remaining $c_n$'s are zero.

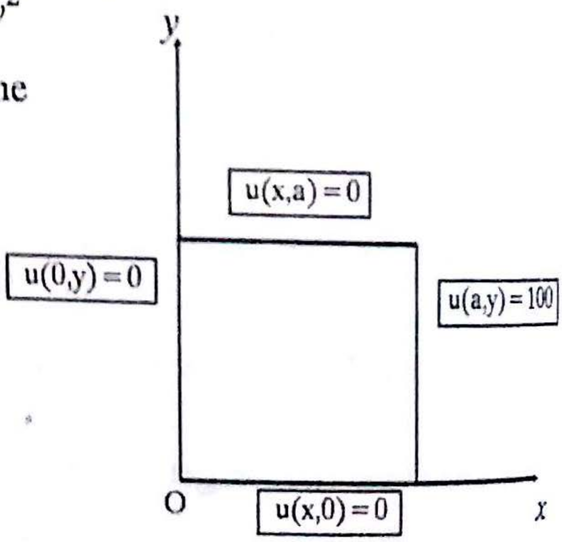
Substituting these values of $c$'s in (8) we get

$$y(x,t)=\frac{3v_0 l}{4\pi a}\sin\frac{\pi x}{l}\sin\frac{\pi at}{l}-\frac{v_0 l}{12\pi a}\sin\frac{3\pi x}{l}\sin\frac{3\pi at}{l}$$

(b) A square plate is bounded by the lines $x = 0$, $x = a$, $y = 0$ and $y = b$. Its surfaces are insulated and the temperature along $y = b$ is kept at 100°C. while the temperature along other three edges are at 0°C. Find the steady - state temperature at any point in the plate. (16)

**Solution :** The equation to be solved is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$




From the gi'en problem, we get the following boundary conditions

(i) $u(x, 0) = 0$

(ii) $u(x, a) = 0$

(iii) $u(0, y) = 0$

(iv) $u(a, y) = 100$

Now, the suitable solution which satisfies our boundary conditions is given by

$$u(x, y) = (A\, e^{px} + B\, e^{-px})\,(C \cos py + D \sin py) \quad \dots (1)$$

Applying condition (i) in equation (1), we get

$$u(x, 0) = (A\, e^{px} + B\, e^{-px})\, [C] = 0$$

Here, $(A\, e^{px} + B\, e^{-px}) \neq 0$ [$\because$ it is defined for all $x$]

$$\therefore \boxed{C = 0}$$

Substitute $C = 0$ in equation (1), we get

$$u(x, y) = (A\, e^{px} + B\, e^{-px})\, D \sin py \qquad \text{... (2)}$$

Applying condition (ii) in equation (2), we get

$$u(x, a) = (A\, e^{px} + B\, e^{-px})\, D \sin pa = 0$$

Here, $A\, e^{px} + B\, e^{-px} \neq 0$ [$\because$ it is defined for all $x$]

$D \neq 0$ [If $D = 0$ already $C = 0$ then we get a trivial solution]

$\therefore \sin pa = 0$

(i.e.,) $\sin pa = \sin n\pi$ [$\because \sin n\pi = 0$]

$$pa = n\pi$$

$$\boxed{p = \frac{n\pi}{a}}$$

Substitute $p = \dfrac{n\pi}{a}$ in equation (2), we get

$$u(x, y) = \left(A\, e^{\frac{n\pi x}{a}} + B\, e^{\frac{-n\pi x}{a}}\right) D \sin \frac{n\pi y}{a} \qquad \text{... (3)}$$

Applying condition (iii) in equation (3), we get

$$u(0, y) = (A + B)\, D \sin \frac{n\pi y}{a} = 0$$

Here, $\sin \dfrac{n\pi y}{a} \neq 0$ [$\because$ it is defined for all $y$]

$D \neq 0$ [If $D = 0$ we have already explained]

$$\therefore \quad A + B = 0$$

$$A = -B$$

$$\text{or} \quad B = -A$$

Substitute $B = -A$ in equation (3), we get

$$u(x, y) = (A e^{\frac{n\pi x}{a}} - A e^{-\frac{n\pi x}{a}}) D \sin\frac{n\pi y}{a}$$

$$= AD (e^{\frac{n\pi x}{a}} - e^{\frac{-n\pi x}{a}}) \sin\frac{n\pi y}{a}$$

$$= 2AD \sinh\frac{n\pi x}{a} \sin\frac{n\pi y}{a} \qquad \text{... (4)}$$

**The most general solution is**

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi x}{a} \sin\frac{n\pi y}{a} \qquad \text{... (5)}$$

**Applying condition (iv) in equation (5), we get**

$$u(a, y) = \sum_{n=1}^{\infty} A_n \sinh n\pi \sin\frac{n\pi y}{a} = 100$$

$$= \sum_{n=1}^{\infty} B_n \sin\frac{n\pi y}{a} = 100 \qquad \text{... (6)}$$

$$\text{where } B_n = A_n \sinh n\pi \qquad \text{... (7)}$$

**To find $B_n$** expand 100 in a Fourier half range sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{a} \qquad \text{...(8)}$$

$$\text{where } b_n = \frac{2}{l}\int_0^l 100 \sin\frac{n\pi y}{a}\, dy$$

From (6) and (8), we get $B_n = b_n$

$$\therefore B_n = \frac{2}{a}\int_0^a 100 \sin\frac{n\pi y}{a}\, dy$$

$$= \frac{200}{a}\left[\frac{-\cos\frac{n\pi y}{a}}{\frac{n\pi}{a}}\right]_0^a = \frac{-200}{a}\,\frac{a}{n\pi}\left[\cos\frac{n\pi y}{a}\right]_0^a$$

$$= \frac{-200}{n\pi}\,[\,(-1)^n - 1\,]$$

$$= \frac{200}{n\pi}\,[\,1 - (-1)^n\,]$$

$B_n = 0$ if $n$ is even.

$B_n = \dfrac{400}{n\pi}$ if $n$ is odd.

$$(7) \Rightarrow A_n = \frac{B_n}{\sinh n\pi} = 0 \text{ if } n \text{ is even.}$$

$$= \frac{400}{n\pi\sinh(n\pi)} \text{ if } n \text{ is odd.}$$

Substitute $A_n$ value in equation (5), we get

$$u(x, y) = \sum_{n = odd} \frac{400}{n\pi\sinh n\pi}\sinh\frac{n\pi x}{a}\sin\frac{n\pi y}{a}$$

$$= \frac{400}{\pi}\sum_{n = odd}^{\infty}\frac{1}{n\sin h\, n\pi}\sinh\frac{n\pi x}{a}\sin\frac{n\pi y}{a}$$

$$= \frac{400}{\pi}\sum_{n=1}^{\infty}\frac{\sinh\frac{(2n-1)\pi x}{a}\sin\frac{(2n-1)\pi y}{a}}{(2n-1)\sinh(2n-1)\pi}$$

14. (a) Find the Fourier transform of $f(x)=\begin{cases}1-|x|, & \text{if } |x|<1\\ 0, & \text{otherwise}\end{cases}$. Hence, deduce the values (i) $\int_0^\infty \frac{\sin^2 t}{t^2}dt$ (ii) $\int_0^\infty \frac{\sin^4 t}{t^4}dt$. (16)

Ans

F.T of $f(x)$ is given by.

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-1}^{1} (1-|x|)\, e^{isx}\, dx\right].$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-1}^{1} (1-|x|)(\cos sx + i\sin sx)\, dx\right].$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-1}^{1}(1-x)\cos sx\, dx + i\int_{-1}^{1}(1-x)\sin sx\, dx\right].$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-1}^{1}(1-x)\cos sx\, dx + 0\right]$$

$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{1}(1-x)\cos sx\, dx.$$

$$= \sqrt{\frac{2}{\pi}}\left[(1-x)\frac{\sin sx}{s} - (-1)\left(-\frac{\cos sx}{s^2}\right)\right]_0^1.$$

$$= \sqrt{\frac{2}{\pi}}\left[\left(0-\frac{\cos s}{s^2}\right) - \left(0-\frac{1}{s^2}\right)\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{1}{s^2} - \frac{\cos s}{s^2}\right]$$

$$F(s) = \sqrt{\frac{2}{\pi}}\left[\frac{1-\cos s}{s^2}\right]$$

By inverse fourier Transform

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-ixs}\, ds.$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s^2}\right)\right)e^{-ixs}\,ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1-\cos s}{s^2}\right)(\cos xs - i\sin xs)\,ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1-\cos s}{s^2}\cos xs\,ds - 0$$

$$= \frac{2}{\pi}\int_{0}^{\infty}\frac{1-\cos s}{s^2}\cos xs\,ds$$

Put $x = 0$ on b.s

$$1-|0| = \frac{2}{\pi}\int_{0}^{\infty}\frac{1-\cos s}{s^2}(1)\,ds$$

$$\frac{\pi}{2} = \int_{0}^{\infty}\frac{1-\cos s}{s^2}\,ds$$

$$= \int_{0}^{\infty}\frac{2\sin^2(s/2)}{s^2}\,ds$$

Put $x = \frac{s}{2}$. $\quad s \to \infty \Rightarrow x \to \infty$

$s = 2x$

$ds = 2dx$. $\quad s \to 0 \Rightarrow x \to 0$.

$$\frac{\pi}{2} = \int_{0}^{\infty}\frac{2\sin^2 x}{(2x)^2}\,2dx$$

$$\frac{\pi}{2} = \int_{0}^{\infty}\frac{\cancel{4}\sin^2 x}{4x^2}\cancel{x}dx$$

$$\therefore \int_{0}^{\infty}\frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}.$$

using Parseval's identity

$$\int_{-\infty}^{\infty}|f(x)|^2dx = \int_{-\infty}^{\infty}|F(s)|^2ds$$

$$\int_{-1}^{1}(1-|x|)^2dx = \int_{-\infty}^{\infty}\left(\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s^2}\right)\right)^2ds$$

$$\int_{-1}^{1}(1-x)^2dx = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{1-\cos s}{s^2}\right)^2ds$$

$$\left[\frac{2(1-x)^3}{-3}\right]_0^1 = \frac{2\times 2}{\pi}\int_{0}^{\infty}\left(\frac{1-\cos s}{s^2}\right)^2ds$$

$$[0]+\left[\frac{2}{3}\right] = \frac{4}{\pi}\int_{0}^{\infty}\left(\frac{1-\cos s}{s^2}\right)^2ds$$

$$\frac{2}{3} = \frac{4}{\pi}\int_{0}^{\infty}\left(\frac{1-\cos s}{s^2}\right)^2ds$$

$$\frac{2\pi}{12} = \int_{0}^{\infty}\left(\frac{1-\cos s}{s^2}\right)^2ds \Rightarrow \frac{2\pi}{3} = \int_{0}^{\infty}\left(\frac{1-\cos s}{s^2}\right)^2ds$$

$$= \int_{0}^{\infty}\left(\frac{2\sin^2 s/2}{s^2}\right)^2ds$$

Put $x = s/2$ $\quad ds = 2dx$ $\quad s \to \infty \Rightarrow x \to \infty$

$s = 2x$ $\quad s \to 0 \Rightarrow x \to 0$.

$$\frac{\cancel{2}\pi}{3\times 2} = \int_0^\infty \left( \frac{2\sin^2 x \cancel{x}}{(2x)^2} \right) \cdot 2\,dx.$$

$$\frac{\cancel{2}\pi}{3\times 2} = \int_0^\infty \cancel{4} \frac{\sin^4 x}{\cancel{16}_2\, x^4} \cdot x\,dx.$$

$$\frac{\pi}{6} = \int_0^\infty \frac{\sin^4 x}{2x^4}\,dx.$$

$$\frac{\pi}{3} = \int_0^\infty \frac{\sin^4 x}{x^4}\,dx.$$

$$\therefore \int_0^\infty \frac{\sin^4 x}{x^4}\,dx = \frac{\pi}{3}$$

(b) (i) Find the Fourier transform of $e^{-a^2x^2}, a > 0$. Hence show that $e^{\frac{-x^2}{2}}$ is self reciprocal under the Fourier transform. (8)

**Solution :** We know that $F[s] = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2 - isx)}\, dx \qquad \dots (1)$$

W.K.T. $(a-b)^2 = a^2 - 2ab + b^2$

$$a^2 - 2ab = (a-b)^2 - b^2$$

Here $a = ax, \qquad 2ab = isx$

$$2(ax)b = isx$$

$$2ab = is$$

$$b = \frac{is}{2a}$$

$$\therefore (ax)^2 - isx = \left[ax - \frac{is}{2a}\right]^2 - \left[\frac{is}{2a}\right]^2$$

$$= \left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2}$$

$$e^{-[(ax)^2 - isx]} = e^{-\left[\left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2}\right]}$$

$$= e^{-\left(ax - \frac{is}{2a}\right)^2} e^{-s^2/4a^2}$$

$$(1) \Rightarrow \quad = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-s^2/4a^2} e^{-\left[ax - \frac{is}{2a}\right]^2} dx$$

$$F(s) = \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[ax - \frac{is}{2a}\right]^2} dx \qquad \dots (2)$$

put $u = ax - \dfrac{is}{2a}$ | $x \to -\infty \Rightarrow u \to -\infty$

$du = a\,dx$ | $x \to \infty \Rightarrow u \to \infty$

$$(2) \Rightarrow F(s) = \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} \frac{1}{a} du$$

$$= \frac{2e^{-s^2/4a^2}}{a\sqrt{2\pi}} \int_{0}^{\infty} e^{-u^2} du \qquad \dots (3)$$

put $t = u^2$ | $u \to 0 \Rightarrow t \to 0$

$dt = 2u\,du$ | $u \to \infty \Rightarrow t \to \infty$

$$du = \frac{1}{2u} dt = \frac{1}{2\sqrt{t}} dt$$

$$(3) \Rightarrow F(s) = \frac{2e^{-s^2/4a^2}}{a\sqrt{2\pi}} \int_{0}^{\infty} e^{-t} \frac{1}{2\sqrt{t}} dt$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \int_{0}^{\infty} e^{-t} t^{-1/2} dt$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \Gamma_{1/2}$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2}\sqrt{\pi}} \sqrt{\pi} \qquad [\because \Gamma_{1/2} = \sqrt{\pi}]$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2}}$$

$$F\left[e^{-a^2x^2}\right] = \frac{1}{a\sqrt{2}} e^{-s^2/4a^2}$$

put $a = \frac{1}{\sqrt{2}}$ we get

$$F\left[e^{-x^2/2}\right] = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)\sqrt{2}} e^{-s^2/4(½)} = e^{-s^2/2}$$

Hence $e^{-x^2/2}$ is self reciprocal under Fourier transform.

(ii) Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$, using Fourier transforms. (8)

**Solution :** Let $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$

then $F_c(f(x)) = F_c[e^{-ax}]$

$$F_c(s) = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]$$

(i.e.,) $F_c[g(x)] = F_c[e^{-bx}]$

$$G_c(s) = \sqrt{\frac{2}{\pi}}\left[\frac{b}{b^2+s^2}\right] \quad \text{(by Qn.No.3 Pg.No. 2.37)}$$

By Parseval's theorem

$$\int_0^\infty f(x)\,g(x)\,dx = \int_0^\infty F_c(s)\,G_c(s)\,ds$$

$$\int_0^\infty e^{-ax} e^{-bx}\,dx = \int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+s^2}\right)\sqrt{\frac{2}{\pi}}\left(\frac{b}{b^2+s^2}\right) ds$$

$$\int_0^\infty e^{-(a+b)x}\,dx = \int_0^\infty \frac{2}{\pi}\frac{ab}{(a^2+s^2)(b^2+s^2)} ds$$

$$\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty = \frac{2ab}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)\ (b^2+s^2)}$$

$$-\frac{1}{a+b}\left[e^{-\infty}-e^0\right] = \frac{2ab}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)\ (b^2+s^2)}$$

$$-\frac{1}{a+b}\ (0-1) = \frac{2ab}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)\ (b^2+s^2)}$$

$$-\frac{1}{a+b}\ (-1) = \frac{2ab}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)\ (b^2+s^2)}$$

$$\frac{1}{a+b} = \frac{2ab}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)\ (b^2+s^2)}$$

Cross multiplying

$$\frac{\pi}{2ab\ (a+b)} = \int_0^\infty \frac{ds}{(a^2+s^2)\ (b^2+s^2)}$$

Changing the variable '$s$' to '$x$'

$$\frac{\pi}{2ab\ (a+b)} = \int_0^\infty \frac{dx}{(a^2+x^2)\ (b^2+x^2)}$$

15. (a) (i) Find $Z(\cos n\theta)$ and $Z(\sin n\theta)$.

**Sol.** Let $a = e^{i\theta}$

$$a^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta$$

We know that $Z\,[a^n] = \dfrac{z}{z-a}, \qquad |z| > |a|$

$$Z\,[(e^{i\theta})^n] = \frac{z}{z-e^{i\theta}} \qquad \text{Here } a = e^{i\theta}$$

$$Z\,[e^{in\theta}] = \frac{z}{z-(\cos\theta + i\sin\theta)} \quad [\because e^{i\theta} = \cos\theta + i\sin\theta]$$

$$Z\,[\cos n\theta + i\sin n\theta] = \frac{z}{(z-\cos\theta) - i\sin\theta}$$

$$Z\,[\cos n\theta] + i\,Z\,[\sin n\theta]$$

$$= \left[\frac{z}{(z-\cos\theta) - i\sin\theta}\right]\left[\frac{(z-\cos\theta)+i\sin\theta}{(z-\cos\theta)+i\sin\theta}\right]$$

$$= \frac{z(z-\cos\theta) + iz\sin\theta}{(z-\cos\theta)^2 + \sin^2\theta}$$

$$= \frac{z(z-\cos\theta) + iz\sin\theta}{z^2 - 2z\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$= \frac{z(z-\cos\theta) + iz\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$= \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1} + i\,\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

Equating the real and imaginary parts we get

$$Z\,\{\cos n\theta\} = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}, \quad |z| > 1$$

$$Z\,\{\sin n\theta\} = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}, \quad |z| > 1$$

(ii) Using $Z$-transforms, solve $u_{n+2} - 3u_{n+1} + 2u_n = 0$ given that $u_0 = 0, u_1 = 1$. (8)

Solve the equation $y_{n+2} - 3y_{n+2} + 2y_n = 0$

given $y_0 = 0$, $y_1 = 1$

Solution :

Given $u(n+2) - 3u(n+1) + 2u(n) = 0$

Taking z-tranform on both sides, we get

$$z[u(n+2)] - 3z[u(n+1)] + 2z[u(n)] = 0$$

$$z^2\left[F[z] - u(0) - \frac{u(1)}{z}\right] - 3z[F[z] - u(0)] + 2F(z) = 0$$

$$z^2\left[F[z] - 0 - \frac{1}{z}\right] - 3z[F[z] - 0] + 2F[z] = 0$$

$$F[z](z^2 - 3z + 2) = z$$

$$F[z] = \frac{z}{(z-2)(z-1)}$$

$$\Rightarrow \frac{F[z]}{z} = \frac{1}{(z-2)(z-1)} = \frac{A}{z-2} + \frac{B}{z-1}$$

$$\Rightarrow 1 = A(z-1) + B(z-2)$$

Put $z = 1 \Rightarrow B = -1$ ; Put $z = 2 \Rightarrow \boxed{A = 1}$

$$\frac{F[z]}{z} = \frac{1}{z-2} - \frac{1}{z-1}$$

$$F[z] = \frac{z}{z-2} - \frac{z}{z-1}$$

Taking Inverse z-transform, we get

$$u(n) = 2^n - (-1)^n.$$

(b) (i) Find the $Z$-transform of $\frac{1}{n(n+1)}$, for $n \geq 1$. (8)

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \quad \ldots (1)$$

$$1 = A(n+1) + B(n)$$

put $n = 0$ we get

$$1 = A$$

put $n = -1$ we get

$$1 = -B$$

(i.e.,) $B = -1$

$$(1) \Rightarrow \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \quad \ldots (2)$$

We know that $Z\left[\frac{1}{n}\right] = \log\frac{z}{z-1}$

$$Z\left[\frac{1}{n+1}\right] = z\log\frac{z}{z-1}$$

Take Z on both sides,

$$\therefore Z\left[\frac{1}{n(n+1)}\right] = Z\left[\frac{1}{n}\right] - Z\left[\frac{1}{n+1}\right] \text{ by linearity}$$

$$= \log\frac{z}{z-1} - z\log\frac{z}{z-1}$$

$$= (1-z)\log\frac{z}{z-1}$$

(ii) Find the inverse $Z$-transform of $\frac{z^2+z}{(z-1)(z^2+1)}$, using partial fraction. (8)

$\frac{z^2+z}{(z-1)(z^2+1)}$

Solution

Let $F[z] = \frac{z(z+1)}{(z-1)(z^2+1)}$

$$\frac{F[z]}{z} = \frac{z+1}{(z-1)(z^2+1)} = \frac{A}{z-1} + \frac{Bz+C}{z^2+1}$$

$$\Rightarrow z+1 = A(z^2+1) + (Bz+C)(z-1)$$

Put $z=1 \Rightarrow 2 = 2A \Rightarrow \boxed{A=1}$

Put $z=0 \Rightarrow 1 = A - C \Rightarrow \boxed{C=0}$

Put $z=-1 \Rightarrow 0 = 2A + 2B - 2C \Rightarrow \boxed{B=-1}$

$$\frac{F[z]}{z} = \frac{1}{z-1} - \frac{z}{z^2+1}$$

$$F[z] = \frac{z}{z-1} - \frac{z^2}{z^2+1}$$

Taking Inverse z- transform, we get

$$Z[n] = z^{-1}\left[\frac{z}{z-1}\right] - z^{-1}\left[\frac{z^2}{z^2+1}\right]$$

$$\boxed{Z[n] = 1^n - \cos\frac{n\pi}{2}} \quad |z|>1.$$