B.E./B.Tech. DEGREE EXAMINATION, APRIL/MAY 2015.

Third Semester

Civil Engineering

MA 6351 — TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS

(Common to all branches except Environmental Engineering, Textile Chemistry, Textile Technology, Fashion Technology and Pharmaceutical Technology)

(Regulation 2013)

Time : Three hours Maximum : 100 marks

Answer ALL questions.

PART A — (10 × 2 = 20 marks)

1. Form the partial differential equation by eliminating the arbitrary constants a and b from $\log(az-1) = x + ay + b$

Solution. $\log(az-1) = x + ay + b$.

Differentiating (1) partially w.r.t x we get

$$\frac{1}{az-1}\cdot a\cdot\frac{\partial z}{\partial x} = 1$$

$$(i.e),\ \frac{ap}{az-1} = 1$$

$$ap = az-1$$

$$az - ap = 1$$

$$a(z-p) = 1$$

$$a = \frac{1}{z-p}.$$

Differentiating (1) partially w.r.t y we get

$$\frac{1}{az-1}\cdot a\cdot\frac{\partial z}{\partial y} = a$$

$$(i.e),\ \frac{aq}{az-1} = a$$

$$q = az-1$$

$$\frac{1+q}{z} = a.$$

From (2) and (3) we obtain

$$\frac{1}{z-p} = \frac{1+q}{z}$$

$$z = (z-p)(1+q)$$

$$z = z + qz - p - pq$$

$$p + pq = qz,$$

which is the required PDE.

2. Find the complete solution of $q = 2px$.

Solution. Given $q = 2px$.

This is of the form $f_1(x, p) = f_2(y, q)$.

Let $q = 2px = a$.

$\therefore q = a$ and $2px = a$

$$p = \frac{a}{2x}.$$

Now, $dz = pdx + qdy$

$$= \frac{a}{2x}dx + ady.$$

Integrating $z = \frac{a}{2}\log x + ay + b$ (1)

which is the complete integral.

There is no singular integral. Put $b = \phi(a)$ in (1) and differentiate partially w.r.t a

The eliminant of a is the general integral.

3. The instantaneous current 'i' at time t of an alternating current wave is given by $i = I_1 \sin(\omega t + \alpha_1) + I_3 \sin(3\omega t + \alpha_3) + I_5 \sin(5\omega t + \alpha_5) + \cdots$. Find the effective value of the current 'i'.

Out of Syllabus.

4. If the Fourier series of the function $f(x) = x, -\pi < x < \pi$ with period 2π is given by $f(x) = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots\right)$, then find the sum of the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$.

Let x = π,

Sum of the series at x = π is equal to $\frac{f(\pi) + f(-\pi)}{2}$.

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\ldots = \frac{f(\pi) + f(-\pi)}{2}$$

$$= \frac{\pi - \pi}{2} = 0$$

5. Classify the partial differential equaiton $(1 - x^2)z_{xx} - 2xyz_{xy} + (1 - y^2)z_{yy} + xz_x + 3x^2yz_y - 2z = 0$.

Here A =1-x^2 , B = -2xy, C = 1-y^2

Δ = B^2 – 4 A C = $4x^2y^2 - 4(1-x^2)(1-y^2) = 4x^2y^2 - 4(1 - x^2 - y^2 + x^2y^2)$

$\Delta = 4x^2y^2 - 4 + 4x^2 + 4y^2 - 4x^2y^2 = -4 + 4x^2 + 4y^2$

If $x^2 + y^2 - 1 > 0 \rightarrow$ when $x, y \in (-\infty, -1) \cup (1, \infty)$ and hence the pde is hyperbolic

If $x^2 + y^2 - 1 < 0 \rightarrow$ when $x \in (-1, 1)$ and $y \in (-1, 1)$ hence the pde is elliptic.

If $x^2 + y^2 - 1 = 0 \rightarrow$ when $(x, y) \in (1,0)$, $(x, y) \in (-1,0)$, $(x, y) \in (0,1)$, and

$(x, y) \in (0,-1)$, hence the pde is parabolic.

6. A rod 30 cm long has its ends A and B kept at 20°C and 80°C respectively until steady state conditions prevail. Find this steady state temperature in the rod.

The solution of heat equation in steady state is u = ax + b

Here u = 20 when x = 0 and u = 80 when x = 30

Using this, we get b = 20 and a = 2

$\therefore$ the solution is u = 2x + 20

7. If the Fourier transform of $f(x)$ is $\Im(f(x)) = F(s)$, then show that $\Im(f(x-a)) = e^{ias}F(s)$.

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\ e^{(t+a)is} dt \qquad \{\text{put } x - a = t \text{ and } dx = dt\}$$

$$= e^{ias} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\ e^{ist} dt$$

$$= e^{ias}\ F(s)$$

8. Find the Fourier sine transform of $1/x$.

$$F_S\,[\,f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x}\,\sin sx\;dx$$

$$= \sqrt{\frac{2}{\pi}} \times \frac{\pi}{2}$$

9. If $Z(x(n)) = X(z)$, then show that $Z(a^n x(n)) = X\left(\frac{z}{a}\right)$.

$$Z\;[\,f(n)\,] = \sum_{n=0}^{\infty} f(n)\left(\frac{1}{z}\right)^n = F(z)$$

$$Z\;[\,a^n f(n)\,] = \sum_{n=0}^{\infty} a^n\, f(n)\left(\frac{1}{z}\right)^n$$

$$= \sum_{n=0}^{\infty} f(n)\left(\frac{a}{z}\right)^n$$

$$= \sum_{n=0}^{\infty} f(n)\left(\frac{1}{\frac{z}{a}}\right)^n$$

$$= F\left(\frac{z}{a}\right)$$

10. State the convolution theorem of Z-transforms.

If W(n) is the convolution of two sequences x(n) and y(n), then

$Z\;[\,W(n)\,] = Z\;[\,x(n)\,]\,.\,Z\;[\,y(n)\,]$

PART B — (5 × 16 = 80 marks)

11. (a) (i) Solve: $(x^2 - yz)p + (y^2 - xz)q = (z^2 - xy)$. (8)

Solution : Given : $(x^2 - yz)\,p + (y^2 - zx)\,q = (z^2 - xy)$ ... (1)

This equation is of the form $P\,p + Q\,q = R$

where $P = x^2 - yz$, $Q = y^2 - zx$, $R = z^2 - xy$

The Lagrange's subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

i.e., $$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$$ ... (2)

Method of grouping is not possible

Using two sets of multipliers x, y, z ; $1, 1, 1$ each of the ratio in (2)

$$= \frac{xdx + ydy + zdz}{x^3 + y^3 + z^3 - 3xyz} = \frac{dx + dy + dz}{x^2 + y^2 + z^2 - xy - yz - zx}$$

$$= \frac{xdx + ydy + zdz}{(x + y + z)\,(x^2 + y^2 + z^2 - xy - yz - zx)} = \frac{dx + dy + dz}{x^2+y^2+z^2-xy-yz-zx}$$

$$= \frac{xdx + ydy + zdz}{x + y + z} = \frac{dx + dy + dz}{1}$$

$$xdx + ydy + zdz = (x + y + z)\,d\,(x + y + z)$$

Integrating on both sides, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{(x + y + z)^2}{2} + \frac{a}{2}$$

$$x^2 + y^2 + z^2 = (x + y + z)^2 + a$$

$$a = x^2 + y^2 + z^2 - (x + y + z)^2$$
$$= x^2 + y^2 + z^2 - x^2 - y^2 - z^2 - 2xy - 2yz - 2zx$$
$$= -2(xy + yz + zx)$$

i.e., $xy + yz + zx = -\frac{a}{2} = u$ [constant]

Using two sets of multipliers $1, -1, 0;\ 0, 1, -1$ each of the ratio in (2).

$$= \frac{dx - dy}{(x^2 - yz) - (y^2 - zx)} = \frac{dy - dz}{(y^2 - zx) - (z^2 - xy)}$$

$$\frac{d(x-y)}{x^2 - yz - y^2 + zx} = \frac{dy - dz}{y^2 - zx - z^2 + xy}$$

$$\frac{d(x-y)}{(x^2 - y^2) + z(x-y)} = \frac{d(y-z)}{(y^2 - z^2) + x(y-z)}$$

$$\frac{d(x-y)}{(x-y)(x+y) + z(x-y)} = \frac{d(y-z)}{(y-z)(y+z) + x(y-z)}$$

$$\frac{d(x-y)}{(x-y)(x+y+z)} = \frac{d(y-z)}{(y-z)(x+y+z)}$$

$$\frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

Integrating on both sides, we get

$$\log(x-y) = \log(y-z) + \log b$$
$$\log(x-y) = \log[b(y-z)]$$
$$x - y = b(y-z)$$
$$\frac{x-y}{y-z} = b = v$$

Hence, the general solution is $f(u, v) = 0$

i.e., $f\left(xy + yz + zx, \frac{x-y}{y-z}\right) = 0$, where f is arbitrary.

(ii) Solve: $(D^2 - 3DD' + 2D'^2)z = (2+4x)e^{x+2y}$. (8)

Solution. The auxiliary equation is

$$m^2 - 3m + 2 = 0$$
$$(m-1)(m-2) = 0$$
$$m = 1, 2.$$
$$\text{C.F} = f_1(y+x) + f_2(y+2x).$$

$$\text{P.I} = \frac{1}{D^2 - 3DD' + 2D'^2}\left[e^{x+2y}(2+4x)\right]$$
$$= \frac{1}{(D-D')(D-2D')}\left[e^{x+2y}(2+4x)\right]$$
$$= \frac{1}{(D-D')}\left[\int e^{x+2(c-2x)}(2+4x)dx\right]_{c\to y+2x} \quad [m=2, y=c-mx=c-2:$$
$$= \frac{1}{(D-D')}\left[\int e^{x+2c-4x}(2+4x)dx\right]_{c\to y+2x}$$
$$= \frac{1}{(D-D')}\left[e^{2c}\int e^{-3x}(2+4x)dx\right]_{c\to y+2x}$$
$$= \frac{1}{(D-D')}\left[e^{2c}\{2\int e^{-3x}dx + 4\int xe^{-3x}dx\}\right]_{c\to y+2x}$$

$$= \frac{1}{(D-D')}\left[e^{2c}\{2\left(\frac{e^{-3x}}{-3}\right) + 4\left(x\left(\frac{e^{-3x}}{-3}\right) - \frac{e^{-3x}}{9}\right)\}\right]_{c\to y+2x}$$
$$= \frac{1}{(D-D')}\left[e^{4x+2y}\left(-\frac{2}{3}e^{-3x} - -\frac{4}{3}xe^{-3x} - \frac{4}{9}e^{-3x}\right)\right]$$
$$= \frac{-1}{(D-D')}e^{x+2y}\left(\frac{2}{3} + \frac{4}{3}x + \frac{4}{9}\right)$$
$$= \frac{-1}{(D-D')}e^{x+2y}\left(\frac{6+12x+4}{9}\right)$$

$$= \frac{-2}{9}\frac{1}{(D-D')}e^{x+2y}(5+16x)$$

$$= \frac{-2}{9}\left[\int e^{x+2c-2x}(5+6x)dx\right]_{c\to x+y} \quad [m = 1, y = c - mx =$$

$$= \frac{-2}{9}\left[\int e^{2c-x}(5+6x)dx\right]_{c\to x+y}$$

$$= \frac{-2}{9}\left[e^{2c}\int e^{-x}(5+6x)dx\right]_{c\to x+y}$$

$$= \frac{-2}{9}\left[e^{2c}\{(5+6x)(-e^{-x})-6e^{-x}\}\right]_{c\to x+y}$$

$$= \frac{2}{9}e^{2x+2y}e^{-x}(5+6x+6)$$

$$= \frac{2}{9}e^{x+2y}(6x+11).$$

The complete solution is $z = C.F + P.I$

$$z = f_1(y+x) + f_2(y+2x) + \frac{2}{9}(6x+11)e^{x+2y}.$$

(b) (i) Obtain the complete solution of $p^2 + x^2y^2q^2 = x^2z^2$. (8)

Solution : Given : $p^2 + x^2y^2q^2 = x^2z^2$

$$\frac{p^2}{x^2} + y^2q^2 = z^2$$

$$x^{-2}p^2 + y^2q^2 = z^2$$

$$(x^{-1}p)^2 + (yq)^2 = z^2 \quad \dots (1)$$

This is of the form $f(x^m p, y^n q, z) = 0$ **[Type 5 case (i) and (ii)]**

Here, $m = -1, n = 1$

Hence, put $X = x^{1-m}$ | put $Y = \log y$

$X = x^{1+1}$ | $\frac{\partial Y}{\partial y} = \frac{1}{y}$

$X = x^2$ | $Q = \frac{\partial z}{\partial Y}$

$$\frac{\partial X}{\partial x} = 2x \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y}$$

$$P = \frac{\partial z}{\partial X} \qquad q = Q\frac{1}{y}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x}$$

$$p = P\,(2x) \qquad yq = Q$$

$$\frac{p}{2x} = P \qquad yq = Q$$

$$x^{-1}p = 2P$$

$$(1) \Rightarrow (2P)^2 + Q^2 = z^2 \qquad \dots (2)$$

This is of the form f (P, Q, z) = 0

Type 3 case (i)

Let $u = X + aY$

$$\frac{\partial u}{\partial X} = 1,$$

$$\frac{\partial u}{\partial Y} = a$$

$$P = \frac{\partial z}{\partial X} = \frac{dz}{du}\frac{\partial u}{\partial X}$$

$$\Rightarrow P = \frac{dz}{du}$$

$$Q = \frac{\partial z}{\partial Y} = \frac{dz}{du}\frac{\partial u}{\partial Y}$$

$$\Rightarrow Q = a\frac{dz}{du}$$

$$(2) \Rightarrow \left(2\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2 = z^2$$

$$(4 + a^2)\left(\frac{dz}{du}\right)^2 = z^2$$

$$(2) \Rightarrow \left(2\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2 = z^2$$

$$(4 + a^2)\left(\frac{dz}{du}\right)^2 = z^2$$

$$\left(\frac{dz}{du}\right)^2 = \frac{z^2}{4 + a^2}$$

$$\frac{dz}{du} = \frac{z}{\sqrt{4 + a^2}}$$

$$\frac{dz}{z} = \frac{1}{\sqrt{4 + a^2}}\,du$$

$$\int \frac{dz}{z} = \int \frac{1}{\sqrt{4 + a^2}}\,du$$

$$\log z = \frac{1}{\sqrt{4 + a^2}}\,u + b$$

$$\log z = \frac{1}{\sqrt{4 + a^2}}\,(X + aY) + b$$

$$= \frac{1}{\sqrt{4 + a^2}}\,[x^2 + a\log y] + b$$

which is the complete integral. Singular and general solu found out as usual.

(ii) Solve $z = px + qy + p^2q^2$ and obtain its singular solution. (8)

Solution : $z = px + qy + p^2 q^2$

It is Clairaut's form

$\therefore$ the complete integral is $z = ax + by + a^2 b^2$ ... (1)

To find the general integral, put $b = \phi(a)$ in (1)

$$\therefore z = ax + \phi(a).y + a^2 (\phi(a))^2 \quad \text{... (2)}$$

Differentiating (2) w.r.to a,

$$0 = x + \phi'(a)y + a^2 . 2\phi(a).\phi'(a) + (\phi(a))^2 . 2a \quad \text{... (3)}$$

Eliminating a from (2) and (3), we get general integral.

To find singular integral

Differentiating the C.I. (1) partially w.r.to a and b, we get

$$0 = x + 2ab^2 \Rightarrow x = -2ab^2 \quad \text{... (4)}$$

and $$0 = y + 2a^2 b \Rightarrow y = -2a^2b \quad \text{... (5)}$$

$$\frac{(4)}{(5)} \Rightarrow \frac{x}{y} = \frac{-2ab^2}{-2a^2b} = \frac{b}{a}$$

$$\Rightarrow \frac{x}{b} = \frac{y}{a} = k$$

$$\Rightarrow \frac{x}{b} = k \Rightarrow \frac{x}{k} = b$$

$$\text{and } \frac{y}{a} = k \Rightarrow \frac{y}{k} = a$$

Substituting in (4), $x = -2\frac{y}{k}\cdot\frac{x^2}{k^2}$

$$\Rightarrow k^3 = -2xy \quad \dots (6)$$

$$\Rightarrow k = -(2xy)^{1/3}$$

Substituting in (1), $z = \frac{xy}{k} + \frac{xy}{k} + \frac{x^2y^2}{k^4} = \frac{2xy}{k} + \frac{x^2y^2}{k^4}$

$$\Rightarrow kz = 2xy + \frac{x^2y^2}{k^3}$$

$$\Rightarrow kz = 2xy - \frac{x^2y^2}{2xy} = 2xy - \frac{xy}{2} = \frac{3xy}{2}$$

Cubing both sides, $k^3z^3 = \frac{27}{8}x^3y^3$

$$\Rightarrow -2xyz^3 = \frac{27}{8}x^3y^3 \quad \text{[Using (6)]}$$

$$\Rightarrow -16z^3 = 27x^2y^2$$

$$\Rightarrow 16z^3 + 27x^2y^2 = 0$$

which is the singular integral.

12. (a) (i) Find the half-range sine series of $f(x) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi - x, & \pi/2 < x < \pi \end{cases}$. Hence deduce the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$. (10)

Step 1 :

The cosine series for the function $f(x)$ in $(0, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \cdots (1)$$

Step 2 : To find a_0

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} (\pi - x)\,dx\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi/2} + \left(\pi x - \frac{x^2}{2}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{8} + \left(\pi^2 - \frac{\pi^2}{2}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right]$$

$$\boxed{a_0 = \frac{\pi}{2}} \qquad \cdots (2)$$

Step 3 : To find a_n

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x \cos nx\,dx + \int_{\pi/2}^{\pi} (\pi - x) \cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\left\{x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right)\right\}_0^{\pi/2}\right.$$

$$\left. + \left\{(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\frac{\pi}{2}\cdot\sin\frac{n\pi}{2}}{n} + \frac{\cos\frac{n\pi}{2}}{n^2} - \frac{1}{n^2} - \frac{\cos n\pi}{n^2} - \frac{\frac{\pi}{2}\sin\frac{n\pi}{2}}{n} + \frac{\cos\frac{n\pi}{2}}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2\cos\frac{n\pi}{2}}{n^2} - \frac{1}{n^2} - \frac{\cos n\pi}{n^2}\right]$$

$$= \frac{2}{n^2\pi}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right]$$

When n is odd, $a_n = 0$, *i.e.*, $a_1 = a_3 = a_5 = \ldots 0$ ... (3)

When n is even

$$a_2 = \frac{2}{2^2\pi}[2\cos\pi - 1 - 1] = -\frac{2}{\pi \cdot 1^2} \quad \ldots (4)$$

$$a_4 = \frac{2}{4^2\pi}[2\cos 2\pi - 1 - 1] = 0 \; (\because \cos 2\pi = 1) \quad \ldots (5)$$

$$a_6 = \frac{2}{6^2\pi}[2\cos 3\pi - 1 - 1] = \frac{-2}{\pi \cdot 3^2} \quad \ldots (6)$$

and so on.

Substituting (2), (3), (4), (5) and (6) in (1) we get

$$\therefore f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \ldots\right]$$

(ii) Find the complex form of the Fourier series of $f(x) = e^{-x}$ in $-1 < x < 1$. (6)

● Solution

The complex form of Fourier Series of $f(x)$ in $(-1, 1)$ is given by

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x} \quad \ldots (1)$$

where

$$c_n = \frac{1}{2}\int_{-1}^{1} e^{-x} \cdot e^{-in\pi x}\,dx = \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x}\,dx$$

$$= \frac{1}{2}\left[\frac{e^{-(1+in\pi)x}}{-(1+in\pi)}\right]_{-1}^{1} = \frac{1}{-2(1+in\pi)}\left[e^{-(1+in\pi)} - e^{(1+in\pi)}\right]$$

$$= \frac{-1}{2(1+in\pi)}\left[e^{-1}(\cos n\pi - i\sin n\pi) - e^{1}(\cos n\pi + i\sin n\pi)\right]$$

$$= -\frac{\cos n\pi}{2(1+in\pi)}[e^{-1}-e^{1}]$$

$$= \frac{\cos n\pi}{2(1+in\pi)} 2 \sin h\, 1 \qquad \left[\because \sin hx = \frac{e^x - e^{-x}}{2}\right]$$

$$c_n = \frac{(-1)^n \sin h\, 1}{1+in\pi} \qquad \ldots (2)$$

Substituting (2) in (1) we get,

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n \sin h 1}{1+in\pi} e^{in\pi x}$$

$$= \sin h\, 1 \sum_{n=-\infty}^{\infty} \frac{(-1)^n (1-in\pi)}{1+n^2\pi^2} e^{in\pi x}$$

(b) (i) Find the Fourier series of $f(x) = |\sin x|$ in $-\pi < x < \pi$ of periodicity 2π. (8)

Solution : Given $f(x) = |\sin x| = \begin{cases} \sin x & 0 \le x < \pi \\ -\sin x & -\pi < x \le 0 \end{cases}$

$$f(-x) = |\sin(-x)| = |-\sin x| = |\sin x| = f(x)$$

Therefore $f(x)$ is an even function. Hence $b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x\,dx = \frac{2}{\pi}\Big[-\cos x\Big]_0^{\pi} = \frac{-2}{\pi}\Big[\cos x\Big]_0^{\pi}$$

$$= \frac{-2}{\pi}[-1-1] = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \cos nx\sin x\,dx = \frac{2}{2\pi}\int_0^{\pi}[\sin(n+1)x - \sin(n-1)x]\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}[\sin(n+1)x - \sin(n-1)x]\,dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right) - \left(\frac{-1}{n+1} + \frac{1}{n-1}\right)\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n+1}}{n-1}\right) - \left(\frac{-1}{n+1} + \frac{1}{n-1}\right)\right]$$

$$= \frac{1}{\pi}\left[(-1)^{n+1}\left(\frac{-1}{n+1} + \frac{1}{n-1}\right) - \left(\frac{-1}{n+1} + \frac{1}{n-1}\right)\right]$$

$$= \frac{1}{\pi}\left(\frac{-1}{n+1} + \frac{1}{n-1}\right)[(-1)^{n+1} - 1] = \frac{1}{\pi}\left(\frac{2}{n^2-1}\right)[-(-1)^n - 1]$$

$$= \frac{-2}{\pi(n^2-1)}[1 + (-1)^n] \text{ if } n \text{ is not equal to } 1$$

$$= 0 \text{ if } n \text{ is odd}$$

$$= \frac{-4}{\pi(n^2-1)} \text{ if } n \text{ is even} \qquad (n \neq 1)$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x\cos x\,dx = \frac{2}{\pi}\int_0^{\pi} \sin 2x\,dx = \frac{1}{\pi}\int_0^{\pi} \sin 2x\,dx = \frac{1}{\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi}$$

$$= \frac{-1}{2\pi}\left[\cos 2x\right]_0^{\pi} = \frac{-1}{2\pi}[1-1] = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx = \frac{2}{\pi} + 0 + \sum_{n=\text{even}}^{\infty} \frac{-4}{\pi(n^2-1)} \cos nx$$

$$= \frac{2}{\pi} - \frac{4}{\pi} \int_{n=\text{even}}^{\infty} \frac{1}{(n^2-1)} \cos nx = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n)^2 - 1} \cos 2n\, x$$

$$= \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n-1)} \cos 2nx$$

(ii) Compute upto the first three harmonics of the Fourier series of $f(x)$ given by the following table: (8)

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
$f(x)$	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Solution :

Here first and last y values are repeated

$\therefore$ we can omit the last value.

$\therefore$ Hence $n = 6$

x	y	$\cos x$	$y\cos x$	$\cos 2x$	$y\cos 2x$	$\cos 3x$	$y\cos 3x$	$\sin x$	$y\sin x$	$\sin 2x$	$y\sin 2x$	$\sin 3x$	$y\sin 3x$
0	1	1	1	1	1	1	1	0	0	0	0	0	0
60	1.4	0.5	0.7	–0.5	–0.7	–1	–1.4	0.866	1.2124	0.866	1.212	0	0
120	1.9	–0.5	–0.95	–0.5	–0.95	1	1.9	0.866	1.6454	–0.866	–1.6454	0	0
180	1.7	–1	–1.7	1	1.7	–1	–1.7	0	0	0	0	0	
240	1.5	–0.5	–0.75	–0.5	–0.75	1	1.5	–0.866	–1.299	0.866	1.299	0	0
300	1.2	0.5	0.6	–0.5	–0.6	–1	–1.2	–0.866	–1.0392	–0.866	–1.0932	0	0
	$\Sigma y = 8.7$		$\Sigma y\cos x = -1.1$		$\Sigma \cos 2x = -0.3$		$\Sigma \cos 3x = 0.1$		$\Sigma \sin x = 0.5196$		$\Sigma y\sin 2x = -0.1736$		$\Sigma y\sin 3x = 0$

$$a_0 = \frac{2}{n}\Sigma y = \frac{2}{6}\times 8.7 = 2.900$$

$$a_1 = \frac{2}{n}\Sigma y\cos x = \frac{2}{6}\times(-1.1) = -0.366$$

$$a_2 = \frac{2}{n}\Sigma y\cos 2x = \frac{2}{6}\times(-0.3) = -0.1$$

$$a_3 = \frac{2}{n}\Sigma y\cos 3x = \frac{2}{6}\times(0.1) = 0.033$$

$$b_1 = \frac{2}{n}\Sigma y\sin x = \frac{2}{6}\times(0.5196) = 0.1732$$

$$b_2 = \frac{2}{n}\Sigma y\sin 2x = \frac{2}{6}\times(-0.1736) = -0.0578$$

$$b_3 = \frac{2}{n}\Sigma y\sin 3x = \frac{2}{6}\times(0) = 0$$

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + b_1\sin x + b_2\sin 2x + b_3\sin 3x$$

$$= \frac{2.9}{2} - 0.366\,cosx - 0.1\cos 2x + 0.033\cos 3x + 0.1732\sin x - 0.0578\sin 2x + 0$$

$$f(x) = 1.45 - 0.366\cos x - 0.1\cos 2x + 0.033\cos 3x + 0.1732\sin x - 0.0578\sin 2x$$

13. (a) Solve $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$ subject to the conditions: $u(0,t) = 0 = u(l,t), t \geq 0$;
$u(x,0) = \begin{cases} x, & 0 \leq x \leq l/2 \\ l-x, & l/2 \leq x \leq l \end{cases}$. (16)

Solution : The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

From the given problem, we get the following boundary conditions

(i) $y(0, t) = 0$ for all $t > 0$

(ii) $y(l, t) = 0$ for all $t > 0$

(iii) $\frac{\partial y}{\partial t}(x, 0) = x(x - l)$ $0 < x < l$

(iv) $y(x, 0) = \begin{cases} x & in \ 0 < x < \frac{l}{2} \\ l - x & in \ \frac{l}{2} < x < l \end{cases}$

Now the suitable solution which satisfies our boundary conditions is given by

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pat + c_4 \sin pat) \quad \text{... (1)}$$

Applying condition (i) in equation (1) we get

$$y(0, t) = (c_1 + 0)(c_3 \cos p\,at + c_4 \sin p\,at) = 0$$

Here $c_3 \cos pat + c_4 \sin pat \neq 0$ [$\because$ It is defined for all t]

Therefore we get $c_1 = 0$

Substitute $c_1 = 0$ in equation (1) we get

$$y(x, t) = c_2 \sin px (c_3 \cos pat + c_4 \sin pat) \quad \text{... (2)}$$

Applying condition (ii) in (2) we get

$$y(l, t) = c_2 \sin pl (c_3 \cos pat + c_4 \sin pat) = 0$$

Here $c_3 \cos pat + c_4 \sin pat \neq 0$ [$\because$ it is defined for all t]

Therefore either $c_2 = 0$ or $\sin pl = 0$

Suppose if we take $c_2 = 0$ and already we have

$c_1 = 0$ then we get a trivial solution.

Therefore we consider $c_2 \neq 0$ and

$$\sin pl = 0$$

$$pl = n\pi \qquad [\because \sin n\pi = 0]$$

$$p = \frac{n\pi}{l} \qquad [n \text{ being } a_n \text{ integer}]$$

Now substituting $p = \frac{n\pi}{l}$ in equation (2) we get

$$y(x,t) = c_2 \sin\frac{n\pi x}{l}\left(c_3 \cos\frac{n\pi at}{l} + c_4 \sin\frac{n\pi at}{l}\right)$$

$$= c_2 c_3 \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} + c_2 c_4 \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l} \quad \ldots (3)$$

The most general solution of (3) can be written as

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \, in \frac{n\pi at}{l} \quad \ldots (4)$$

Where $A_n = c_2 c_3$, $B_n = c_2 c_4$

Before applying condition (iii) diff (4) p.w.r.to 't' we get

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} A_n \left(\frac{n\pi a}{l}\right) \sin\frac{n\pi x}{l}\left(-\sin\frac{n\pi at}{l}\right)$$

$$+ \sum_{n=1}^{\infty} B_n \left(\frac{n\pi a}{l}\right) \sin\frac{n\pi x}{l} \cos\left(\frac{n\pi at}{l}\right)$$

Now applying condition (iii) we get

$$\frac{\partial y}{\partial t}(x, 0) = 0 + \sum_{n=1}^{\infty} B_n \left(\frac{n \pi a}{l}\right) \sin \frac{n \pi x}{l} = x(x - l)$$

We now express $x(x-l)$ as a half range Fourier sine series in $(0, l)$

Let $x(x-l) = \sum_{n=1}^{\infty} c_n \sin \frac{n \pi x}{l}$ when $C_n = B_n \frac{n \pi a}{l}$

$$\text{But } C_n = \frac{2}{l} \int_0^l x(x-l) \sin \frac{n \pi x}{l}\, dx$$

$$= \frac{2}{l} \int_0^l (x^2 - lx) \sin \frac{n \pi x}{l}\, dx$$

$$= \frac{2}{l}\left[(x^2-lx)\left[-\frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}\right] - (2x-l)\left[-\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n^2\pi^2}{l^2}\right)}\right] + (2)\left[\frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n^3\pi^3}{l^3}\right)}\right]\right]_0^l$$

$$= \frac{2}{l}\left[-(x^2 - lx)\left(\frac{l}{n\pi}\right)\cos\left(\frac{n\pi x}{l}\right) + (2x-l)\left(\frac{l^2}{n^2\pi^2}\right)\sin\left(\frac{n\pi x}{l}\right) + 2\left(\frac{l^3}{n^3\pi^3}\right)\cos\frac{n\pi x}{l}\right]_0^2$$

$$= \frac{2}{l}\left[\left(0 - 0 + \frac{2l^3}{n^3\pi^3}(-1)^n\right) - \left(-0 + 0 + \frac{2l^3}{n^3\pi^3}\right)\right]$$

$$= \frac{2}{l}\frac{l^3}{n^3\pi^3}\, 2\left[(-1)^n - 1\right] = \frac{4l^2}{n^3\pi^3}\left[(-1)^n - 1\right]$$

$$c_n = \frac{-8\, l^2}{n^3\pi^3} \quad \text{if } n \text{ is odd}$$

$$= 0 \quad \text{if } n \text{ is even}$$

$$B_n = \frac{l}{n\pi a} c_n = \frac{l}{n\pi a}\left(\frac{-8l^2}{n^3\pi^3}\right) \quad \text{if } n \text{ is odd}$$

$$= \frac{-8l^3}{n^4\pi^4 a} \text{ if } n \text{ is odd}$$

$\therefore$ equation (4) becomes

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} + \sum_{n=\text{odd}}^{\infty} \frac{-8l^3}{n^4\pi^4 a}\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l} \quad \cdots (5)$$

Now we apply condition (iv) in (5) we get

$$y(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} = \begin{cases} x & in \quad 0 < x < \frac{l}{2} \\ l - x & in \quad \frac{l}{2} < x < l \end{cases} \quad \cdots (6)$$

We express $f(x)$ as a half range Fourier sine series in $(0, l)$

$$\text{Let } f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{\pi x}{l} \quad \cdots (7)$$

Comparing (6) & (7) we have $b_n = A_n$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} x\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[(x)\left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^{l/2}$$

$$+ \frac{2}{l}\left[(l-x)\left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_{l/2}^{l}$$

$$= \frac{2}{l}\left[-x\left(\frac{l}{n\pi}\right)\cos\left(\frac{n\pi x}{l}\right)+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_0^{l/2}$$

$$+\frac{2}{l}\left[-(l-x)\frac{l}{n\pi}\cos\frac{n\pi x}{l}-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_{l/2}^{l}$$

$$= \frac{2}{l}\left[-\frac{l}{n\pi}\,\frac{l}{2}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$+\frac{2}{l}\left[(-0-0)+\left(\frac{l}{n\pi}\,\frac{l}{2}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right]$$

$$= \frac{2}{l}\left[-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}+\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{2}{l}\left[2\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$b_n = \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$= 0 \qquad \text{if } n \text{ is odd}$$

$$b_n = \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2} \qquad \text{if } n \text{ is even}$$

$$A_n = \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2} \qquad \text{if } n \text{ is even.}$$

in equation (5) we get

$$y(x,t) = \sum_{n=\text{even}}^{\infty} \frac{4l}{n^2\pi^2} \sin\frac{n\pi}{2} \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l}$$

$$+ \sum_{n=\text{odd}}^{\infty} \frac{-8l^3}{n^4\pi^4 a} \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l}$$

$$= \frac{4l}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi x}{l}\cos\frac{\pi at}{l} - \frac{1}{3^2}\sin\frac{3\pi x}{l}\cos\frac{3\pi at}{l} + \frac{1}{5^2}\sin\frac{5\pi x}{l}\cos\frac{5\pi at}{l} - \ldots\right]$$

$$- \frac{8l^3}{a\pi^4}\left[\frac{1}{1^4}\sin\frac{\pi x}{l}\sin\frac{\pi at}{l} - \frac{1}{3^4}\sin\frac{3\pi x}{l}\sin\frac{3\pi at}{l} + \frac{1}{5^4}\sin\frac{5\pi x}{l}\sin\frac{5\pi at}{l} - \ldots\right]$$

(b) A string is stretched and fastened to two points that are distant l apart. Motion is started by displacing the string into the form $y = k(lx - x^2)$ from which it is released at time t=0. Find the displacement at any point of the string at a distance x from one end at any time t. (16)

Solution : The equation to be solved is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

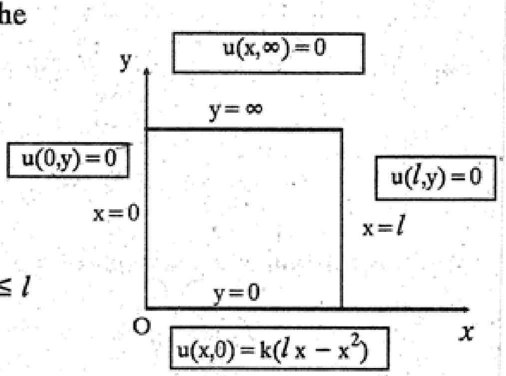
From the given problem, we get the following boundary conditions :

(i) $u(0, y) = 0, \; 0 \le y \le l$

(ii) $u(l, y) = 0, \; 0 \le y \le l$

(iii) $u(x, \infty) = 0, \; 0 \le x \le l$

(iv) $u(x, 0) = k(lx - x^2), \; 0 \le x \le l$



Now, the suitable solution which satisfies our boundary conditions is given by

$$u(x, y) = (A \cos px + B \sin px)(C e^{py} + D e^{-py}) \quad \dots (1)$$

Apply condition (i) in equation (1), we get

$$u(0, y) = A(C e^{py} + D e^{-py}) = 0$$

Here, $C e^{py} + D e^{-py} \neq 0$ [$\because$ it is defined for all y]

$$\therefore \boxed{A = 0}$$

Substitute A = 0 in (1), we get

$$u(x, y) = B \sin px (C e^{py} + D e^{-py}) \quad \dots (2)$$

Apply condition (ii) in equation (2), we get

$$u(l, y) = B \sin lp (C e^{py} + D e^{-py}) = 0$$

Here, $C e^{py} + D e^{-py} \neq 0$ [$\because$ it is defined for all y]

$B \neq 0$ [$\because$ If B=0 we already have A=0 then we get a trivial solution]

$$\therefore \sin lp = 0$$

$\sin lp = \sin n\pi$ [$\because \sin n\pi = 0$]

$$lp = n\pi$$

$$\boxed{p = \frac{n\pi}{l}}$$

Substitute $p = \frac{n\pi}{l}$ in equation (2), we get

$$u(x, y) = B \sin \frac{n\pi}{l} x \left(C e^{\frac{n\pi}{l}y} + D e^{\frac{-n\pi}{l}y}\right) \quad \dots (3)$$

Apply condition (iii) in equation (3), we get

$$u(x, \infty) = B \sin \frac{n\pi}{l} x [C e^{\infty} + D e^{-\infty}] = 0$$

$$B \sin \frac{n\pi}{l} x \, [C e^{\infty}] = 0 \qquad [\because e^{-\infty} = 0]$$

Here, $e^{\infty} \neq 0$

$$\sin \frac{n\pi x}{l} \neq 0 \qquad [\because \text{ it is defined for all } x]$$

$$B \neq 0 \qquad [\because \text{ we already explained}]$$

$$\Rightarrow \boxed{C = 0}$$

Substitute $C = 0$ in equation (3), we get

$$u(x, y) = B D \sin \frac{n\pi x}{l} e^{\frac{-n\pi}{l} y} \qquad \dots (4)$$

The most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} e^{\frac{-n\pi}{l} y} \qquad \dots (5)$$

Applying condition (iv) in equation (5), we get

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = k(lx - x^2) \qquad \dots (6)$$

To find B_n, expand $f(x)$ in a Fourier half range sine series

$$\therefore k(lx - x^2) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \qquad \dots (7)$$

where $B_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$

From (6) and (7) we get, $B_n = b_n$

$$\therefore B_n = \frac{2}{l} \int_0^l k(lx - x^2) \sin \left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2k}{l} \int_0^l (lx - x^2) \sin \left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2k}{l}\left[(lx-x^2)\left[\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)}\right]-(l-2x)\left[\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]+(-2)\left[\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}\right]\right]_0^l$$

$$= \frac{2k}{l}\left[-(lx-x^2)\left(\frac{l}{n\pi}\right)\cos\frac{n\pi x}{l}+(l-2x)\left(\frac{l}{n\pi}\right)^2\sin\frac{n\pi x}{l}-2\left(\frac{l}{n\pi}\right)^3\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{2k}{l}\left[\left(-0+0-2\left(\frac{l}{n\pi}\right)^3\cos n\pi\right)-\left(-0+0-2\left(\frac{l}{n\pi}\right)^3\right)\right]$$

$$= \frac{2k}{l}\left[-2\left(\frac{l}{n\pi}\right)^3(-1)^n+2\left(\frac{l}{n\pi}\right)^3\right]$$

$$= \frac{2k}{l}\,2\left(\frac{l}{n\pi}\right)^3[1-(-1)^n]$$

$$= \frac{2k}{l}\,\frac{2l^3}{n^3\pi^3}[1-(-1)^n]$$

$$= \frac{4kl^2}{n^3\pi^3}[1-(-1)^n]$$

$$= \begin{cases}\dfrac{8kl^2}{n^3\pi^3} & \text{if } n \text{ is odd}\\ 0 & \text{if } n \text{ is even}\end{cases}$$

Substitute the value of B_n in equation (5), we get

$$u(x,y) = \sum_{n=\text{odd}}^{\infty}\frac{8kl^2}{n^3\pi^3}\sin\frac{n\pi x}{l}\,e^{\frac{-n\pi}{l}y}$$

$$= \frac{8kl^2}{\pi^3}\sum_{n=\text{odd}}^{\infty}\frac{1}{n^3}\sin\frac{n\pi x}{l}\,e^{\frac{-n\pi}{l}y}$$

$$= \frac{8kl^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{l}\,e^{\frac{-(2n-1)\pi}{l}y}$$

$$8\sum_{1}^{\infty}\frac{1}{n^4} = \left[\frac{\pi^4}{5}-\frac{\pi^4}{9}\right] \Rightarrow 8\sum_{1}^{\infty}\frac{1}{n^4} = \left[\frac{4\pi^4}{45}\right]$$

$$\sum_{1}^{\infty}\frac{1}{n^4} = \left[\frac{\pi^4}{90}\right]$$

$$\frac{\pi^4}{90} = 1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\ \dots$$

14. (a) Find the Fourier transform of $f(x)=\begin{cases}1, & |x|<a\\ 0, & |x|>a>0\end{cases}$ and hence evaluate $\int_0^\infty \frac{\sin x}{x}dx$. Using Parseval's identity, prove that $\int_0^\infty \frac{\sin^2 t}{t^2}dt=\frac{\pi}{2}$. (16)

Here $f(x) = 1, \quad -a<x<a$

$f(x) = 0, \quad -\infty<x<-a$ and $a<x<\infty$

$$\therefore F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} f(x)\, e^{isx}\, dx + \int_{-a}^{a} f(x)\, e^{isx}\, dx + \int_{a}^{\infty} f(x)\, e^{isx}\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} 0\cdot e^{isx}\, dx + \int_{-a}^{a} 1\cdot e^{isx}\, dx + \int_{a}^{\infty} 0\cdot e^{isx}\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a} e^{isx}\, dx\right] = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{isx}}{is}\right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{isa}}{is} - \frac{e^{-isa}}{is}\right] = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{isa} - e^{-isa}}{is}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2i \sin as}{is}\right] = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin as}{s}$$

(i) Using Fourier inversion formula, we get

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[f(x)] \cdot e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \cdot \frac{\sin as}{s} (\cos sx - i \sin sx)\, ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} \cdot \cos sx\, ds - \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} \sin sx\, ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} \cos sx\, ds$$

[$\because$ $\frac{\sin as}{s}$ $\sin sx$ is an odd function. For let

$$f(s) = \frac{\sin as}{s} \sin sx$$

$$f(-s) = -\frac{\sin as}{s} \sin sx$$

i.e., $f(s) \neq f(-s)$]

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin as}{s} \cos sx\, ds \quad \text{[Even funct}$$

$$\int_0^\infty \frac{\sin as}{s} \cos sx \, ds = \frac{\pi}{2} f(x)$$

$$= \frac{\pi}{2} \text{ when } |x| < a$$

$$= 0 \text{ when } |x| > a$$

In particular if $x = 0$, we get

$$\int_0^\infty \frac{\sin as}{s} ds = \frac{\pi}{2}$$

Putting $as = t$ when $s = 0$, $t = 0$

$\therefore ds = \frac{dt}{a}$ when $s = \infty$, $t = \infty$

$$\therefore \int_0^\infty \frac{\sin t}{t/a} \cdot \frac{dt}{a} = \frac{\pi}{2}$$

i.e.,

$$\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$$

Fourier

(ii) The Parseval's identity is

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |f(s)|^2 \, ds$$

Since $f(x) = 0$ in $-\infty < x < -a$ and $a < x < \infty$.

We have $$\int_{-a}^{a} (1)^2 \, dx = \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right)^2 ds$$

$$(x)_{-a}^{a} = \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right)^2 ds$$

i.e., $$\int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right)^2 ds = \frac{\pi}{2}(a+a) = \frac{\pi}{2} \cdot 2a$$

i.e., $$2\int_{0}^{\infty} \left(\frac{\sin as}{s}\right)^2 ds = \pi a$$

i.e., $$\int_{0}^{\infty} \left(\frac{\sin as}{s}\right)^2 ds = \frac{\pi}{2} \cdot a$$

Put $as = t$ when $s = 0,\ t = 0.$

$\therefore\ ds = \dfrac{dt}{a}$ when $s = \infty,\ t = \infty$

$$\therefore \int_{0}^{\infty} \left(\frac{\sin t}{t/a}\right)^2 \cdot \frac{dt}{a} = \frac{\pi}{2} a$$

i.e., $$\int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^2 \cdot a^2 \cdot \frac{dt}{a} = \frac{\pi}{2} a$$

i.e., $$\int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$$

(b) (i) Show that the function $e^{-x^2/2}$ is self-reciprocal under Fourier transform by finding the Fourier transform of $e^{-a^2x^2}, a > 0$. (10)

Solution : We know that $F[s] = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2 - isx)} \, dx \quad \cdots (1)$$

W.K.T. $(a-b)^2 = a^2 - 2ab + b^2$

$a^2 - 2ab = (a-b)^2 - b^2$

Here $a = ax$, $2ab = isx$

$2(ax)b = is\,x$

$2ab = is$

$b = \frac{is}{2a}$

$$\therefore (ax)^2 - isx = \left[ax - \frac{is}{2a}\right]^2 - \left[\frac{is}{2a}\right]^2$$

$$= \left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2}$$

$$e^{-[(ax)^2 - isx]} = e^{-\left[(ax - \frac{is}{2a})^2 + \frac{s^2}{4a^2}\right]}$$

$$= e^{-\left(ax - \frac{is}{2a}\right)^2} e^{-s^2/4a^2}$$

$$(1) \Rightarrow \quad = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-s^2/4a^2} e^{-\left[ax - \frac{is}{2a}\right]^2} dx$$

$$F(s) = \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[ax - \frac{is}{2a}\right]^2} dx \qquad \ldots (2)$$

put $u = ax - \dfrac{is}{2a}$ $\quad$ | $\quad x \to -\infty \Rightarrow u \to -\infty$

$du = a\,dx$ $\quad$ | $\quad x \to \infty \Rightarrow u \to \infty$

$$(2) \Rightarrow F(s) = \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} \frac{1}{a} du$$

$$= \frac{2e^{-s^2/4a^2}}{a\sqrt{2\pi}} \int_{0}^{\infty} e^{-u^2} du \qquad \ldots (3)$$

put $t = u^2$ $\quad$ | $\quad u \to 0 \Rightarrow t \to 0$

$dt = 2u\,du$ $\quad$ | $\quad u \to \infty \Rightarrow t \to \infty$

$$du = \frac{1}{2u} dt = \frac{1}{2\sqrt{t}} dt$$

$$(3) \Rightarrow F(s) = \frac{2e^{-s^2/4a^2}}{a\sqrt{2\pi}} \int_{0}^{\infty} e^{-t} \frac{1}{2\sqrt{t}} dt$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \int_{0}^{\infty} e^{-t} t^{-1/2} dt$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \Gamma_{1/2}$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2}\sqrt{\pi}} \sqrt{\pi} \qquad [\because \Gamma_{1/2} = \sqrt{\pi}]$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2}}$$

$$F\left[e^{-a^2x^2}\right] = \frac{1}{a\sqrt{2}} e^{-s^2/4a^2}$$

put $a = \frac{1}{\sqrt{2}}$ we get

$$F\left[e^{-x^2/2}\right] = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)\sqrt{2}} e^{-s^2/4(1/2)} = e^{-s^2/2}$$

Hence $e^{-x^2/2}$ is self reciprocal under Fourier transform.

14 (b) (ii) **Find Fourier sine and cosine transform of x^{n-1} and hence prove $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier sine and cosine transforms.**

Solution : (i) $F_c[x^{n-1}] = \sqrt{\frac{2}{\pi}} \int_0^\infty x^{n-1} \cos sx \, dx$

We know that, $\Gamma(n) = \int_0^\infty e^{-y} y^{n-1} dy$, $\quad n > 0$

put $y = ax$, we get $\int_0^\infty e^{-ax} x^{n-1} dx = \frac{\Gamma(n)}{a^n}$, $\quad a > 0$

Let $a = is$

$$\therefore \int_0^\infty e^{-isx} x^{n-1} dx = \frac{\Gamma(n)}{(is)^n}$$

$$\int_0^\infty (\cos sx - i \sin sx) x^{n-1} dx = \frac{\Gamma(n)\, i^{-n}}{s^n}$$

$$= \frac{\Gamma(n)}{s^n}\left[\cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right)\right]^{-n}$$

$$= \frac{\Gamma(n)}{s^n}\left[\cos\frac{n\pi}{2} - i \sin\frac{n\pi}{2}\right]$$

Equating real parts, we get

$$\int_0^\infty x^{n-1} \cos sx\, dx = \frac{\Gamma(n)}{s^n} \cos\left(\frac{n\pi}{2}\right)$$

Using this in (1), we get

$$F_c[x^{n-1}] = \sqrt{\frac{2}{\pi}}\, \frac{\Gamma(n)}{s^n} \cos\left(\frac{n\pi}{2}\right)$$

put $n = \frac{1}{2}$, we get

$$F_c\left[\frac{1}{\sqrt{x}}\right] = \sqrt{\frac{2}{\pi}}\, \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}} \cos\left(\frac{\pi}{4}\right)$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{\sqrt{\pi}}{\sqrt{s}}\, \frac{1}{\sqrt{2}} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$= \frac{1}{\sqrt{s}}$$

Hence, $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier cosine transform.

(ii) $F_s\,[x^{n-1}] = \sqrt{\frac{2}{\pi}} \int_0^\infty x^{n-1} \sin sx\, dx$

$$= \sqrt{\frac{2}{\pi}}\, \frac{\Gamma(n)}{s^n} \sin\frac{n\pi}{2} \qquad \left[\because \int_0^\infty x^{n-1} \sin sx\, dx = \frac{\Gamma(n)}{s^n} \sin\frac{n\pi}{2}\right]$$

Taking $n = \frac{1}{2}$, we get

$$F_s\,[x^{\frac{1}{2}-1}] = \sqrt{\frac{2}{\pi}}\, \frac{\Gamma\frac{1}{2}}{s^{1/2}} \sin\frac{\pi}{4}$$

$$F_s\left[\frac{1}{\sqrt{x}}\right] = \sqrt{\frac{2}{\pi}}\, \frac{\sqrt{\pi}}{\sqrt{s}}\, \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{s}}$$

Hence, $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier sine transform.

15. (a) (i) Find $Z(r^n \cos n\theta)$ and $Z^{-1}\left[\left(1-az^{-1}\right)^{-2}\right]$. (8)

Sol. Let

$$a = re^{i\theta}$$

$$a^n = (re^{i\theta})^n = r^n e^{in\theta}$$

$$= r^n [\cos n\theta + i \sin n\theta]$$

$$= r^n \cos n\theta + i r^n \sin n\theta$$

We know that $Z[a^n] = \frac{z}{z-a}$

$$Z[(re^{i\theta})^n] = \frac{z}{z - re^{i\theta}}$$

$$Z[r^n e^{in\theta}] = \frac{z}{z - r(\cos\theta + i\sin\theta)}$$

$$Z[r^n \cos n\theta + i r^n \sin n\theta] = \frac{z}{z - r\cos\theta - ir\sin\theta}$$

$$= \frac{z[(z - r\cos\theta) + ir\sin\theta]}{[(z - r\cos\theta) - ir\sin\theta][(z - r\cos\theta) + ir\sin\theta)]}$$

$$= \frac{z(z - r\cos\theta) + irz\sin\theta}{(z - r\cos\theta)^2 + r^2\sin^2\theta}$$

$$= \frac{z(z - r\cos\theta) + izr\sin\theta}{z^2 - 2zr\cos\theta + r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= \frac{z(z - r\cos\theta) + izr\sin\theta}{z^2 - 2zr\cos\theta + r^2}$$

$$= \frac{z(z - r\cos\theta)}{z^2 - 2zr\cos\theta + r^2} + i\frac{zr\sin\theta}{z^2 - 2zr\cos\theta + r^2}$$

Equating the real and imaginary parts we get

$$Z\{r^n \cos n\theta\} = \frac{z(z - r\cos\theta)}{z^2 - 2zr\cos\theta + r^2}, \quad |z| > r$$

$$Z\{r^n \sin n\theta\} = \frac{zr\sin\theta}{z^2 - 2zr\cos\theta + r^2}, \quad |z| > r$$

Given that

$$z^{-1}\left[(1-az^{-1})^{-2}\right] = z^{-1}\left[\left(1-\frac{a}{z}\right)^{-2}\right]$$

$$= z^{-1}\left[\frac{1}{(1-a/z)^2}\right]$$

$$= z^{-1}\left[\frac{z^2}{(z-a)^2}\right]$$

We know that

$$z[(n+1)a^n] = \frac{z^2}{(z-a)^2}.$$

(i.e)

$$z^{-1}\left[(1-az^{-1})^{-2}\right] = (n+1)a^n.$$

(ii) Using convolution theorem, find $Z^{-1}\left[\frac{z^2}{(z-1/2)(z-1/4)}\right]$. (8)

Sol. $$Z^{-1}\left[\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\right] = Z^{-1}\left[\frac{z}{z-\frac{1}{2}}\cdot\frac{z}{z-\frac{1}{4}}\right]$$

$$= Z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right] * Z^{-1}\left[\frac{z}{z-\frac{1}{4}}\right]$$

$$= \left(\frac{1}{2}\right)^n * \left(\frac{1}{4}\right)^n$$

$$= \sum_{K=0}^{n}\left(\frac{1}{2}\right)^{n-K}\left(\frac{1}{4}\right)^K$$

$$= \left(\frac{1}{2}\right)^n\sum_{K=0}^{n}\left(\frac{1}{2}\right)^{-K}\left(\frac{1}{4}\right)^K$$

$$= \left(\frac{1}{2}\right)^n\sum_{K=0}^{n}\left(\frac{1}{2}\right)^{-K}\left(\frac{1}{2}\right)^{2K}$$

$$= \left(\frac{1}{2}\right)^n\sum_{K=0}^{n}\left(\frac{1}{2}\right)^{K}$$

$$= \left(\frac{1}{2}\right)^n \left[1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \dots + \left(\frac{1}{2}\right)^n\right]$$

$$= \left(\frac{1}{2}\right)^n \left[\frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}}\right] \text{ by G.P.}$$

$$= \left(\frac{1}{2}\right)^n \left[\frac{1 - \left(\frac{1}{2}\right)^{n+1}}{\frac{1}{2}}\right]$$

$$= \left(\frac{1}{2}\right)^{n-1} \left[1 - \left(\frac{1}{2}\right)^{n+1}\right].$$

(b) (i) Using Z – transform, solve the difference equation $x(n+2) - 3x(n+1) + 2x(n) = 0$ given that $x(0) = 0, x(1) = 1$. (8)

Solve the equation $y_{n+2} - 3y_{n+2} + 2y_n = 0$

given $y_0 = 0$, $y_1 = 1$

Solution :

Given $u(n+2) - 3u(n+1) + 2u(n) = 0$

Taking z - transform on both sides, we get

$$Z[u(n+2)] - 3Z[u(n+1)] + 2Z[u(n)] = 0$$

$$z^2\left[F[z] - u(0) - \frac{u(1)}{z}\right] - 3z[F[z] - u(0)] + 2F(z) = 0$$

$F[z](z^2-3z+2) = z$

$F[z] = \frac{z}{(z-2)(z-1)}$

$\Rightarrow \frac{F[z]}{z} = \frac{1}{(z-2)(z-1)} = \frac{A}{z-2} + \frac{B}{z-1}$

$\Rightarrow 1 = A(z-1) + B(z-2)$

Put $z=1 \Rightarrow B=-1$; Put $z=2 \Rightarrow \boxed{A=1}$

$\frac{F[z]}{z} = \frac{1}{z-2} - \frac{1}{z-1}$

$F[z] = \frac{z}{z-2} - \frac{z}{z-1}$

Taking Inverse z-transform, we get

$u(n) = 2^n - (-1)^n$.

(ii) Using residue method, find $Z^{-1}\left[\frac{z}{z^2-2z+2}\right]$. (8)

$\frac{z^2+z}{(z-1)(z^2+1)}$

Solution

Let $F[z] = \frac{z(z+1)}{(z-1)(z^2+1)}$

$\frac{F[z]}{z} = \frac{z+1}{(z-1)(z^2+1)} = \frac{A}{z-1} + \frac{Bz+C}{z^2+1}$

$\Rightarrow z+1 = A(z^2+1) + (Bz+c)(z-1)$

Put $z=1 \Rightarrow 2 = 2A \Rightarrow \boxed{A=1}$

Put $z = 0 \Rightarrow 1 = A - C \Rightarrow \boxed{C = 0}$

Put $z = -1 \Rightarrow 0 = 2A + 2B - 2C \Rightarrow \boxed{B = -1}$

$$\frac{F[z]}{z} = \frac{1}{z-1} - \frac{z}{z^2+1}$$

$$F[z] = \frac{z}{z-1} - \frac{z^2}{z^2+1}$$

Taking Inverse z-transform, we get

$$z[n] = z^{-1}\left[\frac{z}{z-1}\right] - z^{-1}\left[\frac{z^2}{z^2+1}\right]$$

$$\boxed{z[n] = 1^n - \cos\frac{n\pi}{2}} \quad |z| > 1.$$