B.E./B.Tech. DEGREE EXAMINATION, NOVEMBER/DECEMBER 2015.

Third Semester

Civil Engineering

MA 6351 — TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS

(Common to all branches except Environmental Engineering, Textile Chemistry, Textile Technology, Fashion Technology and Pharmaceutical Technology)

(Regulation 2013)

Time : Three hours   Maximum : 100 marks

Answer ALL questions.

PART A — ($10 \times 2 = 20$ marks)

1. Construct the partial differential equation of all spheres whose centres lie on the Z – axis, by the elimination of arbitrary constants.

Solution : Given that the centres of the spheres lie on the Z-axis.

$\therefore$ Centre is $(0, 0, c)$. Let R be the radius.

$\therefore$ equation of the family of spheres is

$$x^2 + y^2 + (z-c)^2 = R^2 \qquad \dots (1)$$

where c and R are arbitrary constants.

Differentiating (1) partially w.r.to x and y, we get,

$$2x + 2(z-c)\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow x + (z-c)p = 0 \Rightarrow (z-c)p = -x \qquad \dots (2)$$

$$\text{and } 2y + 2(z-c)\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow y + (z-c)q = 0 \quad \Rightarrow (z-c)q = -y \qquad \dots (3)$$

$$\frac{(2)}{(3)} \quad \Rightarrow \frac{p}{q} = \frac{x}{y} \Rightarrow py = qx$$

which is the required partial differential equation.

2. Solve $(D+D'-1)(D-2D'+3)z = 0$.

Comparing this with $(D - m_1 D' - c_1)(D - m_2 D' - c_2) = 0$, we get

$c_1 = 1$, $c_2 = 3$, $m_1 = -1$, $m_2 = 2$

$\therefore$ the solution is $z = e^{c_1 x} f_1(y + m_1 x) + e^{c_2 x} f_2(y + m_2 x)$

$$z = e^{x} f_1(y - x) + e^{-x} f_2(y + 2x)$$

3. Find the root mean square value of $f(x) = x(l-x)$ in $0 \le x \le l$.

$$\text{RMS value} = \sqrt{\frac{\int_0^{2l} [f(x)]^2\, dx}{2l - 0}}$$

$$= \sqrt{\frac{\int_0^{l} (lx^2 - x^3)dx}{l}} = \sqrt{\frac{\left(l\frac{x^3}{3} - \frac{x^4}{4}\right)_0^l}{l}} = \sqrt{\frac{\left(\frac{l^4}{3} - \frac{l^4}{4}\right)}{l}} = \sqrt{\frac{l^4}{12} * \frac{1}{l}} = \sqrt{\frac{l^3}{12}}$$

4. Find the sine series of function $f(x) = 1$, $0 \le x \le \pi$.

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin nx\, dx$$

$$= \frac{2}{n\pi}\left[1 - (-1)^n\right]$$

5. Solve $3x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0$; by method of separation of variables.

Let $u = XY$ be the solution where $X = X(x)$ and $Y = Y(y)$

Then $u_x = X'Y$ and $u_y = XY'$

$\therefore$ $3X'Y + 2XY' = 0$

$3X'Y = -2XY'$

$$3\frac{X'}{X} = -2\frac{Y'}{Y} = k$$

$$\frac{X'}{X} = \frac{k}{3} \qquad \text{and} \qquad \frac{Y'}{Y} = -\frac{k}{2}$$

Integrating, we get

log X = $\frac{k}{3}x$ + logA and logY = $-\frac{k}{2}y$ + logB

X = A $e^{\frac{k}{3}x}$ and Y = B $e^{-\frac{k}{2}y}$

$\therefore$ u = AB $e^{\frac{k}{3}x}$ $e^{-\frac{k}{2}y}$ --------(1)

6. Write all possible solutions of two dimensional heat equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

$$u(x,y) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)\left(C\cos\lambda y + D\sin\lambda y\right)$$

$$u(x,y) = \left(A\cos\lambda x + B\sin\lambda x\right)\left(Ce^{\lambda y} + De^{-\lambda y}\right)$$

$$u(x,y) = (Ax+B)(Cy+D)$$

7. If $F(s)$ is the Fourier Transform of $f(x)$, prove that $F\{f(ax)\} = \frac{1}{a}F\left(\frac{s}{a}\right), a \neq 0$.

$$F\left[f(ax)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)\, e^{isx} dx$$

$$= \frac{1}{a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{i\left(\frac{s}{a}\right)t}\, dt$$ { put ax = t and dx = $\frac{1}{a}$ dt }

$$= \frac{1}{a}F\left(\frac{s}{a}\right)$$

8. Evaluate $\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)} ds$ using Fourier Transforms.

$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx$	$F_s[g(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \sin sx\, dx$
$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx\, dx$	$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-bx} \sin sx\, dx$
$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+a^2}\right]$	$= \sqrt{\frac{2}{\pi}} \frac{s}{s^2+b^2}$

$$f(x)g(x) = e^{-ax}e^{-bx} = e^{-(a+b)x}$$

$$\int_0^\infty f(x)g(x)\,dx = \int_0^\infty e^{-(a+b)x}\,dx = \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$= 0 - \left[\frac{1}{-(a+b)}\right] = \frac{1}{a+b}$$

$$F_s[f(x)]F_s[g(x)] = \frac{2}{\pi} \frac{s^2}{(s^2+a^2)(s^2+b^2)}$$

$$\therefore (1) \Rightarrow \frac{1}{a+b} = \frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}\,ds$$

$$\Rightarrow \int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}\,ds = \frac{\pi}{2(a+b)}$$

$$\Rightarrow \int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{2(a+b)}$$ [∵ s is a dummy variab

9. Find the Z – transform of $\frac{1}{n+1}$.

Solution :

We know that, $Z\{x(n)\} = \sum_{n=0}^{\infty} x(n)\, z^{-n}$

$$Z\left[\frac{1}{n+1}\right] = \sum_{n=0}^{\infty} \frac{1}{n+1} z^{-n} = \sum_{n=0}^{\infty} \frac{1}{n+1} \left(\frac{1}{z}\right)^{n}$$

$$= 1 + \frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{3}\left(\frac{1}{z}\right)^2 + \ldots$$

$$= z\left[\frac{1}{z}\right]\left[1 + \frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{3}\left(\frac{1}{z}\right)^2 + \ldots\right]$$

$$= z\left[\frac{1}{z} + \frac{\left(\frac{1}{z}\right)^2}{2} + \frac{\left(\frac{1}{z}\right)^3}{3} + \ldots\right]$$

$$= z\left[-\log\left(1 - \frac{1}{z}\right)\right]$$

$$= z\left[-\log\left(\frac{z-1}{z}\right)\right]$$

$$= z\left[\log \frac{z}{z-1}\right] = z \log \frac{z}{z-1}$$

Formula :

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$$

Here, $x = \frac{1}{z}$

10. State the final value theorem. In Z transform.

The final value theorem in Z transform is $\underset{n\to\infty}{Lt}\, f(n) = \underset{z\to 1}{Lt}\, (z-1)\, F(z)$

PART B — (5 × 16 = 80 marks)

11. (a) (i) Find complete solution of $z^2(p^2+q^2)=(x^2+y^2)$. (8)

Solution : Given: $z^2(p^2+q^2) = x^2+y^2$ [A.U. CBT A/M 2

$$(zp)^2+(zq)^2 = x^2+y^2 \quad \ldots (1)$$ [A.U.T CH N/D 2

This equation is of the form $f_1(x, z^m p) = f_2(y, z^m q)$ [Type 6]

Here $m \neq -1$

$$\therefore \quad Z = z^{m+1}$$

$$Z = z^2$$

$$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial x}$$

$$P = 2zp$$

$$\frac{P}{2} = zp$$

Similarly, $\dfrac{Q}{2} = zq$

Substitute in equation (1), we get

$$\left(\frac{P}{2}\right)^2 + \left(\frac{Q}{2}\right)^2 = x^2 + y^2$$

$$P^2 + Q^2 = 4(x^2 + y^2)$$

$$P^2 - 4x^2 = 4y^2 - Q^2$$

This equation is of the form $f_1(x, p) = f_2(y, Q)$ [Type 4]

$$\therefore \quad P^2 - 4x^2 = 4y^2 - Q^2 = 4a^2 \text{ (say)}$$

$$P^2 = 4a^2 + 4x^2, \qquad Q^2 = 4y^2 - 4a^2$$

$$P = 2\sqrt{a^2 + x^2}, \qquad Q = 2\sqrt{y^2 - a^2}$$

$$dZ = P\,dx + Q\,dy$$

$$dZ = 2\sqrt{a^2 + x^2}\,dx + 2\sqrt{y^2 - a^2}\,dy$$

$$\int dZ = 2\int \sqrt{a^2 + x^2}\,dx + 2\int \sqrt{y^2 - a^2}\,dy$$

$$Z = 2\left[\frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{y}{2}\sqrt{y^2 - a^2} - \frac{a^2}{2}\cosh^{-1}\frac{y}{a}\right] + b$$

$$z^2 = x\sqrt{x^2 + a^2} + a^2\sinh^{-1}\frac{x}{a} + y\sqrt{y^2 - a^2} - a^2\cosh^{-1}\frac{y}{a} + b$$

$$= x\sqrt{x^2 + a^2} + y\sqrt{y^2 - a^2} + a^2\left[\sinh^{-1}\frac{x}{a} - \cosh^{-1}\frac{y}{a}\right] + b$$

(ii) Find the general solution of $(D^2 + 2DD' + D'^2)z = 2\cos y - x\sin y$. (8)

$(D^2+2DD'+D'^2)z = 2\cos y - x\sin y$

The auxillary eqn is

$m^2+2m+1=0$

$m^2+m+m+1=0$

$m(m+1)+1(m+1)=0$

$(m+1)(m+1)=0$

C.F $f_1(y-x)+xf_2(y-x)$

P.I $= \dfrac{1}{(D+D')(D+D')}(2\cos y - x\sin y)$

$= P.I_1 + P.I_2$

$P.I_1 = \dfrac{1}{D^2+2DD'+D'^2}\,2\cos y.$

Here
$D^2=-a^2=0$
$DD'=0$
$D'^2=-b^2=-1$

$= \dfrac{1}{0+0-1}\,2\cos y$

$P.I_1 = -2\cos y.$

$P.I_2 = \dfrac{1}{D^2+2DD'+D'^2}\,x\sin y.$

$= \dfrac{1}{(D+D')(D+D')}\,x\sin y = \dfrac{x^2}{2!}\int \cancel{(c+x)}\; x\sin(c+x)\,dx$

Let $y = c+x$

$= \dfrac{x^2}{2!}\left[x\sin(c+x)+\cos(c+x)\right]$

Put $c = y-x$

$= \dfrac{x^2}{2!}\left[x\sin y+\cos y\right]$

$\therefore\; z(x,y) = f_1(y-x)+xf_2(y-x) - 2\cos y + \dfrac{x^2}{2}\left[x\sin y+\cos y\right]$

(b) (i) **Solve the equation $(x^2 - y^2 - z^2)\, p + 2xyq = 2xz$**

Solution : This is of the form Lagrange's $Pp + Qq = R$

The subsidiary equation is $\dfrac{dx}{x^2 - y^2 - z^2} = \dfrac{dy}{2xy} = \dfrac{dz}{2xz}$

Considering two equations,

$$\frac{dy}{2xy} = \frac{dz}{2xz}$$

Cancelling '$2x$'

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating both sides,

$$\int \frac{dy}{y} = \int \frac{dz}{z}$$

$$\log y = \log z + \log c_1$$

$$\log y - \log z = \log c_1$$

$$\log \left(\frac{y}{z}\right) = \log c_1$$

$$\frac{y}{z} = c_1$$

Consider the multiplier x, y, z

$$\frac{x\,dx}{x\,(x^2 - y^2 - z^2)} = \frac{y\,dy}{2xy^2} - \frac{z\,dz}{2xz^2}$$

$$\frac{x\,dx + y\,dy + z\,dz}{x\,(x^2 - y^2 - z^2) + y\,(2xy) + z\,(2zx)}$$

$$\frac{x\,dx + y\,dy + z\,dz}{x^3 - xy^2 - xz^2 + 2xy^2 + 2z^2 x}$$

$$\frac{x\,dx + y\,dy + z\,dz}{x\,(x^2 + y^2 + z^2)}$$

Equating the ratios

$$\frac{x\,dx + y\,dy + z\,dz}{x\,(x^2 + y^2 + z^2)} = \frac{dz}{2xz}$$

Cancelling 'x'

$$\frac{2\,(x\,dx + y\,dy + z\,dz)}{x^2 + y^2 + z^2} = \frac{dz}{z}$$

Integrating both sides,

$$\log\,(x^2 + y^2 + z^2) = \log z + \log c_2$$

$$\log x^2 + y^2 + z^2 - \log z = \log c_2$$

$$\log \frac{x^2 + y^2 + z^2}{z} = \log c_2$$

$$\frac{x^2 + y^2 + z^2}{z} = c_2$$

$\therefore$ The solution is $\phi\,(c_1, c_2) = 0$

$$\phi\left(\frac{y}{z}, \frac{x^2 + y^2 + z^2}{z}\right)$$

(ii) Find the general solution of $(D^2+D'^2)z=x^2y^2$. (8)

$(D^2+D'^2)z = x^2y^2$

The auxillary eqn $m^2+1=0$

$m^2=-1$

$m=\pm i$

C.F $f_1(y+ix)+f_2(y-ix)$

$P.I = \frac{1}{(D^2+D'^2)}x^2y^2$

$D(x^2y^2)=2xy^2$ $D'(x^2y^2)=2x^2y$

$D^2(x^2y^2)=2y^2$ $D'^2(x^2y^2)=2x^2$

$= \frac{1}{D^2\left(1-\frac{D'^2}{D^2}\right)}x^2y^2$

$= \frac{1}{D^2}\left(1-\frac{D'^2}{D^2}\right)^{-1}x^2y^2$

$= \frac{1}{D^2}\left[1+\frac{D'^2}{D^2}+\frac{D'^4}{D^4}+\cdots\right]x^2y^2$

$= \frac{1}{D^2}\left[x^2y^2+\frac{D'^2(x^2y^2)}{D^2}\right]$

$= \frac{1}{D^2}\left[x^2y^2+\frac{2x^2}{D^2}\right]$

$= \frac{1}{D^2}\left[x^2y^2+\frac{2}{D}\int x^2\,dx\right]$

$= \frac{1}{D^2}\left[x^2y^2+\frac{2}{D}\left[\frac{x^3}{3}\right]\right]$

$= \frac{1}{D^2}\left[x^2y^2+\int\frac{2x^3}{3}dx\right]$

$= \frac{1}{D^2}\left[x^2y^2+\frac{\cancel{2}x^4}{\cancel{12}\,6}\right]$

$= \frac{1}{D}\int\left(x^2y^2+\frac{x^4}{6}\right)dx$

$= \frac{1}{D}\left[\frac{x^3y^2}{3}+\frac{x^4}{24}\right]$

$PI = \left[\frac{x^4y^2}{12}+\frac{x^5}{120}\right]$

$z = f_1(y+ix)+f_2(y-ix)+\frac{x^4y^2}{12}+\frac{x^5}{120}.$

12. (a) (i) Find the Fourier series expansion the following periodic function of period 4 $f(x)=\begin{cases}2+x & -2\le x\le 0\\ 2-x & 0<x\le 2\end{cases}$ Hence deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8}$. (8)

Sol. Given interval is $(-l, l)$

$$\text{Let } f(x) = \begin{cases}\phi_1(x), & -l \le x \le 0\\ \phi_2(x), & 0 \le x \le l\end{cases}$$

where $\phi_1(x) = l + x$, $\phi_2(x) = l - x$

Here $\phi_1(-x) = l - x = \phi_2(x)$

$\therefore f(x)$ is an even function.

Let the required Fourier series be

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \quad \ldots (1)$$

$$\text{where } a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$a_0 = \frac{2}{l}\int_0^l (l-x)\,dx$$

$$= \frac{2}{l}\left[lx - \frac{x^2}{2}\right]_0^l = \frac{2}{l}\left[\left(l^2 - \frac{l^2}{2}\right) - (0-0)\right]$$

$$= \frac{2}{l}\left[\frac{l^2}{2}\right] = l$$

$$\therefore \boxed{a_0 = l}$$

$$a_n = \frac{2}{l}\int_0^l (l-x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[(l-x)\left[\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)}\right] - (-1)\left[\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]\right]_0^l$$

$$= \frac{2}{l}\left[(l-x)\frac{l}{n\pi}\sin\frac{n\pi x}{l} - \left(\frac{l}{n\pi}\right)^2\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{2}{l}\left[\left(0 - \left(\frac{l}{n\pi}\right)^2 - (-1)^n\right) - \left(0 - \left(\frac{l}{n\pi}\right)^2\right)\right]$$

$$= \frac{2}{l}\left[-\left(\frac{l}{n\pi}\right)^2 (-1)^n + \left(\frac{l}{n\pi}\right)^2\right]$$

$$= \frac{2}{l}\left(\frac{l}{n\pi}\right)^2 [1-(-1)^n]$$

$$= \frac{2l}{n^2\pi^2}[1-(-1)^n]$$

$$\boxed{a_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{4l}{n^2\pi^2} & \text{if } n \text{ is odd} \end{cases}}$$

$$\therefore f(x) = \frac{l}{2} + \sum_{n=\text{odd}}^{\infty} \frac{4l}{n^2\pi^2}\cos\frac{n\pi x}{l}$$

$$= \frac{l}{2} + \sum_{n=1}^{\infty} \frac{4l}{(2n-1)^2\pi^2}\cos\frac{(2n-1)\pi x}{l} \quad \text{... (2)}$$

Put $x = 0$, Here $x = 0$ a point of continuity.

$$\therefore \left.\begin{matrix} f(x) \\ \text{at } x = 0 \end{matrix}\right\} = l$$

$$\therefore l = \frac{l}{2} + \sum_{n=1}^{\infty} \frac{4l}{(2n-1)^2\pi^2}$$

$$l - \frac{l}{2} = \sum_{n=1}^{\infty} \frac{4l}{(2n-1)^2\pi^2}$$

$$\frac{l}{2} = \frac{4l}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{4l}\left[\frac{l}{2}\right] = \frac{\pi^2}{8}$$

(ii) Find the complex form of Fourier series of $f(x)=e^{ax}$ in the interval $(-\pi,\pi)$ where a is a real constant. Hence, deduce that $\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}=\frac{\pi}{a\sinh a\pi}$. (8)

Solution : We know that $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$ where

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax} e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(a-in)x}\, dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{(a-in)x}}{a-in}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(a-in)}\left[e^{(a-in)x}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(a-in)}\left[e^{(a-in)\pi} - e^{-(a-in)\pi}\right]$$

$$= \frac{1}{2\pi(a-in)}\left[e^{a\pi} e^{-in\pi} - e^{-a\pi} e^{in\pi}\right]$$

$$\boxed{\begin{aligned} e^{in\pi} &= \cos n\pi + i\sin n\pi = (-1)^n + i0 = (-1)^n \\ e^{-in\pi} &= \cos n\pi - i\sin n\pi = (-1)^n - i0 = (-1)^n \end{aligned}}$$

$$= \frac{1}{2\pi(a-in)}\left[e^{a\pi}(-1)^n - e^{-a\pi}(-1)^n\right]$$

$$= \frac{(-1)^n}{\pi(a-in)}\left[\frac{e^{a\pi} - e^{-a\pi}}{2}\right]$$

$$= \frac{(-1)^n(a+in)}{\pi(a^2+n^2)}\sinh a\pi$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n(a+in)}{\pi(a^2+n^2)} \sinh a\pi\, e^{inx}$$

$$\text{i.e., } e^{ax} = \frac{\sinh a\pi}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n \frac{a+in}{a^2+n^2} e^{inx} \qquad \ldots (1)$$

(ii) $\cos\alpha x = \frac{1}{2}\left(e^{i\alpha x} + e^{-i\alpha x}\right)$

Put $a = i\alpha$ in (1) we get

$$e^{i\alpha x} = \frac{\sinh i\alpha\pi}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n \frac{i\alpha + in}{-\alpha^2+n^2} e^{inx}$$

$$= \frac{\sinh i\alpha\pi}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n \frac{i(\alpha+n)}{-\alpha^2+n^2} e^{inx}$$

$$= \frac{-i\sinh i\alpha\pi}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n \frac{\alpha+n}{\alpha^2-n^2} e^{inx}$$

$$\boxed{\begin{aligned} &\text{W.K.T } \sin i\theta = \sinh\theta \\ &\text{Put } \theta = i\alpha\pi, \quad \sin i(i\alpha\pi) = i\sinh(i\alpha\pi) \\ &\qquad \sin(-\alpha\pi) = i\sinh(i\alpha\pi) \\ &\qquad -\sin\alpha\pi = i\sinh i\alpha\pi \\ &\qquad \sin\alpha\pi = -i\sinh i\alpha\pi \end{aligned}}$$

(b) (i) Find the half range cosine series of $f(x)=(\pi-x)^2, 0<x<\pi$. Hence find the sum of series $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+....$ (8)

Solution. The half range Fourier cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi - x)^2 dx = \frac{2}{\pi}\left(\frac{(\pi-x)^3}{-3}\right)_0^{\pi} = -\frac{2}{3\pi}(0-\pi^3) = \frac{2}{3}\pi^2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi - x)^2 \cos nx dx$$

$$= \frac{2}{\pi}\left[(\pi-x)^2\cancel{\frac{\sin nx}{n}} - 2(\pi-x)(-1)\left(-\frac{\cos nx}{n^2}\right) + 2\cancel{\frac{-\sin nx}{n^3}}\right]_0^{\pi}$$

$$= -\frac{2}{\pi}\times\frac{2}{n^2}(0-\pi\cos n\pi) = \frac{4}{n^2}(-1)^n$$

∴ The Fourier cosine series of $f(x)$ is

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n \cos nx$$

By Parseval's identity we have

$$\frac{2}{\pi}\int_0^{\pi}(f(x))^2 dx = \frac{{a_0}^2}{2} + \sum_{n=1}^{\infty} {a_n}^2$$

$$\frac{2}{\pi}\int_0^{\pi}(\pi-x)^4dx = \frac{1}{2}\left(\frac{2}{3}\pi^2\right)^2 + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}(-1)^n\right)^2$$

$$\frac{2}{\pi}\left(\frac{(\pi-x)^5}{-5}\right)_0^{\pi} = \frac{1}{2}\frac{4}{9}\pi^4 + \sum_{n=1}^{\infty}\frac{16}{n^4}$$

$$\frac{-2}{5\pi}\left(0-\pi^2\right) = \frac{2}{9}\pi^4 + 16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{2}{5}\pi^4 - \frac{2}{9}\pi^4 = 16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$2\pi^4\left(\frac{1}{5}-\frac{1}{9}\right) = 16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$2\pi^4\times\frac{4}{45} = 16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{\pi^4}{90} = \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots$$

(ii) Determine the first two harmonics of Fourier series for the following data. (8)

x :	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$
$f(x)$:	1.98	1.30	1.05	1.30	–0.88	–0.25

x	y	$y\cos x$	$y\sin x$	$y\cos 2x$	$y\sin 2x$
0°	1.98	1.98	0.000	1.98	0.000
$\frac{\pi}{3}=60^{\circ}$	1.30	0.65	1.126	–0.65	1.126
$\frac{2\pi}{3}=120^{\circ}$	1.05	–0.525	0.909	–0.525	–0.909
$\pi=180^{\circ}$	1.30	–1.30	0.000	1.30	0.000
$\frac{4\pi}{3}=240^{\circ}$	–0.88	0.44	0.762	0.44	–0.762
$\frac{5\pi}{3}=300^{\circ}$	–0.25	–0.125	0.217	0.125	0.2167
	$\Sigma y = 4.5$	$\Sigma y\cos x = 1.12$	$\Sigma y\sin x = 3.014$	$\Sigma y\cos 2x = 2.67$	$\Sigma y\sin 2x = -0.328$

$$a_0 = 2\left[\frac{\Sigma y}{n}\right] = 2\left[\frac{4.5}{6}\right] = 1.5$$

$$a_1 = 2\left[\frac{\Sigma y\cos x}{n}\right] = 2\left[\frac{1.12}{6}\right] = 0.373$$

$$a_2 = 2\left[\frac{\Sigma y\cos 2x}{n}\right] = 2\left[\frac{2.67}{6}\right] = 0.89$$

$$b_1 = 2\left[\frac{\Sigma y\sin x}{n}\right] = 2\left[\frac{3.014}{6}\right] = 1.005$$

$$b_2 = 2\left[\frac{\Sigma y\sin 2x}{n}\right] = 2\left[\frac{-0.328}{6}\right] = -0.109$$

$$y = \frac{a_0}{2} + (a_1\cos x + b_1\sin x) + (a_2\cos 2x + b_2\sin 2x)$$

$$= \frac{1.5}{2} + (0.373\cos x + 1.005\sin x) + (0.89\cos 2x - 0.109\sin 2x)$$

$$= 0.75 + (0.373\cos x + 1.005\sin x) + (0.89\cos 2x - 0.109\sin 2x)$$

13. (a) A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially at rest in its equilibrium position. If it is vibrating by giving to each of its points a velocity $v=\begin{cases}\frac{2kx}{l} & in \quad 0<x<\frac{l}{2} \\ \frac{2k(l-x)}{l} & in \quad \frac{1}{2}<x<l\end{cases}$. Find the displacement of the string at any distance x from one end at any time t. (16)

$\therefore$ the initial position $y(x, 0) = \begin{cases}\frac{2b}{l}x &, 0 \le x \le \frac{l}{2} \\ \frac{2b}{l}(l-x), & \frac{l}{2} < x \le l\end{cases}$

Thus the boundary-value conditions are

(i) $y(0, t) = 0$ and (ii) $y(l, t) = 0 \quad \forall\ t \ge 0$

(iii) $\frac{\partial y}{\partial t}(x, 0) = 0$ and (iv) $y(x, 0) = \frac{2b}{l}x$ if $0 \le x \le \frac{l}{2}$

$= \frac{2b}{l}(l-x)$ if $\frac{l}{2} < x \le l$

The solution of the wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ is

$$y(x, t) = (A \cos \lambda x + B \sin \lambda x)(C \cos \lambda ct + D \sin \lambda ct) \quad \dots (1)$$

Using condition (i) ie. when $x = 0$, $y = 0$ in (1),

we get $(A \cos 0 + B \sin 0)(C \cos \lambda ct + D \sin \lambda ct) = 0$

$\Rightarrow \quad A(C \cos \lambda ct + D \sin \lambda ct) = 0 \Rightarrow A = 0$

$\therefore$ (1) becomes $y(x, t) = B \sin \lambda x (C \cos \lambda ct + D \sin \lambda ct) \quad \dots (2)$

Using condition (ii), ie. when $x = l$, $y = 0$ in (2), we get

$$B \sin \lambda l (C \cos \lambda ct + D \sin \lambda ct) = 0$$

$\Rightarrow \sin \lambda l = 0 \quad [\because B \ne 0,\ C \cos \lambda ct + D \sin \lambda ct \ne 0]$

$\Rightarrow \lambda l = n\pi \Rightarrow \lambda = \frac{n\pi}{l}, n = 1, 2, 3, \dots$

$\therefore$ (2) becomes

$$y(x, t) = B \sin\left(\frac{n\pi x}{l}\right)\left[C \cos\left(\frac{n\pi ct}{l}\right) + D \sin\left(\frac{n\pi ct}{l}\right)\right] \dots (3)$$

Differentiating (3), w.r.to t,

$$\frac{\partial y}{\partial t} = B \sin\left(\frac{n\pi x}{l}\right)\left[-C \sin\left(\frac{n\pi ct}{l}\right).\left(\frac{n\pi c}{l}\right) + D\cos\left(\frac{n\pi ct}{l}\right).\left(\frac{n\pi c}{l}\right)\right]$$

Using condition (iii), ie. when $t = 0$, $\frac{\partial y}{\partial t} = 0$, we get

$$B \sin\left(\frac{n\pi x}{l}\right)\left[0 + D.\frac{n\pi c}{l}\right] = 0$$

$$\Rightarrow \qquad B \sin\left(\frac{n\pi x}{l}\right).D.\frac{n\pi c}{l} = 0 \Rightarrow D = 0$$

$\therefore$ (3) becomes

$$y(x,t) = B\sin\left(\frac{n\pi x}{l}\right).C\cos\left(\frac{n\pi ct}{l}\right)$$

$$= BC\sin\left(\frac{n\pi x}{l}\right).\cos\left(\frac{n\pi ct}{l}\right),\ n = 1, 2, 3, \ldots$$

$\therefore$ the most general solution of the differential equation is the linear combination of these solutions

$\therefore$ the general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right).\cos\left(\frac{n\pi ct}{l}\right) \qquad \ldots (4)$$

Using condition (iv), ie. when $t = 0$,

$$y(x,0) = f(x) = \begin{cases} \frac{2b}{l}x, & 0 \le x \le \frac{l}{2} \\ \frac{2b}{l}(l-x), & \frac{l}{2} \le x \le l \end{cases}$$

$$\therefore\ f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)\cos 0$$

$$= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) \qquad \ldots (5)$$

Since $f(x)$ is in algebraic form, to find B_n, we express $f(x)$ as a Fourier sine series.

Let $f(x)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$ ... (6), where $b_n=\frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

Comparing (5) and (6) we get $B_n = b_n$, $n = 1, 2, 3, \ldots$

$$\text{Now } b_n = \frac{2}{l}\left[\int_0^{l/2} f(x)\sin\left(\frac{n\pi x}{l}\right)dx + \int_{l/2}^{l} f(x)\,.\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{2}{l}\left[\int_0^{l/2} \frac{2bx}{l}\sin\left(\frac{n\pi x}{l}\right)dx + \int_{l}^{l/2} \frac{2b}{l}(l-x)\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{2}{l}\left[\frac{2bx}{l}\cdot\left\{\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right\} - \frac{2b}{l}\cdot\left\{\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right\}\right]_0^{l/2}$$

$$+ \frac{2}{l}\left[\frac{2b}{l}(l-x)\cdot\left\{\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right\} - \frac{2b}{l}(-1).\left\{\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right\}\right]_{l/2}^{l}$$

$$= \frac{2}{l}\cdot\frac{2b}{l}\left[-\frac{l}{n\pi}x\,.\cos\left(\frac{n\pi x}{l}\right) + \frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right)\right]_0^{l/2}$$

$$+ \frac{2}{l}\cdot\frac{2b}{l}\left[-\frac{l}{n\pi}(l-x)\cos\left(\frac{n\pi x}{l}\right) - \frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right)\right]_{l/2}^{l}$$

$$= \frac{4b}{l^2}\left[-\frac{l}{n\pi}\cdot\frac{l}{2}\cdot\cos\left(\frac{n\pi}{l}\cdot\frac{l}{2}\right)+\frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{l}\cdot\frac{l}{2}\right)-0\right]$$

$$+\frac{4b}{l^2}\left[-\frac{l}{n\pi}(l-l)\cdot\cos\left(\frac{n\pi}{l}\cdot l\right)-\frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{l}\cdot l\right)\right.$$

$$\left.-\left\{\frac{-l}{n\pi}\left(l-\frac{l}{2}\right)\cos\left(\frac{n\pi}{l}\cdot\frac{l}{2}\right)-\frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{l}\cdot\frac{l}{2}\right)\right\}\right]$$

$$= \frac{4b}{l^2}\left[-\frac{l^2}{2n\pi}\cdot\cos\left(\frac{n\pi}{2}\right)+\frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$+\frac{4b}{l^2}\left[0+\left\{\frac{l}{n\pi}\cdot\frac{l}{2}\cdot\cos\left(\frac{n\pi}{2}\right)+\frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right\}\right]$$

$$= \frac{4b}{l^2}\left[-\frac{l^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right)+\frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right.$$

$$\left.+\frac{l^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right)+\frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{l}\right)\right]$$

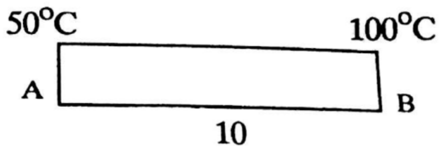
$$b_n = \frac{4b}{l^2}\left[\frac{2l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right] = \frac{8b}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right),\ n = 1, 2, \ldots$$

$$\therefore\ y(x,t) = \sum_{n=1}^{\infty}\frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}\cdot\sin\left(\frac{n\pi x}{l}\right)\cdot\cos\left(\frac{n\pi c t}{l}\right)$$

$$= \frac{8b}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\ \sin\frac{n\pi}{2}\ \sin\left(\frac{n\pi x}{l}\right)\cdot\cos\left(\frac{n\pi c t}{l}\right)$$

(b) A bar 10 cm long with insulated sides has its ends A and B maintained at temperature $50°C$ and $100°C$, respectively, until steady state conditions prevails. The temperature at A is suddenly raised to $90°C$ and at the same time lowered to $60°C$ at B. Find the temperature distributed in the bar at time t. (16)

Solution : The temperature at any point in the bar is given by the heat equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ ... (1)



In steady state, the temperature is

$$u(x) = \frac{\theta_2 - \theta_1}{l}x + \theta_1 \qquad \text{Here } \theta_1 = 50,\ \theta_2 = 100,\ l = 10$$

$$\therefore \quad u(x) = \frac{100 - 50}{10}x + 50 \Rightarrow u(x) = 5x + 50,\ 0 \le x \le 10$$

Then suddenly the temperature at A is raised to 90°C and that at B is lowered to 60°C and maintained. So, the temperature distribution in the bar is changed from steady state to unsteady state.

The temperature $u(x, t)$ is the solution of (1)

The boundary-value conditions of the unsteady state are

(i) $u(0, t) = 90°C$, (ii) $u(10, t) = 60°C \ \forall\ t \ge 0$

and (iii) $u(x, 0) = 5x + 50,\ 0 \le x \le 10$

Since the temperatures at the ends are not 0°C, we split the solution $u(x, t)$ of (1) into two parts in order to get zero boundary conditions.

$u(x, t) = u_1(x) + u_2(x, t)$, where

$u_1(x)$ is the steady state solution of (1)

and $u_2(x, t)$ is the unsteady (or transient) solution of (1)

Since $u(x, 0) \neq 0$, the appropriate solution of (1) is the solution with trigonometric function of x.

$$\therefore u(x,y) = (A\cos\lambda x + B\sin\lambda x)(C e^{\lambda y} + D e^{-\lambda y}) \quad \text{... (2)}$$

First we use the zero conditions.

Using condition (i) ie. when $x = 0$, $u = 0$, in (1) we get

$$(A\cos 0 + B\sin 0)(Ce^{\lambda y} + De^{-\lambda y}) = 0$$

$$\Rightarrow A(C e^{\lambda y} + D e^{-\lambda y}) = 0 \Rightarrow A = 0 \qquad [\because Ce^{\lambda y} + D e^{-\lambda y} \neq 0]$$

$\therefore$ (2) becomes $u(x,y) = B\sin\lambda x\,(Ce^{\lambda y} + De^{-\lambda y})$ ... (3)

Using condition (ii), ie. when $x = a$, $u = 0$, in (3) we get

$$B\sin\lambda a\,(Ce^{\lambda y} + De^{-\lambda y}) = 0$$

$$\Rightarrow \quad \sin\lambda a = 0 \qquad [\because B(Ce^{\lambda y} + De^{-\lambda y} \neq 0\,]$$

$$\Rightarrow \lambda a = n\pi \Rightarrow \lambda = \frac{n\pi}{a},\ n = 1, 2, 3, \ldots$$

$\therefore$ (3) becomes $u(x,y) = B\sin\dfrac{n\pi x}{a}\,(Ce^{\lambda y} + De^{-\lambda y})$ ... (4)

Using condition (iii), ie. when $y = b$, $u = 0$ in (4) we get

$$B\sin\left(\frac{n\pi x}{a}\right)\left[Ce^{\lambda b} + De^{-\lambda b}\right] = 0$$

$$\because B\sin\frac{n\pi x}{a} \neq 0,\ Ce^{\lambda b} + De^{-\lambda b} = 0 \Rightarrow De^{-\lambda b} = -Ce^{\lambda b}$$

$$\Rightarrow D = -Ce^{2\lambda b}$$

$$\therefore \text{(4) becomes } u(x,y) = B\sin\left(\frac{n\pi x}{a}\right)\left(Ce^{\lambda y} - Ce^{2\lambda b}.e^{-\lambda y}\right)$$

$$= BC\sin\left(\frac{n\pi x}{a}\right)\left(e^{\lambda y} - e^{2\lambda b}.e^{-\lambda y}\right)$$

$$= BC \sin\left(\frac{n\pi x}{a}\right) e^{\lambda b}\left[e^{\lambda(y-b)} - e^{-\lambda(y-b)}\right]$$

$$= \left(BC\, e^{\lambda b}\right) 2\sin\left(\frac{n\pi x}{a}\right)\left[\frac{e^{\lambda(y-b)} - e^{-\lambda(y-b)}}{2}\right]$$

$$= k\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\lambda(y-b)\right),\ n = 1, 2, 3, \ldots$$

$\therefore$ the linear combination of these solutions is again a solution of (1).

$\therefore$ the general solution is

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\lambda(y-b)\right) \qquad \ldots (5)$$

Using condition (iv) in (5), ie. when $y = 0$, $u(x, 0) = x(a-x)$, we get

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right).\sinh\left(\lambda(-b)\right)$$

$$= \sum_{n=1}^{\infty} -A_n \sin\left(\frac{n\pi x}{a}\right).\sinh\left(\frac{n\pi b}{a}\right)$$

$$\left[\because \lambda = \frac{n\pi}{a},\ \sinh(-x) = -\sinh x\right]$$

$$= \sum_{n=1}^{\infty}\left(-A_n \sinh\left(\frac{n\pi b}{a}\right)\right)\sin\left(\frac{n\pi x}{a}\right) \qquad \ldots (6)$$

Given $u(x, 0) = f(x) = x(a-x)$. Since it is in algebraic form, to find A_n we express $f(x)$ as a Fourier sine series in $(0, a)$. Here $l = a$

$$\therefore\ u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right) \qquad \ldots (7)$$

where $$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{a}\int_0^a (ax - x^2)\sin\left(\frac{n\pi x}{a}\right)dx$$

$$= \frac{2}{a}\left[(ax - x^2)\left(\frac{-\cos\left(\frac{n\pi x}{a}\right)}{\frac{n\pi}{a}}\right) - (a - 2x)\left(\frac{-\sin\frac{n\pi x}{a}}{\left(\frac{n\pi}{a}\right)^2}\right)\right.$$

$$\left. + (-2)\left(\frac{\cos\left(\frac{n\pi x}{a}\right)}{\left(\frac{n\pi}{a}\right)^3}\right)\right]_0^a$$

$$= \frac{2}{a}\left[\frac{-a}{n\pi}(ax - x^2)\cos\left(\frac{n\pi x}{a}\right) + (a - 2x)\frac{a^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{a}\right)\right.$$

$$\left. - 2\frac{a^3}{n^3\pi^3}\cos\frac{n\pi x}{a}\right]_0^a$$

$$= \frac{2}{a}\left[\frac{-a}{n\pi}(a^2 - a^2)\cos n\pi + (a - 2a)\frac{a^2}{n^2\pi^2}\sin n\pi\right.$$

$$\left. - 2\frac{a^3}{n^3\pi^3}\cos n\pi - \left(0 - \frac{2a^3}{n^3\pi^3}\cos 0\right)\right]$$

$$b_n = \frac{2}{a}\left[0 - \frac{2a^3}{n^3\pi^3}\cos n\pi + \frac{2a^3}{n^3\pi^3}\right] = \frac{4a^2}{n^3\pi^3}[-(-1)^n + 1]$$

When n is even, $(-1)^n = 1 \qquad \therefore -(-1)^n + 1 = -1 + 1 = 0$

When n is odd, $(-1)^n = -1 \qquad \therefore -(-1)^n + 1 = 1 + 1 = 2$

$$\therefore b_n = \frac{4a^2}{n^3\pi^3}\cdot 2 = \frac{8a^2}{n^3\pi^3},\ n = 1,3,5, \ldots$$

Comparing (6) and (7) we find

$$-A_n \sinh\left(\frac{n\pi b}{a}\right) = b_n = \frac{8a^2}{n^3\pi^3}$$

$$\Rightarrow \quad A_n = \begin{cases} \dfrac{-8a^2}{n^3 \pi^3 \sinh\left(\dfrac{n\pi b}{a}\right)}, & n = 1,3,5,\ldots \\ 0 \quad , & \text{if } n \text{ is even} \end{cases}$$

Substituting in (5), we get

$$u(x,y) = \sum_{n=1,3,5,\ldots} \frac{-8a^2}{n^3\pi^3 \sinh\left(\dfrac{n\pi b}{a}\right)} \cdot \sinh\left(\frac{n\pi x}{a}\right) \cdot \sinh\left(\frac{n\pi}{a}(y-b)\right)$$

$$= \frac{-8a^2}{\pi^3} \sum_{n=1,3,5,\ldots} \frac{1}{n^3 \sinh\left(\dfrac{n\pi b}{a}\right)} \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi}{a}(y-b)\right)$$

14. (a) (i) **Find the Fourier sine transform of $f(x) = e^{-x} \cos x$.**

Solution : We know that,

$$\boxed{F_s\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx}$$

$$F_s\,[e^{-x}\cos x] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos x \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin sx \cos x\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \left[\frac{\sin(s+1)x + \sin(s-1)x}{2}\right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_0^\infty e^{-x} \sin(s+1)x\, dx + \int_0^\infty e^{-x} \sin(s-1)x\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{s+1}{(s+1)^2+1} + \frac{s-1}{(s-1)^2+1}\right]$$

$$[\because \int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{s+1}{s^2+2s+2} + \frac{s-1}{s^2-2s+2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{s^3-2s^2+2s+s^2-2s+2+s^3+2s^2+2s-s^2-2s-2}{(s^2+2)^2-(2s)^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2s^3}{s^4+4+2s^2-4s^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\frac{2s^3}{s^2+4-2s^2} = \frac{1}{\sqrt{2\pi}}\frac{2s^3}{(s^2-2)^2}$$

(ii) Find the Fourier cosine transform of the function $f(x) = \dfrac{e^{-ax}-e^{-bx}}{x}, x > 0$. (8)

Solution:

$$F_c[f(x)] = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$F_c[e^{-ax}] = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \left[\frac{e^{-ax}}{a^2+s^2}(-a \cos sx + s \sin sx)\right]_0^\infty$$

$$F_c[e^{-ax}] = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{a}{s^2+a^2}$$

$$\frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{a}{s^2+a^2}$$

Integrating both sides w.r.t a

$$\frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \frac{e^{-ax}}{-x} \cos sx \, dx = \frac{\sqrt{2}}{\sqrt{\pi}} \int \left(\frac{a}{s^2+a^2}\right) da$$

$$\frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \frac{e^{-ax}}{-x} \cos sx \, dx = \frac{-\sqrt{2}}{\sqrt{\pi}} \cdot \frac{1}{2} \log(s^2+a^2)$$

$$F_c\left[\frac{e^{-ax}}{x}\right] = -\frac{1}{\sqrt{2\pi}} \log(s^2+a^2)$$

$$F_c\left[\frac{e^{-ax}-e^{-bx}}{x}\right] = F_c\left[\frac{e^{-ax}}{x}\right] - F_c\left[\frac{e^{-bx}}{x}\right]$$

$$= -\frac{1}{\sqrt{2\pi}} \log(a^2+s^2) + \frac{1}{\sqrt{2\pi}} \log(b^2+s^2)$$

$$= \frac{1}{\sqrt{2\pi}} \log \frac{s^2+b^2}{s^2+a^2}$$

(b) (i) Find the Fourier transform of the function $f(x)=\begin{cases} 1-|x|, & |x|\le 1 \\ 0, & |x|>1 \end{cases}$

Hence deduce that $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$. (8)

Ans. F.T of $f(x)$ is given by.

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^{1} (1-|x|) e^{isx} dx \right].$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^{1} (1-|x|) (\cos sx + i \sin sx) dx \right].$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^{1} (1-x) \cos sx \, dx + i \int_{-1}^{1} (1-x) \sin sx \, dx \right].$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^{1} (1-x) \cos sx \, dx + 0 \right]$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{1} (1-x) \cos sx \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \left[(1-x) \frac{\sin sx}{s} - (-1) \left(-\frac{\cos sx}{s^2} \right) \right]_0^1.$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(0 - \frac{\cos s}{s^2} \right) - \left(0 - \frac{1}{s^2} \right) \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1}{s^2} - \frac{\cos s}{s^2} \right]$$

$$F(s) = \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos s}{s^2} \right]$$

By inverse fourier Transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-ixs} ds.$$

$$\int_{-\infty}^{\infty}\left(\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s^2}\right)\right)e^{-ixs}\,ds.$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1-\cos s}{s^2}\right)(\cos xs-i\sin xs)\,ds$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1-\cos s}{s^2}\cos xs\,ds-0.$$

$$=\frac{2}{\pi}\int_{0}^{\infty}\frac{1-\cos s}{s^2}\cos xs\,ds.$$

Put $x=0$ on b.s

$$1-|0|=\frac{2}{\pi}\int_{0}^{\infty}\frac{1-\cos s}{s^2}(1)\,ds.$$

$$\frac{\pi}{2}=\int_{0}^{\infty}\frac{1-\cos s}{s^2}\,ds$$

$$=\int_{0}^{\infty}\frac{2\sin^2(s/2)}{s^2}\,ds.$$

Put $x=\frac{s}{2}$. $s\to\infty\Rightarrow x\to\infty$

$s=2x$

$ds=2dx$. $s\to 0\Rightarrow x\to 0$.

$$\frac{\pi}{2}=\int_{0}^{\infty}\frac{2\sin^2 x}{(2x)^2}\,2dx.$$

$$\frac{\pi}{2}=\int_{0}^{\infty}\frac{\cancel{2}\sin^2 x}{\cancel{4}x^2}\,\cancel{2}dx$$

$$\therefore\int_{0}^{\infty}\frac{\sin^2 x}{x^2}\,dx=\frac{\pi}{2}.$$

using Parseval's identity

$$\int_{-\infty}^{\infty}|f(x)|^2dx=\int_{-\infty}^{\infty}|F(s)|^2ds.$$

$$\int_{-1}^{1}(1-|x|)^2dx=\int_{-\infty}^{\infty}\left(\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s^2}\right)\right)^2ds.$$

$$\int_{-1}^{1}(1-x)^2dx=\frac{2}{\pi}\int\left(\frac{1-\cos s}{s^2}\right)^2ds.$$

$$\left[\frac{2(1-x)^3}{}\right]_{}^{1}=\frac{2\times 2}{}\int_{}^{\infty}\frac{(1-\cos s)^2}{}ds.$$

$$\frac{\cancel{2}\pi}{3\times 2} = \int_0^\infty \left(\frac{2\sin^2 x}{(2x)^2}\right) \cdot 2\,dx.$$

$$\frac{\cancel{2}\pi}{3\times 2} = \int_0^\infty \cancel{4}\frac{\sin^4 x}{\cancel{16}_2 x^4} \cdot x\,dx.$$

$$\frac{\pi}{6} = \int_0^\infty \frac{\sin^4 x}{2x^4}\,dx.$$

$$\frac{\pi}{3} = \int_0^\infty \frac{\sin^4 x}{x^4}\,dx.$$

$$\therefore \int_0^\infty \frac{\sin^4 x}{x^4}\,dx = \frac{\pi}{3}$$

(ii) Verify the convolution theorem for Fourier transform if $f(x)=g(x)=e^{-x^2}$. (8)

Definition : Convolution theorem for Fourier transforms. The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms.

$$F[f(x) * g(x)] = F\{f(x)\} \cdot F\{g(x)\}$$

Given : $f(x) = g(x) = e^{-x^2}$

We know that, $F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} f(x)\, e^{-isx}\, dx$

$$F[e^{-x^2}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} e^{-isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2 + isx)}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[(x - \frac{is}{2})^2 + \frac{s^2}{4}\right]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(x - \frac{is}{2}\right)^2} e^{-s^2/4}\, dx$$

$$= e^{-s^2/4} \left(\frac{1}{\sqrt{2\pi}}\right) \int_{-\infty}^{\infty} e^{-\left(x - \frac{is}{2}\right)^2}\, dx$$

put $t = x - \frac{is}{2}$ | $x \to \infty \Rightarrow t \to -\infty$

$dt = dx$ | $x \to \infty \Rightarrow t \to \infty$

$$= e^{-s^2/4} \left(\frac{1}{\sqrt{2\pi}}\right) \int_{-\infty}^{\infty} e^{-t^2}\, dt$$

$$= e^{-s^2/4} \left(\frac{1}{\sqrt{2\pi}}\right) 2 \int_{0}^{\infty} e^{-t^2}\, dt$$

put $t^2 = u$ | $t \to 0 \Rightarrow u \to 0$

$2t\, dt = du$ | $t \to \infty \Rightarrow u \to \infty$

$$dt = \frac{1}{2t} du$$

$$= \frac{1}{2\sqrt{u}} du$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right) 2 e^{-s^2/4} \int_{0}^{\infty} e^{-u} \frac{1}{2\sqrt{u}}\, du$$

$$= e^{-s^2/4}\left(\frac{1}{\sqrt{2\pi}}\right)\int_0^\infty e^{-u}u^{-1/2}\,du$$

$$= e^{-s^2/4}\left(\frac{1}{\sqrt{2\pi}}\right)\sqrt{\pi} \qquad \left[\because \int_0^\infty e^{-t}t^{-1/2}\,dt = \sqrt{\pi}\right]$$

$$= \frac{1}{\sqrt{2}}e^{-s^2/4}$$

$$F\,[f(x)] = \frac{1}{\sqrt{2}}e^{-s^2/4}$$

$$F\,[g(x)] = F\,[f(x)] = \frac{1}{\sqrt{2}}e^{-s^2/4}$$

$$\therefore F\,[g(x)]\,.\,F\,[g(x)] = \left(\frac{1}{\sqrt{2}}e^{-s^2/4}\right)\left(\frac{1}{\sqrt{2}}e^{-s^2/4}\right) = \frac{1}{2}e^{-s^2/2} \quad \dots (1)$$

$$f(x) * g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,g(x-u)\,du \quad \text{by convolution definition}$$

$$e^{-x^2} * e^{-x^2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-u^2}e^{-(x-u)^2}\,du$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x-u)^2-u^2}\,du$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-2\left[\left(u-\frac{x}{2}\right)^2+\frac{x^2}{4}\right]}\,du$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-2\left(u-\frac{x}{2}\right)^2} e^{\frac{-x^2}{2}}\,du$$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-2\left(u-\frac{x}{2}\right)^2}\,du$$

put $t = u - \frac{x}{2}$ | $u \to -\infty \Rightarrow t \to -\infty$

$dt = du$ | $u \to \infty \Rightarrow t \to \infty$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2t^2}\, dt$$

$$= \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}\, 2\int_{0}^{\infty} e^{-2t^2}\, dt$$

$$\text{put} \quad t^2 = y \qquad \Big| \quad t \to 0 \Rightarrow y \to 0$$

$$2t\, dt = dy \qquad \Big| \quad t \to \infty \Rightarrow y \to \infty$$

$$dt = \frac{1}{2t} dy$$

$$= \frac{1}{2\sqrt{y}} dy$$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, 2\int_{0}^{\infty} e^{-2y} \frac{1}{2\sqrt{y}}\, dy$$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-2y} y^{-1/2}\, dy$$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}} \frac{\sqrt{\pi}}{\sqrt{2}} \qquad \left[\because \int_{0}^{\infty} e^{-2t} t^{-1/2}\, dt = \frac{\sqrt{\pi}}{\sqrt{2}}\right]$$

$$= \frac{1}{2} e^{-x^2/2}$$

$$F\,[f(x) * g(x)] = F\left[\frac{1}{2} e^{-x^2/2}\right]$$

$$= \frac{1}{2} F\,[e^{-x^2/2}]$$

We know that, $e^{-x^2/2}$ is self reciprocal under Fourier transform.

$$\therefore \quad F\,[e^{-x^2/2}] = e^{-s^2/2}$$

$$\therefore \; F\,[f(x) * g(x)] = \frac{1}{2} e^{-s^2/2} \qquad \dots (2)$$

15. (a) (i) If $U(z)=\frac{z^3+z}{(z-1)^3}$, find the value of u_0, u_1 and u_2. (8)

Solution :

$$\text{Let } F(z) = \frac{z(z+1)}{(z-3)^3}$$

Multiplying both sides by z^{n-1}

$$\therefore z^{n-1} F(z) = z^{n-1} \frac{z(z+1)}{(z-3)^3}$$

$$= \frac{z^n (z+1)}{(z-3)^3} = g(z)$$

Equating the denominator to zero

$$(z-3)^3 = 0$$

$$z = 3, 3, 3$$

$$\therefore z = 3 \text{ is a pole of order three}$$

The residue for the pole of order three is given by

$$\lim_{z \to a} \frac{1}{2!} \frac{d^2}{dz^2} (z-a)^3 g(z)$$

$\therefore$ Residue for $z = 3$ is given by

$$\lim_{z \to 3} \frac{1}{2!} \frac{d^2}{dz^2} (z-3)^3 . \frac{z^n (z+1)}{(z-3)^3}$$

$$\lim_{z \to 3} \frac{1}{2} \frac{d^2}{dz^2} . z^n (z+1)$$

$$\lim_{z \to 3} \frac{1}{2} \frac{d^2}{dz^2} \left(z^{n+1} + z^n\right)$$

$$\lim_{z \to 3} \frac{1}{2} \frac{d}{dz} \left[(n+1) z^{n} + n z^{n-1}\right]$$

$$\lim_{z \to 3} \frac{1}{2} \left[(n+1) n z^{n-1} + n (n-1) z^{n-2}\right]$$

$$\frac{1}{2} \left[(n+1) n\, 3^{n-1} + n (n-1)\, 3^{n-2}\right]$$

(ii) Use convolution theorem to evaluate $z^{-1}\left\{\frac{z^2}{(z-3)(z-4)}\right\}$. (8)

Solution

$$z^{-1}\left[\frac{z^2}{(z-4)(z-3)}\right] = z^{-1}\left[\frac{z}{z-4} \cdot \frac{z}{z-3}\right]$$

$$= z^{-1}\left[\frac{z}{z-4}\right] \cdot z^{-1}\left[\frac{z}{z-3}\right]$$

$$= 4^n * 3^n$$

$$= \sum_{m=0}^{n} 4^m\, 3^{n-m}$$

$$= 3^n \sum_{m=0}^{n} 4^m\, 3^{-m}$$

$$= 3^n \sum_{m=0}^{n} \left(4/3\right)^m$$

$$= 3^n \frac{\left(4/3\right)^{n+1} - 1}{4/3 - 1} \quad \text{being a G.P}$$

$$= 3^n \sum_{m=0}^{n} \left(4/3\right)^m = 3^n \frac{4^{n+1} - 3^{n+1}}{\left(1/3\right)} \left[\frac{1}{3^{n+1}}\right]$$

$$= 4^{n+1} - 3^{n+1}$$

(b) (i) Using the inversion integral method (Residue Theorem), find the inverse Z- transform of $U(z)=\frac{z^2}{(z+2)(z^2+4)}$. (8)

Solution : Let $F(Z) = \frac{z^2}{(z+2)(z^2+4)}$

$$\frac{F(Z)}{z} = \frac{z}{(z+2)(z^2+4)}$$

Let $$\frac{z}{(z+2)(z^2+4)} = \frac{A}{z+2} + \frac{Bz+C}{z^2+4}$$

$$= \frac{A(z^2+4)+(Bz+C)(z+2)}{(z+2)(z^2+4)}$$

$$z = A(z^2+4)+(Bz+C)(z+2) \quad \dots (1)$$

Put $z = -2$

$$-2 = A(4+4)+(Bz+C)(0)$$

$$-2 = 8A$$

$$A = \frac{-2}{8}$$

$$A = -\frac{1}{4}$$

Removing the brackets in (1)

$$z = A(z^2 + 4) + (Bz^2 + 2Bz + Cz + 2C)$$

Comparing the coefficient of z^2

$$A + B = 0$$

$$\frac{-1}{4} + B = 0 \Rightarrow B = \frac{1}{4}$$

Comparing the coefficient of constant term

$$4A + 2C = 0$$

$$4\left(\frac{-1}{4}\right) + 2C = 0$$

$$-1 + 2C = 0$$

$$2C = 1$$

$$C = \frac{1}{2}$$

$$\frac{F(z)}{z} = \frac{\frac{-1}{4}}{z+2} + \frac{\left(\frac{1}{4}\right)z + \frac{1}{2}}{z^2+4}$$

$$= \frac{-\frac{1}{4}}{z+2} + \frac{\frac{1}{4}z}{z^2+4} + \frac{\frac{1}{2}}{z^2+4}$$

$$F(z) = \frac{-\frac{1}{4}z}{z+2} + \frac{\frac{1}{4}z^2}{z^2+4} + \frac{\frac{1}{2}z}{z^2+4}$$

$$Z^{-1}[F(z)] = Z^{-1}\left[\frac{-\frac{1}{4}z}{z+2} + \frac{\frac{1}{4}z^2}{z^2+4} + \frac{\frac{1}{2}z}{z^2+4}\right]$$

$$= -\frac{1}{4} Z^{-1}\left[\frac{z}{z+2}\right] + \frac{1}{4} Z^{-1}\left[\frac{z^2}{z^2+4}\right] + \frac{1}{2} Z^{-1}\left[\frac{z}{z^2+4}\right]$$

$$= -\frac{1}{4}(-2)^n + \frac{1}{4} 2^n \cos\frac{n\pi}{2} + \frac{1}{2} 2^n \sin\frac{n\pi}{2}$$

(ii) Using the Z- transform solve the difference equation $u_{n+2} + 4u_{n+1} + 3u_n = 3^n$ with $u_0 = 0, u_1 = 1$. (8)

Solution : Given : $y_{n+2} + 4y_{n+1} + 3y_n = 3^n$

$$Z[y_{n+2}] + 4Z[y_{n+1}] + 3Z[y_n] = Z[3^n]$$

$$[z^2 Y(z) - z^2 y(0) - zy(1)] + 4[zY(z) - zy(0)] + 3Y(z) = \frac{z}{z-3}$$

$$(z^2 + 4z + 3) Y(z) - z = \frac{z}{z-3}$$

$$(z+1)(z+3) Y(z) = \frac{z}{z-3} + z$$

$$= \frac{z + z(z-3)}{z-3}$$

$$= \frac{z^2 - 2z}{z-3}$$

$$Y(z) = \frac{z(z-2)}{(z+1)(z+3)(z-3)}$$

$$\frac{Y(z)}{z} = \frac{z-2}{(z+1)(z+3)(z-3)} = \frac{A}{z+1} + \frac{B}{z+3} + \frac{C}{z-3} \quad \text{... (1)}$$

Let $z - 2 = A(z+3)(z-3) + B(z+1)(z-3) + C(z+1)(z+3)$

Put $z = -1$, we get

$-3 = -8A$

$\boxed{A = \frac{3}{8}}$

put $z = -3$, we get

$-5 = 12B$

$\boxed{B = \frac{-5}{12}}$

put $z = 3$, we get

$1 = 24C$

$\boxed{C = \frac{1}{24}}$

$$\therefore (1) \Rightarrow \frac{Y(z)}{z} = \frac{(3/8)}{z+1} + \frac{(-5/12)}{z+3} + \frac{(1/24)}{z-3}$$

$$Y(z) = \frac{3}{8}\left(\frac{z}{z+1}\right) - \frac{5}{12}\left(\frac{z}{z+3}\right) + \frac{1}{24}\left(\frac{z}{z-3}\right)$$