B.E./B.Tech. DEGREE EXAMINATION, NOVEMBER/DECEMBER 2016.

Third/Fifth Semester

Civil Engineering

MA 2211/MA 31/MA 1201 A/CK 201/080100008/080210001/10177 MA 301 — TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS/ MATHEMATICS — III

(Common to all branches)

(Regulations 2008/2010)

Time : Three hours Maximum : 100 marks

Answer ALL questions.

PART A — (10 × 2 = 20 marks)

1. If $f(x)=x^2+x$ is expressed as a Fourier series in the interval $-2<x<2$, to which value this series converges at $x=2$.

Since $x = 2$ is an end point, the Fourier series converges to $\frac{f(2)+f(-2)}{2} = 4$

2. Define root mean square value of a function f(x) in $(0,2l)$

$$\text{RMS value} = \sqrt{\frac{\int_0^{2l} [f(x)]^2\, dx}{2l-0}}$$

3. Define Fourier Transform pair.

Definition : The complex (or infinite) Fourier Transform

The complex (or infinite) Fourier Transform of $f(x)$ is given by

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx \quad \dots (1)$$

Then the function $f(x)$ is the inverse Fourier Transform of $F(s)$ and is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds \quad \dots (2)$$

The above (1) & (2) are jointly called Fourier transform pair.

4. Find the Fourier cosine transform of e^{-ax}, $a>0$.

Solution : We know that, $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx\, dx$

$$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2+a^2}\right]$$

Formula :

$$\int_0^{\infty} e^{-ax} \cos bx\, dx = \frac{a}{a^2+b^2}$$

5. Find the partial differential equation by eliminating f from $z = f(x^2 + y^2)$.

Solution: (i) Given $z = f(x^2 + y^2)$

Diff.(1) partially w.r.to 'x' and 'y', we have

$$p = f'(x^2+y^2) \times 2x \Rightarrow \frac{p}{2x} = f'(x^2+y^2)$$

$$q = f'(x^2+y^2) \times 2y \Rightarrow \frac{q}{2y} = f'(x^2+y^2)$$

From (2) & (3)

$$\frac{p}{2x} = \frac{q}{2y}$$

$\Rightarrow$ $py = qx$ is the required PDE

6. Solve $p \tan x + q \tan y = \tan z$.

Solution : Given : $p \tan x + q \tan y = \tan z$

i.e., $(\tan x)\, p + (\tan y)\, q = \tan z$

This equation is of the form $P\, p + Q\, q = R$

where $P = \tan x$, $Q = \tan y$, $R = \tan z$

The Lagrange's subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

i.e., $$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

Take $\dfrac{dx}{\tan x} = \dfrac{dy}{\tan y}$

$$\int \frac{dx}{\tan x} = \int \frac{dy}{\tan y}$$

$$\int \cot x\, dx = \int \cot y\, dy$$

$$\log \sin x = \log \sin y + \log a$$

$$\log \sin x = \log (a \sin y)$$

$$\sin x = a \sin y$$

$$a = \frac{\sin x}{\sin y}$$

i.e., $u = \dfrac{\sin x}{\sin y}$,

Take $\dfrac{dy}{\tan y} = \dfrac{dz}{\tan z}$

$$\int \frac{dy}{\tan y} = \int \frac{dz}{\tan z}$$

$$\int \cot y\, dy = \int \cot z\, dz$$

$$\log \sin y = \log \sin z + \log b$$

$$\log \sin y = \log (b \sin z)$$

$$\sin y = b \sin z$$

i.e., $b = \dfrac{\sin y}{\sin z}$

$$v = \frac{\sin y}{\sin z}$$

Hence, the general solution is $f\left[\dfrac{\sin x}{\sin y}, \dfrac{\sin y}{\sin z}\right] = 0$, which f is arbitrary.

7. **Classify the partial differential equation** $\frac{\partial^2 u}{\partial x^2} = 2\frac{\partial^2 u}{\partial x \partial y}$, $x > 0$, $y > 0$.

Here A = 1, B = 2, C = 0
$\Delta = B^2 - 4AC = 4 - 0 = 4.$
$\therefore$ the pde is Hyperbolic type

8. **Write down all possible solutions of one dimensional heat flow equation.**

$$u(x,y) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)\left(C\cos\lambda y + D\sin\lambda y\right)$$
$$u(x,y) = \left(A\cos\lambda x + B\sin\lambda x\right)\left(Ce^{\lambda y} + De^{-\lambda y}\right)$$
$$u(x,y) = (Ax + B)(Cy + D)$$

9. **Find the Z-transform of n^2.**

$$Z[n^2] = Z[n\,n] = -z\frac{d}{dz}[Z(n)]$$
$$= -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$
$$= -z\left[\frac{(z-1)^2(1) - z[2(z-1)]}{(z-1)^4}\right]$$
$$= -z\left[\frac{z-1-2z}{(z-1)^3}\right] = -z\left[\frac{-1-z}{(z-1)^3}\right]$$
$$= z\frac{(z+1)}{(z-1)^3} = \frac{z^2+z}{(z-1)^3}$$

10. **State the initial and final value theorems of Z-transforms.**

Initial Value Theorem in **Z** transform is $f(0) = \underset{z\to\infty}{Lt} F(z)$ where **Z** [f(n)] = F(z)

The final value theorem in **Z** transform is $\underset{n\to\infty}{Lt} f(n) = \underset{z\to 1}{Lt} (z-1)F(z)$

PART B — ($5 \times 16 = 80$ marks)

11. (a) (i) Expand $f(x) = e^{ax}$ as a Fourier series in $(0, 2\pi)$. (10)

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \ldots (1)$$

Given $f(x) = e^{ax}$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx = \frac{1}{\pi} \int_0^{2\pi} e^{ax}\, dx = \frac{1}{\pi} \left[\frac{e^{ax}}{a} \right]_0^{2\pi}$$

$$= \frac{1}{a\pi} [e^{2a\pi} - 1]$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx = \frac{1}{\pi} \int_0^{2\pi} e^{ax} \cos nx\, dx$$

$$= \frac{1}{\pi} \left[\frac{e^{ax}}{a^2 + n^2} [a \cos nx + n \sin nx] \right]_0^{2\pi}$$

$$= \frac{1}{\pi (a^2 + n^2)} [e^{2a\pi} (a \cos 2n\pi) - a \cos 0]$$

$$= \frac{1}{\pi (a^2 + n^2)} [a e^{2a\pi} - a]$$

$$a_n = \frac{a (e^{2a\pi} - 1)}{\pi (a^2 + n^2)}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx = \frac{1}{\pi} \int_0^{2\pi} e^{ax} \sin nx\, dx$$

$$= \frac{1}{\pi} \left[\frac{e^{ax}}{a^2 + n^2} (a \sin nx - n \cos nx) \right]_0^{2\pi}$$

$$= \frac{1}{\pi (a^2 + n^2)} [e^{2a\pi} (-n \cos 2n\pi) - (-n \cos 0)]$$

$$= \frac{1}{\pi (a^2 + n^2)} [-n e^{2a\pi} + n] = \frac{n (1 - e^{2a\pi})}{\pi (a^2 + n^2)}$$

Substitute a_0, a_n and b_n in (1) we get

$$f(x) = \frac{e^{2a\pi} - 1}{2a\pi} + \sum_{n=1}^{\infty} \left[\frac{a (e^{2a\pi} - 1)}{\pi (a^2 + n^2)} \cos nx + \frac{n (1 - e^{2a\pi})}{\pi (a^2 + n^2)} \sin nx \right]$$

(ii) Find the half-range sine series expansion for $f(x) = k(lx - x^2)$ in $(0, l)$. (6)

Solution: Let $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

$$b_n = \frac{2}{l} \int_0^l K(lx - x^2) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{K2}{l}\left[(lx - x^2)\left(\frac{-l}{n\pi} \cos \frac{n\pi x}{l} \right) - (l - 2x) \times \left(\frac{-l^2}{n^2\pi^2} \sin \frac{n\pi x}{l} \right) + (-2)\left(\frac{+l^3}{n^3\pi^3} \cos \frac{n\pi x}{l} \right) \right]_0^l$$

$$= \frac{K2}{l}\left[-\frac{2l^3}{n^3\pi^3} (\cos n\pi - 1) \right] = \frac{-4l^2K}{n^3\pi^3}[(-1)^n - 1]$$

$$\therefore b_n = \begin{cases} \frac{8l^2K}{n^3\pi^3} & ; \text{ when } n \text{ is odd} \\ 0 & ; \text{ when } n \text{ is even} \end{cases}$$

Or

(b) (i) Find the Fourier series expansion for $f(x) = \begin{cases} l + x, & -l \le x \le 0 \\ l - x, & 0 \le x \le l \end{cases}$ and hence prove that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$. (8)

Solution: $f(x) = \begin{cases} l + x, & -l \le x \le 0 \\ l - x, & 0 \le x \le l \end{cases}$

$$f(-x) = \begin{cases} l - x, & 0 \le x \le l \\ l + x, & -l \le x \le 0 \end{cases} = f(x)$$

$\therefore f(x)$ is an even function of x in $(-l, l)$, $b_n = 0$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$

$$a_0 = \frac{2}{l} \int_0^l (l - x)\, dx = \frac{2}{l}\left[lx - \frac{x^2}{2} \right]_0^l = \frac{2}{l} \cdot \frac{l^2}{2} = l$$

$$a_n = \frac{2}{l} \int_0^l (l - x) \cos \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[(l-x)\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right) - \frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{2}{l}\left[\frac{l^2}{n^2\pi^2}(l - \cos n\pi)\right] = \frac{2l}{n^2\pi^2}[1-(-1)^n]$$

$$\Rightarrow a_n = \begin{cases} \frac{4l}{n^2\pi^2}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$$

$$\therefore\ f(x) = \frac{l}{2} + \frac{4l}{\pi^2}\left[\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \dots\right] \quad \text{...(1)}$$

Deduction:

Put $x = 0$, is a point of continuity in (1)

$$f(0) = \frac{l}{2} + \sum_{n=1}^{\infty} \frac{4l}{\pi^2}\left\{\frac{1}{(2n-1)^2}\right\}$$

$$\Rightarrow \frac{l}{2} = \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{l}{(2n-1)^2} \qquad \left(\because f(x)_{x=0} = l\right)$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{4l}\times\frac{l}{2} = \frac{\pi^2}{8}$$

(ii) Determine the first two harmonic of the Fourier series for the following data : (8)

x:	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
y:	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Solution: The length of the interval is 2π. By omitting the value of $f(x)$ at $x = 2\pi$, we have $n = 6$.

Hence the Fourier series of $f(x)$ is given by

$$y = f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \dots + b_1\sin x + b_2\sin 2x + \dots \quad (1)$$

x	y	$\cos x$	$y\cos 2x$	$y\cos 3x$	$y\sin x$	$y\sin 2x$	$y\sin 3x$
0	1.0	1.0	1.0	1.0	0	0	0
60°	1.4	0.7	– 0.7	– 1.4	1.212	1.212	0
120°	1.9	– 0.95	– 0.95	1.9	1.645	– 1.645	0
180°	1.7	– 1.7	1.7	– 1.7	0	0	0
240°	1.5	– 0.75	– 0.75	1.5	– 1.299	1.299	0
300°	1.2	0.6	– 0.6	– 1.2	1.039	– 1.039	0
Total	8.7	– 1.1	– 0.3	0.1	0.519	– 0.173	0

$$a_0 = \frac{2}{n}\Sigma y = \frac{2}{6}\Sigma y = \frac{1}{3}(8.7) = 2.9$$

$$a_1 = \frac{2}{6}\Sigma y \cos x = \frac{-1.1}{3} = -0.366$$

$$b_1 = \frac{2}{6}\Sigma y \sin x = \frac{0.519}{3} = 0.173$$

$$a_2 = \frac{2}{6}\Sigma y \cos 2x = \frac{1}{3}(-0.3) = -0.10$$

$$b_2 = \frac{2}{6}\Sigma y \sin 2x = \frac{1}{3}(-0.173) = -0.058$$

$$a_3 = \frac{2}{6}\Sigma y \cos 3x = \frac{1}{3}(0.1) = 0.033$$

$$b_3 = \frac{2}{6}\Sigma y \sin 3x = \frac{1}{3}(0) = 0$$

∴ The required Fourier series is

$$y = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + (a_3 \cos 3x + b_3 \sin 3x) + \ldots$$

i.e. $y = 1.45 - 0.37 \cos x + 0.17 \sin x - 0.10 \cos 2x - 0.06 \sin 2x + 0.03 \cos 3x + \ldots$

12. (a) (i) Find the Fourier transform of $f(x)$ given by $f(x) = \begin{cases} 1 - x^2, & \text{for } |x| < 1 \\ 0, & \text{for } |x| > 1 \end{cases}$.

Hence evaluate $\int_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right)\cos\frac{x}{2}\,dx$. (10)

SOLUTION:

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} (1 - x^2)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} (1 - x^2)\, (\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} (1 - x^2)\, \cos sx\, dx + \frac{i}{\sqrt{2\pi}}\int_{-1}^{1} (1 - x^2)\, \sin sx\, dx$$

$$= \frac{1}{\sqrt{2\pi}} 2 \int_0^1 \left(1 - x^2\right) \cos sx \, dx + 0$$

$$\left(\because\ f(x) = \left(1 - x^2\right) \cos x \text{ is even } \& f(x) = \left(1 - x^2\right) \sin x \text{ is odd}\right)$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 \left(1 - x^2\right) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(1 - x^2\right) \left(\frac{\sin sx}{s}\right) - (-2x) \left(\frac{-\cos sx}{s^2}\right) + (-2) \left(\frac{-\sin sx}{s^3}\right) \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(0 - 2\frac{\cos s}{s^2} + \frac{2\sin s}{s^3}\right) - (0 - 0 + 0) \right]$$

$$= 2\sqrt{\frac{2}{\pi}} \left(\frac{-s\cos s + \sin s}{s^3}\right)$$

$$F(s) = 2\sqrt{\frac{2}{\pi}} \left(\frac{\sin s - s\cos s}{s^3}\right).$$

By Fourier inverse transform,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx} \, ds$$

$$\therefore \qquad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}} \left(\frac{\sin s - s\cos s}{s^3}\right) e^{-isx} \, ds$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) (\cos sx - i \sin sx) \, ds$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx \, ds - \frac{2i}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \sin sx \, ds$$

$$f(x) = \frac{4}{\pi} \int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx \, ds - 0$$

$$\therefore \qquad \int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx \, ds = \frac{\pi}{4} f(x)$$

Put $x = \frac{1}{2}$ we get,

$$\int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos\left(\frac{s}{2}\right) ds = \frac{\pi}{4} f\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{4}\left(1 - \left(\frac{1}{2}\right)^2\right)$$

$$= \frac{\pi}{4}\left(1 - \frac{1}{4}\right)$$

$$= \frac{\pi}{4}\left(\frac{3}{4}\right)$$

(i.e.) $$\int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{16}.$$

(ii) Find the Fourier sine transform of $f(x)=\begin{cases} x, & 0<x<1 \\ 2-x, & 1<x<2. \\ 0, & x>2 \end{cases}$ (6)

SOLUTION:

$$F_s(s)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left[\int_0^1 x\sin sx\,dx+\int_1^2(2-x)\sin sx\,dx\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[x\frac{-\cos sx}{s}-(1)\frac{-\sin sx}{s^2}\right]_0^1+\sqrt{\frac{2}{\pi}}\left[(2-x)\frac{-\cos sx}{s}-(-1)\frac{-\sin sx}{s^2}\right]_1^2$$

$$=\sqrt{\frac{2}{\pi}}\left[\left(\frac{-\cos s}{s}+\frac{\sin s}{s^2}\right)-(0+0)\right]+\sqrt{\frac{2}{\pi}}\left[\left(0-\frac{\sin 2s}{s^2}\right)-\left(\frac{-\cos s}{s}-\frac{\sin s}{s^2}\right)\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{-\cos s}{s}+\frac{\sin s}{s^2}-\frac{\sin 2s}{s^2}+\frac{\cos s}{s}+\frac{\sin s}{s^2}\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{2\sin s}{s^2}-\frac{\sin 2s}{s^2}\right]$$

Or

(b) (i) Find the Fourier transform of $xe^{-a|x|}$, $a>0$. (8)

Solution : We know that,

$$F\,[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx$$

$$F\,[e^{-a|x|}]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}\,e^{isx}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}(\cos sx+i\sin sx)\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\,2\int_0^{\infty} e^{-ax}\cos sx\,dx \qquad [\because\ |x|=x \text{ in } (0,\infty)]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2 + a^2}\right] \qquad \boxed{\because \text{ Formula: } \int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2}}$$

Using inversion formula, we get

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[f(x)] \, e^{-isx} \, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \, \frac{a}{s^2 + a^2} \, e^{-isx} \, ds$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \, a \int_{-\infty}^{\infty} \frac{e^{-isx}}{s^2 + a^2} \, ds$$

$$= \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{\cos sx - i \sin sx}{s^2 + a^2} \, ds$$

$$= \frac{a}{\pi} \left[2 \int_0^\infty \frac{\cos sx}{s^2 + a^2} \, ds\right]$$

$$f(x) = \frac{2a}{\pi} \int_0^\infty \frac{\cos sx}{s^2 + a^2} \, ds$$

$$\int_0^\infty \frac{\cos sx}{s^2 + a^2} \, ds = \frac{\pi}{2a} f(x)$$

$$\int_0^\infty \frac{\cos xt}{a^2 + t^2} \, dt = \frac{\pi}{2a} e^{-a|x|} \qquad [\because s \text{ is a dummy variable}]$$

$$F[xe^{-a|x|}] = -i \frac{d}{ds} F(s)$$

$$= -i \frac{d}{ds} F[e^{-a|x|}]$$

$$= -i \frac{d}{ds} \left[\sqrt{\frac{2}{\pi}} \frac{a}{s^2 + a^2}\right]$$

$$= -i \sqrt{\frac{2}{\pi}} \, a \frac{d}{ds} \left[\frac{1}{s^2 + a^2}\right]$$

$$= -i \, a \sqrt{\frac{2}{\pi}} \left[\frac{0 - 2s}{(s^2 + a^2)^2}\right]$$

$$= i \sqrt{\frac{2}{\pi}} \frac{2as}{(s^2 + a^2)^2}$$

(ii) Prove that the Fourier sine transform of a sine transform of a given function is itself. Hence find the Fourier sine transform of $\frac{x}{x^2+a^2}$. (8)

Solution: We know that $F.S.T.$ of e^{-ax} is given by

$$F_s(e^{-ax}) = \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)$$

Using inversion formula for F.S.T, we get

$$e^{-ax} = \sqrt{\frac{2}{\pi}}\int_0^\infty \left[\sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\sin sx\right)\right] ds$$

(ie) $\displaystyle\int_0^\infty \frac{s}{s^2+a^2}\sin sx\, ds = \frac{\pi}{2}e^{-ax}, a>0$

Changing x and s, we get

$$\int_0^\infty \frac{x}{x^2+a^2}\sin sx\, dx = \frac{\pi}{2}e^{-as}$$

Now $F_s\left(\frac{x}{x^2+a^2}\right) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{x}{x^2+a^2}\sin sx\, dx$

$$= \sqrt{\frac{2}{\pi}}\cdot\frac{\pi}{2}e^{-as}$$

$$= \sqrt{\frac{\pi}{2}}\cdot e^{-as}$$

13. (a) A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially in a position given by $y(x,0)=y_0\sin^3\left(\frac{\pi x}{l}\right)$. If it is released from rest from this position, find the displacement y at any distance x from one end at any time t. (16)

Solution : The wave equation is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

The solution is

$$y = (A\cos px + B\sin px)(C\cos pct + D\sin pct) \qquad \dots (1)$$

We have the following boundary conditions

1. $y = 0$ when $x = 0$
2. $y = 0$ when $x = l$
3. $\frac{\partial y}{\partial t} = 0,$ when $t = 0$
4. $y = y_0\sin^3\frac{\pi x}{l}$ when $t = 0$

Applying condition 1. in (1)

$$0 = (A\cos 0 + B\sin 0)(C\cos pct + D\sin pct)$$

$$0 = (A + 0)(C\cos pct + D\sin pct) \qquad [\cos 0 = 1, \sin 0 = 0]$$

$$\Rightarrow A = 0$$

Substitute $A = 0$ in (1)

$$y = B \sin px \,(C \cos pct + D \sin pct) \quad \ldots (2)$$

Applying condition 2. in (2)

$$0 = B \sin pl \,(C \cos pct + D \sin pct)$$

$$\sin pl = 0$$

$$\text{But } \sin n\pi = 0$$

$$\therefore \; pl = n\pi$$

$$p = \frac{n\pi}{l}$$

Substitute $p = \frac{n\pi}{l}$ in (2)

$$y = B \sin \frac{n\pi x}{l} \left(C \cos \frac{n\pi ct}{l} + D \sin \frac{n\pi ct}{l} \right) \quad \ldots (3)$$

To apply condition 3 differentiate (3) partially w.r.t 't'

$$\frac{\partial y}{\partial t} = B \sin \frac{n\pi x}{l} \left(-C \sin \frac{n\pi ct}{l} \cdot \frac{n\pi c}{l} + D \cos \frac{n\pi ct}{l} \frac{n\pi c}{l} \right)$$

Now applying condition (3)

$$0 = B \sin \frac{n\pi x}{l} \left(-C \sin 0 \frac{n\pi c}{l} + D \cos 0 \frac{n\pi c}{l} \right)$$

$$0 = B \sin \frac{n\pi x}{l} \left(0 + D \frac{n\pi c}{l} \right)$$

$$\Rightarrow \quad D = 0$$

Substitute D = 0 in (3)

$$y = B \sin \frac{n\pi x}{l} \left(C \cos \frac{n\pi ct}{l} + 0 \right)$$

$$= B \sin \frac{n\pi x}{l} \left(C \cos \frac{n\pi ct}{l} \right)$$

$$= BC \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}$$

$$= b \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l} \qquad (BC = b)$$

$$y = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l} \qquad \dots (4)$$

Applying condition 4 in (4)

$$y_0 \sin^3\frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cos 0$$

$$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \,.\, 1$$

Here we have to apply $\sin 3\theta$ formula and compare the co-efficients of like terms to get the values of b_1, b_2, b_3, ...

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$\therefore \quad 4\sin^3\theta = 3\sin\theta - \sin 3\theta$$

$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$$

$$\therefore \quad \sin^3\frac{\pi x}{l} = \frac{3}{4}\sin\frac{\pi x}{l} - \frac{1}{4}\sin\frac{3\pi x}{l}$$

Substitute for $\sin^3\frac{\pi x}{l}$ in L.H.S.

$$y_0\left[\frac{3}{4}\sin\frac{\pi x}{l} - \frac{1}{4}\sin\frac{3\pi x}{l}\right] = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\frac{3y_0}{4}\sin\frac{\pi x}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l} = b_1\sin\frac{\pi x}{l} + b_2\sin\frac{2\pi x}{l} + b_3\sin\frac{3\pi x}{l} \dots$$

Comparing the coefficients of $\sin\frac{\pi x}{l}$

$$b_1 = \frac{3y_0}{4}$$

Comparing the coefficients of $\sin \frac{2\pi x}{l}$

$$b_2 = 0$$

Comparing the coefficients of $\sin \frac{3\pi x}{l}$

$$b_3 = \frac{-y_0}{4}$$

$\therefore$ The solution is $y = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot \cos \frac{n\pi ct}{l}$

$$= b_1 \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} + b_2 \sin \frac{2\pi x}{l} \cos \frac{2\pi ct}{l} + b_3 \sin \frac{3\pi x}{l} \cos \frac{3\pi ct}{l} + \ldots$$

$$y = \frac{3y_0}{4} \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l} \cos \frac{3\pi ct}{l}$$

(b) A square plate is bounded by the lines $x = 0$, $y = 0$, $x = l$ and $y = l$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x, l) = lx - x^2$, $0 < x < l$ while the other three edges are kept at 0°C. Find the steady state temperature in the plate. (16)

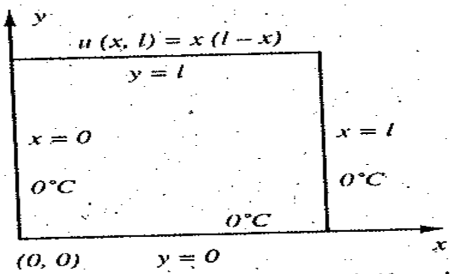
Solution : ***[A.U. Nov./Dec. 2011 Chennai, Nov./Dec. 2011 CBT]***

Let us take the sides of the plate be $l = 20$.

Let $u(x, y)$ be the temperature at any point (x, y).

Then $u(x, y)$ satisfies the Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \ldots (1)$$



From the given problem we have the following boundary conditions.

(i) $u(0, y) = 0$ for $0 < x < l$

(ii) $u(l, y) = 0$ for $0 < y < l$

(iii) $u(x, 0) = 0$ for $0 < x < l$

(iv) $u(x, l) = x(l - x)$ for $0 < x < l$

The correct solution of (1) is

$$u(x, y) = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py}) \quad \ldots (2)$$

After applying conditions (i), (ii) & (iii) in (2), we get the most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} \sin h \frac{n\pi y}{l} \quad \ldots (3)$$

Applying condition (iv) in (3) we get

$$u(x, l) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} \sin h\, n\pi = x(l - x) \quad \ldots (4)$$

To find c_n expand $x(l-x)$ in a half-range Fourier sine series in the interval $0 < x < l$.

$$x(l-x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \ldots (5)$$

From (4) and (5) we get,

$$\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} \sin h\, n\pi = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \ldots (6)$$

Equating like coefficients of (6) we get,

$$c_n \sin h\, n\pi = b_n \text{ (or) } c_n = \frac{b_n}{\sin h\ n\pi} \qquad \ldots (7)$$

$$b_n = \frac{2}{l}\int_0^l x(l-x)\sin\frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}\, dx\right]$$

$$= \frac{2}{l}\left[(lx-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{2}{l}\left[-(lx-x^2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) + (l-2x)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) - 2\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\left\{-(l^2 - l^2)\left(\frac{\cos \cancel{n}\pi}{\frac{\cancel{n}\pi}{l}}\right) + (l - 2l)\left(\frac{\sin \cancel{n}\pi}{\frac{\cancel{n^2}\pi^2}{l^2}}\right) - 2\left(\frac{\cos n\pi}{\frac{n^3\pi^3}{l^3}}\right)\right\} - \left\{-0 + 0 - \frac{2\cos 0}{\frac{n^3\pi^3}{l^3}}\right\}\right]$$

$$= \frac{2}{l}\left[\frac{-2l^3}{n^3\pi^3}\cos n\pi + \frac{2l^3}{n^3\pi^3}\right]$$

$$= \frac{4l^2}{n^3\pi^3}\,[1 - (-1)^n]$$

$$\therefore b_n = \begin{cases} 0, & \text{when '}n\text{' is even} \\ \dfrac{8l^2}{n^3\pi^3}, & \text{when '}n\text{' is odd} \end{cases}$$

$\therefore$ From (7) we get,

$$c_n = \frac{8l^2}{n^3\pi^3 \sin h\, n\pi} \quad \text{when } n \text{ is odd.}$$

Substituting in (3), we get,

$$u(x, y) = \sum_{n = 1,3,5\ldots}^{\infty} \frac{8l^2}{n^3\pi^3 \sin h n\pi} \sin \frac{n\pi x}{l} \sin h \frac{n\pi y}{l}$$

14. (a) (i) Find the differential equation of all spheres whose radii are the same. (6)

☞ **Solution:** The equation any sphere of given radius say (r) is,

$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2, \rightarrow (1)$ where a, b, c are arbitrary constants

Diff.(1) partially w.r.to x, we get

$$2(x-a) + 2(z-c)\cdot p = 0$$

$$\Rightarrow (x-a)+(z-c)\,p=0 \qquad \dots (2)$$

Diff.(1) partially w.r.to 'y', we get

$$2\,(y-b)+2\,(z-c)\,q=0$$

$$\Rightarrow (y-b)+(z-c)\,q=0 \qquad \dots (2)$$

Diff.(2) partially w.r.to 'x', we have

$$1+(z-c)\,\frac{\partial^2 z}{\partial x^2}+p\cdot\frac{\partial z}{\partial x}=0$$

(i.e.,) $$1+(z-c)\,r+p^2=0$$

$$\Rightarrow (z-c)\,r=-(1+p^2) \qquad \dots (4)$$

Also, Diff.(3) partially w.r.to 'y', we have

$$1+(z-c)\,\frac{\partial^2 z}{\partial y^2}+q\cdot\frac{\partial z}{\partial y}=0$$

(i.e.,) $$1+(z-c)\,t+q^2=0$$

$$\Rightarrow (z-c)\,t=-(1+q^2) \qquad \dots (5)$$

$$\frac{(4)}{(5)} \Rightarrow \frac{r}{t}=\frac{1+p^2}{1+q^2}$$

$\Rightarrow r\,(1+q^2)=t\,(1+p^2)$ is the required PDE

(ii) Solve : $\left(D^3-7DD'^2-6D'^3\right)z=e^{3x+y}+\sin(x+2y)$. (10)

Solution: The auxiliary equation is

$$m^3-7m-6=0$$

$$(m+1)\,(m+2)\,(m-3)=0$$

$$m=-1,-2,3$$

$$C.F = f_1(y-x) + f_2(y-2x) + f_3(y+3x)$$

$$P.I_1 = \frac{1}{D^3 - 7DD'^2 - 6D'^3}\sin(x+2y)$$

$$= \frac{1}{-D + 14D' + 24D'}\sin(x+2y)$$

$$= \frac{1}{-D + 38D'}\sin(x+2y)$$

$$= \frac{-D - 38D'}{D^2 - 1444D'^2}\sin(x+2y)$$

$$= \frac{-D - 38D'}{-1 + 5776}\sin(x+2y)$$

$$= \frac{1}{5775}[-\cos(x+2y) - 76\cos(x+2y)]$$

$$= \frac{-1}{5775}[\cos(x+2y) + 76\cos(x+2y)$$

$$= -\frac{77}{5775}\cos(x+2y)$$

$$= -\frac{1}{75}\cos(x+2y)$$

$$P.I_2 = \frac{1}{D^3 - 7DD'^2 - 6D'^3}e^{3x+y}$$

$$= \frac{1}{27 - 21 - 6}e^{3x+y}$$

$$= \frac{1}{0}e^{3x+y}$$

$$\therefore P.I_2 = \frac{x}{3D^2 - 7D'^2}e^{3x+y} = \frac{x}{27-7}e^{3x+y}$$

$$= \frac{x}{20}e^{3x+y}$$

The complete solution is

$$z = C.F + P.I_1 + P.I_2$$

$$= f_1(y-x) + f_2(y-2x) + f_3(y+3x) - \frac{1}{75}\cos(x+2y) + \frac{x}{20}e^{3x+y}$$

Or

(b) (i) Find the singular integral of $z = px + qy + p^2 + q^2$. (8)

Solution: Given $z = px + qy + p^2 + q^2$

This is of the form

$$z = px + qy + f(p, q)$$

(i) Hence the complete integral is

$$z = ax + by + a^2 + b^2$$

(ii) To find the singular integral:

$$z = ax + by + a^2 + b^2$$

Differentiating partially with respect to a, we get

$$0 = x + 2a$$

Hence $a = -\frac{x}{2}$

Differentiating (1) partially with respect to b we get

$$0 = y + 2b$$

Hence $b = -\frac{y}{2}$

Substituting a and b in (1)

$$z = -\frac{x^2}{2} - \frac{y^2}{2} + \frac{x^2}{4} + \frac{y^2}{4}$$

$$z = -\frac{x^2}{4} - \frac{y^2}{4}$$

$x^2 + y^2 + 4z = 0$ is the singular integral.

(iii) To find the **general integral** put $b = \phi(a)$ in (1)

$$z = ax + y\,\phi(a) + a^2 + [\phi(a)]^2 \qquad \text{... (4)}$$

Differentiating (4) with respect to a we get

$$0 = x + y\,\phi'(a) + 2a + 2\,\phi(a)\,\phi'(a) \qquad \text{... (5)}$$

Eliminating a between (4) and (5), we get the general solution

(ii) Solve : $(mz - ny)p + (nx - lz)q = ly - mx$. (8)

Solution: Given $(mz - ny)\,p + (nx - lz)\,q = ly - mx$

This equation is of the form $Pp + Qq = R$

Here $P = mz - ny$

$Q = nx - lz$

$R = ly - mx$

The auxiliary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\text{ie } \frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

using the multipliers x, y, z

$$\text{each ratio } = \frac{xdx + ydy + zdz}{mxz - nxy + nxy - lyz + lyz + mxz}$$

(i.e.) $xdx + ydy + zdz = 0$

Integrating, $\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_1}{2}$

$$\text{(i.e.) } x^2 + y^2 + z^2 = c_1$$

$$\therefore u = x^2 + y^2 + z^2$$

Using the multipliers l, m, n

$$\text{each ratio } = \frac{ldx + mdy + ndz}{lmz - lny + mnx - lmz + lny + mnx}$$

(i.e.) $ldx + mdy + ndz = 0$

Integrating

$$lx + my + nz = c_2$$

(i.e.,) $\quad v = lx + my + nz$

Hence the general solution is, $\phi\,(u, v) = 0$

$$\text{(i.e.,) } \phi\,(x^2 + y^2 + z^2,\; lx + my + nz) = 0$$

15. (a) (i) Find the Z-transform of $\frac{2n+3}{(n+1)(n+2)}$. (6)

Solution : Given : $f(x) = \dfrac{2n+3}{(n+1)(n+2)}$

$$\frac{2n+3}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2} \qquad \dots (1)$$

$$2n+3 = A(n+2) + B(n+1)$$

Put $n = -1$, we get	Put $n = -2$, we get
$-2+3 = A$	$-4+3 = -B$
$A = 1$	$-1 = -B$
	$B = 1$

$$(1) \Rightarrow \frac{2n+3}{(n+1)(n+2)} = \frac{1}{n+1} + \frac{1}{n+2}$$

$$Z[f(n)] = Z\left[\frac{2n+3}{(n+1)(n+2)}\right] = Z\left[\frac{1}{n+1}\right] + Z\left[\frac{1}{n+2}\right] \dots (2)$$

We know that, $Z\left[\dfrac{1}{n+1}\right] = z \log \dfrac{z}{z-1}$ ... (3)

$$Z\left[\frac{1}{n+2}\right] = \sum_{n=0}^{\infty} \frac{1}{n+2} z^{-n}$$

$$= \frac{1}{2} + \frac{1}{3}\left(\frac{1}{z}\right) + \frac{1}{4}\left(\frac{1}{z}\right)^2 + \dots$$

$$= z^2\left[\frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \dots\right]$$

$$= z^2\left[-\log\left(1 - \frac{1}{z}\right) - \frac{1}{z}\right]$$

$$= z^2\left[\log \frac{z}{z-1}\right] - z \qquad \dots (4)$$

$\therefore$ (2) $\Rightarrow Z[f(n)] = z \log \dfrac{z}{z-1} + z^2 \log \dfrac{z}{z-1} - z$ by (3) & (4)

$$= z \log \frac{z}{z-1}\,[1 + z] - z$$

$$= z(z+1) \log \frac{z}{z-1} - z$$

(ii) Solve the difference equation $y(n+2)-3y(n+1)+2y(n)=2^n$, given that $y(0)=3$ and $y(1)=6$, using Z-transform method. (10)

Taking Z transform on both sides of the given difference equation, we get

$$Z[y_{n+2}]-3Z[y_{n+1}]+2Z[y_n]=Z[2^n]$$

$$z^2\left[Y(z)-y_0-\frac{y_1}{z}\right]-3z[Y(z)-y_0]+2Y(z)=\frac{z}{z-2}$$

Using the initial conditions, we get

$$z^2\left[Y(z)-3-\frac{6}{z}\right]-3z[Y(z)-3]+2Y(z)=\frac{z}{z-2}$$

$$Y(z)[z^2-3z+2]-3z^2-6z+9z=\frac{z}{z-2}$$

$$Y(z)[z^2-3z+2]=\frac{z}{z-2}+3z^2-3z$$

$$Y(z)[z^2-3z+2]=\frac{z+3z^3-3z^2-6z^2+6z}{z-2}$$

$$Y(z)=\frac{3z^3-9z^2+7z}{(z-2)^2(z-1)}$$

Taking inverse Z tansform on both sides

$$y_n=Z^{-1}\left[\frac{3z^3-9z^2+7z}{(z-2)^2(z-1)}\right]$$

$$\frac{Y(z)}{z}=\frac{3z^2-9z+7}{(z-2)^2(z-1)}$$

Let $F(z)=\frac{3z^3-9z^2+7z}{(z-2)^2(z-1)}$ and apply partial fraction for $\frac{F(z)}{z}$

$$\frac{F(z)}{z}=\frac{3z^2-9z+7}{(z-2)^2(z-1)}$$

$$\frac{3z^2-9z+7}{(z-2)^2(z-1)}=\frac{A}{z-2}+\frac{B}{(z-2)^2}+\frac{C}{z-1}$$

$$3z^2-9z+7=A(z-2)(z-1)+B(z-1)+C(z-2)^2$$

When $z=1$, $C=1$ and hence $C=1$.

When $z=2$, $B=1$

and when z = 0, 7 = 2A-B+4C

2A = 7-1-4 ➔ A = 2

$$\text{Therefore } \frac{3z^2-18z+23}{(z-2)^2(z-1)} = \frac{2}{z-2} + \frac{1}{(z-2)^2} + \frac{1}{z-1}$$

$$F(z) = \frac{3z^3-18z^2+23z}{(z-2)^2(z-1)} = \frac{z}{z-1} + \frac{2z}{z-2} + \frac{z}{(z-2)^2}$$

$$Z^{-1}\left[\frac{3z^3-18z^2+23z}{(z-2)^2(z-1)}\right] = Z^{-1}\left[\frac{z}{z-1}\right] + 2Z^{-1}\left[\frac{z}{z-2}\right] + Z^{-1}\left[\frac{z}{(z-2)^2}\right]$$

$$= (1^n) - 2(2^n) + n2^n$$

Or

(b) (i) Find $Z^{-1}\left\{\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)}\right\}$ by the partial fraction method. (8)

Solution: Let $\bar{f}(z) = \frac{3z^2-18z+26}{(z-2)(z-3)(z-4)}$

Let $$\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)} = \frac{A}{z-2} + \frac{B}{z-3} + \frac{C}{z-4}$$

$$= \frac{A(z-3)(z-4) + B(z-2)(z-4) + C(z-2)(z-3)}{(z-2)(z-3)(z-4)}$$

$$\therefore 3z^2 - 18z + 26 = A(z-3)(z-4) + B(z-2)(z-4) + C(z-2)(z-3)$$

Put $z = 2$

$2 = 2A \Rightarrow, \quad A = 1$

Put $z = 3$

$-1 = -B \Rightarrow B = 1$

Put $z = 4$

$2 = 2C \qquad \therefore C = 1$

$$\therefore \frac{3z^2 - 18z + 26}{(z-2)(z-3)(z-4)} = \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{z-4}$$

Hence
$$Z^{-1}[\bar{f}(z)] = Z^{-1}\left[\frac{1}{z-2}\right] + Z^{-1}\left[\frac{1}{z-3}\right] + Z^{-1}\left[\frac{1}{z-4}\right]$$
$$= 2^{n-1} + 3^{n-1} + 4^{n-1}$$
$$= \frac{1}{2}2^n + \frac{1}{3}3^n + \frac{1}{4}4^n$$

(ii) Find the Z-transform of $\sin^2\left(\frac{n\pi}{4}\right)$ and $\cos^2 t$. (S)

Solution: $\sin^2\left(\frac{n\pi}{4}\right) = \frac{1}{2}\left(1 - \cos\frac{n\pi}{2}\right)$

$$\therefore Z\left\{\sin^2\left(\frac{n\pi}{4}\right)\right\} = \frac{1}{2}Z(1) - \frac{1}{2}Z\left(\cos\frac{n\pi}{2}\right)$$

$$= \frac{1}{2}\frac{z}{z-1} - \frac{1}{2}\frac{z\left(z - \cos\frac{\pi}{2}\right)}{z^2 - 2z\cos\frac{\pi}{2} + 1}$$

$$= \frac{z}{2(z-1)} - \frac{1}{2}\frac{z^2}{z^2+1}$$

$$= \frac{1}{2}\left[\frac{z}{z-1} - \frac{z^2}{z^2+1}\right]$$

$$= \frac{z(z+1)}{2(z-1)(z^2+1)}$$

Solution: $Z[\cos^2 t] = Z\left[\frac{1+\cos 2t}{2}\right]$ $\left[\because 1+\cos 2t = 2\cos^2 t \quad \therefore \frac{1+\cos 2t}{2} = \cos^2 t\right.$

$$= \frac{1}{2} Z(1+\cos 2t)$$

$$= \frac{1}{2}[Z(1) + Z(\cos 2t)]$$

$$= \frac{1}{2}\left[\left(\frac{z}{z-1}\right) + \frac{z(z-\cos 2T)}{z^2 - 2z\cos 2T + 1}\right]$$

$$= \frac{z}{2}\left[\frac{1}{z-1} + \frac{(z-\cos 2T)}{(z^2 - 2z\cos 2T + 1)}\right]$$