MA 6351 Transforms and Partial Differential Equations May/June 2016

Part A

1. Form the partial differential equation by eliminating the arbitrary function from
$f\left(x^2+y^2, z-xy\right)=0$

Given function is of the form $f(u,v)=0$. Therefore $u=x^2+y$ *and* $v=z-xy$

$$\begin{array}{r|l} u=x^2+y^2 & v=z-xy \\ u_x=2x & v_x=-y \\ u_y=2y & v_y=-x \\ u_z=0 & v_z=1 \end{array}$$

$$P=\begin{vmatrix} u_y & v_y \\ u_z & v_z \end{vmatrix}=\begin{vmatrix} 2y & -x \\ 0 & 1 \end{vmatrix}=2y \quad Q=\begin{vmatrix} u_z & v_z \\ u_x & v_x \end{vmatrix}=\begin{vmatrix} 0 & 1 \\ 2x & -y \end{vmatrix}=-2x$$

$$R=\begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix}=\begin{vmatrix} 2x & -y \\ 2y & -x \end{vmatrix}=-2x^2+2y^2$$

Required p.d.e is $pP+qQ=R$ i.e. $2y\,p-2x\,q=2y^2-2x^2$

2. Find the complete solution of the partial differential equation $p^3-q^3=0$

Given equation is of the form $f(p,q)=0$.

Therefore the solution is $z=ax+by+c$ where $a^3-b^3=0$ and hence $a=b$

Therefore the complete solution is $z=ax+ay+c$

3. Find the value of the Fourier series of $f(x)=\begin{cases} 0 & in\ (-c,0) \\ 1 & in\ (0,c) \end{cases}$ **at the point of discontinuity** $x=0$.

Since $x=0$ is a point of discontinuity, the value of the Fourier series is
$$f(0)=\frac{f(0-)+f(0+)}{2}=\frac{0+1}{2}=\frac{1}{2}$$

4. Find the value of b_n **in the Fourier series expansion of**
$f(x)=\begin{cases} \pi+x & in\ (-\pi,0) \\ \pi-x & in\ (0,\pi) \end{cases}$

Given $f(x)$ is an even function. Therefore $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx=0$

5. Classify the partial differential equation $u_{xx}+u_{yy}=f(x,y)$

Here $A=1$, $B=0$, $C=1$, $\Delta=B^2-4AC=-4<0$. Therefore it is elliptic.

6. Write down all the possible solutions of one dimensional heat equation.

$u(x,t)=\left(Ae^{\lambda x}+Be^{-\lambda x}\right)e^{\alpha^2\lambda^2 t}$ $\quad u(x,t)=\left(A\cos\lambda x+B\sin\lambda x\right)e^{-\alpha^2\lambda^2 t}$ $\quad u(x,t)=(Ax+B)$

7. State Fourier integral theorem.

If $f(x)$ is piecewise continuously differentiable and absolutely integrable in $(-\infty,\infty)$, then

$$f(x)=\frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\,dt\,d\lambda$$

8. Find the Fourier transform of a derivative of the function $f(x)$ if $f(x)\to 0$ as $x\to\pm\infty$

$F[f'(x)]=-is\,F[f(x)]$ if $f(x)\to 0$ as $x\to\pm\infty$

9. Find $Z\left[\frac{1}{n!}\right]$

$$Z\left[\frac{1}{n!}\right]=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{z}\right)^n=1+\frac{1}{1!}\left(\frac{1}{z}\right)+\frac{1}{2!}\left(\frac{1}{z}\right)^2+\frac{1}{3!}\left(\frac{1}{z}\right)^3+\ldots\ldots=e^{\frac{1}{z}}$$

10. Find $Z\left[(\cos\theta+i\sin\theta)^n\right]$

We know that $Z[a^n]=\frac{z}{z-a}$. Therefore

$$Z\left[(\cos\theta+i\sin\theta)^n\right]=Z\left[\left(e^{i\theta}\right)^n\right]=Z\left[e^{in\theta}\right]=\frac{z}{z-e^{i\theta}}$$

Part B

11.a.i. Solve the equation $(x^2-yz)p+(y^2-xz)q=(z^2-xy)$

The auxiliary equation is $\frac{dx}{x^2-yz}=\frac{dy}{y^2-xz}=\frac{dz}{z^2-yx}$

$$\frac{dx-dy}{x^2-yz-y^2+xz}=\frac{dy-dz}{y^2-xz-z^2+yx}=\frac{dz-dx}{z^2-yx-x^2+yz}$$

$$\frac{dx-dy}{(x-y)(x+y)+z(x-y)}=\frac{dy-dz}{(y-z)(y+z)+x(y-z)}=\frac{dz-dx}{(z-x)(z+x)+y(z-x)}$$

$$\frac{dx-dy}{(x-y)(x+y+z)}=\frac{dy-dz}{(y-z)(y+z+x)}=\frac{dz-dx}{(z-x)(z+x+y)}$$

Consider $\frac{dx-dy}{(x-y)(x+y+z)}=\frac{dy-dz}{(y-z)(y+z+x)}$

$$\frac{d(x-y)}{(x-y)}=\frac{d(y-z)}{(y-z)}$$

Integrating, we get $log(x-y)=log(y-z)+K$

$$log\left(\frac{x-y}{y-z}\right)=K$$

$$\frac{x-y}{y-z}=C_1$$

Consider the multipliers x,y,z and 1,1,1. Then

$$\frac{xdx+ydy+zdz}{x^3-xyz+y^3-xyz+z^3-xyz}=\frac{1dx+1dy+1dz}{x^2-yz+y^2-xz+z^2-xy}$$

$$\frac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz}=\frac{1dx+1dy+1dz}{x^2-yz+y^2-xz+z^2-xy}$$

$$\frac{xdx+ydy+zdz}{(x+y+z)(x^2+y^2+z^2-xy-yz-xz)}=\frac{1dx+1dy+1dz}{x^2+y^2+z^2-xy-yz-xz}$$

$$\frac{xdx+ydy+zdz}{(x+y+z)}=\frac{d(x+y+z)}{1}$$

$$xdx+ydy+zdz=(x+y+z)d(x+y+z)$$

Integrating, we get $\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=\frac{(x+y+z)^2}{2}+K$

$$xy+yz+xz=C_2$$

Therefore the solution is of the form $\phi(C_1,C_2)=0$

11.a.ii. Find the singular solution of $z=px+qy+\sqrt{1+p^2+q^2}$

This is of the form clairaut's equation. Therefore the complete solution is

$$z=ax+by+\sqrt{1+a^2+b^2} \qquad ----(1)$$

Differentiate (1) w.r.t 'a' and 'b'

$$0=x+\frac{2a}{2\sqrt{1+a^2+b^2}} \quad \text{and} \quad 0=y+\frac{2b}{2\sqrt{1+a^2+b^2}}$$

$$x=-\frac{a}{\sqrt{1+a^2+b^2}} \quad \text{and} \quad y=-\frac{b}{\sqrt{1+a^2+b^2}} \qquad ----(2)$$

Consider $x^2+y^2=\dfrac{a^2}{1+a^2+b^2}+\dfrac{b^2}{1+a^2+b^2}=\dfrac{a^2+b^2}{1+a^2+b^2}$

$$1-\left(x^2+y^2\right)=1-\frac{a^2+b^2}{1+a^2+b^2}=\frac{1}{1+a^2+b^2}$$

$$1-x^2-y^2=\frac{1}{1+a^2+b^2}$$

$$1+a^2+b^2=\frac{1}{1-x^2-y^2}$$

$$\sqrt{1+a^2+b^2}=\frac{1}{\sqrt{1-x^2-y^2}}$$

From (2), $a=-x\sqrt{1+a^2+b^2}=-\dfrac{x}{\sqrt{1-x^2-y^2}}$

$$b=-y\sqrt{1+a^2+b^2}=-\frac{y}{\sqrt{1-x^2-y^2}}$$

Therefore (1) becomes, $z=-\dfrac{x^2}{\sqrt{1-x^2-y^2}}-\dfrac{y^2}{\sqrt{1-x^2-y^2}}+\dfrac{1}{\sqrt{1-x^2-y^2}}$

$$z=\frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}}$$

$$z=\sqrt{1-x^2-y^2}$$

$$x^2+y^2+z^2=1$$

11.b.i. Solve $\left(D^3-2D^2D'\right)z=2e^{2x}+3x^2y$

The auxiliary equation is $m^3-2m^2=0$

$m^2(m-2)=0$

The roots are m = 0, 0 , 2 and hence the complementary function is

$$f_1(y+0x)+xf_2(y+0x)+f_3(y+2x)$$

P.I$_1$ = $\dfrac{1}{\left(D^3-2D^2D'\right)}2e^{2x}=\dfrac{1}{(8)}2e^{2x}=\dfrac{e^{2x}}{4}$ {putting D=2, D'=0}

$$\text{P.I}_2 = \frac{1}{\left(D^3 - 2D^2D'\right)}3x^3y = \frac{1}{D^3\left(1-2\frac{D'}{D}\right)}3x^3y = \frac{3}{D}\left(1-2\frac{D'}{D}\right)^{-1}3x^3y$$

$$= \frac{3}{D^3}\left(1+2\frac{D'}{D}\right)x^2y = \frac{3}{D^3}\left(x^2y+2\frac{1}{D}x^2\right) = \frac{3}{D^3}\left(x^2y+\frac{2x^3}{3}\right) = 3\left(\frac{x^5y}{60}+\frac{x^6}{180}\right)$$

Therefore the solution is z = CF + PI

11.b.ii. Solve $\left(D^2 + 2DD' + D'^2 - 2D - 2D'\right)z = sin(x+2y)$

Consider the LHS os the given partial differential equation

$$\left(D^2 + 2DD' + D'^2 - 2D - 2D'\right)z = 0$$
$$(D+D')^2 - 2(D+D')z = 0$$
$$(D+D')(D+D'-2)z = 0$$

Compare this with $(D - m_1D' - \alpha_1)(D - m_2D' - \alpha_2) = 0$

Here $m_1 = -1, \quad \alpha_1 = 0, \quad m_2 = -1, \quad \alpha_2 = 2$

Therefore the complementary function is $e^{\alpha_1 x} f_1(y+m_1x) + e^{\alpha_2 x} f_2(y+m_2x)$

i.e. $e^{0x} f_1(y-x) + e^{2x} f_2(y-x)$

$P.I = \dfrac{1}{\left(D^2 + 2DD' + D'^2 - 2D - 2D'\right)} sin(x+2y)$ { put D² = – 1, D'² = – 4, DD' = – 2
}

$$= \frac{1}{(-1-4-4-2D-2D')} sin(x+2y) = -\frac{1}{(9+2D+2D')} sin(x+2y)$$

$$= -\frac{[9-(2D+2D')]}{[9+(2D+2D')][9-(2D+2D')]} sin(x+2y) = -\frac{[9-(2D+2D')]}{[81-(4D^2+4D'^2+8DD')]} sin(x+2y)$$

$$= -\frac{[9-(2D+2D')]}{[81-(-4-16-16)]} sin(x+2y) = -\frac{[9-(2D+2D')]}{117} sin(x+2y)$$

$$= -\frac{1}{117}\{9\,sin(x+2y) - 2\,cos - 4\,cos(x+2y)\}$$

Therefore the solution is z = CF + PI

12.a.i. Find the Fourier series of $f(x) = x \ \ in \ -\pi < x < \pi$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$ { since f(x) is odd function}

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$$ { since f(x) cosnx is odd function}

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$ { since f(x) sinnx is even }

$$= \frac{2}{\pi}\left[(x)\left(-\frac{\cos nx}{n}\right)-(1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi} = \frac{2}{\pi}\left[-\frac{1}{n}x\cos nx+\frac{1}{n^2}\sin nx\right]_0^{\pi} = \frac{2}{\pi}\left[-\frac{\pi}{n}(-1)^n\right] = -\frac{2}{n}(-1)^n$$

The required Fourier series is $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$

12.a.ii. Find the Fourier series expansion of $f(x)=|\cos x|$ $in\ -\pi<x<\pi$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$ { since f(x) is even function}

$$= \frac{2}{\pi}\int_0^{\pi/2}\cos x\,dx\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}-\cos x\,dx\,dx = \frac{2}{\pi}[\sin x]_0^{\pi/2} - \frac{2}{\pi}[\sin x]_{\pi/2}^{\pi} = \frac{2}{\pi}+\frac{2}{\pi} = \frac{4}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$ { since f(x) cosnx is even function}

$$= \frac{2}{\pi}\int_0^{\pi/2}\cos x\cos nx\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}-\cos x\cos nx\,dx$$

$$= \frac{2}{\pi}\frac{1}{2}\int_0^{\pi/2}\cos(n+1)x+\cos(n-1)x\,dx - \frac{2}{\pi}\frac{1}{2}\int_{\pi/2}^{\pi}\cos(n+1)x+\cos(n-1)x\,dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)x}{n+1}+\frac{\sin(n-1)x}{n-1}\right]_0^{\pi/2} - \frac{1}{\pi}\left[\frac{\sin(n+1)x}{n+1}+\frac{\sin(n-1)x}{n-1}\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{1}{n+1}\sin(n+1)\frac{\pi}{2}+\frac{1}{n-1}\sin(n-1)\frac{\pi}{2}\right] - \frac{1}{\pi}\left[-\frac{1}{n+1}\sin(n+1)\frac{\pi}{2}-\frac{1}{n-1}\sin(n-1)\frac{\pi}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{n+1}\sin(n+1)\frac{\pi}{2}+\frac{1}{n-1}\sin(n-1)\frac{\pi}{2}\right]$$

$$= \frac{2}{\pi}\frac{1}{n+1}\left[\cos n\frac{\pi}{2}\right] + \frac{2}{\pi}\frac{1}{n-1}\left[-\cos n\frac{\pi}{2}\right] = \frac{2}{\pi}\cos n\frac{\pi}{2}\left[\frac{1}{n+1}-\frac{1}{n-1}\right] = -\frac{4}{\pi(n+1)(n-1)}\cos n\frac{\pi}{2}$$

When n = 1, we have

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos x\,dx \qquad \{\text{ since f(x) cosnx is even function}\}$$

$$= \frac{2}{\pi}\int_{0}^{\pi/2} \cos x\,\cos x\;dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} -\cos x\cos x\;\;dx = \frac{2}{\pi}\int_{0}^{\pi/2} \cos^2 x\;\;dx - \frac{2}{\pi}\int_{\pi/2}^{\pi} \cos^2 x\;\;dx$$

$$= \frac{2}{\pi}\frac{1}{2}\int_{0}^{\pi/2} 1+\cos 2x\;\;dx - \frac{2}{\pi}\frac{1}{2}\int_{\pi/2}^{\pi} 1+\cos 2x\;\;dx$$

$$= \frac{1}{\pi}\left[x+\frac{\sin 2x}{2}\right]_0^{\pi/2} - \frac{1}{\pi}\left[x+\frac{\sin 2x}{2}\right]_{\pi/2}^{\pi} = \frac{1}{\pi}\left[\frac{\pi}{2}\right] - \frac{1}{\pi}\left[\frac{\pi}{2}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0 \qquad \{\text{ since f(x) sinnx is odd function }\}$$

The required Fourier series is $f(x) = \frac{a_0}{2} + (a_1\cos x + b_1\sin x) + \sum_{n=2}^{\infty}(a_n\cos nx + b_n\sin nx)$

12.b.i. Find the half range sine series of $f(x) = x\cos\pi x \;\; in \;\; (0,1)$

Half range sine series of $f(x)$ in $(0,1)$ is defined $f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$ where

$$b_n = \frac{2}{1}\int_0^1 f(x)\sin n\pi x\;dx = 2\int_0^1 x\;\cos\pi x\sin n\pi x\;dx$$

$$= \int_0^1 x\left[\sin(n+1)\pi x + \sin(n-1)\pi x\right]dx$$

$$= \left[(x)\left(-\frac{\cos(n+1)\pi x}{(n+1)\pi} - \frac{\cos(n-1)\pi x}{(n-1)\pi}\right) - (1)\left(-\frac{\sin(n+1)\pi x}{(n+1)^2\pi^2} - \frac{\sin(n-1)\pi x}{(n-1)^2\pi^2}\right)\right]_0^1$$

$$= \left(-\frac{\cos(n+1)\pi}{(n+1)\pi} - \frac{\cos(n-1)\pi}{(n-1)\pi}\right) = \frac{1}{(n+1)\pi}(-1)^n + \frac{1}{(n-1)\pi}(-1)^n = \frac{(-1)^n}{\pi}\left[\frac{1}{n+1}+\frac{1}{n-1}\right]$$

$$= \frac{(-1)^n}{\pi}\left[\frac{2n}{(n+1)(n-1)}\right]$$

When n = 1, we have

$$b_1 = \frac{2}{1}\int_0^1 f(x)\sin\pi x\;dx = 2\int_0^1 x\;\cos\pi x\sin\pi x\;dx = \int_0^1 x\;\sin 2\pi x\;dx$$

$$= \left[(x)\left(-\frac{\cos 2\pi x}{2\pi} \right) - (1)\left(-\frac{\sin 2\pi x}{4\pi^2} \right) \right]_0^1 = \left[-\frac{1}{2\pi} x \cos 2\pi x + \frac{1}{4\pi^2} \sin 2\pi x \right]_0^1 = -\frac{1}{2\pi}$$

Therefore $f(x) = b_1 \sin \pi x + \sum_{n=2}^{\infty} b_n \sin n\pi x$

12.b.ii. Find the Fourier cosine series up to third harmonic to represent the function given by the following data:

x :	**0**	**1**	**2**	**3**	**4**	**5**
y :	**4**	**8**	**15**	**7**	**6**	**2**

Here $l = 6$. Therefore Fourier cosine series is

$$f(x) = \frac{a_0}{2} + a_1 \cos \frac{\pi}{6} x + a_2 \cos \frac{2\pi}{6} x + a_3 \cos \frac{3\pi}{6} x$$

x	y	$\frac{\pi}{6}x$	$\frac{2\pi}{6}x$	$\frac{3\pi}{6}x$	$y\cos\frac{\pi}{6}x$	$y\cos\frac{2\pi}{6}x$	$y\cos\frac{3\pi}{6}x$
0	4	0	0	0	4	4	4
1	8	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	6.93	4	0
2	15	$\frac{2\pi}{6}$	$\frac{4\pi}{6}$	π	7.5	– 7.5	– 15
3	7	$\frac{3\pi}{6}$	π	$\frac{9\pi}{6}$	0	– 7	0
4	6	$\frac{4\pi}{6}$	$\frac{8\pi}{6}$	2π	– 3	– 3	6
5	2	$\frac{5\pi}{6}$	$\frac{10\pi}{6}$	$\frac{15\pi}{6}$	– 1.73	1	0
	42				13.7	– 8.5	– 5

$$a_0 = 2\left[\frac{\sum y}{n}\right] = 2\left[\frac{42}{6}\right] = 14 \quad a_1 = 2\left[\frac{\sum y\cos\frac{\pi}{6}x}{n}\right] = 2\left[\frac{13.7}{6}\right] = 4.56$$

$$a_2 = 2\left[\frac{\sum y\cos\frac{2\pi}{6}x}{n}\right] = 2\left[-\frac{8.6}{6}\right] = -2.86 \quad a_2 = 2\left[\frac{\sum y\cos\frac{3\pi}{6}x}{n}\right] = 2\left[-\frac{5}{6}\right] = -1.66$$

13.a. Find the displacement of the string stretched between two fixed points at a distance of 2*l*
apart when the string is initially at rest in equilibrium position and points of the string
given initialvelocities v where

$$v = \begin{cases} \dfrac{x}{l} & in\ (0,l) \\ \dfrac{2l-x}{l} & in\ (l,2l) \end{cases}$$ **, x being the distance measured from one end.**

The wave equation is $\dfrac{\partial^2 y}{\partial t^2} = a^2 \dfrac{\partial^2 y}{\partial x^2} \quad ---(1)$

The solution is $y = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \quad ---(2)$

The boundary conditions are

$(i)\quad y = 0 \quad when \quad x = 0 \quad for \quad t \geq 0$
$(ii)\quad y = 0 \quad when \quad x = 2l \quad for \quad t \geq 0$
$(iii)\quad y = 0 \quad when \quad t = 0 \quad for \quad 0 < x < 2l$
$(iv)\quad \dfrac{\partial y}{\partial t} = f(x) \quad when \quad t = 0 \quad for \quad 0 < x < 2l$

Substitute (i) in (2),

$0 = (A)\ (C\cos\lambda at + D\sin\lambda at)$

We get A = 0

Substitute (ii) in (2),

$0 = (B\sin\lambda 2l)(C\cos\lambda at + D\sin\lambda at)$

$\sin\lambda 2l = 0$

But $\sin n\pi = 0$

Therefore $2\lambda l = n\pi$

and hence $\lambda = \dfrac{n\pi}{2l}$

Substitute (iii) in (2),

$$0=\left(A\cos\lambda x+B\sin\lambda x\right)\left(C\right)$$

We get C = 0

Substitute the above results in (2), we get

$$y=\left(B\sin\frac{n\pi}{2l}x\right)\left(D\sin\frac{n\pi a}{2l}t\right)$$

Therefore the most general solution is $y=\sum_{n=0}^{\infty}b_n\ \sin\frac{n\pi}{2l}x\ \sin\frac{n\pi a}{2l}t\ ---(3)$

Differentiate (3) w.r.t 't', we get $\frac{\partial y}{\partial t}=\sum_{n=0}^{\infty}b_n\ \frac{n\pi a}{2l}\ \sin\frac{n\pi}{2l}x\ \cos\frac{n\pi a}{2l}t\ ---(4)$

Using (iv) in (4), we get

$$f(x)=\sum_{n=0}^{\infty}b_n\ \frac{n\pi a}{2l}\ \sin\frac{n\pi}{2l}x$$

This Fourier sine series in $(0,2l)$

Therefore $b_n\ \frac{n\pi a}{2l}=\frac{2}{2l}\int_0^{2l}f(x)\ \sin\frac{n\pi}{2l}x\,dx$

$$b_n\ \frac{n\pi a}{2l}=\frac{2}{2l}\int_0^{2l}\frac{x}{l}\ \sin\frac{n\pi}{2l}x\,dx+\frac{2}{2l}\int_l^{2l}\frac{2l-x}{l}\ \sin\frac{n\pi}{2l}x\,dx$$

$$b_n\ \frac{n\pi a}{2l}=\frac{1}{l^2}\int_0^{2l}x\ \sin\frac{n\pi}{2l}x\,dx+\frac{1}{l^2}\int_l^{2l}(l-x)\ \sin\frac{n\pi}{2l}x\,dx$$

$$b_n\ \frac{n\pi a}{2l}=\frac{1}{l^2}\left[(x)\left(-\frac{\cos\frac{n\pi}{2l}x}{\frac{n\pi}{2l}}\right)-(1)\left(-\frac{\sin\frac{n\pi}{2l}x}{\frac{n^2\pi^2}{4l^2}}\right)\right]_0^l+\frac{1}{l^2}\left[(l-x)\left(-\frac{\cos\frac{n\pi}{2l}x}{\frac{n\pi}{2l}}\right)-(-1)\left(-\frac{\sin\frac{n\pi}{2l}x}{\frac{n^2\pi^2}{4l^2}}\right)\right]_l^{2l}$$

$$b_n\ \frac{n\pi a}{2l}=\frac{1}{l^2}\left[-\frac{2l}{n\pi}x\cos\frac{n\pi}{2l}x+\frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2l}x\right]_0^l+\frac{1}{l^2}\left[-\frac{2l}{n\pi}(l-x)\cos\frac{n\pi}{2l}x-\frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2l}x\right]_l^{2l}$$

$$b_n \frac{n\pi a}{2l} = \frac{1}{l^2}\left[-\frac{2l^2}{n\pi}\cos\frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] + \frac{1}{l^2}\left[\frac{2l^2}{n\pi}(-1)^n + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$b_n \frac{n\pi a}{2l} = \frac{1}{l^2}\left[\frac{8l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$b_n = \frac{2l}{n\pi a}\frac{1}{l^2}\left[\frac{8l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] = \frac{16l}{n^3\pi^3 a}\sin\frac{n\pi}{2}$$

Therefore the most general solution becomes

$$y = \sum_{n=0}^{\infty} \frac{16l}{n^3\pi^3 a} \sin\frac{n\pi}{2} \sin\frac{n\pi}{2l}x \sin\frac{n\pi a}{2l}t$$

13.b. A long rectangular plate with insulated surface is *l* cm wide. If the temperature along
one shortedge is u(x,0)=k(*l*x – x²) for 0<x<*l*, while the other two long edges x=0 and x=*l*
as well as theother short edge are kept at 0C, find the steady state temperature function
u(x,y).

The two dimensional heat equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ---(1)$

The solution is $u = (A\cos\lambda x + B\sin\lambda x)(Ce^{\lambda y} + De^{-\lambda y}) \quad ---(2)$

The boundary conditions are

$(i)\ u = 0 \text{ when } x = 0 \text{ for } 0 < y < \infty$
$(ii)\ u = 0 \text{ when } x = l \text{ for } 0 < y < \infty$
$(iii)\ u = 0 \text{ when } y = \infty \text{ for } 0 < x < l$
$(iv)\ u = k(lx - x^2) \text{ when } y = 0 \text{ for } 0 < x < l$

Applying (i) in (2), we get $0 = (A)(Ce^{\lambda y} + De^{-\lambda y})$

i.e. $A = 0$

Applying (ii) in (2), we get $0 = (B\sin\lambda l)(Ce^{\lambda y} + De^{-\lambda y})$

i.e. $\sin\lambda l = 0$

Therefore $\lambda l = n\pi$

$$\lambda = \frac{n\pi}{l}$$

Applying (iii) in (2), we get $C = 0$

Using these results in (2), we get $u = \left(B\sin\frac{n\pi}{l}x\right)\left(De^{-\frac{n\pi}{l}y}\right)$

Therefore the most general solution is $u = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x\, e^{-\frac{n\pi}{l}y}$

Applying (iv) in most general solution, we get $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x$

This is fourier sine series in the interval (o,l)

Therefore $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi}{l}x\,dx$

$$b_n = \frac{2k}{l}\int_0^l (lx - x^2)\sin\frac{n\pi}{l}x\,dx$$

$$b_n = \frac{2k}{l}\left[(lx - x^2)\left(-\frac{\cos\frac{n\pi}{l}x}{\frac{n\pi}{l}}\right) - (l - 2x)\left(-\frac{\sin\frac{n\pi}{l}x}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos n\frac{\pi}{l}x}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$b_n = \frac{2k}{l}\left[-\frac{2l^3}{n^3\pi^3}(-1)^n + \frac{2l^3}{n^3\pi^3}\right]$$

$$b_n = \frac{4kl^2}{n^3\pi^3}\left[1 - (-1)^n\right]$$

Using this in the most general solution, we get

$$u = \sum_{n=1}^{\infty} \frac{4kl^2}{n^3\pi^3}\left[1 - (-1)^n\right]\sin\frac{n\pi}{l}x\, e^{-\frac{n\pi}{l}y}$$

14.a. Find the Fourier sine and cosine transform of $f(x) = e^{-ax}$ for x ≥0, a>0. Hence deduce the

Integrals $\int_0^\infty \frac{\cos sx}{s^2+a^2}ds$ and $\int_0^\infty \frac{s\sin sx}{s^2+a^2}ds$

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx \quad F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}(-a\sin sx - s\cos sx)\right]_0^\infty = \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}(-a\cos sx + s\sin sx)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right] = F_s(s) = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right] = F_c(s)$$

By inversion formula, By inversion formula,

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin sx\,ds \quad f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(s)\cos sx\,ds$$

$$e^{-ax} = \frac{2}{\pi}\int_0^\infty \frac{s}{a^2+s^2}\sin sx\,ds \quad e^{-ax} = \frac{2}{\pi}\int_0^\infty \frac{a}{a^2+s^2}\cos sx\,ds$$

$$\frac{\pi}{2}e^{-ax} = \int_0^\infty \frac{s}{a^2+s^2}\sin sx\,ds \quad \frac{\pi}{2a}e^{-ax} = \int_0^\infty \frac{1}{a^2+s^2}\cos sx\,ds$$

14.b.i. Find the Fourier transform of $f(x) = e^{-\frac{x^2}{2}}$ in $(-\infty,\infty)$.

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x)e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-a^2x^2}e^{isx}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-(a^2x^2-isx)}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-\left(ax-\frac{is}{2a}\right)^2+\left(\frac{is}{2a}\right)^2}dx = \frac{1}{\sqrt{2\pi}}e^{\left(\frac{is}{2a}\right)^2}\int_{-\infty}^\infty e^{-\left(ax-\frac{is}{2a}\right)^2}dx$$

Put $\left(ax-\frac{is}{2a}\right)=t, \quad then \quad a\,dx=dt$

$$F\left[e^{-a^2x^2}\right]=\frac{1}{a\sqrt{2\pi}}\,e^{-\left(\frac{is}{2a}\right)^2}\int_{-\infty}^{\infty} e^{-t^2}\,dt$$

$$F\left[e^{-a^2x^2}\right]=\frac{1}{a\sqrt{2\pi}}\,e^{-\left(\frac{s^2}{4a^2}\right)}\sqrt{\pi}$$

$$when \quad a=1, \quad F\left[e^{-x^2}\right]=\frac{1}{\sqrt{2}}\,e^{-\left(\frac{s^2}{4}\right)}$$

14.b.ii. Find the Fourier transform of $f(x)=1-|x| \;\; if \;\; |x|<1$ **and hence find the value of**

$$\int_0^{\infty}\frac{\sin^4 t}{t^4}dt$$

$$F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)(\cos sx+i\sin sx)\,dx=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)\cos sx\;dx\,dx+\frac{i}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)\sin sx\;dx\,dx$$

$$=\frac{2}{\sqrt{2\pi}}\int_0^1(1-x)\cos sx\,dx=\frac{2}{\sqrt{2\pi}}\left[(1-x)\left(\frac{\sin sx}{s}\right)-(-1)\left(-\frac{\cos sx}{s^2}\right)\right]_0^1$$

$$=\frac{2}{\sqrt{2\pi}}\left[\frac{1}{s}(1-x)\sin sx-\frac{1}{s^2}\cos sx\right]_0^1=\frac{2}{\sqrt{2\pi}}\left[-\frac{1}{s^2}\cos s+\frac{1}{s^2}\right]$$

$$=\frac{2}{\sqrt{2\pi}}\left[\frac{1-\cos s}{s^2}\right]=\frac{4}{\sqrt{2\pi}}\left[\frac{\sin^2\left(\frac{s}{2}\right)}{s^2}\right]=F(s)$$

By Parseval's identity,

$$\int_{-\infty}^{\infty}|F(s)|^2\,ds=\int_{-\infty}^{\infty}|f(x)|^2\,dx$$

$$\frac{16}{2\pi}\int_{-\infty}^{\infty}\frac{\sin^4\left(\frac{s}{2}\right)}{s^4}\,ds=\int_{-1}^{1}\left(1-|x|\right)^2\,dx$$

$$\frac{32}{2\pi}\int_{0}^{\infty}\frac{\sin^4\left(\frac{s}{2}\right)}{s^4}\,ds=2\int_{0}^{1}\left(1-x\right)^2\,dx$$

$$\frac{32}{2\pi}\int_{0}^{\infty}\frac{\sin^4\left(\frac{s}{2}\right)}{s^4}\,ds=2\left[\frac{(1-x)^3}{-3}\right]_0^1$$

$$\frac{32}{2\pi}\int_{0}^{\infty}\frac{\sin^4\left(\frac{s}{2}\right)}{s^4}\,ds=\frac{2}{3}$$

Put $\frac{s}{2}=t,\quad then\quad s=2t,\quad and\quad ds=2dt$

$$\frac{32(2)}{2\pi(16)}\int_{0}^{\infty}\frac{\sin^4 t}{t^4}\,ds=\frac{2}{3}$$

$$\int_{0}^{\infty}\frac{\sin^4 t}{t^4}\,dt=\frac{\pi}{3}$$

15.a.i. Find the Z transform of $\cos\frac{n\pi}{2}$ and $\frac{1}{n(n+1)}$

(i)

$$Z\left[e^{in\theta}\right]=Z\left[\left(e^{i\theta}\right)^n\right]=\frac{z}{z-e^{i\theta}}=\frac{z}{z-\cos\theta-i\sin\theta}=\frac{z}{(z-\cos\theta)-i\sin\theta}\times\frac{[(z-\cos\theta)+i\sin\theta]}{(z-\cos\theta)+i\sin\theta}$$

$$Z\left[e^{in\theta}\right]=\frac{z[(z-\cos\theta)+i\sin\theta]}{(z-\cos\theta)^2+\sin^2\theta}$$

$$Z[\cos n\theta+i\sin n\theta]=\frac{z[(z-\cos\theta)+i\sin\theta]}{(z-\cos\theta)^2+\sin^2\theta}$$

Equating real parts on both sides, we get

$$Z[\cos n\theta] = \frac{z^2 - z\cos\theta}{z^2 + \cos^2\theta - 2z\cos\theta + \sin^2\theta}$$

$$Z[\cos n\theta] = \frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1}$$

Therefore $Z\left[\cos n\frac{\pi}{2}\right] = \frac{z^2}{z^2+1}$

(ii) Apply partial fraction for $\frac{1}{n(n+1)}$

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$1 = A(n+1) + Bn$$

When n = 0, A = 1 and when n = – 1, B = – 1

Therefore $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$$Z\left[\frac{1}{n}\right] = \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{z}\right)^n = \frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \dots\dots\dots = -\log\left(1-\frac{1}{z}\right) = \log\left(\frac{z}{z-1}\right)$$

$$Z\left[\frac{1}{n+1}\right] = \sum_{n=0}^{\infty} \frac{1}{n+1}\left(\frac{1}{z}\right)^n = 1 + \frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{3}\left(\frac{1}{z}\right)^2 + \dots\dots\dots = z\left[\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \dots\dots\dots\right]$$

$$Z\left[\frac{1}{n+1}\right] = -z\log\left(1-\frac{1}{z}\right) = z\log\left(\frac{z}{z-1}\right)$$

15.a.ii. Using convolution theorem, evaluate $Z^{-1}\left[\frac{z^2}{(z-a)^2}\right]$

$$Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = Z^{-1}\left[\frac{z}{(z-a)}\right] * Z^{-1}\left[\frac{z}{(z-a)}\right]$$

$$= a^n * a^n$$

$$= \sum_{m=0}^{n} a^m a^{n-m}$$

$$= a^n \sum_{m=0}^{n} 1$$
$$= (n+1)a^n$$

15.b.i. Find the inverse Z transform of $\frac{z}{z^2-2z+2}$ by residue method.

Let $F(z) = \frac{z}{z^2-2z+2}$

The singularities of $z^{n-1}F(z) = \frac{z^n}{z^2-2z+2}$ are given by $z^2-2z+2=0$

$z = 1+i$ *and* $z = 1-i$ are the simple poles.

$$Res\left[z^{n-1}F(z); z = 1+i\right] = \underset{z \to 1+i}{Lt} \left[z-(1+i)\right] \frac{z^n}{\left[z-(1+i)\right]\left[z-(1-i)\right]} = \frac{(1+i)^n}{2i}$$

$$Res\left[z^{n-1}F(z); z = 1-i\right] = \underset{z \to 1-i}{Lt} \left[z-(1-i)\right] \frac{z^n}{\left[z-(1+i)\right]\left[z-(1-i)\right]} = \frac{(1-i)^n}{-2i}$$

$$Z^{-1}\left[\frac{z}{z^2-2z+2}\right] = \text{sum of residues} \quad = \frac{1}{2i}\left[(1+i)^n + (1-i)^n\right]$$

15.b.ii. Solve the difference equation $y_{n+2} + y_n = 2$ given that $y_0 = 0$ & $y_1 = 0$ by using Z transform.

Taking Z transform on both sides of the given difference equation, we get

$$Z[y_{n+2}] + Z[y_n] = Z[2]$$

$$z^2\left[Y(z) - y_0 - \frac{y_1}{z}\right] + Y(z) = \frac{2z}{z-1}$$

Using the initial conditions, we get

$$z^2 Y(z) + Y(z) = \frac{2z}{z-1}$$

$$(z^2+1)Y(z) = \frac{2z}{z-1}$$

$$Y(z) = \frac{2z}{(z^2+1)(z-1)}$$

Taking inverse Z tansform on both sides

$$y_n = Z^{-1}\left[\frac{2z}{(z^2+1)(z-1)}\right]$$

Let $F(z) = \frac{2z}{(z^2+1)(z-1)}$ and apply partial fraction for $\frac{F(z)}{z}$

$$\frac{2}{(z^2+1)(z-1)} = \frac{Az+B}{z^2+1} + \frac{C}{z-1}$$

$$2 = (Az+B)(z-1) + C(z^2+1)$$

When z = 1, 2C = 2 and hence C = 1.

When z = 0, 2 = – B + C and hence 2 = – B + 1. Therefore B = – 1

Comparing coefficient of z^2, we get 0 = A + C and hence A = – 1

Therefore $\frac{2}{(z^2+1)(z-1)} = \frac{-z-1}{z^2+1} + \frac{1}{z-1}$

$$\frac{2z}{(z^2+1)(z-1)} = \frac{-z^2-z}{z^2+1} + \frac{z}{z-1}$$

$$Z^{-1}\left[\frac{2z}{(z^2+1)(z-1)}\right] = Z^{-1}\left[\frac{-z^2-z}{z^2+1} + \frac{z}{z-1}\right] = -Z^{-1}\left[\frac{z^2}{z^2+1}\right] - Z^{-1}\left[\frac{z}{z^2+1}\right] + Z^{-1}\left[\frac{z}{z-1}\right]$$

$$= -1^n \cos\frac{n\pi}{2} - 1^n \sin\frac{n\pi}{2} + 1^n$$

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