B.E./B.Tech. DEGREE EXAMINATION, APRIL/MAY 2017.

Third Semester

Mechanical Engineering

MA 6351 — TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS

Answer ALL questions.

PART A — (10 × 2 = 20 marks)

1. Form the partial differential equation by eliminating arbitrary function 'f' from $z = e^{ay} f(x+by)$.

$z = e^{ay} f(x + by)$-----------(1)

$\frac{\partial z}{\partial x} = p = e^{ay} f'(x + by)$------(2)

$\frac{\partial z}{\partial y} = q = e^{ay} f'^{(x+by)} b + a$---------(3)

From (1),(2),(3)

Q$q = pq + az$

2. Solve $(D^3 - D^2D' - 8DD'^2 + 12D'^3)z = 0$.

Soln:put D=m, D'=1

$m^3 - m^2 - 8m + 12 = 0$

m=2,2,-3

CF=$\emptyset_1(y + 2x) + x\emptyset_2(y + 2x) + \emptyset_3(y - 3x)$

The general solution of z=$\emptyset_1(y + 2x) + x\emptyset_2(y + 2x) + \emptyset_3(y - 3x)$

3. State the sufficient condition for a function $f(x)$ to be expressed as a Fourier series.

If a function $f(x)$ is defined in the interval $c \le x \le c+2l$, it can be expressed as the infinite trigonometric series, provided

(i) $f(x)$ is defined and single valued except possibly at a finite number of points in $(c, c+2l)$

(ii) $f(x)$ is continuous or piecewise with finite number of finite discontinuities in $(c, c+2l)$

(iii) $f(x)$ has no or finite number of maxima or minima in $(c, c+2l)$

4. If the Fourier series of the function $f(x) = x + x^2$, in the interval $(-\pi, \pi)$ is $\frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{4}{n^2}\cos nx - \frac{2}{n}\sin nx\right]$, then find the value of the infinite series $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$

Soln: put $x = \pi$ *in given equation*

LHS=$\frac{f(-\pi)+f(\pi)}{2}$ $=\frac{-\pi+\pi^2+\pi+\pi^2}{2}$ $= \pi^2$

RHS=$\frac{\pi^2}{3}+\sum_{n=1}^{\infty}(-1)^n\left[\frac{4}{n^2}\cos n\pi - \frac{2}{n}\sin n\pi\right]$

LHS=RHS

$$\pi^2 = \frac{\pi^2}{3}+\sum_{n=1}^{\infty}(-1)^n\left[\frac{4}{n^2}\cos n\pi - \frac{2}{n}\sin n\pi\right]$$

$$\pi^2 - \frac{\pi^2}{3}=\sum_{n=1}^{\infty}(-1)^n\left[\frac{4}{n^2}\cos n\pi - \frac{2}{n}\sin n\pi\right]$$

$$2\frac{\pi^2}{3} = \sum_{n=1}^{\infty}\frac{4}{n^2}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = 2\frac{\pi^2}{3}\left(\frac{1}{4}\right) = \frac{\pi^2}{6}$$

$$\frac{1}{1^2} + \frac{2}{2^2} + \cdots = \frac{\pi^2}{6}$$

5. Write all possible solutions of one dimensional heat equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$.

1. $u(x, t) = (A_1 e^{px} + A_2 e^{-px}) A_3 e^{c^2 p^2 t}$
2. $u(x, t) = (A_4 \cos px + A_5 \sin px) A_6 e^{-c^2 p^2 t}$
3. $u(x, t) = (A_7 x + A_8) A_9$

6. Using the method of separation of variables, solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ where $u(x, 0) = 6e^{-3x}$.

Solution : Given $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ ... (1)

Here, u is a function of x and t.

Let $u = XT$ ... (2)

where X is a function of x alone and T that of t only.

diff. (2) partially, we get

$$\frac{\partial u}{\partial x} = X' T \quad \text{... (3)}$$

$$\frac{\partial u}{\partial t} = XT' \quad \text{... (4)}$$

(1) $\Rightarrow$ $X' T = 2XT' + XT = X(2T' + T)$

$$\frac{X'}{X} = \frac{2T' + T}{T}$$

$$\frac{X'}{X} = \frac{2T' + T}{T} = k \text{ (say)} \quad \text{... (5)}$$

$$\frac{X'}{X} = k$$

$$\frac{1}{X}\frac{dX}{dx} = k$$

$$\frac{1}{X}\, d X = k dx$$

$$\int \frac{1}{X}\, d X = \int k dx$$

$$\log X = kx + c_1$$

$$X = e^{kx + c_1}$$

$$= e^{kx} e^{c_1}$$

$$X = A\, e^{kx}$$

$$\frac{2T'}{T} + 1 = k$$

$$\frac{2T'}{T} = k - 1$$

$$\frac{1}{T}\frac{dT}{dt} = \frac{1}{2}(k-1)$$

$$\frac{1}{T}\, dT = \frac{1}{2}(k-1)\, dt$$

$$\int \frac{1}{T}\, dT = \int \frac{1}{2}(k-1)\, dt$$

$$\log T = \frac{1}{2}(k-1)\, t + c_2$$

$$T = e^{\frac{1}{2}(k-1)t + c_2}$$

$$= e^{\frac{1}{2}(k-1)t}\, e^{c_2}$$

$$= B\, e^{\frac{1}{2}(k-1)t}$$

$$u = XT$$

$$= A\, e^{kx}\, B\, e^{\frac{1}{2}(k-1)t}$$

$$= AB\, e^{kx + \frac{1}{2}(k-1)t}$$

$$u = c\, e^{kx + \frac{1}{2}(k-1)t},$$

where C and k are arbitrary constants.

Given : $u(x, 0) = 6e^{-3x}$

$$6e^{-3x} = ce^{kx + 0}$$

$$6e^{-3x} = ce^{kx}$$

i.e., $c = 6,\ k = -3$

$$u = 6e^{-3x + \frac{1}{2}(-4)t}$$

$$u = 6e^{-3x - 2t}$$ which is the required solution.

7. If $F(s)$ is the Fourier transform of $f(x)$, prove that $F\{f(x-a)\} = e^{ias}F(s)$.

Given that, $F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)\, e^{isx}\, dx$$

Put $x - a = t,\ \ x = a + t$

$$dx = dt \text{ as } x \to \infty, t \to -\infty$$
$$x \to \infty, t \to \infty$$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is(a+t)}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{isa}\, e^{ist}\, dt$$

$$= e^{isa} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt$$

$$= e^{isa} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

(By changing the dummy variable t by x)

$\therefore F[f(x-a)] = e^{isa} F(f(x))$

or $F[f(x-a)] = e^{isa} F(s)$

8. Find Fourier Sine transform of $\frac{1}{x}$.

Solution: By definition

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx$$

$$\therefore F_s\left(\frac{1}{x}\right) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\sin sx}{x} dx$$

(Put $sx = \theta$, $sdx = d\theta$)

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\sin\theta}{\theta} d\theta$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left(\frac{\pi}{2}\right) \qquad \left(\because \int_0^{\infty} \frac{\sin\theta}{\theta} d\theta = \pi/2\right)$$

$$= \sqrt{\pi/2}$$

9. Find the Z-transform of a^n.

$$Z\{a^n\} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$= \left(1 - \frac{a}{z}\right)^{-1} = \frac{1}{1 - \frac{a}{z}}$$

$$Z[a^n] = \frac{z}{z-a}$$

(i) Put $a = 1$, we get $Z(1) = \dfrac{1}{z-1}$

(ii) put $a = -1$, $Z\{(-1)^n\} = \dfrac{z}{z+1}$

(iii) Also $Z(a^{n-1}) = z^{-1} Z(a^n)$ by property 3

$$= z^{-1} \cdot \left[\frac{z}{z-a}\right] = \frac{1}{z-a}$$

10. State initial and final value theorems on Z-transforms.

Let $Z\{f(n)\} = \bar{f}(z)$. Then $f(0) = \lim_{z \to \infty} \bar{f}(z)$

If $Z\{f(n)\} = \bar{f}(z)$ then $\lim_{n \to \infty} f(n) = \lim_{z \to 1} (z-1)\bar{f}(z)$

PART B — (5 × 16 = 80 marks)

11. (a) (i) Find the general solution of $(z^2 - 2yz - y^2)\,p + (xy + zx)\,q = xy - zx$. (8)

Solution

Solution: Given $(z^2 - 2yz - y^2)\,p + (xy + zx)\,q = xy - zx$

This equation is of the form $Pp + Qq = R$

Here $P = z^2 - 2yz - y^2$

$Q = xy + zx$

$R = xy - zx$

The auxiliary equations are $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

$$\text{ie } \frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{x\,(y+z)} = \frac{dz}{x\,(y-z)}$$

Take
$$\frac{dy}{x\,(y+z)} = \frac{dz}{x\,(y-z)}$$

$$\frac{dy}{y+z} = \frac{dz}{y-z}$$

$$(y-z)\,dy = (y+z)\,dz$$

$$ydy - zdy = ydz + zdz$$

$$ydy - ydz - zdy - zdz = 0$$

$$ydy - d\,(yz) - zdz = 0$$

Integrating,
$$\frac{y^2}{2} - yz - \frac{z^2}{2} = \frac{c_1}{2}$$

$$y^2 - 2yz - z^2 = c_1$$

$$\therefore\ u = y^2 - 2yz - z^2$$

Using the multipliers x, y, z

$$\text{each ratio } = \frac{xdx + ydy + zdz}{xz^2 - 2xyz - xy^2 + xy^2 + xyz + xyz - xz^2}$$

(i.e.) $xdx + ydy + zdz = 0$

Integrating, $\dfrac{x^2}{2} + \dfrac{y^2}{2} + \dfrac{z^2}{2} = \dfrac{c_2}{2}$

$$x^2 + y^2 + z^2 = c_2$$

$$\therefore\ v = x^2 + y^2 + z^2$$

Hence the general solution is

$$\phi\,(y^2 - 2yz - z^2,\ x^2 + y^2 + z^2) = 0$$

(ii) Find the general solution of $(D^2+2DD'+D'^2)z=x^2y+e^{x-y}$. (8)

Solution : $[D^2+2DD'+D'^2]z = x^2y$ [A.U. N/D 2006]

The auxiliary equation is
$$m^2+2m+1 = 0$$
$$(m+1)^2 = 0$$
$$m = -1, -1$$

$$\text{C.F.} = \phi_1(y-x)+x\phi_2(y-x)$$

$$\text{P.I}_1 = \frac{1}{(D+D')^2}x^2y$$

$$= \frac{1}{D^2\left(1+\frac{D'}{D}\right)^2}x^2y = \frac{1}{D^2}\left[1+\frac{D'}{D}\right]^{-2}x^2y$$

$$= \frac{1}{D^2}\left[1-\frac{2D'}{D}+\frac{3D'^2}{D^2}\right](x^2y) = \frac{1}{D^2}\left[x^2y-\frac{2}{D}(x^2)\right]$$

$$= \frac{1}{D^2}\left[x^2y-2\frac{x^3}{3}\right] = \frac{1}{D}\left[\frac{x^3y}{3}-\frac{2x^4}{12}\right] = \frac{x^4y}{12}-\frac{x^5}{30}$$

$$z = \text{C.F.} + \text{P.I}_1$$

$$= \phi_1(y-x)+x\phi_2(y-x)+\frac{x^4y}{12}-\frac{x^5}{30}$$

$$\text{P.I}_2 = \frac{1}{D^2+2DD'+D'^2}e^{x-y}$$

$$= \frac{1}{1-2+1}e^{x-y}$$ Replace D by 1 and D′ by −1

$$= \frac{1}{0}e^{x-y}$$ [ordinary rule fails]

$$= x\frac{1}{2D+2D'}e^{x-y} = \frac{x}{2}\frac{1}{D+D'}e^{x-y}$$

$$= \frac{x}{2}\frac{1}{1-1}e^{x-y}$$ Replace D by 1 an D′ by −1

$$= \frac{x}{2}\frac{1}{0}e^{x-y}$$ [ordinary rule fails]

$$= \left(\frac{x}{2}\right)x\frac{1}{1}e^{x-y} = \frac{x^2}{2}e^{x-y}$$

$$\therefore z = \text{C.F.} + \text{P.I}_1 + \text{P.I}_2$$

$$= \phi_1(y-x)+x\phi_2(y-x)+\frac{x^4y}{12}-\frac{x^5}{30}+\frac{x^2}{2}e^{x-y}$$

(b) (i) Find the general solution of $z = px + qy + p^2 + pq + q^2$. (8)

Solution

Solution: Given $z = px + qy + p^2 + pq + q^2$

It is Clairaut's form, put $p = a, q = b$

The Complete Integral is, $z = ax + by + a^2 + ab + b^2$

Differentiating (1) partially w.r.to 'a' and 'b', we have

$$O = x + 2a + b \Rightarrow 2a + b + x = 0$$

$$O = y + a + 2b \Rightarrow a + 2b + y = 0$$

Solving for 'a' and 'b' by Cross Rule,

$$\frac{a}{\begin{vmatrix} 1 & x \\ 2 & y \end{vmatrix}} = \frac{b}{\begin{vmatrix} x & 2 \\ y & 1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}}$$

$$\Rightarrow \frac{a}{y - 2x} = \frac{b}{x - 2y} = \frac{1}{3}$$

Taking, $\frac{a}{y - 2x} = \frac{1}{3}$ Taking, $\frac{b}{x - 2y} = \frac{1}{3}$

$$\Rightarrow a = \frac{y - 2x}{3} \qquad \Rightarrow b = \frac{x - 2y}{3}$$

Sub in (1), we get

$$z = \left(\frac{y - 2x}{3}\right)x + \left(\frac{x - 2y}{3}\right)y + \left(\frac{y - 2x}{3}\right)^2 + \left(\frac{y - 2x}{3}\right)\left(\frac{x - 2y}{3}\right) + \left(\frac{x - 2y}{3}\right)^2$$

$$\Rightarrow z = \frac{1}{9}[3xy - 6x^2 + 3yx - 6y^2 + y^2 - 4xy + 4x^2 + xy - 2y^2 - 2x^2 + 4xy + x^2 - 4xy - 4y^2]$$

$$\Rightarrow 9z = 3[xy - x^2 - y^2]$$

$\Rightarrow 3z = xy - x^2 - y^2$ is the required S.I.

(ii) Find the general solution of
$(D^2 - 3DD' + 2D'^2 + 2D - 2D')z = \sin(2x + y)$. (8)

(2) The AE is $m^2 - 3m + \cancel{2} + 2m - \cancel{2} = 0$ (3)

$m^2 - m = 0$

$m(m-1) = 0$

$m = 0, m = 1$

$CF = \phi_1(y) + \phi_2(y+x)$

$$PI = \frac{1}{D^2 - 3DD' + 2D'^2 + 2D - 2D'} \sin(2x+y)$$

$$= \frac{1}{-4 - 3(-2) + 2D - 2D' + 2(-1)} \sin(2x+y)$$

$$= \frac{1}{-\cancel{4} + \cancel{6} + 2(D-D') - \cancel{2}} \sin(2x+y)$$

$$= \frac{1}{2(D-D')} \sin(2x+y)$$

$$= \frac{(D+D')}{2(D^2 - D'^2)} \sin(2x+y)$$

$$= \frac{(D+D')}{2(-4-(-1))} \sin(2x+y)$$

$$= \frac{(D+D')}{2(-3)} \sin(2x+y)$$

$$= -\frac{1}{6}(D+D')\sin(2x+y)$$

$$= -\frac{1}{6}\left[2\cos(2x+y) + \cos(2x+y)\right]$$

$$= -\frac{3}{6}\cos(2x+y) = -\frac{1}{2}\cos(2x+y)$$

The complete integral is $z = CF + PI$

(5)

$z = \phi_1(y) + \phi_2(y+x) - \frac{1}{2}\cos(2x+y)$

12. (a) (i) Find the Fourier series of period 2π for the function $f(x) = x\cos x$ in $0 < x < 2\pi$. (8)

Solution

Solution : Let the required Fourier series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{... (1)}$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x\cos x\,dx = \frac{1}{\pi}\left[x[\sin x] - (1)[-\cos x]\right]_0^{2\pi}$$

$$= \frac{1}{\pi}[x\sin x + \cos x]_0^{2\pi} = \frac{1}{\pi}[(0+1)-(0+1)]$$

$$= \frac{1}{\pi}[1-1] = 0$$

$$\therefore \boxed{a_0 = 0}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\cos x\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\cos x\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[\cos(n+1)x + \cos(n-1)x]\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[\cos(n+1)x]\,dx + \frac{1}{2\pi}\int_0^{2\pi} x[\cos(n-1)x]\,dx$$

$$= \frac{1}{2\pi}\left[x\left[\frac{\sin(n+1)x}{(n+1)}\right] - (1)\left[\frac{-\cos(n+1)x}{(n+1)^2}\right]\right]_0^{2\pi} + \frac{1}{2\pi}\left[x\left[\frac{\sin(n-1)x}{(n-1)}\right] + \left[\frac{\cos(n-1)x}{(n-1)^2}\right]\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[x\left[\frac{\sin(n+1)x}{(n+1)}\right] + \left[\frac{\cos(n+1)x}{(n+1)^2}\right]\right]_0^{2\pi} + \frac{1}{2\pi}\left[x\left[\frac{\sin(n-1)x}{(n-1)}\right] + \left[\frac{\cos(n-1)x}{(n-1)^2}\right]\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left[0+\frac{1}{(n+1)^2}\right] - \left[0+\frac{1}{(n+1)^2}\right]\right] + \frac{1}{2\pi}\left[\left[0+\frac{1}{(n-1)^2}\right] - \left[0+\frac{1}{(n-1)^2}\right]\right]$$

$$= \frac{1}{2\pi}\left[\left[\frac{1}{(n+1)^2}\right] - \left[\frac{1}{(n+1)^2}\right]\right] + \frac{1}{2\pi}\left[\left[\frac{1}{(n-1)^2}\right] - \left[\frac{1}{(n-1)^2}\right]\right]$$

$$= \frac{1}{2\pi}[0] + \frac{1}{2\pi}[0] = 0 \ [\text{if } n \neq 1]$$

$$\therefore \boxed{a_n = 0} \quad [\text{if } n \neq 1]$$

$$\therefore a_1 = \frac{1}{\pi}\int_0^{2\pi} x \cos x \cos x \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \cos^2 x \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[1 + \cos 2x]\, dx = \frac{1}{2\pi}\int_0^{2\pi} [x + x\cos 2x]\, dx$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2} + x\frac{\sin 2x}{2} - (1)\left[\frac{-\cos 2x}{4}\right]\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[x^2 + x\sin 2x + \left[\frac{\cos 2x}{2}\right]\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(4\pi^2 + 0 + \frac{1}{2}\right) - \left(0 + 0 + \frac{1}{2}\right)\right]$$

$$= \frac{1}{4\pi}\left[4\pi^2 + \frac{1}{2} - \frac{1}{2}\right] = \frac{1}{4\pi}[4\pi^2]$$

$$\therefore \boxed{a_1 = \pi}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\cos x \sin nx \, dx = \frac{1}{\pi}\int_0^{2\pi} x \sin nx \cos x \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[\sin(n+1)x + \sin(n-1)x]\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[\sin(n+1)x]\, dx + \frac{1}{2\pi}\int_0^{2\pi} x[\sin(n-1)x]\, dx$$

$$= \frac{1}{2\pi}\left[x\left[\frac{-\cos(n+1)x}{(n+1)}\right]-(1)\left[\frac{-\sin(n+1)x}{(n+1)^2}\right]\right]_0^{2\pi}$$

$$+\frac{1}{2\pi}\left[x\left[\frac{-\cos(n-1)x}{(n-1)}\right]-(1)\left[\frac{-\sin(n-1)x}{(n-1)^2}\right]\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-x\left[\frac{\cos(n+1)x}{(n+1)}\right]+\left[\frac{\sin(n+1)x}{(n+1)^2}\right]\right]_0^{2\pi}$$

$$+\frac{1}{2\pi}\left[-x\left[\frac{\cos(n-1)x}{(n-1)}\right]+\left[\frac{\sin(n-1)x}{(n-1)^2}\right]\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left[\frac{-2\pi}{(n+1)}+0\right]-[0+0]\right]+\frac{1}{2\pi}\left[\left[\frac{-2\pi}{(n-1)}+0\right]-[0+0]\right]$$

$$= \frac{1}{2\pi}\left[\frac{-2\pi}{(n+1)}\right]+\frac{1}{2\pi}\left[\frac{-2\pi}{(n-1)}\right]$$

$$= \left[\frac{-1}{(n+1)}\right]+\left[\frac{-1}{(n-1)}\right] = -\left[\frac{1}{(n+1)}+\frac{1}{(n-1)}\right]$$

$$\therefore \boxed{b_n = -\left[\frac{2n}{(n^2-1)}\right]} \qquad [\text{if } n \neq 1]$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\cos x\sin x\,dx = \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\,dx$$

$$= \frac{1}{2\pi}\left[x\left[\frac{-\cos 2x}{2}\right]-(1)\left[\frac{-\sin 2x}{4}\right]\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left[\frac{-x\cos 2x}{2}\right]+\left[\frac{\sin 2x}{4}\right]\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[-x\cos 2x+\left[\frac{\sin 2x}{2}\right]\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[(-2\pi + 0) - (0+0)\right] = \frac{1}{4\pi}\left[-2\pi\right] = -\frac{1}{2}$$

$$\therefore \quad \boxed{b_1 = -\frac{1}{2}}$$

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx + b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

$$f(x) = 0 + \pi \cos x + 0 - \frac{1}{2}\sin x + \sum_{n=2}^{\infty}\left[\frac{-2n}{n^2-1}\right] \sin nx$$

$$f(x)) = \pi \cos x - \frac{1}{2}\sin x - 2\sum_{n=2}^{\infty}\left[\frac{n}{n^2-1}\right] \sin nx$$

(ii) Find the Fourier series expansion for $y = f(x)$ up to second harmonic from the following data: (8)

x:	0	1	2	3	4	5
y:	9	18	24	28	26	20

Solution

Solution : Here, the length of the interval is $2l = 6$, i.e., $l = 3$.

$\therefore$ The Fourier series can be represented by

$$y = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{l} + b_1 \sin\frac{\pi x}{l}\right) + \left(a_2 \cos\frac{2\pi x}{l} + b_2 \sin\frac{2\pi x}{l}\right) + \ldots$$

$$\text{Here } y = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{3} + b_1 \sin\frac{\pi x}{3} \qquad \ldots (1)$$

x	y	$\cos\frac{\pi x}{3}$	$\sin\frac{\pi x}{3}$	$y\cos\frac{\pi x}{3}$	$y\sin\frac{\pi x}{3}$
0	9	1	0	9	0
1	18	0.5	0.866	9	15.588
2	24	–0.5	0.866	–12	20.785
3	28	–1	0	–28	0
4	26	–0.5	–0.866	–13	–22.517
5	20	0.5	–0.866	10	–17.321
	125			–25	–3.465

$$a_0 = \frac{2}{n}\Sigma y = \frac{2}{6}125 = 41.67$$

$$a_1 = \frac{2}{n}\Sigma y\cos\frac{\pi x}{3} = \frac{2}{6}(-25) = -8.33$$

$$b_1 = \frac{2}{n}\Sigma y\sin\frac{\pi x}{3} = \frac{2}{6}(-3.465) = -1.16$$

Substituting the above values in (1), we get

$$y = \frac{41.67}{2} - 8.33\cos\frac{\pi x}{3} - 1.16\sin\frac{\pi x}{3}$$

$$= 20.84 - 8.33\cos\frac{\pi x}{3} - 1.16\sin\frac{\pi x}{3}$$

(b) (i) Find the Fourier half-range cosine series of (8)

$$f(x) = \begin{cases} x, & \text{in } 0 < x < 1 \\ 2 - x, & \text{in } 1 < x < 2 \end{cases}.$$

3) Given $f(x) = \begin{cases} x & \text{in } 0<x<1 \\ 2-x & \text{in } 1<x<2. \end{cases}$

The half range sine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

Here $l = 2$

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)$$

$$a_0 = 2/2 \int_0^2 f(x)\,dx$$

$$= \int_0^2 f(x)\,dx$$

$$= \int_0^1 x\,dx + \int_1^2 (2-x)\,dx$$

$$= \left[x^2/2\right]_0^1 + \left[2x - x^2/2\right]_1^2$$

$$= 1/2 + \left[\left(4 - 4/2\right) - \left(2 - 1/2\right)\right]$$

$$= 1/2\left[(4-2) - 3/2\right]$$

$$= 1/2\left[2 - 3/2\right]$$

$$= 1/2\left[\frac{4-3}{2}\right] = 1/2\left[1/2\right] = 1/4$$

(4)

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \int_0^1 x\cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 (2-x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \left[x\,\frac{\sin\left(\frac{n\pi x}{2}\right)}{n\pi/2} - (1)\frac{\left(-\cos\left(\frac{n\pi x}{2}\right)\right)}{n^2\pi^2/4} \right]_0^1$$

$$+\left[(2-x)\frac{\sin\left(\frac{n\pi x}{2}\right)}{n\pi/2} - (-1)\frac{\left(-\cos\left(\frac{n\pi x}{2}\right)\right)}{\frac{n^2\pi^2}{4}} \right]_1^2$$

$$\left(\frac{\sin n\pi/2}{n\pi/2} + \frac{\cos n\pi/2}{n^2\pi^2/4}\right) - \left(0 + \frac{1}{n^2\pi^2/4}\right)$$

$$+\left(0 - \frac{\cos n\pi}{\frac{n^2\pi^2}{4}}\right) - \left(\frac{\sin\left(n\pi/2\right)}{n\pi/2} - \frac{\cos n\pi/2}{n^2\pi^2/4}\right)$$

$$= \cancel{\frac{\sin n\pi/2}{n\pi/2}} + \frac{\cos n\pi/2}{n^2\pi^2/4} - \frac{1}{\frac{n^2\pi^2}{4}} - \frac{\cos n\pi}{\frac{n^2\pi^2}{4}}$$

$$- \cancel{\frac{\sin n\pi/2}{n\pi/2}} + \frac{\cos n\pi/2}{n^2\pi^2/4}$$

$$= \frac{4}{n^2\pi^2}\times 2\cos n\pi/2 - \frac{4}{n^2\pi^2}\left[1+\cos n\pi\right]$$

$$= \frac{8\cos n\pi/2}{n^2\pi^2} - \frac{4(1+\cos n\pi)}{n^2\pi^2}.$$

$$f(x) = \frac{1}{8} + \sum_{n=1}^{\infty}\left[\frac{8\cos n\pi/2}{n^2\pi^2} - \frac{4(1+\cos n\pi)}{n^2\pi^2}\right]\cos\left(\frac{n\pi x}{2}\right)$$

(ii) Find the complex form of the Fourier series of $f(x) = e^{-ax}$ in, $-l < x < l$. (8)

Solution

Solution : Given $f(x) = e^{-ax}$ in $-l < x < l$

$$\boxed{\begin{aligned} \text{Formula : } f(x) &= \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}} \quad \text{where} \\ C_n &= \frac{1}{2l} \int_{-l}^{l} f(x)\, e^{\frac{-in\pi x}{l}} dx \end{aligned}}$$

$$C_n = \frac{1}{2l} \int_{-l}^{l} e^{-ax}\, e^{\frac{-in\pi x}{l}} dx$$

$$= \frac{1}{2l} \int_{-l}^{l} e^{\left(-a - \frac{in\pi}{l}\right)x} dx$$

$$= \frac{1}{2l} \int_{-l}^{l} e^{\left(\frac{-al - in\pi}{l}\right)x} dx$$

$$= \frac{1}{2l} \left[\frac{e^{\left(\frac{-al - in\pi}{l}\right)x}}{\left(\frac{-al - in\pi}{l}\right)} \right]_{-l}^{l}$$

$$= \frac{1}{2(-al - in\pi)} \left[e^{\left(\frac{-al - in\pi}{l}\right)x} \right]_{-l}^{l}$$

$$= \frac{1}{2(-al - in\pi)} \left[e^{\left(\frac{-al - in\pi}{l}\right) l} - e^{\left(\frac{-al - in\pi}{l}\right)(-l)} \right]$$

$$= \frac{1}{2(-al - in\pi)} \left[e^{-al - in\pi} - e^{+al + in\pi} \right]$$

$$= \frac{1}{2(-al - in\pi)} \left[e^{-al} e^{-in\pi} - e^{+al} e^{in\pi} \right]$$

$$\boxed{\begin{aligned} e^{in\pi} &= \cos n\pi + i\sin n\pi = (-1)^n + i\,0 = (-1)^n \\ e^{-in\pi} &= \cos n\pi - i\sin n\pi = (-1)^n - i\,0 = (-1)^n \end{aligned}}$$

$$= \frac{1}{2(al - in\pi)}\left[\bar{e}^{al}(-1)^n - e^{+al}(-1)^n\right]$$

$$= \frac{(-1)^n}{2(al - in\pi)}\left[\bar{e}^{al} - e^{+al}\right]$$

$$= \frac{(-1)^n}{-al - in\pi}\left[\frac{\bar{e}^{al} - e^{+al}}{2}\right] = \frac{(-1)^n(-1)}{-(al+in\pi)} \quad \frac{e^{al} - e^{-al}}{2}$$

$$\boxed{C_n = \frac{(-1)^n}{al + in\pi}\sin h\, al}$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{al + in\pi} \sinh al\; e^{\frac{in\pi x}{l}}$$

$$= \sinh al \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{al + in\pi}\, e^{\frac{in\pi x}{l}}$$

13. (a) A tightly stretched string of length $2l$ is fastened at $x = 0$ and $x = 2l$. The midpoint of the string is then taken to height 'b' transversely and then released from rest in that position. Find the lateral displacement of the string. (16)

Solution

Solution: Length of the string is $2l$. Let $2l = L$

Since the ends are fixed, we have the **boundary conditions**

(i) $y(0, t) = 0$

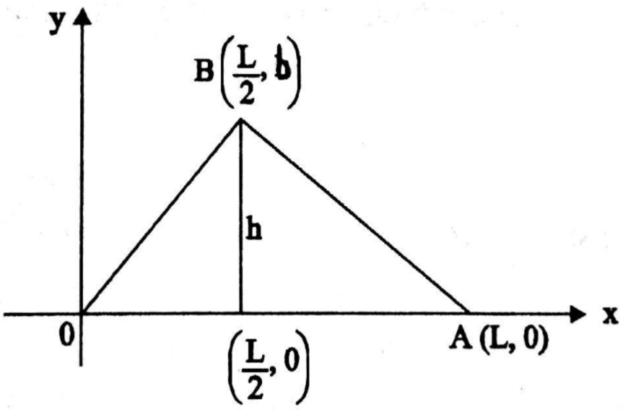
(ii) $y(L, t) = 0$ for all t

Since the string is released from rest, the initial velocity is zero.

(iii) $\frac{\partial y}{\partial t}(x, 0) = 0.$

Here the Midpoint of the string is taken to a height b.

The initial position of the string is given by the equations to the lines OB and BA.



Equation to OB: $\frac{b-0}{y-0} = \frac{\frac{L}{2}-0}{x-0}$

$$xb = y\frac{L}{2} \Rightarrow y = \frac{2b}{L}x \text{ in } \left(0, \frac{L}{2}\right)$$

Equation to BA: $\frac{b-0}{y-0} = \frac{\frac{L}{2}-L}{x-L}$

$$\frac{b}{y} = \frac{-\frac{L}{2}}{x-L} \Rightarrow b(x-L) = y\frac{(-L)}{2} \Rightarrow y = \frac{2b}{L}(L-x)$$

Thus we have the condition

(iv) $y(x,0) = \begin{cases} \dfrac{2b}{L}x \text{ in } \left(0, \dfrac{L}{2}\right) \\ \dfrac{2b}{L}(L-x) \text{ in } \left(\dfrac{L}{2}, L\right) \end{cases} = f(x)$ in $(0, L)$ (say)

The displacement $y(x,t)$ satisfies the equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

It's solution is

$$y(x,t) = (A_1 \cos px + A_2 \sin px)(A_3 \cos cpt + A_4 \sin cpt) \quad \ldots (1)$$

By applying the conditions (i), (ii) and (iii), (1) reduces to

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \cos\frac{cn\pi t}{L} \quad \ldots (2)$$

By the condition (4), $y(x,0) = f(x)$, we have

$\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = f(x)$, which is the half range sine series for $f(x)$ in $(0, L)$

$$\therefore A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\left[\int_0^{\frac{L}{2}} \frac{2b}{L}x\sin\frac{n\pi x}{L}\,dx + \int_{\frac{L}{2}}^{L} \frac{2b}{L}(L-x)\sin\frac{n\pi x}{L}\,dx\right]$$

$$= \frac{4b}{L^2}\left[\left\{x\frac{L}{n\pi}\left(-\cos\frac{n\pi x}{L}\right) - (1)\frac{L^2}{n^2\pi^2}\left(-\sin\frac{n\pi x}{L}\right)\right\}_{x=0}^{\frac{L}{2}}\right.$$

$$\left. + \left\{(L-x)\frac{L}{n\pi}\left(-\cos\frac{n\pi x}{L}\right) - (-1)\frac{L^2}{n^2\pi^2}\left(-\sin\frac{n\pi x}{L}\right)\right\}_{\frac{L}{2}}^{L}\right]$$

$$= \frac{4b}{L^2}\left[\frac{-L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2} - (0+0) \right]$$

$$+ \left[(0+0) - \left(\frac{-L^2}{2n\pi}\cos\frac{n\pi}{2} - \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2} \right) \right]$$

$$= \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\therefore (2) \Rightarrow y(x,t) = \sum_{n=1}^{\infty} \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}$$

$$= \frac{8b}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2l}\cos\frac{n\pi ct}{2l}$$

$$= \frac{8b}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi x}{2l}\cos\frac{\pi ct}{2l} - \frac{1}{3^2}\sin\frac{3\pi x}{2l}\cos\frac{3\pi t}{2l} + \ldots \right]$$

$$= \frac{8b}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{2l}\cos\frac{(2n-1)\pi ct}{2l}.$$

(b) A rectangular plate with insulated surfaces is 20 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature while the other short edge $x = 0$ is given by $u = \begin{cases} 10y & \text{for } 0 \le y \le 10 \\ 10(20-y) & \text{for } 10 \le y \le 20 \end{cases}$ and the two long edges as well as the other short edge are kept at 0°c, find the steady state temperature distribution $u(x, y)$ in the plate. (16)

Solution

From the given problem, we get the foll boundary condition.

(i) $u(x,0) = 0$ for all x

(ii) $u(x,20) = 0$ for all x

(iii) $u(\infty, y) = 0$

(iv) $u(0,y) = \begin{cases} f_1(y) = 10y, & 0 \le y \le 10 \\ f_2(y) = 10(20-y), & 10 \le y \le 20. \end{cases}$

The suitable solution which satisfies our boundary conditions are given by.

$$u(x,y) = (Ae^{px} + Be^{-px})(C\cos py + D\sin py) \quad \text{---(1)}$$

Applying condition (i) in equation (1)

$$u(x,0) = (Ae^{px} + Be^{-px})C = 0$$

$$\Rightarrow C = 0$$

Sub $C = 0$ in (1), we get

$$u(x,y) = (Ae^{px} + Be^{-px}) D\sin py \quad \text{---(2)}$$

Applying condition (ii) in equation (2), we get

$$u(x,20) = (Ae^{px} + Be^{-px}) D\sin 20p = 0$$

But $\because Ae^{px} + Be^{-px} \neq 0$.

(5)

$D \neq 0 \quad \therefore \sin 20p = 0$

$20p = n\pi \Rightarrow \boxed{p = n\pi/20}$

Sub $p = n\pi/20$ in (2), we get

$$u(x,y) = \left(Ae^{\frac{n\pi x}{20}} + Be^{\frac{-n\pi x}{20}}\right) D \sin n\pi y/20 \quad -(3)$$

Applying condn (iii) in (3), we get

$$u(\infty, y) = (Ae^{\infty} + Be^{-\infty}) D \sin \frac{n\pi y}{20} = 0.$$

$\Rightarrow A = 0$

Sub $A = 0$ in (3)

$$u(x,y) = Be^{-\frac{n\pi x}{20}} D \sin\left(\frac{n\pi y}{20}\right)$$

$$= BD e^{\frac{-n\pi x}{20}} \sin(n\pi y/20) \quad -(4)$$

The most general solution is

$$u(x,y) = \sum_{n=1}^{\infty} B_n e^{\frac{-n\pi x}{20}} \sin(n\pi y/20) \quad -(5)$$

Applying Condn (iv) in eqn (5), we get

$$u(0,y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi y}{20} = \begin{cases} 10y, & 0 \le y \le 10 \\ 10(20-y), & 10 \le y \le 20 \end{cases} \quad -(6)$$

To find B_n expand $f(y)$ in a half range Fourier Sine Series in $[0, 20]$.

$$f(y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi y}{20}\right) \quad -(7)$$

where $b_n = \frac{2}{20} \int_0^{20} f(y) \sin\left(\frac{n\pi y}{20}\right) dy.$

From (6) & (7), $\boxed{B_n = b_n}$

$$B_n = \frac{2}{20}\left[\int_0^{10} 10y \sin\frac{n\pi y}{20}\,dy + \int_{10}^{20} 10(20-y)\sin\frac{n\pi y}{20}\,dy\right.$$

$$= \frac{1}{10}\left\{10y\left(\frac{-\cos\left(\frac{n\pi y}{20}\right)}{n\pi/20}\right) - 10\left(\frac{-\sin\left(\frac{n\pi y}{20}\right)}{\frac{n^2\pi^2}{400}}\right)\right]_0^{10}$$

$$+ 10(20-y)\left(\frac{-\cos\left(\frac{n\pi y}{20}\right)}{n\pi/20}\right) - 10(-1)\left(\frac{-\sin\left(\frac{n\pi y}{20}\right)}{\frac{n^2\pi^2}{400}}\right)\Bigg]_{10}^{20}$$

$$= \frac{1}{10}\left[\left(100\left(\frac{-\cos n\pi/2}{n\pi/20}\right) + \frac{10\sin(n\pi/2)}{\frac{n^2\pi^2}{400}}\right) - (0+0)\right.$$

$$+ (0 - 0) - \left(100\left(\frac{-\cos n\pi/2}{n\pi/20}\right) - \frac{10\sin(n\pi/2)}{\frac{n^2\pi^2}{400}}\right)$$

$$= \frac{1}{10}\left[-\cancel{\frac{2000}{n\pi}\cos n\pi/2} + \frac{4000}{n^2\pi^2}\sin(n\pi/2)\right.$$

$$\left.+ \cancel{\frac{2000}{n\pi}\cos n\pi/2} + \frac{4000}{n^2\pi^2}\sin(n\pi/2)\right]$$

$$= \frac{1}{10}\left[\frac{8000}{n^2\pi^2}\sin(n\pi/2)\right] = \frac{800}{n^2\pi^2}\sin(n\pi/2).$$

$$u(x,y) = \sum_{n=0}^{\infty} \frac{800}{n^2\pi^2} \sin\frac{n\pi}{2}\, e^{-\frac{n\pi x}{20}} \sin\frac{n\pi y}{20}. \qquad (6)$$

14. (a) Find the Fourier transform of $f(x)$ given by $f(x) = \begin{cases} 1 & \text{for } |x| < 2 \\ 0 & \text{for } |x| > 2 \end{cases}$ and hence evaluate $\int_0^\infty \frac{\sin x}{x}\,dx$ and $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx$. (16)

Solution

Solution : The given equation can be written as

$$f(x) = 1 \quad \text{if} \quad -2 < x < 2$$

$$= 0 \quad \text{otherwise}$$

$$F(s) = F[f(x)]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} (\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} \cos sx\, dx + \frac{i}{\sqrt{2\pi}} \int_{-2}^{2} \sin sx\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[2\int_{0}^{2} \cos sx\, dx\right] + \frac{i}{\sqrt{2\pi}} [0]$$

[$\because$ $\cos sx$ is an even function in $(-2, 2)$ & $\sin sx$ is an odd function in $(-2, 2)$]

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s}\right]_{x=0}^{x=2}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin 2s}{s} - 0\right]$$

$$F(s) = F[f(x)] = \sqrt{\frac{2}{\pi}} \frac{\sin 2s}{s} \qquad \dots (1)$$

(i) Now, by Fourier inversion formula, we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}}\, \frac{\sin 2s}{s} (\cos sx - i \sin sx)\, ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin 2s}{s} \cos sx\, ds - \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\sin 2s}{s} \sin sx\, dx$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin 2s}{s} \cos sx\, dx - \frac{i}{\pi}(0) \quad \left[\because \frac{\sin 2s}{s} \sin sx \text{ is an odd function in } (-\infty, \infty)\right]$$

$$= \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin 2s}{s} \cos sx\, dx \quad \left[\because \frac{\sin 2s}{s} \cos sx \text{ in an even function in } (-\infty, \infty)\right]$$

$$\text{(i.e.,)} \int_{0}^{\infty} \left(\frac{\sin 2s}{s}\right) \cos sx\, dx = \frac{\pi}{2} f(x)$$

put $x = 0$, we get $\displaystyle\int_{0}^{\infty} \frac{\sin 2s}{s}\, ds = \frac{\pi}{2} f(0) = \frac{\pi}{2}$

$$\therefore \int_{0}^{\infty} \frac{\sin 2s}{s}\, ds = \frac{\pi}{2} \quad [\because f(0) = 1 \text{ in } |x| < 2]$$

Now, put $t = 2s$ | $s \to 0 \Rightarrow t \to 0$

$dt = 2\, ds$ | $s \to \infty \Rightarrow t \to \infty$

$$\therefore \int_{0}^{\infty} \frac{\sin t}{(t/2)} \frac{dt}{2} = \frac{\pi}{2}$$

Hence, $\displaystyle\int_{0}^{\infty} \frac{\sin t}{t}\, dt = \frac{\pi}{2}$. (i.e.,) $\displaystyle\int_{0}^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$ [$\because$ t is a dummy variable]

(ii) Using Parseva's identity, $\displaystyle\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds$

we get $\int_{-2}^{2} dx = \int_{-\infty}^{\infty} \left[\sqrt{\frac{2}{\pi}} \frac{\sin 2s}{s} \right]^2 ds$

$$[x]_{-2}^{2} = \frac{2}{\pi} \int_{-\infty}^{\infty} \left[\frac{\sin 2s}{s}\right]^2 ds$$

$$[2-(-2)] = \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin 2s}{s}\right)^2 ds$$

$$4 = \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin 2s}{s}\right)^2 ds$$

$$\int_{-\infty}^{\infty} \left(\frac{\sin 2s}{s}\right)^2 ds = 2\pi$$

put $t = 2s$, $dt = 2\,ds$ | $s \to -\infty \Rightarrow t \to -\infty$, $s \to \infty \Rightarrow t \to \infty$

$$\int_{-\infty}^{\infty} \left[\frac{\sin t}{t/2}\right]^2 \frac{dt}{2} = 2\pi$$

$$2\int_{-\infty}^{\infty} \left[\frac{\sin t}{t}\right]^2 dt = 2\pi$$

$$\int_{-\infty}^{\infty} \left[\frac{\sin t}{t}\right]^2 dt = \pi$$

$$2\int_{-\infty}^{\infty} \left[\frac{\sin t}{t}\right]^2 dt = \pi \quad \left[\because \left[\frac{\sin t}{t}\right]^2 \text{ is an even function}\right]$$

$$\int_{0}^{\infty} \left[\frac{\sin t}{t}\right]^2 dt = \frac{\pi}{2}$$

Hence, $\int_{0}^{\infty} \left[\frac{\sin x}{x}\right]^2 dx = \frac{\pi}{2}$ $\quad$ [$\because$ t is a dummy variable]

(b) (i) Find the Fourier cosine transform of $e^{-a^2x^2}$ for any $a>0$. (8)

Solution

Solution : We know that,

$$F_c\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx$$

$$F_c\,[e^{-a^2x^2}] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-a^2x^2} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{1}{2} \int_{-\infty}^{\infty} e^{-a^2x^2} \cos sx \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \text{ R.P. } \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \text{ R.P. } \int_{-\infty}^{\infty} e^{-a^2x^2 + isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \text{ R.P. } \int_{-\infty}^{\infty} e^{-(a^2x^2 - isx)} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \text{ R.P. } \int_{-\infty}^{\infty} e^{-\left[\left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2}\right]} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \text{ R.P. } \int_{-\infty}^{\infty} e^{-\left[ax - \frac{is}{2a}\right]^2} e^{\frac{-s^2}{4a^2}} \, dx$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{\frac{-s^2}{4a^2}} \text{ R.P. } \int_{-\infty}^{\infty} e^{-\left[ax - \frac{is}{2a}\right]^2} \, dx$$

put $t = ax - \dfrac{is}{2a}$ | $x \to -\infty \Rightarrow t \to -\infty$

$dt = a\,dx$ | $x \to \infty \Rightarrow t \to \infty$

$$= \frac{1}{\sqrt{2\pi}}\, e^{-s^2/4a^2} \text{ R.P. } \int_{-\infty}^{\infty} e^{-[t]^2} \frac{dt}{a}$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{-s^2/4a^2}\, \frac{1}{a} \text{ R.P. } \int_{-\infty}^{\infty} e^{-t^2} \, dt$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{-s^2/4a^2}\, \frac{1}{a} \text{ R.P } \sqrt{\pi} \quad \left[\because \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}\right]$$

$$= \frac{1}{\sqrt{2}}\, \frac{1}{a}\, e^{-s^2/4a^2}$$

$$F_c\,[e^{-a^2x^2}] = \frac{1}{a\sqrt{2}}\, e^{-s^2/4a^2}$$

(ii) Evaluate $\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)}$ using Fourier transforms. (8)

Solution : Let $f(x) = e^{-ax}$ we know that,

$$F_c[f(x)] = F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2+a^2}\right]$$

Let $g(x) = e^{-bx}$ we know that,

$$F_c[g(x)] = F_c[e^{-bx}] = \sqrt{\frac{2}{\pi}} \left(\frac{b}{s^2+b^2}\right)$$

We know by Parseval's identity,

$$\int_0^\infty f(x)\, g(x)\, dx = \int_0^\infty F_c[f(x)]\, F_c[g(x)]\, ds$$

$$\int_0^\infty e^{-ax} e^{-bx} dx = \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{a}{s^2+a^2}\right) \sqrt{\frac{2}{\pi}} \left(\frac{b}{s^2+b^2}\right) ds$$

$$\int_0^\infty e^{-(a+b)x} dx = \frac{2ab}{\pi} \int_0^\infty \left(\frac{1}{s^2+a^2}\right)\left(\frac{1}{s^2+b^2}\right) ds$$

$$\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty = \frac{2ab}{\pi} \int_0^\infty \frac{1}{(s^2+a^2)(s^2+b^2)} ds$$

$$\left[0 - \left(\frac{1}{-(a+b)}\right)\right] = \frac{2ab}{\pi} \int_0^\infty \frac{1}{(s^2+a^2)(s^2+b^2)} ds \quad \left[\begin{array}{r} \text{since } e^{-\infty} = 0 \\ e^{-0} = 1 \end{array}\right]$$

$$\text{(i.e.,)} \quad \frac{1}{a+b} = \frac{2ab}{\pi} \int_0^\infty \frac{1}{(s^2+a^2)(s^2+b^2)}\, ds$$

$$\text{(i.e.,)} \quad \int_0^\infty \frac{1}{(s^2+a^2)(s^2+b^2)}\, ds = \frac{\pi}{2ab(a+b)}$$

$$\text{(i.e.,)} \quad \int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)}\, dx = \frac{\pi}{2ab(a+b)} \quad [\because s \text{ is a dummy variable}]$$

15. (a) (i) Find Z-transform of $\frac{2n+3}{(n+1)(n+2)}$. (8)

Solution : Given : $f(x) = \frac{2n+3}{(n+1)(n+2)}$

$$\frac{2n+3}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2} \quad \ldots (1)$$

$$2n+3 = A(n+2) + B(n+1)$$

Put $n = -1$, we get	Put $n = -2$, we get
$-2+3 = A$	$-4+3 = -B$
$A = 1$	$-1 = -B$
	$B = 1$

$$(1) \Rightarrow \frac{2n+3}{(n+1)(n+2)} = \frac{1}{n+1} + \frac{1}{n+2}$$

$$Z[f(n)] = Z\left[\frac{2n+3}{(n+1)(n+2)}\right] = Z\left[\frac{1}{n+1}\right] + Z\left[\frac{1}{n+2}\right] \quad \ldots (2)$$

We know that, $Z\left[\frac{1}{n+1}\right] = z \log \frac{z}{z-1}$ ... (3)

$$Z\left[\frac{1}{n+2}\right] = \sum_{n=0}^{\infty} \frac{1}{n+2} z^{-n}$$

$$= \frac{1}{2} + \frac{1}{3}\left(\frac{1}{z}\right) + \frac{1}{4}\left(\frac{1}{z}\right)^2 + \dots$$

$$= z^2\left[\frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \dots\right]$$

$$= z^2\left[-\log\left(1 - \frac{1}{z}\right) - \frac{1}{z}\right]$$

$$= z^2\left[\log \frac{z}{z-1}\right] - z \qquad \dots (4)$$

(2) $\Rightarrow Z[f(n)] = z \log \frac{z}{z-1} + z^2 \log \frac{z}{z-1} - z$ by (3) & (4)

$$= z \log \frac{z}{z-1} [1+z] - z$$

$$= z(z+1) \log \frac{z}{z-1} - z$$

(ii) Using Convolution theorem, find $Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right]$. (8)

Solution : Given : $\frac{8z^2}{(2z-1)(4z-1)} = \frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$

$$Z^{-1}\left[\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\right] = Z^{-1}\left[\frac{z}{z-\frac{1}{2}} \cdot \frac{z}{z-\frac{1}{4}}\right]$$

Same as (4)

$$= Z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right] * Z^{-1}\left[\frac{z}{z-\frac{1}{4}}\right]$$

$$= \left(\frac{1}{2}\right)^n * \left(\frac{1}{4}\right)^n$$

$$= \sum_{K=0}^{n} \left(\frac{1}{2}\right)^{n-K} \left(\frac{1}{4}\right)^{K}$$

$$= \left(\frac{1}{2}\right)^{n} \sum_{K=0}^{n} \left(\frac{1}{2}\right)^{-K} \left(\frac{1}{4}\right)^{K}$$

$$= \left(\frac{1}{2}\right)^{n} \sum_{K=0}^{n} \left(\frac{1}{2}\right)^{-K} \left(\frac{1}{2}\right)^{2K}$$

$$= \left(\frac{1}{2}\right)^{n} \sum_{K=0}^{n} \left(\frac{1}{2}\right)^{K}$$

$$= \left(\frac{1}{2}\right)^{n} \left[1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^{2} + \ldots + \left(\frac{1}{2}\right)^{n}\right]$$

$$= \left(\frac{1}{2}\right)^{n} \left[\frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}}\right] \text{ by G.P.}$$

Formula :

$a + ar + ar^2 + \ldots + ar^n$

$= \dfrac{a\,[1 - r^n]}{1 - r}$, if $r < 1$

$$= \left(\frac{1}{2}\right)^{n} \left[\frac{1 - \left(\frac{1}{2}\right)^{n+1}}{\frac{1}{2}}\right]$$

$$= \left(\frac{1}{2}\right)^{n-1} \left[1 - \left(\frac{1}{2}\right)^{n+1}\right]$$

$$= \left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{2}\right)^{2n}$$

(b) (i) Find $Z^{-1}\left[\frac{4z^3}{(2z-1)^2(z-1)}\right]$, by the method of partial fractions. (8)

Solution

Solution: Let $\bar{f}(z) = \frac{4z^3}{(2z-1)^2(z-1)}$

$$\therefore \frac{\bar{f}(z)}{z} = \frac{4z^2}{(2z-1)^2(z-1)}$$

Let $\frac{4z^2}{(2z-1)^2(z-1)} = \frac{A}{2z-1} + \frac{B}{(2z-1)^2} + \frac{C}{z-1}$

$$\therefore 4z^2 = A(2z-1)(z-1) + B(z-1) + C(2z-1)^2$$

Put $z = 1$

$4 = C; \Rightarrow \quad C = 4$

put $z = 1/2$

$1 = -\frac{1}{2}B \Rightarrow \quad B = -2$

Put $z = 0$

$0 = A - B + C$

$A = B - C = -2 - 4 = -6$

$$\therefore \frac{4z^2}{(2z-1)^2(z-1)} = \frac{-6}{2z-1} - \frac{2}{(2z-1)^2} + \frac{4}{z-1}$$

Then $\quad \bar{f}(z) = -\frac{6z}{2z-1} - \frac{2z}{(2z-1)^2} + \frac{4z}{z-1}$

$$Z^{-1}[\bar{f}(z)] = -6Z^{-1}\left[\frac{z}{2z-1}\right] - 2Z^{-1}\left[\frac{z}{(2z-1)^2}\right] + 4Z^{-1}\left[\frac{z}{z-1}\right]$$

$$= -3Z^{-1}\left[\frac{z}{z-1/2}\right] - \frac{1}{2}Z^{-1}\left[\frac{\frac{1}{2}z}{(z-1/2)^2}\right] + 4Z^{-1}\left[\frac{z}{z-1}\right]$$

$$= -3\left(\frac{1}{2}\right)^n - \frac{1}{2}n\left(\frac{1}{2}\right)^n + 4(1)^n$$

$$= -3\left(\frac{1}{2}\right)^n - \left(\frac{1}{2}\right)^{n+1} + 4$$

$$= 4 - 6\left(\frac{1}{2}\right)^{n+1} - n\left(\frac{1}{2}\right)^{n+1}$$

$$= 4 - (6+n)\left(\frac{1}{2}\right)^{n+1}$$

(ii) Using Z-transforms, solve the equation $y_{n+2} - 7y_{n+1} + 12y_n = 2^n$, given that $y_0 = y_1 = 0$. (8)

Solution

$$y_{n+2} - 7y_{n+1} + 12y_n = 2^n$$

and $y_0 = y_1 = 0$ ... (1)

Taking $Z-$ transform on both sides

$$Z(y_{n+2}) - 7Z(y_{n+1}) + 12Z(y_n) = Z(2^n)$$

$$\therefore z^2[\bar{y}(z) - y_0 - y_1 z^{-1}] - 7z[\bar{y}(z) - y_0] + 12\bar{y}(z) = \frac{z}{z-2}$$ as $y_0 = y_1 = 0$.

$$z^2\bar{y}(z) - 7z\bar{y}(z) + 12\bar{y}(z) = \frac{z}{z-2}$$

$$\bar{y}(z)(z^2 - 7z + 12) = \frac{z}{z-2}$$

$$\bar{y}(z) = \frac{z}{(z-2)(z^2-7z+12)}$$

$$= \frac{z}{(z-2)(z-4)(z-3)}$$

$$\frac{\bar{y}(z)}{z} = \frac{1}{(z-2)(z-3)(z-4)} \quad \text{... (1)}$$

Let $\dfrac{1}{(z-2)(z-3)(z-4)} = \dfrac{A}{z-2} + \dfrac{B}{z-3} + \dfrac{C}{z-4}$

$1 = A(z-3)(z-4) + B(z-2)(z-4) + C(z-2)(z-3)$

put $z = 2$

$1 = 2A$; $A = 1/2$

put $z = 3$

$1 = -B$; $B = -1$

put $z = 4$

$1 = 2C$ $C = 1/2$

$$\therefore \frac{\bar{y}(z)}{z} = \frac{\frac{1}{2}}{z-2} - \frac{1}{z-3} + \frac{\frac{1}{2}}{z-4}$$

$$\bar{y}(z) = \frac{1}{2}\frac{z}{z-2} - \frac{z}{z-3} + \frac{1}{2}\frac{z}{z-4}$$

$$y_n = Z^{-1}(\bar{y}(z))$$

$$= \frac{1}{2}Z^{-1}\left(\frac{z}{z-2}\right) - Z^{-1}\left(\frac{z}{z-3}\right) + \frac{1}{2}Z^{-1}\left(\frac{z}{z-4}\right)$$

$$= \frac{1}{2}(2)^n - 3^n + \frac{1}{2}(4^n)$$

$$= 2^{n-1} - 3^n + \frac{1}{2}(4^n)$$