UNIT IV

APPLICATIONS OF PDE

4.1. ONE DIMENSIONAL WAVE EQUATION:

4.1.1. DEFINITION:

Consider an elastic string tightly stretched between two points O and A (Fig.4.1). Let O be the origin and OA as x-axis. On giving a small displacement to the string, perpendicular to its length (parallel to the y-axis). Let y be the displacement at $p(x,y)$ at any time. The wave equation is

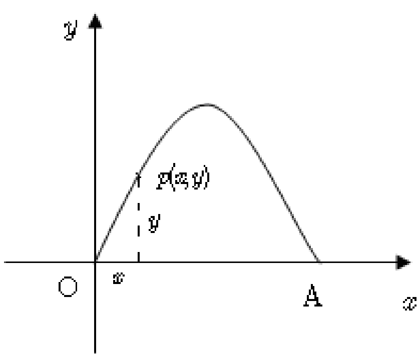

Fig. 4.1.

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

$$\text{where} \quad c^2 = \frac{T}{M} = \frac{\text{Tension}}{\text{Mass}}.$$

4.1.2. SOLUTION OF ONE DIMENSIONAL WAVE EQUATION:

One dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad (1)$$

Let $y(x,t)=X(x).\ T(t)$ ---- (2) be the solution of the given equation, where X is a function of x only and T is a function of t only.

Diff. (2) p.w.r. to x and t, we get,

$$\frac{\partial y}{\partial x} = x'T \qquad \Bigg| \qquad \frac{\partial y}{\partial t} = xT'$$

$$\frac{\partial^2 y}{\partial x^2} = x''T \qquad \Bigg| \qquad \frac{\partial^2 y}{\partial t^2} = xT''.$$

$\therefore$ (1) $\Rightarrow$ $XT'' = c^2 X''T$.

$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X}$$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = k \text{ (say)}$$

$$\therefore \frac{X''}{X} = k \qquad \Bigg| \qquad \frac{1}{c^2}\frac{T''}{T} = k$$

$$X'' = kX \qquad \Bigg| \qquad T'' = c^2 kT$$

$$X'' - kX = 0 \qquad (3) \qquad \Bigg| \qquad T'' - c^2 kT = 0 \qquad (4)$$

Case (i):

Let k be +ve (i.e.) $k = p^2$

(3) $\Rightarrow$ $X'' - p^2 X = 0$

(4) $\Rightarrow$ $T'' - c^2 p^2 T = 0$

Hence we have two ordinary diff. equations.

$\therefore$ The auxiliary equations are

$$m^2 - p^2 = 0 \qquad\qquad m^2 - p^2c^2 = 0$$

$$m^2 = p^2 \qquad\qquad m^2 = p^2c^2$$

$$m = \pm p \qquad\qquad m = \pm cp$$

$$\therefore \quad X = A_1e^{px} + A_2e^{-px}, \qquad T = A_3e^{cpt} + A_4e^{-cpt}$$

$$\therefore \ (2) \quad \Rightarrow \quad y(x,t) = \left(A_1e^{px} + A_2e^{-px}\right)\left(A_3e^{cpt} + A_4e^{-cpt}\right)$$

Case (ii):

Let k be $-$ve (i.e.) $k = -p^2$

$$(3) \quad \Rightarrow \quad X'' + p^2X = 0$$

$$(4) \quad \Rightarrow \quad T'' + c^2p^2T = 0$$

$\therefore$ The auxiliary equations are

$$m^2 + p^2 = 0 \qquad\qquad m^2 - p^2c^2 = 0$$

$$m^2 = -p^2 \qquad\qquad m^2 = p^2c^2$$

$$m = \pm pi \qquad\qquad m = \pm cp$$

$$\therefore \quad X = A_5 \cos px + A_6 \sin px \qquad T = A_7 \cos cpt + A_8 \sin cpt$$

$$\therefore \ (2) \quad \Rightarrow \quad y(x,t) = \left(A_5 \cos px + A_6 \sin px\right)\left(A_7 \cos cpt + A_8 \sin cpt\right)$$

Case (iii):

Let $k = 0$

$$(3) \quad \Rightarrow \quad X'' = 0,$$

$$(4) \quad \Rightarrow \quad T'' = 0.$$

$\therefore$ The auxiliary equations are

$$m^2 = 0 \qquad\qquad m^2 = 0$$

$$m = 0, 0 \qquad\qquad m = 0, 0$$

$\therefore \quad X = A_9 x + A_{10} \qquad T = A_{11}t + A_{12}$

$\therefore \quad (2) \Rightarrow \quad y(x,t) = (A_9 x + A_{10})(A_{11}t + A_{12})$

Thus the various possible solutions of the wave equations are,

$$y(x,t) = (A_1 e^{px} + A_2 e^{-px})(A_3 e^{cpt} + A_4 e^{-cpt}) \tag{5}$$

$$y(x,t) = (A_5 \cos px + A_6 \sin px)\left(A_7 \cos cpt + A_8 \sin cpt\right) \tag{6}$$

$$y(x,t) = (A_9 x + A_{10})(A_{11}t + A_{12}) \tag{7}$$

4.1.3. NOTE:

In this problem the boundary conditions are fixed. (i.e.) $y(0,t) = 0,\ y(l,t) = 0$. If we apply these two conditions in (5) and (7) we get trivial solutions.

$\therefore$ The suitable solution is,

$$y(x,t) = (A_1 \cos px + A_2 \sin px)(A_3 \cos cpt + A_4 \sin cpt)$$

4.1.4. PROBLEMS:

TYPE I: ZERO INITIAL VELOCITY (VIBRATING STRING)

4.1.4.1. PROBLEM:

A string is stretched and fastened to two points x=0 and x=l apart. Motion is started by displacing the string in to the form $y = k(lx - x^2)$ from which it is released at time $t = 0$. Find the displacement of any point on the string at a distance of x from one end at time t.

Solution:

The wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}.$$

From the given problem, we get the following boundary conditions:

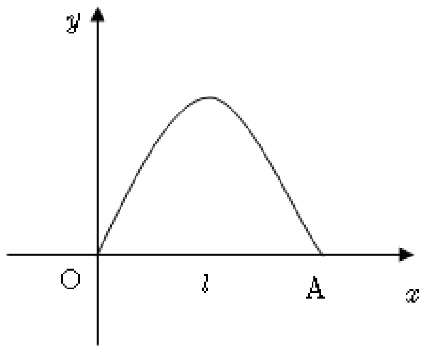

Fig. 4.2.

(i) $y(0,t) = 0, \quad \forall t > 0$

(ii) $y(l,t) = 0, \quad \forall t > 0$

(iii) $\dfrac{\partial y(x,0)}{\partial t} = 0 \quad (\because$ initial velocity zero)

(iv) $y(x,0) = k(lx - x^2), \quad 0 < x < l$

Now the suitable solution is,

$$y(x,t) = \left(c_1 \cos px + c_2 \sin px\right)\left(c_3 \cos pat + c_4 \sin pat\right) \qquad (1)$$

Applying condition (i) in (1), we get,

$$y(0,t) = c_1 \left(c_3 \cos pat + c_4 \sin pat\right) = 0$$

Here $c_3 \cos pat + c_4 \sin pat \neq 0 \Rightarrow c_1 = 0$

Sub. $c_1 = 0$ in (1), we get,

$$y(x,t) = c_2 \sin px \left(c_3 \cos pat + c_4 \sin pat\right) \qquad (2)$$

Applying condition (ii) in (2) we get,

$$y(l,t) = c_2 \sin pl \left(c_3 \cos pat + c_4 \sin pat\right) = 0$$

Here $c_3 \cos pat + c_4 \sin pat \neq 0$,

$\Rightarrow \quad c_2 = 0$ or $\sin pl = 0$.

Suppose if we take $c_2 = 0$ and already we have $c_1 = 0$ then we get trivial solution.

$\therefore \quad c_2 \neq 0 \quad \Rightarrow \quad \sin pl = 0$

Since $\sin n\pi = 0$

$\therefore \quad pl = n\pi \quad$ (n be an integer)

$$p = \frac{n\pi}{l}$$

Put $p = \frac{n\pi}{l}$ in (2) we get,

$$y(x,t) = c_2 \sin\frac{n\pi x}{l}\left(c_3\cos\frac{n\pi at}{l} + c_4\sin\frac{n\pi at}{l}\right) \quad (3)$$

Diff. (3) p.w.r. to 't' we get,

$$\frac{\partial y(x,t)}{\partial t} = c_2\sin\left(\frac{n\pi x}{l}\right)\left(-c_3\sin\left(\frac{n\pi at}{l}\right)\left(\frac{n\pi a}{l}\right) + c_4\cos\left(\frac{n\pi at}{l}\right)\left(\frac{n\pi a}{l}\right)\right)$$

Applying condition (iii), we get,

$$\frac{\partial y(x,0)}{\partial t} = c_2\sin\left(\frac{n\pi x}{l}\right)\left(0 + c_4\left(\frac{n\pi a}{l}\right)\right) = 0$$

Here $c_2 \neq 0,\ \sin\left(\frac{n\pi x}{l}\right) \neq 0,\ \frac{n\pi a}{l} \neq 0$

$\therefore \quad c_4 = 0.$

Sub. $c_4 = 0$ in (3) we get,

$$y(x,t) = c_2\sin\left(\frac{n\pi x}{l}\right)c_3\cos\left(\frac{n\pi at}{l}\right)$$

$$y(x,t) = c_n\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right), \quad \text{where } c_n = c_2c_3. \quad (4)$$

$\therefore$ The most general solution of (4) can be written as

$$y(x,t) = \sum_{n=1}^{\infty} c_n\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right) \quad (5)$$

Applying condition (iv) in (5) we get,

$$y(x,0) = \sum_{n=1}^{\infty} c_n\sin\left(\frac{n\pi x}{l}\right)(1) = k(lx - x^2) \quad (6)$$

To find c_n, expand $k(lx - x^2)$ in a half-range Fourier sine series in the interval $(0,l)$.

(i.e.) $$f(x) = k(lx - x^2) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \qquad (7)$$

$$\text{where } b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right) dx$$

From (6) and (7), we get,

$$c_n = b_n$$

$$\therefore \quad c_n = b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2k}{l}\int_0^l (lx - x^2)\sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2k}{l}\left[(lx - x^2)\left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}\right) - (l-2x)\left(\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right) + (-2)\left(\frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}\right)\right]_0^l$$

$$= \frac{2k}{l}\left[\left(0 + 0 - \frac{-2(-1)^n}{\left(\frac{n\pi}{l}\right)^3}\right) - \left(0 + 0 - \frac{2}{\left(\frac{n\pi}{l}\right)^3}\right)\right]$$

$$= \frac{2k}{l}\left[\frac{-2(-1)^n}{\left(\frac{n\pi}{l}\right)^3} + \frac{2}{\left(\frac{n\pi}{l}\right)^3}\right]$$

$$= \frac{2k}{l}\;\frac{l^3}{n^3\pi^3}\left[-2(-1)^n + 2\right]$$

$$= \frac{4kl^2}{n^3\pi^3}\left[1 - (-1)^n\right]$$

$$= \begin{cases} \dfrac{8kl^2}{n^3\pi^3}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$\therefore$ Sub. c_n in (5) we get,

$$y(x,t) = \sum_{n=\text{odd}}^{\infty} \frac{8kl^2}{n^3\pi^3} \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi at}{l}\right).$$

4.1.4.2. PROBLEM:

A string of length $2l$ is fastened at both ends. The mid point of the string is taken to a height 'b' and then released from rest in that position. Show that the displacement

$$y(x,t) = \frac{8b}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} \sin\left(\frac{(2n-1)\pi x}{2l}\right) \cos\left(\frac{(2n-1)\pi at}{2l}\right)$$

Solution:

The wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}.$$

From the given problem, we get the following boundary conditions:

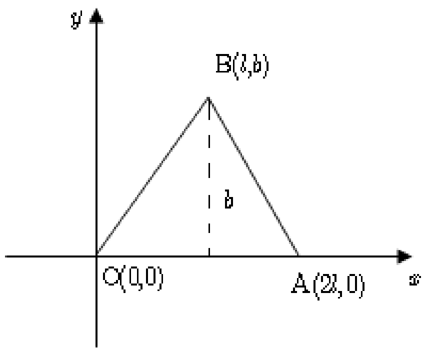

Fig. 4.3.

(i) $y(0,t) = 0, \quad \forall t > 0$

(ii) $y(2l,t) = 0, \quad \forall t > 0$

(iii) $\frac{\partial y(x,0)}{\partial t} = 0 \quad (\because \text{ initial velocity zero})$

Equation of OB is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \qquad \begin{array}{cc} (0,0) & (l,b) \\ (x_1,y_1) & (x_2,y_2) \end{array}$$

$$\frac{y-0}{b-0} = \frac{x-0}{l-0}$$

$$y = \frac{bx}{l}, \qquad 0 < x < l$$

Equation of BA is

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1} \qquad \begin{array}{cc}(l,b) & (2l,0)\\ (x_1,y_1) & (x_2,y_2)\end{array}$$

$$\frac{y-b}{0-b}=\frac{x-l}{2l-l}$$

$$y-b=\frac{x-l}{l}(-b)$$

$$y=b-\frac{xb}{l}+\frac{lb}{l}$$

$$=2b-\frac{xb}{l}$$

$$=\frac{2bl-xb}{l}$$

$$=\frac{b}{l}(2l-x), \qquad l<x<2l$$

$$\therefore \quad \text{(iv)} \quad y(x,0)=\begin{cases}\dfrac{bx}{l}, & 0<x<l\\ \dfrac{b}{l}(2l-x), & l<x<2l\end{cases}$$

Now the suitable solution is,

$$y(x,t)=(c_1\cos px+c_2\sin px)(c_3\cos pat+c_4\sin pat) \qquad (1)$$

Applying condition (i) in (1) we get,

$$y(0,t)=c_1\,(c_3\cos pat+c_4\sin pat)=0$$

Here $c_3\cos pat+c_4\sin pat\neq 0\Rightarrow c_1=0$

Sub. $c_1=0$ in (1), we get,

$$y(x,t)=c_2\sin px\,(c_3\cos pat+c_4\sin pat) \qquad (2)$$

Applying condition (ii) in (2) we get,

$$y(2l,t)=c_2\sin 2pl\,(c_3\cos pat+c_4\sin pat)=0$$

Here $c_3 \cos pat + c_4 \sin pat \neq 0$,

$\Rightarrow \quad c_2 = 0$ or $\sin 2pl = 0$.

Suppose if we take $c_2 = 0$ and already we have $c_1 = 0$ then we get trivial solution.

$\therefore \quad c_2 \neq 0 \quad \Rightarrow \quad \sin 2pl = 0$

Since $\sin n\pi = 0$

$\therefore \quad 2pl = n\pi \quad$ (n be an integer)

$$p = \frac{n\pi}{2l}$$

Put $p = \dfrac{n\pi}{2l}$ in (2) we get,

$$y(x,t) = c_2 \sin\left(\frac{n\pi x}{2l}\right)\left(c_3 \cos\frac{n\pi at}{2l} + c_4 \sin\frac{n\pi at}{2l}\right) \tag{3}$$

Diff. (3) p.w.r. to 't' we get,

$$\frac{\partial y(x,t)}{\partial t} = c_2 \sin\left(\frac{n\pi x}{2l}\right)\left(-c_3 \sin\left(\frac{n\pi at}{2l}\right)\left(\frac{n\pi a}{2l}\right) + c_4 \cos\left(\frac{n\pi at}{2l}\right)\left(\frac{n\pi a}{2l}\right)\right)$$

Applying condition (iii) we get,

$$\frac{\partial y(x,0)}{\partial t} = c_2 \sin\left(\frac{n\pi x}{2l}\right)\left(0 + c_4\left(\frac{n\pi a}{2l}\right)\right) = 0$$

Here $c_2 \neq 0$, $\sin\left(\dfrac{n\pi x}{2l}\right) \neq 0$, $\dfrac{n\pi a}{2l} \neq 0$

$\therefore \quad c_4 = 0$.

Sub. $c_4 = 0$ in (3) we get,

$$y(x,t) = c_2 \sin\left(\frac{n\pi x}{2l}\right) c_3 \cos\left(\frac{n\pi at}{2l}\right)$$

$$y(x,t) = c_n \sin\left(\frac{n\pi x}{2l}\right) \cos\left(\frac{n\pi at}{2l}\right), \qquad \text{where } c_n = c_2 c_3 \tag{4}$$

$\therefore$ The most general solution of (4) can be written as

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{2l}\right) \cos\left(\frac{n\pi at}{2l}\right) \tag{5}$$

Applying condition (iv) in (5) we get,

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{2l}\right) = \begin{cases} \dfrac{bx}{l}, & 0 < x < l \\ \dfrac{b}{l}(2l - x), & l < x < 2l \end{cases} = f(x) \text{ (say)} \tag{6}$$

To find c_n, expand $f(x)$ in a half-range Fourier sine series in the interval $(0, L)$, where $L=2l$.

$$\text{(i.e.)} \quad f(x) = \begin{cases} \dfrac{bx}{l}, & 0 < x < l \\ \dfrac{b}{l}(2l - x), & l < x < 2l \end{cases} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \tag{7}$$

$$\text{where } b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

From (6) and (7), we get,

$$c_n = b_n$$

$$\therefore \quad c_n = b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{2l}\int_0^{2l} f(x) \sin\left(\frac{n\pi x}{2l}\right) dx$$

$$= \frac{1}{l}\left[\int_0^{l} f(x) \sin\left(\frac{n\pi x}{2l}\right) dx + \int_l^{2l} f(x) \sin\left(\frac{n\pi x}{2l}\right) dx\right]$$

$$= \frac{1}{l}\left[\int_0^{l} \frac{bx}{l} \sin\left(\frac{n\pi x}{2l}\right) dx + \int_l^{2l} \frac{b}{l}(2l - x) \sin\left(\frac{n\pi x}{2l}\right) dx\right]$$

$$= \frac{b}{l^2}\int_0^{l} x \sin\left(\frac{n\pi x}{2l}\right) dx + \frac{b}{l^2}\int_l^{2l} (2l - x) \sin\left(\frac{n\pi x}{2l}\right) dx$$

$$= \frac{b}{l^2}\left[x\,\frac{-\cos\left(\frac{n\pi x}{2l}\right)}{\left(\frac{n\pi}{2l}\right)} - (1)\frac{-\sin\left(\frac{n\pi x}{2l}\right)}{\left(\frac{n\pi}{2l}\right)^2}\right]_0^l$$

$$+ \frac{b}{l^2}\left[(2l-x)\,\frac{-\cos\left(\frac{n\pi x}{2l}\right)}{\left(\frac{n\pi}{2l}\right)} - (-1)\frac{-\sin\left(\frac{n\pi x}{2l}\right)}{\left(\frac{n\pi}{2l}\right)^2}\right]_l^{2l}$$

$$= \frac{b}{l^2}\left[\left(-l\,\frac{\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2l}\right)} + \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2l}\right)^2}\right) - (0+0)\right]$$

$$+ \frac{b}{l^2}\left[(0-0) - \left((-l)\,\frac{\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2l}\right)} - \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2l}\right)^2}\right)\right]$$

$$= \frac{b}{l^2}\left[\left(\frac{-l\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2l}\right)} + \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2l}\right)^2} + \frac{l\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2l}\right)} + \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2l}\right)^2}\right)\right]$$

$$= \frac{b}{l^2}\left[2\,\frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2l}\right)^2}\right]$$

$$= \frac{b}{l^2}\,2\,\sin\left(\frac{n\pi}{2}\right)\frac{4l^2}{n^2\pi^2}$$

$$= \frac{8b}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$= \begin{cases} \frac{8b}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right), & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$\therefore \qquad y(x,t) = \sum_{n=\text{odd}}^{\infty} \frac{8b}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{2l}\right)\cos\left(\frac{n\pi at}{2l}\right)$$

$$= \frac{8b}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{(2n-1)\pi}{2}\right)}{(2n-1)^2} \sin\left(\frac{(2n-1)\pi x}{2l}\right) \cos\left(\frac{(2n-1)\pi at}{2l}\right)$$

$$y(x,t) = \frac{8b}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} \sin\left(\frac{(2n-1)\pi x}{2l}\right) \cos\left(\frac{(2n-1)\pi at}{2l}\right)$$

NOTE:

$$\sin\left(\frac{\pi}{2}\right) = 1,\ \ \sin\left(\frac{3\pi}{2}\right) = -1,\ \ \sin\left(\frac{5\pi}{2}\right) = 1, \ldots, \sin\left(\frac{(2n-1)\pi}{2}\right) = (-1)^{n-1}$$

4.1.4.3. PROBLEM:

A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially displaced in a sinusoidal are of length y_0 and then released from rest. Find the displacement y at any distance x from one end at time t.

Solution:

The wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}.$$

From the given problem, we get the following boundary conditions:

(i) $y(0,t) = 0, \quad \forall t > 0$

(ii) $y(l,t) = 0, \quad \forall t > 0$

(iii) $\frac{\partial y(x,0)}{\partial t} = 0 \quad (\because$ initial velocity zero)

(iv) $y(x,0) = f(x) = y_0 \sin\left(\frac{\pi x}{l}\right), \qquad 0 < x < l$

Now the suitable solution is,

$$y(x,t) = \left(c_1 \cos px + c_2 \sin px\right)\left(c_3 \cos pat + c_4 \sin pat\right) \qquad (1)$$

Applying condition (i) in (1), we get,

$$y(0,t) = c_1 \left(c_3 \cos pat + c_4 \sin pat\right) = 0$$

Here $c_3 \cos pat + c_4 \sin pat \neq 0 \Rightarrow c_1 = 0$

Sub. $c_1 = 0$ in (1), we get,

$$y(x,t) = c_2 \sin px \left(c_3 \cos pat + c_4 \sin pat\right) \qquad (2)$$

Applying condition (ii) in (2) we get,

$$y(l,t) = c_2 \sin pl \left(c_3 \cos pat + c_4 \sin pat\right) = 0$$

Here $c_3 \cos pat + c_4 \sin pat \neq 0$,

$\Rightarrow \qquad c_2 = 0$ or $\sin pl = 0$.

Suppose we take $c_2 = 0$ and already we have $c_1 = 0$ then we get trivial solution.

$\therefore \qquad c_2 \neq 0 \qquad \Rightarrow \qquad \sin pl = 0$

Since $\sin n\pi = 0$

$\therefore \qquad pl = n\pi \qquad$ (n be an integer)

$$p = \frac{n\pi}{l}$$

Put $p = \dfrac{n\pi}{l}$ in (2) we get,

$$y(x,t) = c_2 \sin \frac{n\pi x}{l}\left(c_3 \cos \frac{n\pi at}{l} + c_4 \sin \frac{n\pi at}{l}\right) \qquad (3)$$

Diff. (3) p.w.r. to 't' we get,

$$\frac{\partial y(x,t)}{\partial t} = c_2 \sin\left(\frac{n\pi x}{l}\right)\left(-c_3 \sin\left(\frac{n\pi at}{l}\right)\left(\frac{n\pi a}{l}\right) + c_4 \cos\left(\frac{n\pi at}{l}\right)\left(\frac{n\pi a}{l}\right)\right)$$

Applying condition (iii), we get,

$$\frac{\partial y(x,0)}{\partial t} = c_2 \sin\left(\frac{n\pi x}{l}\right)\left(0 + c_4 \left(\frac{n\pi a}{l}\right)\right) = 0$$

Here $c_2 \neq 0,\ \sin\left(\frac{n\pi x}{l}\right) \neq 0,\ \frac{n\pi a}{l} \neq 0$

$\therefore \qquad c_4 = 0.$

Sub. $c_4 = 0$ in (3) we get,

$$y(x,t) = c_2 \sin\left(\frac{n\pi x}{l}\right) c_3 \cos\left(\frac{n\pi at}{l}\right)$$

$$y(x,t) = c_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi at}{l}\right), \qquad \text{where } c_n = c_2 c_3. \tag{4}$$

$\therefore$ The most general solution of (4) can be written as

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi at}{l}\right) \tag{5}$$

Applying condition (iv) in (5) we get,

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) = y_0 \sin\left(\frac{\pi x}{l}\right)$$

To find c_n we expand, we get,

$$c_1 \sin\left(\frac{\pi x}{l}\right) + c_2 \sin\left(\frac{2\pi x}{l}\right) + c_3 \sin\left(\frac{3\pi x}{l}\right) + \cdots = y_0 \sin\left(\frac{\pi x}{l}\right)$$

Equating the coefficients, we get,

$$c_1 = y_0,\ c_2 = 0, \cdots = 0$$

Sub. $c_1 = y_0$ in (5) we get,

$$y(x,t) = y_0 \sin\left(\frac{\pi x}{l}\right) \cos\left(\frac{\pi at}{l}\right).$$

4.1.5. PROBLEMS:

TYPE II: NON-ZERO INITIAL VELOCITY (VIBRATING STRING)

4.1.5.1. PROBLEM:

A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set vibrating string giving each point a velocity $\lambda x(l-x)$. Show that displacement

$$y(x,t) = \frac{8\lambda l^3}{a\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\sin\left(\frac{(2n-1)\pi x}{l}\right)\cos\left(\frac{(2n-1)\pi at}{l}\right).$$

Solution:

The wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}.$$

From the given problem, we get the following boundary conditions:

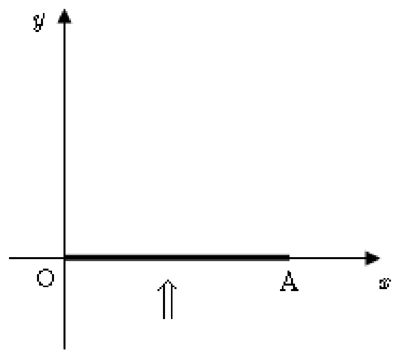

Fig. 4.4.

(i) $y(0,t) = 0, \quad \forall t > 0$

(ii) $y(l,t) = 0, \quad \forall t > 0$

(iii) $y(x,0) = 0, \quad 0 < x < l$

(iv) $\dfrac{\partial y(x,0)}{\partial t} = \lambda x(l-x)$

Now the suitable solution is,

$$y(x,t) = (c_1\cos px + c_2\sin px)(c_3\cos pat + c_4\sin pat) \qquad (1)$$

Applying condition (i) in (1), we get,

$$y(0,t) = c_1\,(c_3\cos pat + c_4\sin pat) = 0$$

Here $c_3\cos pat + c_4\sin pat \neq 0 \Rightarrow c_1 = 0$

Sub. $c_1 = 0$ in (1), we get,

$$y(x,t) = c_2\sin px\,(c_3\cos pat + c_4\sin pat) \qquad (2)$$

Applying condition (ii) in (2) we get,

$$y(l,t) = c_2 \sin pl \left(c_3 \cos pat + c_4 \sin pat\right) = 0$$

Here $c_3 \cos pat + c_4 \sin pat \neq 0$,

$$\Rightarrow \qquad c_2 = 0 \text{ or } \sin pl = 0.$$

Suppose we take $c_2 = 0$ and already we have $c_1 = 0$ then we get trivial solution.

$$\therefore \quad c_2 \neq 0 \Rightarrow \quad \sin pl = 0$$

Since $\sin n\pi = 0$

$$\therefore \qquad pl = n\pi \qquad (n \text{ be an integer})$$

$$p = \frac{n\pi}{l}$$

Put $p = \frac{n\pi}{l}$ in (2) we get,

$$y(x,t) = c_2 \sin \frac{n\pi x}{l}\left(c_3 \cos \frac{n\pi at}{l} + c_4 \sin \frac{n\pi at}{l}\right) \tag{3}$$

Applying condition (iii) in (3) we get,

$$y(x,0) = c_2 \sin\left(\frac{n\pi x}{l}\right)\left(c_3 + 0\right) = 0$$

$$c_2 c_3 \sin\left(\frac{n\pi x}{l}\right) = 0$$

Here $c_2 \neq 0, \sin\left(\frac{n\pi x}{l}\right) \neq 0 \Rightarrow c_3 = 0$

Sub. $c_3 = 0$ in (3) we get,

$$y(x,t) = c_2 \sin\left(\frac{n\pi x}{l}\right) c_4 \sin\left(\frac{n\pi at}{l}\right)$$

$$= c_n \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi at}{l}\right), \qquad \text{where } c_n = c_2 c_4 \tag{4}$$

The most general solution of (4) can be written as

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi at}{l}\right) \quad (5)$$

Diff. (5) p.w.r. to t we get,

$$\frac{\partial y(x,t)}{\partial t} = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi at}{l}\right) . \left(\frac{n\pi a}{l}\right)$$

Applying (iv) we get

$$\frac{\partial y(x,0)}{\partial t} = \sum_{n=1}^{\infty} c_n \left(\frac{n\pi a}{l}\right) \sin\left(\frac{n\pi x}{l}\right) = \lambda x(l-x)$$

(i.e.) $$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) = \lambda x(l-x) \quad (6)$$

where $B_n = \frac{n\pi a}{l} . c_n$

To find B_n, expand $\lambda x(l-x)$ in a half-range Fourier sine series in the interval $(0,l)$.

(i.e.) $$f(x) = \lambda x(l-x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad (7)$$

where $b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

From (6) and (7), we get,

$$B_n = b_n$$

$$\therefore B_n = b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2\lambda}{l}\int_0^l (lx - x^2) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2\lambda}{l}\left[(lx - x^2)\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} - (l-2x)\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2} + (-2)\frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}\right]_0^l$$

$$= \frac{2\lambda}{l}\left[\left(0+0-\frac{2(-1)^n}{\left(\frac{n\pi}{l}\right)^3}\right)-\left(0-0-\frac{2}{\left(\frac{n\pi}{l}\right)^3}\right)\right]$$

$$= \frac{2\lambda}{l}\left[\frac{-2(-1)^n}{\left(\frac{n\pi}{l}\right)^3}+\frac{2}{\left(\frac{n\pi}{l}\right)^3}\right]$$

$$= \frac{4\lambda}{l}\left(\frac{l^3}{n^3\pi^3}\right)\left(1-(-1)^n\right)$$

$$= \frac{4\lambda l^2}{n^3\pi^3}\left(1-(-1)^n\right)$$

Since

$$c_n = \frac{l}{n\pi a}.\, B_n$$

$$= \frac{l}{n\pi a}\cdot\frac{4\lambda l^2}{n^3\pi^3}\left(1-(-1)^n\right)$$

$$= \frac{4\lambda l^3}{an^4\pi^4}\left(1-(-1)^n\right)$$

$$= \begin{cases} \dfrac{8\lambda l^3}{an^4\pi^4}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Sub. c_n in (5) we get,

$$y(x,t) = \sum_{n=\text{odd}}^{\infty}\frac{8\lambda l^3}{an^4\pi^4}\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi at}{l}\right)$$

$$y(x,t) = \frac{8\lambda l^3}{a\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\sin\left(\frac{(2n-1)\pi x}{l}\right)\cos\left(\frac{(2n-1)\pi at}{l}\right).$$

4.1.5.2. PROBLEM:

A string of length l is initially at rest in its equilibrium position and motion is started by giving each of its points a velocity given by

$$v = \begin{cases} cx, & \text{if } 0 \le x \le \frac{l}{2} \\ c(l-x), & \text{if } \frac{l}{2} \le x \le l \end{cases}$$. Find the displacement function $y(x,t)$.

Solution:

The wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}.$$

From the given problem, we get the following boundary conditions:

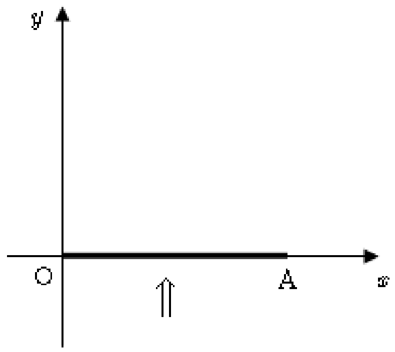

Fig. 4.5.

(i) $y(0,t) = 0, \quad \forall t > 0$

(ii) $y(l,t) = 0, \quad \forall t > 0$

(iii) $y(x,0) = 0, \quad 0 < x < l$

(iv) $$\frac{\partial y(x,0)}{\partial t} = f(x) = \begin{cases} cx, & 0 \le x \le \frac{l}{2} \\ c(l-x), & \frac{l}{2} \le x \le l \end{cases}$$

Now the suitable solution is,

$$y(x,t) = \left(c_1 \cos px + c_2 \sin px\right)\left(c_3 \cos pat + c_4 \sin pat\right) \quad (1)$$

Applying condition (i) in (1), we get,

$$y(0,t) = c_1 \left(c_3 \cos pat + c_4 \sin pat\right) = 0$$

Here $c_3 \cos pat + c_4 \sin pat \ne 0 \Rightarrow c_1 = 0$

Sub. $c_1 = 0$ in (1), we get,

$$y(x,t) = c_2 \sin px\left(c_3 \cos pat + c_4 \sin pat\right) \qquad (2)$$

Applying condition (ii) in (2) we get,

$$y(l,t) = c_2 \sin pl\left(c_3 \cos pat + c_4 \sin pat\right) = 0$$

Here $c_3 \cos pat + c_4 \sin pat \neq 0$,

$\Rightarrow \qquad c_2 = 0$ or $\sin pl = 0$.

Suppose we take $c_2 = 0$ and already we have $c_1 = 0$ then we get trivial solution.

$\therefore \qquad \sin pl = 0$

Since $\sin n\pi = 0$

$\therefore \quad c_2 \neq 0 \ \Rightarrow \quad pl = n\pi \quad$ (n be an integer)

$$p = \frac{n\pi}{l}$$

Put $p = \frac{n\pi}{l}$ in (2) we get,

$$y(x,t) = c_2 \sin \frac{n\pi x}{l}\left(c_3 \cos \frac{n\pi at}{l} + c_4 \sin \frac{n\pi at}{l}\right) \qquad (3)$$

Applying condition (iii) in (3) we get,

$$y(x,0) = c_2 \sin\left(\frac{n\pi x}{l}\right)\left(c_3 + 0\right) = 0$$

$$c_2 c_3 \sin\left(\frac{n\pi x}{l}\right) = 0$$

Here $c_2 \neq 0, \sin\left(\frac{n\pi x}{l}\right) \neq 0 \ \Rightarrow c_3 = 0$

Sub. $c_3 = 0$ in (3) we get,

$$y(x,t) = c_2 \sin\left(\frac{n\pi x}{l}\right) c_4 \sin\left(\frac{n\pi at}{l}\right)$$

$$= c_n \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi at}{l}\right), \qquad (4)$$

where $c_n = c_2 c_4$

The most general solution of (4) can be written as

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi at}{l}\right) \tag{5}$$

Diff. (5) p.w.r. to t we get,

$$\frac{\partial y(x,t)}{\partial t} = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi at}{l}\right).\left(\frac{n\pi a}{l}\right)$$

Applying (iv) we get

$$\frac{\partial y(x,0)}{\partial t} = \sum_{n=1}^{\infty} c_n \left(\frac{n\pi a}{l}\right) \sin\left(\frac{n\pi x}{l}\right) = f(x) = \begin{cases} cx, & 0 \le x \le \frac{l}{2} \\ c(l-x), & \frac{l}{2} \le x \le l \end{cases}$$

(i.e.) $$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) = f(x) \tag{6}$$

where $B_n = \frac{n\pi a}{l}.\, c_n$

To find B_n, expand $f(x)$ in a half-range Fourier sine series in the interval $(0,l)$.

(i.e.) $$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \tag{7}$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right) dx$

From (6) and (7), we get,

$$B_n = b_n$$

$$\therefore B_n = b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{l}\left[\int_0^{\frac{l}{2}} f(x)\sin\left(\frac{n\pi x}{l}\right)dx + \int_{\frac{l}{2}}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{2}{l}\left[\int_0^{\frac{l}{2}} cx\sin\left(\frac{n\pi x}{l}\right)dx + \int_{\frac{l}{2}}^{l} c(l-x)\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{2c}{l}\left[x\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} - (1)\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right]_0^{\frac{l}{2}}$$

$$+\frac{2c}{l}\left[(l-x)\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} - (-1)\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right]_{\frac{l}{2}}^{l}$$

$$= \frac{2c}{l}\left[\left(-\frac{l}{2}\frac{\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{l}\right)} + \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{l}\right)^2}\right) - (0+0)\right]$$

$$+\frac{2c}{l}\left[(0-0) - \left(-\frac{l}{2}\frac{\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{l}\right)} - \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{l}\right)^2}\right)\right]$$

$$= \frac{2c}{l}\left(-\frac{l}{2}\frac{\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{l}\right)} + \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{l}\right)^2} + \frac{l}{2}\frac{\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{l}\right)} + \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{l}\right)^2}\right)$$

$$= \frac{2c}{l}2\sin\left(\frac{n\pi}{2}\right)\ \frac{l^2}{n^2\pi^2}$$

$$B_n = \frac{4cl}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

Since

$$c_n = \frac{l}{n\pi a}.\, B_n$$

$$= \frac{l}{n\pi a}.\, \frac{4cl}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$c_n = \frac{4cl^2}{an^3\pi^3}\sin\left(\frac{n\pi}{2}\right)$$

$\therefore$ The general solution is,

$$y(x,t) = \sum_{n=1}^{\infty}\frac{4cl^2}{an^3\pi^3}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi at}{l}\right)$$

$$y(x,t) = \frac{4cl^2}{a\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi at}{l}\right).$$

4.2. ONE DIMENSIONAL HEAT EQUATION

4.2.1. INTRODUCTION:

The heat equation is $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$

where $c^2 = \frac{\text{Thermal conductivity}}{(\text{density})(\text{specific heat})}$.

Here u is a function of x and t.

4.2.2. SOLUTION OF HEAT EQUATION:

The various possible solutions of the heat equation are

(i) $u(x,t) = \left(A_1e^{px} + A_2e^{-px}\right)\left(A_3e^{c^2p^2t}\right)$

(ii) $u(x,t) = \left(A_4\cos px + A_5\sin px\right)\left(A_6e^{-c^2p^2t}\right)$

(iii) $u(x,t) = \left(A_7x + A_8\right)A_9$.

4.2.3. NOTE:

The suitable solution of heat equation is

$$u(x,t) = (A\cos px + B\sin px)\,e^{-c^2p^2t}.$$

4.2.4. PROBLEMS:

TYPE: I: ZERO BOUNDARY VALUES: (HEAT FLOW)

4.2.4.1. PROBLEM:

A rod of length l with insulated sides is initially at a uniform temperature u_0. Its ends are suddenly cooled to $0^\circ c$ and are kept at the temperature. Find the temperature function $u(x,t)$.

Solution:

The temperature function $u(x,t)$ satisfies the one dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

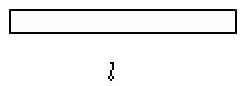

Fig. 4.6.

From the given problem we get the following boundary conditions:

(i) $u(0,t) = 0, \;\; \forall\, t \ge 0$

(ii) $u(l,t) = 0, \;\; \forall\, t \ge 0$

(iii) $u(x,0) = u_0$

Now the suitable solution is

$$u(x,t) = (A\cos px + B\sin px)\,e^{-c^2p^2t} \qquad (1)$$

Applying condition (i) in (1) we get,

$$u(0,t) = (A+0)\,e^{-c^2p^2t} = 0$$

Here $e^{-c^2p^2t} \neq 0 \;\Rightarrow A = 0$

$$\therefore (1) \;\Rightarrow\; u(x,t) = B\sin px\, e^{-c^2p^2t} \tag{2}$$

Applying condition (ii) in (2) we get,

$$u(l,t) = B\sin pl\, e^{-c^2p^2t} = 0$$

Here $e^{-c^2p^2t} \neq 0$, $\therefore$ either $B = 0$ (or) $\sin pl = 0$.

If we take $B = 0$ and already we have $A = 0$, we get trivial solution.

$\therefore\; B \neq 0 \Rightarrow \sin pl = 0.$

Since $\sin n\pi = 0$

$$\Rightarrow pl = n\pi$$

$$p = \frac{n\pi}{l}.$$

Put $p = \dfrac{n\pi}{l}$ in (2) we get,

$$u(x,t) = B\sin\left(\frac{n\pi x}{l}\right) e^{-c^2\frac{n^2\pi^2}{l^2}t} \tag{3}$$

The most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) e^{-c^2\frac{n^2\pi^2}{l^2}t} \tag{4}$$

Applying condition (iii) in (4) we get,

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) = u_0 \tag{5}$$

To find B_n, we expand u_0 in a half-range Fourier sine series in the inverval $(0,l)$,

(i.e.) $$u_0 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \tag{6}$$

$$\text{where } b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

From (5) and (6) we have,

$$B_n = b_n$$

$$\therefore B_n = b_n = \frac{2}{l}\int_0^l u_0 \sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2u_0}{l}\left[\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}\right]_0^l$$

$$= \frac{2u_0}{l}\left[\frac{-(-1)^n}{\left(\frac{n\pi}{l}\right)} - \frac{1}{\left(\frac{n\pi}{l}\right)}\right]$$

$$= \frac{2u_0}{l}\frac{l}{n\pi}\left(1-(-1)^n\right)$$

$$= \begin{cases} \frac{4u_0}{n\pi}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$\therefore$ The most general solution is

$$u(x,t) = \sum_{n=\text{odd}}^{\infty} \frac{4u_0}{n\pi}\sin\left(\frac{n\pi x}{l}\right) e^{-c^2\frac{n^2\pi^2}{l^2}t}$$

$$u(x,t) = \frac{4u_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\left(\frac{(2n-1)\pi x}{l}\right) e^{-c^2\frac{(2n-1)^2\pi^2}{l^2}t}.$$

4.2.4.2. PROBLEM:

A homogeneous rod of conducting material of length l has its ends kept at zero temperature. The temperature at the centre is T and falls uniformly to zero at the two ends. Find the temperature function $u(x,t)$.

Solution:

The temperature function $u(x,t)$ satisfies the one dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

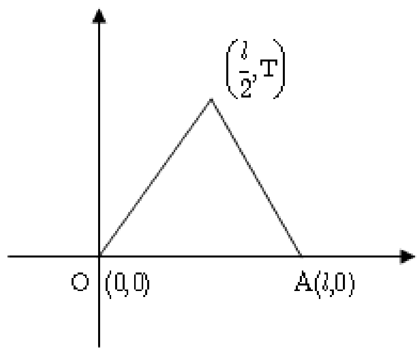

Fig. 4.7.

From the given problem we get the following boundary conditions:

(i) $u(0,t) = 0, \quad \forall\ t \geq 0$

(ii) $u(l,t) = 0, \quad \forall\ t \geq 0$

(iii) $u(x,0) = ?$

Equation of OB is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \qquad \begin{matrix} (0,0) & \left(\frac{l}{2}, T\right) \\ (x_1, y_1) & (x_2, y_2) \end{matrix}$$

$$\frac{y - 0}{T - 0} = \frac{x - 0}{\frac{l}{2} - 0}$$

$$\frac{y}{T} = \frac{2x}{l}$$

$$y = \frac{2Tx}{l}, \qquad 0 \leq x \leq \frac{l}{2}$$

Equation of BA is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \qquad \begin{matrix} \left(\frac{l}{2}, T\right) & (l,0) \\ (x_1, y_1) & (x_2, y_2) \end{matrix}$$

$$\frac{y - T}{0 - T} = \frac{x - \frac{l}{2}}{l - \frac{l}{2}}$$

$$\frac{y-T}{-T} = \frac{2x-l}{l}$$

$$y - T = \frac{-2Tx}{l} + T$$

$$y = 2T - \frac{2Tx}{l}$$

$$y = \frac{2Tl - 2Tx}{l}$$

$$y = \frac{2T}{l}(l-x), \quad \frac{l}{2} \le x \le l$$

$$\therefore \text{ (iii)} \quad u(x,0) = \begin{cases} \dfrac{2Tx}{l}, & 0 \le x \le \dfrac{l}{2} \\ \dfrac{2T}{l}(l-x), & \dfrac{l}{2} \le x \le l \end{cases}$$

Now the suitable solution is

$$u(x,t) = (A\cos px + B\sin px)\, e^{-c^2p^2t} \qquad (1)$$

Applying condition (i) in (1) we get,

$$u(0,t) = (A+0)\, e^{-c^2p^2t} = 0$$

Here $e^{-c^2p^2t} \ne 0 \;\Rightarrow A = 0$

$$\therefore \quad (1) \quad \Rightarrow \quad u(x,t) = B\sin px\; e^{-c^2p^2t} \qquad (2)$$

Applying condition (ii) in (2) we get,

$$u(l,t) = B\sin pl\; e^{-c^2p^2t} = 0$$

Here $e^{-c^2p^2t} \ne 0$, $\therefore$ either $B = 0$ (or) $\sin pl = 0$.

If we take $B = 0$ and already we have $A = 0$, we get trivial solution.

$\therefore \; B \ne 0 \Rightarrow \sin pl = 0.$

Since $\sin n\pi = 0$

$\Rightarrow \quad pl = n\pi$

$$p = \frac{n\pi}{l}.$$

Put $p = \frac{n\pi}{l}$ in (2) we get,

$$u(x,t) = B\sin\left(\frac{n\pi x}{l}\right) e^{-c^2 \frac{n^2\pi^2}{l^2} t} \tag{3}$$

The most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) e^{-c^2 \frac{n^2\pi^2}{l^2} t} \tag{4}$$

Applying condition (iii) in (4) we get,

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) = f(x) = \begin{cases} \frac{2Tx}{l}, & 0 \le x \le \frac{l}{2} \\ \frac{2T}{l}(l-x), & \frac{l}{2} \le x \le l \end{cases} \tag{5}$$

To find B_n, we expand $f(x)$ in a half-range Fourier sine series in the interval $(0,l)$,

(i.e.) $$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \tag{6}$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right) dx$

From (5) and (6) we have,

$$B_n = b_n$$

$$\therefore \quad B_n = b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{l}\int_0^{\frac{l}{2}} \frac{2Tx}{l} \sin\left(\frac{n\pi x}{l}\right) dx + \frac{2}{l}\int_{\frac{l}{2}}^{l} \frac{2T}{l}(l-x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{4T}{l^2}\left[x \frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} - (1)\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2} \right]_0^{\frac{l}{2}}$$

$$+ \frac{4T}{l^2}\left[(l-x) \frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} - (-1)\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2} \right]_{\frac{l}{2}}^{l}$$

$$= \frac{4T}{l^2}\left[\left(\frac{-l}{2}\frac{\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{l}\right)} + \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{l}\right)^2} \right) - (0+0) \right]$$

$$+ \frac{4T}{l^2}\left[(0+0) - \left(\frac{-l}{2}\frac{\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{l}\right)} - \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{l}\right)^2} \right) \right]$$

$$= \frac{4T}{l^2}\left(\frac{2\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{l}\right)^2} \right)$$

$$= \frac{8T}{l^2}\frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$= \frac{8T}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$\therefore$ The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8T}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{l}\right)\; e^{-c^2\frac{n^2\pi^2}{l^2}t}$$

$$u(x,t) = \frac{8T}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\;\sin\left(\frac{n\pi x}{l}\right)\; e^{-c^2\frac{n^2\pi^2}{l^2}t}.$$

NOTE 1:

In unsteady state, the temperature at any point of the body depends on the position of the point and also the time 't'.

In steady state, the temperature at any point depends only on the position of the point and is independent of the time 't'.

NOTE 2:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

In steady state condition, the temperature 'u' depends only on 'x' and not on time 't'.

$\therefore \qquad \dfrac{\partial u}{\partial t} = 0$

$\therefore \qquad (1) \Rightarrow \dfrac{\partial^2 u}{\partial x^2} = 0$

$\Rightarrow \qquad u = ax + b$, where a and b are constants..

NOTE 3: THERMALLY INSULATED ENDS:

If an end of heat conducting body is thermally insulated it means that no heat passing through the section.

4.2.4.3. PROBLEM:

A rod 30 cm long has its ends A and B kept at $20^\circ c$ and $80^\circ c$ respectively until steady state conditions prevail. The temperature at each end is then suddenly reduced to $0^\circ c$ and kept so. Find the resulting temperature function $u(x,t)$, taking $x = 0$ at A.

Solution:

The temperature function $u(x,t)$ is the solution of the one dimensional heat equation

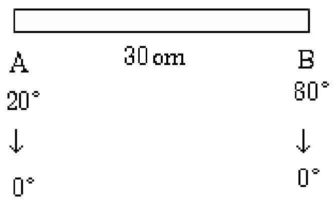

Fig. 4.8.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

When the steady state condition prevails, $\frac{\partial u}{\partial t} = 0$.

$$\therefore \qquad \frac{\partial^2 u}{\partial x^2} = 0$$

We integrating, $\quad u(x) = ax + b$

When $x = 0$, $u(0) = b$

$$20 = b$$

When $x = 30$, $u(30) = 30a + b$

$$80 = 30a + 20$$

$$30a = 60$$

$$a = 2$$

$$\therefore \quad u(x,0) = f(x) = 2x + 20.$$

Hence the conditions are,

(i) $\quad u(0,t) = 0, \quad \forall\ t \geq 0$

(ii) $u(30,t) = 0, \quad \forall\ t \geq 0$

(iii) $u(x,0) = 2x + 20$.

Now the suitable solution is

$$u(x,t) = (A\cos px + B\sin px)\, e^{-c^2p^2t} \tag{1}$$

Applying condition (i) in (1) we get,

$$u(0,t) = (A+0)\, e^{-c^2p^2t} = 0$$

Here $e^{-c^2p^2t} \neq 0 \;\Rightarrow A = 0$

$$\therefore\ (1) \;\Rightarrow\; u(x,t) = B\sin px\ e^{-c^2p^2t} \tag{2}$$

Applying condition (ii) in (2) we get,

$$u(30,t) = B\sin 30p\ e^{-c^2p^2t} = 0$$

Here $e^{-c^2p^2t} \neq 0$, $\therefore$ either $B = 0$ (or) $\sin 30p = 0$.

If we take $B = 0$ and already we have $A = 0$, we get trivial solution.

$\therefore\ B \neq 0 \quad \Rightarrow \qquad \sin 30p = 0$

Since $\sin n\pi = 0$

$\Rightarrow \qquad 30p = n\pi$

$$p = \frac{n\pi}{30}.$$

Put $p = \dfrac{n\pi}{30}$ in (2) we get,

$$u(x,t) = B\sin\left(\frac{n\pi x}{30}\right) e^{-c^2\frac{n^2\pi^2}{30^2}t} \tag{3}$$

The most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{30}\right) e^{-c^2\frac{n^2\pi^2}{30^2}t} \tag{4}$$

Applying condition (iii) in (4) we get,

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{30}\right) = 2x + 20 \tag{5}$$

To find B_n, we expand $f(x)$ in a half-range Fourier sine series in the interval $(0,30)$,

(i.e.) $$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{30}\right) \tag{6}$$

where $b_n = \frac{2}{30}\int_0^{30} f(x)\sin\left(\frac{n\pi x}{30}\right) dx$

From (5) and (6) we have,

$$B_n = b_n$$

$$\therefore \quad B_n = b_n = \frac{2}{30}\int_0^{30} f(x) \sin\left(\frac{n\pi x}{30}\right) dx$$

$$= \frac{1}{15}\int_0^{30} (2x+20) \sin\left(\frac{n\pi x}{30}\right) dx$$

$$= \frac{1}{15}\left[(2x+20)\frac{-\cos\left(\frac{n\pi x}{30}\right)}{\left(\frac{n\pi}{30}\right)} - (2)\frac{-\sin\left(\frac{n\pi x}{30}\right)}{\left(\frac{n\pi}{30}\right)^2}\right]_0^{30}$$

$$= \frac{1}{15}\left[\left((60+20)\frac{-(-1)^n}{\left(\frac{n\pi}{30}\right)} + (2)(0)\right) - \left(\frac{-20}{\left(\frac{n\pi}{30}\right)} + 0\right)\right]$$

$$= \frac{1}{15}\left(-80\frac{(-1)^n}{\left(\frac{n\pi}{30}\right)} + \frac{20}{\left(\frac{n\pi}{30}\right)}\right)$$

$$= \frac{1}{15} 20 \frac{30}{n\pi}\left(1-4(-1)^n\right)$$

$$b_n = \frac{40}{n\pi}\left(1-4(-1)^n\right)$$

$\therefore$ The general solution is,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{40}{n\pi}\left(1-4(-1)^n\right)\sin\left(\frac{n\pi x}{30}\right) e^{-c^2\frac{n^2\pi^2}{30^2}t}$$

$$u(x,t) = \frac{40}{\pi}\sum_{n=1}^{\infty} \frac{\left(1-4(-1)^n\right)}{n}\sin\left(\frac{n\pi x}{30}\right) e^{-c^2\frac{n^2\pi^2}{30^2}t}.$$

4.2.5. PROBLEMS:

TYPE II: (NON-ZERO BOUNDARY VALUES)

4.2.5.1. PROBLEM:

A bar 10 cm long with insulated sides has its ends A and B kept at $20^\circ c$ and $40^\circ c$ respectively until steady state condition prevail. The temperature at A is then suddenly raised to $50^\circ c$ and at the same time at B is lowered to $10^\circ c$ and maintained there after. Find the subsequent temperature distribution in the bar.

Solution:

The temperature function $u(x,t)$ is the solution of the one dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$$

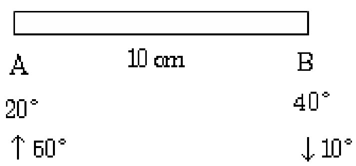

Fig. 4.9.

When the steady state condition prevails, $\frac{\partial u}{\partial t} = 0$.

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = 0$$

We integrating, $\quad u(x) = ax + b$

When $x = 0$, $\quad u(0) = b$

$$20 = b$$

When $x = 10$, $u(10) = 10a + b$

$$40 = 10a + 20$$

$$a = 2$$

$\therefore \quad u(x) = 2x + 20$.

Hence the boundary conditions are

(i) $u(0,t) = 50, \ \forall\, t \geq 0$

(ii) $u(10,t) = 10, \ \forall\, t \geq 0$

(iii) $u(x,0) = 2x + 20$

Now the suitable solution is

$$u(x,t) = (A\cos px + B\sin px)\, e^{-c^2p^2t} \tag{1}$$

Applying condition (i) in (1) we get,

$$u(0,t) = A\, e^{-c^2p^2t} = 50 \tag{2}$$

Applying condition (ii) in (2) we get,

$$u(10,t) = (A\cos 10p + B\sin 10p)\, e^{-c^2p^2t} = 10 \tag{3}$$

From (2) and (3) it is not possible to find the constants A and B, since we have infinite number of values for A and B. Therefore in this case, we split the solution $u(x,t)$ into two parts.

(i.e.) $$u(x,t) = u_S(x) + u_T(x,t) \tag{4}$$

Where $u_S(x)$ is a solution of the equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ and is a function of x alone satisfying the conditions

$$u_S(0) = 50$$

$$u_S(10) = 10 .$$

And $u_T(x,t)$ is a transient solution satisfying equation (4) which decreases as 't' increases.

If $u(x,t)$ is a sub sequent temperature function, the boundary conditions are

(i) $u(0,t) = 50, \;\; \forall\, t \geq 0$

(ii) $u(10,t) = 10, \;\; \forall\, t \geq 0$

(iii) $u(x,0) = 2x + 20$.

To find $u_S(x)$:

$$u_S(x) = Ax + B$$

Put $x = 0$, $u_S(0) = B$

$$50 = B.$$

Put $x = 10$, $u_S(10) = 10A + B$

$$10 = 10A + 50.$$

$$A = -4$$

$\therefore \quad u_S(x) = -4x + 50$

To find $u_T(x,t)$:

(4) $\Rightarrow \quad u(x,t) = u_S(x) + u_T(x,t)$

$$\therefore \quad u_T(x,t) = u(x,t) - u_S(x) \qquad (5)$$

Put $x = 0$, $u_T(0,t) = u(0,t) - u_S(0)$

$$= 50 - 50$$

$$= 0.$$

Put $x = 10$, $u_T(10,t) = u(10,t) - u_S(10)$

$$= 10 - 10$$

$$= 0.$$

Put $t = 0$, $\quad u_T(x,0) = u(x,0) - u_S(x)$

$$= (2x + 20) - (-4x + 50)$$

$$= 6x - 30.$$

$\therefore$ The boundary conditions are,

(i) $\quad u_T(0,t) = 0, \ \ \forall\, t \geq 0$

(ii) $\quad u_T(10,t) = 0, \ \ \forall\, t \geq 0$

(iii) $\quad u_T(x,0) = 6x - 30.$

Now the suitable solution is

$$u_T(x,t) = \left(A\cos px + B\sin px\right)e^{-c^2p^2t} \qquad \text{(I)}$$

Applying condition (i) in (I) we get,

$$u_T(0,t) = A e^{-c^2p^2t} = 0$$

Here $e^{-c^2p^2t} \neq 0 \;\Rightarrow A = 0$

$\therefore$ (I) $\Rightarrow$ $\qquad u_T(x,t) = B\sin px\; e^{-c^2p^2t} \qquad$ (II)

Applying condition (ii) in (II) we get,

$$u_T(10,t) = B\sin 10p\,.\, e^{-c^2p^2t} = 0$$

Here $e^{-c^2p^2t} \neq 0$, $\therefore$ either $B = 0$ (or) $\sin 10p = 0$.

If we take $B = 0$ and already we have $A = 0$, we get trivial solution.

$\therefore \qquad B \neq 0 \Rightarrow \sin 10p = 0.$

Since $\sin n\pi = 0$

$\Rightarrow \qquad 10p = n\pi$

$$p = \frac{n\pi}{10}.$$

Put $p = \dfrac{n\pi}{10}$ in (II) we get,

$$u_T(x,t) = B\sin\left(\frac{n\pi x}{10}\right) e^{-c^2 \frac{n^2\pi^2}{100} t} \qquad \text{(III)}$$

The most general solution is

$$u_T(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{10}\right) \; e^{-c^2 \frac{n^2\pi^2}{100} t} \qquad \text{(IV)}$$

Applying condition (iii) in (IV) we get,

$$u_T(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{10}\right) = 6x - 30 = f(x) \qquad \text{(V)}$$

To find B_n, we expand $f(x)$ in a half-range Fourier sine series in the interval (0,10),

(i.e.) $$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{10}\right) \qquad \text{(VI)}$$

where $b_n = \dfrac{2}{10}\displaystyle\int_0^{10} f(x)\sin\left(\frac{n\pi x}{10}\right) dx$

From (V) and (VI) we have,

$$B_n = b_n$$

$$\therefore \quad B_n = b_n = \frac{2}{10}\int_0^{10} f(x)\sin\left(\frac{n\pi x}{10}\right) dx$$

$$= \frac{1}{5}\int_0^{10} (6x-30)\sin\left(\frac{n\pi x}{10}\right) dx$$

$$= \frac{1}{5}\left[(6x-30)\frac{-\cos\left(\frac{n\pi x}{10}\right)}{\left(\frac{n\pi}{10}\right)} - (6)\frac{-\sin\left(\frac{n\pi x}{10}\right)}{\left(\frac{n\pi}{10}\right)^2}\right]_0^{10}$$

$$= \frac{1}{5}\left[\left((30)\frac{-(-1)^n}{\left(\frac{n\pi}{10}\right)} + (0)\right) - \left((-30)\frac{-1}{\left(\frac{n\pi}{10}\right)} + 0\right)\right]$$

$$= \frac{1}{5}\left(-30\frac{(-1)^n}{\left(\frac{n\pi}{10}\right)} - \frac{30}{\left(\frac{n\pi}{10}\right)}\right)$$

$$= -\frac{1}{5}30\frac{10}{n\pi}\left((-1)^n + 1\right)$$

$$= -\frac{60}{n\pi}\left((-1)^n + 1\right)$$

Sub. B_n in (IV) we get,

$$u_T(x,t) = \sum_{n=1}^{\infty} -\frac{60}{n\pi}\left((-1)^n + 1\right)\sin\left(\frac{n\pi x}{10}\right) \; e^{-c^2\frac{n^2\pi^2}{100}t}$$

Hence

$$u(x,t) = u_S(x) + u_T(x,t)$$

$$u(x,t) = -4x + 50 - \frac{60}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left((-1)^n + 1\right)\sin\left(\frac{n\pi x}{10}\right) \; e^{-c^2\frac{n^2\pi^2}{100}t}.$$

4.3. STEADY STATE SOLUTION OF TWO DIMENSIONAL HEAT EQUATION: (INSULATED EDGES EXCLUDED)

4.3.1. INTRODUCTION:

The differential equation for two-dimensional heat flow for the unsteady case is

$$\frac{\partial u}{\partial t} = \alpha^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Here $u(x,y)$ is the temperature at any point (x,y) at time t.

In the steady-state, u is independent of 't', so that $\frac{\partial u}{\partial t} = 0$. Hence the temperature distribution of the plate in the steady state is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. This is called Laplace equation.

4.3.2. NOTE:

The suitable solution is

$$u(x,y) = \left(A\cos px + B\sin px\right)\left(Ce^{py} + De^{-py}\right)$$

4.3.3. PROBLEMS:

TYPE I: PROBLEMS ON FINITE PLATE:

4.3.3.1. PROBLEM

A square plate is bounded by the lines $x = 0,\ y = 0,\ x = 10,\ y = 10$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x,10) = x(10-x)$, while the other three edges are kept at $0^\circ c$. Find the steady state temperature in the plate.

Solution:

The equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

From the given problem we get the following conditions:

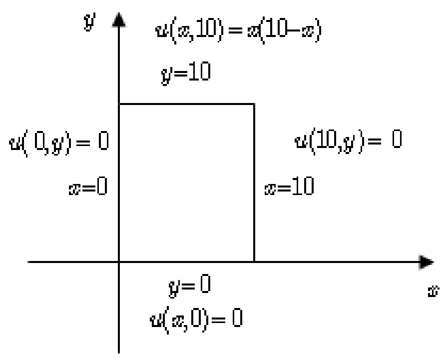

Fig. 4.10.

(i) $u(0,y) = 0,\ \ 0 \le y \le 10$

(ii) $u(10,y)=0, \quad 0 \le y \le 10$

(iii) $u(x,0)=0, \quad 0 \le x \le 10$

(iv) $u(x,10)=x(10-x), \quad 0 \le x \le 10$.

Now, the suitable solution is

$$u(x,y)=\left(A\cos px + B\sin px\right)\left(Ce^{py}+De^{-py}\right) \tag{1}$$

Applying condition (i) in (1) we get,

$$u(0,y)=A\left(Ce^{py}+De^{-py}\right)=0$$

Here $Ce^{py}+De^{-py} \neq 0, \quad \therefore A=0$

Sub. $A=0$ in (1) we get,

$$u(x,y)=B\sin px\left(Ce^{py}+De^{-py}\right) \tag{2}$$

Applying condition (ii) in (2) we get,

$$u(10,y)=B\sin 10p\left(Ce^{py}+De^{-py}\right)=0$$

Since $Ce^{py}+De^{-py} \neq 0, \quad \therefore$ either $B=0$ or $\sin 10p=0$.

Since $B \neq 0 \Rightarrow \sin 10p = 0$

We know that $\sin n\pi = 0$

$\Rightarrow \quad 10p = n\pi$

$\therefore \quad p=\dfrac{n\pi}{10}$

Sub $p=\dfrac{n\pi}{10}$ in (2) we get,

$$u(x,y)=B\sin\left(\frac{n\pi x}{10}\right)\left(Ce^{\frac{n\pi y}{10}}+De^{-\frac{n\pi y}{10}}\right) \tag{3}$$

Applying condition (iii) in (3) we get,

$$u(x,0)=B\sin\left(\frac{n\pi x}{10}\right)(C+D)=0$$

Here ——

Sub. A_n in (5) we get,

$$u(x,y) = \sum_{n=\text{odd}}^{\infty} \frac{800}{n^3\pi^3 \sinh n\pi} \sin\left(\frac{n\pi x}{10}\right).\sinh\left(\frac{n\pi y}{10}\right)$$

$$= \frac{800}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3 \sinh(2n-1)\pi} \sin\left(\frac{(2n-1)\pi x}{10}\right).\sinh\left(\frac{(2n-1)\pi y}{10}\right).$$

4.3.4. PROBLEMS:

TYPE II: PROBLEMS ON INFINITE PLATE:

4.3.4.1. PROBLEM:

A rectangular plate with insulated surface is 10 cm wide so long compared to its width that it may be considered infinite in length. If the temperature at the short edge $y = 0$ is given by $u = \begin{cases} x, & 0 \le x \le 5 \\ 10 - x, & 5 \le x \le 10 \end{cases}$ and the two long edge $x = 0, x = 10$ as well as the other short edge are kept at $0^\circ c$. Find the temperature function $u(x,y)$ at any point of the plate.

Solution:

The two dimensional heat equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

From the given problem we get the following conditions:

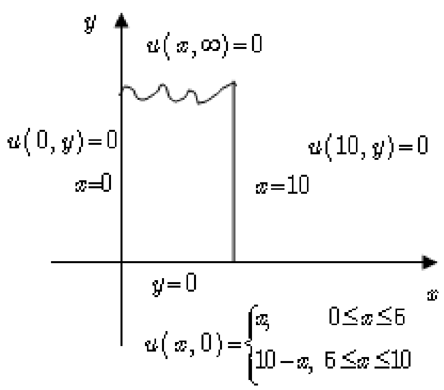

Fig. 4.11.

(i) $u(0,y) = 0, \ \ 0 \le y < \infty$

(ii) $u(10, y) = 0, \;\; 0 \le y < \infty$

(iii) $u(x, \infty) = 0, \;\; 0 \le x \le 10$

(iv) $u(x,0) = \begin{cases} x, & 0 \le x \le 5 \\ 10 - x, & 5 \le x \le 10 \end{cases}.$

Now, the suitable solution is

$$u(x,y) = \left(A\cos px + B\sin px\right)\left(Ce^{py} + De^{-py}\right) \quad (1)$$

Applying condition (i) in (1) we get,

$$u(0,y) = A\left(Ce^{py} + De^{-py}\right) = 0$$

Here $Ce^{py} + De^{-py} \ne 0, \quad \therefore A = 0$

Sub. $A = 0$ in (1) we get,

$$u(x,y) = B\sin px\left(Ce^{py} + De^{-py}\right) \quad (2)$$

Applying condition (ii) in (2) we get,

$$u(10,y) = B\sin 10p\left(Ce^{py} + De^{-py}\right) = 0$$

Since $Ce^{py} + De^{-py} \ne 0, \quad \therefore$ either $B = 0$ or $\sin 10p = 0$.

Since $B \ne 0 \;\Rightarrow \sin 10p = 0$

We know that $\sin n\pi = 0$

$\Rightarrow \quad 10p = n\pi$

$\therefore \quad p = \dfrac{n\pi}{10}$

Sub $p = \dfrac{n\pi}{10}$ in (2) we get,

$$u(x,y) = B\sin\left(\frac{n\pi x}{10}\right)\left(Ce^{\frac{n\pi y}{10}} + De^{-\frac{n\pi y}{10}}\right) \quad (3)$$

Applying condition (iii) in (3) we get,

$$u(x,\infty) = B\sin\left(\frac{n\pi x}{10}\right)\left(Ce^{\infty} + De^{-\infty}\right) = 0$$

Here $\sin\left(\frac{n\pi x}{10}\right) \neq 0,\quad B \neq 0,\ e^{\infty} \neq 0$

$\Rightarrow \quad C = 0$

Put $C = 0$ in (3) we get,

$$u(x,y) = B\sin\left(\frac{n\pi x}{10}\right)De^{-\frac{n\pi y}{10}}$$

$$u(x,y) = B_n \sin\left(\frac{n\pi x}{10}\right)e^{-\frac{n\pi y}{10}} \quad ,\text{where } B_n = BD \qquad (4)$$

The most general solution is given by

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{10}\right)e^{-\frac{n\pi y}{10}} \qquad (5)$$

Applying condition (iv) in (5) we get,

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{10}\right) = f(x) = \begin{cases} x, & 0 \le x \le 5 \\ 10 - x, & 5 \le x \le 10 \end{cases} \qquad (6)$$

To find B_n, expand $f(x)$ in a half-range Fourier sine series in $(0,10)$

(i.e.) $$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{10}\right) \qquad (7)$$

where $b_n = \frac{2}{10}\int_0^{10} f(x)\sin\left(\frac{n\pi x}{10}\right) dx$

From (6) and (7) we get,

$$B_n = b_n$$

$$\therefore \quad B_n = b_n = \frac{2}{10}\int_0^{10} f(x)\sin\left(\frac{n\pi x}{10}\right) dx$$

$$= \frac{1}{5}\left[\int_0^{5} f(x)\sin\left(\frac{n\pi x}{10}\right) dx + \int_5^{10} f(x)\sin\left(\frac{n\pi x}{10}\right) dx\right]$$

$$= \frac{1}{5}\left[\int_0^5 x \sin\left(\frac{n\pi x}{10}\right) dx + \int_5^{10} (10-x) \sin\left(\frac{n\pi x}{10}\right) dx\right]$$

$$= \frac{1}{5}\left(x\,\frac{-\cos\left(\frac{n\pi x}{10}\right)}{\left(\frac{n\pi}{10}\right)} - (1)\,\frac{-\sin\left(\frac{n\pi x}{10}\right)}{\left(\frac{n\pi}{10}\right)^2}\right)_0^5$$

$$+ \frac{1}{5}\left((10-x)\,\frac{-\cos\left(\frac{n\pi x}{10}\right)}{\left(\frac{n\pi}{10}\right)} - (-1)\,\frac{-\sin\left(\frac{n\pi x}{10}\right)}{\left(\frac{n\pi}{10}\right)^2}\right)_5^{10}$$

$$= \frac{1}{5}\left[\left(-5\,\frac{\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{10}\right)} + \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{10}\right)^2}\right) - (0+0)\right]$$

$$+ \frac{1}{5}\left[(0+0) - \left(\frac{-5\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{10}\right)} - \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{10}\right)^2}\right)\right]$$

$$= \frac{1}{5}\left[\frac{2\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{10}\right)^2}\right]$$

$$= \frac{1}{5}\,2\,\frac{100}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$B_n = \frac{40}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$\therefore \quad u(x,y) = \frac{40}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{10}\right) e^{-\frac{n\pi y}{10}}.$$

4.3.4.2. PROBLEM:

An infinitely long rectangular plate with insulated surface is 10 cm wide. The two long edges and one short edge are kept at $0^\circ c$ temperature, while the other short edge $x = 0$ is kept at temperature given by

$u = \begin{cases} 20y, & 0 \le y \le 5 \\ 20(10 - y), & 5 \le y \le 10 \end{cases}$. Find the steady state temperature distribution in the plate.

Solution:

The two dimensional heat equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 .$$

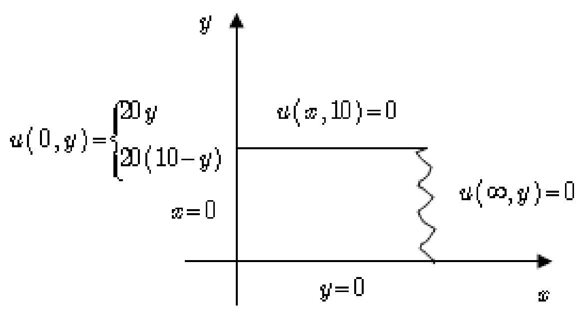

Fig. 4.12.

From the given problem we get the following conditions:

(i) $u(x,0) = 0, \; 0 \le x < \infty$

(ii) $u(x,10) = 0, \; 0 \le x < \infty$

(iii) $u(\infty, y) = 0, \; 0 \le y \le 10$

(iv) $u(0, y) = \begin{cases} 20y, & 0 \le y \le 5 \\ 20(10 - y), & 5 \le y \le 10 \end{cases}$.

Now, the suitable solution is

$$u(x,y) = \left(Ae^{px} + Be^{-px}\right)\left(C\cos py + D\sin py\right) \qquad (1)$$

Applying condition (i) in (1) we get,

$$u(x,0) = \left(Ae^{px} + Be^{-px}\right)C = 0$$

Here $Ae^{px} + Be^{-px} \neq 0 \ \Rightarrow C = 0$

Sub. $C = 0$ in (1) we get,

$$u(x,y) = D\sin py\left(Ae^{px} + Be^{-px}\right) \tag{2}$$

Applying condition (ii) in (2) we get,

$$u(x,10) = D\sin 10p\left(Ae^{px} + Be^{-px}\right) = 0$$

Since $Ae^{px} + Be^{-px} \neq 0$, $\therefore$ either $D = 0$ or $\sin 10p = 0$.

Since $D \neq 0 \ \Rightarrow \sin 10p = 0$

We know that $\sin n\pi = 0$

$\Rightarrow \quad 10p = n\pi$

$\therefore \quad p = \dfrac{n\pi}{10}$

Sub $p = \dfrac{n\pi}{10}$ in (2) we get,

$$u(x,y) = D\sin\left(\frac{n\pi y}{10}\right)\left(Ae^{\frac{n\pi x}{10}} + Be^{-\frac{n\pi x}{10}}\right) \tag{3}$$

Applying condition (iii) in (3) we get,

$$u(\infty,y) = D\sin\left(\frac{n\pi y}{10}\right)\left(Ae^{\infty} + Be^{-\infty}\right) = 0$$

Here $\sin\left(\dfrac{n\pi y}{10}\right) \neq 0$, $D \neq 0$, $e^{\infty} \neq 0$

$\Rightarrow \quad A = 0$

.Put $A = 0$ in (3) we get,

$$u(x,y) = D\sin\left(\frac{n\pi y}{10}\right)Be^{-\frac{n\pi x}{10}}$$

$$u(x,y) = B_n\sin\left(\frac{n\pi y}{10}\right)e^{-\frac{n\pi x}{10}} \qquad \text{where } B_n = BD \tag{4}$$

The most general solution is given by

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi y}{10}\right) e^{-\frac{n\pi x}{10}} \qquad (5)$$

Applying condition (iv) in (5) we get,

$$u(0,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi y}{10}\right) = f(y) = \begin{cases} 20y, & 0 \le y \le 5 \\ 20(10-y), & 5 \le y \le 10 \end{cases} \qquad (6)$$

To find B_n, expand $f(y)$ in a half-range Fourier sine series in $(0,10)$

(i.e.)
$$f(y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi y}{10}\right) \qquad (7)$$

$$\text{where } b_n = \frac{2}{10}\int_0^{10} f(y)\sin\left(\frac{n\pi y}{10}\right) dy$$

From (6) and (7) we get,

$$B_n = b_n$$

$$\therefore \quad B_n = b_n = \frac{2}{10}\int_0^{10} f(y)\sin\left(\frac{n\pi y}{10}\right) dy$$

$$= \frac{1}{5}\left[\int_0^{5} f(y)\sin\left(\frac{n\pi y}{10}\right) dy + \int_5^{10} f(y)\sin\left(\frac{n\pi y}{10}\right) dy\right]$$

$$= \frac{1}{5}\left[\int_0^{5} 20y\sin\left(\frac{n\pi y}{10}\right) dy + \int_5^{10} 20(10-y)\sin\left(\frac{n\pi y}{10}\right) dy\right]$$

$$= 4\left[\int_0^{5} y\sin\left(\frac{n\pi y}{10}\right) dy + \int_5^{10} (10-y)\sin\left(\frac{n\pi y}{10}\right) dy\right]$$

$$= 4\left(y\,\frac{-\cos\left(\frac{n\pi y}{10}\right)}{\left(\frac{n\pi}{10}\right)} - (1)\,\frac{-\sin\left(\frac{n\pi y}{10}\right)}{\left(\frac{n\pi}{10}\right)^2}\right)_0^5$$

$$+ 4\left((10-y)\,\frac{-\cos\left(\frac{n\pi y}{10}\right)}{\left(\frac{n\pi}{10}\right)} - (-1)\,\frac{-\sin\left(\frac{n\pi y}{10}\right)}{\left(\frac{n\pi}{10}\right)^2}\right)_5^{10}$$

(x) $u_2(x,b) = 0$

(xi) $u_2(0,y) = 0$

(xii) $u_2(a,y) = 100$.

To find $u_1(x,y)$:

The suitable solution is

$$u_1(x,y) = \left(A\cos px + B\sin px\right)\left(Ce^{py} + De^{-py}\right) \tag{1}$$

Applying condition (v) in (1) we get,

$$\vdots$$
$$\vdots$$
$$\vdots$$

The most general solution is

$$u_1(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right).\sinh\left(\frac{n\pi y}{a}\right) \tag{5}$$

Applying condition (viii) in (5) we get,

$$u_1(x,b) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right).\sinh\left(\frac{n\pi b}{a}\right) = 100$$

$$= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) = 100, \quad \text{where } A_n = B_n \sinh\left(\frac{n\pi b}{a}\right) \tag{6}$$

To find A_n, expand $f(x) = 100$ in a half-range Fourier sine series in $(0,a)$

(i.e.) $$f(x) = 100 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right) \tag{7}$$

where $b_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right) dx$

From (6) and (7) we get,

$$A_n = b_n$$

$$\therefore \quad A_n = b_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right) dx$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

$$= \frac{200}{n\pi}(1-(-1)^n)$$

$$A_n = \begin{cases} \dfrac{400}{n\pi}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Since $A_n = B_n \sinh\left(\frac{n\pi b}{a}\right)$

$$\therefore \quad B_n = \frac{A_n}{\sinh\left(\frac{n\pi b}{a}\right)}$$

$$= \begin{cases} \dfrac{400}{n\pi \sinh\left(\frac{n\pi b}{a}\right)}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$\therefore \quad u_1(x,y) = \sum_{n=\text{odd}}^{\infty} \frac{400}{n\pi \sinh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi x}{a}\right).\sinh\left(\frac{n\pi y}{b}\right)$$

$$u_1(x,y) = \sum_{n=1}^{\infty} \frac{400}{(2n-1)\pi \sinh\left(\frac{(2n-1)\pi b}{a}\right)} \sin\left(\frac{(2n-1)\pi x}{a}\right).\sinh\left(\frac{(2n-1)\pi y}{b}\right)$$

lll^{ly}

$$u_2(x,y) = \sum_{n=1}^{\infty} \frac{400}{(2n-1)\pi \sinh\left(\frac{(2n-1)\pi a}{b}\right)} \sin\left(\frac{(2n-1)\pi y}{b}\right).\sinh\left(\frac{(2n-1)\pi x}{a}\right)$$

$$\therefore \quad u(x,y) = u_1(x,y) + u_2(x,y).$$