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UNIT I

FOURIER SERIES

1.1. DEFINITION: (PERIODIC FUNCTION)

A function $f(x)$ is said to be periodic, if $f(x+p) = f(x)$ is true for some value of p and every value of x. The smallest value of p for which this equation is true for every value of x will be called the period of the function $f(x)$.

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1.2. EXAMPLE:

(i) $\sin x = \sin(x+2\pi) = \sin(x+4\pi) = \cdots$

$\therefore$ $\sin x$ is a periodic function with period 2π.

(ii) $\cos x$ is a periodic function with period 2π,

(iii) $\tan x$ is a period function with period π.

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1.3. DEFINITION:

The Fourier series for the function $f(x)$ in the interval $0 < x < 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx.$$

The Fourier series for the function $f(x)$ in the interval $-\pi < x < \pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

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1.4. NOTE:

$$\sin n\pi = 0$$

$$\cos n\pi = (-1)^n.$$

Bernoulli's formulae:

$$\int u\ v\ dx = uv_1 - u'v_2 + u''v_3 - \cdots$$

Where

$$u' = \frac{du}{dx} \qquad v_1 = \int v\ dx$$

$$u'' = \frac{d^2u}{dx^2} \qquad v_2 = \int v_1\ dx$$

$$\vdots \qquad \vdots$$

$$\vdots \qquad \vdots$$

1.5. PROBLEMS: TYPE I: PROBLEMS UNDER THE INTERVAL $(0, 2\pi)$:

1.5.1. EXAMPLE:

Find the Fourier expansion of $f(x) = x$ in $0 < x < 2\pi$.

SOLUTION:

The Fourier series for the function $f(x)$ in the interval $0 < x < 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\ dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\ dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\ dx$$

Now,

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$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\, dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left(4\pi^2 - 0\right)\right]$$

$$a_0 = 2\pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\cos nx\, dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(2\pi(0) + \frac{1}{n^2}\right) - \left(0 + \frac{1}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{1}{n^2}\right]$$

$$a_n = 0\,.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin nx\, dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(2\pi\left(-\frac{1}{n}\right) + 0\right) - (0 + 0)\right]$$

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$$= \frac{1}{\pi}\left[\frac{-2\pi}{n}\right]$$

$$b_n = -\frac{2}{n}$$

$\therefore$ The Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{2\pi}{2} + \sum_{n=1}^{\infty} (0) \cos nx + \sum_{n=1}^{\infty} \left(\frac{-2}{n}\right) \sin nx$$

$$f(x) = \pi - 2\sum_{n=1}^{\infty} \left(\frac{1}{n}\right) \sin nx$$

1.5.2. EXAMPLE:

If $f(x) = \frac{1}{2}(\pi - x)$, find the Fourier series of period 2π in the interval $(0, 2\pi)$.

SOLUTION:

The Fourier series for the function $f(x)$ in the interval $(0, 2\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\, dx$$

$$= \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{2\pi}$$

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$$= \frac{1}{2\pi}\left[\left(2\pi^2 - \frac{4\,\pi^2}{2}\right) - (0-0)\right]$$

$$= \frac{1}{2\pi}[0]$$

$$a_0 = 0\,.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\,\cos nx\;dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\,\cos nx\;dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\,\cos nx\;dx$$

$$= \frac{1}{2\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right) - \frac{\cos nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left(-\pi(0) - \frac{1}{n^2}\right) - \left(0 - \frac{1}{n^2}\right)\right]$$

$$= \frac{1}{2\pi}\left[-\frac{1}{n^2} + \frac{1}{n^2}\right]$$

$$a_n = 0\,.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\,\sin nx\;dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\,\sin nx\;dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\,\sin nx\;dx$$

$$= \frac{1}{2\pi}\left[(\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi}$$

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$$= \frac{1}{2\pi}\left[(\pi - x)\left(\frac{-\cos nx}{n}\right) - \frac{\sin nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left((-\pi)\left(-\frac{1}{n}\right) - 0\right) - \left(\pi\left(-\frac{1}{n}\right) - 0\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{\pi}{n} + \frac{\pi}{n}\right]$$

$$= \frac{1}{2\pi}\left[\frac{2\pi}{n}\right]$$

$$b_n = \frac{1}{n}$$

$\therefore$ The Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{(0)}{2} + \sum_{n=1}^{\infty} (0) \cos nx + \sum_{n=1}^{\infty} \left(\frac{1}{n}\right) \sin nx$$

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right) \sin nx .$$

1.5.3. EXAMPLE:

Expand $f(x) = e^{ax}$ in a Fourier series in $(0, 2\pi)$.

SOLUTION:

The Fourier series for the function $f(x)$ in the interval $(0, 2\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{ax}\, dx$$

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$$= \frac{1}{\pi}\left[\frac{e^{ax}}{a}\right]_0^{2\pi}$$

$$a_0 = \frac{1}{\pi a}\left[e^{2\pi a} - 1\right].$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{ax} \cos nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+n^2}(a\cos nx + n\sin nx)\right]_0^{2\pi}$$

$$\left(\because \quad \int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right)$$

$$= \frac{1}{\pi}\left[\frac{1}{a^2+n^2}\left\{e^{2\pi a}(a(1)+0) - 1(a+0)\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{a^2+n^2}\left\{e^{2\pi a}a - a\right\}\right]$$

$$a_n = \frac{a}{\pi\left(a^2+n^2\right)}\left(e^{2\pi a} - 1\right).$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{ax} \sin nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+n^2}(a\sin nx - n\cos nx)\right]_0^{2\pi}$$

$$\left(\because \quad \int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right)$$

$$= \frac{1}{\pi}\left[\frac{1}{a^2+n^2}\left\{e^{2\pi a}(0-n) - (1)(0-n)\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{a^2+n^2}\left\{-ne^{2\pi a} + n\right\}\right]$$

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$$b_n = \frac{n}{\pi\left(a^2+n^2\right)}\left(1-e^{2\pi a}\right).$$

$\therefore$ The Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{1}{2\pi a}\left(e^{2\pi a}-1\right) + \sum_{n=1}^{\infty} \frac{a}{\pi\left(a^2+n^2\right)}\left(e^{2\pi a}-1\right)\cos nx + \sum_{n=1}^{\infty} \frac{n}{\pi\left(a^2+n^2\right)}\left(1-e^{2\pi a}\right)\sin nx$$

$$f(x) = \frac{e^{2\pi a}-1}{2\pi a} + \frac{a\left(e^{2\pi a}-1\right)}{\pi}\sum_{n=1}^{\infty} \frac{1}{\left(a^2+n^2\right)}\cos nx + \frac{\left(1-e^{2\pi a}\right)}{\pi}\sum_{n=1}^{\infty} \frac{n}{\left(a^2+n^2\right)}\sin nx$$

1.5.4. EXAMPLE:

Derive the Fourier series expansion of $\sqrt{1-\cos x}$ in $0 \le x \le 2\pi$ and deduce that $\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$.

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SOLUTION:

$$f(x) = \sqrt{1-\cos x} = \sqrt{2\sin^2\left(\frac{x}{2}\right)} = \sqrt{2}\sin\left(\frac{x}{2}\right) \qquad \left(\because \frac{1-\cos x}{2} = \sin^2 x\right).$$

The Fourier series for the function $f(x)$ in the interval $0 \le x \le 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

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$$= \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\left(\frac{x}{2}\right) dx$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{-\cos\left(\frac{x}{2}\right)}{\left(\frac{1}{2}\right)}\right]_0^{2\pi}$$

$$= \frac{2\sqrt{2}}{\pi}[-(-1)-(-1)]$$

$$a_0 = \frac{4\sqrt{2}}{\pi}.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\left(\frac{x}{2}\right)\cos nx\, dx$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{2\pi} \cos nx \sin\left(\frac{x}{2}\right) dx$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{2\pi} \frac{1}{2}\left[\sin\left(n+\frac{1}{2}\right)x - \sin\left(n-\frac{1}{2}\right)x\right]dx$$

$$\left(\because\ 2\cos A\sin B = \sin(A+B)-\sin(A-B)\right)$$

$$= \frac{1}{\sqrt{2}\pi}\left[\frac{-\cos\left(n+\frac{1}{2}\right)x}{\left(n+\frac{1}{2}\right)} + \frac{\cos\left(n-\frac{1}{2}\right)x}{\left(n-\frac{1}{2}\right)}\right]_0^{2\pi}$$

$$= \frac{1}{\sqrt{2}\pi}\left[\frac{-\cos\left(\frac{2n+1}{2}\right)x}{\left(n+\frac{1}{2}\right)} + \frac{\cos\left(\frac{2n-1}{2}\right)x}{\left(n-\frac{1}{2}\right)}\right]_0^{2\pi}$$

$$= \frac{1}{\sqrt{2}\pi}\left[\left(\frac{-(-1)}{\left(n+\frac{1}{2}\right)} + \frac{(-1)}{\left(n-\frac{1}{2}\right)}\right) - \left(\frac{-1}{\left(n+\frac{1}{2}\right)} + \frac{1}{\left(n-\frac{1}{2}\right)}\right)\right]$$

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$$= \frac{1}{\sqrt{2}\pi}\left(\frac{1}{n+\frac{1}{2}} - \frac{1}{n-\frac{1}{2}} + \frac{1}{n+\frac{1}{2}} - \frac{1}{n-\frac{1}{2}}\right)$$

$$= \frac{1}{\sqrt{2}\pi}\left(\frac{2}{\frac{2n+1}{2}} - \frac{2}{\frac{2n-1}{2}}\right)$$

$$= \frac{4}{\sqrt{2}\pi}\left(\frac{1}{2n+1} - \frac{1}{2n-1}\right)$$

$$= \frac{4}{\sqrt{2}\pi}\left(\frac{2n-1-2n-1}{(2n+1)(2n-1)}\right)$$

$$= \frac{4}{\sqrt{2}\pi}\left(\frac{-2}{4n^2-1}\right)$$

$$a_n = -\frac{4\sqrt{2}}{\pi}\left(\frac{1}{4n^2-1}\right)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\left(\frac{x}{2}\right)\sin nx\, dx$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{2\pi} \sin nx \sin\left(\frac{x}{2}\right) dx$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{2\pi} \frac{1}{2}\left[\cos\left(n-\frac{1}{2}\right)x - \cos\left(n+\frac{1}{2}\right)x\right]dx$$

$$\left(\because\ 2\sin A\sin B = \cos(A-B) - \cos(A+B)\right)$$

$$= \frac{1}{\sqrt{2}\pi}\left[\frac{\sin\left(n-\frac{1}{2}\right)x}{\left(n-\frac{1}{2}\right)} - \frac{\sin\left(n+\frac{1}{2}\right)x}{\left(n+\frac{1}{2}\right)}\right]_0^{2\pi}$$

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$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin\left(\frac{2n-1}{2}\right)x}{\left(n-\frac{1}{2}\right)} - \frac{\sin\left(\frac{2n+1}{2}\right)x}{\left(n+\frac{1}{2}\right)}\right]_0^{2\pi}$$

$$= \frac{1}{\sqrt{2\pi}}[(0-0)-(0-0)]$$

$$b_n = 0$$

$\therefore$ The Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{4\sqrt{2}}{2\pi} + \sum_{n=1}^{\infty}\left(-\frac{4\sqrt{2}}{\pi}\left(\frac{1}{4n^2-1}\right)\right)\cos nx + \sum_{n=1}^{\infty}(0)\sin nx$$

$$f(x) = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos nx$$

Put $x = 0$ $\left(x = 0 \text{ is a point of continuity}\right)$

$$0 = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}(1)$$

$$\Rightarrow \quad \sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{2}.$$

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1.5.5. EXAMPLE:

Find the Fourier series of $f(x) = x\sin x$ in $(0,2\pi)$.

SOLUTION:

The Fourier series for the function $f(x)$ in the interval $(0,2\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where $\quad a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

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$$= \frac{1}{\pi} \int_0^{2\pi} x \sin x \, dx$$

$$= \frac{1}{\pi} [x(-\cos x) - (1)(-\sin x)]_0^{2\pi}$$

$$= \frac{1}{\pi} [-x \cos x + \sin x]_0^{2\pi}$$

$$= \frac{1}{\pi} [(-2\pi(1) + 0) - (0 + 0)]$$

$$= \frac{1}{\pi} (-2\pi)$$

$$a_0 = -2$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \sin x \cos nx \, dx$$

$$= \frac{1}{\pi} \frac{1}{2} \int_0^{2\pi} x \, 2 \cos nx \sin x \quad dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x \, [\sin(n+1)x - \sin(n-1)x] \, dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x \sin(n+1) x \, dx - \frac{1}{2\pi} \int_0^{2\pi} x \sin(n-1) x \, dx$$

$$= \frac{1}{2\pi} \left[x \left(\frac{-\cos(n+1)x}{n+1} \right) - (1) \left(\frac{-\sin(n+1)x}{(n+1)^2} \right) \right]_0^{2\pi}$$

$$- \frac{1}{2\pi} \left[x \left(\frac{-\cos(n-1)x}{n-1} \right) - (1) \left(\frac{-\sin(n-1)x}{(n-1)^2} \right) \right]_0^{2\pi}$$

$$= \frac{1}{2\pi} \left[-x \frac{\cos(n+1)x}{n+1} + \frac{\sin(n+1)x}{(n+1)^2} \right]_0^{2\pi}$$

$$- \frac{1}{2\pi} \left[-x \frac{\cos(n-1)x}{n-1} + \frac{\sin(n-1)x}{(n-1)^2} \right]_0^{2\pi}$$

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$$= \frac{1}{2\pi}\left[\left(-2\pi\frac{1}{n+1}+0\right)-(0+0)\right]-\frac{1}{2\pi}\left[\left(-2\pi\frac{1}{n-1}+0\right)-(0+0)\right]$$

$$= \frac{1}{2\pi}\left(\frac{-2\pi}{n+1}\right)-\frac{1}{2\pi}\left(\frac{-2\pi}{n-1}\right)$$

$$= -\frac{1}{n+1}+\frac{1}{n+1}$$

$$= \frac{-n+1+n+1}{n^2-1}$$

$$a_n = \frac{2}{n^2-1} \qquad \text{if } (n \neq 1)$$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\, dx \qquad (\because\ 2\sin A\cos A = \sin 2A)$$

$$= \frac{1}{2\pi}\left[x\left(\frac{-\cos 2x}{2}\right)-(1)\left(\frac{-\sin 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left(-2\pi\left(\frac{1}{2}\right)+0\right)-(0+0)\right]$$

$$a_1 = \frac{-1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin nx\, dx$$

$$= \frac{1}{\pi}\frac{1}{2}\int_0^{2\pi} x\, 2\sin nx\sin x\ \ dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\cos(n-1)x-\cos(n+1)x\right]dx$$

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$$= \frac{1}{2\pi}\int_0^{2\pi} x\cos(n-1)x\,dx - \frac{1}{2\pi}\int_0^{2\pi} x\cos(n+1)x\,dx$$

$$= \frac{1}{2\pi}\left[x\left(\frac{\sin(n-1)x}{n-1}\right)-(1)\left(\frac{-\cos(n-1)x}{(n-1)^2}\right)\right]_0^{2\pi}$$

$$-\frac{1}{2\pi}\left[x\left(\frac{\sin(n+1)x}{n+1}\right)-(1)\left(\frac{-\cos(n+1)x}{(n+1)^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left(2\pi(0)+\frac{1}{(n-1)^2}\right)-\left(0+\frac{1}{(n-1)^2}\right)\right]$$

$$-\frac{1}{2\pi}\left[\left(0+\frac{1}{(n+1)^2}\right)-\left(0+\frac{1}{(n+1)^2}\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2}-\frac{1}{(n-1)^2}\right]-\frac{1}{2\pi}\left[\frac{1}{(n+1)^2}-\frac{1}{(n+1)^2}\right]$$

$$b_n = 0 \quad \text{if} \quad (n \neq 1)$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin x\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin x\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\left(\frac{1-\cos 2x}{2}\right)dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\,dx - \frac{1}{2\pi}\int_0^{2\pi} x\cos 2x\,dx$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} - \frac{1}{2\pi}\left[x\left(\frac{\sin 2x}{2}\right)-(1)\left(\frac{-\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[4\pi^2-0\right]-\frac{1}{2\pi}\left[\left(0+\frac{1}{4}\right)-\left(0+\frac{1}{4}\right)\right]$$

$$= \frac{1}{4\pi}\left[4\pi^2\right]-\frac{1}{2\pi}(0)$$

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$$b_1 = \pi .$$

$\therefore$ The Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx + b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

$$= \frac{-2}{2} + \left(\frac{-1}{2}\right)\cos x + \sum_{n=2}^{\infty} \frac{2}{n^2 - 1} \cos nx + \pi \sin x + \sum_{n=2}^{\infty} (0) \sin nx$$

$$f(x) = -1 - \frac{\cos x}{2} + 2\sum_{n=2}^{\infty} \frac{\cos nx}{n^2 - 1} + \pi \sin x .$$

1.5.6. EXAMPLE:

Find the Fourier series of periodicity 2π for $f(x) = \begin{cases} x, & (0,\pi) \\ 2\pi - x, & (\pi, 2\pi) \end{cases}$ and hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

SOLUTION:

The Fourier series for the function $f(x)$ in the interval $(0,2\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where $\quad a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx$

$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x)\, dx + \int_{\pi}^{2\pi} f(x)\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\, dx + \int_{\pi}^{2\pi} (2\pi - x)\, dx\right]$$

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$$= \frac{1}{\pi}\left[\left[\frac{x^2}{2}\right]_0^{\pi} + \left[2\pi x - \frac{x^2}{2}\right]_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi^2}{2} - 0\right) + \left[\left(4\pi^2 - \frac{4\pi^2}{2}\right) - \left(2\pi^2 - \frac{\pi^2}{2}\right)\right]\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + 2\pi^2 - \frac{3\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2 + 4\pi^2 - 3\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\pi^2}{2}\right]$$

$$a_0 = \pi .$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x)\cos nx\, dx + \int_{\pi}^{2\pi} f(x)\cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\, dx + \int_{\pi}^{2\pi} (2\pi - x)\cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi} + \frac{1}{\pi}\left[(2\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(\pi(0) + \frac{(-1)^n}{n^2}\right) - \left(0 + \frac{1}{n^2}\right)\right] + \frac{1}{\pi}\left[\left(0 - \frac{1}{n^2}\right) - \left(0 - \frac{(-1)^n}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2} - \frac{1}{n^2} + \frac{(-1)^n}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{2(-1)^n}{n^2} - \frac{2}{n^2}\right]$$

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$$= \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]$$

$$a_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{-4}{\pi n^2}, & \text{if } n \text{ is odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x)\sin nx\, dx + \int_{\pi}^{2\pi} f(x)\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx\, dx + \int_{\pi}^{2\pi} (2\pi - x)\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$+ \frac{1}{\pi}\left[(2\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin x}{n^2}\right)\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(-\pi\frac{(-1)^n}{n} + 0\right) - (0+0)\right] + \frac{1}{\pi}\left[(0-0) - \left(-\pi\frac{(-1)^n}{n} + 0\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n} + \frac{-\pi(-1)^n}{n}\right]$$

$$b_n = 0.$$

$\therefore$ The Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{\pi}{2} + \sum_{n=\text{odd}}^{\infty}\left(\frac{-4}{\pi n^2}\right)\cos nx + \sum_{n=1}^{\infty}(0)\sin nx$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x$$

Put $x = 0$ (point of discont. and end point)

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$$\therefore\ f(x) = \frac{f(0)+f(2\pi)}{2} = \frac{0+0}{2} = 0$$

$$\therefore\ 0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

1.6. NOTE:

$$\sum_{n=\text{odd}}^{\infty} \Rightarrow \sum_{n=1}^{\infty}\ ,\ \text{replace } n \text{ by } (2n-1)$$

$$\sum_{n=\text{even}}^{\infty} \Rightarrow \sum_{n=1}^{\infty}\ ,\ \text{replace } n \text{ by } 2n$$

1.7. DEFINITION: (EVEN FUNCTION)

A function $f(x)$ is said to be even if $f(-x) = f(x)$.

EXAMPLE: x^2, $\cos x$, $|x|$.

(ODD FUNCTION)

A function $f(x)$ is said to be odd if $f(-x) = -f(x)$

EXAMPLE: x^3, $\sin x$

1.8. NOTE:

(i) $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is even} \\ 0, & \text{if } f(x) \text{ is odd} \end{cases}$$

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(ii) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

If $f(x)$ is even then $b_n = 0$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

If $f(x)$ is odd then $a_0 = 0, \quad a_n = 0$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

(iii) (even) (even)=even

(odd)(odd)=even

(odd)(even)=odd.

1.9. TYPE: II: PROBLEMS UNDER THE INTERVAL $(-\pi, \pi)$:

1.9.1. EXAMPLE:

Find the Fourier series expansion of $f(x) = x$ in $-\pi < x < \pi$.

SOLUTION:

$f(x) = x$

$f(-x) = -x = -f(x)$

$\therefore$ $f(x)$ is odd function

Hence $a_0 = 0, a_n = 0$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

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$$= \frac{2}{\pi}\int_0^{\pi} f(x)\,\sin nx\;dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\,\sin nx\;dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(-\pi\frac{(-1)^n}{n} + 0\right) - (0+0)\right]$$

$$= \frac{2}{\pi}\left[\frac{-\pi(-1)^n}{n}\right]$$

$$b_n = \frac{-2(-1)^n}{n}$$

$$\therefore\; f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\,\cos nx + \sum_{n=1}^{\infty} b_n\,\sin nx$$

$$= 0 + 0 + \sum_{n=1}^{\infty}\frac{-2(-1)^n}{n}\sin nx$$

$$f(x) = -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx.$$

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1.9.2. EXAMPLE:

Find the Fourier series for $f(x) = x^2$ in $-\pi \le x \le \pi$ and deduce that

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

SOLUTION:

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$$f(x) = x^2$$

$$f(-x) = (-x)^2 = x^2 = f(x)$$

$\therefore$ $f(x)$ is even

$$\therefore b_n = 0$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\, dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{3\pi}\left[\pi^3 - 0\right]$$

$$a_0 = \frac{2}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(0 + 2\pi\frac{(-1)^n}{n^2} - 0\right) - (0 + 0 + 0)\right]$$

$$= \frac{2}{\pi}\left(2\pi\frac{(-1)^n}{n^2}\right)$$

$$a_n = \frac{4(-1)^n}{n^2}$$

$$\therefore\ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

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$$= \frac{\frac{2}{3}\pi^2}{2} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx + \sum_{n=1}^{\infty} (0) \sin nx$$

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \qquad (1)$$

(i) Put $x = \pi$ (cont. point)

$$\therefore f(\pi) = \pi^2$$

$$\therefore \ \pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (-1)^n$$

$$\pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \left(\because \ (-1)^n (-1)^n = (-1)^{2n} = 1\right)$$

$$\frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \qquad (2)$$

(ii) Put $x = 0$ (cont. point)

$$\therefore f(0) = 0$$

$$\therefore \ 0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{-\pi^2}{12}.$$

$$-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots = \frac{-\pi^2}{12}$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12} \qquad (3)$$

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(iii) $(2)+(3) \Rightarrow \quad \dfrac{2}{1^2}+\dfrac{2}{3^2}+\cdots=\dfrac{\pi^2}{6}+\dfrac{\pi^2}{12}$

$$2\left(\frac{1}{1^2}+\frac{1}{3^2}+\cdots\right)=\frac{2\pi^2+\pi^2}{12}$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\cdots=\frac{3\pi^2}{12\,(2)}$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{8}.$$

1.9.3. EXAMPLE:

Find the Fourier series of $f(x)=x^2+x$ in $(-\pi,\pi)$. Hence deduce that $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}=\dfrac{\pi^2}{6}$.

SOLUTION:

$$f(x)=x^2+x$$

$$f(-x)=(-x)^2-x$$

$$=x^2-x=-\left(-x^2+x\right)$$

$$\neq -f(x)\neq f(x)$$

$\therefore$ $f(x)$ is neither even nor odd.

$$\therefore\ f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

Where

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(x^2+x\right)dx$$

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$$= \frac{1}{\pi}\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi^3}{3}+\frac{\pi^2}{2}\right)-\left(\frac{-\pi^3}{3}+\frac{\pi^2}{2}\right)\right]$$

$$= \frac{1}{\pi}\left(\frac{2\pi^3}{3}\right)$$

$$a_0 = \frac{2\pi^2}{3}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \left(x^2+x\right)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\left(x^2+x\right)\left(\frac{\sin nx}{n}\right)-(2x+1)\left(\frac{-\cos nx}{n^2}\right)+(2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(0+(2\pi+1)\frac{(-1)^n}{n^2}-0\right)-\left(0+(1-2\pi)\frac{(-1)^n}{n^2}+0\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2}+\frac{2\pi(-1)^n}{n^2}-\frac{(-1)^n}{n^2}+\frac{2\pi(-1)^n}{n^2}\right]$$

$$a_n = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \left(x^2+x\right)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\left(x^2+x\right)\left(\frac{-\cos nx}{n}\right)-(2x+1)\left(\frac{-\sin nx}{n^2}\right)+(2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

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$$= \frac{1}{\pi}\left[\begin{array}{l}\left(-\left(\pi^2+\pi\right)\frac{(-1)^n}{n}+0+\frac{2(-1)^n}{n^3}\right) \\ \quad -\left(-\left(\pi^2-\pi\right)\left(-\frac{(-1)^n}{n}\right)+0+\frac{2(-1)^n}{n^3}\right)\end{array}\right]$$

$$= \frac{1}{\pi}\left[-\left(\pi^2+\pi\right)\frac{(-1)^n}{n}+\frac{2(-1)^n}{n^3}+\left(\pi^2-\pi\right)\frac{(-1)^n}{n}-\frac{2(-1)^n}{n^3}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n}\left(-\pi^2-\pi+\pi^2-\pi\right)\right]$$

$$= \frac{-2\pi}{\pi}\left(\frac{(-1)^n}{n}\right)$$

$$b_n = \frac{-2(-1)^n}{n}.$$

$$\therefore\ f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{2\pi^2}{3(2)}+\sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx+\sum_{n=1}^{\infty}\frac{-2(-1)^n}{n}\sin nx$$

$$f(x) = \frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

Put $x = \pi$ (discontinuous point and end point)

$$\therefore\ f(\pi) = \frac{f(-\pi)+f(\pi)}{2} = \frac{\pi^2-\pi+\pi^2+\pi}{2} = \frac{2\pi^2}{2} = \pi^2$$

$$\therefore\ \pi^2 = \frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(-1)^n$$

$$\pi^2-\frac{\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow\ \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

NOTE:

(i) If the series has cos term we put $x = 0$

(ii) If the series has sin term we put $x = \frac{\pi}{2}$

(iii) If the series has alternative sign we put $x = \pi$.

1.9.4. EXAMPLE:

Obtain the Fourier series to represent the function $f(x) = |x|$ in $-\pi < x < \pi$ and deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{8}$.

(OR)

Find the Fourier series of $f(x) = \begin{cases} -x, & -\pi < x \le 0 \\ x, & 0 \le x < \pi \end{cases}$.

SOLUTION:

$$f(x) = |x|$$

$$f(-x) = |-x| = |x| = f(x)$$

$\therefore$ $f(x)$ is even, $\therefore b_n = 0$

$$\therefore \; f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} |x|\, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} x\, dx$$

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$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi^2 - 0\right]$$

$$a_0 = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |x| \cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \cos nx\, dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(0 + \frac{(-1)^n}{n^2}\right) - \left(0 + \frac{1}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]$$

$$= \begin{cases} \dfrac{-4}{\pi n^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$\therefore\ f(x) = \frac{\pi}{2} + \sum_{n=\text{odd}}^{\infty}\left(\frac{-4}{\pi n^2}\right)\cos nx$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)\,x\,.$$

Put $x = 0$ $\left(\text{continuous point}\right)$

$$f(0) = 0$$

$$\therefore 0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

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$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi}{2}$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$$

(i.e.) $$\frac{1}{1^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{8}.$$

1.9.5. EXAMPLE:

Find the Fourier series of $f(x)=|\sin x|$ in $-\pi<x<\pi$.

SOLUTION:

$$f(x)=|\sin x|$$

$$f(-x)=|\sin(-x)|=|-\sin x|=|\sin x|=f(x)$$

$\therefore f(x)$ is even function

$$\therefore b_n=0$$

$$\therefore\ f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx$$

$$=\frac{2}{\pi}\int_{0}^{\pi}|\sin x|\,dx$$

$$=\frac{2}{\pi}\int_{0}^{\pi}\sin x\,dx$$

$$=\frac{2}{\pi}[-\cos x]_0^{\pi}$$

$$=\frac{-2}{\pi}[-1-1]$$

$$a_0=\frac{4}{\pi}.$$

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$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |\sin x| \cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \cos nx \sin x\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos(n+1)x}{(n+1)} - \frac{-\cos(n-1)x}{(n-1)}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right) - \left(\frac{-1}{n+1} + \frac{1}{n-1}\right)\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n+1}}{n-1}\right) - \left(\frac{-1}{n+1} + \frac{1}{n-1}\right)\right] \quad \left(\because\ (-1)^{n+1} = (-1)^{n-1}\right)$$

$$= \frac{1}{\pi}\left[(-1)^{n+1}\left(\frac{-1}{n+1} + \frac{1}{n-1}\right) - \left(\frac{-1}{n+1} + \frac{1}{n-1}\right)\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{-1}{n+1} + \frac{1}{n-1}\right)\left((-1)^{n+1} - 1\right)\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{-n+1+n+1}{n^2-1}\right)\left((-1)^{n+1} - 1\right)\right]$$

$$= \frac{2}{\pi\left(n^2-1\right)}\left((-1)^{n+1} - 1\right)$$

$$a_n = \begin{cases} \dfrac{-4}{\pi\left(n^2-1\right)}, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases} \qquad \left(\text{if } n \neq 1\right)$$

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$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} |\sin x| \cos x\, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} \sin x \cos x\, dx$$

$$= \frac{2}{\pi}\frac{1}{2}\int_{0}^{\pi} 2\sin x \cos x\, dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \sin 2x\, dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi}$$

$$= -\frac{1}{2\pi}[1-1]$$

$$a_1 = 0.$$

$$\therefore\ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$= \frac{4}{\pi\,(2)} + (0) + \sum_{n=\text{even}}^{\infty} \frac{-4}{\pi\left(n^2-1\right)} \cos nx$$

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{\left((2n)^2-1\right)}\ .$$

1.9.6. EXAMPLE:

Expand $f(x) = |\cos x|$ in a Fourier series in the interval $-\pi < x < \pi$

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SOLUTION:

$f(x) = |\cos x|$

$f(-x) = |\cos(-x)| = |\cos x| = f(x)$

$\therefore f(x)$ is even function

$\therefore b_n = 0$

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} |\cos x|\, dx$$

$$= \frac{2}{\pi}\left[\int_{0}^{\pi/2} \cos x\, dx + \int_{\pi/2}^{\pi} (-\cos x)\, dx\right]$$

$$= \frac{2}{\pi}\left[[\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}[(1-0)-(0-1)]$$

$$a_0 = \frac{4}{\pi}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} |\cos x| \cos nx\, dx$$

$$= \frac{2}{\pi}\left[\int_{0}^{\pi/2} \cos x \cos nx\, dx + \int_{\pi/2}^{\pi} (-\cos x)\cos nx\, dx\right]$$

$$= \frac{2}{\pi}\left[\int_{0}^{\pi/2} \cos nx \cos x\, dx - \int_{\pi/2}^{\pi} \cos nx \cos x\, dx\right]$$

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$$= \frac{2}{\pi}\left[\frac{1}{2}\int_0^{\pi/2}[\cos(n+1)x + \cos(n-1)x]\,dx - \frac{1}{2}\int_{\pi/2}^{\pi}[\cos(n+1)x + \cos(n-1)x]\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]_0^{\pi/2} - \frac{1}{\pi}\left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right) - (0+0)\right] - \frac{1}{\pi}\left[(0+0) - \left(\frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos\frac{n\pi}{2}}{n+1} - \frac{\cos\frac{n\pi}{2}}{n-1} + \frac{\cos\frac{n\pi}{2}}{n+1} - \frac{\cos\frac{n\pi}{2}}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\cos\frac{n\pi}{2}}{n+1} - \frac{2\cos\frac{n\pi}{2}}{n-1}\right]$$

$$= \frac{2\cos\frac{n\pi}{2}}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{2\cos\frac{n\pi}{2}}{\pi}\left[\frac{n-1-n-1}{n^2-1}\right]$$

$$a_n = \frac{-4\cos\frac{n\pi}{2}}{\pi\left(n^2-1\right)} \quad (\text{if } \; n \neq 1)$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx$$

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$$= \frac{2}{\pi} \int_0^{\pi} |\cos x| \cos x \, dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} \cos x \cos x \, dx + \int_{\pi/2}^{\pi} (-\cos x) \cos x \, dx \right]$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} \cos^2 x \, dx - \int_{\pi/2}^{\pi} \cos^2 x \, dx \right]$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} \left(\frac{1+\cos 2x}{2} \right) dx - \int_{\pi/2}^{\pi} \left(\frac{1+\cos 2x}{2} \right) dx \right]$$

$$= \frac{1}{\pi} \left[\left[x + \frac{\sin 2x}{2} \right]_0^{\pi/2} - \left[x + \frac{\sin 2x}{2} \right]_{\pi/2}^{\pi} \right]$$

$$= \frac{1}{\pi} \left[\left[\left(\frac{\pi}{2} + 0 \right) - (0+0) \right] - \left[(\pi + 0) - \left(\frac{\pi}{2} + 0 \right) \right] \right]$$

$$= \frac{1}{\pi} \left[\frac{\pi}{2} - \pi + \frac{\pi}{2} \right]$$

$$a_1 = 0.$$

$$\therefore \; f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$= \frac{4}{\pi\,(2)} + (0) + \sum_{n=2}^{\infty} \frac{-4 \cos \frac{n\pi}{2}}{\pi \left(n^2 - 1\right)} \cos nx$$

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=2}^{\infty} \frac{\cos \frac{n\pi}{2}}{\left(n^2 - 1\right)} \cos nx$$

1.9.7. EXAMPLE:

Find the Fourier series expansion of $f(x) = \begin{cases} 0, & -\pi < x \le 0 \\ \sin x, & 0 \le x < \pi \end{cases}$ and evaluate

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(i) $\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\cdots$

(ii) $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\cdots$

SOLUTION:

$$f(x)=\begin{cases}0, & -\pi<x\le 0\\ \sin x, & 0\le x<\pi\end{cases}$$

$$f(-x)=\begin{cases}0, & -\pi<-x\le 0\\ -\sin x, & 0\le -x<\pi\end{cases}$$

$$=\begin{cases}0, & \pi>x\ge 0\\ -\sin x, & 0\ge x>\pi\end{cases}$$

$$=\begin{cases}-\sin x, & -\pi<x\le 0\\ 0, & 0\le x<\pi\end{cases}.$$

Here $f(-x)\ne f(x)$ and $f(-x)\ne -f(x)$

$\therefore$ $f(x)$ is neither even nor odd

$$\therefore\ f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx$$

$$=\frac{1}{\pi}\int_{0}^{\pi}\sin x\,dx$$

$$=\frac{1}{\pi}(-\cos x)_0^{\pi}$$

$$=-\frac{1}{\pi}(-1-1)$$

$$a_0=\frac{2}{\pi}.$$

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$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \sin x \cos nx\, dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \cos nx \sin x\, dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \frac{1}{2}\left[\sin(n+1)x - \sin(n-1)x\right] dx$$

$$= \frac{1}{2\pi}\int_{0}^{\pi} \left[\sin(n+1)x - \sin(n-1)x\right] dx$$

$$= \frac{1}{2\pi}\left[\frac{-\cos(n+1)x}{(n+1)} - \frac{-\cos(n-1)x}{(n-1)}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\left(\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right) - \left(\frac{-1}{n+1} + \frac{1}{n-1}\right)\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n+1}}{n-1}\right) - \left(\frac{-1}{n+1} + \frac{1}{n-1}\right)\right] \quad \left(\because\ (-1)^{n+1} = (-1)^{n-1}\right)$$

$$= \frac{1}{2\pi}\left[(-1)^{n+1}\left(\frac{-1}{n+1} + \frac{1}{n-1}\right) - \left(\frac{-1}{n+1} + \frac{1}{n-1}\right)\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{-1}{n+1} + \frac{1}{n-1}\right)\left((-1)^{n+1} - 1\right)\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{-n+1+n+1}{n^2-1}\right)\left((-1)^{n+1} - 1\right)\right]$$

$$= \frac{2}{2\pi\left(n^2-1\right)}\left((-1)^{n+1} - 1\right)$$

$$a_n = \begin{cases} \dfrac{-2}{\pi\left(n^2-1\right)}, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases} \qquad \left(\text{if } n \neq 1\right)$$

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$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin x \cos x\, dx$$

$$= \frac{1}{\pi}\frac{1}{2}\int_0^{\pi} 2\sin x\cos x\, dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} \sin 2x\, dx$$

$$= \frac{1}{2\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi}$$

$$= -\frac{1}{4\pi}[1-1]$$

$$a_1 = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin x \sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin nx\sin x\, dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} \left[\cos(n-1)x - \cos(n+1)x\right] dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}[(0-0)-(0-0)]$$

$$b_n = 0 \quad (\text{if } n \neq 1)$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\, dx$$

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$$= \frac{1}{\pi}\int_0^{\pi} \sin x \sin x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin^2 x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}[(\pi - 0) - (0 - 0)]$$

$$b_1 = \frac{1}{2}.$$

$$\therefore \; f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx + b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

$$= \frac{2}{\pi(2)} + (0) + \sum_{n=\text{even}}^{\infty} \frac{-2}{\pi\left(n^2 - 1\right)} \cos nx + \frac{1}{2}\sin x + (0)$$

$$= \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{\left((2n)^2 - 1\right)} + \frac{1}{2}\sin x$$

$$f(x) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{(2n-1)(2n+1)} + \frac{1}{2}\sin x.$$

(i) Put $x = 0$ (continuous point)

$$0 = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}$$

(ii) Put $x = \frac{\pi}{2}$ (continuous point)

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$$1 = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)(2n+1)} + \frac{1}{2}(1)$$

$$1 - \frac{1}{\pi} - \frac{1}{2} = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)(2n+1)}$$

$$\frac{\pi-2}{2\pi} = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)(2n+1)}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)(2n+1)} = -\left(\frac{\pi-2}{4}\right)$$

$$-\frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + \cdots = -\left(\frac{\pi-2}{4}\right)$$

$$\text{(i.e.)} \quad \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots = \frac{\pi-2}{4}.$$

1.10. TYPE III: PROBLEMS UNDER THE INTERVAL $(0, 2l)$

1.10.1. DEFINITON:

The Fourier expansion for $f(x)$ in the interval $0 < x < 2l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

Where

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)\,dx$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)\,dx$$

1.10.2. EXAMPLE:

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Find the Fourier series expansion of period $2l$ for the function $f(x) = (l - x)^2$ in the range $(0, 2l)$. Deduce that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

SOLUTION:

The Fourier series be,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

Where

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\, dx$$

$$= \frac{1}{l}\int_0^{2l} (l - x)^2\, dx$$

$$= \frac{1}{l}\left[\frac{(l - x)^3}{-3}\right]_0^{2l}$$

$$= -\frac{1}{3l}\left[(-l)^3 - l^3\right]$$

$$a_0 = \frac{2l^2}{3}.$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{1}{l}\int_0^{2l} (l - x)^2 \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{1}{l}\left[(l - x)^2 \frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} - [2(l - x)(-1)]\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2} + [-2(-1)]\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}\right]_0^{2l}$$

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$$= \frac{1}{l}\left[(l-x)^2 \frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} - 2(l-x)\frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2} - 2\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}\right]_0^{2l}$$

$$= \frac{1}{l}\left[\left((0) - 2(-l)\frac{1}{\left(\frac{n\pi}{l}\right)^2} - 0\right) - \left(0 - 2l\frac{1}{\left(\frac{n\pi}{l}\right)^2} - 0\right)\right]$$

$$= \frac{1}{l}\left[\frac{2l}{\left(\frac{n\pi}{l}\right)^2} + \frac{2l}{\left(\frac{n\pi}{l}\right)^2}\right]$$

$$= \frac{1}{l}\left[\frac{4l}{\left(\frac{n\pi}{l}\right)^2}\right]$$

$$a_n = \frac{4l^2}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)\,dx$$

$$= \frac{1}{l}\int_0^{2l} (l-x)^2 \sin\left(\frac{n\pi x}{l}\right)\,dx$$

$$= \frac{1}{l}\left[(l-x)^2 \frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} - [2(l-x)(-1)]\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2} + [-2(-1)]\frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}\right]_0^{2l}$$

$$= \frac{1}{l}\left[(l-x)^2 \frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} - 2(l-x)\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2} + 2\frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}\right]_0^{2l}$$

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$$= \frac{1}{l}\left[\left(-l^2 \frac{1}{\left(\frac{n\pi}{l}\right)} - 0 + 2\frac{1}{\left(\frac{n\pi}{l}\right)^3}\right) - \left(-l^2 \frac{1}{\left(\frac{n\pi}{l}\right)} + 0 + 2\frac{1}{\left(\frac{n\pi}{l}\right)^3}\right)\right]$$

$$= \frac{1}{l}\left[\frac{-l^2}{\left(\frac{n\pi}{l}\right)} + \frac{2}{\left(\frac{n\pi}{l}\right)^3} + \frac{l^2}{\left(\frac{n\pi}{l}\right)} - \frac{2}{\left(\frac{n\pi}{l}\right)^3}\right]$$

$$= \frac{1}{l}(0)$$

$$b_n = 0$$

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{2l^2}{3(2)} + \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2} \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} (0) \sin\left(\frac{n\pi x}{l}\right)$$

$$f(x) = \frac{l^2}{3} + \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left(\frac{n\pi x}{l}\right)$$

put $x = 0$ $\left(\text{dis. cont. and end point}\right)$

$$\therefore \quad \frac{f(0) + f(2l)}{2} = \frac{l^2}{3} + \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{l^2 + l^2}{2} = \frac{l^2}{3} + \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

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1.10.3. EXAMPLE:

Find the Fourier Series of $f(x) = \begin{cases} \frac{x}{l}, & 0 < x < l \\ \frac{2l - x}{l}, & l < x < 2l \end{cases}$ in the range $(0, 2l)$.

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SOLUTION:

The Fourier series be,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

Where

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$= \frac{1}{l}\left[\int_0^{l} f(x)\,dx + \int_l^{2l} f(x)\,dx\right]$$

$$= \frac{1}{l}\left[\int_0^{l}\left(\frac{x}{l}\right)dx + \int_l^{2l}\left(\frac{2l-x}{l}\right)dx\right]$$

$$= \frac{1}{l^2}\left[\frac{x^2}{2}\right]_0^l + \frac{1}{l^2}\left[\frac{(2l-x)^2}{-2}\right]_l^{2l}$$

$$= \frac{1}{2l^2}\left[l^2 - 0\right] - \frac{1}{2l^2}\left[0 - l^2\right]$$

$$= \frac{1}{2l^2}l^2 + \frac{1}{2l^2}l^2$$

$$a_0 = 1.$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\left[\int_0^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx + \int_l^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{1}{l}\left[\int_0^{l}\left(\frac{x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx + \int_l^{2l}\left(\frac{2l-x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{1}{l^2}\left[x\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} - (1)\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right]_0^l + \frac{1}{l^2}\left[(2l-x)\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} - (-1)\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right]_l^{2l}$$

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$$= \frac{1}{l^2}\left[\left(0+\frac{(-1)^n}{\left(\frac{n\pi}{l}\right)^2}\right)-\left(0+\frac{1}{\left(\frac{n\pi}{l}\right)^2}\right)\right]+\frac{1}{l^2}\left[\left(0-\frac{1}{\left(\frac{n\pi}{l}\right)^2}\right)-\left(0-\frac{(-1)^n}{\left(\frac{n\pi}{l}\right)^2}\right)\right]$$

$$= \frac{1}{l^2}\left[\frac{(-1)^n}{\left(\frac{n\pi}{l}\right)^2}-\frac{1}{\left(\frac{n\pi}{l}\right)^2}-\frac{1}{\left(\frac{n\pi}{l}\right)^2}+\frac{(-1)^n}{\left(\frac{n\pi}{l}\right)^2}\right]$$

$$= \frac{1}{l^2}\left[\frac{2(-1)^n}{\left(\frac{n\pi}{l}\right)^2}-\frac{2}{\left(\frac{n\pi}{l}\right)^2}\right]$$

$$= \frac{2}{l^2}\frac{l^2}{n^2\pi^2}\left((-1)^n-1\right)$$

$$= \frac{2}{n^2\pi^2}\left((-1)^n-1\right)$$

$$a_n = \begin{cases} \dfrac{-4}{n^2\pi^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\left[\int_0^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx+\int_l^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{1}{l}\left[\int_0^{l}\left(\frac{x}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx+\int_l^{2l}\left(\frac{2l-x}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

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$$= \frac{1}{l^2}\left[x\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} - (1)\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2} \right]_0^l$$

$$+ \frac{1}{l^2}\left[(2l-x)\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} - (-1)\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2} \right]_l^{2l}$$

$$= \frac{1}{l^2}\left[\left(-l\frac{(-1)^n}{\left(\frac{n\pi}{l}\right)} + 0\right) - (0+0)\right] + \frac{1}{l^2}\left[(0+0) - \left(-l\frac{(-1)^n}{\left(\frac{n\pi}{l}\right)} - 0\right)\right]$$

$$= \frac{1}{l^2}\left[\frac{-l(-1)^n}{\left(\frac{n\pi}{l}\right)} + \frac{l(-1)^n}{\left(\frac{n\pi}{l}\right)}\right]$$

$b_n = 0$.

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{1}{2} + \sum_{n=\text{odd}}^{\infty}\left(\frac{-4}{n^2\pi^2}\right)\cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty}(0)\sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{(2n-1)^2}\right)\cos\left(\frac{(2n-1)\pi x}{l}\right)$$

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1.10.4. EXAMPLE:

Obtain the Fourier series expansion of $f(x)$ if

$$f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 2, & 1 \le x \le 2 \end{cases}, \text{ and } f(x+2) = f(x).$$

SOLUTION:

Since $(0, 2l)$

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Here $(0,2)$

$\Rightarrow 2l = 2$

$\Rightarrow l = 1$

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$= \frac{1}{1}\int_0^{2} f(x)\,dx$$

$$= \int_0^1 (1)\,dx + \int_1^2 (2)\,dx$$

$$= [x]_0^1 + 2[x]_1^2$$

$$= (1-0) + 2(2-1)$$

$$a_0 = 3.$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{1}{1}\int_0^{2} f(x)\cos(n\pi x)\ dx$$

$$= \int_0^1 (1)\cos(n\pi x)\ dx + \int_1^2 (2)\cos(n\pi x)\ dx$$

$$= \left[\frac{\sin n\pi x}{n\pi}\right]_0^1 + 2\left[\frac{\sin n\pi x}{n\pi}\right]_1^2$$

$$= \frac{1}{n\pi}(0+0) + 2\frac{1}{n\pi}(0+0)$$

$$a_n = 0.$$

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$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{1}{1}\int_0^{2} f(x)\sin(n\pi x)\ dx$$

$$= \int_0^1 (1)\sin(n\pi x)\ dx + \int_1^2 (2)\sin(n\pi x)\ dx$$

$$= \left[\frac{-\cos n\pi x}{n\pi}\right]_0^1 + 2\left[\frac{-\cos n\pi x}{n\pi}\right]_1^2$$

$$= \frac{-1}{n\pi}\left[(-1)^n - 1\right] - \frac{2}{n\pi}\left[1-(-1)^n\right]$$

$$= \frac{-1}{n\pi}\left[(-1)^n - 1\right] + \frac{2}{n\pi}\left[(-1)^n - 1\right]$$

$$= \frac{1}{n\pi}\left[(-1)^n - 1\right]$$

$$b_n = \begin{cases} \dfrac{-2}{n\pi}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$\therefore \qquad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

$$= \frac{3}{2} + \sum_{n=1}^{\infty} (0)\cos(n\pi x) + \sum_{n=\text{odd}}^{\infty}\left(\frac{-2}{n\pi}\right)\sin(n\pi x)$$

$$f(x) = \frac{3}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin((2n-1)\pi x)}{(2n-1)}.$$

1.10.5. EXAMPLE:

Find the Fourier series expansion for $f(x) = \begin{cases} (l-x), & 0 \le x \le l \\ 0, & l \le x \le 2l \end{cases}$.

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$$a_0 = \frac{l}{2}$$

$$a_n = \begin{cases} \frac{2l}{n^2\pi^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$b_n = \frac{1}{n\pi}.$$

1.10.6. EXAMPLE:

Find the Fourier series of $f(x) = \pi x$ in $0 < x < 2$.

SOLUTION:

$$a_0 = 2\pi$$

$$a_n = 0$$

$$b_n = \frac{-2}{n}.$$

1.11. TYPE IV: PROBLEMS UNDER THE INTERVAL $(-l, l)$:

1.11.1. EXAMPLE:

If $f(x) = x$ is defined in $-l < x < l$ with period $2l$, find the Fourier expansion of $f(x)$.

SOLUTION:

$$f(x) = x$$

$$f(-x) = -x = -f(x)$$

$\therefore f(x)$ is odd.

$$\therefore a_0 = 0,\ a_n = 0.$$

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$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

Where

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{1}{l}\int_{-l}^{l} x\sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{l}\int_{0}^{l} x\ \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{1}{l}\left[x\,\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} - (1)\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right]_0^l$$

$$= \frac{2}{l}\left[\left(\frac{-l(-1)^n}{\left(\frac{n\pi}{l}\right)} + 0\right) - (0+0)\right]$$

$$= \frac{2}{l}.\frac{-l(-1)^n}{\left(\frac{n\pi}{l}\right)}$$

$$b_n = \frac{-2l}{n\pi}(-1)^n$$

$$\therefore \quad f(x) = \frac{-2l}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi x}{l}\right).$$

1.11.2. EXAMPLE:

Obtain the Fourier series for the function

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$$f(x)=\begin{cases}1+\dfrac{2x}{l}, & -l\le x\le 0\\ 1-\dfrac{2x}{l}, & 0\le x\le l\end{cases}$$, and hence deduce that $\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\cdots=\dfrac{\pi^2}{8}$.

SOLUTION:

$$f(x)=\begin{cases}1+\dfrac{2x}{l}, & -l\le x\le 0\\ 1-\dfrac{2x}{l}, & 0\le x\le l\end{cases}$$

$$f(-x)=\begin{cases}1-\dfrac{2x}{l}, & -l\le -x\le 0\\ 1+\dfrac{2x}{l}, & 0\le -x\le l\end{cases}$$

$$=\begin{cases}1-\dfrac{2x}{l}, & l\ge x\ge 0\\ 1+\dfrac{2x}{l}, & 0\ge x\ge -l\end{cases}$$

$$=\begin{cases}1+\dfrac{2x}{l}, & -l\le x\le 0\\ 1-\dfrac{2x}{l}, & 0\le x\le l\end{cases}$$

$$=f(x)$$

$\therefore$ $f(x)$ is even.

$\therefore$ $b_n=0$.

$$\therefore\quad f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{l}\right)$$

$$a_0=\frac{2}{l}\int_0^l f(x)\,dx$$

$$=\frac{2}{l}\int_0^l\left(1-\frac{2x}{l}\right)dx$$

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$$= \frac{2}{l}\left[\frac{\left(1-\frac{2x}{l}\right)^2}{2\left(\frac{-2}{l}\right)}\right]_0^l$$

$$= \frac{-1}{2}[1-1]$$

$$a_0 = 0.$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l \left(1-\frac{2x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[\left(1-\frac{2x}{l}\right)\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} - \left(-\frac{2}{l}\right)\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right]_0^l$$

$$= \frac{2}{l}\left[\left(0-\left(\frac{2}{l}\right)\frac{(-1)^n}{\left(\frac{n\pi}{l}\right)^2}\right) - \left(0-\left(\frac{2}{l}\right)\frac{1}{\left(\frac{n\pi}{l}\right)^2}\right)\right]$$

$$= \frac{2}{l}\left[-\frac{2}{l}\frac{(-1)^n}{\left(\frac{n\pi}{l}\right)^2} + \frac{2}{l}\frac{1}{\left(\frac{n\pi}{l}\right)^2}\right]$$

$$= \frac{2}{l}\left[\frac{2}{l}\frac{l^2}{n^2\pi^2}\right]\left(1-(-1)^n\right)$$

$$= \begin{cases} \frac{8}{n^2\pi^2} & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$= 0 + \sum_{n=\text{odd}}^{\infty} \frac{8}{n^2\pi^2}\cos\left(\frac{n\pi x}{l}\right)$$

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$$f(x) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left(\frac{(2n-1)\pi x}{l}\right)$$

put $x = 0$: (continuous point)

$$f(0) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$1 = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

1.11.3. EXAMPLE:

Find the Fourier series of $f(x)$ in $(-2,2)$ which is defined by $f(x) = \begin{cases} 0, & -2 < x < -1 \\ k, & -1 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$.

SOLUTION:

Here $(-2,2)$

$(-l,l)$

$\Rightarrow \quad l = 2.$

$$f(-x) = \begin{cases} 0, & -2 < -x < -1 \\ k, & -1 < -x < 1 \\ 0, & 1 < -x < 2 \end{cases}$$

$$= \begin{cases} 0, & 2 > x > 1 \\ k, & 1 > x > -1 \\ 0, & -1 > x > -2 \end{cases}$$

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$$= \begin{cases} 0, & -2 < x < -1 \\ k, & -1 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$$

$$= f(x)$$

$\therefore$ $f(x)$ is even.

$\therefore$ $b_n = 0$.

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{2}\int_0^2 f(x)\,dx$$

$$= \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

$$= \int_0^1 k\,dx + \int_1^2 (0)\,dx$$

$$= k\,[x]_0^1$$

$$= k(1-0)$$

$$a_0 = k.$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^1 f(x)\cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

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$$= \int_0^1 k\cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 (0)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= k\left[\frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right]_0^1$$

$$= \frac{2k}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right) - 0\right]$$

$$a_n = \frac{2k}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x}{2}\right).$$

1.12. HALF RANGE SERIES:

1.12.1. FORMULA:

Half range cosine series in $(0,\pi)$,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

in $(0,l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

Where $a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx$

$$a_n = \frac{2}{l}\int_0^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

Half range sine series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$b_n = \frac{2}{l}\int_0^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

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1.12.2. EXAMPLE:

Expand the function $f(x) = x,\ 0 < x < \pi$ in Fourier sine series.

SOLUTION:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

$$= \frac{-2}{n}(-1)^n$$

1.12.3. EXAMPLE:

Find the Fourier cosine series for $f(x) = x^2$ in $0 < x < \pi$.

SOLUTION:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\ dx = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx = \frac{4}{n^2}(-1)^n.$$

1.12.4. EXAMPLE:

Find the half range sine series $f(x) = x(\pi - x)$ in $(0,\pi)$. Hence deduce that

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots = \frac{\pi^3}{32}.$$

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SOLUTION:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

$$= \frac{4}{n^3\pi}\left(1-(-1)^n\right)$$

$$= \begin{cases} \frac{8}{n^3\pi}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$\therefore \quad f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \sum_{n=\text{odd}}^{\infty} \frac{8}{n^3\pi}\sin nx$$

$$f(x) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin(2n-1)x$$

$$f(x) = \frac{8}{\pi}\left[\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \cdots\right]$$

$$\text{put } x = \frac{\pi}{2} \ (\text{cont. point})$$

$$\frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{8}{\pi}\left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots\right]$$

$$\Rightarrow \quad \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots = \frac{\pi^3}{32}$$

1.12.5. EXAMPLE:

Find the Fourier sine series of $f(x) = l - x$ in $(0, l)$

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$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{2l}{n\pi}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2l}{n\pi}\sin\left(\frac{n\pi x}{l}\right).$$

1.12.6. EXAMPLE:

Find the Half-range cosine series for $f(x) = (x-1)^2$ in $(0,1)$. Hence show that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{24}.$$

SOLUTION:

Here $(0,l)$

$(0,1)$

$\Rightarrow$ $l = 1$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\, dx = \frac{2}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{4}{n^2\pi^2}$$

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$f(x) = \frac{2}{3(2)} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{l}\right)$$

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\cos\left(\frac{n\pi x}{l}\right)$$

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put $x = 0$ (dis. cont. point)

$$\therefore \quad \frac{f(0)+f(1)}{2} = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{1+0}{2} = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \quad \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{24}.$$

1.13. PARSEVAL'S THEOREM:

1.13.1. STATEMENT:

Let $f(x)$ be a periodic function with period 2π defined in the interval $(-\pi,\pi)$, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x))^2dx = \frac{a_0{}^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right),$$

where a_0, a_n and b_n are Fourier coefficient of $f(x)$.

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1.13.2. DEFINITION: ROOT MEAN SQUARE (OR) EFFECTIVE VALUE OF $f(x)$:

Let $f(x)$ be a function defined in (a,b) then $\sqrt{\frac{\int_a^b (f(x))^2\, dx}{b-a}}$ is called the root mean square and is denoted by $\bar{y}$. Hence

$$\bar{y}^2 = \frac{1}{b-a}\int_a^b (f(x))^2\, dx.$$

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1.13.3. NOTE:

Parseval's theorem gives the value of root mean square (RMS) of $f(x)$ in terms of its Fourier coefficients.

1.13.4. EXAMPLE:

Find the Fourier series of x^2 in $(-\pi,\pi)$. Use parseval's identity to prove that

$$\frac{\pi^4}{90}=\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots$$

SOLUTION:

$$f(x)=x^2$$

$f(x)$ is even, $\therefore\ b_n=0$

$$\therefore\quad f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$$

$$a_0=\frac{2}{\pi}\int_0^{\infty}f(x)\,dx=\frac{2\pi^2}{3}$$

$$a_n=\frac{2}{\pi}\int_0^{\infty}f(x)\cos nx\,dx=\frac{4}{n^2}(-1)^n$$

$$\therefore\qquad f(x)=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx$$

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By Parseval's theorem,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx=\frac{a_0{}^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}x^4\,dx=\frac{4\pi^4/9}{4}+\frac{1}{2}\sum_{n=1}^{\infty}\left(\left(\frac{4}{n^2}(-1)^n\right)^2+0^2\right)$$

$$\frac{2}{2\pi}\int_0^{\pi}x^4\,dx=\frac{\pi^4}{9}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4}(-1)^{2n}$$

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$$\frac{1}{\pi}\left[\frac{x^5}{5}\right]_0^{\pi} = \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{1}{\pi}\left[\frac{\pi^5 - 0}{5}\right] = \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{\pi^4}{5} = \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\Rightarrow \qquad \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

1.13.5. EXAMPLE:

Find the sine series for $f(x) = x$ in $0 < x < \pi$, then using R.M.S. value show that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

SOLUTION:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{-2}{n}(-1)^n$$

$$\therefore\ f(x) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

Using RMS value,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx = \frac{a_0{}^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}x^2\,dx = 0 + \frac{1}{2}\sum_{n=1}^{\infty}\left(0 + \frac{4}{n^2}(1)\right)$$

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$$\frac{2}{2\pi}\int_0^{\pi} x^2\, dx = 2\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = 2\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{1}{3\pi}\left(\pi^3 - 0\right) = 2\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\therefore \qquad \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

1.13.6. EXAMPLE:

Obtain the Fourier series of $f(x) = x^2$ in $(-l, l)$. Hence find the value of the series

$\frac{1}{1^4} + \frac{1}{2^4} + \cdots$.

SOLUTION:

$f(x) = x^2$

$f(x)$ is even

$\therefore \quad b_n = 0$

$$a_0 = \frac{2}{l}\int_0^{l} f(x)\, dx = \frac{2l^3}{3}$$

$$a_n = \frac{2}{l}\int_0^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{4l^2(-1)^n}{n^2\pi^2}$$

By Parseval's theorem,

$$\frac{1}{2l}\int_{-l}^{l}(f(x))^2\, dx = \frac{{a_0}^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

Hence we have $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$.

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1.14. HARMONIC ANALYSIS:

1.14.1. DEFINITION:

The Fourier series be,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where

$$a_0 = 2\left(\frac{\sum f(x)}{n}\right)$$

$$a_n = 2\left(\frac{\sum f(x)\cos nx}{n}\right)$$

$$b_n = 2\left(\frac{\sum f(x)\sin nx}{n}\right)$$

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1.15. TYPE I: (GIVEN DATA ARE IN π FORM)

1.15.1. EXAMPLE:

Find the Fourier series up to two harmonic for $y = f(x)$ in $(0, 2\pi)$ defined by the table of values given below:

$x:$	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
$y:$	1.98	1.30	1.05	1.30	– 0.88	– 0.25	1.98

SOLUTION:

Since the last value of y is a repetition of the first, only the first six values will be used.

The Fourier series be,

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$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{2} a_n \cos nx + \sum_{n=1}^{2} b_n \sin nx$$

$$= \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x$$

x	y	$\cos x$	$\sin x$	$\cos 2x$	$\sin 2x$	$y \cos x$	$y \sin x$	$y \cos 2x$	$y \sin 2x$
0	1.98	1	0	1	0	1.98	0	1.98	0
$\pi/3$	1.3	0.5	0.866	–0.5	0.866	0.65	1.126	–0.65	1.126
$2\pi/3$	1.05	–0.5	0.866	–0.5	–0.866	–0.525	0.909	–0.525	–0.909
π	1.3	–1	0	1	0	–1.3	0	1.3	0
$4\pi/3$	–0.88	–0.5	–0.866	–0.5	0.866	0.44	0.762	0.44	–0.762
$5\pi/3$	–0.25	0.5	–0.866	–0.5	–0.866	–0.125	0.217	0.125	0.2167
$\sum$	4.5					1.12	3.014	2.67	–0.328

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Now,

$$a_0 = 2\left(\frac{\sum y}{n}\right) = 2\left(\frac{4.5}{6}\right) = 1.5$$

$$a_1 = 2\left(\frac{\sum y \cos x}{n}\right) = 2\left(\frac{1.12}{6}\right) = 0.373$$

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$$a_2 = 2\left(\frac{\sum y\cos 2x}{n}\right) = 2\left(\frac{2.67}{6}\right) = 0.89$$

$$b_1 = 2\left(\frac{\sum y\sin x}{n}\right) = 2\left(\frac{3.014}{6}\right) = 1.005$$

$$b_2 = 2\left(\frac{\sum y\sin 2x}{n}\right) = 2\left(\frac{-0.328}{6}\right) = -0.109$$

$$\therefore \quad y = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + b_1\sin x + b_2\sin 2x$$

$$= \frac{1.5}{2} + (0.373)\cos x + (0.89)\cos 2x + (1.005)\sin x + (-0.109)\sin 2x$$

$$y = 0.75 + (0.373)\cos x + (0.89)\cos 2x + (1.005)\sin x - (0.109)\sin 2x$$

1.16. TYPE II: (GIVEN DATA ARE IN DEGREE FORM)

1.16.1. EXAMPLE:

Find the Fourier series up to first harmonic for the following values:

x in degree:	0	60	120	180	240	300	360
$y = f(x)$:	40	31	–13.7	20	3.7	–21	40

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SOLUTION:

Since the last value of y is a repetition of the first, only the first six values will be used.

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x	y	$\cos x$	$\sin x$	$y \cos x$	$y \sin x$
0	40	1	0	40	0
60	31	0.5	0.866	15.5	26.846
120	–13.7	–0.5	0.866	6.85	–11.864
180	20	–1	0	– 20.00	0
240	3.7	–0.5	–0.866	–1.85	–3.204
300	–21	0.5	–0.866	– 10.5	18.186
$\sum$	60			30	29.964

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Now,

$$a_0 = 2\left(\frac{\sum y}{n}\right) = 2\left(\frac{60}{6}\right) = 20$$

$$a_1 = 2\left(\frac{\sum y \cos x}{n}\right) = 2\left(\frac{30}{6}\right) = 10$$

$$b_1 = 2\left(\frac{\sum y \sin x}{n}\right) = 2\left(\frac{29.964}{6}\right) = 9.988$$

$$\therefore \quad f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x$$

$$f(x) = \frac{20}{2} + 10 \cos x + (9.988) \sin x \,.$$

1.17. TYPE: III: (GIVEN DATA ARE IN T FORM)

1.17.1. FORMULAE:

$$\theta = \frac{2\pi x}{T}$$

1.17.2. EXAMPLE:

The values of x and the corresponding values of $f(x)$ over a period T are given below, show that $f(x) = 0.75 + 0.37\cos\theta + 1.004\sin\theta$, where $\theta = \frac{2\pi x}{T}$

x :	0	T/6	T/3	T/2	2T/3	5T/6	T
$y = f(x)$	1.98	1.3	1.05	1.3	–0.88	–0.25	1.98

SOLUTION:

First and last value are same. Hence we omit the last value.

The Fourier series be $f(x) = \frac{a_0}{2} + a_1\cos\theta + b_1\sin\theta$

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Where $a_0 = 2\left(\frac{\sum y}{n}\right)$

$$a_1 = 2\left(\frac{\sum y\cos\theta}{n}\right)$$

$$b_1 = 2\left(\frac{\sum y\sin\theta}{n}\right).$$

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x	$\theta = \frac{2\pi x}{T}$	y	$\cos\theta$	$\sin\theta$	$y\cos\theta$	$y\sin\theta$
0	0	1.98	1	0	1.98	0
T/6	$\pi/3$	1.3	0.5	0.866	0.65	1.1258
T/3	$2\pi/3$	1.05	–0.5	0.866	–0.525	0.9093
T/2	π	1.3	–1	0	–1.3	0
2T/3	$4\pi/3$	–0.88	–0.5	–0.866	0.44	0.762
5T/6	$5\pi/3$	–0.25	0.5	–0.866	–0.125	0.2165
$\sum$		4.5			1.12	3.013

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$$a_0 = 2\left(\frac{\sum y}{n}\right) = 2\left(\frac{4.5}{6}\right) = 1.5$$

$$a_1 = 2\left(\frac{\sum y\cos\theta}{n}\right) = 2\left(\frac{1.12}{6}\right) = 0.37$$

$$b_1 = 2\left(\frac{\sum y\sin\theta}{n}\right) = 2\left(\frac{3.013}{6}\right) = 1.005$$

$\therefore\ f(x) = 0.75 + 0.37\cos\theta + 1.005\sin\theta$.

1.18. TYPE IV: (GIVEN DATA ARE IN l FORM)

1.18.1. EXAMPLE:

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Find the constant term and the coeff. of the first sine and cosine terms in the Fourier expansion of y as given in the following table:

x:	0	1	2	3	4	5
y:	9	18	24	28	26	20

SOLUTION:

Here the length of the interval is $2l = 6$.

$\therefore \quad l = 3$.

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right)$$

Here $f(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{3}\right) + b_1 \sin\left(\frac{\pi x}{3}\right)$

x	y	$\cos\left(\frac{\pi x}{3}\right)$	$\sin\left(\frac{\pi x}{3}\right)$	$y\cos\left(\frac{\pi x}{3}\right)$	$y\sin\left(\frac{\pi x}{3}\right)$
0	9	1	0	9	0
1	18	0.5	0.866	9	15.588
2	24	–0.5	0.866	–12	20.785
3	28	–1	0	–28	0
4	26	–0.5	–0.866	–13	–22.517
5	20	0.5	–0.866	10	–17.321
$\sum$	125			–25	–3.465

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$$a_0 = 2\left(\frac{\sum y}{n}\right) = 2\left(\frac{125}{6}\right) = 41.67$$

$$a_1 = 2\left(\frac{\sum y\cos\left(\frac{\pi x}{3}\right)}{n}\right) = 2\left(\frac{-25}{6}\right) = -8.33$$

$$b_1 = 2\left(\frac{\sum y\sin\left(\frac{\pi x}{3}\right)}{n}\right) = 2\left(\frac{-3.465}{6}\right) = -1.16$$

$$\therefore\ f(x) = \frac{41.67}{2} + (-8.33)\cos\left(\frac{\pi x}{3}\right) + (-1.16)\sin\left(\frac{\pi x}{3}\right).$$

1.19. COMPLEX FORM OF FOURIER SERIES:

1.19.1. DEFINITION:

Let $f(x)$ be a periodic function with period 2π. Then the complex form of Fourier series in the interval $(0,2\pi)$ is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

Where $$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx.$$

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1.19.2. NOTE:

(i) In the interval $-\pi < x < \pi$,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \qquad \text{where } c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

(ii) In the interval $(0,2l)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}} \qquad \text{where } c_n = \frac{1}{2l}\int_0^{2l} f(x)e^{\frac{-in\pi x}{l}}dx$$

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(iii) In the interval $-l < x < l$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{\frac{in\pi x}{l}}, \quad \text{where} \quad c_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{\frac{-inx}{l}}\, dx.$$

1.19.3. EXAMPLE:

Find the complex form of Fourier series of the function $f(x) = e^x$ when $-\pi < x < \pi$ and $f(x+2\pi) = f(x)$.

SOLUTION:

The fourier series be $f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{inx}$

Where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx} dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x \, e^{-inx} dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-in)x} dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{(1-in)x}}{(1-in)}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{(1-in)\pi} - e^{-(1-in)\pi}}{(1-in)}\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{\pi} e^{-in\pi} - e^{-\pi} e^{in\pi}}{(1-in)}\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{\pi}(-1)^n - e^{-\pi}(-1)^n}{(1-in)}\right] \qquad \left(\begin{aligned} \because e^{-n\pi} &= \cos n\pi + i \sin n\pi \\ &= (-1)^n + 0 \end{aligned}\right)$$

$$= \frac{(-1)^n}{\pi(1-in)}\left(\frac{e^{\pi} - e^{-\pi}}{2}\right)$$

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$$= \frac{(-1)^n}{\pi(1-in)} \sinh \pi$$

$$\therefore \quad f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{inx}$$

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\pi(1-in)} \sinh \pi \, e^{inx}$$

1.19.4. EXAMPLE:

Find the Complex form of the Fourier series of $f(x) = e^{-x}$ in $-1 < x < 1$.

SOLUTION:

The Fourier series be

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{\frac{in\pi x}{l}},$$

where

$$c_n = \frac{1}{2l} \int_{-l}^{l} f(x) e^{\frac{-in\pi x}{l}} dx$$

$$= \frac{1}{2} \int_{-1}^{1} f(x) e^{-in\pi x} dx$$

$$= \frac{1}{2} \int_{-1}^{1} e^{-x} \, e^{-in\pi x} dx$$

$$= \frac{1}{2} \int_{-1}^{1} e^{-(1+in\pi)x} dx$$

$$= \frac{1}{2} \left[\frac{e^{-(1+in\pi)x}}{-(1+in\pi)} \right]_{-1}^{1}$$

$$= \frac{1}{2} \left[\frac{e^{-(1+in\pi)} - e^{(1+in\pi)}}{-(1+in\pi)} \right]$$

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$$= \frac{1}{2}\left[\frac{e^{-1}e^{-in\pi} - e^{1}e^{in\pi}}{-(1+in\pi)}\right]$$

$$= \frac{1}{2}\left[\frac{e^{-1}(-1)^n - e^{1}(-1)^n}{-(1+in\pi)}\right]$$

$$= \frac{(-1)^n}{-(1+in\pi)}\left[\frac{e^{-1} - e^{1}}{2}\right]$$

$$= \frac{(-1)^n}{(1+in\pi)}\left[\frac{e^{1} - e^{-1}}{2}\right]$$

$$= \frac{(-1)^n}{(1+in\pi)}\sinh 1$$

$$\therefore \quad f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{\frac{in\pi x}{l}}$$

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(1+in\pi)}\sinh 1\, e^{\frac{in\pi x}{l}}.$$

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