UNIT II

FOURIER TRANSFORM

2.1. FORMULA:

1. Fourier Transform:

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

2. Fourier Inverse Transform:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

3. Fourier Sine Transform:

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin sx\, dx$$

4. Fourier Inverse Sine Transform:

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$$

5. Fourier Cosine Transform:

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

6. Fourier Inverse Cosine Transform:

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds .$$

2.2. EXAMPLES:

2.2.1. EXAMPLE:

Find the Fourier transform of $f(x) = \begin{cases} x, & |x| \le a \\ 0, & |x| > a \end{cases}$.

SOLUTION:

$$f(x) = \begin{cases} x, & |x| \le a \\ 0, & |x| > a \end{cases} \quad \text{(i.e.)} \; f(x) = \begin{cases} x, & -a \le x \le a \\ 0, & -\infty < x < -a \;\; \& \;\; a < x < \infty \end{cases}$$

The Fourier transform,

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x \;\; (\cos sx + i \sin sx) \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x \;\; \cos sx \, dx + \frac{i}{\sqrt{2\pi}} \int_{-a}^{a} x \;\; \sin sx \, dx$$

$$= 0 + \frac{i}{\sqrt{2\pi}} 2 \int_0^a x \;\; \sin sx \, dx$$

$(\because \;\; f(x) = x \cos x \;$ is odd $\& \; f(x) = x \sin x \;$ is even$)$

$$= i\sqrt{\frac{2}{\pi}}\left[x\left(\frac{-\cos sx}{s}\right) - (1)\left(\frac{-\sin sx}{s^2}\right)\right]_0^a$$

$$= i\sqrt{\frac{2}{\pi}}\left[\left(\frac{-a\cos sa}{s} + \frac{\sin sa}{s^2}\right) - (0+0)\right]$$

$$= i\sqrt{\frac{2}{\pi}}\left(\frac{-as\cos sa + \sin sa}{s^2}\right)$$

$$F(s) = i\sqrt{\frac{2}{\pi}}\left(\frac{\sin sa - as\cos sa}{s^2}\right).$$

2.2.2. EXAMPLE:

Find the Fourier transform of $f(x)$ defined by $f(x) = \begin{cases} 0, & x < a \\ 1, & a < x < b \\ 0, & x > b \end{cases}$.

SOLUTION:

The Fourier transform,

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_a^b (1)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{e^{isx}}{is}\right)_a^b$$

$$F(s) = \frac{1}{\sqrt{2\pi}}\frac{1}{is}\left(e^{isb} - e^{isa}\right).$$

2.2.3. EXAMPLE:

Find the Fourier transform of $f(x) = \begin{cases} 0, & x < a \\ e^{ikx}, & a < x < b \\ 0, & x > b \end{cases}$.

SOLUTION:

The Fourier transform,

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\ dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{a}^{b} e^{ikx}\ e^{isx}\ dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{a}^{b} e^{i(k+s)x}\ dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{i(k+s)x}}{i(k+s)} \right)_a^b$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{i(k+s)} \left(e^{i(k+s)b} - e^{i(k+s)a} \right).$$

2.2.4. EXAMPLE:

Find the Fourier transform of $f(x) = \begin{cases} |x| & \text{for } |x| \le a \\ 0 & \text{for } |x| > a \end{cases}, \quad a > 0$

SOLUTION:

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\ dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} |x|\ e^{isx}\ dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} |x| \ \ (\cos sx + i \sin sx)\ dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} |x| \ \ \cos sx\ dx + \frac{i}{\sqrt{2\pi}} \int_{-a}^{a} |x| \ \ \sin sx\ dx$$

$$= \frac{1}{\sqrt{2\pi}} 2 \int_{0}^{a} |x| \ \ \cos sx\ dx + 0$$

($\because$ $f(x) = |x| \cos x$ is even & $f(x) = |x| \sin x$ is odd)

$$= \sqrt{\frac{2}{\pi}} \int_0^a x \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[x \left(\frac{\sin sx}{s} \right) - (1) \left(\frac{-\cos sx}{s^2} \right) \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(a \frac{\sin sa}{s} + \frac{\cos sa}{s^2} \right) - \left(0 + \frac{1}{s^2} \right) \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{sa \sin sa + \cos sa - 1}{s^2} \right].$$

2.2.5. EXAMPLE:

Find the Fourier transform of the function $f(x)$ defined by

$$f(x) = \begin{cases} 1 - x^2, & \text{if } |x| \le 1 \\ 0 \quad , & \text{if } |x| > 1 \end{cases}.$$

Hence prove that $\int_0^\infty \left(\frac{\sin s - s \cos s}{s^3} \right) \cos\left(\frac{s}{2} \right) ds = \frac{3\pi}{16}$.

SOLUTION:

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \, e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \left(1 - x^2 \right) e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \left(1 - x^2 \right) \left(\cos sx + i \sin sx \right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \left(1 - x^2 \right) \cos sx \, dx + \frac{i}{\sqrt{2\pi}} \int_{-1}^{1} \left(1 - x^2 \right) \sin sx \, dx$$

$$= \frac{1}{\sqrt{2\pi}} 2 \int_0^1 \left(1 - x^2\right) \cos sx \, dx + 0$$

$$(\because \; f(x) = \left(1 - x^2\right) \cos x \; \text{is even} \; \& \; f(x) = \left(1 - x^2\right) \sin x \; \text{is odd})$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 \left(1 - x^2\right) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(1 - x^2\right) \left(\frac{\sin sx}{s}\right) - (-2x) \left(\frac{-\cos sx}{s^2}\right) + (-2) \left(\frac{-\sin sx}{s^3}\right) \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(0 - 2\frac{\cos s}{s^2} + \frac{2\sin s}{s^3}\right) - (0 - 0 + 0) \right]$$

$$= 2\sqrt{\frac{2}{\pi}} \left(\frac{-s\cos s + \sin s}{s^3}\right)$$

$$F(s) = 2\sqrt{\frac{2}{\pi}} \left(\frac{\sin s - s\cos s}{s^3}\right).$$

By Fourier inverse transform,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) \, e^{-isx} \, ds$$

$$\therefore \qquad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}} \left(\frac{\sin s - s\cos s}{s^3}\right) e^{-isx} \, ds$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) (\cos sx - i\sin sx) \, ds$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx \, ds - \frac{2i}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \sin sx \, ds$$

$$f(x) = \frac{4}{\pi} \int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx \, ds - 0$$

$$\therefore \qquad \int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx \; ds = \frac{\pi}{4} f(x)$$

Put $x = \frac{1}{2}$ we get,

$$\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right) \cos\left(\frac{s}{2}\right) ds = \frac{\pi}{4} f\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{4}\left(1 - \left(\frac{1}{2}\right)^2\right)$$

$$= \frac{\pi}{4}\left(1 - \frac{1}{4}\right)$$

$$= \frac{\pi}{4}\left(\frac{3}{4}\right)$$

(i.e.) $$\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right) \cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{16}.$$

2.2.6. EXAMPLE:

Find the Fourier transform of $f(x)$ if $f(x) = \begin{cases} a - |x|, & \text{if } |x| \le a \\ 0 \quad , & \text{if } |x| > a \end{cases}$. Hence deduce that $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$.

SOLUTION:

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} \; dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a - |x|)\, e^{isx} \; dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a - |x|) \; (\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a - |x|) \ \cos sx \ dx - \frac{i}{\sqrt{2\pi}} \int_{-a}^{a} (a - |x|) \ \sin sx \ dx$$

$$= \frac{1}{\sqrt{2\pi}} 2 \int_{0}^{a} (a - |x|) \ \cos sx \ dx + 0$$

$$(\because \ f(x) = (a - |x|) \cos x \text{ is even } \& f(x) = (a - |x|) \sin x \text{ is odd})$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{a} (a - |x|) \ \cos sx \ dx$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{a} (a - x) \ \cos sx \ dx$$

$$= \sqrt{\frac{2}{\pi}} \left[(a - x) \left(\frac{\sin sx}{s} \right) - (-1) \left(\frac{-\cos sx}{s^2} \right) \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(0 - \frac{\cos sa}{s^2} \right) - \left(0 - \frac{1}{s^2} \right) \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-\cos sa}{s^2} + \frac{1}{s^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{1 - \cos sa}{s^2} \right)$$

$$F(s) = \sqrt{\frac{2}{\pi}} \left(\frac{2 \sin^2 \left(\frac{sa}{2} \right)}{s^2} \right). \qquad \left(\begin{array}{l} \because \ \cos 2A = 1 - 2\sin^2 A \\ \Rightarrow \ \sin^2 A = \dfrac{1 - \cos 2A}{2} \end{array} \right)$$

By Fourier inverse transform,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) \ e^{-isx} \ ds$$

$$\therefore \qquad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left(\frac{2 \sin^2 \left(\frac{sa}{2} \right)}{s^2} \right) e^{-isx} \ ds$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin\left(\frac{sa}{2}\right)}{s} \right)^2 (\cos sx - i \sin sx)\, ds$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin\left(\frac{sa}{2}\right)}{s} \right)^2 \cos sx\, ds - \frac{2i}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin\left(\frac{sa}{2}\right)}{s} \right)^2 \sin sx\, ds$$

$$= \frac{2}{\pi} 2 \int_{0}^{\infty} \left(\frac{\sin\left(\frac{sa}{2}\right)}{s} \right)^2 \cos sx\, ds - 0$$

$$f(x) = \frac{4}{\pi} \int_{0}^{\infty} \left(\frac{\sin\left(\frac{sa}{2}\right)}{s} \right)^2 \cos sx\, ds$$

$$\therefore \quad \int_{0}^{\infty} \left(\frac{\sin\left(\frac{sa}{2}\right)}{s} \right)^2 \cos sx\, ds = \frac{\pi}{4} f(x)$$

Put $x = 0$ we get,

$$\int_{0}^{\infty} \left(\frac{\sin\left(\frac{sa}{2}\right)}{s} \right)^2 ds = \frac{\pi}{4} f(0)$$

$$= \frac{\pi}{4}(a)$$

Put $\frac{sa}{2} = t \Rightarrow sa = 2t$ when $s = 0 \Rightarrow t = 0$

$a\, ds = 2\, dt$ $\quad s = \infty \Rightarrow t = \infty$

$$\therefore \int_0^\infty \left(\frac{\sin t}{\frac{2t}{a}} \right)^2 \frac{2}{a} dt = \frac{\pi}{4}(a)$$

$$\int_0^\infty \left(\frac{\sin t}{t} \right)^2 . \frac{a^2}{4} . \frac{2}{a} dt = \frac{\pi}{4}(a)$$

$$\Rightarrow \qquad \int_0^\infty \left(\frac{\sin t}{t} \right)^2 dt = \frac{\pi}{2}.$$

DEFINITION: (SELF RECIPROCAL)

If a transformation of a function $f(x)$ is equal to $f(s)$ then the function $f(x)$ is called self reciprocal.

2.2.7. EXAMPLE:

Show that the Fourier transform of $e^{-x^2/2}$ is $e^{-s^2/2}$.

SOLUTION:

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}+isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(x^2-2isx\right)}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(x^2-2isx+(is)^2-(is)^2\right)}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left((x-is)^2+s^2\right)}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-is)^2} \; e^{-\frac{s^2}{2}} \, dx$$

$$= \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-is)^2} \, dx$$

Put $t = x - is$ if $x = -\infty \Rightarrow t = -\infty$

$dt = dx$ if $x = \infty \Rightarrow t = \infty$

$$= \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} \, dt$$

$$= \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}} \sqrt{2\pi} \qquad \left(\because \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \, dx = \sqrt{2\pi} \right)$$

$$= e^{-\frac{s^2}{2}}.$$

2.2.8. EXAMPLE:

Find the Fourier cosine transform of $f(x) = \begin{cases} \cos x & \text{if } 0 < x < a \\ 0 & \text{if } x \geq a \end{cases}$

SOLUTION:

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{a} \cos x \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{a} \cos sx \cos x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{a} \frac{1}{2} \Big[\cos(s+1)x + \cos(s-1)x \Big] \, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s+1)x}{s+1} + \frac{\sin(s-1)x}{s-1}\right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left(\frac{\sin(s+1)a}{s+1} + \frac{\sin(s-1)a}{s-1}\right) - (0+0)\right]$$

$$F_c(s) = \frac{1}{\sqrt{2\pi}}\left(\frac{\sin(s+1)a}{s+1} + \frac{\sin(s-1)a}{s-1}\right)$$

2.2.9. EXAMPLE:

Find the Fourier cosine transform of e^{-ax} , $a > 0$.

SOLUTION:

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2}(-a\cos sx + s\sin sx)\right]_0^\infty$$

$$\left(\because \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right)$$

$$= \sqrt{\frac{2}{\pi}} \left[(0) - \frac{1}{a^2+s^2}(-a(1)+0)\right]$$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}.$$

2.2.10. EXAMPLE:

Find the Fourier transform of $f(x) = \begin{cases} 1, & \text{if } 0 < x < a \\ 0, & \text{if } x \geq a \end{cases}$.

SOLUTION:

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a (1) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s} \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sa}{s} - 0 \right]$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{\sin sa}{s} \right).$$

2.2.11. EXAMPLE:

Find the Fourier cosine transform of $e^{-\frac{x^2}{2}}$.

SOLUTION:

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2/2} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{2} \int_{-\infty}^\infty e^{-x^2/2} \cos sx \, dx \qquad (\because \text{ Function is even})$$

$$= \frac{1}{\sqrt{2\pi}} \text{Real part of} \int_{-\infty}^\infty e^{-x^2/2} e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \text{Real part of} \int_{-\infty}^\infty e^{\frac{-x^2}{2} + isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \text{Real part of} \int_{-\infty}^\infty e^{-\frac{1}{2}\left(x^2 - 2isx\right)} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \text{Real part of} \int_{-\infty}^\infty e^{-\frac{1}{2}\left(x^2 - 2isx + (is)^2 - (is)^2\right)} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \text{Real part of} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left((x-is)^2+s^2\right)} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \text{Real part of} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-is)^2} \, e^{-\frac{s^2}{2}} \, dx$$

$$= \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}} \text{Real part of} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-is)^2} \, dx$$

Put $t = x - is$ if $x = -\infty \Rightarrow t = -\infty$

$dt = dx$ if $x = \infty \Rightarrow t = \infty$

$$= \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}} \text{Real part of} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} \, dt$$

$$= \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}} \text{Real part of } \sqrt{2\pi} \qquad \left(\because \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \, dx = \sqrt{2\pi} \right)$$

$$F_c(s) = e^{-\frac{s^2}{2}}.$$

2.2.12. EXAMPLE:

Find the Fourier cosine transform of $e^{-|x|}$ and deduce that $\int_0^{\infty} \frac{\cos xt}{1+t^2} \, dt = \frac{\pi}{2} e^{-|x|}$.

SOLUTION:

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-|x|} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{1^2 + s^2} (-\cos sx + s \sin sx) \right]_0^\infty$$

$$\left(\because \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) \right)$$

$$= \sqrt{\frac{2}{\pi}} \left[(0) - \frac{1}{1^2 + s^2} (-(1) + 0) \right]$$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \frac{1}{1 + s^2}$$

By inverse Fourier cosine transform,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{1}{1 + s^2} \cos sx \, ds$$

$$= \frac{2}{\pi} \int_0^\infty \frac{1}{1 + s^2} \cos sx \, ds$$

$$\therefore \int_0^\infty \frac{1}{1 + s^2} \cos sx \, ds = \frac{\pi}{2} f(x)$$

$$= \frac{\pi}{2} e^{-|x|}.$$

2.2.13. EXAMPLE:

Find the Fourier sine transform of $f(x) = \begin{cases} \sin x & \text{if } 0 \le x \le a \\ 0 & \text{if } x \ge a \end{cases}$.

SOLUTION:

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \sin x \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \sin sx \, \sin x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \frac{1}{2}\left[\cos(s-1)x - \cos(s+1)x\right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s-1)x}{s-1} - \frac{\sin(s+1)x}{s+1}\right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(\frac{\sin(s-1)a}{s-1} - \frac{\sin(s+1)a}{s+1}\right) - (0-0)\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{\sin(s-1)a}{s-1} - \frac{\sin(s+1)a}{s+1}\right).$$

2.2.14. EXAMPLE:

Find the Fourier sine transform of $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}.$

SOLUTION:

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 x \sin sx \, dx + \int_1^2 (2-x) \sin sx \, dx\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[x\frac{-\cos sx}{s} - (1)\frac{-\sin sx}{s^2}\right]_0^1 + \sqrt{\frac{2}{\pi}} \left[(2-x)\frac{-\cos sx}{s} - (-1)\frac{-\sin sx}{s^2}\right]_1^2$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(\frac{-\cos s}{s} + \frac{\sin s}{s^2}\right) - (0+0)\right] + \sqrt{\frac{2}{\pi}} \left[\left(0 - \frac{\sin 2s}{s^2}\right) - \left(\frac{-\cos s}{s} - \frac{\sin s}{s^2}\right)\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-\cos s}{s} + \frac{\sin s}{s^2} - \frac{\sin 2s}{s^2} + \frac{\cos s}{s} + \frac{\sin s}{s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2\sin s}{s^2} - \frac{\sin 2s}{s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{2\sin s - \sin 2s}{s^2}\right).$$

2.2.15. EXAMPLE:

Find the Fourier sine transform of $f(x) = \frac{1}{x}e^{-ax}$.

SOLUTION:

$$F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x}e^{-ax}\sin sx\, dx \qquad (1)$$

Diff. w.r.to 's'

$$\frac{d}{ds}F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial}{\partial s}\left(\frac{1}{x}e^{-ax}\sin sx\right)dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x}e^{-ax}\cos sx.\, x\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right)$$

Integrating w.r.to 's' we get,

$$F_s(s) = \sqrt{\frac{2}{\pi}}.a\int\frac{1}{s^2+a^2}ds$$

$$= \sqrt{\frac{2}{\pi}}.a\,\frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right)+c \qquad (2)$$

Put $s = 0$ in (1) we get,

$$F_s(0) = 0$$

Put $s = 0$ in (2) we get,

$$F_s(0) = \sqrt{\frac{2}{\pi}} \tan^{-1}(0) + c$$

$$0 = \sqrt{\frac{2}{\pi}} (0) + c$$

$$\therefore \quad c = 0$$

$$\therefore \quad F_s(s) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{s}{a}\right).$$

2.2.16. EXAMPLE:

Find the Fourier sine transform of $e^{-ax}, a > 0$. Hence deduce that

$$\int_0^\infty \frac{s}{s^2 + a^2} \sin sx \, ds = \frac{\pi}{2} e^{-ax}$$

SOLUTION:

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2 + a^2}\right).$$

Inverse Fourier sine transform,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{s}{s^2 + a^2} \sin sx \, ds$$

$$= \frac{2}{\pi} \int_0^\infty \frac{s}{s^2 + a^2} \sin sx \, ds$$

$$\therefore \quad \int_0^\infty \frac{s}{s^2 + a^2} \sin sx \, ds = \frac{\pi}{2} f(x)$$

$$= \frac{\pi}{2} e^{-ax}.$$

2.3. PARSEVAL'S IDENTITY:

If $F(s)$ is the Fourier transform of $f(x)$, then

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |F(s)|^2 \, ds$$

2.4. EXAMPLES:

2.4.1. EXAMPLE:

Find the Fourier transform of $f(x)$ if $f(x) = \begin{cases} 1, & |x| \le a \\ 0, & |x| > a \end{cases}$. Hence deduce that

(i) $\displaystyle\int_0^{\infty} \left(\frac{\sin t}{t}\right) dt = \frac{\pi}{2}$

(ii) $\displaystyle\int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$.

SOLUTION:

The Fourier transform,

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (1)\, e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (\cos sx + i \sin sx) \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \cos sx \, dx + \frac{i}{\sqrt{2\pi}} \int_{-a}^{a} \sin sx \, dx$$

$$= \frac{1}{\sqrt{2\pi}} 2 \int_{0}^{a} \cos sx \, dx + 0 \quad (\because \quad f(x) = \cos x \text{ is even } \& \, f(x) = \sin x \text{ is odd})$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s} \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sa}{s} - 0 \right]$$

$$F(s) = \sqrt{\frac{2}{\pi}} \left(\frac{\sin sa}{s} \right)$$

By inverse Fourier transform,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$\therefore \qquad = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left(\frac{\sin sa}{s} \right) e^{-isx}\, ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin sa}{s} \right) (\cos sx - i \sin sx)\, ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin sa}{s} \right) \cos sx\, ds - \frac{i}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin sa}{s} \right) \sin sx\, ds$$

$$= \frac{1}{\pi} 2 \int_{0}^{\infty} \left(\frac{\sin sa}{s} \right) \cos sx\, ds - 0$$

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \left(\frac{\sin sa}{s} \right) \cos sx\, ds$$

$$\therefore \qquad \int_{0}^{\infty} \left(\frac{\sin sa}{s} \right) \cos sx\, ds = \frac{\pi}{2} f(x)$$

Put $x = 0$ we get,

$$\int_{0}^{\infty} \left(\frac{\sin sa}{s} \right) ds = \frac{\pi}{2} f(0)$$

$$= \frac{\pi}{2} (1)$$

Put $sa = t$ when $s = 0 \Rightarrow t = 0$

$a\,ds = dt$ $s = \infty \Rightarrow t = \infty$

$$\therefore \int_0^\infty \left(\frac{\sin t}{\frac{t}{a}}\right)\frac{1}{a}dt = \frac{\pi}{2}$$

$$\Rightarrow \int_0^\infty \left(\frac{\sin t}{t}\right)dt = \frac{\pi}{2}.$$

Using Parseval's identity,

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(s)|^2\,ds$$

$$\int_{-a}^{a} (1)^2 dx = \int_{-\infty}^{\infty}\left(\sqrt{\frac{2}{\pi}}\left(\frac{\sin sa}{s}\right)\right)^2 ds$$

$$[x]_{-a}^{a} = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin sa}{s}\right)^2 ds$$

$$a-(-a) = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin sa}{s}\right)^2 ds$$

$$\Rightarrow \int_{-\infty}^{\infty}\left(\frac{\sin sa}{s}\right)^2 ds = \frac{\pi}{2}2a$$

$$\Rightarrow 2\int_{0}^{\infty}\left(\frac{\sin sa}{s}\right)^2 ds = \pi a \qquad (\because \text{ function is even})$$

$$\int_{0}^{\infty}\left(\frac{\sin sa}{s}\right)^2 ds = \frac{\pi a}{2}$$

Put $sa = t$ when $s = 0 \Rightarrow t = 0$

$a\,ds = dt$ $s = \infty \Rightarrow t = \infty$

$$\therefore \int_0^\infty \left(\frac{\sin t}{\frac{t}{a}}\right)^2\frac{1}{a}dt = \frac{\pi a}{2}$$

$$\Rightarrow \qquad \int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}.$$

2.4.2. EXAMPLE:

Find the Fourier transform of $f(x) = \begin{cases} 1-|x|, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$. Hence deduce that

$$\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}.$$

SOLUTION:

The Fourier transform,

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)\,(\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|) \cos sx\, dx - \frac{i}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|) \sin sx\, dx$$

$$= \frac{1}{\sqrt{2\pi}} 2\int_{0}^{1} (1-|x|) \cos sx\, dx + 0$$

$(\because\ f(x) = (1-|x|) \cos x$ is even & $f(x) = (1-|x|) \sin x$ is odd$)$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 (1-|x|)\ \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 (1-x)\ \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[(1-x)\frac{\sin sx}{s} - (-1)\frac{-\cos sx}{s^2}\right]_0^1$$

$$= \sqrt{\frac{2}{\pi}}\left[\left(0 - \frac{\cos s}{s^2}\right) - \left(0 - \frac{1}{s^2}\right)\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{-\cos s}{s^2} + \frac{1}{s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s^2}\right)$$

$$F(s) = \sqrt{\frac{2}{\pi}}\left(\frac{2\sin^2\left(\frac{s}{2}\right)}{s^2}\right). \qquad \left(\begin{array}{l}\because \cos 2A = 1 - 2\sin^2 A \\ \Rightarrow \sin^2 A = \frac{1-\cos 2A}{2}\end{array}\right)$$

By Parseval's identity,

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds$$

$$\int_{-1}^{1} (1-|x|)^2\, dx = \int_{-\infty}^{\infty} \left(2\sqrt{\frac{2}{\pi}}\left(\frac{\sin^2\left(\frac{s}{2}\right)}{s^2}\right)\right)^2 ds$$

$$2\int_{0}^{1} (1-|x|)^2\, dx = \frac{8}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin\left(\frac{s}{2}\right)}{s}\right)^4 ds \qquad (\because f(x) = (1-|x|)^2 \text{ is even})$$

$$\left[\frac{(1-x)^3}{-3}\right]_0^1 = \frac{4}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin\left(\frac{s}{2}\right)}{s}\right)^4 ds$$

$$\frac{1}{-3}[0-1] = \frac{4}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin\left(\frac{s}{2}\right)}{s}\right)^4 ds$$

$$\Rightarrow \qquad \int_{-\infty}^{\infty} \left(\frac{\sin\left(\frac{s}{2}\right)}{s} \right)^4 ds = \frac{\pi}{12}$$

$$2\int_{0}^{\infty} \left(\frac{\sin\left(\frac{s}{2}\right)}{s} \right)^4 ds = \frac{\pi}{12}$$

$$\therefore \qquad \int_{0}^{\infty} \left(\frac{\sin\left(\frac{s}{2}\right)}{s} \right)^4 ds = \frac{\pi}{24}$$

Put $\frac{s}{2} = t$ when $s = 0 \Rightarrow t = 0$

$\frac{1}{2} ds = dt$ $\qquad s = \infty \Rightarrow t = \infty$

$$\therefore \qquad \int_{0}^{\infty} \left(\frac{\sin t}{2t} \right)^4 2dt = \frac{\pi}{24}$$

$$\int_{0}^{\infty} \left(\frac{\sin t}{t} \right)^4 \frac{1}{16} 2dt = \frac{\pi}{24}$$

$$\Rightarrow \qquad \int_{0}^{\infty} \left(\frac{\sin t}{t} \right)^4 dt = \frac{\pi}{3}.$$

2.4.3. EXAMPLE:

Find the Fourier transform of $f(x)$ if $f(x) = \begin{cases} 1, & |x| \le 2 \\ 0, & |x| > 2 \end{cases}$. Hence evaluate

(i) $\int_{0}^{\infty} \left(\frac{\sin x}{x} \right) dx$

(ii) $\int_{0}^{\infty} \left(\frac{\sin x}{x} \right)^2 dx$.

SOLUTION:

The Fourier transform,

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} (1)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} (\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} \cos sx\, dx + \frac{i}{\sqrt{2\pi}} \int_{-2}^{2} \sin sx\, dx$$

$$= \frac{1}{\sqrt{2\pi}} 2 \int_{0}^{2} \cos sx\, dx + 0$$

$$(\because \quad f(x) = \cos x \text{ is even } \& \, f(x) = \sin x \text{ is odd})$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s}\right]_0^2$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin 2s}{s} - 0\right]$$

$$F(s) = \sqrt{\frac{2}{\pi}} \left(\frac{\sin 2s}{s}\right)$$

By inverse Fourier transform,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$\therefore \quad = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left(\frac{\sin 2s}{s}\right) e^{-isx}\, ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin 2s}{s}\right) (\cos sx - i \sin sx)\, ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin 2s}{s}\right) \cos sx\, ds - \frac{i}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin 2s}{s}\right) \sin sx\, ds$$

$$= \frac{1}{\pi} 2 \int_0^\infty \left(\frac{\sin 2s}{s}\right) \cos sx \; ds - 0$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \left(\frac{\sin 2s}{s}\right) \cos sx \; ds$$

$$\therefore \quad \int_0^\infty \left(\frac{\sin 2s}{s}\right) \cos sx \; ds = \frac{\pi}{2} f(x)$$

Put $x = 0$ we get,

$$\int_0^\infty \left(\frac{\sin 2s}{s}\right) \; ds = \frac{\pi}{2} f(0)$$

$$= \frac{\pi}{2}(1)$$

Put $2s = t$ when $s = 0 \Rightarrow t = 0$

$2ds = dt$ $s = \infty \Rightarrow t = \infty$

$$\therefore \quad \int_0^\infty \left(\frac{\sin t}{\frac{t}{2}}\right) \frac{1}{2} dt = \frac{\pi}{2}$$

$$\Rightarrow \quad \int_0^\infty \left(\frac{\sin t}{t}\right) dt = \frac{\pi}{2}.$$

Using Parseval's identity,

$$\int_{-\infty}^\infty |f(x)|^2 \; dx = \int_{-\infty}^\infty |F(s)|^2 \; ds$$

$$\int_{-2}^2 (1)^2 dx = \int_{-\infty}^\infty \left(\sqrt{\frac{2}{\pi}} \left(\frac{\sin 2s}{s}\right)\right)^2 ds$$

$$(x)_{-2}^2 = \frac{2}{\pi} \int_{-\infty}^\infty \left(\frac{\sin 2s}{s}\right)^2 ds$$

$$2 - (-2) = \frac{2}{\pi} \int_{-\infty}^\infty \left(\frac{\sin 2s}{s}\right)^2 ds$$

$$\Rightarrow \qquad \int_{-\infty}^{\infty} \left(\frac{\sin 2s}{s}\right)^2 ds = \frac{\pi}{2} 4$$

$$\Rightarrow \qquad 2\int_{0}^{\infty} \left(\frac{\sin 2s}{s}\right)^2 ds = 2\pi \qquad (\because \text{ function is even})$$

$$\int_{0}^{\infty} \left(\frac{\sin 2s}{s}\right)^2 ds = \pi$$

Put $\quad 2s = t \qquad$ when $\quad s = 0 \quad \Rightarrow \quad t = 0$

$\qquad 2ds = dt \qquad\qquad s = \infty \quad \Rightarrow \quad t = \infty$

$$\therefore \qquad \int_{0}^{\infty} \left(\frac{\sin t}{\frac{t}{2}}\right)^2 \frac{1}{2} dt = \pi$$

$$\Rightarrow \qquad \int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}.$$

2.4.4. EXAMPLE:

Show that the Fourier transform of the function $f(x) = \begin{cases} a^2 - x^2, & \text{if } |x| \le a \\ 0 \quad , & \text{if } |x| > a \end{cases}$ is $2\sqrt{\frac{2}{\pi}}\left(\frac{\sin sa - sa \sin sa}{s^3}\right)$. Hence deduce that $\int_{0}^{\infty}\left(\frac{\sin t - t\cos t}{t^3}\right)dt = \frac{\pi}{4}$. Using Parseval's identity prove that $\int_{0}^{\infty}\left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}$.

SOLUTION:

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \left(a^2 - x^2\right) e^{isx}\, dx$$

⋮

⋮

⋮

$$F(s) = \frac{4}{\sqrt{2\pi}}\left(\frac{\sin sa - sa\sin sa}{s^3}\right)$$

By Fourier inverse transform,

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$\therefore \quad f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{4}{\sqrt{2\pi}}\left(\frac{\sin sa - sa\sin sa}{s^3}\right) e^{-isx}\, ds$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin sa - sa\sin sa}{s^3}\right)(\cos sx - i\sin sx)\, ds$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin sa - sa\sin sa}{s^3}\right)\cos sx\, ds - \frac{2i}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin sa - sa\sin sa}{s^3}\right)\sin sx\, ds$$

$$f(x) = \frac{4}{\pi}\int_{0}^{\infty}\left(\frac{\sin sa - sa\sin sa}{s^3}\right)\cos sx\, ds - 0$$

$$\therefore \quad \int_{0}^{\infty}\left(\frac{\sin sa - sa\sin sa}{s^3}\right)\cos sx\, ds = \frac{\pi}{4} f(x)$$

Put $x = 0$ we get,

$$\int_{0}^{\infty}\left(\frac{\sin sa - sa\sin sa}{s^3}\right) ds = \frac{\pi}{4} f(0)$$

$$\int_{0}^{\infty}\left(\frac{\sin sa - sa\sin sa}{s^3}\right) ds = \frac{\pi}{4} a^2$$

Put $sa = t$ when $s = 0 \Rightarrow t = 0$

$a\, ds = dt$ $s = \infty \Rightarrow t = \infty$

$$\therefore \quad \int_0^\infty \left(\frac{\sin t - t\sin t}{\left(\frac{t}{a}\right)^3} \right) \frac{1}{a} dt = \frac{\pi}{4} a^2$$

$$\int_0^\infty \left(\frac{\sin t - t\sin t}{t^3} \right) dt = \frac{\pi}{4}.$$

By parseval's identity,

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |F(s)|^2 \, ds$$

$$\int_{-a}^{a} \left(a^2 - x^2\right)^2 dx = \int_{-\infty}^{\infty} \frac{16}{2\pi} \left(\frac{\sin sa - sa\sin sa}{s^3} \right)^2 ds$$

$$2\int_0^a \left(a^2 - x^2\right)^2 dx = \frac{8}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin sa - sa\sin sa}{s^3} \right)^2 ds$$

$$\int_0^a \left(a^4 + x^4 - 2a^2x^2\right) dx = \frac{4}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin sa - sa\sin sa}{s^3} \right)^2 ds$$

$$\left[a^4 x + \frac{x^5}{5} - 2a^2 \frac{x^3}{3} \right]_0^a = \frac{4}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin sa - sa\sin sa}{s^3} \right)^2 ds$$

$$\left[\left(a^5 + \frac{a^5}{5} - 2\frac{a^5}{3} \right) - (0) \right] = \frac{4}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin sa - sa\sin sa}{s^3} \right)^2 ds$$

$$\frac{15a^5 + 3a^5 - 10a^5}{15} = \frac{4}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin sa - sa\sin sa}{s^3} \right)^2 ds$$

$$\frac{8a^5}{15} = \frac{4}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin sa - sa\sin sa}{s^3} \right)^2 ds$$

$$\int_{-\infty}^{\infty} \left(\frac{\sin sa - sa\sin sa}{s^3} \right)^2 ds = \frac{2\pi a^5}{15}$$

$$2\int_0^{\infty} \left(\frac{\sin sa - sa\sin sa}{s^3} \right)^2 ds = \frac{2\pi a^5}{15} \qquad (\because \text{ function is even})$$

$$\int_0^{\infty} \left(\frac{\sin sa - sa\sin sa}{s^3} \right)^2 ds = \frac{\pi a^5}{15}$$

Put $sa = t$ when $s = 0 \Rightarrow t = 0$

$a\, ds = dt$ $s = \infty \Rightarrow t = \infty$

$$\therefore \int_0^\infty \left(\frac{\sin t - t \sin t}{\left(\frac{t}{a}\right)^3} \right)^2 \frac{1}{a} dt = \frac{\pi a^5}{15}$$

$$\Rightarrow \int_0^\infty \left(\frac{\sin t - t \sin t}{t^3} \right)^2 dt = \frac{\pi}{5}.$$

2.4.5. EXAMPLE:

Evaluate $\int_0^\infty \frac{dx}{\left(x^2+a^2\right)\left(x^2+b^2\right)}$ using transforms.

SOLUTION:

Let $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$ then we have,

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}} \left(\frac{a}{s^2+a^2} \right) \text{ and } F_c(g(x)) = \sqrt{\frac{2}{\pi}} \left(\frac{b}{s^2+b^2} \right).$$

Hence by the property of Fourier cosine transform we have,

$$\int_0^\infty f(x) . g(x)\, dx = \int_0^\infty F_c(f(x)) . F_c(g(x))\, ds$$

$$\int_0^\infty e^{-ax} . e^{-bx} dx = \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{a}{s^2+a^2} \right) . \sqrt{\frac{2}{\pi}} \left(\frac{b}{s^2+b^2} \right) ds$$

$$\int_0^\infty e^{-(a+b)x}\, dx = \frac{2ab}{\pi} \int_0^\infty \frac{1}{\left(s^2+a^2\right)\left(s^2+b^2\right)} ds$$

$$\left[\frac{e^{-(a+b)x}}{-(a+b)} \right]_0^\infty = \frac{2ab}{\pi} \int_0^\infty \frac{1}{\left(s^2+a^2\right)\left(s^2+b^2\right)} ds$$

$$\left[0 - \frac{1}{-(a+b)} \right] = \frac{2ab}{\pi} \int_0^\infty \frac{1}{\left(s^2+a^2\right)\left(s^2+b^2\right)} ds$$

$$\therefore \quad \int_0^\infty \frac{1}{(s^2+a^2)(s^2+b^2)}\, ds = \frac{\pi}{2ab(a+b)}$$

$$\text{(i.e.)} \int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)}\, dx = \frac{\pi}{2ab(a+b)}.$$

4.6. EXAMPLE:

Using Parseval's identity calculate,

$$\text{(i)} \int_0^\infty \frac{dx}{(x^2+a^2)^2} \qquad \text{(ii)} \int_0^\infty \frac{x^2}{(x^2+a^2)^2}\, dx, \quad \text{if } a > 0.$$

SOLUTION:

(i) Let $f(x) = e^{-ax}$ then we have, $F_c(f(x)) = \sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right)$.

Now applying Parseval's identity of Fourier cosine transform we have,

$$\int_0^\infty |f(x)|^2\, dx = \int_0^\infty |F_c(f(x))|^2\, ds$$

$$\int_0^\infty (e^{-ax})^2\, dx = \int_0^\infty \left|\sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right)\right|^2 ds$$

$$\int_0^\infty e^{-2ax}\, dx = \frac{2}{\pi}\int_0^\infty \frac{a^2}{(s^2+a^2)^2} ds$$

$$\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{2}{\pi}\int_0^\infty \frac{a^2}{(s^2+a^2)^2} ds$$

$$\left[0 - \frac{1}{-2a}\right] = \frac{2}{\pi}\int_0^\infty \frac{a^2}{(s^2+a^2)^2} ds$$

$$\Rightarrow \quad \int_0^\infty \frac{1}{(s^2+a^2)^2} ds = \frac{\pi}{4a^3}$$

$$\text{(i.e.)} \quad \int_0^\infty \frac{1}{(x^2+a^2)^2} dx = \frac{\pi}{4a^3}$$

(ii) Let $f(x) = e^{-ax}$ then we have, $F_s(f(x)) = \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)$.

Now applying Parseval's identity of Fourier sine transform we have,

$$\int_0^\infty |f(x)|^2\, dx = \int_0^\infty |F_s(f(x))|^2\, ds$$

$$\int_0^\infty (e^{-ax})^2\, dx = \int_0^\infty \left|\sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)\right|^2 ds$$

$$\int_0^\infty e^{-2ax}\, dx = \frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)^2}\, ds$$

$$\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)^2}\, ds$$

$$\left[0 - \frac{1}{-2a}\right] = \frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)^2}\, ds$$

$$\Rightarrow \quad \int_0^\infty \frac{s^2}{(s^2+a^2)^2}\, ds = \frac{\pi}{4a}$$

$$\text{(i.e.)} \quad \int_0^\infty \frac{s^2}{(x^2+a^2)^2}\, dx = \frac{\pi}{4a}.$$

2.5. APPLICATION OF FOURIER TRANSFORM FOR SOLVING INTEGRAL EQUATIONS:

2.5.1. EXAMPLE:

Solve the integral equation $\int_0^\infty f(x)\cos\lambda x\, dx = e^{-\lambda}$.

SOLUTION:

Replace λ by s,

$$\int_0^\infty f(x)\cos sx\, dx = e^{-s}$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\, dx = \sqrt{\frac{2}{\pi}}\, e^{-s}$$

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}}\, e^{-s}.$$

By inverse Fourier cosine transform,

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(s)\cos sx\, ds$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\, e^{-s}\cos sx\, ds$$

$$= \frac{2}{\pi}\int_0^\infty e^{-s}\cos sx\, ds$$

$$= \frac{2}{\pi}\left[\frac{e^{-s}}{1^2+x^2}(-\cos sx - x\sin sx)\right]_0^\infty \qquad \left(\text{here } a=-1,\, b=x\right)$$

$$= \frac{2}{\pi}\left[(0) - \frac{1}{1^2+x^2}(-1-0)\right]$$

$$f(x) = \frac{2}{\pi}\left(\frac{1}{1+x^2}\right).$$

2.5.2. EXAMPLE:

Solve the integral equation $\int_0^\infty f(x)\cos\lambda x\, dx = \begin{cases} 1-\lambda, & 0\le\lambda\le 1 \\ 0, & \lambda > 1\end{cases}$, and hence deduce that $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$.

SOLUTION:

Replace λ by s,

$$\int_0^\infty f(x)\cos sx\, dx = \begin{cases} 1-s, & 0\le s\le 1 \\ 0, & s>1\end{cases}$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\, dx = \sqrt{\frac{2}{\pi}}\begin{cases} 1-s, & 0\le s\le 1 \\ 0, & s>1\end{cases}$$

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}} \begin{cases} 1-s, & 0 \le s \le 1 \\ 0, & s > 1 \end{cases}.$$

By inverse Fourier cosine transform,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(s) \cos sx \, ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{1} \sqrt{\frac{2}{\pi}} (1-s) \cos sx \, ds$$

$$= \frac{2}{\pi} \int_0^{1} (1-s) \cos sx \, ds$$

$$= \frac{2}{\pi} \left[(1-s) \frac{\sin sx}{x} - (-1) \frac{-\cos sx}{x^2} \right]_0^1$$

$$= \frac{2}{\pi} \left[\left(0 - \frac{\cos x}{x^2} \right) - \left(0 - \frac{1}{x^2} \right) \right]$$

$$= \frac{2}{\pi} \left(-\frac{\cos x}{x^2} + \frac{1}{x^2} \right)$$

$$f(x) = \frac{2}{\pi} \left(\frac{1-\cos x}{x^2} \right)$$

Since $$\int_0^{\infty} f(x) \cos \lambda x \, dx = \begin{cases} 1-\lambda, & 0 \le \lambda \le 1 \\ 0, & \lambda > 1 \end{cases}$$

Put $\lambda = 0$,

$$\int_0^{\infty} f(x)\,(1)\, dx = 1$$

$$\frac{2}{\pi} \int_0^{\infty} \left(\frac{1-\cos x}{x^2} \right) dx = 1$$

$$\int_0^{\infty} \frac{2\sin^2\left(\frac{x}{2}\right)}{x^2} dx = \frac{\pi}{2}$$

$$\int_0^{\infty} \left(\frac{\sin\left(\frac{x}{2}\right)}{x} \right)^2 dx = \frac{\pi}{4}$$

Put $t = \frac{x}{2}$ when $x = 0,\ t = 0$

$dt = \frac{1}{2} dx$ $x = \infty,\quad t = \infty$

$$\int_0^\infty \left(\frac{\sin t}{2t}\right)^2 2dt = \frac{\pi}{4}$$

$$\int_0^\infty \left(\frac{\sin t}{t}\right)^2 \frac{2}{4}\, dt = \frac{\pi}{4}$$

$$\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}.$$

2.5.3. EXAMPLE:

Find $f(x)$ from $\int_0^\infty f(x) \sin xt\ dt = \begin{cases} 1, & 0 \le t \le 1 \\ 2, & 1 \le t \le 2 \\ 0, & t > 2 \end{cases}$.

SOLUTION:

$$f(x) = \frac{2}{\pi x}(1 + \cos x - 2\cos 2x).$$

2.6. LIST OF FORMULA:

1. Fourier integral theorem:

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t) \cos \lambda(t - x)\ dt\ d\lambda$$

2. Fourier sine integral formula:

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin \lambda x \int_0^\infty f(t) \sin \lambda t\ dt\ d\lambda$$

3. Fourier cosine integral formula:

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos \lambda x \int_0^\infty f(t) \cos \lambda t\ dt\ d\lambda.$$

2.6.1. EXAMPLE:

Express the function for $f(x)=\begin{cases}1, \text{ for} & |x|\le 1\\ 0 \text{ for} & |x|>1\end{cases}$ as a Fourier integral. Hence evaluate

$\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda$ and find the value of $\int_0^\infty \frac{\sin\lambda}{\lambda}\,d\lambda$.

SOLUTION:

The Fourier integral formula,

$$f(x)=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t)\cos\lambda(t-x)\,dt\,d\lambda$$

Here $f(t)=\begin{cases}1, \text{ for} & |t|\le 1\\ 0, \text{ for} & |t|>1\end{cases}$

$$\therefore \qquad f(x)=\frac{1}{\pi}\int_0^\infty\left(\int_{-1}^1 (1)\cos\lambda(t-x)\,dt\right)d\lambda$$

$$=\frac{1}{\pi}\int_0^\infty\left(\frac{\sin\lambda(t-x)}{\lambda}\right)_{-1}^1 d\lambda$$

$$=\frac{1}{\pi}\int_0^\infty\frac{1}{\lambda}[\sin\lambda(1-x)-\sin\lambda(-1-x)]\,d\lambda$$

$$=\frac{1}{\pi}\int_0^\infty\frac{1}{\lambda}[\sin(\lambda-\lambda x)+\sin(\lambda+\lambda x)]\,d\lambda$$

$$=\frac{1}{\pi}\int_0^\infty\frac{1}{\lambda}(2\sin\lambda.\cos\lambda x)\,d\lambda$$

$$\therefore \qquad \int_0^\infty\left(\frac{\sin\lambda.\cos\lambda x}{\lambda}\right)d\lambda=\frac{\pi}{2}f(x)$$

$$\int_0^\infty\left(\frac{\sin\lambda.\cos\lambda x}{\lambda}\right)d\lambda=\frac{\pi}{2}\begin{cases}1, \text{ for} & |x|\le 1\\ 0, \text{ for} & |x|>1\end{cases}$$

Put $x=0$:

$$\int_0^\infty\frac{\sin\lambda}{\lambda}\,d\lambda=\frac{\pi}{2}(1)=\frac{\pi}{2}.$$

2.6.2. EXAMPLE:

Find the Fourier cosine integral of e^{-ax} and hence evaluate $\int_0^\infty \frac{\cos \lambda x}{1+\lambda^2}\, d\lambda$.

SOLUTION:

The Fourier cosine integral,

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos \lambda x \; d\lambda \int_0^\infty f(t) \cos \lambda t \; dt$$

Here $f(t) = e^{-at}$.

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty \cos \lambda x \; d\lambda \int_0^\infty e^{-at} \cos \lambda t \; dt$$

$$= \frac{2}{\pi}\int_0^\infty \cos \lambda x \; d\lambda \left(\frac{a}{a^2+\lambda^2}\right)$$

$$f(x) = \frac{2a}{\pi}\int_0^\infty \frac{\cos \lambda x}{a^2+\lambda^2}\, d\lambda$$

$$\Rightarrow \quad \int_0^\infty \frac{\cos \lambda x}{a^2+\lambda^2}\, d\lambda = \frac{\pi}{2a} f(x)$$

Put $a = 1$:

$$\Rightarrow \quad \int_0^\infty \frac{\cos \lambda x}{1+\lambda^2}\, d\lambda = \frac{\pi}{2} e^{-x}.$$

2.6.3. EXAMPLE:

Express $f(x) = \begin{cases} 1, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$ as a Fourier sine integral and hence evaluate $\int_0^\infty \frac{1-\cos \pi\lambda}{\lambda} \sin \lambda x \; d\lambda$.

SOLUTION:

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin \lambda x \int_0^\infty f(t) \sin \lambda t \; dt \; d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \sin \lambda x \left(\int_0^\pi (1) \sin \lambda t \; dt\right) d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \sin \lambda x \left(\frac{-\cos \lambda t}{\lambda}\right)_0^\pi d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \sin \lambda x \left[\frac{-1}{\lambda}(\cos \pi\lambda - 1)\right] d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \sin \lambda x \left(\frac{1-\cos \pi\lambda}{\lambda}\right) d\lambda$$

$$\Rightarrow \quad \int_0^\infty \sin \lambda x \left(\frac{1-\cos \pi\lambda}{\lambda}\right) d\lambda = \frac{\pi}{2} f(x)$$

$$= \frac{\pi}{2}\begin{cases} 1, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$$

2.7. CONVOLUTION OF TWO FUNCTIONS:

2.7.1. DEFINITION:

The convolution of two functions $f(x)$ and $g(x)$ is defined as

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt\,.$$

2.7.2. CONVOLUTION THEOREM FOR FOURIER TRANSFORM:

If $F(s)$ and $G(s)$ are the Fourier transform of $f(x)$ and $g(x)$ respectively then the Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transform.

(i.e.) $$F\big(f(x) * g(x)\big) = F(s).\, G(s)\,.$$

PROOF:

$$F\left[(f * g)(x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (f * g)(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt\right] e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x-t)\, e^{isx}\, dx\right] dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-t)\, e^{isx}\, e^{ist} . e^{-ist}\, dx \right] dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-t)\, e^{is(x-t)}\, d(x-t) \right] e^{ist}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, G(s)\, e^{ist}\, dt$$

$$= G(s) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt$$

$$= G(s).F(s)$$

(i.e.) $F\left(f(x) * g(x)\right) = F(s).\, G(s)$.

2.8. PROPERTIES OF FOURIER TRANSFORM:

2.8.1. PROPERTY: (LINEAR PROPERTY:)

$$F\left[af(x) + bg(x)\right] = aF(f(x)) + bF(g(x))$$

2.8.2. PROPERTY:

If $F(f(x)) = F(s)$ the $F\left(e^{iax} f(x)\right) = F(s+a)$.

PROOF:

$$F\left(e^{iax}\, f(x)\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(s+a)x} f(x)\, dx$$

$$F\left(e^{iax}\, f(x)\right) = F(s+a).$$

2.8.3. PROPERTY: (SHIFTING THEOREM:)

If $F(f(x)) = F(s)$ then $F(f(x-a)) = e^{isa}\, F(s)$

PROOF:

$$F(f(x-a)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)\, e^{isx}\, dx$$

Put $x - a = t$ If $x \to -\infty$, $t \to -\infty$

$dx = dt$ $x \to \infty$, $t \to \infty$

$$\therefore \quad = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is(t+a)}\, dt$$

$$= e^{isa} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt$$

$$F\left(f(x-a)\right) = e^{isa}\ F(s).$$

2.8.4. PROPERTY: (CHANGE OF SCALE PROPERTY:)

If $F\left(f(x)\right) = F(s)$ then $F\left(f(ax)\right) = \frac{1}{a}\ F\left(\frac{s}{a}\right),\ \ a > 0$.

PROOF:

$$F\left(f(ax)\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\, e^{isx}\, dx$$

Put $ax = t$ If $x \to -\infty$, $t \to -\infty$

$a\ dx = dt$ $x \to \infty$, $t \to \infty$

$$\therefore \quad = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is\frac{t}{a}}\, \frac{dt}{a}$$

$$= \frac{1}{a}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\left(\frac{s}{a}\right)t}\, dt\right]$$

$$F\left(f(ax)\right) = \frac{1}{a}\ F\left(\frac{s}{a}\right).$$

2.8.5. PROPERTY: (MODULATION PROPERTY:)

If $F\left(f(x)\right) = F(s)$ then $F\left(f(x)\cos ax\right) = \frac{1}{2}\ \left[F(s+a) + F(s-a)\right]$

PROOF:

$$F\left(f(x)\cos ax\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos ax\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\left(\frac{e^{iax} + e^{-isx}}{2}\right) e^{isx}\, dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(s+a)x}\,dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(s-a)x}\,dx\right]$$

$$F\left(f(x)\cos ax\right) = \frac{1}{2}\left[F(s+a) + F(s-a)\right].$$

2.8.6. PROPERTY:

If $F(f(x)) = F(s)$ then $F\left(x^n f(x)\right) = (-i)^n \frac{d^n}{ds^n}(F(s))$

PROOF:

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

Diff. w.r.to 's', n times,

$$\frac{d^n}{ds^n}(F(s)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial^n}{\partial s^n}\left(f(x)\, e^{isx}\right) dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\,(ix)^n\, dx$$

$$= \frac{(i)^n}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \left(f(x)\, x^n\right) e^{isx}\, dx$$

$$= (i)^n F\left(x^n f(x)\right)$$

$$\Rightarrow \qquad F\left(x^n f(x)\right) = \frac{1}{(i)^n}\frac{d^n}{ds^n}(F(s))$$

$$F\left(x^n f(x)\right) = (-i)^n \frac{d^n}{ds^n}(F(s)).$$

2.8.7. PROPERTY:

If $F(f(x)) = F(s)$ then $F\left(f'(x)\right) = -is\left(F(s)\right)$, where $f(x) \to 0$ as $x \to \pm\infty$.

PROOF:

$$F\left(f'(x)\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}\, f'(x)\, dx$$

Let $u = e^{isx}$ $\qquad dv = f'(x)\, dx$

$du = e^{isx}(is)\, dx$ $\qquad v = f(x)$

$$= \frac{1}{\sqrt{2\pi}}\left[\left(e^{isx}\ f(x)\right)_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\ e^{isx}\ (is)\ dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[(0-0) - is\int_{-\infty}^{\infty} f(x)\ e^{isx}\ dx\right]$$

$$F\left(f'(x)\right) = (-is)\left(F(s)\right)$$

lll^{ly} $\quad F\left(f^{n}(x)\right) = (-is)^{n}\left(F(s)\right)$, if $\quad f, f', f'', \cdots f^{n-1} \to 0$ as $x \to \pm\infty$.

2.8.8. PROPERTY:

If $F\left(f(x)\right) = F(s)$ then $F\left[\int_{a}^{x} f(x)\ dx\right] = \frac{F(s)}{(-is)}$.

PROOF:

Let $\phi(x) = \int_{a}^{x} f(x)\ dx$ then $\phi'(x) = f(x)$.

$\therefore \quad F\left[\phi'(x)\right] = (-is)\ F\left(\phi(x)\right)$ (by property (7))

$$= (-is)\ F\left[\int_{a}^{x} f(x)\ dx\right]$$

$\therefore \quad F\left[\int_{a}^{x} f(x)\ dx\right] = \frac{1}{-is} F\left(\phi'(x)\right)$

$$F\left[\int_{a}^{x} f(x)\ dx\right] = \frac{1}{(-is)} F(f(x))$$

(i.e.) $\quad F\left[\int_{a}^{x} f(x)\ dx\right] = \frac{F(s)}{(-is)}$.

2.8.9. PROPERTY:

If $F\left(f(x)\right) = F(s)$ then $F\left(\overline{f(x)}\right) = \overline{F(-s)}$.

PROOF:

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\ e^{isx}\ dx$$

$$\therefore \qquad F(-s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$$

Taking complex conjugate on both sides we get,

$$\therefore \qquad \overline{F(-s)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(x)}\, e^{isx}\, dx$$

$$= F\left(\overline{f(x)}\right).$$

2.8.10. PROPERTY:

If $F(f(x)) = F(s)$ then $F\left(\overline{f(-x)}\right) = \overline{F(s)}$.

PROOF:

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

Taking conjugate,

$$\therefore \qquad \overline{F(s)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(x)}\, e^{-isx}\, dx$$

Put $x = -t$ If $x \to -\infty, t \to \infty$

$dx = -dt$ $x \to \infty \quad , t \to -\infty$

$$\therefore \qquad \overline{F(s)} = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} \overline{f(-t)}\, e^{ist}\, (-dt)$$

$$= -\frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} \overline{f(-t)}\, e^{ist}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(-t)}\, e^{ist}\, dt$$

$$\overline{F(s)} = F\left(\overline{f(-t)}\right).$$

2.9. PROPERTIES OF SINE AND COSINE TRANSFORM:

2.9.1. PROPERTY: (LINEAR PROPERTY)

(i) $F_s\left[af(x) + bg(x)\right] = aF_s(f(x)) + bF_s(g(x))$

(ii) $F_c\left[af(x)+bg(x)\right]=aF_c(f(x))+bF_c(g(x))$

2.9.2. PROPERTY: (MODULATION PROPERTY)

(i) $F_s\left(f(x)\cos ax\right)=\frac{1}{2}\left[F_s(s+a)+F_s(s-a)\right]$

(ii) $F_c\left(f(x)\cos ax\right)=\frac{1}{2}\left[F_c(s+a)+F_c(s-a)\right]$

(iii) $F_s\left(f(x)\sin ax\right)=\frac{1}{2}\left[F_c(s-a)-F_s(s+a)\right]$

(iv) $F_c\left(f(x)\sin ax\right)=\frac{1}{2}\left[F_s(s+a)-F_s(s-a)\right]$

2.9.3. PROPERTY:

(i) $F_s\left(f(ax)\right)=\frac{1}{a}F_s\left(\frac{s}{a}\right),\quad a>0$

(ii) $F_c\left(f(ax)\right)=\frac{1}{a}F_c\left(\frac{s}{a}\right),\quad a>0$

2.9.4. PROPERTY:

(i) $F_s\left(f'(x)\right)=-s\left(F_c(s)\right)$, where $f(x)\to 0$ as $x\to\pm\infty$

(ii) $F_c\left(f'(x)\right)=-\sqrt{\frac{2}{\pi}}f(0)+s\,F_s(s)$, where $f(x)\to 0$ as $x\to\pm\infty$

2.9.5. PROPERTY:

(i) $F_s\left(x\,f(x)\right)=-\frac{d}{ds}\left(F_c(s)\right)$

(ii) $F_c\left(x\,f(x)\right)=\frac{d}{ds}\left(F_s(s)\right)$

2.10. EXAMPLES:

2.10.1. EXAMPLE:

Find the Fourier sine transform of $x\,e^{-x^2/2}$.

SOLUTION:

We know that $F_s\left(x\, f(x)\right) = -\frac{d}{ds}\left(F_c(s)\right)$

$$\therefore \quad F_s\left(x\, e^{-x^2/2}\right) = -\frac{d}{ds} F_c\left(e^{-x^2/2}\right) \qquad (1)$$

We know that,

$$F_c\left(e^{-x^2/2}\right) = e^{-s^2/2}$$

$$F_s\left(x\, e^{-x^2/2}\right) = -\frac{d}{ds} F_c\left(e^{-s^2/2}\right)$$

$$= -e^{-s^2/2}\left(\frac{-2s}{2}\right)$$

$$F_s\left(x\, e^{-x^2/2}\right) = s\, e^{-s^2/2}.$$

2.10.2. EXAMPLE:

Find Fourier cosine transform of $e^{-a^2x^2}$ and hence find $F_s\left(x\, e^{-a^2x^2}\right)$.

SOLUTION:

We know that $F_s\left(x\, f(x)\right) = -\frac{d}{ds}\left(F_c(s)\right)$

$$\therefore \quad F_s\left(x\, e^{-a^2x^2}\right) = -\frac{d}{ds} F_c\left(e^{-a^2x^2}\right) \qquad (1)$$

We know that,

$$F_c\left(e^{-a^2x^2}\right) = \frac{1}{\sqrt{2}}\frac{1}{a}\; e^{-s^2/4a^2}$$

$$F_s\left(x\, e^{-a^2x^2}\right) = -\frac{d}{ds} F_c\left(e^{-a^2x^2}\right)$$

$$= -\frac{d}{ds}\left(\frac{1}{\sqrt{2}}\frac{1}{a}\; e^{-s^2/4a^2}\right)$$

$$= -\frac{1}{\sqrt{2}}\frac{1}{a}\; e^{-s^2/4a^2}\left(\frac{-2s}{4a^2}\right)$$

$$F_s\left(x\, e^{-a^2x^2}\right) = \frac{s}{2\sqrt{2}a^3}\, e^{-s^2/4a^2}.$$