Homogeneous Linear partial differential equation

A homogeneous linear pde is of the form

$$a_0\frac{\partial^n z}{\partial x^n}+a_1\frac{\partial^n z}{\partial x^{n-1}\partial y}+a_2\frac{\partial^n z}{\partial x^{n-2}\partial y^2}\cdot\cdot\cdot+a_n\frac{\partial^n z}{\partial y^n}=F(x,y)\quad --------(1)$$

Where $a_0,a_1,a_2,\cdot\cdot\cdot,a_n$ are constants.

Equation (1) can be written as

$$(a_0D^n+a_1D^{n-1}D'+a_2D^{n-2}D'^2+\cdot\cdot\cdot+a_nD'^n)z=F(x,y)\quad ----------(2)$$

$$ie,\ f(D,D')z=F(x,y)\quad --------(3)$$

where $D\equiv\frac{\partial}{\partial x},\quad D'\equiv\frac{\partial}{\partial y}$

Solution of Homogeneous Linear partial differential equation

The complete solution has two parts

(1). Complementary Function (C.F)

(2). Particular Integrals (P.I)

Complementary Function (C.F)

The complementary function is the solution of the equation

$$(a_0D^n+a_1D^{n-1}D'+a_2D^{n-2}D'^2+\cdot\cdot\cdot+a_nD'^n)z=F(x,y)\quad ----------(4)$$

put $D=m\ \ and\ \ D'=1$, then we get an equation which is called the auxiliary equation (A.E.) .

The A.E. of (4) is $a_0m^n+a_1m^{n-1}+a_2m^{n-2}+\cdot\cdot\cdot+a_n=0\quad ----------(5)$

Let the roots of the equation be $m_1,m_2,\ .\ .\ .,m_n$

(i) If the roots are real or imaginary and different say $m_1\neq m_2\neq\ .\ .\ .\neq m_n$, then the CF is $z=\phi_1(y+m_1x)+\phi_2(y+m_2x)+\cdots+\phi_n(y+m_nx)$

(ii) Two two roots of (5) are equal and others are distinct, then the CF is
$z=\phi_1(y+m_1x)+x\phi_2(y+m_1x)+\phi_3(y+m_3x)+\cdots+\phi_n(y+m_nx)$

(iii) 'r' of the roots of (5) ar equal and others distinct.
$ie,\ m_1=m_2=\ .\ .\ .=m_r=m\ \ (say)$, then the CF is
$z=\phi_1(y+mx)+x\phi_2(y+mx)+x^2\phi_3(y+mx)+\cdots+x^{r-1}\phi_r(y+mx)$

Particular Integrals

The P.I. of $f(D,D')z=F(x,y)$

Rule:1 If the RHS function of a given pde is of the form $F(x,y)=e^{ax+by}$, then

$$P.I=\frac{1}{f(D,D')}e^{ax+by}\qquad \text{Re}\ place\ \ D\ by\ a\ \ and\ \text{Re}\ place\ D'\ by\ b$$

$$P.I = \frac{1}{f(a,b)} e^{ax+by} \quad provided \quad f(a,b) \neq 0$$

Rule:2 If the RHS function of a given pde is of the form $F(x,y) = \sin(mx+ny) \;\; or \;\; \cos(mx+ny)$, then

$$P.I = \frac{1}{f(D,D')} \sin(mx+ny) \;\; or \;\; \cos(mx+ny)$$

Replace $D^2 \;\; by \;\; -m^2, D'^2 \;\; by \;\; -n^2 \;\; and \;\; DD' \;\; by \;\; -mn$

In $f(D,D')$ provided the $DR \neq 0$

Rule:3 is of the form $F(x,y) = x^m y^n$, then

$$P.I = \frac{1}{f(D,D')} x^m y^n$$

If the RHS function of a given pde

$$= f(D,D')^{-1} \; x^m y^n$$

Expand $f(D,D')^{-1}$ by using Binomial theorem and then operate $x^m y^n$

Note:1 $\frac{1}{D}$ denote integration w.r.to x and $\frac{1}{D'}$ denote integration w.r.to y.

Note:2 **If** $x^m y^n \;\; if \;\; m < n \;\; then \;\; try \;\; to \;\; write \;\; f(D,D') \;\; as \;\; f\left(\frac{D}{D'}\right)$ **and**

$x^m y^n \;\; if \;\; m > n \;\; then \;\; try \;\; to \;\; write \;\; f(D,D') \;\; as \;\; f\left(\frac{D'}{D}\right)$

Rule:4 If the RHS function of a given pde $F(x,y)$ is any other function (other than in Rule-1,2,3). Resolve $f(D,D')$ into linear factor say $(D-m_1D')(D-m_2D')$, etc, then $P.I = \frac{1}{(D-m_1D')(D-m_2D')} F(x,y)$

Note: $\frac{1}{(D-mD')} F(x,y) = \int F(x, c-mx)dx$ where $y = c - mx$

Problems based on RHS=0

1. Solve $(D^3 - 3DD'^2 + 2D'^3)z = 0$

Sol.

The A.E is $m^3 - 3m + 2 = 0$ $\quad$ Re $placeD \;\; by \;\; m \;\; \& \;\; D' \;\; by \;\; 1$

$(m-1)(m^2+m-2) = 0$

$(m-1)(m+2)(m-1) = 0$

$m = 1, 1, -2$

$\therefore C.F = \phi_1(y+x) + x\phi_2(y+x) + \phi_3(y-2x)$

The solution $z = \phi_1(y+x) + x\phi_2(y+x) + \phi_3(y-2x)$

2. Solve $(D^3 + D^2D' + DD'^2 + D'^3)z = 0$

Sol.

The A.E is $m^3 + m^2 + m + 1 = 0$ $\quad$ Re $placeD\ by\ m\ \&\ D'\ by\ 1$

$(m+1)(m^2+1) = 0$

$m = -1, \quad m^2 = -1 \Rightarrow m = \pm i$

$m = -1, i, -i$

$\therefore C.F = \phi_1(y-x) + \phi_2(y+ix) + \phi_3(y-ix)$

The solution $z = \phi_1(y-x) + \phi_2(y+ix) + \phi_3(y-ix)$

(OR)

$(m+1)^3 = 0$

$m = -1, -1, -1$

$z = \phi_1(y-x) + x\phi_2(y-x) + x^2\phi_3(y-x)$

3. Find the general solution of $4\dfrac{\partial^2 z}{\partial x^2} - 12\dfrac{\partial^2 z}{\partial y \partial x} + 9\dfrac{\partial^2 z}{\partial y^2} = 0$

Sol.

The A.E is $4m^2 - 12m + 9 = 0$ $\quad$ Re $placeD\ by\ m\ \&\ D'\ by\ 1$

$$m = \frac{12 \pm \sqrt{144 - 144}}{8}$$

$$\Rightarrow m = \frac{3}{2}, \frac{3}{2}$$

$$z = \phi_1(y - \frac{3}{2}x) + x\phi_2(y - \frac{3}{2}x)$$

4. Solve $(2D^2 + 5DD' + 2DD'^2)z = 0$

Sol.

The A.E is $2m^2 + 5m + 2 = 0$

$(m+2)(2m+1) = 0$

$m = -2, \quad m = -\frac{1}{2}$

The roots are different. The solution is

$$z = \phi_1(y - 2x) + x\phi_2(y - \frac{1}{2}x)$$

5. Solve $(D^4 - D'^4)z = 0$

Sol.

$m^4 - 1 = 0$

$(m^2 - 1)(m^2 + 1) = 0$

$m^2 = 1, \quad m^2 = -1$

$m = \pm 1, \quad m = \pm i$

The solution is

$z = \phi_1(y+x) + \phi_2(y-x) + \phi_3(y+ix) + \phi_4(y-ix)$

Type-I Problems based on $F(x,y)=e^{ax+by}$

1. Solve $\frac{\partial^2 z}{\partial x^2}-5\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}=e^{x+y}$

Sol.

The given equation can be written as

$(D^2-5DD'+6D'^2)z=e^{x+y}$

The A.E. is $m^2-5m+6=0$

$(m-3)(m-2)=0$

$m=3,2$

$CF=\phi_1(y+3x)+\phi_2(y+2x)$

$P.I=\frac{1}{D^2-5DD'+6D'^2}e^{x+y}$

$=\frac{1}{1-5+6}e^{x+y}$ *Here* $a=1,\ b=1$

Replace D by a=1
Replace D' by b=1

$=\frac{1}{2}e^{x+y}$

The complete solution is $z=C.F+P.I$

$$z=\phi_1(y+3x)+\phi_2(y+2x)+\frac{1}{2}e^{x+y}$$

2. Solve $(D^4-2D^3D'+2DD'^3-D'^4)z=e^{2x+3y}$

Sol.

The A.E is $m^4-2m^3+2m-1=0$

$(m+1)(m-1)^3=0$

$m=-1,1,1,1$

$CF=\phi_1(y-x)+\phi_2(y+x)+x\phi_3(y+x)+x^2\phi_4(y+x)$

$P.I=\frac{1}{D^4-2D^3D'+2DD'^3-D'^4}e^{2x+3y}$

$=\frac{1}{16-48+108-81}e^{2x+3y}$ *Here* $a=2,\ b=3$

Replace D by a=2
Replace D' by b=3

$=-\frac{1}{5}e^{2x+3y}$

The complete solution is $z=C.F+P.I$

$$z=\phi_1(y-x)+\phi_2(y+x)+x\phi_3(y+x)+x^2\phi_4(y+x)-\frac{1}{5}e^{2x+3y}$$

3. Solve $(D^2-3DD'+2D'^2)z=2\cosh(3x+4y)$

Sol.

The A.E. is $m^2-3m+2=0$

$(m-1)(m-2)=0$

$m=1,2$

$CF=\phi_1(y+x)+\phi_2(y+2x)$

$$P.I=\frac{1}{D^2-3DD'+2D'^2}2\cosh(3x+4y)$$

$$=\frac{1}{D^2-3DD'+2D'^2}2\frac{\left[e^{(3x+4y)}+e^{-(3x+4y)}\right]}{2}$$

$$=\frac{1}{D^2-3DD'+2D'^2}e^{(3x+4y)}+\frac{1}{D^2-3DD'+2D'^2}e^{-(3x+4y)}$$

$$=\frac{1}{3^2-3(3)(4)+2(4)^2}e^{(3x+4y)}+\frac{1}{(-3)^2-3(-3)(-4)+2(-4)^2}e^{-(3x+4y)}$$

$$=\frac{1}{9-36+32}e^{(3x+4y)}+\frac{1}{9-36+32}e^{-(3x+4y)}$$

$$=\frac{1}{5}e^{(3x+4y)}+\frac{1}{5}e^{-(3x+4y)}$$

$$=\frac{1}{5}\left[e^{(3x+4y)}+e^{-(3x+4y)}\right]$$

$$P.I=\frac{1}{5}2\cosh(3x+4y)$$

The complete solution is $z=C.F+P.I$

$$z=\phi_1(y+x)+\phi_2(y+2x)+\frac{2}{5}\cosh(3x+4y)$$

Type-II $RHS=F(x,y)=\cos(mx+ny)$ *or* $\sin(mx+ny)$

1. Solve $(D^2-2DD'+D'^2)z=cos(x-3y)$

Sol.

The A.E is $m^2-2m+1=0$

$(m-1)(m-1)=0$

$m=1,1$

$CF=\phi_1(y+x)+x\phi_2(y+x)$

$$P.I = \frac{1}{D^2 - 2DD' + D'^2} \cos(x - 3y)$$

$$= \frac{1}{-1 - 2(3) + (-9)} \cos(x - 3y)$$

Here $m = 1, n = -3$

Replace $D^2 = -m^2 = -(1)^2 = -1$

Replace $D'^2 = -n^2 = -(-3)^2 = -9$

Replace $DD' = -mn = -(1)(-3) = 3$

$$= -\frac{1}{16} \cos(x - 3y)$$

The complete solution is $z = C.F + P.I$

$$z = \phi_1(y + x) + \phi_2(y + x) - \frac{1}{16} \cos(x - 3y)$$

2. Solve $(D^3 - 4D^2D' + 4DD'^2)z = 6\sin(3x + 6y)$

Sol.

The. A.E is $m^3 - 4m^2 + 4m = 0$

$\Rightarrow m(m^2 - 4m + 4) = 0 \; m = 0, \; (m - 2)^2 = 0$

$m = 0, 2, 2$

$$CF = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x)$$

$$P.I = \frac{1}{(D^3 - 4D^2D' + 4DD'^2)} 6\sin(3x + 6y)$$

$$= \frac{1}{D(D^2 - 4DD' + 4D'^2)} 6\sin(3x + 6y)$$

Here $m = 3, n = 6$

Replace $D^2 = -m^2 = -(3)^2 = -9$

Replace $D'^2 = -n^2 = -(-6)^2 = -36$

Replace $DD' = -mn = -(3)(6) = -18$

$$= \frac{6}{D}\left[\frac{\sin(3x + 6y)}{-9 + 72 - 144}\right]$$

$$= \frac{6}{D}\left[\frac{\sin(3x + 6y)}{-81}\right]$$

$$= -\frac{2}{27}\int \sin(3x + 6y)dx \qquad \therefore \frac{1}{D} = \int^{te} w.r.to \; x$$

$$= -\frac{2}{27}\left[\frac{-\cos(3x + 6y)}{3}\right]$$

$$= \frac{2}{81}\cos(3x + 6y)$$

The complete solution is $z = C.F + P.I$

$$z = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x) + \frac{2}{81}\cos(3x + 6y)$$

3. Solve $(D^2+3DD'-4D'^2)z=\sin(y)$

Sol.

The A.E is $m^2+3m-4=0$

$(m-1)(m+4)=0$

$m=-4,1$

$CF=\phi_1(y-4x)+\phi_2(y+x)$

$$P.I=\frac{1}{D^2+3DD'-4D'^2}\sin y$$

Here $m=0, n=1$

Re $place$ $D^2=-m^2=0$

Re $place$ $D'^2=-n^2=-(1)^2=-1$

Re $place$ $DD'=-mn=-(0)(1)=0$

$$=\frac{1}{0+0-4(-1)}\sin y$$

$$=\frac{1}{4}\sin y$$

The complete solution is $z=C.F+P.I$

$$z=\phi_1(y-4x)+\phi_2(y+x)+\frac{1}{4}\sin y$$

4. Solve $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}=\cos 2x\cos y$

Sol.

Given equation can be written as

$$(D^2+DD')z=\cos 2x\cos y$$

$$=\frac{1}{2}\cos(2x+y)+\cos(2x-y)$$

The A.E is $m^2+m=0$

$m(m+1)=0$

$m=0,\ m=-1$

$C.F=\phi_1(y)+\phi_2(y-x)$

$$P.I=\frac{1}{D^2+DD'}\cos(2x+y)+\frac{1}{D^2+DD'}\cos(2x-y)$$

$$=\frac{1}{-4-2}\cos(2x+y)+\frac{1}{-4+2}\cos(2x-y)$$

$Here\ m=2,\ n=1$

Re $place$ $D^2=-m^2=-4$

$D'^2=-n^2=-1$ & $D'^2=-n^2=-(-1)=1$

$$=\frac{1}{-6}\cos(2x+y)+\frac{1}{-2}\cos(2x-y)$$

The complete solution is $z=C.F+P.I$

$$z=\phi_1(y)+\phi_2(y-x)-\frac{1}{6}\cos(2x+y)-\frac{1}{2}\cos(2x-y)$$

Type-III $RHS=F(x,y)=x^m y^n$

1. Solve $(D^2+3DD'+2D'^2)z=x+y$

Sol.

The A.E is $m^2+3m+2=0$

$(m+1)(m+2)$

$m=-1, m=-2$

$C.F=\phi_1(y-x)+\phi_2(y-2x)$

$$P.I=\frac{1}{D^2+3DD'+2D'^2}(x+y)$$

$$=\frac{1}{D^2}\left[1+\left(\frac{3D'}{D}+\frac{2D'2}{D^2}\right)\right]^{-1}(x+y)$$

$$=\frac{1}{D^2}\left[1-\frac{3D'}{D}\right](x+y)$$

$$=\left[\frac{1}{D^2}-\frac{3D'}{D^3}\right](x+y)$$

$$=\left[\frac{1}{D^2}(x+y)-\frac{3D'}{D^3}(x+y)\right] \qquad \because \frac{1}{D}=\int^{te} w.r.to\ \ x$$

$$=\left(\frac{x3}{6}+\frac{yx2}{2}\right)-\frac{3(0+1)}{D^3}$$

$$=\left(\frac{x^3}{6}+\frac{yx^2}{2}\right)-\frac{3x^3}{6}$$

$$=\frac{yx^2}{2}-\frac{2x^3}{6}=\frac{1}{6}\ 3x^2y-2x^3$$

The complete solution is $z=C.F+P.I$

$$z=\phi_1(y-x)+\phi_2(y-2x)-\frac{x^2}{6}(3y-2x)$$

Type – IV $F(x,y)=\phi(x)e^{ax+by}$ where $\phi(x)=\begin{cases}Cos(mx+ny)\ or\ \sin(mx+ny) & or\\ x^m y^n\end{cases}$

1. Solve $(D^2-2DD'+D'^2)z=x^2y^2e^{x+y}$

Sol.

The A.E is $m^2-2m+1=0$

$(m-1)(m-1)=0$

$m=1, m=1$

$C.F=\phi_1(y+x)+x\phi_2(y+x)$

$$P.I = \frac{1}{D^2 - 2DD' + D'^2}(x^2y^2e^{x+y})$$

$$= \frac{1}{(D-D')^2}(x^2y^2e^{x+y})$$

Here $a = 1$ & $b = 1$

Re *place* $D = D + 1$

$D' = D' + 1$

$$= e^{x+y}\frac{1}{(D+1)-(D'+1)^{2}}x^2y^2$$

$$= e^{x+y}\frac{1}{(D+1-D'-1)^{2}}x^2y^2$$

$$= e^{x+y}\frac{1}{D-D'^{2}}x^2y^2$$

$$= e^{x+y}\frac{1}{D^2\left[1-\frac{D'}{D}\right]^2}x^2y^2$$

$$= e^{x+y}\frac{1}{D^2}\left[1-\left(\frac{D'}{D}\right)\right]^{-2}(x^2y^2)$$

$$= e^{x+y}\frac{1}{D^2}\left[1+2\frac{D'}{D}+3\left(\frac{D'}{D}\right)^2\right](x^2y^2)$$

$$= e^{x+y}\left[\frac{1}{D^2}+2\frac{D'}{D^3}+3\frac{D'^2}{D^4}\right](x^2y^2)$$

$$= e^{x+y}\left[\frac{1}{D^2}(x^2y^2)+2\frac{D'}{D^3}(x^2y^2)+3\frac{D'^2}{D^4}(x^2y^2)\right]$$

$$= e^{x+y}\left[\frac{x^4y^2}{12}+4\frac{x^2y}{D^3}+6\frac{x^2}{D^4}\right]$$

$$= e^{x+y}\left[\frac{x^4y^2}{12}+4\frac{x^5y}{60}+6\frac{x^6}{360}\right]$$

$$= e^{x+y}\left[\frac{x^4y^2}{12}+\frac{x^5y}{15}+\frac{x^6}{60}\right]$$

The complete solution is

$$z = \phi_1(y+x) + x\phi_2(y+x) + e^{x+y}\left[\frac{x^4y^2}{12}+\frac{x^5y}{15}+\frac{x^6}{60}\right]$$