UNIT – III
PARTIAL DIFFERENTIAL EQUATIONS (P.D.E.)

An equation containing pde co-efficients is called pde. Here we deal with pdes contains only two independent variables. Z will be taken as the dependent variable and x and y the independent variables, so that z=f(x,y) . We will use the partial derivatives

$$\frac{\partial z}{\partial x} = p,\ \frac{\partial z}{\partial y} = q, \qquad \frac{\partial^2 z}{\partial x^2} = r,\ \frac{\partial^2 z}{\partial x \partial y} = s,\ \frac{\partial^2 z}{\partial y^2} = t$$

The order of a pde is that of the highest order derivative occurring in it.

Formation of PDE

PDEs can be formed by eliminating the arbitrary constants or arbitrary functions from the functional relations satisfied by the dependent and independent variables.

Note

1. If the number of arbitrary constants to b eliminated is equal to the number of independent variables, the process of elimination results in a pde of the first order.
2. If the number of arbitrary constants to be eliminated is more than the number of independent variables, the process of elimination will lead to a pde of second or higher orders.
3. If the pde is formed by eliminating arbitrary functions, the order of the equation will be equal to the number of arbitrary functions eliminated.

Elimination of arbitrary constants

Consider the functional relation among $x, y, z.\ ie,\ f(x, y, z, a, b) = 0$---------(1) where a and b are arbitrary constants to be eliminated.

Diff. (1) p.w.r.to x and y, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p = 0 \text{-----------------------------(2)}$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q = 0 \text{-----------------------------(3)}$$

Equation (2) and (3) will contain a and b from equation (1), (2) and (3), we get pde (involving p and q) of the first order.

TYPE-I (Number of a.c. < Number of independent variables)

1. Form the pde by eliminating the arbitrary constants a and b from $z = ax + by$.

   Sol.

   Given $z = ax + by$ ----------(1)

   Diff (1) p.w.r.to x and y, we get

   $$\frac{\partial z}{\partial x} = a \quad ie,\ p = a$$

   $$\frac{\partial z}{\partial y} = b \quad ie,\ q = b$$

   Substitute in (1) we get $z = px + qy$.

2. Eliminate the arbitrary constants a & b from $z = (x^2 + a)(y^2 + b)$

   Sol.

   $z = (x^2 + a)(y^2 + b)$ --------(1)

   Diff (1) p.w.r.to x and y, we get

   $$\frac{\partial z}{\partial x} = 2x(y^2 + b)$$

   $$p = 2x(y^2 + b) \Rightarrow y^2 + b = \frac{p}{2x} \text{------}(2)$$

   $$\frac{\partial z}{\partial y} = 2y(x^2 + a)$$

   $$q = 2y(x^2 + a) \Rightarrow (x^2 + a) = \frac{q}{2y} \text{--------}(3)$$

   Sub. (2) and (3) in (1), we get

   $$z = \frac{q}{2y}\frac{p}{2x} \quad ie,\ 4xyz = pq$$

3. Form the pde by eliminating a and b from $(x-a)^2 + (y-b)^2 = z^2 \cot^2 \alpha$

   Sol.

   Given $(x-a)^2 + (y-b)^2 = z^2 \cot^2 \alpha$ ---------------(1)

   Diff (1) w.r.to x and y

   $$2z\frac{\partial z}{\partial x}\cot^2 \alpha = 2(x-a)$$

$$zp\cot^2\alpha=(x-a)$$
$$zp\cot^2\alpha=(x-a)\ -----(2)$$

$$2z\frac{\partial z}{\partial y}\cot^2\alpha=2(y-a)$$
$$zq\cot^2\alpha=(y-a)\ ---------(3)$$

Sub (2) and (3) in (1), we get

$$(zp\cot^2\alpha)^2+(zq\cot^2\alpha)^2=z^2\cot^2\alpha$$

$$(z^2p^2\cot^4\alpha)+(z^2q^2\cot^4\alpha)=z^2\cot^2\alpha$$

$$z^2\cot^4\alpha(p^2+q^2)=z^2\cot^2\alpha$$
$$p^2+q^2=\frac{1}{\cot^2\alpha}$$
$$p^2+q^2=\tan^2\alpha$$

4. find the pde of all sphere whose centre lie on the z-axis.

Sol.

Let the centre of the sphere be (0,0,c) a point on the z-axis and 'r' its rdius.

Its equation is $(x-0)^2+(y-0)^2+(z-c)^2=r^2$

$$x^2+y^2+(z-c)^2=r^2\ ------(1)$$

Here c and r are constants.

Diff (1) p.w.r.to x and y, we get

$$2x+2(z-c)\frac{\partial z}{\partial x}=0$$
$$x+(z-c)p=0$$
$$(z-c)=-\frac{x}{p}\ ------(2)$$
$$2y+2(z-c)\frac{\partial z}{\partial y}=0$$
$$y+(z-c)q=0$$
$$(z-c)=-\frac{y}{q}\ ------(3)$$

From (2) and (3), we have

$$-\frac{x}{p}=-\frac{y}{q}$$
$$xq=yp$$

$\Rightarrow qx - py = 0$

5. Find the pde of all spheres of radius c having their centres in the XOY plane.

Sol.

Let the centre of the sphere be (a,b,0) a point in XOY plane, c is the given radius.

The equation of the sphere is $(x-a)^2 + (y-b)^2 + (z-0)^2 = c^2$ ------(1)
Here a and b are the two arbitrary constants
Diff (1) p.w.r.to x and y

$$2(x-a) + 2z\frac{\partial z}{\partial x} = 0$$
$$(x-a) = -zp \quad ------(2)$$
$$2(y-b) + 2z\frac{\partial z}{\partial y} = 0$$
$$(y-b) = -zq \quad ------(3)$$

Sub (2) and (3), we get

$$(-zp)^2 + (-zq)^2 + z^2 = c^2$$

$z^2(p^2 + q^2 + 1) = c^2$ which is required pde.

6. Find the pde of the family of spheres having the centers on the line x=y=z.

Sol.

The equation of such sphere is

$$(x-a)^2 + (y-a)^2 + (z-a)^2 = r^2 \quad -------(1)$$

Diff (1) p.w.r.to x and y, we get

$$2(x-a) + 2(z-a)\frac{\partial z}{\partial x} = 0$$
$$(x-a) + (z-a)p = 0$$
$$x - a + zp - ap = 0$$
$$(x - zp) - a(1+p) = 0$$
$$a = \frac{x+zp}{1+p} \quad ------(2)$$
$$2(y-a) + 2(z-a)\frac{\partial z}{\partial y} = 0$$
$$(y-a) + (z-a)q = 0$$
$$y - a + zq - aq = 0$$
$$(y - zq) - a(1+q) = 0$$
$$a = \frac{y+zq}{1+q} \quad ------(3)$$

From (2) and (3) we get

$$\frac{x+zp}{1+p}=\frac{y+zq}{1+q}$$ which is the required pde.

7. Find the pde of all planes cutting equal intercepts from the x and y axes.

Sol.

The equation of the plane who intercepts on the x-axis, y-axis and z-axis are a,b,c is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, given that x and y axis having equal intercepts.

The equation becomes $\frac{x}{a}+\frac{y}{a}+\frac{z}{c}=1$ --------(1)

Diff (1) p.w.r.to. x and y, we get

$$\frac{1}{a}+\frac{1}{c}\frac{\partial z}{\partial x}=0$$

$$\Rightarrow \frac{1}{a}=-\frac{1}{c}p \quad ------(2)$$

$$\frac{1}{a}+\frac{1}{c}\frac{\partial z}{\partial y}=0$$

$$\Rightarrow \frac{1}{a}=-\frac{1}{c}q \quad ------(3)$$

From (2) and (3), we get

$$-\frac{1}{c}p=-\frac{1}{c}q$$

$\therefore\ p=q$ which is required pde.

8. Form the pde by eliminating the arbitrary constants a & b from $\log(az-1)=x+ay+b$.

Sol.

Given $\log(az-1)=x+ay+b$ ----(1)

Diff. (1) p.w.r.to x and y

$$\frac{1}{az-1}a\frac{\partial z}{\partial x}=1 \quad \text{ie,} \quad \frac{1}{az-1}ap=1$$

$$\Rightarrow ap=az-1,\ a(p-z)=-1$$

$$\therefore a=\frac{1}{z-p} \quad --------(2)$$

$$\frac{1}{az-1}a\frac{\partial z}{\partial y}=a \quad \text{ie,} \quad \frac{1}{az-1}aq=a$$

$$\Rightarrow aq=a(az-1)$$

$$q = (az - 1) \quad ------- (3)$$

Sub (2) in (3), we get

$$q = \frac{z}{z-p} - 1$$

$$= \frac{z - z + p}{z - p}$$

$$q = \frac{p}{z-p}$$

$$p = q(z-p), \quad p - qz + pq = 0$$

$$p(q+1) = qz \ \ which\ is\ required\ \ pde$$

9. Form the pde by eliminating the arbitrary constants a and b from $\log z = a\log x + \sqrt{1-a^2}\log y + b$.

Sol.

Given $\log z = a\log x + \sqrt{1-a^2}\log y + b \ ----- (1)$

Diff (1) p.w.r.to x

$$\frac{1}{z}\frac{\partial z}{\partial x} = \frac{a}{x}$$

$$\frac{1}{z}p = \frac{a}{x} \qquad \Rightarrow az = px$$

$$\therefore \quad a = \frac{px}{z} \quad ---------- (2)$$

Diff (1) p.w.r.to y

$$\frac{1}{z}\frac{\partial z}{\partial y} = \frac{\sqrt{1-a^2}}{y}$$

$$\frac{1}{z}q = \frac{\sqrt{1-a^2}}{y} \quad \Rightarrow qy = z\sqrt{1-a^2}$$

$$\frac{qy}{z} = \sqrt{1-a^2} \quad ------------ (3)$$

Sub (2) in (3), we get

$$\sqrt{1-(px/z)^2} = \frac{qy}{z}$$

$$1 - \frac{p^2x^2}{z^2} = \frac{q^2y^2}{z^2}$$

$$z^2 - p^2x^2 - q^2y^2 = 0$$

$p^2x^2 + q^2y^2 = z^2$ which is the required pde.

10. Form the pde by eliminating the arbitrary constants a & b from $\sqrt{1+a^2}\,\log(z+\sqrt{z^2-1}) = x+ay+b$.

Sol.

Given $\sqrt{1+a^2}\,\log(z+\sqrt{z^2-1}) = x+ay+b$ -----(1)

Diff (1) p. w. r to. X

$$\sqrt{1+a^2}\,\frac{1}{z+\sqrt{z^2-1}}\left(1+\frac{2z}{2\sqrt{z^2-1}}\right)\frac{\partial z}{\partial x}=1$$

$$\sqrt{1+a^2}\,\frac{1}{z+\sqrt{z^2-1}}\left(1+\frac{2z}{2\sqrt{z^2-1}}\right)p=1$$

$$\sqrt{1+a^2}\,\frac{1}{z+\sqrt{z^2-1}}\left(\frac{\sqrt{z^2-1}+z}{\sqrt{z^2-1}}\right)p=1$$

$$\sqrt{1+a^2}\left(\frac{p}{\sqrt{z^2-1}}\right)=1 \text{ ----------(2)}$$

Diff (1) p.w.r.to y

$$\sqrt{1+a^2}\,\frac{1}{z+\sqrt{z^2-1}}\left(1+\frac{2z}{2\sqrt{z^2-1}}\right)\frac{\partial z}{\partial y}=a$$

$$\sqrt{1+a^2}\,\frac{1}{z+\sqrt{z^2-1}}\left(1+\frac{2z}{2\sqrt{z^2-1}}\right)q=a$$

$$\sqrt{1+a^2}\,\frac{1}{z+\sqrt{z^2-1}}\left(\frac{\sqrt{z^2-1}+z}{\sqrt{z^2-1}}\right)q=a$$

$$\sqrt{1+a^2}\left(\frac{q}{\sqrt{z^2-1}}\right)=a \text{ ----------(3)}$$

$$\frac{(2)}{(3)} \Rightarrow \frac{p}{q}=\frac{1}{a}$$

$$\therefore\ a=\frac{q}{p} \text{ ----------(4)}$$

Sub (4) in (2), we get

$$\sqrt{1+\left(\frac{q}{p}\right)^2}\left(\frac{p}{\sqrt{z^2-1}}\right)=1$$

$$\frac{\sqrt{p^2+q^2}}{p}\frac{p}{\sqrt{z^2-1}} = 1$$

$$\sqrt{p^2+q^2} = \sqrt{z^2+1}$$

$p^2+q^2+1=z^2$ which is required pde.

TYPE-II (Number of a.c > Number of independent variables)

1. Obtain the pde by eliminating a,b,c from $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Sol.

Given $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ----------(1)

Diff. (1) p.w.r.to x

$$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{\partial z}{\partial x}=0$$

$$\frac{2x}{a^2}+\frac{2z}{c^2}p=0 \text{ ----------(2)}$$

Diff. (1) p.w.r.to y

$$\frac{2y}{b^2}+\frac{2z}{c^2}\frac{\partial z}{\partial y}=0$$

$$\frac{2y}{b^2}+\frac{2z}{c^2}q=0 \text{ ----------(3)}$$

Again Diff. (2) p.w.r.to x

$$\frac{1}{a^2}+\frac{1}{c^2}\left(z\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial x}\frac{\partial z}{\partial x}\right)=0$$

$$\frac{1}{a^2}+\frac{1}{c^2}\left(zr+p^2\right)=0 \text{ ----------(4)}$$

Again Diff. (3) p.w.r.to y

$$\frac{1}{b^2}+\frac{1}{c^2}\left(z\frac{\partial^2 z}{\partial y^2}+\frac{\partial z}{\partial y}\frac{\partial z}{\partial y}\right)=0$$

$$\frac{1}{b^2}+\frac{1}{c^2}\left(zt+q^2\right)=0 \text{ ----------(5)}$$

Again Diff. (2) p.w.r.to y

$$0+\frac{1}{c^2}\left(z\frac{\partial^2 z}{\partial y\partial x}+\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\right)=0$$

$$\left(z\frac{\partial^2 z}{\partial y\partial x}+\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\right)=0$$

$zs+pq=0$ which is required pde.

2. Find the differential equation of all spheres whose radii are the same.

Sol.

The equation of all spheres with equal radius can be taken as $(x-a)^2+(y-b)^2+(z-c)^2=r^2$ ---------(1) where a,b,c are arbitrary constants and r is a given constant.

Diff (1) p.w.r.to x

$$2(x-a)+2(z-c)\frac{\partial z}{\partial x}=0$$

$$(x-a)+(z-c)p=0 \text{ ---------(2)}$$

Diff (1) p.w.r.to y

$$2(y-b)+2(z-c)\frac{\partial z}{\partial y}=0$$

$$(y-b)+(z-c)q=0 \text{ ----------(3)}$$

Again, Diff (2) p.w.r.to x

$$1+(z-c)\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial x}\frac{\partial z}{\partial x}=0$$

$$1+(z-c)r+p^2=0 \text{ ----------(4)}$$

Again, Diff (3) p.w.r.to y

$$1+(z-c)\frac{\partial^2 z}{\partial y^2}+\frac{\partial z}{\partial y}\frac{\partial z}{\partial y}=0$$

$$1+(z-c)t+q^2=0 \text{ -----------(5)}$$

$$(4)\Rightarrow(z-c)=\frac{-1-p^2}{r} \text{ ------(6)}$$

$$(5) \Rightarrow (z-c) = \frac{-1-q^2}{t} \text{ ------}(7)$$

From (6) and (7)

$$\frac{-(1+p^2)}{r} = \frac{-(1+q^2)}{t}$$

$$(1+p^2)t = (1+q^2)r$$

$$\frac{1+p^2}{1+q^2} = \frac{r}{t}$$ which is the required pde.

Elimination of arbitrary functions

Consider the relation f(u,v)=0, where u and v are function of x,y,z and f is an arbitrary function to be eliminated.

TYPE-I (elimination of one arbitrary function, gives a pde of order one)

1. Eliminate f from $z = f(x^2 - y^2)$.

Sol.

Given $z = f(x^2 - y^2)$ ---------------------------(1)

Diff. (1) p.w.r.to x

$$\frac{\partial z}{\partial x} = f'(x^2 - y^2)(2x)$$

$$p = f'(x^2 - y^2)(2x) \text{ ----------}(2)$$

Diff. (1) p.w.r.to y

$$\frac{\partial z}{\partial y} = f'(x^2 - y^2)(-2y)$$

$$q = f'(x^2 - y^2)(-2y) \text{ ----------}(3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{p}{q} = \frac{f'(x^2-y^2)(2x)}{f'(x^2-y^2)(-2y)}$$

$$\frac{p}{q} = \frac{x}{-y} \Rightarrow -py = qx$$

$\therefore\ py + qx = 0$ which is required pde.

2. Form the pde by eliminating the arbitrary function from the relation $z = xy + f(x^2 + y^2)$.

Sol.

Given $z = xy + f(x^2+y^2)$ ---------------(1)

Diff (1) p.w.r.to x

$$\frac{\partial z}{\partial x} = y + f'(x^2+y^2)(2x)$$

$$\frac{p-y}{2x} = f'(x^2+y^2) \text{ ---------(2)}$$

Diff (1) p.w.r.to y

$$\frac{\partial z}{\partial y} = x + f'(x^2+y^2)(2y)$$

$$\frac{q-x}{2y} = f'(x^2+y^2) \text{ ---------(3)}$$

$$\frac{(2)}{(3)} \Rightarrow \frac{(p-y)/2x}{(q-x)/2y} = \frac{f'(x^2+y^2)}{f'(x^2+y^2)}$$

$$\frac{(p-y)2y}{(q-x)2x} = 1$$

$y(p-y) - x(q-x) = 0$ which is required pde.

3. Eliminate of arbitrary function f from $z = f\left(\frac{xy}{z}\right)$ and form a pde.

Sol.

Given $z = f\left(\frac{xy}{z}\right)$ ---------------(1)

$$\frac{\partial z}{\partial x} = f'\left(\frac{xy}{z}\right) y \left(\frac{z(1) - x\frac{\partial z}{\partial x}}{z^2}\right)$$

$$p = f'\left(\frac{xy}{z}\right)\left(\frac{yz - xyp}{z^2}\right) \text{ ----------(2)}$$

$$\frac{\partial z}{\partial y} = f'\left(\frac{xy}{z}\right)x\left(\frac{z(1) - y\frac{\partial z}{\partial y}}{z^2}\right)$$

$$q = f'\left(\frac{xy}{z}\right)\left(\frac{xz - yxq}{z^2}\right) \text{-----------}(3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{p}{q} = \frac{f'\left(\frac{xy}{z}\right)\left(\frac{yz - xyp}{z^2}\right)}{f'\left(\frac{xy}{z}\right)\left(\frac{xz - xyq}{z^2}\right)}$$

$$p(xz - xyq) = q(yz - xyp)$$

$$pxz - xypq = qyz - xypq$$

$px - qy = 0$ is required pde

4. Form the pde by eliminating f from $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

Sol.

Given $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$ -----------(1)

Diff (1) p.w.r.to. x

$$\frac{\partial z}{\partial x} = 2f'\left(\frac{1}{x} + \log y\right)\left(-\frac{1}{x^2}\right)$$

$$p = 2f'\left(\frac{1}{x} + \log y\right)\left(-\frac{1}{x^2}\right)$$

$$-px^2 = 2f'\left(\frac{1}{x} + \log y\right) \text{----------}(2)$$

$$\frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right)\left(\frac{1}{y}\right)$$

$$q = 2y + \left(\frac{2}{y}\right)f'\left(\frac{1}{x} + \log y\right)$$

$$qy - 2y^2 = 2f'\left(\frac{1}{x} + \log y\right) \text{-----------}(3)$$

From (2) & (3)

$$qy - 2y^2 = -px^2$$

$px^2 + qy = 2y^2$ is required pde.

5. Form the pde by eliminating f from $z = e^{ay} f(ax+by)$.

Sol.

Given $z = e^{ay} f(ax+by)$ ------------(1)

$$\frac{\partial z}{\partial x} = e^{ay} f'(ax+by)a$$

$$p = ae^{ay} f'(ax+by) \text{ -----------(2)}$$

$$\frac{\partial z}{\partial y} = e^{ay} f'(ax+by)b + ae^{ay} f(ax+by)$$

$$q = be^{ay} f'(ax+by) + ae^{ay} f(ax+by) \text{ ------(3)}$$

From (1) $f(ax+by) = \dfrac{z}{e^{ay}}$

From (2) $f'(ax+by) = \dfrac{p}{ae^{ay}}$

$$\therefore q = be^{ay} \frac{p}{ae^{ay}} + ae^{ay}\left(\frac{z}{e^{ay}}\right)$$

$q = \dfrac{bq}{a} + az$ is required pde.

6. Form the pde by eliminating f from $f(xy+z^2, x+y+z) = 0$

Sol.

Given $f(xy+z^2, x+y+z) = 0$ is of the form $f(u,v) = 0$ ------(1)

Let $u = z^2 - xy$, $v = x+y+z$

The elimination of f from (1) gives

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0 \quad \text{ie,} \quad \begin{vmatrix} y+2zp & 1+p \\ x+2zq & 1+q \end{vmatrix} = 0$$

$(1+q)(y+2zp) - (1+p)(x+2zq) = 0$

$(2z-x)p + (y-2z)q = x-y$ is required pde.

7. Form the pde by eliminating arbitrary function ϕ from $\phi(xy+z^2, x+y+z) = 0$.

Sol.

Given $\phi(xy+z^2, x+y+z) = 0$ is of the form

$\phi(u,v) = 0$ -----------(1)

Let $u = z^2 - xy$, $v = x+y+z$

The elimination of ϕ from (1) gives

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0 \quad \text{ie,} \quad \begin{vmatrix} 2zp - y & \frac{z - px}{z^2} \\ 2zq - x & -\frac{xq}{z^2} \end{vmatrix} = 0$$

$$(2zp - y)\left(\frac{-xq}{z^2}\right) - \left(\frac{z - px}{z^2}\right)(2zq - x) = 0$$

$px^2 - q(xy - 2z^2) = zx$ which is required pde.

8. Form the pde by eliminating arbitrary function ϕ from $\phi(x^2 + y^2 + z^2, xyz) = 0$.

Sol.

Given $\phi(x^2 + y^2 + z^2, xyz) = 0$ is of the form

$\phi(u,v) = 0$ −−−−−−−−−−−(1)

Let $u = x^2 + y^2 + z^2$, $v = xyz$

The elimination of ϕ from (1) gives

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0 \quad \text{ie,} \quad \begin{vmatrix} 2x + 2zp & y(xp + z) \\ 2y + 2zq & x(yq + z) \end{vmatrix} = 0$$

$$(2x + 2zp)x(yq + z) - (2y + 2zq)(xp + z) = 0$$

$$px(z^2 - y^2) + qy(x^2 - z^2) = z(y^2 - x^2)$$

TYPE – II (Elimination of two arbitrary function and get pde of 2nd order)

1. Form the pde by eliminating f and ϕ from $z = xf\left(\frac{y}{x}\right) + y\phi(x)$.

Sol.

Given $z = xf\left(\frac{y}{x}\right) + y\phi(x)$ −−−−−−−−−−(1)

Diff (1) w.r.to x

$$\frac{\partial z}{\partial x} = xf'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + f\left(\frac{y}{x}\right) + y\phi'(x)$$

$$p = f'\left(\frac{y}{x}\right)\left(-\frac{y}{x}\right) + f\left(\frac{y}{x}\right) + y\phi'(x) \text{ −−−−−−−−(2)}$$

Diff (1) w.r.to y

$$\frac{\partial z}{\partial y} = xf'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) + \phi(x)$$

$$q = f'\left(\frac{y}{x}\right) + \phi(x) \text{ --------(3)}$$

Again Diff (3) w.r.to x

$$s = \frac{\partial^2 z}{\partial x \partial y} = f''\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + \phi'(x) \text{ ------(4)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = f''\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) \text{ ----------(5)}$$

$(2) \times x + (3) \times y \Rightarrow$

$$px + qy = -yf'\left(\frac{y}{x}\right) + xf\left(\frac{y}{x}\right) + xy\phi'(x) + yf'\left(\frac{y}{x}\right) + y\phi(x)$$

$$= xy\phi'(x) + xf\left(\frac{y}{x}\right) + y\phi(x)$$

$$= xy\phi'(x) + z$$

$$use\,(5)\,in\,(4) \Rightarrow s = -\frac{y}{x}t + \phi'(x)$$

$$\phi'(x) = s + \frac{y}{x}t = \frac{sx + ty}{x}$$

$$\therefore px + qy = xy\left(\frac{sx + ty}{x}\right) + z$$

$$= xys + y^2 t + z$$

$$\therefore z = px + qy - xys - y^2 t$$