Solution of standard types of first order PDE

The general form of a first order PDE is $F(x, y, z, p, q) = 0$ where $p = \frac{\partial z}{\partial x}$ *and* $q = \frac{\partial z}{\partial y}$.

There are three types of solution of a PDE.

- A solution which contains as many arbitrary constants as there are independent variables is called a **complete solution or integrals.**

- A solution of a PDE which contains as many arbitrary functions as the order of the equation is called a **general solution**.

General solution

Let the PDE be $F(x, y, z, p, q) = 0$ ---------(1)

Let the complete solution of (1) be $\phi(x, y, z, a, b) = 0$ --------(2)

Where a and b are arbitrary constants.

Let $b = f(a)$, where f is an arbitrary function.

Then (2) becomes $\phi(x, y, z, a, f(a)) = 0$ --------(3)

Diff. (2) p.w.r.to a, we get

$$\frac{\partial \phi}{\partial a} + \frac{\partial \phi}{\partial b} f'(a) = 0 \text{ ---------(4)}$$

The elimination of 'a' between (3) and (4) if exists is called the general solution of (1).

Note: A solution obtained by giving particular values to the arbitrary constants in the complete solution or to the arbitrary function in the general solution is called a particular solution or particular integral of the PDE.

Singular solution

Let the PDE be $F(x, y, z, p, q) = 0$ ---------(1)

It is complete solution is $\phi(x, y, z, a, b) = 0$ --------(2)

Diff. (2) p.w.r.to a and b, we have

$\frac{\partial \phi}{\partial a} = 0$ ----(3) $\frac{\partial \phi}{\partial b} = 0$ ----(4)

The elimination of a and b from (2),(3) and (4), if exists, is the singular solution of (1).

Type-I $F(p, q) = 0$ (The equation contains p & q only)

Suppose that $z = ax + by + c$ is the a trial solution of $F(p, q) = 0$. Then

$p = \frac{\partial z}{\partial x} = a \quad q = \frac{\partial z}{\partial x} = b$, we get $F(a, b) = 0$.

Hence the complete Integral of a given equation is $z = ax + by + c$, where $F(a, b) = 0$.

Solving for b from $F(a, b) = 0$, we get $b = \phi(a)\ (say)$.

Then $z = ax + \phi(a) y + c$ -------(1) is the C.I of the given equation.

Since it contains two arbitrary constants singular integral is got by eliminating a and c from (1).

Diff. (1) p.w.r.to a we get $0 = x + \phi'(a) y$

Diff. (1) p.w.r.to c we get 0 = 1 (absurd)

There is no singular integral for the given PDE. To find the general integral, put c=f(a). f being arbitrary function. Then $z = ax + y\phi(a) + f(a)$ --------(2).

Diff (2) p.w.r.to a $0 = x + y\phi'(a) + f'(a)$ -------(3)

Eliminating a between (2) and (3) we get the general solution.

1. Solve $pq = k$

Sol.

Given $pq = k$. This is of the type $F(p,q)=0$. Let a solution of the given equation be $z = ax+by+c$ ––––––––(1).

From (1), we get p=a and q=b

Since (1) is the solution of the given equation, $ab = k$ $b = \frac{k}{a}$ ––––––––––(2)

Using (2) in (1), the required complete solution of the equation

$$z = ax + \frac{k}{a}y + c \text{ ––––––––––(3)}$$

Diff (3) p.w.r.to c, we get 0=1 (absurd), Hence there is no singular solution exists for the given equation.

To find the general solution, we put $c = f(a)$ in (3), where f is arbitrary function.

Ie, $$z = ax + \frac{k}{a}y + f(a) \text{ ––––––––––(4)}$$

Diff (4) p.w.r.to a, we get

$$0 = x - \frac{k}{a^2}y + f'(a) \text{ ––––––––––(5)}$$

Eliminating 'a' between (4) and (5), we get the general solution.

2. Solve $\sqrt{p}+\sqrt{q}=1$

Sol.

Given $\sqrt{p}+\sqrt{q}=1$. This is of the type $F(p,q)=0$. Let a solution of the given equation be $z = ax+by+c$ ––––––––(1).

From (1), we get $p = a \ \ and \ \ q = b$

Since (1) is the solution of the given equation

$$\sqrt{a}+\sqrt{b}=1 \Rightarrow \sqrt{b}=1-\sqrt{a}$$

$$b = \ 1-\sqrt{a}\ ^{2} \text{ –––––––––––(2)}$$

Using (2) in (1), the required complete solution of the equation

$$z = ax + \ 1-\sqrt{a}\ ^{2}\ y + c \ ---------(3)$$

Diff (3) p.w.r.to c, we get 0=1 (absurd), Hence there is no singular solution exists for the given equation.

To find the general solution, we put $c = f(a)$ in (3), where f is arbitrary function.

Ie, $$z = ax + \ 1-\sqrt{a}\ ^{2}\ y + f(a) ---------(3)$$

Diff (4) p.w.r.to a, we get

$$0 = x + 2(1-\sqrt{a})\left(-\frac{1}{2\sqrt{a}}\right)y + f'(a) \ ---------(5)$$

Eliminating 'a' between (4) and (5), we get the general solution.

3. Solve $pq + p + q = 0$

Sol.

Given $pq + p + q = 0$. This is of the type $F(p,q) = 0$. Let a solution of the given equation be $z = ax + by + c \ -------(1)$.

From (1), we get $p = a \ \ and \ \ q = b$

Since (1) is the solution of the given equation

$$ab + a + b = 0$$

$$b = -\frac{a}{a+1} \ ---------(2)$$

Using (2) in (1), the required complete solution of the equation

$$z = ax + \left(-\frac{a}{a+1}\right)y + c \ ---------(3)$$

Diff (3) p.w.r.to c, we get 0=1 (absurd), Hence there is no singular solution exists for the given equation.

To find the general solution, we put $c = f(a)$ in (3), where f is arbitrary function.

Ie, $$z = ax + \left(-\frac{a}{a+1}\right)y + f(a) \ ---------(3)$$

Diff (4) p.w.r.to a, we get

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$$0 = x - \frac{1}{a+1^{\,2}} y + f'(a) ----------(5)$$

Eliminating ‘a’ between (4) and (5), we get the general solution.

4. Solve $p^2 + q^2 = 4pq$.

Sol.

Given $p^2 + q^2 = 4pq$. This is of the type $F(p,q) = 0$**.** Let a solution of the given equation be $z = ax + by + c --------(1)$.

From (1), we get $p = a \ \ and \ \ q = b$

Since (1) is the solution of the given equation

$$a^2 + b^2 = 4ab \quad \Rightarrow a^2 + b^2 - 4ab = 0$$

Solving for b, we get

$$b = \frac{4a \pm \sqrt{16a^2 - 4a^2}}{2}$$

$$b = \frac{4a \pm \sqrt{12a^2}}{2}$$

$$= \ 2 \pm \sqrt{3} \ \ a$$

Using (2) in (1), the required complete solution of the equation

$$z = ax + \ 2 \pm \sqrt{3} \ ay + c ----------(3)$$

Diff (3) p.w.r.to c, we get 0=1 (absurd), Hence there is no singular solution exists for the given equation.

To find the general solution, we put $c = f(a)$in (3), where f is arbitrary function.

Ie, $$z = ax + \ 2 \pm \sqrt{3} \ \ ay + f(a) ----------(3)$$

Diff (4) p.w.r.to a, we get

$$0 = x + \ 2 \pm \sqrt{3} \ \ y + f'(a) ----------(3)$$

Eliminating ‘a’ between (4) and (5), we get the general solution.

Type – II $z = px + qy + f(p,q)$ (Clairaut's form)

Suppose that the given equation is of the form

$z = px + qy + f(p,q)$ ----------(1) we can prove that

$z = ax + by + f(a,b)$ ---------(2) is the complete solution of (1) where a, b are arbitrary constants.

Diff 2) p.w.r.to a and b we get

$$x + \frac{\partial f}{\partial a} = 0 \text{ ------(3)} \quad y + \frac{\partial f}{\partial b} = 0 \text{ ------(4)}$$

By eliminating a & b from (2),(3) & (4), we get the singular solution of (1)

Taking $b = \phi(a)$, (2) becomes

$$z = ax + \phi(a)y + f(a, \phi(a)) \text{ ---------(5)}$$

Diff (5) p.w.r.to a, we get

$$0 = x + \phi'(a)y + f'(a) \text{ ---------(6)}$$

Eliminating 'a' between (5) and (6), we get the general solution of (1).

1. Solve $z = px + qy + p^2q^2$

Sol.

Given $z = px + qy + p^2q^2$ ----------(1)

This is Clairaut's form. The complete Solution of (1) is

$$z = ax + by + a^2b^2 \text{ ----------(2)}$$

Diff (2) p.w.r.to a and b, we get

$$0 = x + 2ab^2 \quad \Rightarrow x = -2ab^2 \quad ie, -2ab = \frac{x}{b} \text{ ----(2)}$$

$$0 = y + 2ba^2 \quad \Rightarrow y = -2ba^2 \quad ie, -2ab = \frac{y}{a} \text{ -----(3)}$$

From (2) and (3), we have

$$\frac{x}{b} = \frac{y}{a} = -2ab = \frac{1}{k} \ (say)$$

$$ie,\ a = ky, \quad b = kx$$

Put in (2), we get

$$x = -2k^3yx^2 \ \Rightarrow k^3 = -\frac{1}{2xy}$$

Put a and b in (1)

$$z = kxy + kxy + k^4x^2y^2$$

$$= 2kxy + kx^2y^2k^3$$

$$= 2kxy + kx^2y^2\left(-\frac{1}{2xy}\right)$$

$$= 2kxy - \frac{k}{2}xy$$

$$= \frac{3}{2}kxy$$

$$z^3 = \frac{27}{8}k^3x^3y^3$$

$$= \frac{27}{8}x^3y^3\left(-\frac{1}{2xy}\right)$$

$$= -\frac{27}{16}x^2y^2$$

$16z^3 + 27x^2y^2 = 0$ is the singular solution

Put $b = \phi(a)$ in (1)

$$z = ax + \phi(a)y + a^2 \ \phi(a)^{\ 2} \ ----------(4)$$

Diff (4) w.r.to a, we get

$$z = ax + \left[\phi'(a) + a^2 \ \phi(a)\phi'(a)\ \right]y + 2a \ \phi(a)^{\ 2} ----------(5)$$

Eliminate 'a' between (4) and (5), we get the general solution of (1).

2. Solve $z = px + qy + p^2 - q^2$

Sol.

Given $z = px + qy + p^2 - q^2$ ----------(1)

This is Clairaut's form. The complete Solution of (1) is

$$z = ax + by + a^2 - b^2 \text{ ----------(2)}$$

Diff (2) p.w.r.to a and b, we get

$$0 = x + 2a \quad \Rightarrow a = -\frac{x}{2} \text{ --------(3)}$$

$$0 = y - 2b \quad \Rightarrow b = \frac{y}{2} \text{ --------(4)}$$

Sub in (1), we get

$$z = -\frac{x^2}{2} + \frac{y^2}{2} + \frac{x^2 - y^2}{4}$$

$4z = y^2 - x^2$ is the singular solution.

Put $b = \phi(a)$ in (1)

$$z = ax + \phi(a)y + a^2 - \phi(a)^{2} \text{ ----------(5)}$$

Diff (5) p.w.r.to a

$$z = x + \phi'(a)y + 2a - 2\phi(a)\phi'(a) \text{ ----------(6)}$$

Eliminating 'a' between (5) and (6), we get the general solution.

3. Solve $z = px + qy + \sqrt{1 + q^2 + p^2}$

Sol.

Given $z = px + qy + \sqrt{1 + q^2 + p^2}$ ---------(1)

This is Clairaut's form. The complete Solution of (1) is

$$z = ax + by + \sqrt{1 + b^2 + a^2} \text{ ---------(2)}$$

Where a and b are arbitrary constants.

Diff (2) p.w.r.to a, we get

$$0 = x + \frac{a}{\sqrt{1 + a^2 + b^2}} \qquad ie, \ x = -\frac{a}{\sqrt{1 + a^2 + b^2}} \text{ --------(3)}$$

Diff (2) p.w.r.to b, we get

$$0 = y + \frac{b}{\sqrt{1+a^2+b^2}} \qquad ie,\ y = -\frac{b}{\sqrt{1+a^2+b^2}} \text{ --------}(4)$$

$$(3)^2 + (4)^2 \Rightarrow x^2 + y^2 = \frac{a^2+b^2}{1+a^2+b^2}$$

$$1-\ x^2+y^2\ = 1 - \frac{a^2+b^2}{1+a^2+b^2}$$

$$1 - x^2 - y^2 = \frac{1+a^2+b^2-a^2-b^2}{1+a^2+b^2}$$

$$1 - x^2 - y^2 = \frac{1}{1+a^2+b^2}$$

$$\sqrt{1-x^2-y^2} = \frac{1}{\sqrt{1+a^2+b^2}} \text{ --------}(a)$$

$$\sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1-x^2-y^2}} \text{ --------}(b)$$

$$(3) \Rightarrow x = -a\sqrt{1-x^2-y^2}\ \ by\,(a)$$

$$ie,\ a = -\frac{x}{\sqrt{1-x^2-y^2}}$$

$$(4) \Rightarrow y = -b\sqrt{1-x^2-y^2}\ \ by\,(a)$$

$$ie,\ b = -\frac{y}{\sqrt{1-x^2-y^2}}$$

Sub in (2), we get

$$z = -\frac{x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}\ \ by\ (b)$$

$$= \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}} = \sqrt{1-x^2-y^2}$$

$$z^2 = 1 - x^2 - y^2$$

$x^2+y^2+z^2=1$ is the singular solution.

Put $b = \phi(a)$ in (2)

$$z = ax + \phi(a)y + \sqrt{1+a^2+\ \phi(a)\ ^2} \text{ --------}(5)$$

Diff (5) p.w.t.r.to a, we get

$$0 = x + \phi'(a)y + \frac{2a + 2\phi(a)\phi'(a)}{2\sqrt{1 + a2 + \phi(a)^2}} \text{-------}(6)$$

Eliminating 'a' between (5) and (6), we get the general solution.

3. Solve $z = px + qy + \left(\frac{q}{p} - p\right)$

Sol.

Given $z = px + qy + \left(\frac{q}{p} - p\right)$ ----------(1)

This is Clairaut's form. The complete Solution of (1) is

$$z = ax + by + \left(\frac{b}{a} - a\right) \text{----------}(2)$$

Diff (2) p.w.r.to a

$$0 = x - \frac{b}{a^2} - 1 \text{----------}(3)$$

Diff (2) p.w.r.to b

$$0 = y + \frac{1}{a} \Rightarrow a = -\frac{1}{y} \text{-------}(4)$$

Sub (4) in (3), we get

$$x - by^2 - 1 = 0$$

$$b = \frac{x-1}{y^2}$$

Sub a and b in (2), we get

$$z = -\frac{x}{y} + \frac{x-1}{y^2}y + \left(\frac{x-1}{y^2}\right) - y + \frac{1}{y}$$

$$= -\frac{x}{y} + \frac{x-1}{y} - \frac{x-1}{y} + \frac{1}{y}$$

$$z = \frac{1-x}{y} \Rightarrow yz = 1 - x$$

$x + yz = 1$ is the singular solution of (1).

Put $b=\phi(a)$ in (2)

$$z = ax + \phi(a)y + \left(\frac{\phi(a)}{a} - a\right) ---------(5)$$

Diff (5) p.w.t.r.to a, we get

$$0 = x + \phi'(a)y + \left(\frac{a\phi'(a) - \phi(a)}{a^2}\right) - 1 \quad ---------(6)$$

Eliminating 'a' between (5) and (6), we get the general solution.

4. Solve $(pq-p-q)(z-px-qy)=pq$

Sol.

Rewrite the given equation as $z-px-qy=\dfrac{pq}{pq-p-q}$

$$z = px + qy + \frac{pq}{pq-p-q} \quad ----------(1)$$

This is Clairaut's form. The complete Solution of (1) is

$$z = ax + by + \frac{ab}{ab-a-b} \quad --------(2)$$

Diff (2) p.w.r.to a

$$0 = x + \frac{(ab-a-b)b - ab(b-1)}{ab-a-b^{\ 2}}$$

$$= x + \frac{ab^2 - ab - b^2 - ab^2 - ab}{ab-a-b^{\ 2}}$$

$$= x - \frac{b^2}{ab-a-b^{\ 2}}$$

$$x = \frac{b^2}{ab-a-b^{\ 2}} \quad --------(2)$$

Diff (2) p.w.r.to b

$$0 = y + \frac{(ab-a-b)a - ab(a-1)}{ab-a-b^{\ 2}}$$

$$= y + \frac{a^2b - a^2 - ab - a^2b + ab}{(ab - a - b)^2}$$

$$= y - \frac{a^2}{(ab - a - b)^2}$$

$$y = \frac{a^2}{(ab - a - b)^2} \quad --------(3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{x}{y} = \frac{b^2}{a^2} \Rightarrow a^2x = b^2y$$

$$a\sqrt{x} = b\sqrt{y},\ ie,\ \frac{a}{\sqrt{y}} = \frac{b}{\sqrt{x}} = k(say)$$

$$a = k\sqrt{y} \quad and \quad b = k\sqrt{x}$$

Using these values in (3), we get

$$k^2x - (k^2\sqrt{xy} - k\sqrt{y} - k\sqrt{x})^2 x = 0$$

$$k(\sqrt{xy} - \sqrt{x} - \sqrt{y}) = 1$$

$$k = \frac{1 + \sqrt{x} + \sqrt{y}}{\sqrt{xy}}$$

Hence $a = \frac{1 + \sqrt{x} + \sqrt{y}}{\sqrt{x}}$ and $b = \frac{1 + \sqrt{x} + \sqrt{y}}{\sqrt{y}}$

Also $\frac{ab}{ab - a - b} = \frac{1}{1 - \frac{1}{b} - \frac{1}{a}} = \frac{1}{1 - \frac{\sqrt{y}}{1 + \sqrt{x} + \sqrt{y}} - \frac{\sqrt{x}}{1 + \sqrt{x} + \sqrt{y}}}$

$$= 1 + \sqrt{x} + \sqrt{y}$$

Using these values in (2), the singular solution of (1) is

$$z = \sqrt{x}(1 + \sqrt{x} + \sqrt{y}) + \sqrt{y}(1 + \sqrt{x} + \sqrt{y}) + (1 + \sqrt{x} + \sqrt{y})$$

$$ie, \quad z = (1 + \sqrt{x} + \sqrt{y})^2$$

Type-III

Equation of the type $F(z,p,q)=0$ ie, Equation not containing y and z. The solution will be of the form $z=\phi(x+ay)$, where 'a' is an arbitrary constant and ϕ is a specific function to be found out.

Putting $u=x+ay$, (2) becomes $z=\phi(u)\ or\ z(u)$

$$p=\frac{dz}{du}\frac{\partial u}{\partial x}=\frac{dz}{du}$$ and

$$q=\frac{dz}{du}\frac{\partial u}{\partial y}=a\frac{dz}{du}$$

Sub p,q in $F(z,p,q)=0$ ie $F\left(z,\frac{dz}{du},a\frac{dz}{dy}\right)=0$

Which is the ODE of the first order, solving $\frac{dz}{du}$, we obtain $\frac{dz}{du}=\phi(z,a)\ (say)$

$$\frac{dz}{\phi(z,a)}=du\ --------(1)$$

Integrate on both sides , we get $f(z,a)=u+c=x+ay+c$. This is complete solution. The general and singular solutions of given equation found out by the usual methods.

1. Solve the equation $z^2\ \ p^2+q^2+1\ \ =c^2$, where c is constant.

Sol.

Given $z^2\ \ p^2+q^2+1\ \ =c^2\ \ --------(1)$

This is of Type-III. Therefore (1) has a solution of the form

$z=z(u)=z(x+ay)\ --------(2)$

From (2), we have $p=\frac{dz}{du}\ \ and\ \ p=a\frac{dz}{du}$

Since (2) is a solution of (1), we get

$$z^2\left[\left(\frac{dz}{du}\right)^2+a^2\left(\frac{dz}{du}\right)^2+1\right]=c^2$$

$$ie,\qquad (1+a^2)\left(\frac{dz}{du}\right)^2=\frac{c^2}{z^2}-1$$

ie, $\sqrt{(1+a^2)}\left(\frac{dz}{du}\right)=\frac{\sqrt{c^2-z^2}}{z}$

ie, $\sqrt{(1+a^2)}\frac{zdz}{\sqrt{c^2-z^2}}=du \quad --------(3)$

Integrating (3), the complete solution of (1) is

$$-\frac{1}{2}\sqrt{1+a^2}\int\frac{-2zdz}{\sqrt{c^2-z^2}}=u+b$$

$$-\frac{1}{2}\sqrt{1+a^2}\sqrt{c^2-z^2}=x+ay+b$$

$(1+a^2)(c^2-z^2)=(x+ay+)^2$ is the complete solution of (1).

2. Solve $p(1-q^2)=q(1-z)$

Sol.

Given $p(1-q^2)=q(1-z) \quad --------(1)$

This is of Type-III. Therefore (1) has a solution of the form

$z=z(u)=z(x+ay) \quad --------(2)$

From (2), we have $p=\frac{dz}{du}$ *and* $p=a\frac{dz}{du}$

Since (2) is a solution of (1), we get

$$\frac{dz}{du}\left[1-a^2\left(\frac{dz}{du}\right)^2\right]=a\frac{dz}{du}(1-z)$$

$$\frac{dz}{du}\left[1-a^2\left(\frac{dz}{du}\right)^2-a+az\right]=0$$

$$a^2\left(\frac{dz}{du}\right)^2=az+1-a$$

$$a\left(\frac{dz}{du}\right)=\sqrt{az+1-a} \quad ----------(3)$$

Solving (3), we get $a\int\frac{dz}{\sqrt{az+1-a}}=u+b$

$$2\sqrt{az+1-a} = x+ay+b$$

$$4(az+1-a) = (x+ay+b)^2$$

Is the complete solution of (1).

Type –IV Separable equation

First order PDE's are separable. It can be written as $f(x,p) = \phi(y,q)$

Put $f(x,p) = \phi(y,q)$=a (say)

Solving for p and q, we get $p = f_1(x,a) \quad and \quad q = \phi_1(y,b)$

But $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$

Hence $dz = pdx + qdy$

$$= f_1(x,a)\,dx + \phi_1(y,b)dy$$

$$z = \int f_1(x,a)\,dx + \int \phi_1(y,b)dy + b$$

This equation contains tow arbitrary constants and hence it is the complete solution.

1. Solve $p^2 y(1+x^2) = qx^2$

   Sol.

   Given equation can be reduced to the type $f(x,p) = \phi(y,q)$

$$\frac{p^2(1+x^2)}{x^2} = \frac{q}{y} = a\,(say)$$

$$\frac{p^2(1+x^2)}{x^2} = a\,, \qquad \frac{q}{y} = a$$

$$p = \frac{x\sqrt{a}}{\sqrt{1+x^2}}, \qquad q = ay$$

$$WKT,\; dz = pdx + qdy$$

$$\int dz = \int \frac{x\sqrt{a}}{\sqrt{1+x^2}}\,dx + \int ay\,dy$$

$$z = \sqrt{a(1+x^2)} + \frac{1}{2}ay^2 + b$$

Is the complete solution of the given equation.

2. Solve $q = px + p^2$

Sol.

Given $q = px + p^2$, This is of the type $f(x,p) = \phi(y,q)$.

Let $q = px + p^2 = \frac{k^2}{4}$ $(Cons\tan t)$

$$q = \frac{k^2}{4} ------(1)$$

$$px + p^2 = \frac{k^2}{4}, \qquad px + p^2 - \frac{k^2}{4} = 0 --------(2)$$

$$p = \frac{-x \pm \sqrt{x^2 + \frac{4k^2}{4}}}{2} = \frac{-x \pm \sqrt{x^2 + k^2}}{2}$$

$$WKT,\ dz = pdx + qdy$$

$$dz = \frac{-x \pm \sqrt{x^2 + k^2}}{2}\,dx + \frac{k^2}{4}\,dy$$

$$\int dz = \int \frac{-x \pm \sqrt{x^2 + k^2}}{2}\,dx + \int \frac{k^2}{4}\,dy$$

$$z = -\frac{x^2}{4} \pm \frac{1}{2}\left[\frac{k^2}{2}\sinh^{-1}\left(\frac{x}{k}\right) + \frac{x\sqrt{x^2 + k^2}}{2}\right] + \frac{k^2 y}{4} + b$$

Which is the complete solution.

3. Solve $p^2 + q^2 = x^2 + y^2$

Sol.

Given $p^2+q^2=x^2+y^2$. This is of the type $f(x,p)=\phi(y,q)$.

Let $p^2-x^2=y^2-q^2=a^2$

$p^2-x^2=a^2, \qquad y^2-q^2=a^2$

$p=\sqrt{a^2+x^2}$ --------(1) $\quad q=\sqrt{y^2-a^2}$ --------(2)

$WKT,\ dz=pdx+qdy$

$dz=\sqrt{a^2+x^2}dx+\sqrt{y^2-a^2}dy$

$\int dz=\int\sqrt{a^2+x^2}dx+\int\sqrt{y^2-a^2}dy$

$$z=\frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{x^2+a^2}}{2}+\frac{y\sqrt{y^2-a^2}}{2}-\frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right)+b$$

Which is complete integral.

EQUATIONS REDUCIBLE TO STANDARD TYPES – TRANSFORMATION

Type – A

Equation of the form $f(x^m p, y^n q)=0$ --------(1) or $f(x^m p, y^n q, z)=0$ --------(2)

Case (i) If $m\neq 1$ *and* $n\neq 1$ then put $X=x^{1-m},\ Y=y^{1-n}$

$X=x^{1-m} \qquad Y=y^{1-n}$

$\frac{\partial X}{\partial x}=(1-m)x^{-m} \qquad \frac{\partial Y}{\partial y}=(1-n)y^{-n}$

$P=\frac{\partial z}{\partial X} \qquad Q=\frac{\partial z}{\partial Y}$

$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}\frac{\partial X}{\partial x} \qquad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y}$

$p=P(1-m)x^{-m} \qquad q=Q(1-n)y^{-n}$

$x^m p=P(1-m) \qquad y^n q=Q(1-n)$

Sub in (1) we get $F(P,Q)=0 \quad (Type-I)$

Sub in (2) we get $F(z,P,Q)=0 \quad (Type-III)$

Case (ii) **If** $m=n=1$ *then put* $X=\log x,\ Y=\log y$

$X=\log x$ $\qquad\qquad$ $Y=\log y$

$\frac{\partial X}{\partial x}=\frac{1}{x}$ $\qquad\qquad$ $\frac{\partial Y}{\partial y}=\frac{1}{y}$

$P=\frac{\partial z}{\partial X}$ $\qquad\qquad$ $Q=\frac{\partial z}{\partial Y}$

$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}\frac{\partial X}{\partial x}$ $\qquad\qquad$ $\frac{\partial z}{\partial y}=\frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y}$

$p=P\frac{1}{x}$ $\qquad\qquad$ $q=Q\frac{1}{y}$

$xp=P$ $\qquad\qquad$ $yq=Q$

Sub in (1) we get $F(P,Q)=0 \quad (Type-I)$

Sub in (2) we get $F(z,P,Q)=0 \quad (Type-III)$

1.Solve $x^2p+y^2q=0$

Sol.

Given $x^2p+y^2q=0$ ----------(1)

This is of the form $f(x^m p, y^n q)=0$ $case-(i)$

Here $m=2\neq 1$ *and* $n=2\neq 1$

$X=x^{1-m}$ $\qquad\qquad$ $Y=y^{1-n}$

$put\ X=x^{1-2}$ $\qquad\qquad$ $Y=y^{1-2}$

$X=x^{-1}$ $\qquad\qquad$ $Y=y^{-1}$

$\frac{\partial X}{\partial x} = -x^{-2}$ $\frac{\partial Y}{\partial y} = -y^{-2}$

$P = \frac{\partial z}{\partial X}$ $Q = \frac{\partial z}{\partial Y}$

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x}$ $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y}$

$p = -Px^{-2}$ $q = -Qy^{-2}$

$x^2 p = -P$ $y^2 q = -Q$

Sub in (1) we get $F(-P,-Q) = 0 \quad (Type - I)$

$ie, \quad P + Q = 0 \quad ----------(1)$

Hence the C.I is $z = ax + by + c \quad ---------(2)$

$$P = \frac{\partial z}{\partial X} = a, \quad Q = \frac{\partial z}{\partial Y} = b$$

$(1) \Rightarrow a + b = 0, \quad b = -a$

$(2) \Rightarrow z = aX - aY + c$

$$= a\left(\frac{1}{x}\right) - a\left(\frac{1}{y}\right) + c \qquad \because X = \frac{1}{x}, Y = \frac{1}{y}$$

$$z = \frac{a}{x} - \frac{a}{y} + c \quad -------(4)$$

Diff (4) p.w.r.to c, we get 0=1(absurd), Hence no singular solution exists.

Let $c = f(a)$ where f is arbitrary function

$$(4) \Rightarrow z = \frac{a}{x} - \frac{a}{y} + f(a) \quad --------(5)$$

Diff (5) p.w.r.to a, we get

$$0 = \frac{1}{x} - \frac{1}{y} + f'(a) \quad --------(6)$$

Eliminating ‘a’ between (5) and (6), we get the general solution.

3. Solve $xp + yq = 0$

Sol.

Given $xp + yq = 0$ ----------(1)

This is of the form $f(x^m p, y^n q) = 0$ $case-(ii)$

$Here \ m=1, n=1$

$X = \log x$ $Y = \log y$

$P = \frac{\partial z}{\partial X}$ $Q = \frac{\partial z}{\partial Y}$

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x}$ $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y}$

$p = P\frac{1}{x}$ $q = Q\frac{1}{y}$

$xp = P$ $yq = Q$

Sub in (1) we get $P + Q = 0$ --------(2), $F(P,Q) = 0$ $(Type-I)$

Hence the C.I is $z = aX + bY + c$ ----------(2)

$P = \frac{\partial z}{\partial X} = a, \quad Q = \frac{\partial z}{\partial Y} = b$

$(1) \Rightarrow a + b = 0, \ \ ie, b = -a$

$(2) \Rightarrow z = aX - aY + c$ ----------(3)

$z = a\log x - a\log y + c$ ----------(4)

Diff (4) p.w.r.to c, we get 0=1(absurd), Hence no singular solution exists.

Let $c = f(a)$ where f is arbitrary function

$z = a\log x - a\log y + f(a)$ ----------(5)

Diff (5) p.w.r.to a, we get

$0 = \log x - \log y + f'(a)$ ----------(6)

Eliminating ‘a’ between (5) and (6), we get the general solution.

4. Solve $q^2 y^2 = z(z - px)$

Sol.

Given $zpx + q^2 y^2 = z^2$

$$z(xp)+(yq)^2=z^2 \quad ----------(1)$$

This is of Type-A case (ii)

$Here\ \ m=1,\ n=1$

$X=\log x$ $\qquad$ $Y=\log y$

$P=\dfrac{\partial z}{\partial X}$ $\qquad$ $Q=\dfrac{\partial z}{\partial Y}$

$\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial X}\dfrac{\partial X}{\partial x}$ $\qquad$ $\dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial Y}\dfrac{\partial Y}{\partial y}$

$p=P\dfrac{1}{x}$ $\qquad$ $q=Q\dfrac{1}{y}$

$xp=P$ $\qquad$ $yq=Q$

Sub in (1), we get $zP+Q^2=z^2 \quad ----------(2)$

Thisi is $F(z,P,Q)=0 \quad (Type-III)$

Let $z=z(u)=z(X+aY)$ $\dfrac{\partial u}{\partial X}=1,\ \dfrac{\partial u}{\partial Y}=a$

$P=\dfrac{dz}{du}\dfrac{\partial u}{\partial X}=\dfrac{dz}{du}\Rightarrow P=\dfrac{dz}{du}$ and

$Q=\dfrac{dz}{du}\dfrac{\partial u}{\partial Y}=a\dfrac{dz}{du}\Rightarrow Q=a\dfrac{dz}{du}$

Sub P,Q in (2), we get

$$a^2\left(\frac{dz}{du}\right)^2+z\frac{dz}{du}-z^2=0$$

$$\frac{dz}{du}=\frac{-z\pm\sqrt{z^2+4z^2a^2}}{2a^2}=z\left[\frac{-1\pm\sqrt{1+4a^2}}{2a^2}\right]$$

$$\int\frac{dz}{z}=\frac{-1\pm\sqrt{1+4a^2}}{2a^2}\int du$$

$$\log z = \frac{-1 \pm \sqrt{1+4a^2}}{2a^2} u + b$$

$$= \frac{-1 \pm \sqrt{1+4a^2}}{2a^2} (X + aY) + b$$

$$\log z = \frac{-1 \pm \sqrt{1+4a^2}}{2a^2} (\log x + a \log y) + b$$

Which is the complete solution.

5. solve $p^2 x + q^2 y = z$

Sol.

Given equation is rewritten as $\left(x^{\frac{1}{2}} p\right)^2 + \left(y^{\frac{1}{2}} q\right)^2 = z$ ---------- (1)

This is of the form $f(x^m p, y^n q, z) = 0$

Here $m = \frac{1}{2} \neq 1$ *and* $n = \frac{1}{2} \neq 1$

$X = x^{1-m}$ $\quad Y = y^{1-n}$

put $X = x^{1-\frac{1}{2}}$ $\quad Y = y^{1-\frac{1}{2}}$

$X = x^{\frac{1}{2}}$ $\quad Y = y^{\frac{1}{2}}$

$\frac{\partial X}{\partial x} = \frac{1}{2} x^{-\frac{1}{2}}$ $\quad \frac{\partial Y}{\partial y} = \frac{1}{2} y^{-\frac{1}{2}}$

$P = \frac{\partial z}{\partial X}$ $\quad Q = \frac{\partial z}{\partial Y}$

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \frac{\partial X}{\partial x}$ $\quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \frac{\partial Y}{\partial y}$

$p = \frac{1}{2} P x^{-\frac{1}{2}}$ $\quad q = \frac{1}{2} Q y^{-\frac{1}{2}}$

$x^{\frac{1}{2}} p = \frac{P}{2}$ $\quad y^{\frac{1}{2}} q = \frac{Q}{2}$

Sub in (1), we get $\left(\frac{P}{2}\right)^2+\left(\frac{Q}{2}\right)^2=z$

$P^2+Q^2=4z$ ----------(3) $(Type-III)$

Let $z=z(u)=z(X+aY)$ $\frac{\partial u}{\partial X}=1, \frac{\partial u}{\partial Y}=a$

$P=\frac{dz}{du}\frac{\partial u}{\partial X}=\frac{dz}{du}\Rightarrow P=\frac{dz}{du}$ and

$Q=\frac{dz}{du}\frac{\partial u}{\partial Y}=a\frac{dz}{du}\Rightarrow Q=a\frac{dz}{du}$

Sub P,Q in (3), we get

$$\left(\frac{dz}{du}\right)^2+\left(a\frac{dz}{du}\right)^2=4z$$

$$\left(\frac{dz}{du}\right)^2(1-a^2)+=4z$$

$$\frac{dz}{du}=\frac{2\sqrt{z}}{\sqrt{1+a^2}}$$

$$\int\frac{dz}{\sqrt{z}}=2\int\frac{du}{\sqrt{1+a^2}}$$

$$2\sqrt{z}=2\frac{u}{\sqrt{1+a^2}}+b$$

$$\sqrt{z}=\frac{(X+aY)}{\sqrt{1+a^2}}+b$$

$$\sqrt{z}=\frac{\sqrt{x}+a\sqrt{y}}{\sqrt{1+a^2}}+b$$

6.Solve $p^2+x^2y^2q^2=z^2$

Sol.

Given $p^2 + x^2 y^2 q^2 = z^2$

$$\div \; by \; x^2 \Rightarrow \frac{p^2}{x^2} + y^2 q^2 = z^2$$

$$\left(x^{-1}p\right)^2 + \left(yq\right)^2 = z^2 \; -------(1)$$

Here $m = -1 \neq 1 \; and \; n = 1$

$X = x^{1-m}$ $\qquad$ $Y = \log y$

$put \; X = x^{1+1}$ $\qquad$ $Y = y^{1-2}$

$X = x^2$

$\frac{\partial X}{\partial x} = 2x$ $\qquad$ $\frac{\partial Y}{\partial y} = \frac{1}{y}$

$P = \frac{\partial z}{\partial X}$ $\qquad$ $Q = \frac{\partial z}{\partial Y}$

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x}$ $\qquad$ $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y}$

$p = P(2x)$ $\qquad$ $q = Q\frac{1}{y}$

$x^{-1}p = 2P$ $\qquad$ $yq = Q$

$$(1) \Rightarrow \; \left(2P\right)^2 + Q^2 = z^2 \; ----------(2)$$

This is of the form $F(P,Q,z) = 0 \quad (Type - III)$

Let $z = z(u) = z(X + aY)$ $\; \frac{\partial u}{\partial X} = 1, \frac{\partial u}{\partial Y} = a$

$P = \frac{dz}{du}\frac{\partial u}{\partial X} = \frac{dz}{du} \Rightarrow P = \frac{dz}{du}$ and

$Q = \frac{dz}{du}\frac{\partial u}{\partial Y} = a\frac{dz}{du} \Rightarrow Q = a\frac{dz}{du}$

Sub P,Q in (2), we get

$$\left(2\frac{dz}{du}\right)^2+\left(a\frac{dz}{du}\right)^2=z^2$$

$$\left(\frac{dz}{du}\right)^2=\frac{z^2}{4+a^2}$$

$$\left(\frac{dz}{du}\right)=\frac{z}{\sqrt{4+a^2}}$$

$$\int\frac{dz}{z}=\int\frac{du}{\sqrt{a^2+4}}$$

$$\log z=\frac{u}{\sqrt{a^2+4}}+b$$

$$=\frac{X+aY}{\sqrt{a^2+4}}+b$$

$$\log z=\frac{x^2+a\log y}{\sqrt{a^2+4}}+b$$ is the complete solution of (1).

Type-VI Equation of the type $f(z^m p, z^m q)=0$ ----------(1)

Or $f_1(x, z^m p)=f_2(y, z^m q)=0$ -----(2)

Case (i) *If* $m\neq -1$ *then*

Put $Z=z^{m+1}$

$$\frac{\partial Z}{\partial x}=\frac{\partial Z}{\partial z}\frac{\partial z}{\partial x} \qquad \frac{\partial Z}{\partial y}=\frac{\partial Z}{\partial z}\frac{\partial z}{\partial y}$$

$$P=(m+1)z^m p \qquad Q=(m+1)z^m q$$

$$\frac{P}{(m+1)}=z^m p \qquad \frac{Q}{(m+1)}=z^m q$$

Sub in (1), we get $F(P,Q)=0,\ Type-I$

Sub in (2), we get $F_1(x,P)=F_2(y,Q),\ Type-IV$

Case(ii) *If* $m=-1$ *then put* $Z=\log z$

$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial x}$ $\frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial y}$

$P = \frac{1}{z}p$ $Q = \frac{1}{z}q$

$zP = p$ $zQ = q$

Sub in (1), we get $F(P,Q) = 0, \; Type - I$

Sub in (2), we get $F_1(x,P) = F_2(y,Q), \; Type - IV$

1. Solve $z^2(p^2 + q^2) = x + y$

Sol.

Given $z^2(p^2 + q^2) = x + y$

$$(zp)^2 + (zq)^2 = x + y \text{ ----------- } (1)$$

This is of Type – B

Here $m \neq -1, \therefore Z = z^{m+1}, \quad Z = z^2$

$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial x}$ $\frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial y}$

$P = 2zp$ $Q = 2zq$

$zp = \frac{P}{2}$ $zq = \frac{Q}{2}$

$$(1) \Rightarrow \left(\frac{P}{2}\right)^2 + \left(\frac{Q}{2}\right)^2 = x + y$$

$$P^2 + Q^2 = 4(x + y)$$

$$P^2 - 4x = 4y - Q^2$$

This equation of the form $F_1(x,P) = F_2(y,Q), \; Type - IV$

$$P^2 - 4x = 4y - Q^2 = 4a \; say$$

$P^2 - 4x = 4a,$ $4y - Q^2 = 4a$

$P^2 = 4a + 4x$ $Q^2 = 4y - 4a$

$P = 2\sqrt{a+x}$ $\qquad Q = 2\sqrt{y-a}$

$dZ = Pdx + Qdy$

$dZ = 2\sqrt{a+x}dx + 2\sqrt{y-a}dy$

$\int dZ = 2\int \sqrt{a+x}dx + 2\int \sqrt{y-a}dy$

$$Z = 2\frac{(a+x)^{3/2}}{3/2} + 2\frac{(y-a)^{3/2}}{3/2} + b$$

$$z^2 = \frac{4}{3}(a+x)^{3/2} + \frac{4}{3}(y-a)^{3/2} + b \qquad \because Z = z^2$$

Which is complete solution.

2. Solve $p^2 + q^2 = z^2(x^2+y^2)$

Sol.

Given $p^2 + q^2 = z^2(x^2+y^2)$

$$\frac{p^2}{z^2} + \frac{q^2}{z^2} = (x^2+y^2)$$

$$\left(\frac{p}{z}\right)^2 + \left(\frac{q}{z}\right)^2 = (x^2+y^2) \text{ ---------- (1)}$$

This is of Type – B

Here $m = -1$ *put* $Z = \log z$

$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial x}$ $\qquad \frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial y}$

$P = \frac{1}{z}p$ $\qquad Q = \frac{1}{z}q$

$z^{-1}p = P$ $\qquad z^{-1}q = Q$

$(1) \Rightarrow P^2 + Q^2 = x^2 + y^2$

$P^2 - x^2 = y^2 - Q^2 = a^2$ *say*

$P^2 - x^2 = a^2, \qquad y^2 - Q^2 = a^2$

$P^2 = a^2 + x^2, \qquad Q^2 = y^2 - a^2$

$P = \sqrt{a^2 + x^2}, \qquad Q = \sqrt{y^2 - a^2}$

$$dZ = Pdx + Qdy$$

$$\int dZ = \int \sqrt{a^2 + x^2}\,dx + \int \sqrt{y^2 - a^2}\,dy$$

$$Z = \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 + x^2} + \frac{y}{2}\sqrt{y^2 - a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right) + b$$

$$\log z = \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 + x^2} + \frac{y}{2}\sqrt{y^2 - a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right) + b$$

Is the complete solution.