2. Solve $(D^2 - D'^2)z = e^{x-y}\sin(2x+3y)$

Sol.

$$m^2 - 1 = 0,\ ie,\ (m+1)(m-1) = 0$$
$$m = -1, 1$$
$$C.F = \phi_1(y-x) + \phi_2(y+x)$$

$$P.I = \frac{1}{D^2 - D'^2} e^{x-y}\sin(2x+3y)$$

$$= e^{x-y}\frac{1}{(D+1)^2 - (D'-1)^2}\sin(2x+3y) \qquad \text{Re } place\ D = D+1\ \&\ D' = D'-1$$

$$= e^{x-y}\frac{1}{(D+1)+(D'-1)\ \ (D+1)-(D'-1)}\sin(2x+3y)$$

$$= e^{x-y}\frac{1}{D+1+D'-1\ \ D+1-D'+1}\sin(2x+3y)$$

$$= e^{x-y}\frac{1}{D+D'\ \ (D-D')+2}\sin(2x+3y)$$

$$= e^{x-y}\frac{1}{(D^2 - D'^2) + 2(D+D')}\sin(2x+3y)$$

$$= e^{x-y}\frac{1}{-4-(-9)+2(D+D')}\sin(2x+3y) \quad \text{Re } place\ D^2 = -4,\ D'^2 = -9\ \&\ DD' = -6$$

$$= e^{x-y}\frac{1}{2(D+D')+5}\sin(2x+3y)$$

$$= e^{x-y}\frac{2(D+D')-5}{2(D+D')+5\ \ 2(D+D')-5}\sin(2x+3y)$$

$$= e^{x-y}\frac{2(D+D')-5}{4(D+D')^2-25}\sin(2x+3y)$$

$$= e^{x-y}\frac{2D(\sin(2x+3y)) + 2D'(\sin(2x+3y))\ \ -5\sin(2x+3y)}{4\ \ -4+2(-6)-9\ \ -25}$$

$$= e^{x-y}\frac{4\cos(2x+3y) + 6\cos(2x+3y)\ \ -5\sin(2x+3y)}{-125}$$

$$= e^{x-y}\left[\frac{5\sin(2x+3y)}{125} - \frac{10\cos(2x+3y)}{125}\right]$$

$$= \frac{e^{x-y}}{25}\ 5\sin(2x+3y) - 10\cos(2x+3y)$$

The complete solution is $z = C.F + P.I$

$$z = \phi_1(y+x) + \phi_2(y-x) + \frac{e^{x-y}}{25}\ 5\sin(2x+3y) - 10\cos(2x+3y)$$

Problems Based on above Four Types

1. Solve $(D^2 - DD' - 2D'^2)z = 2x + 3y + e^{3x+4y}$

   Sol.

   The A.E is $m^2 - m - 2 = 0$

$$(m-2)(m+1) = 0$$

$$m = 2, -1$$

$$C.F = \phi_1(y+2x) + \phi_2(y-x)$$

$$P.I_1 = \frac{1}{D^2 - DD' - 2D'^2}(2x+3y)$$

$$= \frac{1}{D^2\left[1 - \left(\frac{DD' + 2D'^2}{D^2}\right)\right]}(2x+3y)$$

$$= \frac{1}{D^2}\left[1 - \left(\frac{DD' + 2D'^2}{D^2}\right)\right]^{-1}(2x+3y)$$

$$= \frac{1}{D^2}\left[1 + \frac{DD' + 2D'^2}{D^2} + \cdots\right](2x+3y)$$

$$= \left[\frac{1}{D^2} + \frac{DD' + 2D'^2}{D^4}\right](2x+3y)$$

$$= \left[\frac{1}{D^2} + \frac{D'}{D^3} + \frac{2D'^2}{D^4}\right](2x+3y)$$

$$= \frac{1}{D^2}(2x+3y) + \frac{D'}{D^3}(2x+3y) + \frac{2D'^2}{D^4}(2x+3y)$$

$$= \frac{x^3}{3} + \frac{3x^2y}{2} + \frac{3}{D^3}$$

$$= \frac{x^3}{3} + \frac{3x^2y}{2} + \frac{1}{2}x^3$$

$$P.I_1 = \frac{5x^3}{6} + \frac{3x^2y}{2}$$

$$P.I_2 = \frac{1}{D^2 - DD' - 2D'^2}e^{3x+4y}$$

$$= \frac{1}{3^2 - 3\times 4 - 2(4)^2}e^{3x+4y}$$

$$=\frac{1}{-35}e^{3x+4y}$$

The complete solution is $z = C.F + P.I_1 + P.I_2$

$$z=\phi_1(y+2x)+\phi_2(y-x)+\frac{5x^3}{6}+\frac{3x^2y}{2}-\frac{e^{3x+4y}}{35}$$

2. Solve $(D^3-7DD'^2-6D'^3)z=\sin(x+2y)+e^{3x+y}$

Sol.

The A.E. is $m^3-7m-6=0$

$$(m+1)(m^2-m-6)=0$$

$$(m+1)(m-3)(m+2)=0$$

$$m=-1,3,-2$$

$$C.F=\phi_1(y-x)+\phi_2(y-2x)+\phi_3(y+3x)$$

$$P.I_1=\frac{1}{D^3-7DD'^2-6D'^3}\sin(x+2y)$$

$$=\frac{1}{D^2D-7DD'D'-6D'^2D'}\sin(x+2y)$$ Re place $D^2=-1,\ D'^2=-4\ \&\ DD'=-2$

$$=\frac{1}{-D+14D'-24D'}\sin(x+2y)$$

$$=\frac{1}{-D+38D'}\sin(x+2y)$$

$$=\frac{-D-38D'}{-D+38D'\quad -D-38D'}\sin(x+2y)$$

$$=\frac{-D-38D'}{(-D)^2-(38D')^2}\sin(x+2y)$$

$$=\frac{-D-38D'}{D^2-1444D'^2}\sin(x+2y)$$

$$=\frac{-D-38D'}{-1-1444(-4)}\sin(x+2y)$$

$$=\frac{-D-38D'}{5775}\sin(x+2y)$$

$$=-\frac{D\sin(x+2y)\ +38D'\sin(x+2y)}{5775}$$

$$=-\frac{77\cos(x+2y)}{5775}$$

$$=-\frac{1}{75}\cos(x+2y)$$

$$P.I_2 = \frac{1}{D^3 - 7DD'^2 - 6D'^3} e^{3x+y}$$

$$= \frac{1}{27-21-6} e^{3x+y} \qquad \text{Re } place \ D = a = 3 \ \& \ D' = b = 1$$

$$= \frac{1}{0} e^{3x+y}$$

$$= \frac{x}{3D^2 - 7D'^2} e^{3x+y}$$

$$= \frac{x}{3(3)^2 - 7(1)^2} e^{3x+y}$$

$$= \frac{x}{20} e^{3x+y}$$

The complete solution is $z = C.F + P.I_1 + P.I_2$

$$z = \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x) - \frac{1}{75}\cos(x+2y) + \frac{x}{20} e^{3x+y}$$

3. Solve $(D^2 + 2DD' + D'^2)z = x^2 y + e^{x-y}$

Sol.

The A.E is $m^2 + 2m + 1 = 0$

$$(m+1)(m+1) = 0$$

$$m = -1, -1$$

$$C.F = \phi_1(y-x) + x\phi_2(y-x)$$

$$P.I_1 = \frac{1}{D^2 + 2DD' + D'^2} x^2 y$$

$$= \frac{1}{D^2\left[1 + \left(\frac{2DD' + D'^2}{D^2}\right)\right]} x^2 y$$

$$= \frac{1}{D^2}\left[1 + \left(\frac{2DD' + D'^2}{D^2}\right)\right]^{-1} (x^2 y)$$

$$= \left[\frac{1}{D^2} - \frac{2DD'}{D^4} - \frac{D'^2}{D^4}\right](x^2 y)$$

$$= \left[\frac{1}{D^2}(x^2 y) - \frac{2DD'}{D^4}(x^2 y) - \frac{D'^2}{D^4}(x^2 y)\right]$$

$$= \left[\frac{x^4 y}{12} - \frac{2x^2}{D^3} - 0\right]$$

$$= \left[\frac{x^4 y}{12} - \frac{2x^5}{60}\right]$$

$$P.I_2 = \frac{1}{D^2 + 2DD' + D'^2} e^{x-y}$$

$$= \frac{1}{1+2+(-1)^2} e^{x-y} \qquad \text{Replace } D \text{ by } 1 \text{ \& } D' \text{ by } -1$$

$$= \frac{x}{2D+2D'} e^{x-y} \qquad \because (ORF)$$

$$= \frac{x}{2-2} e^{x-y}$$

$$= \frac{x^2}{2} e^{x-y} \qquad \because (ORF)$$

The complete solution is $z = C.F + P.I_1 + P.I_2$

$$z = \phi_1(y-x) + \phi_2(y-x) + \frac{x^4 y}{12} - \frac{x^5}{30} + \frac{x^2}{2} e^{x-y}$$

Type-V

1. Solve $r + s - 6t = y\cos x$

Sol.

Given equation can be written as

$(D^2 + DD' - 6D'^2)z = y\cos x$

The A.E is $m^2 + m - 6 = 0$

$(m+3)(m-2) = 0$

$m = -3, 2$

$C.F = \phi_1(y-3x) + x\phi_2(y+2x)$

$$P.I = \frac{1}{(D+3D')(D-2D')} y\cos x \qquad \begin{aligned} &Let\ \ y = c - mx \\ &ie,\ \ y = c - 2x \\ &c = y + 2x \end{aligned}$$

$$= \frac{1}{(D+3D')} \int (c-2x)\cos x \, dx$$

$$= \frac{1}{(D+3D')} (c-2x)\sin x - (-2)(-\cos x)$$

$$= \frac{1}{(D+3D')} (y+2x-2x)\sin x - 2\cos x$$

$$= \frac{1}{(D+3D')} y\sin x - 2\cos x$$

$$= \frac{1}{D-(-3D')} y\sin x - 2\cos x \qquad \begin{aligned} &Let\ \ y = c - mx \\ &ie,\ \ y = c + 3x \\ &c = y - 3x \end{aligned}$$

$$= \int (c+3x)\sin x - 2\cos x \;\; dx$$
$$= (c+3x)(-\cos x) - 3(-\sin x) - 2\sin x$$
$$= -(y-3x+3x)\cos x + \sin x$$
$$= -y\cos x + \sin x$$

The complete solution is $z = C.F + P.I$

$$z = \phi_1(y-3x) + \phi_2(y+2x) + \sin x - y\cos x$$

2. Solve $(4D^2 - 4DD' + D'^2)z = 16\log(x+2y)$

Sol.

The A.E. is $4m^2 - 4m + 1 = 0$

$$(2m-1)(2m-1) = 0$$
$$m = \frac{1}{2}, \frac{1}{2}$$
$$C.F = \phi_1(y + \frac{1}{2}x) + x\phi_2(y + \frac{1}{2}x)$$