UNIT V

Z – TRANSFORMS AND DIFFERENCE EQUATIONS

5.1. DEFINITIONS:

5.1.1. DEFINITION: (ONE-SIDED OR UNILATERAL)

Let $\{f(n)\}$ be a sequence defined for all positive integers $n = 0,1,2,\ldots\infty$, then Z-transform of $\{f(n)\}$ is defined as

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n} \quad , \quad \text{where } z \text{ is an arbitrary complex variable.}$$

5.1.2. DEFINITION: (Z – TRANSFORM FOR DISCRETE VALUES OF t)

If $f(t)$ is a function defined for discrete values of t, where $t = nT$, $n = 0,1,2,\ldots$, T being the sampling period, then Z-transform of $f(t)$ is defined as

$$Z\left(f(t)\right) = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$$

5.2. NOTE:

(i) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$ if $|x| < 1$

(ii) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

(iii) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

(iv) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

(v) $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \cdots$

(vi) $e^{-x} = 1 - \dfrac{x}{1!} + \dfrac{x^2}{2!} - \cdots$

(vii) $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$ if $|x| < 1$

(viii) $-\log(1-x) = x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4} + \cdots$

(ix) $1 + a + a^2 + \cdots + a^r = \dfrac{a^{r+1} - 1}{a - 1}$.

5.3. ELEMENTARY PROPERTIES:

5.3.1. PROPERTY:

$$Z(1) = \frac{z}{z-1}$$

PROOF:

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$$

$\therefore$
$$Z(1) = \sum_{n=0}^{\infty} (1)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^n}$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots$$

$$= \left(1 - \frac{1}{z}\right)^{-1} \qquad , \left|\frac{1}{z}\right| \le 1 \quad \left(\because \quad (1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots\right)$$

$$= \left(\frac{z-1}{z}\right)^{-1} \qquad |z| > 1$$

$$= \left(\frac{z}{z-1}\right).$$

5.3.2. PROPERTY:

$$Z(a^n) = \frac{z}{z-a} \quad \text{if } |z| > |a|$$

PROOF:

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$$

$$Z\left(a^n\right) = \sum_{n=0}^{\infty} a^n\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$= 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \cdots$$

$$= \left(1 - \frac{a}{z}\right)^{-1}, \qquad \text{if } \left|\frac{a}{z}\right| < 1$$

$$= \left(\frac{z-a}{z}\right)^{-1}$$

$$= \left(\frac{z}{z-a}\right), \qquad \text{if } |a| < |z|.$$

5.3.3. PROPERTY:

$$Z(n) = \frac{z}{(z-1)^2}$$

PROOF:

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$$

$$\therefore \qquad Z(n) = \sum_{n=0}^{\infty} n\, z^{-n}$$

$$= 0 + \frac{1}{z} + \frac{2}{z^2} + \cdots$$

$$= \frac{1}{z}\left(1 + \frac{2}{z} + \frac{3}{z^2} + \cdots\right)$$

$$= \frac{1}{z}\left(1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + \cdots\right)$$

$$= \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-2}$$

$$= \frac{1}{z}\left(\frac{z-1}{z}\right)^{-2}$$

$$= \frac{1}{z}\left(\frac{z}{z-1}\right)^{2}$$

$$Z(n) = \frac{z}{(z-1)^2} \ .$$

5.3.4. PROPERTY:

$$Z\left(\frac{1}{n}\right) = \log\left(\frac{z}{z-1}\right), \quad \text{if } \ |z| > 1, \ \ n > 0$$

PROOF:

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$$

$$\therefore \quad Z\left(\frac{1}{n}\right) = \sum_{n=1}^{\infty} \frac{1}{n} z^{-n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{nz^n}$$

$$= \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \cdots$$

$$= \left(\frac{1}{z}\right) + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \cdots$$

$$= -\log\left(1 - \frac{1}{z}\right) \qquad \left(\because \ -\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right)$$

$$= -\log\left(\frac{z-1}{z}\right)$$

$$= \log\left(\frac{z}{z-1}\right)$$

5.3.5. PROPERTY:

$$Z\left(\frac{1}{n+1}\right) = z \log\left(\frac{z}{z-1}\right)$$

PROOF:

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$$

$$\therefore \quad Z\left(\frac{1}{n+1}\right) = \sum_{n=0}^{\infty} \frac{1}{n+1} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)\, z^n}$$

$$= 1 + \frac{1}{2z} + \frac{1}{3z^2} + \cdots$$

$$= z\left(\frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \cdots\right) \qquad (\times \text{ and } \div \text{ by } z)$$

$$= z\left(\left(\frac{1}{z}\right) + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \cdots\right)$$

$$= z\left(-\log\left(1-\frac{1}{z}\right)\right)$$

$$= -z\log\left(\frac{z-1}{z}\right)$$

$$= z\log\left(\frac{z}{z-1}\right).$$

5.3.6. PROPERTY:

$$Z\left(\frac{1}{n-1}\right) = \frac{1}{z}\log\left(\frac{z}{z-1}\right), \qquad n > 1$$

PROOF:

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$$

$$\therefore \qquad Z\left(\frac{1}{n-1}\right) = \sum_{n=2}^{\infty} \frac{1}{n-1}\quad z^{-n}$$

$$= \sum_{n=2}^{\infty} \frac{1}{(n-1)\,z^{n}}$$

$$= \frac{1}{z^2} + \frac{1}{2z^3} + \frac{1}{3z^4} + \cdots$$

$$= \frac{1}{z}\left(\left(\frac{1}{z}\right) + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \cdots\right)$$

$$= \frac{1}{z}\left(-\log\left(1-\frac{1}{z}\right)\right)$$

$$= \frac{1}{z}\left(-\log\left(\frac{z-1}{z}\right)\right)$$

$$= \frac{1}{z}\log\left(\frac{z}{z-1}\right)$$

5.3.7. PROPERTY:

$$Z\left(\frac{1}{n!}\right) = e^{1\backslash z}$$

PROOF:

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$$

$$\therefore \quad Z\left(\frac{1}{n!}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \; z^{-n}$$

$$\therefore \quad = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z^n}\right)$$

$$= 1 + \frac{1}{1!}\left(\frac{1}{z}\right) + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \cdots$$

$$\left(\because \quad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right)$$

$$= e^{1\backslash z}$$

5.3.8. PROPERTY:

$$Z\left(\frac{1}{(n+1)!}\right) = ze^{1\backslash z} - z$$

PROOF:

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$$

$$\therefore \quad Z\left(\frac{1}{(n+1)!}\right) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \; z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\left(\frac{1}{z^n}\right)$$

$$= \frac{1}{1!} + \frac{1}{2!}\left(\frac{1}{z}\right) + \frac{1}{3}\left(\frac{1}{z}\right)^2 + \cdots$$

$$= z\left(\frac{1}{1!}\left(\frac{1}{z}\right) + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \cdots\right)$$

$$= z\left(e^{1\backslash z} - 1\right) \qquad \left(\because \quad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right)$$

$$= \left(ze^{1\backslash z} - z\right).$$

5.3.9. PROPERTY:

$$Z(n\ a^n) = \frac{az}{(z-a)^2}$$

PROOF:

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\ z^{-n}$$

$$Z\left(n\ a^n\right) = \sum_{n=0}^{\infty} n\ a^n\ z^{-n}$$

$$= \sum_{n=0}^{\infty} n\left(\frac{a}{z}\right)^n$$

$$= 0 + \left(\frac{a}{z}\right) + 2\left(\frac{a}{z}\right)^2 + \cdots$$

$$= \frac{a}{z}\left(1 + 2\left(\frac{a}{z}\right) + 3\left(\frac{a}{z}\right)^2 + \cdots\right)$$

$$= \frac{a}{z}\left(1 - \frac{a}{z}\right)^{-2}$$

$$= \frac{a}{z}\left(\frac{z-a}{z}\right)^{-2}$$

$$= \frac{a}{z}\frac{z^2}{(z-a)^2}$$

$$= \frac{az}{(z-a)^2}$$

5.3.10. PROPERTY: (LINEAR PROPERTY)

$$Z\left(af(n) + bg(n)\right) = aZ\left(f(n)\right) + bZ\left(g(n)\right)$$

$$= a\ F(z) + b\ G(z)$$

5.4. NOTE:

(i) $Z(1) = \dfrac{z}{z-1}$

(ii) $Z(a^n) = \dfrac{z}{z-a}$, if $|z| > |a|$

5.5. EXAMPLES:

5.5.1. EXAMPLE:

Find $Z(k)$

SOLUTION:

$$Z(k) = k\ Z(1)$$

$$= k\left(\frac{z}{z-1}\right)$$

5.5.2. EXAMPLE:

Find $Z\left((-1)^n\right)$

SOLUTION:

Since $Z(a^n) = \dfrac{z}{z-a}$

$\therefore$ $$Z((-1)^n) = \frac{z}{z-(-1)}$$

$$= \frac{z}{z+1}$$

5.5.3. EXAMPLE:

Prove that $Z\left(e^{-an}\right) = \dfrac{z}{z-e^{-a}}$

PROOF:

Since $Z(a^n) = \dfrac{z}{z-a}$

$$\therefore \qquad Z((e^{-a})^n) = \frac{z}{z - e^{-a}}.$$

5.5.4. EXAMPLE:

Find $Z(\cos n\theta)$ and $Z(\sin n\theta)$.

SOLUTION:

Let $\qquad a = e^{i\theta}$

$$a^n = e^{in\theta} = \cos n\theta + i \sin n\theta$$

We know that, $\qquad Z(a^n) = \dfrac{z}{z-a}$

$$\therefore \qquad Z(a^n) = Z\left((e^{i\theta})^n\right)$$

$$= \frac{z}{z - e^{i\theta}}$$

$$Z(\cos n\theta + i \sin n\theta) = \frac{z}{z - (\cos\theta + i\sin\theta)}$$

$$= \frac{z}{(z - \cos\theta) - i\sin\theta}$$

$$= \frac{z}{(z-\cos\theta) - i\sin\theta}\,\frac{(z-\cos\theta)+i\sin\theta}{(z-\cos\theta)+i\sin\theta}$$

$$= \frac{z\,(z-\cos\theta) + iz\sin\theta}{(z-\cos\theta)^2 + \sin^2\theta}$$

$$= \frac{z\,(z-\cos\theta) + iz\sin\theta}{z^2 + \cos^2\theta - 2z\cos\theta + \sin^2\theta}$$

$$= \frac{z\,(z-\cos\theta) + iz\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$Z(\cos n\theta) + iZ(\sin n\theta) = \frac{z\,(z-\cos\theta)}{z^2 - 2z\cos\theta + 1} + \frac{iz\sin\theta}{z^2 - 2z\cos\theta + 1}$$

Equating real and imaginary parts,

$$Z(\cos n\theta) = \frac{z\,(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$$

$$Z(\sin n\theta) = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

REMARK:

We know that $Z\left(f(t)\right) = \sum_{n=0}^{\infty} f(nT)\ z^{-n}$

$$\therefore \qquad Z\left(\sin at\right) = \sum_{n=0}^{\infty} \sin anT \quad z^{-n}$$

$$= \sum_{n=0}^{\infty} \sin n\theta \ .\ z^{-n}, \quad \text{where} \quad \theta = aT$$

$$= Z(\sin n\theta)$$

$$= \frac{z\ \sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$= \frac{z\ \sin aT}{z^2 - 2z\cos aT + 1}$$

$$lll^{ly} \quad Z(\cos at) = \frac{z(z - \cos aT)}{z^2 - 2z\cos aT + 1}.$$

5.5.5. EXAMPLE:

Find $Z(r^n \cos n\theta)$ and $Z(r^n \sin n\theta)$

SOLUTION:

Hints: Let $a = re^{i\theta}$

$$a^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta)$$

ANSWER:

$$Z(r^n \cos n\theta) = \frac{z\ (z - r\cos\theta)}{z^2 - 2zr\cos\theta + r^2}$$

$$Z(r^n \sin n\theta) = \frac{zr\sin\theta}{z^2 - 2zr\cos\theta + r^2}.$$

5.5.6. EXAMPLE:

Find $Z(t)$

SOLUTION:

$$Z(f(t)) = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$$

$$\therefore \quad Z(t) = \sum_{n=0}^{\infty} nT\, z^{-n}$$

$$= T\sum_{n=0}^{\infty} n\, z^{-n}$$

$$= T\, Z(n)$$

$$= T\,\frac{z}{(z-1)^2} \qquad \left(\because Z(n) = \frac{z}{(z-1)^2}\right).$$

5.5.7. EXAMPLE:

Find $Z(e^{-at})$

SOLUTION:

$$Z(f(t)) = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$$

$$\therefore \quad Z\left(e^{-at}\right) = \sum_{n=0}^{\infty} e^{-anT}\ z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(e^{-aT}\right)^{n}\ z^{-n}$$

$$= Z\left(e^{-aT}\right)^{n}$$

$$= \frac{z}{z-e^{-aT}} \qquad \left(\because \quad Z(a^n) = \frac{z}{z-a}\right)$$

5.5.8. EXAMPLE:

Find $Z\left(\dfrac{1}{n(n+1)}\right)$

SOLUTION:

Now $$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$= \frac{A(n+1)+B(n)}{n(n+1)}$$

$\Rightarrow \quad A(n+1)+B(n) = 1$

Put $n = 0$

$$A(1)+B(0) = 1 \;\Rightarrow\; A = 1$$

Put $n = -1$:

$$A(0)+B(-1) = 1 \;\Rightarrow\; B = -1$$

$\therefore$ $$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$\therefore$ $$Z\left(\frac{1}{n(n+1)}\right) = Z\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= Z\left(\frac{1}{n}\right) - Z\left(\frac{1}{n+1}\right)$$

$$= \log\left(\frac{z}{z-1}\right) - z\log\left(\frac{z}{z-1}\right)$$

$$= (1-z)\log\left(\frac{z}{z-1}\right)$$

5.5.9. EXAMPLE:

Find $Z\left(\cos^2 t\right)$

SOLUTION:

$$Z\left(\cos^2 t\right) = Z\left(\frac{1+\cos 2t}{2}\right)$$

$$= \frac{1}{2}[Z(1) + Z(\cos 2t)]$$

$$= \frac{1}{2}\left[\frac{z}{z-1} + \frac{z(z-\cos 2T)}{z^2 - 2z\cos 2T + 1}\right]$$

5.5.10. EXAMPLES:

1. Find $Z(\sin^2 t)$
2. $Z(\cos^3 t)$
3. $Z(\sin^3 t)$

5.6. NOTE:

(i) $\sin 3A = 3\sin A - 4\sin^3 A$

(ii) $\cos 3A = 4\cos^3 A - 3\cos A$

5.7. INVERSE *Z*-TRANSFORM:

5.7.1. DEFINITION:

If $Z(f(n)) = F(z)$ then inverse Z-transform is defined as

$$f(n) = Z^{-1}(F(z))$$

5.7.2. REMARK:

(i) $Z(a^n) = \frac{z}{z-a} \quad \Rightarrow \quad Z^{-1}\left(\frac{z}{z-a}\right) = a^n$

(ii) $Z(na^{n-1}) = \frac{z}{(z-a)^2} \quad \Rightarrow \quad Z^{-1}\left(\frac{z}{(z-a)^2}\right) = na^{n-1}$

(iii) $Z(a^{n-1}) = \frac{1}{z-a} \quad \Rightarrow \quad Z^{-1}\left(\frac{1}{z-a}\right) = a^{n-1}$

5.8. EXAMPLES: TYPE: I (METHOD OF PARTIAL FRACTION)

5.8.1. EXAMPLE:

Find $Z^{-1}\left[\dfrac{10z}{(z-1)(z-2)}\right]$

SOLUTION:

$$\text{Let}\quad F(z)=\frac{10z}{(z-1)(z-2)}$$

$$\therefore\quad \frac{F(z)}{z}=\frac{10}{(z-1)(z-2)}$$

$$\therefore\quad \frac{F(z)}{z}=\frac{10}{(z-1)(z-2)}=\frac{A}{z-1}+\frac{B}{z-2}$$

$$=\frac{A(z-2)+B(z-1)}{(z-1)(z-2)}$$

$$\Rightarrow\quad A(z-2)+B(z-1)=10$$

put $z=1$:

$$A(-1)=10 \Rightarrow A=-10$$

put $z=2$:

$$B(1)=10 \Rightarrow B=10$$

$$\therefore\quad \frac{F(z)}{z}=\frac{-10}{z-1}+\frac{10}{z-2}$$

$$\Rightarrow\quad F(z)=\frac{-10z}{z-1}+\frac{10z}{z-2}$$

Taking Z^{-1} on both sides,

$$Z^{-1}(F(z))=-10Z^{-1}\left(\frac{z}{z-1}\right)+10\,Z^{-1}\left(\frac{z}{z-2}\right)$$

$$=-10\,(1^n)+10\,(2^n) \qquad \left(\therefore\ Z^{-1}\left(\frac{z}{z-a}\right)=a^n\right)$$

$$f(n)=10\,(2^n-1).$$

5.8.2. EXAMPLE:

Find $Z^{-1}\left[\dfrac{z^3}{(z-1)^2(z-2)}\right]$ using partial fraction

SOLUTION:

Let $F(z) = \dfrac{z^3}{(z-1)^2\,(z-2)}$

$$\therefore \quad \frac{F(z)}{z} = \frac{z^2}{(z-1)^2(z-2)}$$

$$\therefore \quad \frac{F(z)}{z} = \frac{z^2}{(z-1)^2(z-2)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z-2}$$

$$= \frac{A(z-1)(z-2) + B(z-2) + C(z-1)^2}{(z-1)^2(z-2)}$$

$$\Rightarrow \quad A(z-1)(z-2) + B(z-2) + C(z-1)^2 = z^2$$

Put $z = 1$:

$$B(-1) = 1 \quad \Rightarrow \quad B = -1$$

Put $z = 2$:

$$C(1) = 4 \quad \Rightarrow \quad C = 4$$

Equating the Coeff. of z^2:

$$A + C = 1$$

$$\Rightarrow \quad A = 1 - C = 1 - 4 = -3$$

$$\therefore \quad \frac{F(z)}{z} = \frac{-3}{z-1} + \frac{-1}{(z-1)^2} + \frac{4}{z-2}$$

$$\Rightarrow \quad F(z) = \frac{-3z}{z-1} - \frac{z}{(z-1)^2} + \frac{4z}{z-2}$$

Taking Z^{-1} on both sides,

$$Z^{-1}(F(z)) = -3Z^{-1}\left(\frac{z}{z-1}\right) - Z^{-1}\left(\frac{z}{(z-1)^2}\right) + 4\,Z^{-1}\left(\frac{z}{z-2}\right)$$

$$= -3(1^n) - n(1^{n-1}) + 4\,(2^n)$$

$$f(n) = -3 - n + 4\,(2^n).$$

5.8.3. EXAMPLE:

Find $Z^{-1}\left[\dfrac{z-4}{(z+2)(z+3)}\right]$

ANSWER:

$$Z^{-1}\left[\frac{z-4}{(z+2)(z+3)}\right] = Z^{-1}\left[\frac{-6}{z+2} + \frac{7}{z+3}\right] = -6(-2)^{n-1} + 7(-3)^{n-1}$$

5.8.4. EXAMPLE:

Find $Z^{-1}\left[\dfrac{z-4}{(z-1)(z-2)^2}\right]$

ANSWER:

$$Z^{-1}\left[\frac{z-4}{(z-1)(z-2)^2}\right] = Z^{-1}\left[\frac{-3}{z-1} - \frac{2}{(z-2)^2} + \frac{3}{z-2}\right]$$

$$= -3(1^{n-1}) - (n-1)\left(2^{n-1}\right) + 3\left(2^{n-1}\right)$$

5.9. EXAMPLES: TYPE: II (METHOD OF RESIDUES)

(CAUCHY'S RESIDUE THEOREM)

5.9.1. FORMULAE:

If $Z(f(n)) = F(z)$ then

$$f(n) = Z^{-1}(F(z))$$

$$= \frac{1}{2\pi i}\int_c z^{n-1}F(z)\,dz$$

Where c is the closed contour which encloses all the poles of the integrand.

Where $\int_c z^{n-1}F(z)\,dz = 2\pi i \begin{pmatrix} \text{sum of the residues of } z^{n-1}F(z) \\ \text{at each of its poles} \end{pmatrix}$

5.9.2. NOTE:

(i) If $z=a$ is a simple pole of $f(z)$ then residue at $z=a$ is

$$\lim_{z\to a}(z-a)f(z)$$

(ii) If $z=a$ is a pole of order m of $f(z)$ then residue at $z=a$ is

$$\frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}(z-a)^m\; f(z).$$

5.10. EXAMPLES:

5.10.1. EXAMPLE:

Find inverse Z-transform of $\frac{z}{(z-1)(z-2)}$ using residue theorem.

SOLUTION:

$$Z^{-1}\left[\frac{z}{(z-1)(z-2)}\right] = f(n)$$

Then

$$f(n) = \frac{1}{2\pi i}\int_c z^{n-1}F(z)\,dz$$

$$= \frac{1}{2\pi i}\int_c z^{n-1}\frac{z}{(z-1)(z-2)}\,dz$$

$$= \frac{1}{2\pi i}\int_c \frac{z^n}{(z-1)(z-2)}\,dz \qquad (1)$$

To find $\int_c \frac{z^n}{(z-1)(z-2)}\,dz$:

$$\int_c \frac{z^n}{(z-1)(z-2)}\,dz = 2\pi i \begin{pmatrix} \text{sum of the residues of } \phi(z) \\ \text{at each of its poles} \end{pmatrix}$$

where $\phi(z) = \dfrac{z^n}{(z-1)(z-2)}$.

The poles are $z = 1,\ z = 2$.

Res. at $z = 1$:

$$\text{Residue } [\phi(z)]_{z=1} = \lim_{z\to 1}(z-1)\phi(z)$$

$$= \lim_{z\to 1}(z-1)\frac{z^n}{(z-1)(z-2)}$$

$$= \lim_{z\to 1}\frac{z^n}{(z-2)}$$

$$= \frac{1^n}{-1}$$

$$= -1.$$

Res. at $z = 2$:

$$\text{Residue } [\phi(z)]_{z=2} = \lim_{z\to 2}(z-2)\phi(z)$$

$$= \lim_{z\to 2}(z-2)\frac{z^n}{(z-1)(z-2)}$$

$$= \lim_{z\to 2}\frac{z^n}{(z-1)}$$

$$= \frac{2^n}{1}$$

$$= 2^n.$$

$$\therefore \quad \int_c \frac{z^n}{(z-1)(z-2)}\,dz = 2\pi i\left[-1+2^n\right] \qquad (2)$$

Sub. (2) in (1) we get,

$$f(n) = \frac{1}{2\pi i} 2\pi i \left[-1 + 2^n\right]$$

$$f(n) = 2^n - 1 .$$

5.10.2. EXAMPLE:

Find $Z^{-1}\left[\frac{z(z+1)}{(z-1)^3}\right]$

SOLUTION:

Let $Z^{-1}\left[\frac{z(z+1)}{(z-1)^3}\right] = f(n)$

Then $$f(n) = \frac{1}{2\pi i}\int_c z^{n-1} F(z)\, dz$$

$$= \frac{1}{2\pi i}\int_c z^{n-1} \frac{z(z+1)}{(z-1)^3}\, dz$$

$$= \frac{1}{2\pi i}\int_c \frac{z^n(z+1)}{(z-1)^3}\, dz \qquad (1)$$

To find $\int_c \frac{z^n(z+1)}{(z-1)^3}\, dz$:

$$\int_c \frac{z^n(z+1)}{(z-1)^3}\, dz = 2\pi i \begin{pmatrix} \text{sum of the residues of } \phi(z) \\ \text{at each of its poles} \end{pmatrix}$$

where $\phi(z) = \frac{z^n(z+1)}{(z-1)^3}$.

Here $z = 1$ is a pole of order 3.

$$\text{Residue } [\phi(z)]_{z=1} = \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}(z-1)^3 \frac{z^n(z+1)}{(z-1)^3}$$

$$= \frac{1}{2}\lim_{z\to 1}\frac{d^2}{dz^2}\left(z^n(z+1)\right)$$

$$= \frac{1}{2} \lim_{z \to 1} \frac{d^2}{dz^2}\left(z^{n+1} + z^n\right)$$

$$= \frac{1}{2} \lim_{z \to 1} \frac{d}{dz}\left((n+1)z^n + nz^{n-1}\right)$$

$$= \frac{1}{2} \lim_{z \to 1}\left((n+1)\ n\ z^{n-1} + n\ (n-1)\ z^{n-2}\right)$$

$$= \frac{1}{2}\left((n+1)\ n + n\ (n-1)\right)$$

$$= \frac{1}{2}\left((n^2 + n + n^2 - n\right)$$

$$= \frac{1}{2}\left(2n^2\right)$$

$$= n^2$$

$$\therefore \qquad \int_c \frac{z^n\ (z+1)}{(z-1)^3}\ dz = 2\pi i\ \left(n^2\right) \qquad (2)$$

$$\therefore \qquad (1) \Rightarrow f(n) = \frac{1}{2\pi i}\ 2\pi i\ (n^2) = n^2 .$$

5.10.3. EXAMPLE:

Find $Z^{-1}\left[\dfrac{z^2}{z^2+9}\right]$.

5.11. DEFINITION: (CONVOLUTION)

The convolution of two sequence $\{f(n)\}$ and $\{g(n)\}$ is defined as

$$\{f(n) * g(n)\} = \sum_{r=0}^{n} f(r)\ g(n-r) .$$

The convolution of two functions $f(t)$ and $g(t)$ is defined as

$$f(t) * g(t) = \sum_{r=0}^{n} f(rT)\ g(n-r)T , \quad \text{where } T \text{ is the sampling period.}$$

5.12. CONVOLUTION THEOREM:

(i) $Z\left(f(n) * g(n)\right) = F(z).\, G(z)$

where $Z\left(f(n)\right) = F(z)$ and $Z\left(g(n)\right) = G(z)$

(ii) $Z\left(f(t) * g(t)\right) = F(z).\, G(z)$

where $Z\left(f(t)\right) = F(z)$ and $Z\left(g(t)\right) = G(z)$.

PROOF:

(i) $$F(z) = Z\left(f(n)\right) = \sum_{n=0}^{\infty} f(n)\, z^{-n}$$

$$G(z) = Z\left(g(n)\right) = \sum_{n=0}^{\infty} g(n)\, z^{-n}$$

$$\therefore \quad F(z).\, G(z) = \sum_{n=0}^{\infty} f(n)\, z^{-n} \sum_{n=0}^{\infty} g(n)\, z^{-n}$$

$$= \left(f(0) + f(1)\, z^{-1} + f(2)\, z^{-2} + \cdots f(n)\, z^{-n} + \cdots\right)$$

$$\left(g(0) + g(1)\, z^{-1} + g(2)\, z^{-2} + \cdots g(n)\, z^{-n} + \cdots\right)$$

$$= \left(f(0)\, g(0) + \left[f(0)\, g(1) + f(1) g(0)\right] z^{-1} + \cdots + \left[\sum_{r=0}^{n} f(r) g(n-r)\right] z^{-n} + \cdots\right)$$

$$= \sum_{n=0}^{\infty} \left[\sum_{r=0}^{n} f(r) g(n-r)\right] z^{-n}$$

$$= \sum_{n=0}^{\infty} \left[f(n) * g(n)\right] z^{-n}$$

$$= Z\left(f(n) * g(n)\right)$$

(i.e.) $Z\left(f(n) * g(n)\right) = F(z)\, G(z)$.

(ii) $$F(z) = Z\left(f(t)\right) = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$$

$$G(z) = Z\left(g(t)\right) = \sum_{n=0}^{\infty} g(nT)\, z^{-n}$$

$\therefore$ $$F(z).\, G(z) = \sum_{n=0}^{\infty} f(nT)\, z^{-n} \sum_{n=0}^{\infty} g(nT)\, z^{-n}$$

$$= \left(f(0T) + f(1T)\, z^{-1} + f(2T)\, z^{-2} + \cdots f(nT)\, z^{-n} + \cdots\right)$$

$$\left(g(0T) + g(1T)\, z^{-1} + g(2T)\, z^{-2} + \cdots g(nT)\, z^{-n} + \cdots\right)$$

$$= \Bigg(f(0T)\, g(0T) + \Big[f(0T)\, g(1T) + f(1T) g(0T)\Big]\, z^{-1} +$$

$$\cdots + \left[\sum_{r=0}^{n} f(rT) g(n-r)T\right] z^{-n} + \cdots\Bigg)$$

$$= \sum_{n=0}^{\infty} \left[\sum_{r=0}^{n} f(rT) g(n-r)T\right] z^{-n}$$

$$= \sum_{n=0}^{\infty} [f(t) * g(t)]\, z^{-n}$$

$$= Z\left(f(t) * g(t)\right)$$

(i.e.) $Z\left(f(t) * g(t)\right) = F(z)\, G(z)$.

5.13. NOTE:

(i) $Z\left(f(n) * g(n)\right) = F(z)\, G(z)$

$\Rightarrow$ $Z^{-1}\left(F(z)\, G(z)\right) = f(n) * g(n)$

(ii) $Z\left(f(t) * g(t)\right) = F(z)\, G(z)$

$\Rightarrow$ $Z^{-1}\left(F(z)\, G(z)\right) = f(t) * g(t)$.

5.14. EXAMPLES: TYPE III (CONVOLUTION METHOD)

5.14.1. EXAMPLE:

Using convolution theorem evaluate $Z^{-1}\left[\dfrac{z^2}{(z-1)(z-3)}\right]$.

SOLUTION:

$$Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = Z^{-1}\left[\frac{z}{z-1}\cdot\frac{z}{z-3}\right]$$

$$= Z^{-1}\left[\frac{z}{z-1}\right] * Z^{-1}\left[\frac{z}{z-3}\right]$$

$$= 1^n * 3^n$$

$$= \sum_{r=0}^{n} 1^r\ 3^{n-r}$$

$$= 3^n + 3^{n-1} + 3^{n-2} + \cdots + 3^1 + 1$$

$$= 1 + 3 + 3^2 + \cdots + 3^n$$

$$= \frac{3^{n+1}-1}{3-1} \left(\because\ = 1 + a + a^2 + \cdots + a^n = \frac{a^{n+1}-1}{a-1}, |a| < 1\right)$$

$$= \frac{3^{n+1}-1}{2}.$$

5.14.2. EXAMPLE:

Find $Z^{-1}\left[\dfrac{z^2}{(z-a)^2}\right]$

SOLUTION:

$$Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = Z^{-1}\left[\frac{z}{z-a}\cdot\frac{z}{z-a}\right]$$

$$= Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-a}\right]$$

$$= a^n * a^n$$

$$= \sum_{r=0}^{n} a^r \; a^{n-r}$$

$$= a^n + a\; a^{n-1} + a^2\; a^{n-2} + \cdots$$

$$= (n+1)\; a^n .$$

5.15. SOLUTION OF DIFFERENCE EQUATIONS USING Z-TRANSFORMS:

5.15.1. FORMULA:

$$Z\left(y(k)\right) = F(z)$$

$$Z\left(y(k+1)\right) = zF(z) - zy(0)$$

$$Z\left(y(k+2)\right) = z^2F(z) - z^2y(0) - zy(1)$$

$$Z\left(y(k+3)\right) = z^3F(z) - z^3y(0) - z^2y(1) - zy(2)$$

5.16. EXAMPLES:

5.16.1. EXAMPLE:

Solve the difference equation $y(k+2) - 4\; y(k+1) + 4\; y(k) = 0$,

where $y(0) = 1,\;\; y(1) = 0$.

SOLUTION:

Given $y(k+2) - 4\; y(k+1) + 4\; y(k) = 0$

Taking Z-transform on both sides,

$$Z\left[\, y(k+2) - 4\; y(k+1) + 4\; y(k) \right] = Z(0)$$

$$Z[y(k+2)] - 4\,Z[y(k+1)] + 4\,Z[\,y(k)] = Z(0)$$

$$\left[z^2F(z) - z^2y(0) - zy(1)\right] - 4\,[zF(z) - zy(0)] + 4\,F(z) = 0$$

$$\left[z^2F(z) - z^2\,(1) - z\,(0)\right] - 4\left[zF(z) - z\,(1)\right] + 4\,F(z) = 0$$

$$F(z)\left[z^2 - 4z + 4\right] = z^2 - 4z$$

$$F(z) = \frac{z^2 - 4z}{z^2 - 4z + 4}$$

$$Z(y(k)) = \frac{z^2 - 4z}{z^2 - 4z + 4}$$

$$\Rightarrow \qquad y(k) = Z^{-1}\left(\frac{z^2 - 4z}{z^2 - 4z + 4}\right)$$

$$= Z^{-1}\left(\frac{z\,(z-4)}{(z-2)^2}\right)$$

To find $Z^{-1}\left(\frac{z\,(z-4)}{(z-2)^2}\right)$

Let $\qquad F(z) = \dfrac{z\,(z-4)}{(z-2)^2}$

$$\therefore \qquad \frac{F(z)}{z} = \frac{z-4}{(z-2)^2} = \frac{A}{z-2} + \frac{B}{(z-2)^2}$$

$$= \frac{A(z-2) + B}{(z-2)^2}$$

$$\Rightarrow \qquad A(z-2) + B = z - 4$$

put $z = 2$:

$$A(0) + B = 2 - 4$$

$$B = -2$$

Equating the constant term:

$$-2A + B = -4$$

$$-2A = -4 - B$$

$$-2A = -4 + 2$$

$$A = 1$$

$$\therefore \qquad \frac{F(z)}{z} = \frac{1}{z-2} - \frac{2}{(z-2)^2}$$

$$\Rightarrow \qquad F(z) = \frac{z}{z-2} - \frac{2z}{(z-2)^2}$$

Taking Z^{-1} on both sides,

$$Z^{-1}(F(z)) = Z^{-1}\left[\frac{z}{z-2}\right] - Z^{-1}\left[\frac{2z}{(z-2)^2}\right]$$

$$= 2^k - k2^k \qquad \left(\because\ Z^{-1}\left[\frac{az}{(z-a)^2}\right] = k\ a^k\right)$$

$$y(k) = 2^k(1-k).$$

5.16.2. EXAMPLE:

Solve $y(n+2) - 3\,y(n+1) + 2\,y(n) = 2^n$, given that $y(0) = 0, y(1) = 0$.

SOLUTION:

Given $y(n+2) - 3\,y(n+1) + 2\,y(n) = 2^n$

Taking Z-transform on both sides,

$$Z\left[y(n+2) - 3\,y(n+1) + 2\,y(n)\right] = Z(2^n)$$

$$Z\left[y(n+2)\right] - 3\,Z\left[y(n+1)\right] + 2\,Z\left[\,y(n)\right] = Z(2^n)$$

$$\left[z^2F(z) - z^2\,y(0) - z\,y(1)\right] - 3\left[z\,F(z) - z\,y(0)\right] + 2\,F(z) = \frac{z}{z-2}$$

$$\left[z^2F(z) - z^2\,(0) - z\,(0)\right] - 3\left[z\,F(z) - z\,(0)\right] + 2\,F(z) = \frac{z}{z-2}$$

$$F(z)\left[z^2-3z+2\right]=\frac{z}{z-2}$$

$$F(z)=\frac{z}{(z-2)\left(z^2-3z+2\right)}$$

$$Z(y(n))=\frac{z}{(z-2)(z-2)(z-1)}$$

$$Z(y(n))=\frac{z}{(z-2)^2(z-1)}$$

$$\Rightarrow \qquad y(n)=Z^{-1}\left(\frac{z}{(z-2)^2(z-1)}\right)$$

To find $Z^{-1}\left(\dfrac{z}{(z-2)^2(z-1)}\right)$

Let $\qquad F(z)=\dfrac{z}{(z-2)^2(z-1)}$

$$\therefore \qquad \frac{F(z)}{z}=\frac{1}{(z-1)(z-2)^2}=\frac{A}{z-1}+\frac{B}{z-2}+\frac{C}{(z-2)^2}$$

$$=\frac{A(z-2)^2+B(z-1)(z-2)+C(z-1)}{(z-1)(z-2)^2}$$

$$\Rightarrow \quad A(z-2)^2+B(z-1)(z-2)+C(z-1)=1$$

put $z=2$:

$C=1$

put $z=1$:

$A=1$

Equating the coeff. of z^2:

$A+B=0$

$B=-1$

$$\therefore \quad \frac{F(z)}{z}=\frac{1}{z-1}+\frac{-1}{z-2}+\frac{1}{(z-2)^2}$$

$$\Rightarrow \qquad F(z) = \frac{z}{z-1} - \frac{z}{z-2} + \frac{z}{(z-2)^2}$$

Taking Z^{-1} on both sides,

$$Z^{-1}(F(z)) = Z^{-1}\left[\frac{z}{z-1}\right] - Z^{-1}\left[\frac{z}{z-2}\right] + Z^{-1}\left[\frac{z}{(z-2)^2}\right]$$

$$y(n) = 1^n - 2^n + n2^{n-1}$$

$$\left(\because \ Z^{-1}\left[\frac{z}{z-a}\right] = a^n, \ Z^{-1}\left[\frac{z}{(z-a)^2}\right] = n \ a^{n-1}\right)$$

5.16.3. EXAMPLE:

Solve the difference equation $y(n) + 3\,y(n-1) - 4\,y(n-2) = 0, \ \ n \geq 2$, given that $y(0) = 3, \ y(1) = -2$.

SOLUTION:

Changing n in to $n+2$, then given equation becomes

$$y(n+2) + 3\,y(n+1) - 4\,y(n) = 0, \quad n \geq 0$$

Taking Z-transform on both sides,

$$Z\left[y(n+2) + 3\,y(n+1) - 4\,y(n)\right] = Z(0)$$

$$Z[y(n+2)] + 3\,Z[y(n+1)] - 4\,Z[\ y(n)] = Z(0)$$

$$\left[z^2F(z) - z^2\ y(0) - z\ y(1)\right] + 3\left[z\ F(z) - z\ y(0)\right] - 4\ F(z) = 0$$

$$\left[z^2F(z) - z^2\ (3) - z\ (-2)\right] + 3\left[z\ F(z) - z\ (3)\right] - 4\ F(z) = 0$$

$$F(z)\left[z^2 + 3z - 4\right] = 3z^2 - 2z + 9z$$

$$F(z) = \frac{3z^2 + 7z}{z^2 + 3z - 4}$$

$$F(z) = \frac{z(3z+7)}{(z+4)(z-1)}$$

$$Z(y(n)) = \frac{z(3z+7)}{(z+4)(z-1)}$$

$$\Rightarrow \quad y(n) = Z^{-1}\left(\frac{z(3z+7)}{(z+4)(z-1)}\right)$$

To find $Z^{-1}\left(\frac{z(3z+7)}{(z+4)(z-1)}\right)$

Let $F(z) = \frac{z(3z+7)}{(z+4)(z-1)}$

$$\therefore \quad \frac{F(z)}{z} = \frac{(3z+7)}{(z+4)(z-1)} = \frac{A}{z+4} + \frac{B}{z-1}$$

$$= \frac{A(z-1)+B(z+4)}{(z+4)(z-1)}$$

$$\Rightarrow \quad A(z-1)+B(z+4) = 3z+7$$

put $z = -4$:

$$-5A = -5$$

$$A = 1$$

put $z = 1$:

$$5B = 10$$

$$B = 2$$

$$\therefore \quad \frac{F(z)}{z} = \frac{(3z+7)}{(z+4)(z-1)} = \frac{1}{z+4} + \frac{2}{z-1}$$

$$\Rightarrow \quad F(z) = \frac{z}{z+4} + 2\frac{z}{z-1}$$

Taking Z^{-1} on both sides,

$$\Rightarrow \quad Z^{-1}(F(z)) = Z^{-1}\left[\frac{z}{z+4}\right] + 2Z^{-1}\left[\frac{z}{z-1}\right]$$

$$y(n) = (-4)^n + 2(-1)^n \qquad \left(\because\ Z^{-1}\left[\frac{z}{z-a}\right] = a^n\right).$$

5.16.4. EXAMPLE:

Solve $u_{n+2} - 5u_{n+1} + 6u_n = (-1)^n$, where $u_0 = u_1 = 0$.

SOLUTION:

$$u_{n+2} - 5u_{n+1} + 6u_n = (-1)^n$$

Taking Z-transform on both sides,

$$Z\left[u_{n+2} - 5u_{n+1} + 6u_n\right] = Z\left[(-1)^n\right]$$

$$Z\left[u_{n+2}\right] - 5Z\left[u_{n+1}\right] + 6Z\left[u_n\right] = Z\left[(-1)^n\right]$$

$$\left[z^2F(z) - z^2\ u(0) - z\ u(1)\right] - 5\left[z\ F(z) - z\ u(0)\right] + 6\ F(z) = \frac{z}{z+1}$$

$$\left[z^2F(z) - z^2\ (0) - z\ (0)\right] - 5\left[z\ F(z) - z\ (0)\right] + 6\ F(z) = \frac{z}{z+1}$$

$$F(z)\left[z^2 - 5z + 6\right] = \frac{z}{z+1}$$

$$F(z) = \frac{z}{(z+1)\left(z^2 - 5z + 6\right)}$$

$$Z(u_n)) = \frac{z}{(z+1)(z-3)(z-2)}$$

$$\Rightarrow \qquad u_n = Z^{-1}\left(\frac{z}{(z+1)(z-3)(z-2)}\right)$$

To find $Z^{-1}\left(\frac{z}{(z+1)(z-3)(z-2)}\right)$

Let $F(z) = \frac{z}{(z+1)(z-3)(z-2)}$

$$\therefore \qquad \frac{F(z)}{z} = \frac{z}{(z+1)(z-3)(z-2)} = \frac{A}{z+1} + \frac{B}{z-3} + \frac{C}{z-2}$$

$$= \frac{A(z-3)(z-2) + B(z+1)(z-2) + C(z+1)(z-3)}{(z+1)(z-3)(z-2)}$$

$$\Rightarrow \qquad A(z-3)(z-2) + B(z+1)(z-2) + C(z+1)(z-3) = 1$$

put $z=-1$:

$$A(-4)(-3)=1 \quad \Rightarrow \quad A=\frac{1}{12}$$

put $z=3$:

$$B(4)(1)=1 \quad \Rightarrow \quad B=\frac{1}{4}$$

put $z=2$:

$$C(3)(-1)=1 \quad \Rightarrow \quad C=-\frac{1}{3}$$

$$\therefore \qquad \frac{F(z)}{z}=\left(\frac{1}{12}\right)\frac{1}{z+1}+\left(\frac{1}{4}\right)\frac{1}{z-3}+\left(\frac{-1}{3}\right)\frac{1}{z-2}$$

$$\Rightarrow \qquad F(z)=\left(\frac{1}{12}\right)\frac{z}{z+1}+\left(\frac{1}{4}\right)\frac{z}{z-3}-\left(\frac{1}{3}\right)\frac{z}{z-2}$$

Taking Z^{-1} on both sides,

$$\Rightarrow \qquad Z^{-1}(F(z))=\left(\frac{1}{12}\right)Z^{-1}\left[\frac{z}{z+1}\right]+\left(\frac{1}{4}\right)Z^{-1}\left[\frac{z}{z-3}\right]-\left(\frac{1}{3}\right)Z^{-1}\left[\frac{z}{z-2}\right]$$

$$u_n=\frac{1}{12}(-1)^n+\frac{1}{4}3^n-\frac{1}{3}2^n \qquad \left(\because\ Z^{-1}\left[\frac{z}{z-a}\right]=a^n\right).$$

5.17. FIRST SHIFTING THEOREM:

If $Z(f(t))=F(z)$ then

(i) $Z\left(e^{-at}\ f(t)\right)=F(z\ e^{aT})$

(ii) $Z\left(e^{at}\ f(t)\right)=F(z\ e^{-aT})$

(iii) $Z\left(a^n\ f(t)\right)=F\left(\frac{z}{a}\right)$

(iv) $Z\left(a^n\ f(n)\right)=F\left(\frac{z}{a}\right)$

PROOF:

(i) We know that

$$Z\left(f(t)\right) = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$$

$$Z\left(e^{-at}\, f(t)\right) = \sum_{n=0}^{\infty} e^{-anT}\, f(nT)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT)\left(ze^{aT}\right)^{-n}$$

$$= Z\left(f(t)\right)_{z \to z\, e^{aT}}$$

$$= \left[F(z)\right]_{z \to z^{aT}}$$

$$Z\left(e^{-at}\, f(t)\right) = F\left(ze^{aT}\right).$$

(iii) We know that

$$Z\left(f(t)\right) = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$$

$$Z\left(a^{n}\, f(t)\right) = \sum_{n=0}^{\infty} a^{n}\, f(nT)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT)\left(\frac{z}{a}\right)^{-n}$$

$$= Z\left(f(t)\right)_{z \to \frac{z}{a}}$$

$$= \left[F(z)\right]_{z \to \frac{z}{a}}$$

$$Z\left(a^{n}\, f(t)\right) = F\left(\frac{z}{a}\right).$$

5.18. DIFFERENTIATION IN Z-DOMAIN:

$$Z\left(n\, f(n)\right) = -z\frac{d}{dz}\left(F(z)\right),\ \text{where}\ F(z) = Z\left(f(n)\right).$$

PROOF:

$$F(z) = Z\left(f(n)\right)$$

$$F(z) = \sum_{n=0}^{\infty} f(n)\ z^{-n}$$

$$\frac{d}{dz}\left(F(z)\right) = \sum_{n=0}^{\infty} f(n)\ (-n)\ z^{-n-1}$$

$$= -\frac{1}{z}\sum_{n=0}^{\infty} f(n)\ n\ z^{-n}$$

$$-z\,\frac{d}{dz}\left(F(z)\right) = \sum_{n=0}^{\infty} n\ f(n)\ z^{-n}$$

$$= Z\left(n\ f(n)\right)$$

$$\therefore \qquad Z\left(n\ f(n)\right) = -z\frac{d}{dz}\left(F(z)\right)$$

5.19. SECOND SHIFTING THEOREM:

(i) $Z\left(f(n+1)\right) = z\ F(z) - zf(0)$

PROOF:

$$Z\left(f(n+1)\right) = \sum_{n=0}^{\infty} f(n+1)\ z^{-n}$$

$$= \sum_{n=0}^{\infty} f(n+1)\ z^{-n}\ z\ z^{-1}$$

$$Z\left(f(n+1)\right) = z\sum_{n=0}^{\infty} f(n+1)\ z^{-(n+1)}$$

put $n+1 = m$

$$= z\sum_{m=1}^{\infty} f(m)\ z^{-m}$$

$$= z\left[\sum_{m=0}^{\infty} f(m)\, z^{-m} - f(0)\right]$$

$$= z\left(F(z) - f(0)\right)$$

$$Z\left(f(n+1)\right) = z\, F(z) - zf(0)\,.$$

(ii) $Z\left(f(t+T)\right) = z\, F(z) - zf(0)$

PROOF:

$$Z\left(f(t+T)\right) = \sum_{n=0}^{\infty} f(nT+T)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT+T)\, z^{-n}\, z\, z^{-1}$$

$$= z \sum_{n=0}^{\infty} f\left((n+1)T\right)\; z^{-(n+1)}$$

put $n+1 = m$

$$= z \sum_{m=1}^{\infty} f(mT)\, z^{-m}$$

$$= z\left[\sum_{m=0}^{\infty} f(mT)\, z^{-m} - f(0)\right]$$

$$= z\left(F(z) - f(0)\right)$$

$$Z\left(f(t+T)\right) = z\, F(z) - zf(0)\,.$$

5.20. INITIAL VALUE THEOREM:

If $Z\left(f(t)\right) = F(z)$ then $f(0) = \lim_{z\to\infty} F(z)$

PROOF:

$$F(z) = Z\left(f(t)\right)$$

$$= \sum_{n=0}^{\infty} f(nT)\, z^{-n}$$

$$= f(0.\,T) + \frac{f(1.\,T)}{z} + \frac{f(2.\,T)}{z^2} + \cdots$$

$$= f(0) + \frac{f(T)}{z} + \frac{f(2T)}{z^2} + \cdots$$

$$\lim_{z \to \infty} F(z) = \lim_{z \to \infty} \left[f(0) + \frac{f(T)}{z} + \frac{f(2T)}{z^2} + \cdots \right]$$

$$= f(0)$$

(i.e.) $f(0) = \lim_{z \to \infty} F(z)$.

5.21. NOTE:

If $Z(f(n)) = F(z)$ then $f(0) = \lim_{z \to \infty} F(z)$

5.22. FINAL VALUE THEOREM:

If $Z(f(t)) = F(z)$ then $\lim_{t \to \infty} f(t) = \lim_{z \to 1} (z-1) F(z)$

PROOF:

$$Z[f(t+T) - f(t)] = \sum_{n=0}^{\infty} [f(nT+T) - f(nT)]\, z^{-n}$$

$$Z(f(t+T)) - Z(f(t)) = \sum_{n=0}^{\infty} [f(nT+T) - f(nT)]\, z^{-n}$$

$$zF(z) - zf(0) - F(z) = \sum_{n=0}^{\infty} [f(nT+T) - f(nT)]\, z^{-n}$$

$$(z-1)F(z) - z\, f(0) = \sum_{n=0}^{\infty} [f(nT+T) - f(nT)]\, z^{-n}$$

Taking limit as $z \to 1$ we get

$$\lim_{z \to 1}\left[(z-1)F(z) - z\ f(0)\right] = \lim_{z \to 1}\sum_{n=0}^{\infty}[f(nT+T) - f(nT)]\ z^{-n}$$

$$\lim_{z \to 1}(z-1)F(z) - f(0) = \sum_{n=0}^{\infty}[f(nT+T) - f(nT)]$$

$$= \lim_{z \to \infty}[f(T) - f(0) + f(2T) - f(T)$$

$$+f(3T) - f(2T) + \cdots + f(n+1)T - f(nT)]$$

$$= \lim_{z \to \infty}\left[f(n+1)T - f(0)\right]$$

$$= \lim_{z \to \infty} f(n+1)T - f(0)$$

$$\lim_{z \to 1}(z-1)F(z) - f(0) = f(\infty) - f(0) = \lim_{t \to \infty} f(t).$$

(i.e.) $\lim_{t \to \infty} f(t) = \lim_{z \to 1}(z-1)F(z)$.

5.23. NOTE:

If $Z(f(n)) = F(z)$ then $\lim_{n \to \infty} f(n) = \lim_{z \to 1}(z-1)F(z)$

5.24. EXAMPLES:

5.24.1. EXAMPLE:

Find $Z\left(e^{-at}\ t\right)$.

SOLUTION:

We know that

$$Z\left(e^{-at}\ f(t)\right) = Z\left[f(t)\right]_{z \to z\, e^{aT}}$$

$$= F(z)_{z \to z\, e^{aT}}$$

Since $\quad Z(f(t)) = \sum_{n=0}^{\infty} f(nT)\ z^{-n}$

$$Z(t) = \sum_{n=0}^{\infty} nT \quad z^{-n}$$

$$= T\sum_{n=0}^{\infty} n\ z^{-n}$$

$$= T\left(0 + \frac{1}{z} + \frac{2}{z^2} + \cdots\right)$$

$$= T\left(\frac{1}{z} + 2\left(\frac{1}{z}\right)^2 + \cdots\right)$$

$$= T\left(\frac{1}{z}\right)\left(1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + \cdots\right)$$

$$= \frac{T}{z}\left(1 - \frac{1}{z}\right)^{-2}$$

$$= \frac{T}{z}\left(\frac{z-1}{z}\right)^{-2}$$

$$= \frac{T}{z}\left(\frac{z}{z-1}\right)^{2}$$

$$= \frac{T\ z}{(z-1)^2}.$$

$$\therefore \qquad Z\left(e^{-at}\ t\right) = \left[\frac{T\ z}{(z-1)^2}\right]_{z \to z\ e^{aT}}$$

$$\therefore \qquad Z\left(e^{-at}\ t\right) = \frac{T\ z\ e^{aT}}{\left(z\ e^{aT} - 1\right)^2}.$$

5.24.2. EXAMPLE:

Find $Z\left(a^n\ n\right)$.

SOLUTION:

We know that

$$Z\left(a^n\ f(n)\right) = F\left(\frac{z}{a}\right)$$

Now, $$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\ z^{-n}$$

$\therefore$ $$Z\,(n) = \sum_{n=0}^{\infty} n\ z^{-n}$$

$$= 0 + \frac{1}{z} + \frac{2}{z^2} + \cdots$$

$$= \frac{1}{z}\left(1 + \frac{2}{z} + \frac{3}{z^2} + \cdots\right)$$

$$= \frac{1}{z}\left(1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + \cdots\right)$$

$$= \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-2}$$

$$= \frac{1}{z}\left(\frac{z-1}{z}\right)^{-2}$$

$$= \frac{1}{z}\left(\frac{z}{z-1}\right)^{2}$$

$$Z(n) = \frac{z}{(z-1)^2}.$$

$\therefore$ $$Z\left(a^n\ n\right) = \left[\frac{z}{(z-1)^2}\right]_{z \to \frac{z}{a}}$$

$$= \left[\frac{\frac{z}{a}}{\left(\frac{z}{a} - 1\right)^2}\right]$$

$$= \left[\frac{\frac{z}{a}}{\left(\frac{z-a}{a}\right)^2}\right]$$

$$= \frac{z}{a}\left[\frac{a^2}{(z-a)^2}\right]$$

$$Z\left(a^n\ n\right) = \frac{az}{(z-a)^2}.$$

5.24.3. EXAMPLE:

Find $Z\left(n^2\right)$.

SOLUTION:

We know that

$$Z\left(n\ f(n)\right) = -z\frac{d}{dz}(F(z))$$

$$Z\left(n^2\right) = Z\left(n\ .n\right) = -z\frac{d}{dz}(Z(n))$$

Now,

$$Z(n) = \sum_{n=0}^{\infty} n\ z^{-n}$$

$$= 0 + \frac{1}{z} + \frac{2}{z^2} + \cdots$$

$$= \frac{1}{z}\left(1 + \frac{2}{z} + \frac{3}{z^2} + \cdots\right)$$

$$= \frac{1}{z}\left(1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + \cdots\right)$$

$$= \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-2}$$

$$= \frac{1}{z}\left(\frac{z-1}{z}\right)^{-2}$$

$$= \frac{1}{z}\left(\frac{z}{z-1}\right)^{2}$$

$$Z(n) = \frac{z}{(z-1)^2}.$$

$$\therefore \quad Z\left(n^2\right) = -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$= -z\left[\frac{(z-1)^2(1) - z\,2(z-1)}{(z-1)^4}\right]$$

$$= -z\left[\frac{(z-1)\,(z-1-2z)}{(z-1)^4}\right]$$

$$= z\left[\frac{(z+1)}{(z-1)^3}\right]$$

$$= \frac{z^2+z}{(z-1)^3}.$$