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UNIT-I

SEQUENCES, SERIES & MEAN VALUE THEOREMS

Sequences: An order set of Real Numbers $a_1, a_2, a_3 \ldots$ a_n is called a sequence and is denoted by "a_n".

If the Number of terms is unlimited, then the Sequence is said to be an infinite sequence and "a_n" is its General term.

Ex: $1, \frac{1}{2}, \frac{1}{3} \ldots \frac{1}{n}, \ldots$ is infinite sequence.

Limit: A sequence is said to tend to a limit "l", if for every $\epsilon > 0$, a value "N" of "n" can be found such that $|a_n - l| < \epsilon$, for $n \geq N$.

$$\lim_{n \to \infty} a_n = l, \text{ (or) } a_n \to l \text{ as } n \to \infty.$$

Convergence: If a sequence (a_n) has a finite limit, it is called a convergent sequence.

If a_n is not convergent, it is said to be divergent.

Convergence, Divergence, and oscillation.

If $\lim_{n \to \infty} a_n = l$ is finite and unique then the sequence is said to be convergent.

if $\lim_{n \to \infty} a_n$ is infinite $(\pm \infty)$, the sequence is said to be divergent.

if $\lim_{n \to \infty} (a_n)$ is not unique, then a_n is said to be oscillatory.

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Ex: The series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges.

Sol: Here $u_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ and

$S_n = u_1 + u_2 + \cdots + u_n$

$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$

$\therefore \{S_n\}$ converges to "1" and hence the series converges to "1" $\therefore \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

Ratio Test:

D'ALEMBERT'S Ratio test:

Theorem: if Σu_n is a series of positive terms such that $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = l$ then

(a) Σu_n convergent, if $l > 1$

(b) Σv_n divergent, if $l < 1$

(c) The test fails to decide the nature of the series if $l = 1$

Ex: Test for convergence of the series $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$

Sol: Here $u_0 = 1$, and $u_n = \frac{1}{n!} > 0$ for all $n \in z^+$

$u_{n+1} = \frac{1}{(n+1)!}$ and $\frac{u_n}{u_{n+1}} = \frac{(n+1)!}{n!} = n+1$

$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} n+1 = \infty > 1$

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By D' Alembert's Ratio test, $\sum u_n$ converges

Note: The sum of the above convergent series is denoted by 'e'

$$\therefore 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \alpha = \sum_{n=0}^{\alpha} \frac{1}{n!} = e$$

Ex: Show that $1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + \cdots$ is convergent for all values of "p".

Sol Here $u_n = \frac{n^p}{n!}$

Then $u_{n+1} = \frac{(n+1)^p}{(n+1)!}$

Now $\frac{u_n}{u_{n+1}} = \frac{n^p}{n!} \times \frac{(n+1)!}{(n+1)^p} = \frac{n^p}{(n+1)^{p-1}} = \frac{n^p}{n^{p-1}\left(1+\frac{1}{n}\right)^{p-1}}$

$\therefore \lim_{n\to\alpha} \frac{u_n}{u_{n+1}} = \alpha$, for all values of "p"

$\therefore$ By D'Alembert's ratio test the series is convergent for all values of "p".

Ex: Test for convergence of the series $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \cdots$

(or) $\sum_{n=1}^{\alpha} \frac{x^{2n}}{(n+1)\sqrt{n}}$

Sol Here $u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}} > 0$ for all "n".

$\therefore u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$

Now $\frac{u_n}{u_{n+1}} = \frac{x^{2n-2}}{(n+1)\sqrt{n}} \cdot \frac{(n+2)\sqrt{n+1}}{x^{2n}}$

$= \frac{n\sqrt{n}\left(1+\frac{2}{n}\right)\sqrt{1+\frac{1}{n}}}{n\sqrt{n}\left(1+\frac{1}{n}\right)} \cdot \frac{1}{x^2}$

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$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x^2}$

By D'Alembert's ratio test Σu_n is convergent if $\frac{1}{x^2} > 1$ and divergent if $\frac{1}{x^2} < 1$ (ie convergent if $x^2 < 1$ and divergent if $x^2 > 1$)

if $x^2 = 1$ the test fails.

when $x^2 = 1$, $u_n = \frac{1}{(n+1)\sqrt{n}} = \frac{1}{n^{3/2}} \frac{1}{\left(1+\frac{1}{n}\right)}$

Take $v_n = \frac{1}{n^{3/2}}$; $\frac{u_n}{v_n} = \frac{1}{n^{3/2}} \frac{1}{\left(1+\frac{1}{n}\right)} . n^{3/2}$

$= \frac{1}{1+\frac{1}{n}}$

$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{1+0} = 1$

$\therefore$ By comparison test, Σu_n and Σv_n will both converge (or) both diverge

Here $\Sigma v_n = \Sigma \frac{1}{n^{3/2}}$ with $p = \frac{3}{2} > 1$

$\therefore \Sigma v_n$ is convergent

$\therefore \Sigma u_n$ is also convergent

$\therefore$ Hence the Given series converges if "$x^2 \leq 1$" and diverges when "$x^2 > 1$"

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Ex! Test for the convergence of $1+\frac{2}{5}x+\frac{6}{9}x^2+\cdots+\frac{2^n-2}{2^n+1}x^{n-1}+\cdots$ $(x>0)$.

$$\frac{u_n}{u_{n+1}} = \frac{2^n-2}{2^n+1}x^{n-1}\cdot\frac{2^{n+1}+1}{2^{n+1}-2}\cdot\frac{1}{x^n}$$

$$= \frac{1-\frac{2}{2^n}}{1+\frac{1}{2^n}}\cdot\frac{2+\frac{1}{2^n}}{2-\frac{2}{2^n}}\cdot\frac{1}{x}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1-0}{1+0}\cdot\frac{2+0}{2-0}\cdot\frac{1}{x}$$

$$= \frac{1}{x}$$

Thus by Ratio test, Σu_n converges for $x^{-1}>1$

i.e for $x<1$ diverges for $x>1$, But it fails for $x=1$

when $x=1$, $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{2^n-2}{2^n+1}$

$$= \lim_{n\to\infty}\frac{1-\frac{2}{2^n}}{1+\frac{1}{2^n}} = 1\neq 0$$

$\therefore \Sigma u_n$ diverges for $x=1$, Hence the Given Series converges for $x<1$ and diverges for $x\geq 1$

Ex: Test for convergence $\sum_{n=1}^{\infty}\frac{n^4}{n!}$

Sol Here $u_n = \frac{n^4}{n!} > 0$ for all 'n' and

$$u_{n+1} = \frac{(n+1)^4}{(n+1)!}$$

Now $\frac{u_n}{u_{n+1}} = \frac{n^4}{n!}\frac{(n+1)!}{(n+1)^4} = \frac{n^4}{(n+1)^3}$

$\therefore \lim_{n\to\infty}\frac{n^4}{(n+1)^3} = \lim_{n\to\infty}\frac{n}{(n+1)^3} = \infty > 1$

$\therefore \Sigma u_n$ is convergent

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Comparision Tests:

If two Positive term Series Σu_n and Σv_n be such that

1, Σv_n converges

2, $u_n \leq v_n$ for all values of 'n', then Σu_n also converges.

Proof

since Σv_n is convergent

$$\lim_{n\to\infty} (v_1 + v_2 + v_3 + \cdots v_n) = \text{a finite quantity 'K' say}$$

Also since $u_1 \leq v_1$, $u_2 \leq v_2$ $\cdots$ $u_n \leq v_n$

$\therefore$ Adding $u_1 + u_2 + \cdots + u_n \leq v_1 + v_2 + \cdots v_n$

$$\therefore \lim_{n\to\infty} u_1 + u_2 + \cdots + u_n \leq \lim_{n\to\infty} v_1 + v_2 + \cdots + v_n = K$$

$\therefore$ Hence the series Σu_n also converges.

Ex Test for convergence $\sum_{n=1}^{\infty} \frac{2n-1}{n(n+1)(n+2)}$

Sol let $u_n = \frac{2n-1}{n(n+1)(n+2)} > 0, \forall n$

This is a series of Positive terms

Take $v_n = \frac{n}{n^3} = \frac{1}{n^2}$

we know that $\Sigma v_n = \Sigma \frac{1}{n^2}$

$= \Sigma \frac{1}{n^p}$ with $P = 2 > 1$ is convergent by P-test

we have $\frac{u_n}{v_n} = \frac{\frac{2n-1}{n(n+1)(n+2)}}{\frac{1}{n^2}} = \frac{n\left(2-\frac{1}{n}\right)}{n.n.n\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} \cdot n^2$

$= \frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$

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$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$= \frac{2}{1\cdot 1} = 2 \neq 0$$

By comparison test Σu_n is convergent.

Prob Test for convergence $\sum_{n=1}^{\infty} \{\sqrt[3]{n^3+1} - n\}$

Sol Let $u_n = \sqrt[3]{n^3+1} - n$

since $u_n > 0, \forall n$ this is a series of positive terms

$$u_n = (n^3+1)^{1/3} - n = n\left(1+\frac{1}{n^3}\right)^{1/3} - n$$

$$= n\left[1 + \frac{1}{3}\cdot\frac{1}{n^3} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2}\cdot\frac{1}{n^6} + \cdots\right] - n$$

$$= \frac{1}{3n^2} - \frac{1}{9n^5} + \cdots$$

Take $v_n = \frac{1}{n^2}$

We know that Σv_n is convergent

Now $\frac{u_n}{v_n} = \frac{\frac{1}{n^2}\left[\frac{1}{3} - \frac{1}{9n^3} + \cdots\right]}{\frac{1}{n^2}}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{3} \neq 0$$

∴ By comparison test Σu_n and Σv_n behave in the same way.

But $\Sigma v_n = \Sigma \frac{1}{n^2}$ is convergent

∴ By comparison test, Σu_n is also convergent

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Pno Test for convergence of $\sum \frac{1}{\sqrt{n}+\sqrt{n+1}}$

Sol Let $u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}}$

Take $v_n = \frac{1}{\sqrt{n}}$

Now $\frac{u_n}{v_n} = \frac{1}{\sqrt{n}+\sqrt{n+1}} \cdot \sqrt{n} = \frac{\sqrt{n}}{\sqrt{n}+\sqrt{n+1}}$

$= \frac{\sqrt{n}}{\sqrt{n}\left(1+\sqrt{1+\frac{1}{n}}\right)}$

$\lim_{n\to\infty} \frac{1}{1+\sqrt{1+\frac{1}{n}}} = \frac{1}{1+1} = \frac{1}{2} \neq 0$

∴ By comparison test, $\sum u_n$ and $\sum v_n$ will have same property

But $\sum v_n = \sum \frac{1}{n^{1/2}} = \sum \frac{1}{n^p}$ is divergent

∴ Hence $\sum u_n$ is also divergent

INTEGRAL TEST:

Let f be a non-negative decreasing function on $[1,\infty)$. Then the series $\sum_{n=1}^{\infty} f(n)$ and the improper integral $\int_1^{\infty} f(x)\,dx$ converge (or) diverge together

Ex: Prove that $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges

Sol Let $f(x) = \frac{1}{x^2+1}$ for $x \in [1,\infty)$

Then $\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$

We have $f(x) \geq 0$ and $f(x)$ is decreasing on $[1,\infty)$

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$\therefore \lim_{t\to\infty} F(t) = \lim_{t\to\infty} \left[\tan^{-1}t - \frac{\pi}{4}\right]$

$= \frac{\pi}{2} - \frac{\pi}{4} = \pi/4.$

$\therefore \int_1^\infty f(x)\,dx$ converges to $\pi/4$.

$\therefore$ By integral test $\Sigma f(n)$ also converges.

Ex: Discuss the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n\log n}$

Sol Let $f(x) = \frac{1}{x\log x}$, $x \geq 2$

$f(x) > 0$ and $f(x)$ is decreasing in $[2, \infty)$

$$F(t) = \int_2^t \frac{1}{x\log x}\,dx = \log(\log x)\Big]_2^t$$

$$= \log(\log t) - \log(\log 2)$$

$$\lim_{t\to\infty} F(t) = \lim_{t\to\infty} \log(\log t) - \log(\log 2)$$

$$= \infty$$

$\therefore$ By Integral test $\Sigma f(n)$ diverges

Ex: using integral test, determine the convergence of the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

Sol Let $f(x) = \frac{x}{x^2+1}$ for $x \in [1, \infty)$. Then

$$\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

We have $f(x) > 0$ and $f(x)$ is decreasing on $[1, \infty)$

$$F(t) = \int_1^t f(x)\,dx$$

$$= \int_1^t \frac{x}{x^2+1}\,dx$$

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$$= \frac{1}{2}\log(x^2+1)_1^t$$

$$= \frac{1}{2}\log(t^2+1) - \frac{1}{2}\log 2$$

$\therefore \lim_{t\to\infty} F(t) = \infty$.

Hence $\int_1^\infty f(x)\,dx$ is divergent

$\therefore$ By Integral test, $\sum_{n=1}^{\infty} f(n)$ is also divergent

CAUCHY'S n^{th} ROOT TEST :

If Σu_n is a series of Positive terms such that $\lim_{n\to\infty} u_n^{1/n} = l$ then

(a) Σu_n converges if $l < 1$

(b) Σu_n diverges if $l > 1$ and

(c) The test fails to decide the Nature if $l = 1$

Ex:1 Test for the convergence of $\Sigma \frac{2^n}{n^3}$

Sol This is a series of Positive terms

Let $u_n = \frac{2^n}{n^3}$. Then

$$\therefore \lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty} \frac{(2^n)^{\frac{1}{n}}}{(n^3)^{\frac{1}{n}}} = \frac{2}{n^{3/n}} = 2 > 1$$

By Cauchy's n^{th} root Test

Σu_n is divergent

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Ex Test for the convergence of $\sum \left(1+\frac{1}{n}\right)^{-n^2}$

Sol This is a series of positive terms

let $u_n = \left(1+\frac{1}{n}\right)^{-n^2}$

$$\therefore u_n^{1/n} = \left[\left(1+\frac{1}{n}\right)^{-n^2}\right]^{1/n} = \left(1+\frac{1}{n}\right)^{-n}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$\lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1 \quad (\because 2<e<3)$$

$\therefore$ By Cauchy's root test, $\sum u_n$ converges

Ex Test the convergence of the series

$$\frac{2}{1^2}x + \frac{3^2}{2^3}x^2 + \cdots + \frac{(n+1)^n}{n^{n+1}}x^n + \cdots \quad x>0$$

Sol This is a series of positive terms

$$u_n = \frac{(n+1)^n}{n^{n+1}}\cdot x^n$$

$$\therefore u_n^{1/n} = \frac{n+1}{n\cdot n^{1/n}}\cdot x$$

$$= \frac{1+\frac{1}{n}}{n^{1/n}}\cdot x$$

$$\lim_{n\to\infty} u_n^{1/n} = x \quad \left[\because \lim_{n\to\infty} n^{1/n} = 1\right]$$

By Cauchy's root test, $\sum u_n$ is convergent if $x<1$ and $\sum u_n$ is divergent if $x>1$ when $x=1$, the root test fails

when $x=1$, $u_n = \frac{(n+1)^n}{n^{n+1}} = \frac{n^n\left(1+\frac{1}{n}\right)^n}{n^n\cdot n} = \frac{\left(1+\frac{1}{n}\right)^n}{n}$

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Take $v_n = \frac{1}{n}$, $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \neq 0$

$\therefore$ By comparision test

Σu_n and Σv_n behave in the same way,

But Σv_n is divergent

$\therefore$ Σu_n is divergent

$\therefore$ Hence the Given series is convergent when $x<1$ and divergent when $x \geq 1$

Ex Test the convergence of the series $\Sigma \frac{1}{(\log(\log n))^n}$

Given series is $\Sigma u_n = \Sigma \frac{1}{(\log \log n)^n}$

n^{th} term $= u_n = \frac{1}{(\log \log n)^n}$

Now $(u_n)^{1/n} = \frac{1}{\log(\log n)}$

$\therefore \lim_{n\to\infty} (u_n)^{1/n} = 0 < 1$

By Cauchy's n^{th} root test, the series is convergent

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ALTERNATING SERIES:

A series whose terms are alternatively positive and negative is called an alternating series.

An Alternating series may be written as

$$u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n-1} u_n + \cdots$$

where each u_n is positive. it is denoted as

$$\sum_{n=1}^{\infty} (-1)^{n-1} u_n$$

Ex:1: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$

Ex:2: $1 - \frac{2}{\log 2} + \frac{3}{\log 3} - \frac{4}{\log 4} + \cdots = 1 + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{n}{\log n}$

LEIBNITZ'S TEST:

If $\{u_n\}$ is a sequence of positive terms such that

(a) $u_1 \geq u_2 \geq u_3 \geq \cdots \geq u_n \geq u_{n+1} \cdots$

(b) $\lim_{n \to \infty} u_n = 0$

then the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ is convergent

Ex: Prove that $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \cdots$ converges.

Sol: This is an alternating series and $u_n = \frac{1}{n!}$

Clearly $u_n > 0$ and $u_n > u_{n+1}$ for all "n"

Also $\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{n!} = 0$

$\therefore$ By Leibnitz's test $\sum \frac{(-1)^n}{n!}$ converges.

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x! Test for convergence of the series

$$\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \cdots \quad (0<x<1)$$

y This is an alternating series.

Consider $u_n = \frac{x^n}{1+x^n}$

Now $u_n - u_{n+1} = \frac{x^n}{1+x^n} - \frac{x^{n+1}}{1+x^{n+1}}$

$$= \frac{x^n(1+x^{n+1}) - (1+x^n)(x^{n+1})}{(1+x^n)(1+x^{n+1})}$$

$$= \frac{x^n(1-x)}{(1+x^n)(1+x^{n+1})} > 0 \quad \text{for } 0<x<1$$

$\therefore u_n > u_{n+1}$ i.e $u_1 > u_2 > u_3 - \cdots$

Also $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{x^n}{1+x^n} = \frac{0}{1+0} = 0 \quad [\because \lim_{n\to\infty} x^n = 0]$

By Leibnitz's test, the series is convergent.

x Test for convergence of the following series.

$$\frac{1}{6} - \frac{2}{11} + \frac{3}{16} - \frac{4}{21} + \frac{5}{26} - \cdots$$

Let the Given series be $\sum_{n=0} u_n$, where

$$u_n = (-1)^{n-1} \frac{n}{5n+1}$$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (-1)^{n-1} \frac{1}{5+\frac{1}{n}} \neq 0$

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MEAN VALUE THEOREMS

ROLLE'S THEOREM :

Let $f(x)$ be a function such that

① it is continuous in closed interval $[a\ b]$

② it is differentiable in open interval (a,b) and

③ $f(a) = f(b)$

Ex Verify Rolle's theorem for $f(x) = 2x^3+x^2-4x-2$ in $[-\sqrt{3},\sqrt{3}]$

Sol $f(x) = 2x^3+x^2-4x-2$ is a polynomial in x.

(1) $f(x)$ is continuous for every value of x.

in Particular, $f(x)$ is continuous in $[-\sqrt{3},\sqrt{3}]$

② $f'(x) = 6x^2+2x-4$ exists for every $x \in (-\sqrt{3},\sqrt{3})$

$\Rightarrow f(x)$ is derivable in $(-\sqrt{3},\sqrt{3})$

③ $f(-\sqrt{3}) = 2(-\sqrt{3})^3+(-\sqrt{3})^2-4(-\sqrt{3})-2$

$= -6\sqrt{3}+3+4\sqrt{3}-2$

$= 1-2\sqrt{3}$

and $f(\sqrt{3}) = 2(\sqrt{3})^3+(\sqrt{3})^2-4(\sqrt{3})-2$

$= 6\sqrt{3}+3-4\sqrt{3}-2$

$= 1+2\sqrt{3}$

since $f(-\sqrt{3}) \neq f(\sqrt{3})$, therefore

Rolle's theorem is not applicable for $f(x)$ in $[-\sqrt{3},\sqrt{3}]$

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Verify Rolle's theorem for the function

$$\log\left(\frac{x^2+ab}{x(a+b)}\right) \text{ in } [a,b],\ a>0,\ b>0$$

Let $f(x) = \log\left(\frac{x^2+ab}{x(a+b)}\right) = \log(x^2+ab) - \log(x(a+b))$

$$= \log(x^2+ab) - \log x - \log(a+b)$$

① since $f(x)$ is a composite function of continuous functions in $[a,b]$ it is continuous in $[a,b]$

② $f'(x) = \frac{1}{x^2+ab}\,2x - \frac{1}{x} = \frac{x^2-ab}{x(x^2+ab)}$

$\therefore f'(x)$ exists for all $x \in (a,b)$

③ $f(a) = \log\left(\frac{a^2+ab}{a^2+ab}\right) = \log 1 = 0$

$f(b) = \log\left(\frac{b^2+ab}{b^2+ab}\right) = \log 1 = 0$

$\therefore f(a) = f(b)$

Thus $f(x)$ satisfies all the three conditions of Rolle's theorem.

$\therefore$ There exists $c \in (a,b)$ such that $f'(c) = 0$

$\Rightarrow \frac{c^2-ab}{c(c^2+ab)} = 0 \Rightarrow c^2 = ab$

$\Rightarrow c = \pm\sqrt{ab}$

$c = \sqrt{ab} \in (a,b)$

Hence Rolle's theorem is verified

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Ex Verify Rolle's theorem for $f(x) = |x|$ in $[-1, 1]$

Sol we have $f(x) = |x|$

$f(x) = x$, for $x \geq 0$

$= -x$ for $x < 0$

① $f(x)$ is continuous for every value of "x"

$\therefore$ $f(x)$ is continuous in the closed interval $[-1, 1]$

② $f(x)$ is not derivable at $x = 0$

we have $f(0) = |0| = 0$

$$L\cdot H\cdot D = L\,f'(0) = \lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0-} \frac{|x| - 0}{x}$$

$$= \lim_{x \to 0-} \frac{-x}{x}$$

$$= \lim_{x \to 0-} (-1) = -1$$

$$R\cdot H\cdot D = R\,f'(0) = \lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0+} \frac{|x| - 0}{x}$$

$$= \lim_{x \to +} \frac{x}{x} = \lim_{x \to 0+} (1) = 1$$

Since $L\,f'(0) \neq R\,f'(0)$, therefore

$f(x)$ is not derivable at $x = 0$

$\therefore$ $f(x)$ is not derivable in the open interval $(-1, 1)$

Hence Rolle's theorem is not applicable to $f(x) = |x|$ in $[-1, 1]$

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LAGRANGE'S MEAN VALUE THEOREM

Let $f(x)$ be a function such that

① it is continuous in closed interval $[a,b]$ and

② differentiable in open interval (a,b)

then there exists at least one point "c" in open (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Ex Verify lagrange's mean value theorem for $f(x) = x^3 - x^2 - 5x + 3$ in $[0,4]$

Sol $f(x) = x^3 - x^2 - 5x + 3$ is a polynomial in x.

∴ it is continuous and derivable for every value of "x"

in particular, $f(x)$ is continuous in closed $[0,4]$ and derivable in open $(0,4)$. Hence by lagrange's mean value theorem, there exists a point "c" in open $(0,4)$ such that.

$$f'(c) = \frac{f(4)-f(0)}{4-0}$$

$$3c^2 - 2c - 5 = \frac{f(4)-f(0)}{4}$$

Now $f(4) = 4^3 - 4^2 - 5(4) + 3$

$= 31$

and $f(0) = 3$

$$\therefore \frac{f(4)-f(0)}{4} = \frac{31-3}{4} = \frac{28}{4} = 7$$

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from ① we have

$$3c^2 - 2c - 5 = 7 \Rightarrow 3c^2 - 2c - 12 = 0$$

$$\therefore c = \frac{2 \pm \sqrt{4+144}}{6}$$

$$= \frac{2 \pm \sqrt{148}}{6}$$

$$= \frac{1 \pm \sqrt{37}}{3}$$

We see that $\frac{1+\sqrt{37}}{3}$ lies in open (a,b) and then Lagrange's mean value theorem is verified.

Ex: If $a<b$ prove that $\frac{b-a}{(1+b^2)} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{(1+a^2)}$ using Lagrange's mean value theorem. Deduce the following

① $\frac{\pi}{4} + \frac{3}{35} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$

② $\frac{5\pi+4}{20} < \tan^{-1}2 < \frac{\pi+2}{4}$

Sol: Consider $f(x) = \tan^{-1}x$ in $[a,b]$ for $0<a<b<1$

since $f(x)$ is continuous in closed $[a,b]$ and derivable in open (a,b), we can apply Lagrange's mean value theorem here.

Hence there exists a point 'c' in open (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Here $f'(x) = \frac{1}{1+x^2}$ and hence $f'(c) = \frac{1}{1+c^2}$

Thus there exists a point c, $a<c<b$ such that

$$\frac{1}{1+c^2} = \frac{\tan^{-1}b - \tan^{-1}a}{b-a}$$

We have $1+a^2 < 1+c^2 < 1+b^2$

$\Rightarrow$ $\frac{1}{1+a^2}$ $\frac{1}{1+c^2}$ $\frac{1}{1+b^2}$

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using ① and ② we have

$$\frac{1}{1+a^2} > \frac{\tan^{-1}b - \tan^{-1}a}{b-a} > \frac{1}{1+b^2}$$

(or) $$\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$$

① Taking $b = 4/3$, $a = 1$ we get

$$\frac{\frac{4}{3}-1}{1+16/9} < \tan^{-1}4/3 - \tan^{-1}1 < \frac{\frac{4}{3}-1}{1+1^2}$$

$$\Rightarrow \frac{4-3}{\frac{25}{9}} < \tan^{-1}4/3 - \frac{\pi}{4} < \frac{\frac{1}{3}}{2}$$

$$\Rightarrow \frac{3}{25} + \frac{\pi}{4} < \tan^{-1}4/3 < \frac{\pi}{4} + \frac{1}{6}$$

② Taking $b=2$ and $a=1$ we get

$$\frac{2-1}{1+2^2} < \tan^{-1}2 - \tan^{-1}1 < \frac{2-1}{1+1^2}$$

$$\Rightarrow \frac{1}{5} < \tan^{-1}2 - \frac{\pi}{4} < \frac{1}{2}$$

$$\Rightarrow \frac{1}{5} + \frac{\pi}{4} < \tan^{-1}2 < \frac{1}{2} + \frac{\pi}{4}$$

$$\Rightarrow \frac{4+5\pi}{20} < \tan^{-1}2 < \frac{2+\pi}{4}$$

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CAUCHY'S MEAN VALUE THEOREM

if $f:[a,b] \to R$, $g:[a,b] \to R$ are such that

① f, g are continuous on $[a,b]$

② f, g are differentiable on (a,b) and

③ $g'(x) \neq 0 \quad \forall \; x \in (a,b)$

then there exists a point $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Another form of cauchy's Mean Value Theorem:

if $f:[a,a+h] \to R$, $g:[a,a+h] \to R$ are such that

① f, g are continuous on $[a,a+h]$

② f, g are differentiable on $(a,a+h)$

③ $g'(x) \neq 0, \quad \forall \; x \in (a,a+h)$

then there exists $0 < \theta < 1$ such that

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)}$$

Prob Verify cauchy's mean value theorem for $f(x) = x^2$, $g(x) = x^3$ in $[1,2]$

Sol We have $f(x) = x^2$, $g(x) = x^3$ are of the form $x^n, n \in Z$

$\therefore$ f, g are continuous on $[1,2]$ and f, g are differentiable on $(1,2)$

Also $g'(x) = 3x^2 \neq 0 \quad \forall \; x \in (1,2)$

$\therefore$ f, g are satisfy the conditions of cauchy's mean value theorem

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There exists $c \in (a,b)$ such that

$$\frac{f(2)-f(1)}{g(2)-g(1)} = \frac{f'(c)}{g'(c)} \Rightarrow \frac{4-1}{8-1} = \frac{2c}{3c^2} \Rightarrow c = \frac{14}{9} \in (1,2)$$

This verifies cauchy's mean value theorem

TAYLOR'S THEOREM :

If $f:[a,b] \to R$ is such that

(a) f^{n-1} is continuous on $[a,b]$

(b) f^{n-1} is derivable on (a,b) (or)

f^n exists on (a,b) and $p \in z^+$ then there exist a point $c \in (a,b)$ such that

$$f(b) = f(a) + \frac{b-a}{1!}f'(a) + \frac{(b-a)^2}{2!}f''(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!}f^{n-1}(a) + R_n$$

where $R_n = \dfrac{(b-a)^p (b-c)^{n-p} \cdot f^n(c)}{(n-1)!\,p}$

Prob: Verify Taylor's theorem for $f(x) = (1-x)^{5/2}$ with lagrange's form of remainder upto 2 terms in the interval $[0,1]$

Sol: Consider $f(x) = (1-x)^{5/2}$ in $[0,1]$

① $f(x)$, $f'(x)$ are continuous in $[0,1]$

② $f''(x)$ is differentiable in $(0,1)$

Thus $f(x)$ satisfies the conditions of Taylor's theorem.

We consider Taylor's theorem with lagrange's form of Remainder

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$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(\theta x) \quad \text{with} \quad 0<\theta<1$$

Here $n = p = 2$, $a = 0$, and $x = 1$

$$f(x) = (1-x)^{5/2} \Rightarrow f(0) = 1$$

$$f'(x) = \frac{-5}{2}(1-x)^{3/2} \Rightarrow f'(0) = \frac{-5}{2}$$

$$f''(x) = \frac{15}{4}(1-x)^{1/2} \Rightarrow f''(\theta x) = \frac{15}{4}(1-\theta x)^{1/2}$$

$$f''(\theta) = \frac{15}{4}(1-\theta)^{1/2}$$

and $f(1) = 0$

we have $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(\theta x)$

substituting the above values we get

$$0 = 1 + 1\left(\frac{-5}{2}\right) + \frac{1^2}{2!} \cdot \frac{15}{4}(1-\theta)^{1/2}$$

$$\Rightarrow \theta = \frac{9}{25} = 0.36$$

$\therefore \theta$ lies between "0 and 1"

Thus the Taylor's theorem is verified

MACLAURIN'S THEOREM

if $f:[0,x] \to R$ is such that

① f^{n-1} is continuous on $[0,x]$

② f^{n-1} is derivable on $(0,x)$ and $p \in z^+$ then there exists a real number $\theta \in (0,1)$ such that

$$f(x) = f(0) + x f'(0) + \frac{x}{2!} f(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n (1-\theta)^{n-p}}{p(n-1)!} f^n(\theta x)$$

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Prove: show that $\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \cdots$ and hence deduce that

$$\frac{e^x}{e^x+1} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \cdots$$

Sol

$$f(x) = \log(1+e^x) \Rightarrow f(0) = \log 2$$

$$f'(x) = \frac{1}{1+e^x} e^x \Rightarrow f'(0) = \frac{1}{2}$$

$$f''(x) = \frac{(1+e^x)e^x - e^x(e^x)}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2} \Rightarrow f''(0) = \frac{1}{4}$$

$$f'''(x) = \frac{(1+e^x)^2 e^x - e^x \cdot 2(1+e^x)e^x}{(1+e^x)^4}$$

$$= \frac{(1+e^x)e^x - 2e^{2x}\cdot e^x}{(1+e^x)^3}$$

$$= \frac{e^x(1+e^x-2e^x)}{(1+e^x)^3} \Rightarrow f'''(0) = 0$$

$$f^4(x) = \frac{(1+e^x)^3(e^x-2e^{2x}) - (e^x-e^{2x})3(1+e^x)^2 e^x}{(1+e^x)^6}$$

$$= \frac{(1+e^x)^3(e^x-2e^{2x}) - 3e^x(e^x-e^{2x})}{(1+e^x)^4}$$

$$= \frac{e^{3x} - 4e^{2x} + e^x}{(1+e^x)^4}$$

$$f^4(0) = \frac{-2}{16} = -\frac{1}{8}$$

using Maclaurins series

$$f(x) = f(0) + f'(0) + \frac{f''(0)}{2!} + \cdots$$

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$\therefore \log(1+e^x) = \log 2 + \frac{1}{2}x + \frac{x^2}{8} - \frac{x^4}{192} + \cdots \infty$

Diff w.r.t 'x' on both sides, we get

$\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \cdots$ Hence the result

Prob: ~~show that~~ obtain the Maclaurin's series Expansion for $f(x) = (1+x)^n$

Sol Let $f(x) = (1+x)^n$, Then

$f'(x) = n(1+x)^{n-1}$; $f''(x) = n(n-1)(1+x)^{n-2}$

$f'''(x) = n(n-1)(n-2)(1+x)^{n-3}$ etc

Hence $f^k(x) = n(n-1)(n-2)\cdots(n-k+1)(1+x)^{n-k}$

$f^k(0) = n(n-1)\cdots(n-k+1)$

Hence $f(x) = \sum_{k=0}^{\infty} f^k(0)\frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{n(n-1)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdots k} x^k$

i.e $(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 + \cdots$

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