SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS.

Method 1:

Bisection Method,

Consider an equation $f(x) = 0$, find a, b, So that $f(a), f(b)$ have opposite signs

$$x_1 = \frac{a+b}{2}$$

i) If $f(x_1) < 0$ is negative then root lies b/w x_1 & b

$$f(x_2) = \frac{x_1+b}{2}$$

ii) If $f(x_1) > 0$ is positive, then root lies b/w a and x_1

$$f(x_2) = \frac{a+x_1}{2}$$

problem:

1) Find the positive root of $x^3 - x - 1 = 0$ correct two decimal places by bisection method.

Sol: let $f(x) = x^3 - x - 1 = 0$

Consider $f(x) = x^3 - x - 1$

$f(0) = 0 - 0 - 1 = -1$ (−ve)

$f(1) = 1 - 1 - 1 = -1$ (−ve)

$f(2) = 8 - 2 - 1 = 5$ (+ve)

Since $f(1) < 0$ and $f(2) > 0$

∴ The root lies b/w 1 and 2.

let $a=1$, $b=2$

by bisection method,

$$x_1 = \frac{a+b}{2} = \frac{1+2}{2} = \frac{3}{2} = 1.5$$

$$f(x_1) = f(1.5) = (1.5)^3 - (1.5) - 1 = 0.875 \ (+ve)$$

Since $f(1) < 0$ and $f(2) > 0$

$\therefore$ The root lies b/w 1 & 1.5

$$x_2 = \frac{x_1 + b}{2} = \frac{1.5+1}{2} = \frac{2.5}{2} = 1.25 \ (+ve)$$

$$f(x_2) = f(1.25) = (1.25)^3 - (1.25) - 1$$
$$= -0.296 \ (-ve)$$

Since $f(1.25) < 0$ and $f(1.5) > 0$

$\therefore$ The root lies b/w 1.25 and 1.5

$$\therefore x_3 = \frac{1.25 + 1.5}{2} = \frac{2.75}{2} = 1.375 \ (+ve)$$

$$f(x_3) = f(1.375) = (1.375)^3 - (1.375) - 1$$
$$= 0.2246 \ (+ve)$$

Since $f(1.25) < 0$ and $f(1.375) > 0$

$\therefore$ The root lies b/w 1.25 & 1.375

$$\therefore x_4 = \frac{1.25 + 1.375}{2} = \frac{2.625}{2} = 1.3125 \ (+ve)$$

$$f(x_4) = f(1.3125) = (1.3125)^3 - (1.3125) - 1$$
$$= -0.0515 \ (-ve)$$

Since $f(1.3125) < 0$ and $f(1.375) > 0$.

Let $a = 1$, $b = 2$

by bisection method,

$x_1 = \frac{a+b}{2} = \frac{1+2}{2} = \frac{3}{2} = 1.5$

$f(x_1) = f(1.5) = (1.5)^3 - (1.5) - 1 = 0.875$ (+ve)

Since $f(1) < 0$ and $f(2) > 0$

$\therefore$ The root lies b/w 1 & 1.5

$x_2 = \frac{x_1 + b}{2} = \frac{1.5+1}{2} = \frac{2.5}{2} = 1.25$ (+ve)

$f(x_2) = f(1.25) = (1.25)^3 - (1.25) - 1$
$= -0.296$ (-ve)

Since $f(1.25) < 0$ and $f(1.5) > 0$

$\therefore$ The root lies b/w 1.25 and 1.5

$\therefore x_3 = \frac{1.25 + 1.5}{2} = \frac{2.75}{2} = 1.375$ (+ve)

$f(x_3) = f(1.375) = (1.375)^3 - (1.375) - 1$
$= 0.2246$ (+ve)

Since $f(1.25) < 0$ and $f(1.375) > 0$

The root lies b/w 1.25 & 1.375

$\therefore x_4 = \frac{1.25 + 1.375}{2} = \frac{2.625}{2} = 1.3125$ (+ve)

$f(x_4) = f(1.3125) = (1.3125)^3 - (1.3125) - 1$
$= -0.0515$ (-ve)

Since $f(1.3125) < 0$ and $f(1.375) > 0$.

$\therefore$ The root lies b/w 1.3125 & 1.375

$\therefore x_5 = \frac{1.3125 + 1.375}{2} = \frac{2.6875}{2} = 1.343$ (+ve)

$f(x_5) = f(1.343) = (1.343)^3 - (1.343) - 1 = 0.082$ (+ve)

Since $f(1.343) > 0$ and $f(1.3125) < 0$

$\therefore$ The root lies b/w 1.3125 & 1.343

$\therefore x_6 = \frac{1.3125 + 1.343}{2} = \frac{2.6563}{2} = 1.3281$

$f(x_6) = f(1.3281) = (1.3281)^3 - (1.3281) - 1 = 0.0145$

Since $f(1.3125) < 0$ and $f(1.3281) > 0$

$\therefore$ The root lies b/w 1.3125 & 1.3281

$\therefore x_7 = \frac{1.3125 + 1.3281}{2} = \frac{2.6406}{2} = 1.3203$

upto 2 decimals places x_7, x_6 are same.

$\therefore$ The root of the given Eq is 1.32.

Method 2: Method of false position:-

Consider an Equation $f(x) = 0$, find a, b, so that $f(a), f(b)$ have opposite sign

$$x = \frac{a \cdot f(b) - b \cdot f(a)}{f(b) - f(a)}$$

Problem:

1\. Find the root of the Eqn $x \log_{10} x = 1.2$. Using false position method.

Sol: Let $f(x) = x \log_{10} x - 1.2$

Consider

$f(2) = 2 \times \log_{10}(2) - 1.2 = 2 \times 0.3010 - 1.2 = -0.5979$ (-ve)

$f(3) = 3 \times \log_{10}(3) - 1.2 = 3 \times 0.47712 - 1.2 = 0.23136$ (+ve)

Since $f(2) < 0$ and $f(3) > 0$ have opposite signs

$\therefore$ The root lies b/w 2 and 3.

By false position method.

$a = 2, \; b = 3$.

$$x = \frac{a \cdot f(b) - b\, f(a)}{f(b) - f(a)} = \frac{2 \times (0.23136) - 3 \times (-0.59794)}{0.23136 - (-0.59794)}$$

$$= 2.7210$$

$f(x) = f(2.7210) = 2.7210 \log_{10}(2.7210) - 1.2$

$= -0.0171$

$\therefore$ The root lies b/w 2.721 & 3

$$x_1 = \frac{2.721 \times 0.23136 - 3 \times (-0.0171)}{0.23136 - (-0.0171)} = 2.740$$

$f(x_1) = f(2.740) = 2.740 \log_{10}(2.740) - 1.2$

$= -0.00056$

The root lies b/w 2.740 and 3.

$$\therefore x_2 = \frac{2.740 \times (0.23168) - 3 \times (-0.00056)}{0.23136 - (-0.00056)} = 2.7406$$

upto two decimal places x_1, x_2 are same

$\therefore$ The root is 2.74

Method 3:- Iteration method:-

Given equation $f(x) = 0$

we write $x = \varphi(x)$, so that $|\varphi'(x)| < 1$

find a,b so that $f(a)$, $f(b)$ have opposite signs

Initial approximation by bisection method,

$$x_0 = \frac{a+b}{2}$$

1st approximation $x_1 = \varphi(x_0)$

2nd approximation $x_2 = \varphi(x_1)$

$\vdots$

nth approximation $x_n = \varphi(x_{n-1})$

Problem:

Find the positive root of $x^4 - x - 10 = 0$ by iteration method.

Given that $x^4 - x - 10 = 0$

$$x = x^4 - 10$$

$$x = \frac{10}{x^3 - 1}$$

It can be written as $x = \varphi(x)$

$x = (x+10)^{1/4}$, satisfies the converge criteria

$|\varphi'(x)| < 1$

we take iteration formula as

$$x_{i+1} = (x_i + 10)^{1/4} ; \quad i = 0, 1, 2, \ldots$$

we observe $f(1) < 0$, $f(2) > 0$ for the given equation

Hence the root lies b/w 1 and 2.

$$x_0 = \frac{1+2}{2} = 1.5$$

$$x_1 = (1.5 + 10)^{1/4} = 1.8415$$

$$x_2 = (1.8415 + 10)^{1/4} = 1.8550$$

$$x_3 = (1.8550 + 10)^{1/4} = 1.8556$$

$$x_4 = (1.8556 + 10)^{1/4} = 1.8556.$$

Hence the root is 1.8556 correct to '4' decimal places.

Method 4: Newton - Raphson method:

Given $f(x) = 0$

Find a, b, so that f(a), f(b) have opposite sign.

Find initial approximation, $x_0 = \frac{a+b}{2}$

1^{st} approximation, $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

2^{nd} approximation $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$\vdots$

n^{th} approximation $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$

Problem

Find the real root of the equation $x \cdot e^x - \cos x = 0$ using Newton-Raphson method.

Given $xe^x - \cos x = 0$.

let $f(x) = xe^x - \cos x$

find x_1 and x_2 such that $f(x_1)$ and $f(x_2)$ are of opposite signs

$\therefore$ Root of the equation lies b/w x_1 and x_2.

$f(x) = xe^x - \cos x$

$f'(x) = (x+1)e^x + \sin x$

Now $f(0) = 0 - \cos 0 = -1$ (-ve)

$f'(0) = 1 + \sin 0 = 1$

$f(1) = e - \cos 1 = 2.17797$ (+ve)

$f'(1) = 6.27807$.

$\therefore$ The root lies b/w 0 and 1.

let $x_0 = \frac{x_1 + x_2}{2} = \frac{0+1}{2} = 0.5$

by Newton Raphson method.

$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$

put $i = 0$

$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= 0.5 - \frac{(-0.0532)}{\ldots} = 0.5310$

$f(x_1) = 0.0407, \quad f'(x_1) = 3.1106$

put $i=1$,

$\therefore x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 0.5310 - \dfrac{0.4073}{3.1106}$

$= 0.5179$

$f(x_2) = 0.0004, \quad f'(x_2) = 3.0428.$

put $i=2$,

$\therefore x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$

$= 0.5179 - \dfrac{0.0004}{3.0428}$

$= 0.5177$

$f(x_3) = 0.0000011068 73, \quad f'(x_3) = 3.042121853.$

put $i=3$,

$\therefore x_4 = x_3 - \dfrac{f(x_3)}{f'(x_3)}$

$= 0.517757 - \dfrac{0.000001106873}{3.042121853}$

$= 0.517757$

$\therefore$ Since $x_3 = x_4 = 0.517757$.

$\therefore$ The Root of the equation is 0.517757.

⇒ obtain Newton – Raphson formula to find the $\sqrt{N}$ and hence deduce the value of $\sqrt{8}$

A. Let $x = \sqrt{N}$

$\Rightarrow x^2 = N$ (squaring on bothsides)

$\Rightarrow x^2 - N = 0$.

$\Rightarrow f(x) = x^2 - N$

$f'(x) = 2x$.

Newton-Raphson formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$x_{n+1} = x_n - \frac{x_n^2 - N}{2x_n}.$$

$$= \frac{2x_n^2 - (x_n^2 - N)}{2x_n}$$

$$x_{n+1} = \frac{x_n^2 + N}{2x_n} \quad \text{— (1)}.$$

To find $\sqrt{8}$, take $N = 8$ in (1).

$f(x) = x^2 - N \Rightarrow f(2.8) = (2.8)^2 - 8 = -0.16 < 0$

$f(2.9) = (2.9)^2 - 8 = 0.14 > 0$.

root lies b/w 2.8 and 2.9

Take $x_0 = \dfrac{2.8 + 2.9}{2} = 2.85$

From (1) Put $n=0$; $x_0 = 2.85$, $N = 8$

$$x_1 = \frac{x_0^2 + 8}{2x_0} = \frac{(2.85)^2 + 8}{2(2.85)} = \frac{16.1225}{5.7} = 2.8285.$$

$$x_2 = \frac{(2.8285)^2 + 8}{2(2.8285)} = \frac{16.0004}{5.657} = 2.8284$$

x_1 & x_2 are equal upto 2 decimal places.

⇒ Obtain Newton-Raphson formula to find $\sqrt[3]{N}$ and hence deduce the value of $\sqrt[3]{12}$.

A). Let $x = (N)^{1/3} \Rightarrow x^3 = N \Rightarrow x^3 - N = 0$

Let $f(x) = x^3 - N$

$f'(x) = 3x^2$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 - N}{3x_n^2} = \frac{3x_n^3 - (x_n^3 - N)}{3x_n^2}$$

$$x_{n+1} = \frac{2x_n^3 + N}{3x_n^2} \quad — (1)$$

To find $\sqrt[3]{12}$, put $N = 2$ in (1).

$$x_0 = \frac{2.2 + 2.3}{2} = 2.25$$

Put $n = 0$, $N = 12$, $x_0 = 2.25$ in (1).

$$x_1 = \frac{2(2.25)^3 + 12}{3(2.25)^2} = \frac{34.7812}{15.1875} = 2.2901$$

$$x_2 = \frac{2(2.2901)^3 + 12}{3(2.2901)^2} = \frac{36.0211}{15.7336} = 2.2894$$

$$x_3 = \frac{2(2.2894)^3 + 12}{3(2.2894)} = \frac{35.9991}{15.724} = 2.2894$$

x_2 and x_3 are equal upto 4 decimals.

cube root of 12 = 2.2894.

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pb:- Obtain Newton-Raphson formula for reciprocal of N and hence deduce the value of 1/31

Sol:- Let $x = \frac{1}{N} \Rightarrow N = \frac{1}{x}$

$f(x) = N - \frac{1}{x} = 0 \Rightarrow f'(x) = \frac{1}{x^2}$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{\left(N - \frac{1}{x_n}\right)}{\frac{1}{x_n^2}} = x_n - \left(\frac{Nx_n - 1}{\cancel{x_n}}\right) x_n^{\cancel{2}}$$

$$x_{n+1} = x_n - (Nx_n - 1)x_n$$

$$= x_n - Nx_n^2 + x_n$$

$$x_{n+1} = 2x_n - Nx_n^2 \quad \text{--- (1)}.$$

To find the value of $\frac{1}{31}$ put $N = 31$

$f(x) = 31 - \frac{1}{x}$

$$x_{n+1} = 2x_n - 31x_n^2$$

root lies b/w 0.03 & 0.04

$$x_0 = \frac{0.03 + 0.04}{2} = 0.035.$$

Put $n = 0$, $x_0 = 0.035$, $N = 31$ in (1).

$$x_1 = 2x_0 - 31x_0^2 = 2(0.035) - 31(0.035)^2$$

$$x_1 = 0.0320.$$

$$x_2 = 2(0.0320) - 31(0.0320)^2$$

$$= 0.0323.$$

$$x_3 = 2(0.0323) - 31(0.0323)^2$$

$$= 0.0323.$$

→ Find a real root for $x\tan x + 1 = 0$ using Newton-Raphson method.

Sol: Given $f(x) = x\tan x + 1 = 0$.

$f'(x) = x\sec^2 x + \tan x$.

Now $f(2) = 2\tan 2 + 1 = -3.370079 < 0$.

$f(3) = 3\tan 3 + 1 = 0.572370 > 0$.

root lies b/w '2' and '3'.

Let $x_0 = \frac{2+3}{2} = 2.5$.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.5 - \frac{-0.86755}{3.14808} = 2.77558.$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.77558 - \frac{(-0.06383)}{2.80004} = 2.798.$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.798 - \frac{-0.0010803052}{2.7983} = 2.798$$

Since $x_2 = x_3$; the real root is 2.798.

→ Errors in polynomial Interpolation

Let the function $y(x)$ defined by $(n+1)$ points (x_i, y_i) $i = 0,1,2,\ldots n$ be continuous and differentiable $(n+1)$ times and let $y(x)$ be approximated by a polynomial $\phi_n(x)$ of degree not exceeding 'n' such that

$$\phi_n(x_i) = y_i \; ; \; i = 0,1,2,\ldots n$$

required expression for error is

$$y(x) - \phi_n(x) = \frac{\Pi_{n+1}(x)}{(n+1)!} y^{n+1}(\xi), \; x_0 < \xi < x_n$$

Newton – Raphson Method for Solution of Simultaneous Equations

Let (x_0, y_0) be an initial approximation to the root of the equation. If (x_0+h, y_0+k) is the root of the system, then we must have $f(x_0+h, y_0+k) = 0$, $g(x_0+h, y_0+k) = 0$ — ①

Assuming that f and g are sufficiently differentiable, we expand both the functions in ①, by Taylor's series to obtain

$$\left.\begin{aligned} f_0 + h\frac{\partial f}{\partial x_0} + k\frac{\partial f}{\partial y_0} + \cdots &= 0 \\ g_0 + h\frac{\partial g}{\partial x_0} + k\frac{\partial g}{\partial y_0} + \cdots &= 0 \end{aligned}\right\} \text{— ②}$$

where $\frac{\partial f}{\partial x_0} = \left[\frac{\partial f}{\partial x}\right]_{x=x_0}$, $f_0 = f(x_0, y_0)$ etc

Neglecting the second and higher-order derivative terms, we obtain the following system of linear equations.

$$\left.\begin{aligned} h\frac{\partial f}{\partial x_0} + k\frac{\partial f}{\partial y_0} &= -f_0 \\ \text{and} \quad h\frac{\partial g}{\partial x_0} + k\frac{\partial g}{\partial y_0} &= -g_0 \end{aligned}\right\} \text{— ③}$$

Equation ③ possesses a unique solution if.

$$D = \begin{vmatrix} \frac{\partial f}{\partial x_0} & \frac{\partial f}{\partial y_0} \\ \frac{\partial g}{\partial x_0} & \frac{\partial g}{\partial y_0} \end{vmatrix} \neq 0 \quad \text{— ④}$$

By crammer's rule, the solution of Eq. ③ is given by.

$$h = \frac{1}{D}\begin{vmatrix} -f_0 & \frac{\partial f}{\partial y_0} \\ -g_0 & \frac{\partial g}{\partial y_0} \end{vmatrix} \quad \text{and} \quad k = \frac{1}{D}\begin{vmatrix} \frac{\partial f}{\partial x_0} & -f_0 \\ \frac{\partial g}{\partial x_0} & -g_0 \end{vmatrix}.$$

~~The new approximations are $x_1 = x_0+h$, and $y_1 = y_0+k$~~

The process is to be repeated till we obtain the roots

⇒ Solve the system of equations $3yx^2 - 10x + 7 = 0$, $y^2 - 5y + 4 = 0$. by using Newton-Raphson method.

A) we have $f(x) = 3yx^2 - 10x + 7 = 0$, $g(x) = y^2 - 5y + 4 = 0$.

Then $\frac{\partial f}{\partial x} = 6yx - 10$, $\frac{\partial f}{\partial y} = 3x^2$, $\frac{\partial g}{\partial x} = 0$, $\frac{\partial g}{\partial y} = 2y - 5$.

Taking $x_0 = y_0 = 0.5$, we obtain

$\frac{\partial f}{\partial x_0} = -8.5$, $\frac{\partial f}{\partial y_0} = 0.75$, $f_0 = 2.375$.

$\frac{\partial g}{\partial x_0} = 0$, $\frac{\partial g}{\partial y_0} = -4$, $g_0 = 1.75$.

Hence $D = \begin{vmatrix} -8.5 & 0.75 \\ 0 & -4 \end{vmatrix} = 34 \neq 0$.

$$h = \frac{1}{34}\begin{vmatrix} -2.375 & 0.75 \\ -1.75 & -4 \end{vmatrix} = 0.3180$$

$$k = \frac{1}{34}\begin{vmatrix} -8.5 & -2.375 \\ 0 & -1.75 \end{vmatrix} = 0.4375$$

It follows that $x_1 = 0.5 + 0.3180 = 0.8180$

$y_1 = 0.5 + 0.4375 = 0.9375$.

For the second approximation, we have

$f_1 = 0.7019$, $g_1 = 0.1914$

$\frac{\partial f_1}{\partial x} = -5.3988$, $\frac{\partial f}{\partial y_1} = 2.0074$

$\frac{\partial g}{\partial x_1} = 0$, $\frac{\partial g}{\partial y_1} = -3.125$.

$$D = \begin{vmatrix} -5.3988 & 2.0074 \\ 0 & -3.125 \end{vmatrix} = 16.8712 \neq 0.$$

$$h = \frac{1}{16.8712}\begin{vmatrix} -0.7019 & 2.0074 \\ -0.1914 & -3.125 \end{vmatrix} = 0.1528.$$

$$k = \frac{1}{16.8712}\begin{vmatrix} -5.3918 & -0.7019 \\ 0 & -0.1914 \end{vmatrix} = 0.0612$$

It follows that

$$x_2 = 0.8180 + 0.1528 = 0.9708$$

$$y_2 = 0.9375 + 0.0612 = 0.9987$$

⇒ solve the system $x^2+y^2=1$ and $y=x^2$ by Newton-Raphson method.

Sol Let $f = x^2+y^2-1$ and $g = y-x^2$

From the graphs of the curves, we find that there are two points of intersection, one each in the first and second quadrants. we shall approximate to the solution in the first quadrant. we have.

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y, \quad \frac{\partial g}{\partial x} = -2x, \quad \frac{\partial g}{\partial y} = 1.$$

we start with $x_0 = y_0 = 0.7071$ obtained the approximation $y=x$, Then we compute.

$$\frac{\partial f}{\partial x_0} = 1.4142, \quad \frac{\partial f}{\partial y_0} = 1.4142$$

$$\frac{\partial g}{\partial x_0} = -1.4142, \quad \frac{\partial g}{\partial y_0} = 1.$$

$$D = \begin{vmatrix} 1.4142 & 1.4142 \\ -1.4142 & 1 \end{vmatrix} = 3.4142 \neq 0; \quad f_0 = 0$$

$$h = \frac{1}{3.4142}\begin{vmatrix} 0 & 1.4142 \\ -0.2071 & 1 \end{vmatrix} = 0.0858$$

$$k = \frac{1}{3.4142}\begin{vmatrix} 1.4142 & 0 \\ -1.4142 & -0.2071 \end{vmatrix} = -0.0858$$

It follows, therefore

$$x_1 = 0.7071 + 0.0858 = 0.7858.$$

$$y_1 = 0.7071 - 0.0858 = 0.6213.$$

The process can be repeated to obtain a better approximation.

⇒ Solve the system $\sin x - y = -0.9793$
$\cos y - x = -0.6703$

with $x_0 = 0.5$ and $y_0 = 1.5$ as the initial approximation.

Sol:

$$f(x,y) = \sin x - y + 0.9793$$
$$g(x,y) = \cos y - x + 0.6703.$$

$$f_0 = -0.0413, \quad g_0 = 0.2410.$$

$$D = f_x g_y - g_x f_y = \cos(0.5)(-\sin 1.5) - (1) = -1.8754.$$

$$h = \frac{f g_y - g f_y}{D} = 0.1505, \quad K = \frac{g f_x - f g_x}{D} = -0.0908$$

$$x = 0.5 + 0.1505 = 0.6505.$$
$$y = 1.5 + 0.0908 = 1.5908.$$

For the second Iteration, we have $x_0 = 0.6505$, and $y_0 = 1.5908$; we obtain $D = -1.7956$.

$h = -0.003181$ and $K = 0.003384$.

Hence the new approximation is

$$x = 0.6505 + 0.0032 = 0.6537.$$

INTERPOLATION

* Forward Differences

Consider a function $y = f(x)$ of an idependent variable 'x'. Let $y_0, y_1, y_2, ---, y_r$ be the value of y corresponding to the values $x_0, x_1, x_2, ---, x_r$ of x respectively. Then the differences $y_1 - y_0, y_2 - y_1, ----$ are called the first forward Differences of y, and we denoted them by $\Delta y_0, \Delta y_1, ---$; that is $\Delta y_0 = y_1 - y_0$, $\Delta y_1 = y_2 - y_1, ---$

In general, $\boxed{\Delta y_r = y_{r+1} - y_r}$, $r = 0, 1, 2 ----$

Here, the symbol 'Δ' is called the Forward Difference Operator.

This is Rewrite

$$\boxed{\Delta f(x) = f(x+h) - f(x)}$$

* Backward Differences

Consider a function $y = f(x)$ of an idependent Variable 'x'

Let $y_0, y_1, y_2 ---, y_r$ be the value of y corresponding to the values $x_0, x_1, x_2, ---, x_r$ of x respectively. Then

$\nabla y_1 = y_1 - y_0$, $\nabla y_2 = y_2 - y_1$, $\nabla y_3 = y_3 - y_2, -----$ are called the first backward Differences.

In general, $\nabla y_r = y_r - y_{r-1}$, $r = 1, 2, 3 ----$

The Symbol '∇' is called Backward Difference operator.

$$\boxed{\nabla f(x) = f(x) - f(x-h)}$$

* Newton's Forward Formula:-

We write x_0, x, h find $\boxed{P = \frac{x - x_0}{h}}$

$$y(x) = y(x_0 + Ph) = y_0 + \frac{P}{1!}\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \cdots$$

Problems:-

1. If $y_0 = 1, y_1 = 0, y_2 = 5, y_3 = 22, y_4 = 57$. Find $y_{0.5}$ by using newton Forward interpolation Formula.

Sol Forward Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		-1			
1	0		6		
		5		6	
2	5		12		0
		17		6	
3	22		18		
		35			
4	57				

Given that $x = 0.5, x_0 = 0, h = 1$

$$P = \frac{x - x_0}{h} = \frac{0.5 - 0}{1} = 0.5$$

$P = 0.5$

By Newton's Forward interpolation Formula

$$y_x = f(x) = f(x_0 + Ph)$$

$$= y_0 + \frac{P}{1!}\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \cdots$$

$$y_{0.5} = 1 + \frac{0.5}{1}(-1) + \frac{0.5(-0.5)}{2}(6) + \frac{0.5(-0.5)(-1.5)}{6}(6)$$

$$y_{0.5} = 0.125$$

* Newton's Backward Formula:-

We write x_n, x, h find $\boxed{P = \frac{x - x_n}{h}}$

$$y(x) = y(x_n + ph)$$
$$= y_n + \frac{p}{1!}\nabla y_0 + \frac{p(p+1)}{2!}\nabla^2 y_0 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_0 + \cdots$$

Problems:-

1\. Apply Newton Backward Interpolation Formula to compute the $y_{5.5}$ given that $y_5 = 2.236$, $y_6 = 2.449$, $y_7 = 2.648$, $y_8 = 2.828$

Sol Backward Difference Table

x	$y = \sqrt{x}$	∇y	$\nabla^2 y$	$\nabla^3 y$
5	2.236			
		0.213		
6	2.449		-0.04	
		0.199		-0.005
7	2.648		-0.019	
		0.18		
8	2.828			

Given that

$x_n = x_3 = 8$, $x = 5.5$, $h = 1$

$$p = \frac{x - x_n}{1} = \frac{5.5 - 8}{1} = -2.5$$

By Newton Backward Interpolation Formula,

$$y_{5.5} = f(5.5) = y_3 + \frac{p}{1!}\nabla y_3 + \frac{p(p+1)}{2!}\nabla^2 y_3 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_3$$

$$= 2.828 - \frac{2.5}{1}(0.180) + \frac{(-2.5)(-2.5+1)}{2!}(0.014)$$

$$+ \frac{(-2.5)(-2.5+1)(-2.5+2)}{3!}(0.005)$$

$$= 2.828 - 0.4500 + 0.0263 + 0.0016$$

$$= 2.4059$$

* **Gauss Forward Interpolation:-**

Let x_0 near to x find $p = \frac{x - x_0}{h}$

$$y(x) = y(x_0 + ph) = y_0 + \frac{p}{1!}\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \cdots\cdots$$

Problem:-

Q Find the value of y at $x = 1936$ using the following data by Gauss Forward interpolation.

x	1901	1911	1921	1931	1941	1951
y	12	15	20	27	39	52

Sol: Given that $x = 1936$, $x_0 = 1931$, $h = 10$

$$p = \frac{x - x_0}{h} = \frac{1936 - 1931}{10} = 0.5$$

12.

27

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1901	12					
		3				
			2			
1911	15			0		
		5			3	
1921	20		2			−10
		7		3		
					−7	
1931	27		5			
		12		−4		
1941	39		1			
		13				
1951	52					

Now Gauss Forward Interpolation Formula.

$$y_x = f(x) = f(x_0+ph)$$

$$= y_0 + \frac{p}{1!}\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$= 8 + 0.5\times 19 + \frac{0.5(-0.5)}{2}(12) + \frac{(1.5)(0.5)(-0.5)6}{6}$$

$$= 8 + 9.5 - 1.5 - 0.3750$$

$$= 15.6250.$$

Gauss Backward Interpolation :-

$$y(x) = y(x_0+ph) = y_0 + \frac{p}{1!}\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \cdots$$

where $p = \dfrac{x - x_0}{h}$

Problem :-

Q) Using Gauss Backward Interpolation Formula. Given that

$\sqrt{6500} = 80.6223$, $\sqrt{6510} = 80.6846$, $\sqrt{6520} = 80.7456$

Sol

x	$y=\sqrt{x}$	Δy	$\Delta^2 y$	$\Delta^3 y$
6500	80.6223			
		0.0623		
6510	80.6846		-0.0013	
		0.0610		0.003
6520	80.7456		-0.0018	
		0.0628		
6530	80.8084			

Given that

$x = 6526$, $x_0 = 6520$, $h = 10$

$$p = \frac{x - x_0}{h} = \frac{6526 - 6520}{10} = 0.6$$

Now Gauss Backward Interpolation Formula,

$$y_x = f(x) = f(x_0 + ph)$$

$$= y_0 + \frac{p}{1!}\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$= 80.7456 + 0.6\,(0.0610) + \frac{(1.6)(0.6)(0.0018)}{2}$$

$$+ \frac{(1.6)(0.6)(-0.4)}{6} \times 0.003$$

$$= 80.7456 + 0.0366 + 0.0009 - 0.0002$$

$$= 80.7829$$

Lagrange's Interpolation Formula:-

Used for Un Equally spread data

$y_0, y_1, y_2, y_3 \text{-----} \rightarrow x_0, x_1, x_2, x_3 \text{----}$ Then

$$y_x = y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$+ \text{--------}$$

Problem:-

① Using Lagrange's Formula, find $f(3)$

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

Sol: Given $x_0 = 0$ $f(x_0) = 1$

$x_1 = 1$ $f(x_1) = 14$

$x_2 = 2$ $f(x_2) = 15$

$x_3 = 4$ $f(x_3) = 5$

$x_4 = 5$ $f(x_4) = 6$

$x_5 = 6$ $f(x_5) = 19$

From Lagrange's Interpolation Formula

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)} f(x_2)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)} f(x_3)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_5)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)} f(x_4)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)} f(x_5)$$

Here $x = 3$

$$f(3) = \frac{(3-1)(3-2)(3-4)(3-5)(3-6)}{(0-1)(0-2)(0-4)(0-5)(0-6)} \times 1 + \frac{(3-0)(3-2)(3-4)(3-5)(3-6)}{(1-0)(1-2)(1-4)(1-5)(1-6)} \times 14$$

$$+ \frac{(3-0)(3-1)(3-4)(3-5)(3-6)}{(2-0)(2-1)(2-4)(2-5)(2-6)} \times 15 + \frac{(3-0)(3-1)(3-2)(3-5)(3-6)}{(4-0)(4-1)(4-2)(4-5)(4-6)} \times 5$$

$$+ \frac{(3-0)(3-1)(3-2)(3-4)(3-6)}{(5-0)(5-1)(5-2)(5-4)(5-6)} \times 6 + \frac{(3-0)(3-1)(3-2)(3-4)(3-5)}{(6-0)(6-1)(6-2)(6-4)(6-5)} \times 19$$

$$= \frac{12}{240} - \frac{18}{16} \times 14 + \frac{36}{48} \times 15 + \frac{36}{48} \times 5 - \frac{18}{60} \times 6 + \frac{12}{240} \times 19$$

$$= 0.05 - 4.2 + 11.25 + 3.75 - 1.8 + 0.95$$

$$= 10$$

$$f(3) = 10$$

Problem

Find the parabola passing through the points (0, 1), (1, 3) and (3, 55) using Lagrange's interpolation formula.

Sol:- Given points are (0, 1), (1, 3), (3, 55)

Lagrange's interpolation formula is

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2).$$

$$f(x) = \frac{(x-1)(x-3)}{(0-1)(0-3)} (1) + \frac{(x-0)(x-3)}{(1-0)(1-3)} (3)$$

$$+ \frac{(x-0)(x-1)}{(3-0)(3-1)} (55).$$

$$= \frac{x^2-4x+3}{3} + \frac{x^2-3x}{(-2)}(3) + \frac{x^2-x}{6}(55)$$

$$= \frac{2x^2-8x+6-9x^2+27x+55x^2-55x}{6}$$

$$= \frac{1}{6}\left[48x^2-36x+6\right]$$

$$f(x) = 8x^2-6x+1$$

Interpolation and Extrapolation

Suppose, we are given the values of $y=f(x)$ for a set of values of x:

x : x_0 x_1 x_2 - - - - - - x_n
y : y_0 y_1 y_2 - - - - - - y_n.

The process of the find the value of 'y' corresponding to any value of $x=x_i$ between x_0 and x_n is called interpolation.

Hence, interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable while the process of computing the value of the function outside the given range is called extrapolation.

Central Differences:-

with $y_0, y_1, y_2 \cdots y_r$ as the values of a function $y = f(x)$ corresponding to the values $x_1, x_2, x_3 \cdots x_r$ of x, we define the first central differences $\delta y_{1/2}, \delta y_{3/2}, \delta_{5/2} \cdots$ as follows $\delta y_{1/2} = y_1 - y_0$, $\delta y_{3/2} = y_2 - y_1$, $\delta y_{5/2} = y_3 - y_2 \cdots$

$$\delta y_{r-\frac{1}{2}} = y_r - y_{r-1}.$$

The symbol δ is called central difference operation. This operator is a linear operator.

shift operator:- The shift operator 'E' is defined by the equation $Ey_r = y_{r+1}$ $[\, Ef(x) = f(x+h)]$.

$E^2 y_r = E[Ey_r] = E[y_{r+1}] = y_{r+2}$; Generally $E^n y_r = y_{r+n}$.

Inverse operator E^{-1} is defined as $E^{-1} y_r = y_{r-1}$.

In general $E^{-n} y_r = y_{r-n}$

Relation b/w Δ an E

$\Delta f(x) = f(x+h) - f(x)$.

$\Delta f(x) = Ef(x) - f(x)$

$\Delta f(x) = [E-1] f(x)$

$\boxed{\Delta = E-1}$ or $\boxed{E = 1+\Delta}$

Relation b/w '∇' and 'E'

$\nabla f(x) = f(x) - f(x-h)$

$= f(x) - E^{-1} f(x)$.

$\nabla f(x) = [1 - E^{-1}] f(x)$

$\boxed{\nabla = 1 - E^{-1}}$

Some more Relations

$\Delta^3 y_0 = [E-1]^3 y_0 = (E^3 - 3E^2 + 3E - 1) y_0$

$= E^3 y_0 - 3E^2 y_0 + 3E y_0 - y_0 = y_3 - 3y_2 + 3y_1 - y_0$.

$\Delta^4 y_0 = [E-1]^4 y_0 = (E^4 - 4E^3 + 6E^2 - 4E + 1) y_0$.

Central Difference operater :- The central difference operator 'δ' is defined by $\delta f(x) = f(x+\frac{h}{2}) - f(x-\frac{h}{2})$

Relation b/w 'E' and 'δ'

$$\delta f(x) = f(x+\tfrac{h}{2}) - f(x-\tfrac{h}{2})$$

$$E^{1/2} f(x) = f(x+\tfrac{h}{2})$$

$$E^{-1/2} f(x) = f(x-\tfrac{h}{2})$$

$$= E^{1/2} f(x) - E^{-1/2} f(x)$$

$$\delta f(x) = [E^{1/2} - E^{-1/2}] f(x)$$

$$\boxed{\delta = E^{1/2} - E^{-1/2}}$$

Average operator (μ) :-

The Average operator 'μ' is defined as

$$\mu f(x) = \frac{f(x+\frac{h}{2}) + f(x-\frac{h}{2})}{2}.$$

Relation b/w 'μ' and 'E' :-

$$\mu f(x) = \frac{f(x+\frac{h}{2}) + f(x-\frac{h}{2})}{2}$$

$$= \frac{E^{1/2} f(x) + E^{-1/2} f(x)}{2}$$

$$\mu = \frac{E^{1/2} + E^{-1/2}}{2}.$$

Differential operator :-

The Differential operator 'D' is defined by

$$Dy_x = \frac{d}{dx}[y_x] = y'(x).$$

$$D f(x) = \frac{d}{dx}[f(x)].$$

Relation b/w 'D' and 'E'

by using Taylor's formula.

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \cdots$$

$$y(x+h) = y(x) + \frac{h}{1!} y'(x) + \frac{h^2}{2!} y''(x) + \frac{h^3}{3!} y'''(x) + \cdots$$

This can be written as.

$$Ey_x = y_x + \frac{h}{1!} Dy_x + \frac{h^2}{2!} D^2 y_x + \frac{h^3}{3!} D^3 y_x + \cdots$$

$$= \left[1 + \frac{hD}{1!} + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \cdots\right] y_x.$$

$$E = 1 + \frac{hD}{1!} + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \cdots$$

$$\boxed{E = e^{hD}} \qquad \left[e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right]$$

Pb Show that $\mu^2 = 1 + \frac{\delta^2}{4}$ (or) $\mu = \sqrt{1 + \frac{\delta^2}{4}}$.

Sol:- $\mu = \frac{E^{1/2} + E^{-1/2}}{2}$ and $\delta = E^{1/2} - E^{-1/2}$.

$$\mu^2 = \left[\frac{E^{1/2} + E^{-1/2}}{2}\right]^2 = \frac{\left[E^{1/2} + E^{-1/2}\right]^2}{4}.$$

$$(a+b)^2 = (a-b)^2 + 4ab.$$

$$\mu^2 = \frac{\left(E^{1/2} - E^{-1/2}\right)^2 + 4E^{1/2}E^{-1/2}}{4}$$

$$= \frac{\left[E^{1/2} - E^{-1/2}\right]^2 + 1}{4}$$

$$= \frac{\delta^2 + 4}{4} \text{ (or) } \frac{\delta^2}{4} + 1.$$

Pb- Evaluate (i) $\Delta \cos x$ (ii) $\Delta \log f(x)$ (iii) $\Delta^2 \sin(px+q)$ (iv) $\Delta \tan^{-1} x$ (v) $\Delta^n e^{ax+b}$.

(i) $\Delta \cos x = \cos(x+h) - \cos x$ $\left[\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}\right]$

$$= -2\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}$$

(ii) $\Delta \log f(x) = \log f(x+h) - \log f(x)$

$$= \log\left[\frac{f(x+h)}{f(x)}\right]$$

$$= \log\left[\frac{f(x)+\Delta f(x)}{f(x)}\right] = \log\left[1+\frac{\Delta f(x)}{f(x)}\right]$$

(iii) $\Delta \sin px+q = \sin(p(x+h)+q) - \sin(px+q)$

$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$$= 2\cos\frac{px+ph+q+px+q}{2}\sin\frac{px+ph+q-px-q}{2}$$

$$= 2\cos\left(px+q+\frac{ph}{2}\right)\sin\frac{ph}{2}.$$

$$= 2\sin\frac{ph}{2}\sin\left(\frac{\pi}{2}+px+q+\frac{ph}{2}\right)$$

$$\Delta^2 \sin(px+q) = 2\sin\frac{ph}{2}\,\Delta\left[\sin\left(px+q+\tfrac{1}{2}(\pi+ph)\right)\right]$$

$$= \left[2\sin\frac{ph}{2}\right]^2 \sin\left[px+q+2\cdot\tfrac{1}{2}(\pi+ph)\right]$$

Proceeding in the same manner, we get

$$\Delta^3 \sin(px+q) = \left(2\sin\frac{ph}{2}\right)^3 \sin\left(px+q+\frac{3(\pi+ph)}{2}\right)$$

$$\Delta^n \sin(px+q) = \left(2\sin\frac{ph}{2}\right)^n \sin\left[px+q+\frac{n(\pi+ph)}{2}\right]$$

(iv) $\Delta \tan^{-1} x = \tan^{-1}(x+h) - \tan^{-1} x$.

$$= \tan^{-1}\left[\frac{x+h-x}{1+x(x+h)}\right] \qquad \left[\tan^{-1}A - \tan^{-1}B = \tan^{-1}\frac{A-B}{1+AB}\right]$$

$$= \tan^{-1}\left[\frac{h}{1+x(x+h)}\right].$$

(v) $\Delta e^{ax+b} = e^{a(x+h)+b} - e^{ax+b}$.

$$= e^{ax+ah+b} - e^{ax+b}.$$

$$= e^{(ax+b)+ah} - e^{ax+b}.$$

$$= e^{ax+b}\left[e^{ah}-1\right].$$

$$\Delta^2 e^{ax+b} = \Delta\left[\Delta e^{ax+b}\right]$$

$$= \Delta\left[e^{ax+b}(e^{ah}-1)\right]$$

$$= (e^{ah}-1)\Delta e^{ax+b}.$$

$$= (e^{ah}-1)\left[(e^{ah}-1)e^{ax+b}\right].$$

$$= (e^{ah}-1)^2 e^{ax+b}. \qquad \left[e^{ah}-1 \text{ is constant}\right]$$

Proceeding in this manner, we get.

$$\Delta^3 e^{ax+b} = (e^{ah}-1)^3 e^{ax+b}.$$

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$$\Delta^n e^{ax+b} = (e^{ah}-1)^n e^{ax+b}.$$

Problem:- If the interval of differencing is ~~unity, Prove that $\Delta \tan^{-1}\left(\frac{n-1}{n}\right) = \tan^{-1}\left(\frac{1}{2n^2}\right)$.~~

Sol:- we know that $\Delta \tan^{-1} x = \tan^{-1}(x+h) - \tan^{-1} x$.

$$\Delta \tan^{-1}\left(\frac{n-1}{n}\right) = \Delta \tan^{-1}\left(1-\frac{1}{n}\right)$$

$$= \tan^{-1}\left(1-\frac{1}{n+1}\right) - \tan^{-1}\left(1-\frac{1}{n}\right)$$

$$= \tan^{-1}\left[\frac{\left(1-\frac{1}{n+1}\right)-\left(1-\frac{1}{n}\right)}{1+\left(1-\frac{1}{n+1}\right)\left(1-\frac{1}{n}\right)}\right] = \tan^{-1}\left[\frac{\left(\frac{1}{n}-\frac{1}{n+1}\right)}{1+\left(\frac{n}{n+1}\right)\left(\frac{n-1}{n}\right)}\right]$$

$$= \tan^{-1}\left(\frac{1}{2n^2}\right) \text{ on simplification}$$

Problem:- using the method of separation of symbols, show that $\Delta^n u_{x-n} = u_x - n u_{x-1} + \frac{n(n-1)}{2} u_{x-2} + \cdots + (-1)^n u_{x-n}$

Solution:- To prove the result, we start with the right hand side.

$$u_x - n u_{x-1} + \frac{n(n-1)}{2} u_{x-2} + \cdots + (-1)^n u_{x-n}$$

$$= u_x - n E^{-1} u_x + \frac{n(n-1)}{2} E^{-2} u_x + \cdots + (-1)^n E^{-n} u_x.$$

$$= \left[1 - nE^{-1} + \frac{n(n-1)}{2} E^{-2} + \cdots + (-1)^n E^{-n}\right] u_x.$$

$$= \left[1 - E^{-1}\right]^n u_x. = \left(1-\frac{1}{E}\right)^n u_x = \frac{(E-1)^n}{E^n} u_x.$$

$$= \frac{\Delta^n}{E^n} u_x = \Delta^n E^{-n} u_x.$$

$$= \Delta^n u_{x-n}. \text{ which is left hand side.}$$

Problem:- Show that $\Delta f_i^2 = (f_i + f_{i+1}) \Delta f_i$

Solution:- $\Delta f_k = f_{k+1} - f_k$.

$$\Delta f_i^2 = f_{i+1}^2 - f_i^2 = (f_{i+1} + f_i)(f_{i+1} - f_i)$$

$$= (f_{i+1} + f_i) \Delta f_i$$

Problem :- Show that $e^{x}\left(u_0 + x\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \cdots\right)$

$$= u_0 + u_1 x + u_2 \frac{x^2}{2!} + \cdots$$

Solution:- $e^{x}\left(u_0 + x\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \cdots\right) = e^{x}\left(1 + x\Delta + \frac{x^2}{2}\Delta^2 + \cdots\right] u_0.$

$$= e^{x} \cdot e^{x\Delta} u_0 = e^{x(1+\Delta)} u_0 = e^{xE} u_0.$$

$$= \left[1 + xE + \frac{x^2 E^2}{2!} + \cdots\right] u_0.$$

$$= u_0 + xEu_0 + \frac{x^2}{2!}E^2 u_0 + \cdots$$

$$= u_0 + x u_1 + \frac{x^2}{2!} u_2 + \cdots$$

Problem :- Evaluate (i) $\Delta\left[f(x)g(x)\right]$ (ii) $\Delta\left[\frac{f(x)}{g(x)}\right]$.

(i) $\Delta f(x)g(x) = f(x+h)g(x+h) - f(x)g(x).$

$$= f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x).$$

$$= f(x+h)\left[g(x+h) - g(x)\right] + g(x)\left[f(x+h) - f(x)\right]$$

$$= f(x+h)\Delta g(x) + g(x)\Delta f(x).$$

(ii) $\Delta \frac{f(x)}{g(x)} = \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}.$

$$= \frac{f(x+h)g(x) - g(x+h)f(x)}{g(x+h)g(x)}.$$

$$= \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - g(x+h)f(x)}{g(x+h)g(x)}.$$

$$= \frac{g(x)\left[f(x+h) - f(x)\right] + f(x)\left[g(x) - g(x+h)\right]}{g(x+h)g(x)}$$

$$= g(x)\left[f(x+h) - f(x)\right] - f(x)\left[g(x+h) - g(x)\right]$$

$$= \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x+h)\,g(x)}.$$

Problem:- Show that $\sum_{k=0}^{n-1} \Delta^2 f_k = \Delta f_n - \Delta f_0$.

Solution:- $\sum_{k=0}^{n-1} \Delta^2 f_k = \Delta^2 f_0 + \Delta^2 f_1 + \Delta^2 f_2 + \cdots + \Delta^2 f_{n-1}$ — ①.

$$\left[\begin{array}{l} \Delta f_0 = f_1 - f_0 \Rightarrow \Delta^2 f_0 = \Delta f_1 - \Delta f_0 \\ \Delta f_1 = f_2 - f_1 \Rightarrow \Delta^2 f_1 = \Delta f_2 - \Delta f_1. \end{array}\right]$$

From ①

$$\sum_{k=0}^{n-1} \Delta^2 f_k = \cancel{\Delta f_1} - \Delta f_0 + \cancel{\Delta f_2} - \cancel{\Delta f_1} + \Delta f_3 - \cancel{\Delta f_2} + \cdots + \Delta f_n - \Delta f_{n-1}.$$

$$= \Delta f_n - \Delta f_0.$$

Problem:- If the interval of differencing is unity, Prove that $\Delta \frac{1}{f(x)} = \frac{-\Delta f(x)}{f(x)\,f(x+1)}$.

Sol:- we know that $\Delta\left(\frac{1}{f(x)}\right) = \frac{1}{f(x+h)} - \frac{1}{f(x)}$.

$$= \frac{f(x) - f(x+h)}{f(x+h)f(x)}$$

$$= -\left[\frac{f(x+h) - f(x)}{f(x+h)\,f(x)}\right] = \frac{-\Delta f(x)}{f(x)\,f(x+h)}$$

Taking $h=1$, we get.

$$\Delta \frac{1}{f(x)} = \frac{-\Delta f(x)}{f(x)\,f(x+1)}$$

Prove the results

(i) $E\nabla = \Delta = \nabla E$.

$$(E\nabla)f(x) = E[\nabla f(x)]$$
$$= E[f(x) - f(x-h)]$$
$$= Ef(x) - Ef(x-h).$$
$$= f(x+h) - f(x).$$
$$= \Delta f(x)$$
$$\therefore \; E\nabla = \Delta.$$

Also $(\nabla E) f(x) = \nabla [E f(x)]$

$$= \nabla f(x+h).$$
$$= f(x+h) - f(x)$$
$$= \Delta f(x)$$
$$\nabla E = \Delta.$$
$$\therefore \; E\nabla = \Delta = \nabla E.$$

(ii) $\delta E^{1/2} = \Delta$.

$$\delta f(x+\tfrac{h}{2}) = (E^{1/2} - E^{-1/2}) f(x+\tfrac{h}{2})$$
$$= E^{1/2} f(x+\tfrac{h}{2}) - E^{-1/2} f(x+\tfrac{h}{2})$$
$$= E^{1/2} y_{x+\frac{h}{2}} - E^{-1/2} y_{x+\frac{h}{2}}.$$
$$= y_{x+h} - y_x = \Delta y_x.$$
$$\delta f(x+\tfrac{h}{2}) = \Delta f(x)$$
$$\delta E^{1/2} f(x) = \Delta f(x)$$
$$\boxed{\delta E^{1/2} = \Delta}$$

(iii). $hD = \log(1+\Delta) = -\log(1-\nabla)$
$= \sinh^{-1}(\mu\delta)$

Sol:- we know $e^{hD} = E = 1+\Delta$.

Taking logarithm

$$hD \log_e = \log(1+\Delta).$$

Also $\nabla = 1 - E^{-1} \Rightarrow E^{-1} = 1-\nabla$.

$$e^{-hD} = (1-\nabla)$$

Taking logarithms.

$$-hD = \log(1-\nabla)$$
$$hD = -\log(1-\nabla).$$
$$\sinh(hD) = \frac{e^{hD} - e^{-hD}}{2}$$
$$= \frac{E - E^{-1}}{2}$$
$$= \left[\frac{E^{1/2} + E^{-1/2}}{2}\right]\left[E^{1/2} - E^{-1/2}\right] = \mu\delta.$$
$$hD = \sinh^{-1}(\mu\delta)$$

(iv). $\nabla\Delta = \Delta - \nabla = \delta^2$.

$$\nabla\Delta = (1-E^{-1})(E-1)$$
$$= E - 1 - EE^{-1} + E^{-1}.$$
$$= E + E^{-1} - 2$$
$$= [E^{1/2} - E^{-1/2}]^2 = \delta^2$$

Also $\Delta - \nabla$

$$= (E-1) - (1-E^{-1})$$
$$= E + E^{-1} - 2 = \delta^2.$$

$$\nabla\Delta = \Delta - \nabla = \delta^2.$$

(v) $(1+\Delta)(1-\nabla) = 1.$

$(1+\Delta)(1-\nabla) = E[1-(1-E^{-1})]$

$= E[\cancel{1}-\cancel{1}+E^{-1}]$

$= EE^{-1} = 1.$

(vi) $\frac{1}{2}(\Delta+\nabla) = \mu\delta.$

$\frac{1}{2}(\Delta+\nabla) = \frac{1}{2}[E-\cancel{1}+\cancel{1}-E^{-1}]$

$= \frac{1}{2}[E-E^{-1}]$

$= \mu\delta.$

(vii) $1+\mu^2\delta^2 = \left(1+\frac{\delta^2}{2}\right)^2.$

$1+\mu^2\delta^2 = 1+\left(\frac{E^{1/2}+E^{-1/2}}{2}\right)^2\left(E^{1/2}-E^{-1/2}\right)^2$

$= 1+\left(\frac{E-E^{-1}}{4}\right)^2$

$= \frac{4+(E-E^{-1})^2}{4} = \left[\frac{E+E^{-1}}{2}\right]^2 \quad -①$

Now $\left(1+\frac{\delta^2}{2}\right)^2 = \left[1+\frac{1}{2}\left[E^{1/2}-E^{-1/2}\right]^2\right]^2$

$= \left[1+\frac{1}{2}(E+E^{-1}-2)\right]^2$

$= \left[\frac{E+E^{-1}}{2}\right]^2 \quad -②$

From ① and ②.

$1+\mu^2\delta^2 = \left(1+\frac{\delta^2}{2}\right)^2$

(viii) $\mu+\frac{\delta}{2} = \frac{(E^{1/2}+E^{-1/2})}{2}+\frac{(E^{1/2}-E^{-1/2})}{2}$

$= \frac{\cancel{2}E^{1/2}}{\cancel{2}} = E^{1/2}.$

(ix) $\mu-\frac{\delta}{2} = E^{-1/2}$

$\mu-\frac{\delta}{2} = \frac{E^{1/2}+E^{-1/2}}{2}-\frac{(E^{1/2}-E^{-1/2})}{2}$

$= \frac{\cancel{2}E^{-1/2}}{\cancel{2}} = E^{-1/2}$

(x) $\mu\delta = \frac{1}{2}\Delta E^{-1}+\frac{1}{2}\Delta$

$\frac{1}{2}\Delta E^{-1}+\frac{1}{2}\Delta = \frac{1}{2}\Delta[E^{-1}+1]$

$= \frac{1}{2}(E-1)(E^{-1}+1)$

$= \frac{1}{2}(E-E^{-1})$

$= \mu\delta$

(xi) $\Delta = \frac{\delta^2}{2}+\delta\sqrt{1+\frac{\delta^2}{4}}.$

$\frac{\delta^2}{2}+\delta\sqrt{1+\frac{\delta^2}{4}} = \frac{1}{2}\delta\left[\delta+2\sqrt{1+\frac{\delta^2}{4}}\right]$

$= \frac{1}{2}\delta\left[\delta+\sqrt{4+\delta^2}\right]$

$= \frac{1}{2}\delta\left[(E^{1/2}-E^{-1/2})+\sqrt{4+(E^{1/2}-E^{-1/2})^2}\right]$

$= \frac{1}{2}\delta\left[(E^{1/2}-E^{-1/2})+\sqrt{(E^{1/2}+E^{-1/2})^2}\right]$

$= \frac{1}{2}\left[(E^{1/2}-E^{-1/2})\left[E^{1/2}-\cancel{E^{-1/2}}+E^{1/2}+\cancel{E^{-1/2}}\right]\right]$

$= \frac{1}{\cancel{2}}\left[(E^{1/2}-E^{-1/2})\cancel{2}E^{1/2}\right]$

$= E-E^{0} = E-1 = \Delta$

$\frac{\delta^2}{2}+\delta\sqrt{1+\frac{\delta^2}{4}} = \Delta.$

(xii) $\Delta^n\cos(ax+b).$

$= \left(2\sin\frac{ah}{2}\right)^n\cos\left[ax+b+n\frac{(ah+\pi)}{2}\right]$

$\lll y$ we can prove that.

Problem:- Estimate the missing term in the following table.

x :	0	1	2	3	4
$f(x)$:	1	3	9	?	81.

Solution:- we are given four values.

$$\Delta^4 f(x) = 0 \Rightarrow \Delta^4 y_0 = 0.$$

$$\Rightarrow (E-1)^4 y_0 = 0$$

$$\Rightarrow (E^4 - 4E^3 + 6E^2 - 4E + 1) y_0 = 0.$$

$$E^4 y_0 - 4E^3 y_0 + 6E^2 y_0 - 4Ey_0 + y_0 = 0.$$

$$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0.$$

$$81 - 4y_3 + 6(9) - 4(3) + 1 = 0.$$

$$81 - 4y_3 + 54 - 12 + 1 = 0.$$

$$4y_3 = 123 \Rightarrow y_3 = 31.$$

Problem:- Estimate the production for 1964 and 1966 from the following data.

Year :	1961	1962	1963	1964	1965	1966	1967
population :	200	220	260	—	350	—	430.

Solution:- Since ~~five~~ five figures are known, assume fifth differences as zero. ie $\Delta^5 y_0 = 0$ and $\Delta^5 y_1 = 0$.

$$(E-1)^5 y_0 = 0. \quad \text{and } (E-1)^5 y_1 = 0.$$

$$(E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1) y_0 = 0.$$

$$E^5 y_0 - 5E^4 y_0 + 10E^3 y_0 - 10E^2 y_0 + 5Ey_0 - y_0 = 0.$$

3

$(E-1)^5 y_1 = 0 \Rightarrow (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1) y_1 = 0.$

$E^5 y_1 - 5E^4 y_1 + 10E^3 y_1 - 10E^2 y_1 + 5E y_1 - y_1 = 0.$

$y_6 - 5y_5 + 10y_4 - 10y_3 + 5y_2 - y_1 = 0.$

Substituting the known values, we get.

$y_5 - 1750 + 10y_3 - 2600 + 1100 - 200 = 0.$

$430 - 5y_5 + 3500 - 10y_3 + 1300 - 220 = 0.$

$y_5 + 10y_3 = 3450. \quad -(1)$

$-5y_5 - 10y_3 = -5010. \quad -(2).$

Adding (1) and (2). we get.

$-4y_5 = -1560 \Rightarrow y_5 = 390.$

From (1). $390 + 10y_3 = 3450.$

$10y_3 = 3060 \Rightarrow y_3 = 306.$

Hence production for year 1964 = 306.

and production for year 1966 = 390.

Problem:- $\Delta^3 y_2 = \nabla^3 y_5.$

$\Delta^3 y_2 = (E-1)^3 y_2 = (E^3 - 3E^2 + 3E - 1) y_2$

$E^3 y_2 - 3E^2 y_2 + 3E y_2 - y_2$

$y_5 - 3y_4 + 3y_3 - y_2$

$\nabla^3 y_5 = (1-E^{-1})^3 y_5 = (1 - 3E^{-1} + 3E^{-2} - E^{-3}) y_5$

$y_5 - 3E^{-1} y_5 + 3E^{-2} y_5 - E^{-3} y_5 = y_5 - 3y_4 + 3y_3 - y_2$

Problem: Given $u_0 + u_8 = 1.9243$, $u_1 + u_7 = 1.9590$.

$u_2 + u_6 = 1.9823$, $u_3 + u_5 = 1.9956$ find u_4.

Sol:- Taking $(E-1)^8 u_0 = 0$.

$$(E^8 - 8E^7 + 28E^6 - 56E^5 + 70E^4 - 56E^3 + 28E^2 - 8E + 1)u_0 = 0$$

$$E^8 u_0 - 8E^7 u_0 + 28E^6 u_0 - 56E^5 u_0 + 70E^4 u_0 - 56E^3 u_0 + 28E^2 u_0 - 8Eu_0 + u_0 = 0.$$

$$u_8 - 8u_7 + 28u_6 - 56u_5 + 70u_4 - 56u_3 + 28u_2 - 8u_1 + u_0 = 0$$

$$(u_0 + u_8) - 8(u_7 + u_1) + 28(u_6 + u_2) - 56(u_5 + u_3) + 70u_4 = 0.$$

$$1.9243 - 8(1.9590) + 28(1.983) - 56(1.9956) + 70u_4 = 0$$

$$u_4 = 0.99996.$$

Problem:- For the second difference of the polynomial $x^4 - 12x^3 + 42x^2 - 30x + 9$ with interval of differencing $h = 2$.

Solution:- Let $f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$.

First difference is given by $= \Delta f(x)$

$$f(x+h) - f(x) = f(x+2) - f(x).$$

$$= (x+2)^4 - 12(x+2)^3 + 42(x+2)^2 - 30(x+2) + 9$$
$$- x^4 + 12x^3 - 42x^2 + 30x - 9.$$

$$= 8x^3 - 48x^2 + 56x + 28.$$

Second difference $\Delta^2 f(x) = \Delta[\Delta f(x)]$

$$= 8(x+2)^3 - 48(x+2)^2 + 56(x+2) + 28$$

$$= -8x^3 + 48x^2 - 56x - 28$$

Prove the results. (i) $\Delta\nabla = \Delta - \nabla$ (ii) $\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} = \Delta + \nabla$

(i) $\Delta\nabla = (E-1)(1-E^{-1})$

$= E - EE^{-1} - 1 + E^{-1}$.

$= E - 1 - 1 + E^{-1}$.

$= (E-1) - (1-E^{-1}) = \Delta - \nabla$.

$\therefore \Delta\nabla = \Delta - \nabla$.

(ii) $\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} = \frac{\Delta^2 - \nabla^2}{\Delta\nabla}$

$= \frac{(\Delta+\nabla)(\Delta-\nabla)}{\Delta\nabla}$

$= \frac{(\Delta+\nabla)\cancel{(\Delta-\nabla)}}{\cancel{(\Delta-\nabla)}}$ From (i)

$= \Delta + \nabla$

$\therefore \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} = \Delta + \nabla$.

Problem:- $\Delta^2 \cos 2x$.

$\Delta^2 \cos 2x = \Delta[\cos 2(x+h) - \cos 2x]$

$= [\cos 2(x+2h) - \cos 2(x+h)] - [\cos 2(x+h) - \cos 2x]$

$= -2\sin(2x+3h)\sin h + 2\sin(2x+h)\sin h$.

$= -2\sin h[\sin(2x+3h) - \sin(2x+h)]$

$= -2\sin h[2\cos(2x+2h)\sin h]$ $\left[\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}\right]$

$= -4\sin^2 h \cos 2(x+h)$

Prove the results. (i) $\Delta\nabla = \Delta - \nabla$ (ii) $\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} = \Delta + \nabla$

(i) $\Delta\nabla = (E-1)(1-E^{-1})$

$= E - EE^{-1} - 1 + E^{-1}$.

$= E - 1 - 1 + E^{-1}$.

$= (E-1) - (1-E^{-1}) = \Delta - \nabla$.

$\therefore \Delta\nabla = \Delta - \nabla$.

(ii) $\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} = \frac{\Delta^2 - \nabla^2}{\Delta\nabla}$

$= \frac{(\Delta+\nabla)(\Delta-\nabla)}{\Delta\nabla}$

$= \frac{(\Delta+\nabla)\cancel{(\Delta-\nabla)}}{\cancel{(\Delta-\nabla)}}$ From (i)

$= \Delta + \nabla$

$\therefore \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} = \Delta + \nabla$.

Problem:- $\Delta^2 \cos 2x$.

$\Delta^2 \cos 2x = \Delta[\cos 2(x+h) - \cos 2x]$

$= [\cos 2(x+2h) - \cos 2(x+h)] - [\cos 2(x+h) - \cos 2x]$

$= -2\sin(2x+3h)\sin h + 2\sin(2x+h)\sin h$.

$= -2\sin h[\sin(2x+3h) - \sin(2x+h)]$

$= -2\sin h[2\cos(2x+2h)\sin h]$ $\left[\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}\right]$

$= -4\sin^2 h \cos 2(x+h)$

NUMERICAL SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

* Taylor's series:

Given Differential Equation is $\frac{dy}{dx} = f(x,y)$ with initial condition is that $y = y_0$ at $x = x_0$

By Taylor's series

$$y(x) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

* problem:

1. Using Taylor's series method to find $y(0.1)$ and $y(0.2)$ if $y' = x - y^2$, $y = 1$ and $x = 0$.

Sol: Given $y' = f(x,y)$. $x_1 = x_0 + h$

$y' = x - y^2$, $x_0 = 0$, $y_0 = 1$. $x_1 = 0 + 0.1 = 0.1$

$y' = x - y^2$, $y_0' = x_0 - y_0^2 = 0 - (1)^2 = 1$.

$y'' = 1 - 2yy' \Rightarrow y_0'' = 1 - 2y_0y_0' = 3$.

$y''' = 0 - 2[yy'' + y'y'] \Rightarrow y_0''' = -2[y_0y_0'' + y_0'y_0'] = -8$

$y^{IV} = -2[yy''' + y''y' + y'y'' + y'y'']$

$\Rightarrow y_0^{IV} = -2[y_0y_0''' + 3y_0'y_0''] = 34$.

$y^{V} = -2[yy^{IV} + y'''y' + y''y'' + y'y''' + 2(y'y''' + y''y'')]$

$= -2[yy^{IV} + 4y'y''' + 3(y'')^2]$

$\Rightarrow y_0^{V} = -2[y_0y_0^{IV} + 4y_0'y_0''' + 3(y_0'')^2]$

$= -186$

By using Taylor's series.

$$y(x) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{IV} + \frac{h^5}{5!}y_0^{V} + \cdots$$

$$y_1 = y(0.1) = 1 + (0.1)(-1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(-8) + \frac{(0.1)^4}{24}(34) + \frac{(0.1)^5}{120}(-186).$$

$$y_1 = 1 - 0.1 + 0.015 - 0.00133 + 0.0001416 - 0.0000155$$

$$y(0.1) = 0.9138.$$

$x_1 = 0.1,\ h = 0.1,\ y_1 = 0.9138.$

$$y_1' = x_1 - y_1^2 = 0.1 - (0.9138)^2 = 0.1 - 0.8350487 = -0.735.$$

$$y_1'' = 1 - 2y_1y_1' = 1 - 2(0.9138)(-0.735) = 2.3433$$

$$y_1''' = -2[(y_1')^2 + y_1y_1''] = -2[(-0.735)^2 + (0.9138)(2.3433)] = -5.363112$$

$$y_1^{iv} = -2[3y_1'y_1'' + y_1y_1'''] = -2[3(-0.735)(2.3433) + (0.9138)(2.3433)]$$

$$= 20.133953.$$

$$y_2 = y(0.2) = y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \frac{h^4}{4!}y_1^{iv} + \frac{h^5}{5!}y_1^{v}$$

$$y_2 = y(0.2) = 0.9138 + (0.1)(-0.735) + \frac{(0.1)^2}{2}(2.3433) + \frac{(0.1)^3}{6}(-5.363112) + \frac{(0.1)^4}{24}(20.133953) + \cdots$$

$$= 0.9138 - 0.0735 + 0.0117 - 0.00089 + 0.00008$$

$$y_2 = 0.8512$$

* Picard's method of successive Approximations:

Given Differential Equation is $\frac{dy}{dx} = f(x,y)$

$y = y_0$ at $x = x_0$.

1st Approximation $y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$.

2nd Approximation $y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\,dx$.

3rd Approximation $y^{(3)} = y_0 + \int_{x_0}^{x} f(x, y^{(2)})\,dx$.

⋮

proceeding in this way after Finite no. of steps, we get the value of y when x is given.

* problem:

Find an approximate value of y for $x = 0.2$ if $\frac{dy}{dx} = x+y$, $y(0) = 1$ using picards method compare the numerical solution obtained with Exact solution.

Given Equation $\frac{dy}{dx} = x + y$, $x_0 = 0$, $y_0 = 1$.

1st approximation, $y^{(1)} = y_0 + \int_{x_0}^{x} f(x + y_0)\,dx$.

$= y_0 + \int_{x_0}^{x} (x+1)\,dx$

$y^{(1)} = 1 + x^2/2 + x = 1 + x + x^2/2$.

2nd approximation, $y^{(2)} = y_0 + \int_{x_0}^{x} (x - y^{(1)})\,dx$

$= y_0 + \int_{x_0}^{x} x + (1 + x + x^2/2)\,dx$.

$= 1 + x^2/2 + x + x^2/2 + x^3/6$.

$$y^{(2)} = 1 + x + x^2 + x^3/6.$$

3^{rd} approximation $y^{(3)} = y_0 + \int_{x_0}^{x} \left(x + y^{(2)}\right) dx$

$$= y_0 + \int_{x_0}^{x} \left(x + 1 + x + x^2 + x^3/6\right) dx$$

$$= 1 + x^2/2 + x + x^2/2 + x^3/3 + \frac{x^4}{24}$$

$$= 1 + x + x^2 + x^3/3 + x^4/24.$$

when $x = 0.2$

$$y^{(1)} = 1 + (0.2) + \frac{(0.2)^2}{2} = 1.22$$

$$y^{(2)} = 1 + (0.2) + (0.2)^2 + \frac{(0.2)^3}{6} = 1.2413$$

$$y^{(3)} = 1 + (0.2) + (0.2)^2 + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{24} = 1.2427$$

$\therefore$ The value of y when $x = 0.2$ is 1.2427.

* Exact value:

Given Equation is $\frac{dy}{dx} = x + y$.

$$\frac{dy}{dx} - y = x.$$

comparing this Equation with $\frac{dy}{dx} + Py = Q$.

$$P = -1, \quad Q = x.$$

$$I.F = e^{\int P dx} = e^{-\int 1 dx} = e^{-x}.$$

* General solution:

$$y(I.F) = \int Q(I.F)\, dx + C$$

$$y e^{-x} = \int x e^{-x} dx + C \qquad u = x \quad v = -e^{-x}$$

$$du = dx$$

$$= -x e^{-x} - e^{-x} + C$$

$$y e^{-x} = -e^{-x}[x+1] + C$$

$$y = -(x+1) + Ce^{x}$$

put $x=0, y=1$.

$\therefore 1 = -(0+1) + ce^0$

$c = 2$.

$\therefore y e^{-x} = (-x+1)e^{-x} + 2$

$y = -(x+1) + 2e^x$

put $x = 0.2$; $y = 1.2427$

* Eulers method:

Given Differential Equation is $y' = f(x,y)$.

at $y = y_0$ at $x = x_0$.

1st approximation $y_1 = y_0 + h f(x_0, y_0)$.

2nd approximation $y_2 = y_1 + h f(x_1, y_1)$, $x_1 = x_0 + h$.

proceeding in this way after a Finite no. of steps we get the value of y when x is given.

* problem: using Euler's method Find an approximate value of y when $x=1$ given that $\frac{dy}{dx} = x+y$ and $y = 1$ when $x=0$ and $h = 0.25$

Sol: Given that $\frac{dy}{dx} = x+y$.

$f(x,y) = x+y$, $h = 0.25$.

$x_0 = 0$, $y_0 = 1$,

1st approximation, $y_1 = y_0 + h f(x_0, y_0)$

$= 1 + (0.25)(x_0 + y_0)$

$= 1 + (0.25)(0+1)$

$= 1 + 0.25$

$= 1.25$

2nd approximation, $y_2 = y_1 + h f(x_1, y_1)$ $\quad x_1 = x_0 + h$

$= 1.25 + (0.25)(x_1 + y_1)$ $\quad = 0 + 0.25$

$= 1.625$ $\quad = 0.25$

3rd approximation, $y_3 = y_2 + h f(x_2, y_2)$

$= 1.625 + (0.25)(0.5 + 1.625)$ $\quad \left[\begin{aligned} x_2 &= x_1 + h \\ &= 0.25 + 0.25 \\ &= 0.5 \end{aligned}\right]$

$= 2.156.$

4th approximation, $y_4 = y_3 + h f(x_3, y_3)$

$= 2.156 + (0.25)(0.75 + 2.156)$ $\quad \left[\begin{aligned} x_3 &= x_2 + h \\ &= 0.5 + 0.25 \\ &= 0.75 \end{aligned}\right]$

$= 2.8825$

5th approximation, $y_5 = y_4 + h f(x_4, y_4)$

$= 2.8825 + (0.25)(1 + 2.8825)$

$= 3.853.$

$\therefore$ The value of y when $x = 0$ is 3.853.

* Eulers Modified method:

Given Differential Equation is $y' = f(x, y)$.

$y = y_0$ at $x = x_0$.

1st approximation, $y_1^{(1)} = y_0 + h f(x_0, y_0)$

2nd approximation, $y_1^{(2)} = y_0 + h/2 \left[f(x_0, y_0) + f(x_1, y_1^{(1)}) \right]$

3rd approximation, $y_1^{(3)} = y_0 + h/2 \left[f(x_0, y_0) + f(x_1, y_1^{(2)}) \right]$

$(x_1 = x_0 + h, \quad x_2 = x_1 + h, \quad x_3 = x_2 + h).$

* Runge - Kutta Method:

* RK Method of 1st order:

Eulers Method is nothing but RK method of 1st order

$y_1 = y_0 + h f(x_0, y_0).$

* RK method of 2nd order:

Eulers modified method is nothing but RK method of 2nd order. $y_1^{(1)} = y_0 + h f(x_0, y_0)$

$y_1 = y_0 + 1/2 (K_1 + K_2)$

where $K_1 = h\,f(x_0, y_0)$

$K_2 = h\,f(x_0+h, y_0+K)$.

* RK Method of 3rd order:

Given that $y' = f(x,y)$, $y = y_0$ at $x = x_0$.

$K_1 = h\,f(x_0, y_0)$, $K_2 = h\,f(x_0+h/2, y_0+K_1/2)$

$K_3 = h\,f(x_0+h, y_0+K')$ (or) $h\,f(x_0+h, y_0+2K_2-K_1)$

where $K' = f(x_0+h, y_0+K_1)$.

Finally $y_1 = y_0 + \frac{1}{6}(K_1+4K_2+K_3)$

* RK Method of 4th order:

Given that $\frac{dy}{dx} = f(x,y)$, $y = y_0$ at $x = x_0$.

$K_1 = h\,f(x_0, y_0)$, $K_2 = h\,f(x_0+h/2, y_0+K_1/2)$.

$K_3 = h\,f(x_0+h/2, y_0+K_2/2)$.

$K_4 = h\,f(x_0+h, y_0+K_3)$.

$K = \frac{1}{6}(K_1+2K_2+2K_3+K_4)$.

$\therefore y_1 = y_0 + K$.

$y_1 = y_0 + \frac{1}{6}(K_1+2K_2+2K_3+K_4)$.

* problem:

Apply RK method of 4th order Find an approximate value of y when $x = 0.2$ given that $\frac{dy}{dx} = x+y$ and $y = 1$ when $x = 0$.

sol:

Given Equation

$f(x,y) = x+y$.

$x_0 = 0$, $y_0 = 1$, $h = 0.2$

4th order:

$K_1 = h\,f(x_0, y_0)$

$= h(x+y)$

$= (0.2)(0+1) = 0.2$

$$K_2 = h\,f\left[\left(x_0+h/2,\ y_0+K_1/2\right)\right]$$
$$= 0.2\,f\left[0+\frac{0.2}{2},\ 1+\frac{0.2}{2}\right]$$
$$= 0.2\ [0.1+1.1]$$
$$= 0.24.$$

$$K_3 = h\left[f\left(x_0+h/2,\ y_0+K_2/2\right)\right]$$
$$= 0.2\,f\left[0+\frac{0.2}{2},\ 1+\frac{0.24}{2}\right]$$
$$= 0.2\ [0.1+1+0.12]$$
$$= 0.244.$$

$$K_4 = h\,f(x_0+h,\ y_0+K_3]$$
$$= 0.2\,f[0+0.2,\ 1+0.244)$$
$$= 0.2\ [0.2+1.244)$$
$$= 0.2888.$$

$$\therefore K = \frac{1}{6}(K_1+2K_2+2K_3+K_4).$$
$$= 1/6\left(0.2+2(0.24)+2(0.244)+0.2888)\right]$$
$$= 0.2428$$

$$y_1 = y_0+K$$
$$= 1+0.2428$$
$$= 1.2428.$$

2:- solve the differential equation $\frac{dy}{dx} = x^2+y$, $y(0)=1$ by modified Euler's Method, and compute $y(0.02)$ and $y(0.04)$

Sol:- Here $f(x,y) = x^2+y$, $x_0 = 0$, $y_0 = 1$, $h = 0.02$.

To find y_1 i.e $y(0.02)$.

$$f(x_0, y_0) = f(0,1) = 0+1 = 1.$$

using Euler's formula $y_1^{(0)} = y_0 + hf(x_0,y_0) = 1+0.02(1) = 1.02$.

Now $x_1 = 0.02$. and $f(x_1, y_1^{(0)}) = f(0.02, 1.02) = (0.02)^2 + 1.02 = 1.0204$.

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1^{(0)})\right] = 1+0.01\left[1+1.0204\right] = 1.0202$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1^{(1)})\right] = 1+0.01\left[1+f(0.02, 1.0202)\right]$$

$$= 1+0.01\left[1+(0.02)^2+1.0202\right]$$

$$= 1.0202.$$

$$y_1^{(1)} = y_1^{(2)} = 1.0202$$

we take $y_1 = y(0.02) = 1.0202$.

To find y_2 i.e $y(0.04)$.

Now $x_1 = 0.02$, $y_1 = 1.0202$, $x_2 = 0.04$, and $h = 0.02$.

$$f(x_1, y_1) = f(0.02, 1.0202) = (0.02)^2 + 1.0202 = 1.0206.$$

using Euler's formula $y_2^{(0)} = y_1 + hf(x_1,y_1)$.

$$= 1.0202 + 0.02(1.0206) = 1.0406$$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2,y_2^{(0)})\right]$$

$$= 1.0202 + (0.01)\left[1.0206 + f(0.04, 1.0406)\right]$$

$$= 1.0202 + 0.01\left[1.0206 + (0.04)^2 + 1.0406\right]$$

$$= 1.0408.$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2,y_2^{(1)})\right]$$

$$= 1.0202 + (0.01)\left[1.0206 + f(0.04, 1.0408)\right]$$

Si

problem:- using Runge-kutta method of second order, compute $y(2.5)$ from $\frac{dy}{dx} = \frac{x+y}{x}$, $y(2) = 2$ taking $h=0.25$

Solution:- $f(x,y) = \frac{x+y}{x}$

$x_0 = 2,\ y_0 = 2$ and $h = 0.25$

$K_1 = h f(x_0, y_0) = 0.25 f(2,2) = 0.25(2) = 0.5$

$K_2 = h f(x_0+h, y_0+K_1) = 0.25\left[f(2.25, 2.5)\right]$

$= 0.25\left[\frac{2.25+2.5}{2.25}\right] = 0.528.$

$y_1 = y(2.25) = y_0 + \frac{1}{2}(K_1+K_2)$

$= 2+\frac{1}{2}(0.5+0.528)$

$y_1 = 2.514.$

Here $x_1 = x_0+h = 2+0.25 = 2.25$

$y_1 = 2.514,\ h = 0.25$

$K_1 = h f(x_1, y_1) = 0.25 f(2.25, 2.594)$

$= 0.251\left(\frac{2.25+2.514}{2.25}\right)$

$K_1 = 0.5293$

$K_2 = h f(x_1+h, y_1+K_1)$

$= 0.25\left[f(2.25)+0.25,\ 2.514+0.5293\right]$

$= 0.25\left[f(2.5, 3.0433)\right]$

$= 0.25\left[\frac{2.5+3.0433}{2.5}\right] = 0.55433$

$y_2 = y(2.5) = y_1 + \frac{1}{2}(K_1+K_2) = 2.514+\frac{1}{2}(0.5293+0.55433)$

TRAPEZOIDAL RULE

The formula is $\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})\right]$

Thus $\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\Big[$(sum of the first and last ordinates) + 2 (sum of the remaining ordinates)$\Big]$

Problem :- Evaluate $\int_0^{\pi/2} e^{\sin x}\,dx$ taking $h=\pi/6$

Solution :- Let $y = e^{\sin x}$.

Length of interval is $\left(\frac{\pi}{2}-0\right)=\frac{\pi}{2}$

The values of 'y' are calculated as follows

taking $h=\pi/6$.

x	:	0	$\pi/6$	$\frac{2\pi}{6}=\frac{\pi}{3}$	$\frac{3\pi}{6}=\frac{\pi}{2}$
$y=e^{\sin x}$	:	1	1.6487	2.3774	2.71828
		y_0	y_1	y_2	y_3

Here $n=3$ we will use Trapezoidal Rule.

By Trapezoidal Rule,

$$\int_0^{\pi/2} e^{\sin x}\,dx = \frac{h}{2}\left[(y_0+y_3)+2(y_1+y_2)\right]$$

$$= \frac{\pi}{12}\cdot\left[(1+2.71828)+2(1.6487+2.3774)\right]$$

$$= \frac{\pi}{12}(11.77048) = 3.0815$$

$\int^{\pi/2} e^{\sin x}\,dx = 3.0815$

SIMPSON'S $\frac{1}{3}$ - RULE

The formula is

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\Big[(y_0+y_n)+4(y_1+y_3+y_5+\cdots+y_{n-1}) + 2(y_2+y_4+y_6+\cdots+y_{n-2})\Big]$$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\Big[(\text{sum of the first and last ordinates}) + 4(\text{sum of the odd ordinates}) + 2(\text{sum of the remaining even ordinates})\Big]$$

Problem:- Evaluate $\int_0^2 e^{-x^2}\,dx$ using Simpson's $\frac{1}{3}$-Rule taking $h=0.25$.

Solution:- The values of $y=f(x)=e^{-x^2}$ are given below.

x :	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
x^2 :	0.0625	0.25	0.5625	1.00	1.5625	2.25	3.0625	4.00
y :	0.93941	0.7788	0.56978	0.36788	0.20961	0.1054	0.04677	0.0183
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7

By Simpson's $\frac{1}{3}$ - Rule, we have.

$$\int_0^2 e^{-x^2}dx = \frac{h}{3}\Big[(y_0+y_7)+4(y_1+y_3+y_5)+2(y_2+y_4+y_6)\Big].$$

$$= \frac{0.25}{3}\Big[[(0.93941)+(0.0183)] + 4(0.7788+0.36788+0.1054) + 2[(0.56978)+0.20961+0.04677)\Big]$$

$= \frac{0.25}{3}$ [0.957 [illegible] 1.65232].

$= \frac{0.25}{3}[7.61835] = 0.6348$

Problem :-

A train is moving at the speed of 30 m/sec. suddenly brakes are applied. The speed of the train per second after 't' seconds is given by.

Time (t) :	0	5	10	15	20	25	30	35	40	45
speed (v) :	30	24	19	16	13	11	10	8	7	5

Apply simpson's $\frac{3}{8}$ - rule to determine the distance moved by the train in 45 seconds.

Sol:- If 's' metres is the distance covered in 't' seconds then $\frac{ds}{dt} = v \Rightarrow [s]_{t=0}^{t=45} = \int_0^{45} v\,dt$.

Since the no. of sub intervals is 9 (a multiple of 3) hence by using simpson's $\frac{3}{8}$th rule.

$$\int_0^{45} v\,dt = \frac{3h}{8}\left[(v_0+v_9)+3(v_1+v_2+v_4+v_5+v_7+v_8) + 2(v_3+v_6)\right]$$

$$= \frac{15}{8}\left[(30+5)+3(24+19+13+11+8+7) + 2(16+10)\right]$$

$$= 624.375 \text{ metres.}$$

Hence the distance moved by the train in 45 seconds is 624.375 metres.