

UNIT-1

* SEQUENCES, SERIES & MEAN VALUE THEOREMS *

sequences: An ordered set of Real numbers $a_1, a_2, a_3, \dots$ is called a sequence and is denoted by " a_n ".

If the Number of terms is unlimited, then the sequence is said to be an infinite sequence and " a_n " is its General term.

Ex: $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$ is infinite sequence.

Limit: A sequence is said to tend to a limit " l ", if for every $\epsilon > 0$, a value " N " of " n " can be found such that $|a_n - l| < \epsilon$, for $n \geq N$.

$$\lim_{n \rightarrow \infty} a_n = l, \text{ (or) } a_n \rightarrow l \text{ as } n \rightarrow \infty.$$

Convergence: If a sequence (a_n) has a finite limit, it is called a convergent sequence.

If a_n is not convergent, it is said to be divergent.

Convergence, Divergence, and oscillation.

If $\lim_{n \rightarrow \infty} a_n = l$ is finite and unique then the sequence is said to be convergent.

if $\lim_{n \rightarrow \infty} a_n$ is infinite ($\pm \infty$), the sequence is said to be divergent.

if $\lim_{n \rightarrow \infty} (a_n)$ is not unique, then a_n is said to be

Oscillatory.

Ex: The series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges.

Sol: Here $u_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ and

$$S_n = u_1 + u_2 + \dots + u_n$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1}\right) = 1$$

$\therefore \{S_n\}$ converges to 1 and hence the series converges to 1
 $\therefore \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

Ratio Test:

D'ALEMBERT'S Ratio test:

Theorem: if $\sum u_n$ is a series of positive terms such that

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = l \text{ then}$$

(a) $\sum u_n$ convergent, if $l > 1$

(b) $\sum u_n$ divergent, if $l < 1$

(c) The test fails to decide the nature of the series if $l = 1$

Ex: Test for convergence of the series $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$

Sol: Here $u_0 = 1$, and $u_n = \frac{1}{n!} > 0$ for all $n \in \mathbb{Z}^+$

$$u_{n+1} = \frac{1}{(n+1)!} \text{ and } \frac{u_n}{u_{n+1}} = \frac{(n+1)!}{n!} = n+1$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} = \lim_{n \rightarrow \infty} (n+1) = \infty > 1$$

By D'Alembert's Ratio test, $\sum u_n$ Converges

Note: The sum of the above convergent series is denoted by 'e'

$$\therefore 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \alpha = \sum_{n=0}^{\infty} \frac{1}{n!} = e$$

Ex: show that $1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + \dots$ is convergent for all values of 'p'.

Sol Here $u_n = \frac{n^p}{n!}$

$$\text{Then } u_{n+1} = \frac{(n+1)^p}{(n+1)!}$$

$$\text{Now } \frac{u_n}{u_{n+1}} = \frac{n^p}{n!} \times \frac{(n+1)!}{(n+1)^p} = \frac{n^p}{(n+1)^{p-1}} = \frac{n^p}{n^{p-1} \left(1 + \frac{1}{n}\right)^{p-1}}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \alpha, \text{ for all values of 'p'}$$

$\therefore$ By D'Alembert's ratio test the series is convergent for all values of 'p'.

Ex: Test for convergence of the series $\frac{1}{2\sqrt{n}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \dots$

$$(08) \sum_{n=1}^{\infty} \frac{x^{2n}}{(n+1)\sqrt{n}}$$

Sol Here $u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}} > 0$ for all 'n'.

$$\therefore u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

$$\text{Now } \frac{u_n}{u_{n+1}} = \frac{x^{2n-2}}{(n+1)\sqrt{n}} \times \frac{(n+2)\sqrt{n+1}}{x^{2n}} = \frac{(n+2)\sqrt{n+1}}{(n+1)\sqrt{n} \cdot x^2}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \frac{1}{x^2}$$

By D'Alembert's ratio test $\sum u_n$ is convergent if $\frac{1}{x^2} > 1$ and divergent if $\frac{1}{x^2} < 1$ (i.e. convergent if $x^2 < 1$ and divergent if $x^2 > 1$)

if $x^2 = 1$ the test fails.

$$\text{when } x^2 = 1, \quad u_n = \frac{1}{(n+1)\sqrt{n}} = \frac{1}{n^{3/2}} \cdot \frac{1}{\left(1 + \frac{1}{n}\right)}$$

$$\begin{aligned} \text{Take } v_n &= \frac{1}{n^{3/2}}; \quad \frac{u_n}{v_n} = \frac{1}{n^{3/2}} \cdot \frac{1}{\left(1 + \frac{1}{n}\right)} \cdot n^{3/2} \\ &= \frac{1}{1 + \frac{1}{n}} \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \frac{1}{1+0} = 1$$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ will both converge
(or) both diverge

$$\text{Here } \sum v_n = \sum \frac{1}{n^{3/2}} \text{ with } p = \frac{3}{2} > 1$$

$\therefore \sum v_n$ is convergent

$\therefore \sum u_n$ is also convergent

$\therefore$ Hence the given series converges if " $x^2 \leq 1$ " and diverges when " $x^2 > 1$ "

Ex: Test for the convergence of $1 + \frac{2}{5}x + \frac{6}{9}x^2 + \dots + \frac{2^n - 2}{2^n + 1}x^{n-1} + \dots$ ($x > 0$).

$$\frac{u_n}{u_{n+1}} = \frac{2^n - 2}{2^n + 1} x^{n-1} \cdot \frac{2^{n+1} + 1}{2^{n+1} - 2} \cdot \frac{1}{x^n}$$

$$= \frac{1 - \frac{2}{2^n}}{1 + \frac{1}{2^n}} \cdot \frac{2 + \frac{1}{2^n}}{2 - \frac{2}{2^n}} \cdot \frac{1}{x}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \frac{1-0}{1+0} \cdot \frac{2+0}{2-0} \cdot \frac{1}{x}$$

$$= \frac{1}{x}$$

Thus by Ratio test, $\sum u_n$ converges for $x > 1$

i.e. for $x < 1$ diverges for $x > 1$, But it fails for $x = 1$

$$\begin{aligned} \text{When } x=1, \quad \lim_{n \rightarrow \infty} u_n &= \lim_{n \rightarrow \infty} \frac{2^n - 2}{2^n + 1} \\ &= \lim_{n \rightarrow \infty} \frac{1 - \frac{2}{2^n}}{1 + \frac{1}{2^n}} = 1 \neq 0 \end{aligned}$$

$\therefore \sum u_n$ diverges for $x=1$, Hence the Given series converges for $x < 1$ and diverges for $x \geq 1$

Ex: Test for convergence $\sum_{n=1}^{\infty} \frac{n^4}{n!}$

Sol: Here $u_n = \frac{n^4}{n!} > 0$ for all 'n' and

$$u_{n+1} = \frac{(n+1)^4}{(n+1)!}$$

$$\text{Now } \frac{u_n}{u_{n+1}} = \frac{n^4}{n!} \cdot \frac{(n+1)!}{(n+1)^4} = \frac{n^4}{(n+1)^3}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{n^4}{(n+1)^3} = \lim_{n \rightarrow \infty} \frac{n}{(n+1)^3} = \infty > 1$$

Comparison Tests :

If two positive term series $\sum u_n$ and $\sum v_n$ be such that

1, $\sum v_n$ converges

2, $u_n \leq v_n$ for all values of 'n', then $\sum u_n$ also converges.

Proof

Since $\sum v_n$ is convergent

$$\lim_{n \rightarrow \infty} (v_1 + v_2 + v_3 + \dots + v_n) = \text{a finite quantity "K" say}$$

Also since $u_1 \leq v_1, u_2 \leq v_2, \dots, u_n \leq v_n$

$$\therefore \text{Adding } u_1 + u_2 + \dots + u_n \leq v_1 + v_2 + \dots + v_n$$

$$\therefore \lim_{n \rightarrow \infty} u_1 + u_2 + \dots + u_n \leq \lim_{n \rightarrow \infty} v_1 + v_2 + \dots + v_n = K$$

$\therefore$ Hence the series $\sum u_n$ also converges.

Ex Test for convergence $\sum_{n=1}^{\infty} \frac{2n-1}{n(n+1)(n+2)}$

Sol let $u_n = \frac{2n-1}{n(n+1)(n+2)} > 0, \forall n$

This is a series of positive terms.

Take $v_n = \frac{1}{n^3} = \frac{1}{n^2}$

We know that $\sum v_n = \sum \frac{1}{n^2}$

$$= \sum \frac{1}{n^p} \text{ with } p=2 > 1 \text{ is convergent by P-test}$$

$$\text{We have } \frac{u_n}{v_n} = \frac{\frac{2n-1}{n(n+1)(n+2)}}{\frac{1}{n^2}} = \frac{n(2-\frac{1}{n})}{n \cdot n \cdot (1+\frac{1}{n})(1+\frac{2}{n})} \cdot n^2$$

$$= \frac{2-\frac{1}{n}}{(1+\frac{1}{n})(1+\frac{2}{n})}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{2 - \frac{1}{n}}{\left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right)}$$

$$= \frac{2}{1 \cdot 1} = 2 \neq 0$$

By comparison test $\sum u_n$ is convergent.

Prob Test for convergence $\sum_{n=1}^{\infty} \{ \sqrt[3]{n^3+1} - n \}$

Sol let $u_n = \sqrt[3]{n^3+1} - n$

since $u_n > 0, \forall n$ this is a series of positive terms

$$u_n = (n^3+1)^{\frac{1}{3}} - n = n \left(1 + \frac{1}{n^3}\right)^{\frac{1}{3}} - n$$

$$= n \left[1 + \frac{1}{3} \cdot \frac{1}{n^3} + \frac{\frac{1}{3}(\frac{1}{3}-1)}{2} \cdot \frac{1}{n^6} + \dots \right] - n$$

$$= \frac{1}{3n^2} - \frac{1}{9n^5} + \dots$$

Take $v_n = \frac{1}{n^2}$

We know that $\sum v_n$ is convergent

$$\text{now } \frac{u_n}{v_n} = \frac{\frac{1}{n^2} \left[\frac{1}{3} - \frac{1}{9n^3} + \dots \right]}{\frac{1}{n^2}}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \frac{1}{3} \neq 0$$

$\therefore$ By comparison test $\sum u_n$ and $\sum v_n$ behave in the same way.

But $\sum v_n = \sum \frac{1}{n^2}$ is convergent.

$\therefore$ By comparison test, $\sum u_n$ is also convergent

Pro Test for convergence of $\sum \frac{1}{\sqrt{n} + \sqrt{n+1}}$

Sol Let $u_n = \frac{1}{\sqrt{n} + \sqrt{n+1}}$

Take $v_n = \frac{1}{\sqrt{n}}$

Now $\frac{u_n}{v_n} = \frac{1}{\sqrt{n} + \sqrt{n+1}} \cdot \sqrt{n} = \frac{\sqrt{n}}{\sqrt{n} + \sqrt{n+1}}$

$$= \frac{\sqrt{n}}{\sqrt{n} (1 + \sqrt{1 + \frac{1}{n}})}$$

$$\lim_{n \rightarrow \infty} \frac{1}{1 + \sqrt{1 + \frac{1}{n}}} = \frac{1}{1 + 1} = \frac{1}{2} \neq 0$$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ will have same property.

But $\sum v_n = \sum \frac{1}{n^{1/2}} = \sum \frac{1}{n^p}$ is divergent

$\therefore$ Hence $\sum u_n$ is also divergent

INTEGRAL TEST:

Let 'f' be a non-negative decreasing function on $[1, \infty)$. Then the series $\sum_{n=1}^{\infty} f(n)$ and the improper integral $\int_1^{\infty} f(x) dx$ converge (or) diverge together.

Ex: Prove that $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges

Sol Let $f(x) = \frac{1}{x^2+1}$ for $x \in [1, \infty)$

Then $\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$

We have $f(x) > 0$ and $f(x)$ is decreasing on $[1, \infty)$

$$\therefore \lim_{t \rightarrow \infty} F(t) = \lim_{t \rightarrow \infty} \left[\tan^{-1} t - \frac{\pi}{4} \right]$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$$\therefore \int_1^{\infty} f(x) dx \text{ converges to } \frac{\pi}{4}$$

$\therefore$ By integral test $\sum f(n)$ also converges.

Ex: Discuss the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n \log n}$

Sol let $f(x) = \frac{1}{x \log x}$, $x \geq 2$

$f(x) > 0$ and $f(x)$ is decreasing in $[2, \infty)$

$$F(t) = \int_2^t \frac{1}{x \log x} dx = \log(\log x) \Big|_2^t$$

$$= \log(\log t) - \log(\log 2)$$

$$\lim_{t \rightarrow \infty} F(t) = \lim_{t \rightarrow \infty} \log(\log t) - \log(\log 2)$$

$$= \infty$$

$\therefore$ By Integral test $\sum f(n)$ diverges.

Ex: using integral test, determine the convergence of the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

Sol Let $f(x) = \frac{x}{x^2+1}$ for $x \in [1, \infty)$. Then

$$\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

We have $f(x) > 0$ and $f(x)$ is decreasing on $[1, \infty)$

$$F(t) = \int_1^t f(x) dx$$

$$= \int_1^t \frac{x}{x^2+1} dx$$

$$= \frac{1}{2} \log (x^2+1)_1^t$$

$$= \frac{1}{2} \log (t^2+1) - \frac{1}{2} \log 2$$

$$\therefore \lim_{t \rightarrow \infty} F(t) = \infty$$

Hence $\int_1^{\infty} f(x) dx$ is divergent

$\therefore$ By Integral test, $\sum_{n=1}^{\infty} f(n)$ is also divergent

CAUCHY'S n^{th} ROOT TEST:

If $\sum u_n$ is a series of Positive terms such that

$$\lim_{n \rightarrow \infty} u_n^{1/n} = l \text{ then}$$

- (a) $\sum u_n$ converges if $l < 1$
- (b) $\sum u_n$ diverges if $l > 1$ and
- (c) The test fails to decide the Nature if $l = 1$

Ex:1 Test for the convergence of $\sum \frac{2^n}{n^3}$

Sol This is a series of Positive terms.

$$\text{Let } u_n = \frac{2^n}{n^3}, \text{ Then}$$

$$\therefore \lim_{n \rightarrow \infty} u_n^{1/n} = \lim_{n \rightarrow \infty} \frac{(2^n)^{1/n}}{(n^3)^{1/n}} = \frac{2}{n^{3/n}} = 2 > 1$$

By Cauchy's n^{th} root Test

$\sum u_n$ is divergent

Ex Test for the convergence of $\sum \left(1 + \frac{1}{n}\right)^{n^2}$

Sol This is a series of positive terms

$$\text{let } u_n = \left(1 + \frac{1}{n}\right)^{n^2}$$

$$\therefore u_n^{1/n} = \left[\left(1 + \frac{1}{n}\right)^{n^2}\right]^{1/n} = \left(1 + \frac{1}{n}\right)^n$$

$$= \frac{1}{\left(1 + \frac{1}{n}\right)^n}$$

$$\lim_{n \rightarrow \infty} u_n^{1/n} = \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} = \frac{1}{e} < 1 \quad (\because 2 < e < 3)$$

$\therefore$ By Cauchy's root test, $\sum u_n$ converges.

Ex Test the convergence of the series

$$\frac{2}{1^2}x + \frac{3^2}{2^3}x^2 + \dots + \frac{(n+1)^n}{n^{n+1}}x^n + \dots, x > 0$$

Sol This is a series of positive terms.

$$u_n = \frac{(n+1)^n}{n^{n+1}} x^n$$

$$\therefore u_n^{1/n} = \frac{n+1}{n \cdot n^{1/n}} x$$

$$= \frac{1 + \frac{1}{n}}{n^{1/n}} x$$

$$\lim_{n \rightarrow \infty} u_n^{1/n} = x \quad \left[\because \lim_{n \rightarrow \infty} \frac{n+1}{n \cdot n^{1/n}} = 1 \right]$$

By Cauchy's root test, $\sum u_n$ is convergent if $x < 1$ and $\sum u_n$ is divergent if $x > 1$. When $x = 1$, the root test fails.

$$\text{when } x = 1, u_n = \frac{(n+1)^n}{n^{n+1}} = \frac{n^n \left(1 + \frac{1}{n}\right)^n}{n^{n+1}} = \frac{\left(1 + \frac{1}{n}\right)^n}{n}$$

Take $v_n = \frac{1}{n}$, $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e \neq 0$

$\therefore$ By comparison test

$\sum u_n$ and $\sum v_n$ behave in the same way,

But $\sum v_n$ is divergent

$\therefore \sum u_n$ is divergent

$\therefore$ Hence the Given series is convergent when $x < 1$ and divergent when $x \geq 1$

Ex Test the convergence of the series $\sum \frac{1}{(\log(\log n))^n}$

Given series is $\sum u_n = \sum \frac{1}{(\log \log n)^n}$

$$n^{\text{th}} \text{ term} = u_n = \frac{1}{(\log \log n)^n}$$

$$\text{Now } (u_n)^{1/n} = \frac{1}{\log(\log n)}$$

$$\therefore \lim_{n \rightarrow \infty} (u_n)^{1/n} = 0 < 1$$

By Cauchy's n^{th} root test, the series is convergent.

ALTERNATING SERIES:

A series whose terms are alternatively positive and negative is called an alternating series.

An Alternating series may be written as,

$$u_1 - u_2 + u_3 - u_4 + \dots + (-1)^{n-1} u_n + \dots$$

where each u_n is positive, it is denoted as

$$\sum_{n=1}^{\infty} (-1)^{n-1} u_n$$

Ex: 1: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$

Ex: 2: $1 - \frac{2}{\log 2} + \frac{3}{\log 3} - \frac{4}{\log 4} + \dots = 1 + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{n}{\log n}$

LEIBNITZ'S TEST:

If $\{u_n\}$ is a sequence of positive terms such that

(a) $u_1 \geq u_2 \geq u_3 \geq \dots \geq u_n \geq u_{n+1} \geq \dots$

(b) $\lim_{n \rightarrow \infty} u_n = 0$

then the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ is convergent

Ex: Prove that $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots$ converges.

sol This is an alternating series and $u_n = \frac{1}{n!}$

Clearly $u_n > 0$ and $u_n > u_{n+1}$ for all n .

Also $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{n!} = 0$

$\therefore$ By Leibnitz's test $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ converges.

x) Test for convergence of the series

$$\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \dots \quad (0 < x < 1)$$

y) This is an alternating series.

consider $u_n = \frac{x^n}{1+x^n}$

Now $u_n - u_{n+1} = \frac{x^n}{1+x^n} - \frac{x^{n+1}}{1+x^{n+1}}$

$$= \frac{x^n(1+x^{n+1}) - (1+x^n)x^{n+1}}{(1+x^n)(1+x^{n+1})}$$

$$= \frac{x^n(1-x)}{(1+x^n)(1+x^{n+1})} > 0 \quad \text{for } 0 < x < 1$$

$$\therefore u_n > u_{n+1} \quad \text{i.e. } u_1 > u_2 > u_3 - \dots$$

Also $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{x^n}{1+x^n} = \frac{0}{1+0} = 0 \quad \left[\because \lim_{n \rightarrow \infty} x^n = 0 \right]$

By Leibnitz's test, the series is convergent.

x) Test for convergence of the following series.

$$\frac{1}{6} - \frac{2}{11} + \frac{3}{16} - \frac{4}{21} + \frac{5}{26} - \dots$$

Let the given series be $\sum_{n=0}^{\infty} u_n$ where

$$u_n = (-1)^{n-1} \frac{n}{5n+1}$$

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} (-1)^{n-1} \frac{1}{5+\frac{1}{n}} \neq 0$$

MEAN VALUE THEOREMSROLLE'S THEOREM :

Let $f(x)$ be a function such that

- ① it is continuous in closed interval $[a, b]$
- ② it is differentiable in open interval (a, b) and
- ③ $f(a) = f(b)$

Ex. Verify Rolle's theorem for $f(x) = 2x^3 + x^2 - 4x - 2$ in $[-\sqrt{3}, \sqrt{3}]$

Sol. $f(x) = 2x^3 + x^2 - 4x - 2$ is a polynomial in " x "

(1) $f(x)$ is continuous for every value of " x "
in particular, $f(x)$ is continuous in $[-\sqrt{3}, \sqrt{3}]$

② $f'(x) = 6x^2 + 2x - 4$ exists for every $x \in (-\sqrt{3}, \sqrt{3})$
 $\Rightarrow f(x)$ is derivable in $(-\sqrt{3}, \sqrt{3})$

$$\begin{aligned}\textcircled{3} \quad f(-\sqrt{3}) &= 2(-\sqrt{3})^3 + (-\sqrt{3})^2 - 4(-\sqrt{3}) - 2 \\ &= -6\sqrt{3} + 3 + 4\sqrt{3} - 2 \\ &= 1 - 2\sqrt{3}\end{aligned}$$

$$\begin{aligned}\text{and } f(\sqrt{3}) &= 2(\sqrt{3})^3 + (\sqrt{3})^2 - 4(\sqrt{3}) - 2 \\ &= 6\sqrt{3} + 3 - 4\sqrt{3} - 2 \\ &= 1 + 2\sqrt{3}\end{aligned}$$

Since $f(-\sqrt{3}) \neq f(\sqrt{3})$, therefore

Rolle's theorem is not applicable for $f(x)$ in $[-\sqrt{3}, \sqrt{3}]$

Ex: Verify Rolle's Theorem for the function

$$\log \left(\frac{x^2+ab}{x(a+b)} \right) \text{ in } [a, b], a > 0, b > 0$$

sol) Let
$$f(x) = \log \left(\frac{x^2+ab}{x(a+b)} \right) = \log(x^2+ab) - \log(x(a+b))$$
$$= \log(x^2+ab) - \log x - \log(a+b)$$

① Since $f(x)$ is a composite function of continuous functions in $[a, b]$ it is continuous in $[a, b]$

$$② f'(x) = \frac{1}{x^2+ab} \cdot 2x - \frac{1}{x} = \frac{x^2-ab}{x(x^2+ab)}$$

$\therefore f'(x)$ exists for all $x \in (a, b)$

$$③ f(a) = \log \left(\frac{a^2+ab}{a^2+ab} \right) = \log 1 = 0$$

$$f(b) = \log \left(\frac{b^2+ab}{b^2+ab} \right) = \log 1 = 0$$

$$\therefore f(a) = f(b)$$

Thus $f(x)$ satisfies all the three conditions of Rolle's theorem.

$\therefore$ There exists $c \in (a, b)$ such that $f'(c) = 0$

$$\Rightarrow \frac{c^2-ab}{c(c^2+ab)} = 0 \Rightarrow c^2 = ab$$

$$\Rightarrow c = \pm \sqrt{ab}$$

$$c = \sqrt{ab} \in (a, b)$$

Ex Verify Rolle's theorem for $f(x) = |x|$ in $[-1, 1]$

sol we have

$$f(x) = |x|$$

$$f(x) = x, \text{ for } x \geq 0$$

$$= -x \text{ for } x < 0$$

① $f(x)$ is continuous for every value of " x "

$\therefore f(x)$ is continuous in the closed interval $[-1, 1]$

② $f(x)$ is not derivable at $x=0$

$$\text{we have } f(0) = |0| = 0$$

$$L.H.D = L f'(0) = \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \rightarrow 0^-} \frac{|x| - 0}{x}$$

$$= \lim_{x \rightarrow 0^-} \frac{-x}{x}$$

$$= \lim_{x \rightarrow 0^-} (-1) = -1$$

$$R.H.D = R f'(0) = \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \rightarrow 0^+} \frac{|x| - 0}{x}$$

$$= \lim_{x \rightarrow 0^+} \frac{x}{x} = \lim_{x \rightarrow 0^+} (1) = 1$$

Since $L f'(0) \neq R f'(0)$, therefore

$f(x)$ is not derivable at $x=0$

$\therefore f(x)$ is not derivable in the open interval $(-1, 1)$

Hence Rolle's theorem is not applicable to
 $f(x) = |x|$ in $[-1, 1]$

LAGRANGE'S MEAN VALUE THEOREM

Let $f(x)$ be a function such that

- ① it is continuous in closed interval $[a, b]$ and
- ② differentiable in open interval (a, b)

then there exists at least one point " c " in open (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Ex. Verify Lagrange's mean value theorem for

$$f(x) = x^3 - x^2 - 5x + 3 \text{ in } [0, 4]$$

Sol. $f(x) = x^3 - x^2 - 5x + 3$ is a polynomial in x .

$\therefore$ it is continuous and derivable for every value of " x ".

In particular, $f(x)$ is continuous in closed $[0, 4]$ and derivable in open $(0, 4)$. Hence by Lagrange's mean value theorem, there exists a point " c " in open $(0, 4)$ such that.

$$f'(c) = \frac{f(4) - f(0)}{4 - 0}$$

$$3c^2 - 2c - 5 = \frac{f(4) - f(0)}{4}$$

$$\text{Now } f(4) = 4^3 - 4^2 - 5(4) + 3$$

$$= 31$$

$$\text{and } f(0) = 3$$

$$\therefore \frac{f(4) - f(0)}{4} = \frac{31 - 3}{4} = \frac{28}{4} = 7$$

from ① we have

$$3C^2 - 2C - 5 = 7 \Rightarrow 3C^2 - 2C - 12 = 0$$

$$\therefore C = \frac{2 \pm \sqrt{4 + 144}}{6}$$

$$= \frac{2 \pm \sqrt{148}}{6}$$

$$= \frac{1 \pm \sqrt{37}}{3}$$

We see that $\frac{1 + \sqrt{37}}{3}$ lies in open $(0, 4)$ and then

Lagrange's mean value theorem is verified.

Ex: If $a < b$ prove that $\frac{b-a}{(1+b^2)} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{(1+a^2)}$ using Lagrange's mean value theorem. Deduce the following.

$$\textcircled{1} \quad \frac{\pi}{4} + \frac{3}{35} < \tan^{-1} \frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$$

$$\textcircled{2} \quad \frac{5\pi + 4}{20} < \tan^{-1} 2 < \frac{\pi + 2}{4}$$

Sol: Consider $f(x) = \tan^{-1}x$ in $[a, b]$ for $0 < a < b < 1$

since $f(x)$ is continuous in closed $[a, b]$ and derivable in open (a, b) , we can apply Lagrange's mean value theorem here.

Hence there exists a point c in open (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\text{Here } f'(x) = \frac{1}{1+x^2} \text{ and hence } f'(c) = \frac{1}{1+c^2}$$

Thus there exists a point c , $a < c < b$ such that

$$\frac{1}{1+c^2} = \frac{\tan^{-1}b - \tan^{-1}a}{b-a}$$

$$\text{we have } 1+a^2 < 1+c^2 < 1+b^2$$

$$\Rightarrow \frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$$

using ① and ② we have

$$\frac{1}{1+a^2} > \frac{\tan^{-1}b - \tan^{-1}a}{b-a} > \frac{1}{1+b^2}$$

$$(or) \frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$$

① Taking $b = \frac{4}{3}$, $a=1$ we get

$$\frac{\frac{4}{3}-1}{1+\frac{16}{9}} < \tan^{-1}\frac{4}{3} - \tan^{-1}1 < \frac{\frac{4}{3}-1}{1+1^2}$$

$$\Rightarrow \frac{\frac{4-3}{3}}{\frac{25}{9}} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{\frac{1}{3}}{\frac{2}{2}}$$

$$\Rightarrow \frac{3}{25} + \frac{\pi}{4} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$$

② Taking $b=2$ and $a=1$ we get

$$\frac{2-1}{1+2^2} < \tan^{-1}2 - \tan^{-1}1 < \frac{2-1}{1+1^2}$$

$$\Rightarrow \frac{1}{5} < \tan^{-1}2 - \frac{\pi}{4} < \frac{1}{2}$$

$$\Rightarrow \frac{1}{5} + \frac{\pi}{4} < \tan^{-1}2 < \frac{1}{2} + \frac{\pi}{4}$$

$$\Rightarrow \frac{4+5\pi}{20} < \tan^{-1}2 < \frac{2+\pi}{4}$$

CAUCHY'S MEAN VALUE THEOREM

if $f: [a, b] \rightarrow \mathbb{R}$, $g: [a, b] \rightarrow \mathbb{R}$ are such that

- ① f, g are continuous on $[a, b]$
- ② f, g are differentiable on (a, b) and
- ③ $g'(x) \neq 0 \quad \forall x \in (a, b)$

then there exists a point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Another form of Cauchy's Mean Value Theorem:

if $f: [a, a+h] \rightarrow \mathbb{R}$, $g: [a, a+h] \rightarrow \mathbb{R}$ are such that

- ① f, g are continuous on $[a, a+h]$
- ② f, g are differentiable on $(a, a+h)$
- ③ $g'(x) \neq 0, \quad \forall x \in (a, a+h)$

then there exists $0 < \theta < 1$ such that

$$\frac{f(a+h) - f(a)}{g(a+h) - g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)}$$

Pro Verify Cauchy's mean value theorem for $f(x) = x^2$, $g(x) = x^3$ in $[1, 2]$

Sol we have $f(x) = x^2$, $g(x) = x^3$ are of the form $x^n, n \in \mathbb{Z}$

$\therefore f, g$ are continuous on $[1, 2]$ and f, g are differentiable on $(1, 2)$

$$\text{Also } g'(x) = 3x^2 \neq 0 \quad \forall x \in (1, 2)$$

$\therefore f, g$ satisfy the conditions of Cauchy's mean value theorem

There exists $c \in (a, b)$ such that

$$\frac{f(2) - f(1)}{g(2) - g(1)} = \frac{f'(c)}{g'(c)} \Rightarrow \frac{4-1}{8-1} = \frac{2c}{3c^2} \Rightarrow c = \frac{14}{9} \in (1, 2)$$

This verifies Cauchy's mean value theorem.

TAYLOR'S THEOREM:

If $f: [a, b] \rightarrow \mathbb{R}$ is such that

(a) $f^{(n-1)}$ is continuous on $[a, b]$

(b) $f^{(n-1)}$ is derivable on (a, b) (or)

f^n exists on (a, b) and $P \in \mathbb{Z}^+$ then there exist a point $c \in (a, b)$ such that

$$f(b) = f(a) + \frac{b-a}{1!} f'(a) + \frac{(b-a)^2}{2!} f''(a) + \dots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n$$

$$\text{where } R_n = \frac{(b-a)^p (b-c)^{n-p}}{(n-1)! p} f^{(n)}(c)$$

Ex 10: Verify Taylor's theorem for $f(x) = (1-x)^{5/2}$ with Lagrange's form of remainder upto 2 terms in the interval $[0, 1]$

Sol: Consider $f(x) = (1-x)^{5/2}$ in $[0, 1]$

(1) $f(x)$, $f'(x)$ are continuous in $[0, 1]$

(2) $f''(x)$ is differentiable in $(0, 1)$

Thus $f(x)$ satisfies the conditions of Taylor's theorem.

We consider Taylor's theorem with Lagrange's form of Remainder

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(\theta x) \quad \text{with } 0 < \theta < 1$$

Here $n = p = 2$, $a = 0$, and $x = 1$

$$f(x) = (1-x)^{5/2} \Rightarrow f(0) = 1$$

$$f'(x) = -\frac{5}{2} (1-x)^{3/2} \Rightarrow f'(0) = -\frac{5}{2}$$

$$f''(x) = \frac{15}{4} (1-x)^{1/2} \Rightarrow f''(\theta x) = \frac{15}{4} (1-\theta x)^{1/2}$$

$$f''(0) = \frac{15}{4} (1-0)^{1/2}$$

$$\text{and } f(1) = 0$$

$$\text{we have } f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(\theta x)$$

substituting the above values we get

$$0 = 1 + 1\left(-\frac{5}{2}\right) + \frac{1^2}{2!} \cdot \frac{15}{4} (1-\theta)^{1/2}$$

$$\Rightarrow \theta = \frac{9}{25} = 0.36$$

$\therefore \theta$ lies between "0 and 1"

Thus the Taylor's theorem is verified.

MAGLAURIN'S THEOREM

if $f: [a, x] \rightarrow \mathbb{R}$ is such that

① $f^{(n-1)}$ is continuous on $[a, x]$

② $f^{(n-1)}$ is derivable on (a, x) and $p \in \mathbb{Z}^+$ then there exists a real number $\theta \in (0, 1)$ such that

$$f(x) = f(a) + x f'(a) + \frac{x^2}{2!} f''(a) + \dots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{x^n (1-\theta)^{n-p}}{p(n-1)!} f^{(n)}(\theta x)$$

Prove: show that $\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \dots$ and hence deduce that

$$\frac{e^x}{e^x + 1} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \dots$$

sol

$$f(x) = \log(1+e^x) \Rightarrow f(0) = \log 2$$

$$f'(x) = \frac{1}{1+e^x} e^x \Rightarrow f'(0) = \frac{1}{2}$$

$$f''(x) = \frac{(1+e^x)e^x - e^x(e^x)}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2} \Rightarrow f''(0) = \frac{1}{4}$$

$$f'''(x) = \frac{(1+e^x)^2 e^x - e^x \cdot 2(1+e^x)e^x}{(1+e^x)^4}$$

$$= \frac{(1+e^x)e^x - 2e^{2x}}{(1+e^x)^3}$$

$$= \frac{e^x(1+e^x-2e^x)}{(1+e^x)^3} \Rightarrow f'''(0) = 0$$

$$f^{(4)}(x) = \frac{(1+e^x)^3(e^x - 2e^{2x}) - (e^x - e^{2x})3(1+e^x)^2 \cdot e^x}{(1+e^x)^6}$$

$$= \frac{(1+e^x)^3(e^x - 2e^{2x}) - 3e^x(e^x - e^{2x})(1+e^x)^2}{(1+e^x)^4}$$

$$= \frac{e^{3x} - 4e^{2x} + e^x}{(1+e^x)^4}$$

$$f^{(4)}(0) = \frac{-2}{16} = -\frac{1}{8}$$

using Maclaurin's series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$$

$$\therefore \log(1+e^x) = \log 2 + \frac{1}{2}x + \frac{x^2}{8} - \frac{x^4}{192} + \dots \propto$$

Diff w.r.t 'x' on both sides, we get

$$\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \dots \quad \text{Hence the result}$$

Q10! ~~show~~ that obtain the Maclaurin's series expansion for $f(x) = (1+x)^n$

Sol Let $f(x) = (1+x)^n$, Then

$$f'(x) = n(1+x)^{n-1}; \quad f''(x) = n(n-1)(1+x)^{n-2}$$

$$f'''(x) = n(n-1)(n-2)(1+x)^{n-3} \text{ etc}$$

$$\text{Hence } f^k(x) = n(n-1)(n-2) \dots (n-k+1)(1+x)^{n-k}$$

$$f^k(0) = n(n-1) \dots (n-k+1)$$

$$\text{Hence } f(x) = \sum_{k=0}^{\infty} f^k(0) \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{n(n-1) \dots (n-k+1)}{1 \cdot 2 \cdot 3 \dots k} \cdot x^k$$

$$\text{i.e. } (1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1 \cdot 2}x^2 + \dots$$