UNIT-I

LINEAR SYSTEM OF EQUATIONS

SUB-MATRIX OF A MATRIX

Suppose A is any matrix of the type $m \times n$, then a matrix obtained by deleting some rows & columns or both from given matrix A is called a sub-matrix of A.

Eg: Let $A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & -1 & 4 & 6 \\ 3 & 5 & 6 & 7 \end{bmatrix}_{3\times4}$

then Submatrix of A is $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \end{bmatrix}$ (By delating 3rd row and 4th column from A)

MINOR OF A MATRIX

Let A be $m \times n$ matrix. The determinant of a square sub-matrix of A is said to be the minor of the matrix.

Eg:- let $A = \begin{bmatrix} 1 & 4 & 5 & 8 \\ 2 & -1 & 6 & 9 \\ 6 & 9 & 3 & 4 \end{bmatrix}$

This matrix has 3-rowed, 2-rowed and 1-rowed minors

$\begin{vmatrix} 1 & 4 & 5 \\ 2 & -1 & 6 \\ 6 & 9 & 3 \end{vmatrix}, \begin{vmatrix} 4 & 5 & 8 \\ -1 & 6 & 9 \\ 9 & 3 & 4 \end{vmatrix}$ etc are 3-rowed minors of A

$\begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix}, \begin{vmatrix} 4 & 5 \\ -1 & 6 \end{vmatrix}, \begin{vmatrix} 5 & 8 \\ 6 & 9 \end{vmatrix}$ etc are 2-rowed minors of A

Each element of the matrix is called 1-rowed minors of A.

ELEMENTARY TRANSFORMATIONS (OR OPERATIONS) ON A MATRIX

There are three types of elementary transformations on a matrix

~~(i) Interchanging ith and jth rows, denoted by $R_i \leftrightarrow R_j$~~

ii, Multiplication of any row with a non-zero scalar K, denoted by $R_i \to KR_i$

iii, Addition to any row say i^{th} with K times the corresponding elements of j^{th} row, denoted by $R_i \to R_i + KR_j$

ll^y, the elementary column operations are given by

$$C_i \leftrightarrow C_j$$
$$C_i \to KC_i$$
$$C_i \to C_i + KC_j$$

ZERO ROW AND NON-ZERO ROW

If all the elements in a row of a matrix are zeros, then it is called a zero row and if there is atleast one non-zero element in a row, then it is called a non-zero row.

Eg: $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \to \text{non-zero rows} \\ \} \to \text{zero row.} \end{matrix}$

EQUIVALENT MATRICES

Two matrices A and B are said to be Equivalent if any one of the matrix say B can be obtained by applying elementary operations on A. It is denoted by

$A \sim B$.

RANK OF A MATRIX

A matrix is said to be of rank r if

(i) it has atleast one non-zero minor of order r

and ii, Every minor of order higher than r zero.

Briefly, The rank of a matrix is the highest order of any non-zero minor of the matrix.

The rank of A is denoted by $\rho(A)$

Properties

(i) Rank of a matrix is always unique.

ii, Every matrix will have rank.

iii, Rank of a null matrix is zero.

iv, Rank of the Identity matrix I_n is n

(v) if A is a matrix of order n and A is non-singular then $\rho(A) = n$

vi, The rank of singular matrix of order n is $< n$

(vii) if A is a matrix of order $m \times n$, then the rank of $A = \rho(A) \leq \min(m,n)$

viii $\rho(A) = \rho(A^T)$

METHODS TO FIND THE RANK OF A MATRIX

Method 1: The rank can be determined by finding the highest order non-zero minor of matrix.

Method 2: Echelon form (Triangular form of a matrix)

Method 3: Normal form

Method 4: Normal form of the type PAQ

Method I:

Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$

Soln: Minor of order 3 = $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{vmatrix} = 0$

The rank of the matrix cannot be 3 since third order minor is zero.

Minor of second order $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2 \neq 0$

The second order minor is not equal to zero.

$\therefore$ The rank is 2

Find the rank of the matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 6 \\ 2 & 1 & 3 \end{bmatrix}$

Soln:

Minor of the third order = 0

Minor of second order

$$\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 0 \quad \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 0 \quad \begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} = 0 \quad \begin{vmatrix} 2 & 6 \\ 1 & 3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0 \quad \begin{vmatrix} 4 & 6 \\ 2 & 3 \end{vmatrix} = 0$$

All the possible minors of second order are zero.

$\therefore$ The rank of the matrix is 1.

Method 2 : Echelon form

A matrix is said to be in Echelon form if it has the following three properties are satisfied

(i) Zero rows, if any, must be below the non-zero rows.

ii, The first non-zero element of a non-zero is equal to 1.

iii, The number of zero rows before the first non zero element of a row is less than such zero's in the next row.

NOTE: After reducing the matrix to Echelon form, the number of non-zero rows gives the rank of the matrix.

Reduce the matrix $A = \begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$ into Echelon form and hence find its rank.

Soln:

Given matrix A is $A = \begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & -3 & -1 \\ 3 & -2 & 4 \\ 2 & 3 & 7 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$; $R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 7 & 7 \\ 0 & 9 & 9 \end{bmatrix}$$

$R_2 \to \frac{R_2}{7}$ $\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 9 & 9 \end{bmatrix}$

$R_3 : R_3 - 9R_2$

$$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This is in Echelon form

$\therefore$ The number of non-zero rows is 2

Hence the rank of A = 2

Reduce to Echelon form and hence find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & -4 & 5 \\ 2 & -1 & 3 & 6 \\ 8 & 1 & 9 & 7 \end{bmatrix}$

Soln: Given matrix is $A = \begin{bmatrix} 1 & 2 & -4 & 5 \\ 2 & -1 & 3 & 6 \\ 8 & 1 & 9 & 7 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 8R_1$

$$\sim \begin{bmatrix} 1 & 2 & -4 & 5 \\ 0 & -5 & 11 & -4 \\ 0 & -15 & 41 & -33 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$\sim \begin{bmatrix} 1 & 2 & -4 & 5 \\ 0 & -5 & 11 & -4 \\ 0 & 0 & 8 & 21 \end{bmatrix}$ This is in Echelon form

Hence the number of non-zero rows = 3

$\rho(A) = 3$

... value of K the matrix $A = \begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ K & 2 & 2 & 2 \\ 9 & 9 & K & 3 \end{bmatrix}$ has rank 3

Soln:

Given matrix is $A = \begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ K & 2 & 2 & 2 \\ 9 & 9 & K & 3 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ K & 2 & 2 & 2 \\ 9 & 9 & K & 3 \end{bmatrix}$$

$R_2 : R_2 - 4R_1$; $R_3 : R_3 - KR_1$, $R_4 : R_4 - 9R_1$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 2-K & 2+K & 2 \\ 0 & 0 & K+9 & 3 \end{bmatrix}$$

Since the given matrix is of rank 3. then its 4th order minor is zero

ie $|A| = 0$

$$\begin{vmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 2+K & 2+K & 2 \\ 0 & 0 & K+9 & 3 \end{vmatrix} = 0$$

Expand along C_1,

$$\begin{vmatrix} 0 & 1 & 1 \\ 2-K & 2+K & 2 \\ 0 & K+9 & 3 \end{vmatrix} = 0$$

Again Expand along C_1,

$$-(2-K) \begin{vmatrix} 1 & 1 \\ K+9 & 3 \end{vmatrix} = 0$$

$\Rightarrow -(2-K)[3-K-9] = 0$

$\Rightarrow (-2+K)(-K-6) = 0$

$\Rightarrow K-2 = 0$ or $-K-6 = 0$

$\Rightarrow K = 2$ or -6

Method 3: Normal form or canonical form of a matrix

(Applying elementary row and column operations)

Every $m\times n$ matrix of rank r can be reduced by a finite number of elementary transformations to the form

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

where I_r is the r-rowed unit matrix

The form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ is called the normal form (or) canonical form.

The various normal forms are $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$, $[I_3 \; 0]$, $[I_4]$ etc

Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$ to canonical form and find its rank.

Soln:- Given matrix is $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$

$R_2 : R_2 + 2R_1$; $R_3 : R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix}$$

$C_2 : C_2 - 2C_1$; $C_3 : C_3 - C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix}$$

$R_2 : \frac{R_2}{8}$ $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{5}{8} & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix}$

$R_3 : R_3 + 2R_2$ $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5/8 & 0 \\ 0 & 0 & 1+\cancel{2}\left(\frac{5}{\cancel{8}4}\right) & -8 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{3}{8} & 0 \\ 0 & 0 & \frac{9}{4} & -8 \end{bmatrix}$$

$C_3 : C_3 - \frac{3}{8}C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{9}{4} & -8 \end{bmatrix}$$

$R_3 \to \frac{4}{9}R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{32}{9} \end{bmatrix}$$

$C_4 : C_4 + \frac{32}{9}C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim [I_3 \; 0]$$

This is the canonical form of the matrix A.

Hence the Rank of A = 3

Find the rank of the matrix $\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ by reducing to canonical form

Soln: Given matrix is $A = \begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$

$R_1 \Rightarrow R_3$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 4 & 2 & 0 & 2 \\ 2 & -2 & 0 & 6 \\ 1 & -2 & 1 & 2 \end{bmatrix}$$

$R_2 : R_2 - 4R_1$; $R_3 : R_3 - 2R_1$; $R_4 : R_4 - R_1$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

$C_2 : C_2 - 2C_1 \;;\; C_4 : C_4 - 3C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

$R_3 \rightleftarrows R_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & -10 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \rightleftarrows R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to (-1)R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 : R_3 - 6R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 6 & -16 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 : C_3 + C_2 \;;\; C_4 : C_4 - C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 6 & -16 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 : \frac{R_3}{6}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -16/6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -8/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 : C_4 + \frac{8}{3} C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$ This is the normal form

Rank of $A = 3$

Method: Normal form of the type PAQ

Every $m\times n$ matrix of rank r can be transformation to the form $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by elementary transformations where P and Q are non-singular matrices of orders m and n.

NOTE: if A be $m\times n$ matrix, then P and Q are square matrices of order m and n respectively.

Procedure:-

(i) write $A = I_m A_{m\times n} I_n$

ii, Reduce the matrix A on L.H.S to normal form by applying elementary transformations

iii, Every elementary row transformations on A must be accompained by the same transformation on the pre factor of R.H.S

(iv) Every elementary column transformation on A must be accompained by the same transformations on the post factor on R.H.S

Find the non-singular matrices P and Q such that PAQ is in the normal form of the matrix and find the rank of a matrix $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$

Soln: Given matrix is $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}_{3\times 4}$

$$A = I_3 A I_4$$

$$\begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & -6 & -5 & 7 \\ 0 & -6 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1$; $C_3 \to C_3 - 3C_1$; $C_4 : C_4 + 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -5 & 7 \\ 0 & -6 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 : R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -5 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to \left(\frac{-1}{6}\right) C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{1}{3} & -3 & 2 \\ 0 & -\frac{1}{6} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 : C_3 + 5C_2$; $C_4 : C_4 - 7C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{1}{3} & -\frac{4}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{6} & -\frac{5}{6} & \frac{7}{6} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

where $P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{4}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{6} & -\frac{5}{6} & \frac{7}{6} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

are non-singular matrices of A. $[\because |P| \neq 0$, $|Q| \neq 0]$

$\therefore$ Rank of $A = 2$

$A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 4 & 2 & 2 & -1 \\ 2 & 2 & 0 & 2 \end{bmatrix}$, find two non-Singular matrices P and Q Such that PAQ is in normal form.

Soln: Given matrix is $A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 4 & 2 & 2 & -1 \\ 2 & 2 & 0 & -2 \end{bmatrix}_{3\times 4}$

$$A = I_3 A I_4$$

$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ 4 & 2 & 2 & -1 \\ 2 & 2 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 : R_2 - 4R_1$, $R_3 : R_3 - 2R_1$

$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 6 & 6 & -9 \\ 0 & 4 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 : C_2 + C_1$; $C_3 : C_3 + C_1$; $C_4 : C_4 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 6 & -9 \\ 0 & 4 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 : \frac{R_2}{6}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -9/6 \\ 0 & 4 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4/6 & 1/6 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 : R_3 - 4R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & \frac{-3}{2} \\ 0 & 0 & -2 & -6+4\left(\frac{3}{2}\right) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{6} & 0 \\ -2+\frac{8}{3} & 0-\frac{4}{6} & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -\frac{3}{2} \\ 0 & 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & \frac{1}{6} & 0 \\ \frac{2}{3} & -\frac{2}{3} & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{3}{2} \\ 0 & 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1/6 & 0 \\ \frac{2}{3} & -\frac{2}{3} & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 : C_4 + \frac{3}{2} C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & \frac{1}{6} & 0 \\ 2/3 & -2/3 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 & -2+\frac{3}{2}(1) \\ 0 & 1 & -1 & 0+\frac{3}{2}(1) \\ 0 & 0 & 1 & 0+0 \\ 0 & 0 & 0 & 1+0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & \frac{1}{6} & 0 \\ 2/3 & -2/3 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 & -1/2 \\ 0 & 1 & -1 & 3/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 : \frac{R_3}{-2}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & \frac{1}{6} & 0 \\ -1/3 & \frac{1}{3} & -\frac{1}{2} \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 & -1/2 \\ 0 & 1 & -1 & 3/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[I_3 \; 0] = PAQ$$

where $P = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & \frac{1}{6} & 0 \\ -1/3 & \frac{1}{3} & -\frac{1}{2} \end{bmatrix}$ $Q = \begin{bmatrix} 1 & 1 & 0 & -1/2 \\ 0 & 1 & -1 & 3/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$[\because |P| \neq 0 \quad |Q| \neq 0]$

$\therefore$ The rank of $A = 3$.

THE INVERSE OF A NON-SINGULAR MATRIX BY ELEMENTARY TRANFORMATIONS (GAUSS-JORDAN METHOD)

Suppose A is a non-singular matrix of order n.

Then we write $A = I_n A$ —— ①

We reduce the matrix A on the L.H.S to be Identity matrix I_n by applying elementary row transformations only. Each elementary row transformation will be applied to the prefactor I_n of the R.H.S of Equation ①

ie ① becomes $I_n = BA$

The B is the inverse of A

ie $B = A^{-1}$

Find the Inverse of matrix $\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ using elementary operations.

Soln: Let $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}_{3\times3}$

$A = I_n A$

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 = R_2 - 2R_1$; $R_3 = R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & -1 & 1 \end{bmatrix} A$$

$R_2 : \frac{R_2}{3}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{3} \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & -1 & 1 \end{bmatrix} A$$

$R_3 : R_3 + 2R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 \\ -\frac{2}{3} & \frac{1}{3} & 1 \end{bmatrix} A$$

$R_3 : (-\frac{3}{2}) R_3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1/3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1/3 & 2/3 & 0 \\ 1 & -1/2 & -3/2 \end{bmatrix} A$$

$R_2 : R_2 + \frac{1}{3} R_3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1/2 & -1/2 \\ 1 & -1/2 & -3/2 \end{bmatrix} A$$

$R_1 : R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -1/2 \\ 1 & -1/2 & -3/2 \end{bmatrix} A$$

$R_1 : R_1 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & \frac{1}{2} & -1/2 \\ 1 & -1/2 & -3/2 \end{bmatrix} A$$

$$I_3 = BA$$

$\therefore$ where $B = A^{-1} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & \frac{1}{2} & -1/2 \\ 0 & -1/2 & -3/2 \end{bmatrix}$

Find the inverse of the matrix $\begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix}$ by using E - transformations

Soln:

$$\text{Let } A = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix}_{4\times4}$$

$$A = I_4 A$$

$$\begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$R_3 : R_3 - 2R_1$; $R_4 : R_4 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -2 & -3 \\ 0 & 1 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} A$$

$R_1 : R_1 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -2 & -3 \\ 0 & 1 & -1 & -3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} A$$

$R_4 : R_4 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & -3 & -5 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{bmatrix} A$$

$R_2 : R_2 + R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & -3 & -5 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{bmatrix} A$$

$R_3 : R_3 - R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -3 & -5 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 0 & 1 & -1 \\ -1 & -2 & 0 & 1 \end{bmatrix} A$$

$R_4 : R_4 + 3R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 0 & 1 & -1 \\ 2 & -2 & 3 & -2 \end{bmatrix} A$$

$R_1 : R_1 - R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 3 & -3 & 2 \\ 1 & -2 & 1 & 0 \\ 1 & 0 & 1 & -1 \\ 2 & -2 & 3 & -2 \end{bmatrix} A$$

$R_2 : R_2 + R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 4 & -2 \\ 1 & 0 & 1 & -1 \\ 2 & -2 & 3 & -2 \end{bmatrix} A$$

$R_3 : R_3 - 2R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 4 & -2 \\ -3 & 4 & -5 & 3 \\ 2 & -2 & 3 & -2 \end{bmatrix} A$$

$$I_4 = BA$$

where $B = A^{-1} = \begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 4 & -2 \\ -3 & 4 & -5 & 3 \\ 2 & -2 & 3 & -2 \end{bmatrix}$

LINEAR SYSTEM OF EQUATIONS

Consider the System of m linear Equations in n unknowns $x_1, x_2, \dots x_n$ as given below

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots\dots + a_{mn}x_n = b_m \end{array}\right\} \text{—— (1)}$$

where all the a's and b's are constant.

Matrix form is

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ - & - & - & - \\ - & - & - & - \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$A \quad X = B \quad \text{—— (2)}$$

Where $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ - & - & - & - \\ - & - & - & - \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$ is called the coefficient Matrix

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is the matrix of unknown.

$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ is the column matrix of constants.

if $B=0$, the system (2) is said to be homogeneous.

if atleast one of $b_1, b_2, \dots b_n$ is not zero, the system (2) is said to be non-homogeneous.

Consistent:

A System of Equations ① is said to be consistent, if the system has a solution.

if the given system is consistent, it can have either unique solution (or) infinitely many solutions.

Inconsistent:

A system of Equations ① is said to be inconsistent, if the system has no solution.

Augmented Matrix

A matrix which is obtained by adding the column matrix B as the last column in the coefficient matrix A and is denoted by [AB] is called augmented matrix

ie

$$[AB] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ - & - & - & - & - \\ - & - & - & - & - \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

SOLUTION OF SYSTEM OF LINEAR EQUATIONS

Method : 1 : RANK METHOD

1. Non-Homogeneous linear Equations

Consider m linear non-Homogeneous in "n" unknowns as given below

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\cdots\cdots\cdots$$
$$\cdots\cdots\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

Matrix form is $AX = B$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Here A is called the coefficient matrix and [AB] is called the augmented matrix.

1. Condition of consistency

The system of equation $AX = B$ is consistent if and only if rank of A = rank of [AB]

2. Nature of Solution

The system of equations $AX = B$ is said to be

(i) consistent if $\rho(A) = \rho(AB)$

ii, consistent and an unique solution if $\rho(A) = \rho(AB) = r = n$

where r is the rank and 'n' is the number of unknowns.

iii, consistent and an infinite number of solutions if $\rho(A) = \rho(AB) = r < n$

In this case, we have to give arbitrary values to $n-r$ variables and the remaining variables expressed in terms of these arbitrary values.

(iv) Inconsistent if $\rho(A) \neq \rho(AB)$

3. Method of finding the rank A and rank [AB]

Reduce the augmented matrix [AB] to Echelon form by elementary row transformations. Then the matrix A will be reduced to the Echelon form.

Test whether the following Equations are consistent, if so solve them

$x+y+2z=4$; $2x-y+3z=9$; $3x-y-z=2$

Soln: Given system of Equations can be written in matrix form

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$$

$$A \quad X = B$$

Augmented matrix is $[AB] = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & 3 & 9 \\ 3 & -1 & -1 & 2 \end{bmatrix}$

$R_2 : R_2 - 2R_1$; $R_3 : R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & -4 & -7 & -10 \end{bmatrix}$$

$R_2 : \frac{R_2}{-3}$

$$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & -4 & -7 & -10 \end{bmatrix}$$

$R_3 : R_3 + 4R_1$

$$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & -7+\frac{4}{3} & -10+4(-\frac{1}{3}) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & -\frac{17}{3} & -\frac{34}{3} \end{bmatrix}$$

$R_3 : \left(\frac{-3}{17}\right) R_3$
$$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

This is in Echelon form

rank of A = 3

rank of AB = 3

number of unknowns (n) = 3

$\rho(A) = \rho(AB) = n$

The given system is consistent and it has a unique solution.

$$x + y + 2z = 4 \quad \text{—} (1)$$

$$y + \frac{1}{3}z = -\frac{1}{3} \quad \text{—} (2)$$

$$z = 2 \quad \text{—} (3)$$

$$y + \frac{1}{3}(2) = -\frac{1}{3}$$

$$\Rightarrow y = -\frac{1}{3} - \frac{2}{3} = \frac{-1-2}{3} = -\frac{\cancel{3}}{\cancel{3}}$$

$$\Rightarrow \boxed{y = -1}$$

$$x - 1 + 4 = 4$$

$$\Rightarrow x + 3 = 4$$

$$\Rightarrow x = 4 - 3 = 1$$

$$\therefore \boxed{x = 1}$$

$\therefore$ $x = 1;\ y = -1;\ z = 2$ is the unique solution of given system of Equations.

Test for consistency and solve

$$5x + 3y + 7z = 4$$

$$3x + 26y + 2z = 9$$

$$7x + 2y + 10z = 5$$

Soln: Given system of Equations can be written in the matrix form

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

$$A \quad X = B$$

Augmented matrix is $[AB] = \begin{bmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{bmatrix}$

$R_1 : \frac{R_1}{5}$ $\begin{bmatrix} 1 & \frac{3}{5} & \frac{7}{5} & \frac{4}{5} \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{bmatrix}$

$R_2 : R_2 - 3R_1$

$$\begin{bmatrix} 1 & \frac{3}{5} & \frac{7}{5} & \frac{4}{5} \\ 3-3 & 26-\frac{9}{5} & 2-\frac{21}{5} & 9-\frac{12}{5} \\ 7 & 2 & 10 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{3}{5} & \frac{7}{5} & \frac{4}{5} \\ 0 & \frac{121}{5} & \frac{-11}{5} & \frac{33}{5} \\ 7 & 2 & 10 & 5 \end{bmatrix}$$

$R_3 : R_3 - 7R_1$

$$\begin{bmatrix} 1 & 3/5 & 7/5 & \frac{4}{5} \\ 0 & \frac{121}{5} & \frac{-11}{5} & 33/5 \\ 7-7 & 2-\frac{21}{5} & 10-\frac{49}{5} & 5-\frac{28}{5} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3/5 & 7/5 & 4/5 \\ 0 & \frac{121}{5} & \frac{-11}{5} & 33/5 \\ 0 & -\frac{11}{5} & \frac{1}{5} & -3/5 \end{bmatrix}$$

$R_2 : \left(\frac{5}{121}\right) R_2$

$$\begin{bmatrix} 1 & \frac{3}{5} & 7/5 & \frac{4}{5} \\ 0 & 1 & \frac{-1}{11} & \frac{3}{11} \\ 0 & -\frac{11}{5} & \frac{1}{5} & -3/5 \end{bmatrix}$$

$R_3 : R_3 + \frac{11}{5}(R_2)$

$$\begin{bmatrix} 1 & 3/5 & 7/5 & \frac{4}{5} \\ 0 & 1 & -1/11 & 3/11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank of $A = 2$

rank of $[AB] = 2$

number of unknowns $(n) = 3$

$\rho(A) = \rho(B) = 2 < 3 = n$

The given system is consistent and it has an infinite number of solutions.

$x + \frac{3}{5}y + 7/5\,z = 4/5$ — (1)

$y - 1/11\,z = 3/11$ — (2)

$n - r = 3 - 2$

put $z = k$

$$y - \frac{1}{11}k = \frac{3}{11}$$

$$\Rightarrow y = \frac{3}{11} + \frac{k}{11}$$

$$\Rightarrow y = \frac{3+k}{11}$$

$$x + \frac{3}{5}\left(\frac{3+k}{11}\right) + \frac{7}{5}(k) = \frac{4}{5}$$

$$\Rightarrow \frac{55x + 9 + 3k + 77k}{55} = \frac{4}{5}$$

$$\Rightarrow 55x + 9 + 80k = \frac{4}{\cancel{5}}(\cancel{55}^{11})$$

$$\Rightarrow 55x = 44 - 9 + 80k$$

$$\Rightarrow x = \frac{35 - 80k}{55} = \frac{\cancel{5}(7-16k)}{\cancel{55}_{11}}$$

$$\Rightarrow x = \frac{7-16k}{11}$$

$\therefore$ $x = \frac{7-6k}{11}$, $y = \frac{3+k}{11}$, $z = k$ is the unique solution of given Equation,

Discuss for what values of λ, μ the Simultaneous Equations $x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z = \mu$ have (i) no solution ii, a unique solution iii, An infinite number of solutions.

Soln: Given system of Equations can be written in the Matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$$A \quad X = B$$

The augmented matrix is $[AB] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$

$R_2: R_2 - R_1$ $R_3: R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{bmatrix}$$

$R_3: R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$$

i) The given system has no solution if $\lambda=3$ and $\mu \neq 10$ then $\rho(A)=2$ and $\rho(AB)=3$

$\therefore$ For $\lambda=3$ and $\mu \neq 10$; the system has no solution

ii) The given system has a unique solution if $\lambda \neq 3$ and for any value of μ. then $\rho(A) = \rho(AB) = 3$

iii) The given system has infinite number of solutions if $\lambda = 3$ and $\mu = 10$ then $\rho(A)=\rho(AB) = 2<3$

II. Homogeneous linear Equations:

consider 'm' homogeneous linear Equations in "n" unknowns as given below

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0$$
$$\dots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = 0$$

The given system of Equations can be written in the matrix form

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{O}$$

Consistency: Here the coefficient matrix A and the augmented matrix [AB] are same. then the system of Equations is always consistent.

Method of finding the rank of A: Reduce the coefficient matrix A is Echelon form and determine the rank of A.

Let r be the rank of the coefficient matrix and n be the number of unknowns.

Trivial Solution: The solution $x_1 = x_2 = \cdots = x_n = 0$ is said to be trivial solution (zero solution) of the system of homogeneous Equation $AX = 0$

Solution procedure:

(i) if $\rho(A) = r = n$ then the given system of Equations $AX=0$ possesses only a trivial (zero solution) ie $x_1 = x_2 = x_3 = \cdots = x_n = 0$

ii, if $\rho(A) = r < n$ then the given system of Equations possesses an infinite number of solutions.

NOTE:

(1) if A is a non-Singular matrix, then the linear system $AX=0$ has only a zero solution.

(2) if A is a Singular then the linear system $AX=0$ has a non-zero solution. ie we get an infinite number of solutions.

... and solve $x+y=0$; $y+z=0$; $z+x=0$

Soln: Given System of Equations can be written in the matrix form

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A \quad X = O$$

coefficient matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$R_3 : R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$R_3 : R_3 + R_2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

rank of $A = 3$

number of unknowns $(n) = 3$

rank of A = number of unknowns

The given system of Equations has a trivial solution

i.e $x = y = z = 0$

Solve completely the system of Equations $x+3y-2z=0$, $2x-y+4z=0$; $x-11y+14z=0$

Soln: Given System of Equations can be written in the matrix form

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A \quad X = O$$

Coefficient matrix $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$

$R_2 : R_2 - 2R_1$
$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 1 & -11 & 14 \end{bmatrix}$$

$R_3 : R_3 - R_1$
$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$$

$R_2 : \frac{R_2}{-7}$
$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -\frac{8}{7} \\ 0 & -14 & 16 \end{bmatrix}$$

$R_3 : R_3 + 14R_2$
$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -\frac{8}{7} \\ 0 & 0 & 0 \end{bmatrix}$$

rank of $A = 2$

number of unknowns $(n) = 3$.

$r = 2 < 3 = n$

The given system of Equations has an infinite number solutions.

$n - r = 3 - 2 = 1$

$x + 3y - 2z = 0$ —①

$y - \frac{8}{7}z = 0$ —②

Let $z = k$

$y - \frac{8}{7}k = 0 \Rightarrow \boxed{y = \frac{8}{7}k}$

$x + \frac{24}{7}k - 2k = 0$

$\Rightarrow x = 2k - \frac{24k}{7} = \frac{-10k}{7}$

$\therefore x = \frac{-10k}{7} ; y = \frac{8}{7}k ; z = k$ is the general solution of the given system.

Show that the system of Equations

$$2x_1-2x_2+x_3=\lambda x_1$$
$$2x_1-3x_2+2x_3=\lambda x_2$$
$$-x_1+2x_2=\lambda x_3$$

can possess a non-trivial solution only if $\lambda=1$; $\lambda=-3$ and obtain the general solution in each case.

Soln:- Given system of Equations can be written in the matrix form

$$\begin{bmatrix} 2-\lambda & -2 & 1 \\ 2 & -3-\lambda & 2 \\ -1 & 2 & -\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

if the given system possesses a non-trivial solution then A is singular.

ie $\begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -3-\lambda & 2 \\ -1 & 2 & -\lambda \end{vmatrix}=0$

$\Rightarrow (2-\lambda)\left[(3+\lambda)(\lambda)-4\right]+2\left[-2\lambda+2\right]+1\left[4-3-\lambda\right]=0$

$\Rightarrow (2-\lambda)\left[\lambda^2+3\lambda-4\right]-4(\lambda-1)-1(\lambda-1)=0$

$\Rightarrow (2-\lambda)(\lambda-1)(\lambda+4)-4(\lambda-1)-1(\lambda-1)=0$

$\Rightarrow (\lambda-1)\left[(2-\lambda)(\lambda+4)-4-1\right]=0$

$\Rightarrow (\lambda-1)\left[-\lambda^2-2\lambda+8-5\right]=0$

$\Rightarrow (\lambda-1)\left[-\lambda^2-2\lambda+3\right]=0$

$\Rightarrow (\lambda-1)\left[\lambda^2+2\lambda-3\right)=0$

$\Rightarrow (\lambda-1)(\lambda-1)(\lambda+3)=0$

$\Rightarrow \lambda=1,1,-3$

The given system has non-trivial solution for $\lambda=1$ and $\lambda=3$.

case (i) $\lambda = 1$

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{bmatrix}$$

$R_2 : R_2 - 2R_1$; $R_3 : R_3 + R_1$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

rank of $A = 1$

$n = 3$

$n - r = 3 - 1 = 2$

$x - 2y + z = 0$ —(1)

Let $z = k_1$ —(2)

$y = k_2$ —(3)

$x - 2k_2 + k_1 = 0$

$\Rightarrow x = -k_1 + 2k_2$

$\therefore x = 2k_2 - k$; $y = k_2$; $z = k_1$ is the general solution of the given system of equations if $\lambda = 1$

case (ii) $\lambda = -3$

$$A = \begin{bmatrix} 5 & -2 & 1 \\ 2 & 0 & 2 \\ -1 & 2 & 3 \end{bmatrix}$$

$R_1 \rightleftarrows R_2$
$$\begin{bmatrix} -1 & 2 & 3 \\ 2 & 0 & 2 \\ 5 & -2 & 1 \end{bmatrix}$$

$R_2 : R_2 + 2R_1$; $R_3 : R_3 + 5R_1$

$$\sim \begin{bmatrix} -1 & 2 & 3 \\ 0 & 4 & 8 \\ 0 & 8 & 16 \end{bmatrix}$$

$R_2 : \frac{R_2}{4}$

$$\sim \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 8 & 16 \end{bmatrix}$$

$R_3 : R_3 - 8R_2$

$$\sim \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

rank of $A = r = 2$

$n = 3.$

$n - r = 3 - 2 = 1.$

$-x + 2y + 3z = 0$ —— (1)

$y + 2z = 0$ —— (2)

$z = k$ —— (3)

$y + 2k = 0$

$\Rightarrow \boxed{y = -2k}$

$-x - 4k + 3k = 0$

$\Rightarrow -x - k = 0$

$\Rightarrow \boxed{x = -k}$

$\therefore$ $x = -k$; $y = -2k$; $z = k$ is the general Solution of the given system of Equations.

if $\lambda = -3$.

METHOD 2: GAUSS - ELIMINATION METHOD

Consider the System of Equations be

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Matrix form is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$A \quad X = B$$

The augmented matrix is $[AB] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$

Reduce the augmented matrix $[AB]$ to Echelon form by applying E-row operations only.

Let the Equivalent matrix is $[AB] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & 0 & a''_{33} & b''_3 \end{bmatrix}$

Its Equivalent matrix Equation is given by

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \end{bmatrix}$$

By solving this, we obtained the values of x_1, x_2, x_3 by back substitution.

Solve the equations $3x+y+2z=3$, $2x-3y-z=-3$, $x+2y+z=4$ by Gauss-Elimination method.

Soln: The given system of Equations can be written in the matrix form

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

A $X = B$

Augmented matrix is $[AB] = \begin{bmatrix} 3 & 1 & 2 & 3 \\ 2 & -3 & -1 & -3 \\ 1 & 2 & 1 & 4 \end{bmatrix}$

$R_1 \rightleftarrows R_3$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & -3 & -1 & -3 \\ 3 & 1 & 2 & 3 \end{bmatrix}.$$

$R_2 - 2R_1$; $R_3 : R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & -5 & -1 & -9 \end{bmatrix}$$

$R_2 : \frac{R_2}{-7}$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & \frac{3}{7} & \frac{11}{7} \\ 0 & -5 & -1 & -9 \end{bmatrix}$$

$R_3 : R_3 + 5R_2$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & \frac{3}{7} & \frac{11}{7} \\ 0 & 0 & \frac{8}{7} & -\frac{8}{7} \end{bmatrix}$$

$R_3 : \left(\frac{7}{8}\right) R_3$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & \frac{3}{7} & \frac{11}{7} \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

This is in Echelon form

$x + 2y + z = 4$ — (1)

$y + \frac{3}{7}z = \frac{11}{7}$ — (2)

$z = -1$ — (3)

$z = -1$ Substitute in (2)

$$y + \frac{3}{7}(-1) = \frac{11}{7}$$

$$y = \frac{11}{7} + \frac{3}{7} = \frac{14}{7} = 2$$

$$\therefore \boxed{y = 2}$$

$y = 2$, $z = -1$ Substitute in (1)

$$x + 2(2) - 1 = 4$$

$$x + 3 = 4$$

$$x = 4 - 3 = 1$$

$$\therefore \boxed{x = 1}$$

$\therefore$ The Solution is $x = 1$; $y = 2$; $z = -1$

METHOD : 3 GAUSS - JORDAN METHOD

Consider given system of linear Equations in Matrix form $AX = B$. Reduce the augmented matrix $[A\,B]$ by applying E-row operations only such that the coefficient matrix A is in diagonal form $[D : B']$. The solution can be obtained directly without any backward substitution.

Solve the system of Equations $x + y + z = 8$, $2x + 3y + 2z = 19$, $4x + 2y + 3z = 23$ using Gauss-Jordan method.

Soln: The given system of Equations can be written in the matrix form

$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 4 & 2 & 3 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} 8 \\ 19 \\ 23 \end{bmatrix}}$$

The augmented matrix is $[AB] = \begin{bmatrix} 1 & 1 & 1 & 8 \\ 2 & 3 & 2 & 19 \\ 4 & 2 & 3 & 23 \end{bmatrix}$

[illegible]

$$\approx \begin{bmatrix} 1 & 1 & 1 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & -2 & -1 & -9 \end{bmatrix}$$

$R_3 : R_3 + 2R_2$

$$\approx \begin{bmatrix} 1 & 1 & 1 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

$R_1 : R_1 - R_2$

$$\approx \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

$R_3 : (-1)R_3$

$$\approx \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$R_1 : R_1 - R_3$

$$\approx \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$x = 2,\ y = 3,\ z = 3$

$\therefore$ the solution is $x = 2,\ y = 3,\ z = 3$.

METHOD II: GAUSS-SEIDEL ITERATION METHOD

Gauss-Seidel method is an iterative method of solving a system of equations. This method is also called "method of successive correction" & Gauss Jacobi method.

Consider the system of equations

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}\right\} \quad \text{——— (2)}$$

Condition for applying iterative methods is that diagonal coefficient are not zero and large in absolute value compared to other coefficients.

The coefficient matrix should be diagonally dominant. Such System is called a "diagonally dominant System". if not interchange the Equations to make it diagonally dominant.

The System of Equations ① can be written as

$$x = \frac{1}{a_{11}}\left[b_1 - a_{12}y - a_{13}z\right] \quad \text{—— ②}$$

$$y = \frac{1}{a_{22}}\left[b_2 - a_{21}x - a_{23}z\right] \quad \text{—— ③}$$

$$z = \frac{1}{a_{23}}\left[b_3 - a_{31}x - a_{32}y\right] \quad \text{—— ④}$$

Step: 1 As a first approximation, initially we take $y=0$ and $z=0$ in ② we get $x = \frac{1}{a_{11}}(b_1)$

Then in Equation ③, put $x = \frac{b_1}{a_{11}}$ and $z=0$ then find the value of y.

$$y = \frac{1}{a_{22}}\left[b_2 - \frac{a_{21}b_1}{a_{11}}\right]$$

Then in Equation ④, we put these values of x and y to obtain the value of z.

$$z = \frac{1}{a_{33}}\left[b_3 - a_{31}\left(\frac{b_1}{a_{11}}\right) - \frac{a_{32}}{a_{22}}\left[b_2 - \frac{a_{21}b_1}{a_{11}}\right]\right]$$

Step: 2 we repeated the above procedure, in the Equation ② we put the values of y and z obtained in step ① and redetermine x. By using the new value redetermine y and so on.

This method uses the latest values of unknowns in each step.

The process is stopped, when two successive solutions are Equal (or) nearly same.

Solve the System of Equations using Gauss-Seidel method

$10x+y+z=12$; $2x+10y+z=13$; $2x+2y+10z=14$

Soln: The numbers 10, 10, 10 are in the leading diagonals are the largest in the respective Equations. The given system is diagonally dominant.

we apply Gauss-Seidal method to Solve the above System of Equations. The given Equations can be written as

$$x = \frac{1}{10}[12-y-z] \quad \text{--- (1)}$$

$$y = \frac{1}{10}[13-2x-z] \quad \text{--- (2)}$$

$$z = \frac{1}{10}[14-2x-2y] \quad \text{--- (3)}$$

Ist approximation

put $y=z=0$ in (1)

then $x = \frac{12}{10} = 1.2$ $\boxed{x=1.2}$

again putting $x=1.2$ & $z=0$ in (2)

we get $y = \frac{1}{10}[13-2(1.2)]$

$= \frac{13-2.4}{10} = \frac{10.6}{10} = 1.06$

$\therefore \boxed{y=1.06}$

put $y=1.06$; $x=1.2$ in Eqn (3)

we get $z = \frac{1}{10}[14-2(1.2)-2(1.06)]$

$= \frac{14-2.4-2.12}{10} = 0.95$

$\therefore \boxed{z=0.95}$

2nd approximation:

put $y=1.06$ and $z=0.95$ in (1)

$x = \frac{1}{10}[12-1.06-0.95] = 0.999$

$y = \frac{1}{10}[13-1.998-0.95] = 1.005$

$z = \frac{1}{10}[14-1.998-2.010] = 0.999.$

approximation

$$x = \frac{1}{10}[12 - 1.005 - 0.999] = 0.9996 = 1.00$$

$$y = \frac{1}{10}[13 - 2.0 - 0.999] = 1.0001 = 1.00$$

$$z = \frac{1}{14}[14 - 2.0 - 2.0] = 1.00$$

4th approximation:

$$x = \frac{1}{10}[12 - 1 - 1] = 1.00$$

$$y = \frac{1}{10}[13 - 2 - 1] = 1.00$$

$$z = \frac{1}{10}[14 - 2 - 2] = 1.00$$

Since the value of 3rd and 4th iterations are similar, we get the solution as $x = 1$; $y = 1$; $z = 1$

Solve using Gauss-Seidal Iteration method

$$x_1 + 10x_2 + x_3 = 6$$
$$10x_1 + x_2 + x_3 = 6$$
$$x_1 + x_2 + 10x_3 = 6$$

Soln:- The coefficient matrix is not diagonally dominant. if we interchange the first and second Equations, we get a diagonally dominant system.

$$10x_1 + x_2 + x_3 = 6$$
$$x_1 + 10x_2 + x_3 = 6$$
$$x_1 + x_2 + 10x_3 = 6$$

The above Equation can be written as

$$x_1 = \frac{1}{10}[6 - x_2 - x_3] \quad \text{—} (1)$$

$$x_2 = \frac{1}{10}[6 - x_3 - x_1] \quad \text{—} (2)$$

$$x_3 = \frac{1}{10}[6 - x_1 - x_2] \quad \text{—} (3)$$

Ist approximation

put $x_2 = x_3 = 0$ in (1)

$$x_1 = \frac{1}{10}(6) = 0.6$$

putting $x_1 = 0.6$ and $x_3 = 0$ in Eqn (2)

$$x_2 = \frac{1}{10}[6 - 0 - 0.6] = \frac{5.4}{10} = 0.54$$

putting $x_1 = 0.6$ and $x_2 = 0.54$ in Eqn (3)

$$x_3 = \frac{1}{10}[6 - 0.6 - 0.54] = \frac{4.86}{10} = 0.486$$

2nd approximation : Again start from Eqn (1) putting $x_2 = 0.54$ and $x_3 = 0.486$ as obtained in iteration 1.

$$x_1 = \frac{1}{10}[6 - 0.54 - 0.486] = 0.4974$$

proceeding like this, we get tabular form

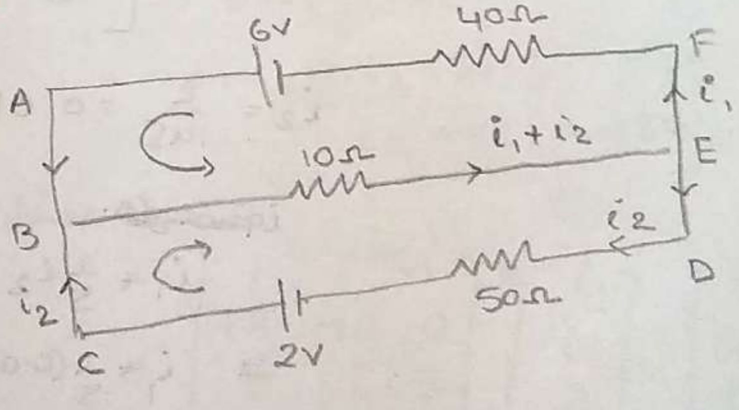
Iterations	x_1	x_2	x_3
1	0.6	0.54	0.486
2	0.4974	0.5017	0.4992
3	0.4999	0.5	0.5

Thus the solution is $x_1 = 0.5$; $x_2 = 0.5$; $x_3 = 0.5$

APPLICATIONS TO MATRIX ANALYSIS

Finding the current in Electrical circuits

Find the current in each cell considering the circuit given below



Soln: According to Kirchoff's voltage law in the loop ABEFA

$$40 i_1 + 10(i_1 + i_2) = 6$$

$$\Rightarrow 40 i_1 + 10 i_1 + 10 i_2 = 6$$

$$\Rightarrow 50 i_1 + 10 i_2 = 6 \quad \text{--- (1)}$$

In the loop BCDEB

$$10(i_1+i_2)+50i_2=2$$

$$\Rightarrow 10i_1+10i_2+50i_2=2$$

$$\Rightarrow 10i_1+60i_2=2 \quad \text{——} \ (2)$$

From (1) & (2)

Matrix form is

$$\begin{bmatrix} 50 & 10 \\ 10 & 60 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

$$A \quad X = B$$

Augmented matrix is $[AB] = \begin{bmatrix} 50 & 10 & 6 \\ 10 & 60 & 2 \end{bmatrix}$

$R_1 : \frac{R_1}{50}$

$$\begin{bmatrix} 1 & \frac{1}{5} & \frac{3}{25} \\ 10 & 60 & 2 \end{bmatrix}$$

$R_2 : R_2 - 10R_1$

$$\begin{bmatrix} 1 & \frac{1}{5} & \frac{3}{25} \\ 0 & 58 & \frac{4}{5} \end{bmatrix}$$

$R_2 : \frac{R_2}{58}$

$$\begin{bmatrix} 1 & \frac{1}{5} & \frac{3}{25} \\ 0 & 1 & \frac{2}{145} \end{bmatrix}$$

$$i_2 = \frac{2}{145} = 0.01379 \text{ Amp}$$

$$i_1 + \frac{1}{5}i_2 = \frac{3}{25}$$

$$\Rightarrow i_1 + \frac{1}{5}(0.01379) = \frac{3}{25}$$

$$\Rightarrow i_1 + 0.002758 = 0.12$$

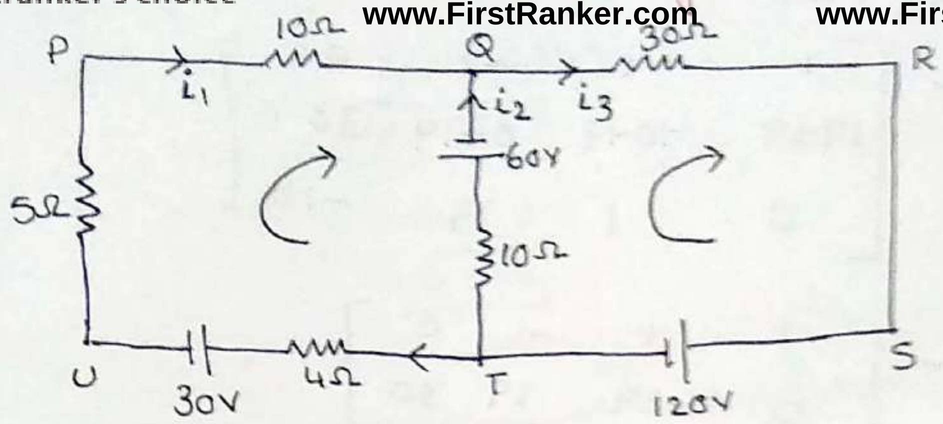
$$\Rightarrow i_1 = 0.12 - 0.002758$$

$$i_1 = 0.1172 \text{ Amp}$$

$$\therefore i_1 = 0.1172 \text{ Amp}$$

$$i_2 = 0.01379 \text{ Amp}$$

Find the current in the following circuit



Soln: At the Junction Q

By Kirchoff's current law

$$i_1 + i_2 = i_3$$

$$\Rightarrow i_1 + i_2 - i_3 = 0 \quad \text{--- (1)}$$

Around the loop PQTUP

$$10i_1 - 60 + 10i_2 + 4i_1 + 5i_1 + 30 = 0$$

$$19i_1 - 10i_2 - 30 = 0$$

$$\Rightarrow 19i_1 - 10i_2 = 30 \quad \text{--- (2)}$$

Around the loop QRSTU

$$30i_3 + 120 + 10i_2 + 60 = 0$$

$$\Rightarrow 30i_3 + 10i_2 + 180 = 0$$

$$\Rightarrow 10i_2 + 30i_3 = -180$$

$$\Rightarrow i_2 + 3i_3 = -18 \quad \text{--- (3)}$$

Matrix form is

$$\begin{bmatrix} 1 & 1 & -1 \\ 19 & -10 & 0 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 30 \\ -18 \end{bmatrix}$$

$$A \quad X = B$$

Augmented matrix is $[AB] = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 19 & -10 & 0 & 30 \\ 0 & 1 & 3 & -18 \end{bmatrix}$

$R_2 : R_2 - 19R_1$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 19-19 & -10-19 & 0+19 & 30-0 \\ 0 & 1 & 3 & -18 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -29 & 19 & 30 \\ 0 & 1 & 3 & -18 \end{bmatrix}$$

$R_2 : \frac{R_2}{-29}$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & \frac{-19}{29} & \frac{-30}{29} \\ 0 & 1 & 3 & -18 \end{bmatrix}$$

$R_3 : R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & \frac{-19}{29} & \frac{-30}{29} \\ 0 & 0 & 3+\frac{19}{29} & -18+\frac{30}{29} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & \frac{-19}{29} & \frac{-30}{29} \\ 0 & 0 & \frac{106}{29} & \frac{-492}{29} \end{bmatrix}$$

$R_3 : \left(\frac{29}{106}\right) R_3$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & \frac{-19}{29} & \frac{-30}{29} \\ 0 & 0 & 1 & \frac{-492}{106} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & \frac{-19}{29} & \frac{-30}{29} \\ 0 & 0 & 1 & -4.6415 \end{bmatrix}$$

$$i_1 + i_2 - i_3 = 0 \quad \text{— (1)}$$

$$i_2 - \frac{19}{29} i_3 = \frac{-30}{29} \quad \text{— (2)}$$

$$i_3 = -4.6415 \quad \text{— (3)}$$

$$i_2 - \frac{19}{29}(-4.6415) = -\frac{30}{29}$$

$$\Rightarrow i_2 + 3.0409 = -1.0344$$

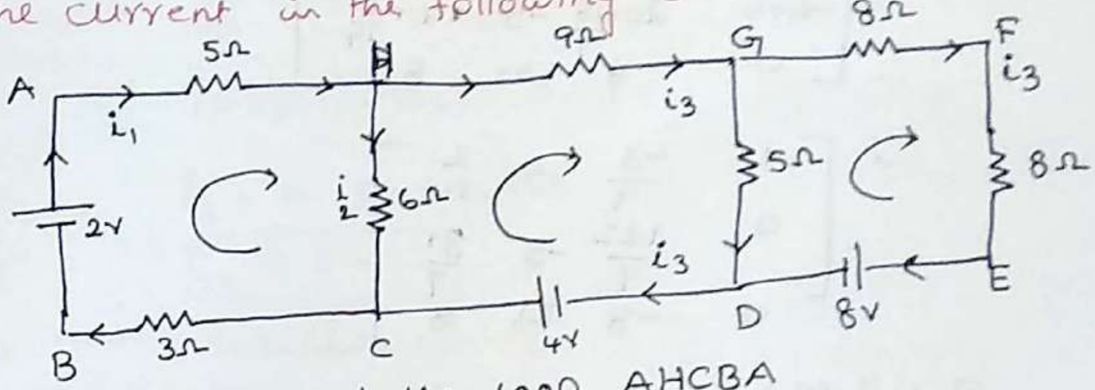
$$\boxed{i_2 = -4.0753}$$

i_2, i_3 values substitute in ①

$$i_1 - 4.0753 + 4.6415 = 0$$

$$\boxed{i_1 = -0.566}$$

Find the current in the following circuit



Soln:- By KVL around the loop AHCBA

$$5i_1 + 6(i_1 - i_2) + 3i_1 - 2 = 0$$

$$\Rightarrow 5i_1 + 6i_1 - 6i_2 + 3i_1 = 2$$

$$\Rightarrow 14i_1 - 6i_2 = 2 \quad \text{——— ①}$$

Around the loop HGDCH

$$9i_2 + 5(i_2 - i_3) - 4 + 6(i_2 - i_1) = 0$$

$$\Rightarrow 9i_2 + 5i_2 - 5i_3 - 4 + 6i_2 - 6i_1 = 0$$

$$\Rightarrow -6i_1 + 20i_2 - 5i_3 = 4 \quad \text{——— ②}$$

Around the loop GFEDG

$$8I_3 + 8I_3 + 8 + 5(I_3 - I_2) = 0$$

$$\Rightarrow 16i_3 + 8 + 5i_3 - 8i_2 = 0$$

$$\Rightarrow -8i_2 + 21i_3 = -8 \quad \text{——— ③}$$

Matrix form

$$\begin{bmatrix} 14 & -6 & 0 & 2 \\ -6 & 20 & -5 & 4 \\ 0 & -8 & 21 & -8 \end{bmatrix}$$

$$R_1 : \frac{R_1}{14} \quad \begin{bmatrix} 1 & -6/14 & 0 & \frac{2}{14} \\ -6 & 20 & -5 & 4 \\ 0 & -8 & 21 & -8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -\frac{3}{7} & 0 & \frac{1}{14} \\ -6 & 20 & -5 & 4 \\ 0 & -8 & 21 & 8 \end{bmatrix}$$

$R_2 : R_2 + 6R_1$

$$\sim \begin{bmatrix} 1 & -\frac{3}{7} & 0 & \frac{1}{14} \\ -6+6 & 20-\frac{18}{7} & -5 & 4+\frac{6}{14} \\ 0 & -8 & 21 & 8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -\frac{3}{7} & 0 & \frac{1}{14} \\ 0 & \frac{122}{7} & -5 & \frac{31}{7} \\ 0 & -8 & 21 & 8 \end{bmatrix}$$

$R_2 : \frac{7}{122}(R_2)$

$$\begin{bmatrix} 1 & -\frac{3}{7} & 0 & \frac{1}{14} \\ 0 & 1 & \frac{7}{122}(-5) & \frac{31}{122} \\ 0 & -8 & 21 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{3}{7} & 0 & \frac{1}{14} \\ 0 & 1 & -\frac{35}{122} & \frac{31}{122} \\ 0 & -8 & 21 & 8 \end{bmatrix}$$

$R_3 : R_3 + 8R_2$

$$\begin{bmatrix} 1 & -\frac{3}{7} & 0 & \frac{1}{14} \\ 0 & 1 & -\frac{35}{122} & \frac{31}{122} \\ 0 & 0 & \frac{2282}{122} & \frac{1224}{122} \end{bmatrix}$$

$R_3 : \frac{122}{2282}(R_3)$

$$\begin{bmatrix} 1 & -\frac{3}{7} & 0 & \frac{1}{14} \\ 0 & 1 & -\frac{35}{122} & \frac{31}{122} \\ 0 & 0 & 1 & \frac{1224}{2282} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{3}{7} & 0 & \frac{1}{14} \\ 0 & 1 & -\frac{35}{122} & \frac{31}{122} \\ 0 & 0 & 1 & 0.536 \end{bmatrix}$$

$$i_1 - \frac{3}{7} i_2 = \frac{1}{14} \quad \text{—— (1)}$$

$$i_2 - \frac{35}{122} i_3 = \frac{31}{122} \quad \text{—— (2)}$$

$$i_3 = 0.536 \quad \text{—— (3)}$$

i_3 value substitute in (2)

$$i_2 - \frac{35}{122}(0.536) = \frac{31}{122}$$

$$i_2 - (0.286)(0.536) = 0.254$$

$$i_2 = 0.254 + 0.153$$

$$\boxed{i_2 = 0.407}$$

i_2 value substitute in (1)

$$i_1 - \frac{3}{7}(0.407) = \frac{1}{14}$$

$$i_1 - 0.174 = 0.071$$

$$i_1 = 0.071 + 0.174$$

$$\boxed{i_1 = 0.245}$$

$i_1 = 0.245\,A$; $i_2 = 0.407\,A$ $i_3 = 0.536\,A$

UNIT - II

EIGEN VALUES - EIGEN VECTORS AND QUADRATIC FORM

Def: Let A be an $n \times n$ matrix, there Exists a non-zero vector X such that $AX = \lambda X$ then λ is called Eigen value of the matrix and corresponding vector X is called Eigen vector.

Eigen values are also called as proper values (or) Latent values (or) Spectral values (or) characteristic values.

Eigen vectors also called as proper vectors (or) Latent vectors (or) spectral vectors (or) characteristic vectors.

PROCEDURE TO FIND EIGEN VALUES OF A MATRIX

Let A be an $n \times n$ matrix and λ be the Eigen value of A and X be the corresponding Eigen vector.

By the definition, we have $AX = \lambda X$ $(X \neq 0)$

$$\Rightarrow AX - \lambda I X = 0$$

$$\Rightarrow (A - \lambda I) X = 0$$

which is a homogeneous system of Equations.

The System posses non-zero solutions if the coefficient matrix is Singular.

ie if $|A - \lambda I| = 0$

$|A - \lambda I| = 0$ is called characteristic Equation.

$(A - \lambda I)$ is called characteristic polynomial.

Procedure to find Eigen values and Eigen vectors

(1) Let the matrix $A = [a_{ij}]_{n\times n}$

(2) Write the characteristic Equation $|A-\lambda I| = 0$

(3) Solve the Equation, we get 'n' roots

(4) At Each $\lambda = \lambda_i$, we solve the Equation $(A-\lambda_i I)X = 0$ we get the corresponding vectors.

Find the Eigen values and Eigen vectors of the matrix

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix}$$

Soln: Given matrix is $A = \begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix}$

The characteristic Equation of A is $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 2 \\ 1 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)(5-\lambda) - 2 = 0$$

$$\Rightarrow 20 - 9\lambda + \lambda^2 - 2 = 0$$

$$\Rightarrow \lambda^2 - 9\lambda + 18 = 0$$

$$\Rightarrow \lambda^2 - 6\lambda - 3\lambda + 18 = 0$$

$$\Rightarrow \lambda(\lambda-6) - 3(\lambda-6) = 0$$

$$\Rightarrow (\lambda-3)(\lambda-6) = 0$$

$$\Rightarrow \lambda = 3, 6$$

$\therefore$ The Eigen values of A are 3, 6.

To find Eigen vectors of A corresponding to Eigen values

Consider $(A-\lambda I)X = 0$ Here $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

ie $$\begin{bmatrix} 4-\lambda & 2 \\ 1 & 5-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{—— (1)}$$

Case i) $\lambda = 3$

The Eigen vector X is given by $(A-3I)X_1 = 0$

$$\begin{bmatrix} 4-3 & 2 \\ 1 & 5-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The coefficient matrix is in Echelon form.

The rank of the matrix $= r = 1$

number of unknowns $(n) = 2$

$n - r = 2 - 1 = 1$

Give arbitrary constant to one variable

Let $x_2 = K$

$x_1 + 2x_2 = 0$

$x_1 + 2K = 0$

$x_1 = -2K$

$\therefore X_1 = \begin{bmatrix} -2K \\ K \end{bmatrix} = K\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ where $K \neq 0$

$\therefore \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is the Eigen vector of A corresponding to the Eigen value $\lambda = 3$

Case ii, $\lambda = 6$

The Eigen vector X is given by $(A-6I)X = 0$

$$\begin{bmatrix} 4-6 & 2 \\ 1 & 5-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_1 : \frac{R_1}{-2}$ $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$R_2 - R_1$

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The coefficient matrix is in Echelon form

The rank of the coefficient matrix $(r) = 1$

number of unknowns $(n) = 2$

$n - r = 2 - 1 = 1$

Give arbitrary constant to one variable

$x_1 - x_2 = 0$ — ①

Let $x_2 = K$ — ②

$\therefore x_1 - K = 0$

$\Rightarrow x_1 = K$

$$\therefore X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} K \\ K \end{bmatrix} = K\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ where } K \neq 0$$

$\therefore \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the Eigen vector of A corresponding to Eigen value $\lambda = 6$.

Find the characteristic roots and the corresponding characteristic vectors of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Soln: Let $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

The characteristic Equation of A is $|A - \lambda I| = 0$

$$\text{ie } \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$R_2 : R_2 + R_3$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2+2 & 3-\lambda-1 & -1+3-\lambda \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ 0 & 2-\lambda & 2-\lambda \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$= (2-\lambda) \begin{vmatrix} 6-\lambda & -2 & 2 \\ 0 & 1 & 1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow 2-\lambda = 0 \quad \& \quad \begin{vmatrix} 6-\lambda & -2 & 2 \\ 0 & 1 & 1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow 2 = \lambda \quad \& \quad (6-\lambda) \begin{vmatrix} 1 & 1 \\ -1 & 3-\lambda \end{vmatrix} + 2 \begin{vmatrix} -2 & 2 \\ 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (6-\lambda)\left[3-\lambda+1\right] + 2\left[-2-2\right] = 0$$

$$\Rightarrow (6-\lambda)\left[4-\lambda\right] + 2(-4) = 0$$

$$\Rightarrow 24 - 6\lambda - 4\lambda + \lambda^2 - 8 = 0$$

$$\Rightarrow \lambda^2 - 10\lambda + 16 = 0$$

$$\Rightarrow \lambda^2 - 8\lambda - 2\lambda + 16 = 0$$

$$\Rightarrow \lambda(\lambda-8) - 2(\lambda-8) = 0$$

$$\Rightarrow (\lambda-2)(\lambda-8) = 0$$

$$\Rightarrow \lambda = 2 \;\&\; 8$$

$$\therefore \lambda = 2, 2, 8$$

$\therefore$ Eigen values are 2, 2, 8.

To find the Eigen vectors of A corresponding to Eigen values

Consider $(A-\lambda I)X=0$ Here $X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{—— (1)}$$

case i, $\lambda = 2$

The Eigen vector X of A is given by

$(A-2I)X=0$

$$\begin{bmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1: \frac{R_1}{4}$ $$\begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2: R_2+2R_1$; $R_3: R_3-2R_1$

$$\begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The coefficient matrix is in Echelon form.

coefficient matrix rank $(r) = 1$

number of unknowns $(n) = 3$

$n-r = 3-1=2$

Give arbitrary constants to two variables

$$x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_3 = 0 \quad \text{—— (2)}$$

Let $x_3 = k_1$ ——— ③

$x_2 = k_2$ ——— ④

$$x_1 - \frac{1}{2}k_2 + \frac{1}{2}k_1 = 0$$

$$\Rightarrow \quad x_1 = -\frac{1}{2}k_1 + \frac{1}{2}k_2$$

$$\therefore \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}k_1 + \frac{1}{2}k_2 \\ k_2 \\ k_1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2}k_1 \\ 0 \\ k_1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}k_2 \\ k_2 \\ 0 \end{bmatrix}$$

$$= k_1 \underbrace{\begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}}_{X_1} + k_2 \underbrace{\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}}_{X_2}$$

$\therefore$ The Eigen vectors of A corresponding to $\lambda = 2$

is $X_1 = \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}$

Case ii, $\lambda = 8$

The Eigen vector X of A is given by

$$(A - 8I)X = 0$$

$$\Rightarrow \begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 : \frac{R_1}{2}$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 : R_2 + 2R_1 \quad ; R_3 : R_3 - 2R_1$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 : \frac{R_2}{-3}$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 : R_3 + 3R_2$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The coefficient matrix is in Echelon form

coefficient matrix rank $(r) = 2$

number of unknowns $(n) = 3$

$n - r = 3 - 2 = 1$

Give arbitrary constant to one variable

$$x_1 + x_2 - x_3 = 0 \quad \text{--- (5)}$$

$$x_2 + x_3 = 0 \quad \text{--- (6)}$$

Let $x_3 = k$ --- (7)

$$x_2 + k = 0$$

$$\Rightarrow \boxed{x_2 = -k}$$

$$x_1 - k - k = 0$$

$$\Rightarrow x_1 = -2k$$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2k \\ -k \\ k \end{bmatrix} = k \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

~~$X_3 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$ is the eigen vector of A corresponding to Eigen value $\lambda = 8$~~

PROPERTIES OF EIGEN VALUES

① if λ is a Eigen value of matrix A corresponding to the Eigen vector X, then λ^2 is the Eigen value of A^2 corresponding to the Eigen vector X.

Proof:- if λ is Eigen value of a matrix A, then $AX = \lambda X$ —— ①

pre multiply Equation ① with matrix A, we get

$$A(AX) = A(\lambda X)$$
$$\Rightarrow A^2X = A\lambda X$$
$$\Rightarrow A^2X = \lambda(AX)$$
$$\Rightarrow A^2X = \lambda(\lambda X) \qquad \boxed{\text{From ①}}$$
$$\Rightarrow A^2X = \lambda^2 X$$
$\Rightarrow \lambda^2$ is the Eigen value of A^2.

② if λ is a Eigen value of the matrix A then $K\lambda$ is a Eigen value of KA corresponding to same Eigen vector X.

proof: if λ is Eigen value of a matrix A then $AX = \lambda X$ —— ①

Multiply Equation ① with $K (K \neq 0)$ on both Sides

$$(KA)X = (K\lambda)X$$
$\Rightarrow K\lambda$ is a Eigen value of KA

③ The matrix A and A^T have same Eigen values

proof: Let λ be a Eigen value of A then $AX = \lambda x$ and it Satisfy $|A - \lambda I| = 0$ —— ①

Take $(A-\lambda I)^T = A^T - (\lambda I)^T = A^T - \lambda I^T = A^T - \lambda I$

$$|(A-\lambda I)^T| = |A^T - \lambda I|$$
$$\Rightarrow |A - \lambda I| = |A^T - \lambda I| \qquad \because |A| = |A^T|$$
$$\Rightarrow 0 = |A^T - \lambda I| \qquad \text{From ①}$$
$\Rightarrow \lambda$ is the eigen value of A^T

... the Eigen value of A and A^T are same

(4) If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the Eigen values of A then $\lambda_1^n, \lambda_2^n, \ldots \lambda_n^n$ are Eigen values of A^n.

proof: If A is $n \times n$ matrix and $\lambda_1, \lambda_2, \ldots \lambda_n$ are Eigen values of matrix A then $AX_i = \lambda_i X_i$ $(i = 1, 2, \ldots n)$ — (1)

Multiplying Equation (1) with "A" on both sides

$$A(AX_i) = A(\lambda_i X_i)$$

$$\Rightarrow A^2 X_i = \lambda_i (AX_i)$$

$$\Rightarrow A^2 X_i = \lambda_i (\lambda_i X_i) \quad (\text{From } (1))$$

$$\Rightarrow A^2 X_i = \lambda_i^2 X_i \quad \text{— (2)}$$

Again multiplying Equation (2) with A on both sides

$$A(A^2 X_i) = A(\lambda_i^2 X_i)$$

$$\Rightarrow A^3 X_i = \lambda_i^2 (AX_i)$$

$$\Rightarrow A^3 X_i = \lambda_i^2 (\lambda_i X_i)$$

$$\Rightarrow A^3 X_i = \lambda_i^3 X_i \quad \text{— (3)}$$

Repeat above process of 'n' times, we get

$$A^n X_i = \lambda_i^n X_i$$

$\therefore \lambda_1^n, \lambda_2^n \ldots \lambda_n^n$ are Eigen values of A^n.

(5) If X is a Eigen vector of A then X cannot correspond to more than one Eigen value.

proof: Let λ_1, λ_2 be two Eigen values of A corresponding to same Eigen vector X then

$$AX = \lambda_1 X \quad \text{— (1)}$$

$$AX = \lambda_2 X \quad \text{— (2)}$$

From (1) & (2), we have

$$\lambda_1 X = \lambda_2 X$$

$\Rightarrow \lambda_1 x - \lambda_2 x = 0$

$\Rightarrow (\lambda_1 - \lambda_2) x = 0$

Since x is non-zero vector

$\therefore \lambda_1 - \lambda_2 = 0$

$\Rightarrow \lambda_1 = \lambda_2$

$\therefore x$ cannot correspond to more than one Eigen value

⑥ if $\lambda_1, \lambda_2, \ldots \lambda_n$ are the Eigen values of A then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots \frac{1}{\lambda_n}$ are the Eigen values of A^{-1}

proof:- if A is $n \times n$ matrix and $\lambda_1, \lambda_2, \ldots \lambda_n$ are the Eigen value of A then $AX_i = \lambda_i X_i$ $(i = 1, 2, \ldots n)$ — ①

multiplying Equation ① by A^{-1} on both sides

$$A^{-1}(AX_i) = A^{-1}(\lambda_i X_i)$$

$$\Rightarrow (A^{-1}A) X_i = \lambda_i (A^{-1} X_i)$$

$$\Rightarrow I X_i = \lambda_i (A^{-1} X_i)$$

$$\Rightarrow \frac{1}{\lambda_i} X_i = A^{-1} X_i$$

$$\Rightarrow A^{-1} X_i = \frac{1}{\lambda_i} X_i$$

$\therefore \frac{1}{\lambda_i}$ are the Eigen values of A^{-1} $(i = 1, 2, \ldots n)$

$(i = 1, 2, \ldots n)$

ie $\frac{1}{\lambda_1}, \frac{1}{\lambda_2} \ldots \frac{1}{\lambda_n}$ are the Eigen values of A^{-1}.

⑦ The Eigen values of diagonal matrix are its diagonal elements.

Soln: Let A be the diagonal matrix of order n

ie $A = \begin{bmatrix} a_{11} & 0 & \ldots & 0 \\ 0 & a_{22} & \ldots & 0 \\ - & - & \ldots & 0 \\ - & - & \ldots & \\ 0 & 0 & \ldots & a_{nn} \end{bmatrix}$

The characteristic equation of A is $|A-\lambda I|=0$

is $$\begin{vmatrix} a_{11}-\lambda & 0 & \cdots & 0 \\ 0 & a_{22}-\lambda & \cdots & 0 \\ - & - & - & - \\ - & - & - & - \\ 0 & 0 & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

$\Rightarrow (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)=0$

$\Rightarrow \lambda = a_{11}, a_{22} \cdots a_{nn}$ are the diagonal element of the matrix A.

(8) The Eigen values of traingular matrix are its diagonal elements.

Proof. Case i. Let A be an upper triangular matrix of order 3.

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

The characteristic Equation of A is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ 0 & a_{22}-\lambda & a_{23} \\ 0 & 0 & a_{33}-\lambda \end{vmatrix} = 0$$

$\Rightarrow (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda)=0$

$\Rightarrow \lambda = a_{11}, a_{22}, a_{33}$ are the diagonal elements of the matrix A.

Case ii. Let A be the Lower triangular matrix of order 3

$$\text{Let } A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

characteristic Equation of A is $|A-\lambda I|=0$

$$\begin{vmatrix} a_{11}-\lambda & 0 & 0 \\ a_{21} & a_{22}-\lambda & 0 \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$$

$\Rightarrow (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda)=0$

$\Rightarrow \lambda = a_{11}, a_{22}, a_{33}$ are the diagonal elements of the matrix A.

$\therefore$ The Eigen values of any triangular matrix are its diagonal elements.

Similar matrix : Two matrices A and B are said to be Similar. if $B = P^{-1}AP$ where P is a non-Singular matrix.

(9) Eigen values of two Similar matrices are Same.

proof:- Let A and B be two similar matrices So that $B = P^{-1}AP$

To prove that A and $P^{-1}AP$ have the Same Eigen values.

Let λ be the Eigen value of B then $|B-\lambda I|=0$

$\Rightarrow |P^{-1}AP - \lambda I| = 0$

$\Rightarrow |P^{-1}AP - \lambda P^{-1}P| = 0$

$\Rightarrow |P(A-\lambda I)P^{-1}| = 0$

$\Rightarrow |P||A-\lambda I||P^{-1}| = 0$

$\boxed{\because |P||P^{-1}| = 1}$

$\Rightarrow |P||P^{-1}||A-\lambda I| = 0$

$\Rightarrow 1\ |A-\lambda I| = 0$

$\Rightarrow |A-\lambda I| = 0$

$\Rightarrow |B^{-1}(B-\lambda I)| = 0 \Rightarrow |A-\lambda I| = 0$

Hence A and B have the same Eigen values.

(10) if A and B are two square invertible matrices then BA and AB have the same Eigen values.

Proof.

$$AB = IAB$$
$$= (B^{-1}B)AB$$
$$= B^{-1}(BA)B$$
$$AB = B^{-1}(BA)B \quad \text{--- (1)}$$

By the property (9), matrices AB, and $B^{-1}(BA)B$ have the same Eigen values.

Now by (1), the matrices AB and BA have the same Eigen values.

(11) If λ is a Eigen values of A then $\frac{|A|}{\lambda}$ is also Eigen value of adj A.

proof:- if λ is the Eigen value of A then $AX = \lambda X$ --- (1)

Multiply Equation (1) with adj A on both Sides

$$(\text{adj} A)AX = (\text{adj} A)\lambda X$$
$$\Rightarrow (\text{adj} A \cdot A)X = \lambda(\text{adj} A \cdot X)$$
$$\Rightarrow |A|X = \lambda(\text{adj} A \cdot X)$$
$$\Rightarrow \frac{|A|}{\lambda}X = (\text{adj} A)X$$
$$\Rightarrow (\text{adj} A)X = \frac{|A|}{\lambda} \cdot X$$

$\Rightarrow \frac{|A|}{\lambda}$ is a Eigen value of adj A.

(12) The product of the Eigen values of a matrix A is Equal to its determinant

proof: Let $\lambda_1, \lambda_2, \ldots \lambda_n$ be the Eigen values of $A_{n\times n}$

then $|A-\lambda I| = (-1)^n (\lambda-\lambda_1)(\lambda-\lambda_2) \ldots (\lambda-\lambda_n) = 0$

put $\lambda = 0$

$|A| = (-1)^n (-\lambda_1)(-\lambda_2) \ldots (-\lambda_n)$

$\Rightarrow |A| = (-1)^n (-1)^n \lambda_1 \lambda_2 \ldots \lambda_n$

$\Rightarrow |A| = (-1)^{2n} \lambda_1 \lambda_2 \ldots \lambda_n$

$\Rightarrow |A| = \lambda_1 \lambda_2 \ldots \lambda_n$

$\therefore$ The product of the Eigen values of a matrix A is Equal to its determinant

(13) The sum of the Eigen values of a matrix is its trace of the matrix.

~~sta~~ proof: Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix of order 3.

Let λ be the Eigen value of the matrix

The characteristic Equation is $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$$

$$(a_{11}-\lambda)[(a_{22}-\lambda)(a_{33}-\lambda)-a_{23}a_{32}]-a_{12}[a_{21}(a_{33}-\lambda)-a_{23}a_{31}]$$
$$+a_{13}[a_{21}a_{32}-a_{31}(a_{22}-\lambda)]=0$$

$$\Rightarrow (a_{11}-\lambda)[a_{22}a_{33}-a_{22}\lambda-a_{33}\lambda+\lambda^2-a_{23}a_{32}]-a_{12}[a_{21}a_{33}-a_{21}\lambda-a_{23}a_{31}]$$
$$+a_{13}[a_{21}a_{32}-a_{31}a_{22}+a_{31}\lambda]=0$$

$$\Rightarrow a_{11}a_{22}a_{33}-a_{11}a_{22}\lambda-a_{11}a_{33}\lambda+a_{11}\lambda^2-a_{11}a_{23}a_{32}-a_{22}a_{33}\lambda+a_{22}\lambda^2+a_{33}\lambda^2$$
$$-\lambda^3+a_{23}a_{32}\lambda-a_{12}a_{21}a_{33}+a_{12}a_{21}\lambda+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}$$
$$-a_{13}a_{31}a_{22}+a_{13}a_{31}\lambda=0$$

$$\Rightarrow -\lambda^3+\lambda^2(a_{11}+a_{22}+a_{33})+\lambda(-a_{11}a_{22}-a_{11}a_{33}-a_{22}a_{33}+a_{23}a_{32}+a_{12}a_{21})$$
$$+(a_{11}a_{22}a_{33}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32})=0$$

Sum of the Eigen values = Sum of roots

$$=\frac{-[\text{coefficient of } \lambda^2]}{\text{coefficient of } \lambda^3}$$

$$=\frac{-[a_{11}+a_{22}+a_{33}]}{-1}$$

$$=a_{11}+a_{22}+a_{33}$$

= Sum of the diagonal elements of A

= Trace of A

Hence the sum of the Eigen values of a matrix A is equal to its Trace.

LINEAR COMBINATION : A vector $\bar{x}$ is said to be a linear combination of the vectors $\bar{x}_1, \bar{x}_2, \bar{x}_3 - - - - \bar{x}_n$ etc if there exists scalars $k_1, k_2, - - - k_n$ such that

$$\bar{x} = k_1\bar{x}_1 + k_2\bar{x}_2 + - - - - + k_n\bar{x}_n$$

LINEARLY DEPENDENT : A set of vectors $\bar{x}_1, \bar{x}_2 - - - - \bar{x}_n$ is said to be linearly dependent if there exists n scalars $k_1, k_2 - - - - - k_n$ not all zero such that

$$k_1\bar{x}_1 + k_2\bar{x}_2 + - - - - - + k_n\bar{x}_n = 0$$

where '0' denotes the n-vector whose components are all zero.

LINEARLY INDEPENDENT : A set of vectors $\bar{x}_1, \bar{x}_2 - - - \bar{x}_n$ is said to be linearly independent if every relation of the type $k_1\bar{x}_1 + k_2\bar{x}_2 + - - - - + k_n\bar{x}_n = 0$ implies $k_1 = 0, k_2 = 0 - - - - - k_n = 0$.

(16) Two distinct Eigen vectors corresponding to two distinct Eigen values are linearly independent.

proof: Let x_1, x_2 be two distinct Eigen vectors of A corresponding to two distinct Eigen values λ_1, λ_2

ie
$$\left.\begin{aligned} Ax_1 &= \lambda_1 x_1 \\ Ax_2 &= \lambda_2 x_2 \end{aligned}\right\} \text{——— (1)}$$

To prove that x_1 and x_2 are linearly independent.

Suppose that x_1 and x_2 are linearly dependent.

Consider two constants c_1 and c_2 (none are zero's)

Such that $c_1x_1 + c_2x_2 = 0$ ——— (2)

$\because$ Multiply by A on both sides

$$A(c_1x_1) + A(c_2x_2) = 0$$

$$\Rightarrow c_1(Ax_1) + c_2(Ax_2) = 0$$

$c_1(\lambda_1 x_1) + c_2(\lambda_2 x_2) = 0$

$\Rightarrow (c_1\lambda_1)x_1 + (c_2\lambda_2)x_2 = 0$ ——— (3)

(3) $\Rightarrow (c_1\lambda_1)x_1 + (c_2\lambda_2)\cancel{x_2} = 0$

(2) $\times \lambda_2 \Rightarrow (c_1\lambda_2)x_1 + (c_2\lambda_2)x_2 = 0$

$c_1\lambda_1 x_1 - c_1\lambda_2 x_1 = 0$

$\Rightarrow c_1(\lambda_1)x_1 - c_1(\lambda_2)x_1 = 0$

$\Rightarrow c_1[\lambda_1 - \lambda_2)x_1 = 0$

Since $x_1 \neq 0$; $\lambda_1 - \lambda_2 \neq 0$

$\Rightarrow c_1 = 0$

lly

(3) $\Rightarrow (c_1\lambda_1)\cancel{x_1} + (c_2\lambda_2)x_2 = 0$

(2) $\times \lambda_1 \Rightarrow (c_1\lambda_1)x_1 + (c_2\lambda_1)x_2 = 0$

$c_2\lambda_2 x_2 - c_2\lambda_1 x_2 = 0$

$\Rightarrow c_2[\lambda_2 - \lambda_1)x_2 = 0$

Since $x_2 \neq 0$; $\lambda_2 - \lambda_1 \neq 0$

$\therefore c_2 = 0$

This is contradicts that our assumption that c_1 and c_2 are not zero.

$\therefore x_1$ and x_2 are linearly independent.

lly n distinct Eigen vectors corresponding to 'n' distinct Eigen values are linearly independent.

NOTE: The Eigen vectors corresponding to repeated Eigen values may be linearly independent (or) linearly dependent.

(17) Zero is Eigen value of a matrix if and only if it is Singular.

Proof: Let A be a matrix of order n

Let $\lambda = 0$ be the Eigen value of A then $|A - \lambda I| = 0$

$$\Rightarrow |A - (0)I| = 0$$
$$\Rightarrow |A| = 0$$
$\Rightarrow$ A is singular matrix

Suppose A is Singular matrix

$$\Rightarrow |A| = 0$$
$$\Rightarrow |A - (0)I| = 0$$
$\Rightarrow$ 0 is the Eigen value of A

$\therefore$ Zero is the Eigen value of a matrix iff it is Singular.

(18) if λ is the Eigen value of A then the Eigen value of $B = a_0A^2 + a_1A + a_2I$ is $a_0\lambda^2 + a_1\lambda + a_2$

Proof:- λ is the Eigen value of $A \Rightarrow AX = \lambda X$ —— ①

Also λ^2 is the Eigen value of $A^2 \Rightarrow A^2X = \lambda^2 X$ —— ②

Now $B = a_0A^2 + a_1A + a_2I$

$$BX = (a_0A^2 + a_1A + a_2I)X$$
$$\Rightarrow BX = a_0A^2X + a_1AX + a_2IX$$
$$\Rightarrow BX = (a_0\lambda^2X + a_1\lambda X + a_2X)$$
$$\Rightarrow BX = (a_0\lambda^2 + a_1\lambda + a_2)X$$

$\Rightarrow$ $a_0\lambda^2 + a_1\lambda + a_2$ is the Eigen value of B

Hence if λ is the Eigen value of A then the Eigen value of $B = a_0A^2 + a_1A + a_2I$ is given by $a_0\lambda^2 + a_1\lambda + a_2$

If λ is the Eigen value of the matrix A then $\lambda+K$ is the Eigen value of the matrix $A+KI$

Proof: Let λ be the Eigen value of the matrix A and let X be the Eigen ~~value~~ vector of the matrix A such that $AX = \lambda X$ —— ①

Now $(A+KI)X = AX + KIX$

$$= \lambda X + KX$$

$$= (\lambda+K)X$$

$\therefore (A+KI)X = (\lambda+K)X$

$\therefore \lambda+K$ is the Eigen value of $A+KI$.

Q Determine the Eigen values of A^{-1}, where $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Soln: Given matrix is $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

The characteristic Equation of A is $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[(2-\lambda)(3-\lambda)-2\right] - 1\left[2-2(2-\lambda)\right] = 0$$

$$\Rightarrow (1-\lambda)\left[6-2\lambda-3\lambda+\lambda^2-2\right] - \left[2-4+2\lambda\right] = 0$$

$$\Rightarrow (1-\lambda)\left[\lambda^2-5\lambda+4\right] - \left[2\lambda-2\right] = 0$$

$$\Rightarrow (1-\lambda)\left[\lambda^2-5\lambda+4\right] + 2(1-\lambda) = 0$$

$$\Rightarrow (1-\lambda)\left[\lambda^2-5\lambda+4+2\right] = 0$$

$$\Rightarrow (1-\lambda)\left[\lambda^2-5\lambda+6\right] = 0$$

$$\Rightarrow \lambda = 1, 2, 3$$

We know that λ is Eigen value of A then $\frac{1}{\lambda}$ is the Eigen value of A^{-1}

$\therefore$ The Eigen values of A are 1, 2, 3

The Eigen values of A^{-1} are $1, \frac{1}{2}, \frac{1}{3}$

Find the Eigen values of adj A ; if $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

Soln: Given matrix is $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

The Eigen values of A are $\lambda = 2, 3, 4$

The Eigen values of adj A are $\frac{|A|}{\lambda}$

Now $|A| = 2(12) - 3(0) + 4(0)$

$\Rightarrow |A| = 24$

The Eigen values of adj A are $\frac{24}{2}, \frac{24}{3}, \frac{24}{4}$

$= 12, 8, 6$

if 2, 3, 5 are the Eigen values of A, then find the Eigen values of $2A^3 + 3A^2 + 5A + 3I$

Soln: Given that the Eigen values of A are 2, 3, 5

By the property "if λ is the Eigen value of A then Eigen value of $a_0A^2 + a_1A + a_2I$ is $a_0\lambda^2 + a_1\lambda + a_2$"

$\lambda = 2 \Rightarrow 2(2)^3 + 3(2)^2 + 5(2) + 3 = 41$

$\lambda = 3 \Rightarrow 2(3)^3 + 3(3)^2 + 5(3) + 3 = 99$

$\lambda = 5 \Rightarrow 2(5)^3 + 3(5)^2 + 5(5) + 3 = 353$

$\therefore$ Hence the Eigen values of $2A^3 + 3A^2 + 5A + 3I$ are 41, 99, 353.

Algebraic Multiplicity: if λ is an Eigen value of A repeated "r" times, then r is called algebraic multiplicity of the Eigen value λ.

Eg: if $A = \begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$ then $|A - \lambda I| = (\lambda - 1)^3 = 0$

Algebraic multiplicity of $\lambda = 1$ is 3.

Geometric multiplicity : The number of linearly independent Eigen vectors corresponding to the repeated eigenvalues of λ is called Geometric multiplicity of a matrix

Eg: Now Eigen vector corresponding to $\lambda = 1$ is $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

only one Eigen vector corresponding to $\lambda = 1$

$\therefore$ Geometric multiplicity is one.

NOTE: Always Geometric multiplicity $\leq$ Algebraic multiplicity

CAYLEY - HAMILTON THEOREM

STATEMENT : Every Square matrix Satisfies its own characteristic Equation.

proof : Let A be the square matrix of order n

ie $A = [a_{ij}]_{n \times n}$

The characteristic Equation of A is $|A - \lambda I| = 0$

Let $|A - \lambda I| = (-1)^n [\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots + a_n]$ —— ①

We know that if the degree of $(A - \lambda I)$ is atmost n then the degree of adj $(A - \lambda I)$ is atmost $(n-1)$ & less

Let adj $(A - \lambda I) = B_0 \lambda^{n-1} + B_1 \lambda^{n-2} + B_2 \lambda^{n-3} + \cdots + B_{n-2} \lambda + B_{n-1}$ —— ②

We know that

$(A - \lambda I) \operatorname{adj}(A - \lambda I) = |A - \lambda I| I$ —— ③

Substitute ①, ② in ③, we get

$$(A - \lambda I)[B_0 \lambda^{n-1} + B_1 \lambda^{n-2} + \cdots + B_{n-2}\lambda + B_{n-1}]$$
$$= (-1)^n [\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots + a_n] I$$

$$\Rightarrow AB_0 \lambda^{n-1} + AB_1 \lambda^{n-2} + \cdots + AB_{n-2}\lambda + AB_{n-1} - \lambda^n B_0 - \lambda^{n-1} B_1 - \cdots$$
$$\cdots - \lambda^2 B_{n-2} - \lambda B_{n-1} = (-1)^n \lambda^n I + (-1)^n a_1 \lambda^{n-1} I + (-1)^n a_2 \lambda^{n-2} I + \cdots + (-1)^n a_n I$$

$\Rightarrow -\lambda^n B_0 + \lambda^{n-1}(AB_0 - B_1) + \cdots + \lambda(AB_{n-2} - B_{n-1}) + AB_{n-1}$

$= (-1)^n \lambda^n I + \lambda^{n-1}(-1)^n a_1 I + \cdots + (-1)^n a_n I$

Equating coefficients on both sides

coefficients of λ^n

$$-B_0 = (-1)^n I \quad \text{--- (4)}$$

coefficients of λ^{n-1}

$$AB_0 - B_1 = (-1)^n a_1 I \quad \text{--- (5)}$$

– – – – – – –

– – – – –

constants

$$AB_{n-1} = (-1)^n a_n I \quad \text{--- (6)}$$

(4) $\times A^n \Rightarrow -A^n B_0 = (-1)^n I A^n$

$\Rightarrow -A^n B_0 = (-1)^n A^n$ --- (7)

(5) $\times A^{n-1} \Rightarrow A^{n-1}(AB_0 - B_1) = A^{n-1}[(-1)^n a_1 I]$

$\Rightarrow A^n B_0 - A^{n-1} B_1 = (-1)^n a_1 A^{n-1}$ --- (8)

– – – – – – – – –

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(6) $\times I \Rightarrow I(AB_{n-1}) = I[(-1)^n a_n I]$

$\Rightarrow AB_{n-1} = (-1)^n a_n I$ --- (9)

Adding (7), (8) ------ (9), we get

$$-\cancel{A^n B_0} + \cancel{A^n B_0} - \cancel{A^{n-1} B_1} + \cdots + \cancel{AB_{n-1}}$$

$$= (-1)^n A^n + (-1)^n a_1 A^{n-1} + \cdots + (-1)^n a_n I$$

$\Rightarrow 0 = (-1)^n [A^n + a_1 A^{n-1} + \cdots + a_n I]$

$\therefore (-1)^n [A^n + a_1 A^{n-1} + \cdots + a_n I] = 0$

$\therefore$ Hence A satisfies its characteristic equation.

APPLICATIONS OF CAYLEY - HAMILTON THEOREM

The Important applications of cayley - Hamilton theorem are (i) To find the Inverse of a matrix
(ii) To find higher powers of the matrix

Verify Cayley - Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ hence find A^{-1} and A^4

Soln: Given matrix is $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

The characteristic Equation of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 3 \\ 2 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[(1-\lambda)^2 - 1\right] + 3\left[-2-1+\lambda\right] = 0$$

$$\Rightarrow (1-\lambda)\left[1+\lambda^2-2\lambda-1\right] + 3\left[-3+\lambda\right] = 0$$

$$\Rightarrow (1-\lambda)\left[\lambda^2-2\lambda\right] - 9 + 3\lambda = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - \lambda^3 + 2\lambda^2 - 9 + 3\lambda = 0$$

$$\Rightarrow -\lambda^3 + 3\lambda^2 + \lambda - 9 = 0$$

$$\Rightarrow \lambda^3 - 3\lambda^2 - \lambda + 9 = 0$$

To verify the cayley - Hamilton theorem, we have to show that $A^3 - 3A^2 - A + 9I = 0$

Verification

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{bmatrix}$$

Now

$A^3 - 3A^2 - A + 9I$

$$= \begin{bmatrix} 4 & -9 & 21 \\ 11 & 2 & -11 \\ 1 & -7 & 7 \end{bmatrix} + \begin{bmatrix} -12 & 9 & -18 \\ -9 & -6 & -12 \\ 0 & 6 & -15 \end{bmatrix} + \begin{bmatrix} -1 & 0 & -3 \\ -2 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Hence Cayley-Hamilton theorem is verified.

To find A^{-1}

$|A| = 1(1-1) - 3(-2-1) = 9 \neq 0$

$\Rightarrow A^{-1}$ Exists

Since $A^3 - 3A^2 - A + 9I = 0$

multiplying with A^{-1}

$A^{-1}(A^3 - 3A^2 - A + 9I) = 0$

$\Rightarrow A^2 - 3A - I + 9A^{-1} = 0$

$\Rightarrow 9A^{-1} = -A^2 + 3A + I$

$$\Rightarrow 9A^{-1} = \begin{bmatrix} -4 & 3 & -6 \\ -3 & -2 & -4 \\ 0 & 2 & -5 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 9 \\ 6 & 3 & -3 \\ 3 & -3 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$$

To find A^4

Since $A^3 - 3A^2 - A + 9I = 0$

multiplying by A

$A(A^3 - 3A^2 - A + 9I) = 0$

$\Rightarrow A^4 - 3A^3 - A^2 + 9A = 0$

$\Rightarrow A^4 = 3A^3 + A^2 - 9A$

$$A^4 = \begin{bmatrix} 12 & -27 & 63 \\ 33 & -6 & 33 \\ 3 & -21 & 21 \end{bmatrix} + \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} + \begin{bmatrix} -9 & 0 & -27 \\ -18 & -9 & 9 \\ -9 & 9 & -9 \end{bmatrix} = \begin{bmatrix} 7 & -30 & 42 \\ 18 & -13 & 46 \\ -6 & -14 & 17 \end{bmatrix}$$

DIAGONALISATION OF A MATRIX

A matrix "A" is said to be diagonalizable if it is similar to a diagonal matrix.

i.e if there exist an invertible matrix P such that $D = P^{-1}AP$ where D is a diagonal matrix.

The matrix P is said to diagonalize A.

Working rule

1. Let the given matrix be $A = [a_{ij}]_{n \times n}$
2. Find the Eigen values of A
3. Find the Eigen vectors corresponding to Eigen value of A
4. choose the Modal matrix $P = [x_1 x_2 ---- x_n]$
5. calculate $P^{-1} = \frac{adj P}{|P|}$
6. Calculate $P^{-1}AP = D$, where D is the diagonal matrix whose diagonal elements are Eigen values of A

Powers of matrix A^n

if A is diagonalizable

then $D = P^{-1}AP$

$D^2 = P^{-1}A^2P$

$D^3 = P^{-1}A^3P$

lly $D^n = P^{-1}A^nP$ —— (1)

To obtain A^n, pre-multiply (1) by P and post multiply by P^{-1};

$$PD^nP^{-1} = PP^{-1}A^nPP^{-1}$$

$$\Rightarrow PD^nP^{-1} = IA^nI$$

$$\Rightarrow PD^nP^{-1} = A^n$$

$$\therefore \boxed{A^n = PD^nP^{-1}}$$

Diagonalize the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ and find A^n

Soln: Given matrix is $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

The characteristic Equation of A is $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)[(2-\lambda)(3-\lambda)-2] - 1[2-2(2-\lambda)] = 0$$

$$\Rightarrow (1-\lambda)[6-5\lambda+\lambda^2-2] - 1[2-4+2\lambda] = 0$$

$$\Rightarrow (1-\lambda)[\lambda^2-5\lambda+4] - 1[-2+2\lambda] = 0$$

$$\Rightarrow (1-\lambda)[\lambda^2-5\lambda+4) - 1(-2)[1-\lambda] = 0$$

$$\Rightarrow (1-\lambda)[\lambda^2-5\lambda+4) + 2(1-\lambda) = 0$$

$$\Rightarrow (1-\lambda)[\lambda^2-5\lambda+4+2] = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2-5\lambda+6) = 0$$

$$\Rightarrow (1-\lambda)(\lambda-2)(\lambda-3) = 0$$

$$\Rightarrow \lambda = 1, 2, 3$$

Thus the Eigen values of A are $\lambda = 1, 2, 3$

To find the Eigen vectors of A corresponding to Eigen values:

consider $(A-\lambda I)X = 0$ Here $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

ie $$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{——— } (1)$$

case i, $\lambda = 1$

$$\begin{bmatrix} 1-1 & 0 & -1 \\ 1 & 2-1 & 1 \\ 2 & 2 & 3-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} & & \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \rightleftarrows R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 : R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

rank $(r) = 2$

$n = 3$

$n - r = 3 - 2 = 1$

Give arbitrary constant to one variable

$-x_3 = 0$

$\Rightarrow x_3 = 0$ —— (2)

$x_1 + x_2 + x_3 = 0$ —— (3)

let $x_2 = K$ —— (4)

$x_1 + K + 0 = 0$

$\Rightarrow x_1 = -K$

$$X_1 = \begin{bmatrix} -K \\ K \\ 0 \end{bmatrix} = K\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Hence $X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is Eigen vector of A corresponding to the Eigen value $\lambda = 1$

Case ii, $\lambda = 2$

$$\begin{bmatrix} 1-2 & 0 & -1 \\ 1 & 2-2 & 1 \\ 2 & 2 & 3-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \rightleftarrows R_2$

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 : R_2 + R_1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 : R_3 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightleftarrows R_3$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\text{rank}(r) = 2$

$n = 3$

$n - r = 3 - 2 = 1.$

$x_1 + x_3 = 0$ ——— (5)

$2x_2 - x_3 = 0$ ——— (6)

$x_3 = K$ ——— (7)

$2x_2 - K = 0$

$\Rightarrow \boxed{x_2 = K/2}$

$x_1 + K = 0 \Rightarrow \boxed{x_1 = -K}$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -K \\ K/2 \\ K \end{bmatrix} = \begin{bmatrix} -2K \\ K \\ 2K \end{bmatrix} = K\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

$\therefore X_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$ is the Eigen vector of A corresponding to the Eigen value $\lambda = 2$

Case iii, $\lambda = 3$

$$\begin{bmatrix} 1-3 & 0 & -1 \\ 1 & 2-3 & 1 \\ 2 & 2 & 3-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ -2 & 0 & -1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 : R_2 + 2R_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 : R_3 - 2R_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

rank $(r) = 2$

$n = 3$

$n - r = 3 - 2 = 1$

$x_1 - x_2 + x_3 = 0$ — (8)

$-2x_2 + x_3 = 0$ — (9)

Let $x_3 = K$ — (10)

$-2x_2 + K = 0$

$\Rightarrow -2x_2 = -K$

$\Rightarrow \boxed{x_2 = \frac{K}{2}}$

$x_1 - \frac{K}{2} + K = 0$

$\Rightarrow x_1 + K/2 = 0$

$\Rightarrow x_1 = -K/2$

$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -K/2 \\ K/2 \\ K \end{bmatrix} = \frac{+K}{2}\begin{bmatrix} -1 \\ +1 \\ 2 \end{bmatrix}$$

$\therefore X_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ is the Eigen vector of A corresponding to Eigen value $\lambda = 3$.

Modal matrix $P = [X_1\ X_2\ X_3] = \begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$

$|P| = -1[2-2) + 2[2-0] - 1[2-0] = 2 \neq 0$

so P^{-1} exists

$P^{-1} = \frac{\text{adj}P}{|P|}$

adj P

1st row

co-factor of $-1 = +\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0$

co-factor of $-2 = -\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -2$

co-factor of $-1 = +\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$

2nd row

co-factor of $1 = -\begin{vmatrix} -2 & -1 \\ 2 & 2 \end{vmatrix}$

$= -[-4+2]$

$= 2$

co-factor of $1 = +\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix}$

$= -2$

co-factor of $1 = -\begin{vmatrix} -1 & -2 \\ 0 & 2 \end{vmatrix}$

$= 2$

3rd row

co-factor of $0 = +\begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix}$

$= (-2)+1$

$= -1$

co-factor of $2 = -\begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix}$

$= 0$

co-factor of $2 = +\begin{vmatrix} -1 & -2 \\ 1 & 1 \end{vmatrix}$

$= -1+2$

$= 1$

co-factors matrix $B = \begin{bmatrix} 0 & -2 & 2 \\ 2 & -2 & 2 \\ -1 & 0 & 1 \end{bmatrix}$

$\text{adj}P = B^T = \begin{bmatrix} 0 & 2 & -1 \\ -2 & -2 & 0 \\ 2 & 2 & 1 \end{bmatrix}$

$P^{-1} = \frac{\text{adj}P}{|P|} = \frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ -2 & -2 & 0 \\ 2 & 2 & 1 \end{bmatrix}$

$P^{-1}AP = \frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ -2 & -2 & 0 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ -2 & -2 & 0 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} -1 & -4 & -3 \\ 1 & 2 & 3 \\ 0 & 4 & 6 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D$

$\therefore P^{-1}AP = D$

Find A^4

$$A^4 = PD^4P^{-1}$$

$$= \begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 0 & 2 & -1 \\ -2 & -2 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix} \begin{bmatrix} 0 & 2 & -1 \\ -2 & -2 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 & -1 \\ -32 & -32 & 0 \\ 162 & 162 & 81 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 64-162 & -2+64-162 & 1+0-81 \\ -32+162 & 2-32+162 & -1+0+81 \\ 0-64+324 & 0-64+324 & 0+0+162 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} -98 & -100 & -80 \\ 130 & 132 & 80 \\ 260 & 260 & 162 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} -49 & -50 & -40 \\ 65 & 66 & 40 \\ 130 & 130 & 81 \end{bmatrix}$$

Show that the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is not diagonalisable.

Soln:- Given matrix is $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Since given matrix is traingular, the Eigen values are its diagonal elements.

ie $\lambda = 1, 1, 3$.

To find the Eigen vectors of A corresponding to Eigen values

Consider $(A-\lambda I)X = 0$ Here $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

ie $\begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ — (1)

$$\begin{bmatrix} 1-1 & 2 & 3 \\ 0 & 3-1 & 1 \\ 0 & 0 & 1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 : R_2 - R_1$

$$\sim \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x_2 + 3x_3 = 0 \quad \text{--- (2)}$$

$$-2x_3 = 0 \quad \text{--- (3)}$$

$$\Rightarrow \boxed{x_3 = 0}$$

$$\boxed{x_2 = 0}$$

let $x_1 = k$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the Eigen vector corresponding to $\lambda = 1$

The algebraic multiplicity of Eigen values $\lambda = 1$ is 2

and geometric multiplicity of Eigen value $\lambda = 1$ is 1

Since Algebraic multiplicity $\neq$ Gemometric multiplicity

$\therefore$ The matrix cannot be diagonalizable.

NOTE: if Eigen values are repeated, check correspon-ding Eigen vectors that are linearly Independent, So that $P^{-1}AP$ is diagonalizable.

DIAGONALIZATION BY ORTHOGONAL TRANSFORMATION

Working rule:-

1. Diagonalization by orthogonal transformation is possible in the case of real symmetric matrices only

2. Let $A = [a_{ij}]_{n\times n}$ be a real Symmetric matrix

(i) Find the Eigen values of A

(ii) if all the Eigen values of A are distinct then the corresponding Eigen vectors are linearly independent and pair wise orthogonal, the matrix is diagonalizable

(iii) For repeated Eigen values choose corresponding Eigen vectors are pair wise orthogonal.

(iv) The normalised Eigen vector given by

$$\frac{X_1}{\|X_1\|}, \frac{X_2}{\|X_2\|}, \ldots \frac{X_n}{\|X_n\|}.$$

(v) The modal matrix is $P = \left[\frac{X_1}{\|X_1\|}, \frac{X_2}{\|X_2\|} \ldots \frac{X_n}{\|X_n\|}\right]$

then P will be an orthogonal matrix.

(vi) calculate $P^{-1} = P^T$

(vii) Find $P^{-1}AP = P^TAP = D$ which is diagonal matrix.

Find an orthogonal matrix that will diagonalize the real symmetric matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$. Also find the resulting diagonal matrix

Soln: Given matrix is $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

The characteristic Equation of A is $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left[(3-\lambda)^2-1\right]=0$$

$$\Rightarrow (3-\lambda)\left[9+\lambda^2-6\lambda-1\right]=0$$

$$\Rightarrow (3-\lambda)\left[\lambda^2-6\lambda+8\right]=0$$

$$\Rightarrow (3-\lambda)(\lambda-2)(\lambda-4)=0$$

$$\Rightarrow \lambda = 2, 3, 4$$

Thus the Eigen values of A are $\lambda = 2, 3, 4$

To find the Eigen vectors of A corresponding to Eigen Values

Consider $(A-\lambda I)X = 0$ Here $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

ie
$$\begin{bmatrix} 3-\lambda & 0 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{——— } (1)$$

Case (i) put $\lambda = 2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 : R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_2 + x_3 = 0$$

$$x_1 = 0$$

let $x_3 = K$

$$x_2 + K = 0$$

$$\Rightarrow x_2 = -K$$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -K \\ K \end{bmatrix} = K\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$X_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is the Eigen vector corresponding to Eigen value $\lambda = 2$

$\lambda = 3$

$$\begin{bmatrix} 3-3 & 0 & 0 \\ 0 & 3-3 & 1 \\ 0 & 1 & 3-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \rightleftarrows R_3$ $$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_2 = 0$

$x_3 = 0$

let $x_1 = k$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} = k\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$X_2 : \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the Eigen vector corresponding to Eigen value $\lambda = 3$

<u>case iii</u> $\lambda = 4$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \to (-1)R_1$ $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to (-1)R_2$ $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 : R_3 - R_2$ $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = 0$

$x_2 - x_3 = 0$

let $x_3 = k$

$x_2 - K = 0$

$\Rightarrow x_2 = K$

$$\therefore X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ K \\ K \end{bmatrix} = K\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is the Eigen vector corresponding to Eigen value $\lambda = 4$

$$X_1^T X_2 = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0$$

$$X_2^T X_3 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0$$

$$X_3^T X_1 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = 0 - 1 + 1 = 0$$

$\therefore X_1, X_2$ and X_3 are pairwise orthogonal

Normalize vectors are

$$X_1 = \frac{X_1}{\|X_1\|} = \frac{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}{\sqrt{0+1+1}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$X_2 = \frac{X_2}{\|X_2\|} = \frac{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}{\sqrt{1+0+0}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$X_3 = \frac{X_3}{\|X_3\|} = \frac{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}{\sqrt{0+1+1}} = \frac{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}{\sqrt{2}} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\therefore X_3 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Modal Matrix $P = \left[\frac{X_1}{\|X_1\|}, \frac{X_2}{\|X_2\|}, \frac{X_3}{\|X_3\|} \right]$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$P^{-1} = P^T = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$P^{-1}AP = P^TAP = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 3 & 0 \\ -\frac{2}{\sqrt{2}} & 0 & \frac{4}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} & 0 & \frac{4}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= D$$

Determine the diagonal matrix orthogonally similar to the symmetric matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Soln:- Given matrix is $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Eigen values of A are $\lambda = 2, 2, 8$ and corresponding Eigen vectors are $x_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ $x_3 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

Since $x_2^T x_3 \neq 0$

$\Rightarrow$ These two vectors x_2, x_3 are not orthogonal.

To choose another Eigen vector corresponding to $\lambda = 2$ and orthogonal to $x_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$\Rightarrow$ choose α, β such that $\alpha \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ is orthogonal to $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \alpha - \beta \\ 2\alpha \\ 2\beta \end{bmatrix}$ is orthogonal to $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$\Rightarrow (\alpha-\beta)1 + 2(2\alpha) + 2\beta(0) = 0$

$\Rightarrow \quad \beta = 5\alpha$

The orthogonal vector to $\begin{bmatrix}1\\2\\0\end{bmatrix}$ is $\begin{bmatrix}-4\alpha\\2\alpha\\10\alpha\end{bmatrix} = 2\alpha\begin{bmatrix}2\\1\\5\end{bmatrix}$

Take $X_3 = \begin{bmatrix}2\\1\\5\end{bmatrix}$

$\therefore X_1 = \begin{bmatrix}2\\-1\\1\end{bmatrix}$ $X_2 = \begin{bmatrix}1\\2\\0\end{bmatrix}$ and $X_3 = \begin{bmatrix}2\\1\\5\end{bmatrix}$ are the required Eigen vectors.

Normalizing we get $P = \begin{bmatrix}\frac{2}{\sqrt6} & \frac{1}{\sqrt5} & \frac{-2}{\sqrt{30}}\\ \frac{-1}{\sqrt6} & \frac{2}{\sqrt5} & \frac{1}{\sqrt{30}}\\ \frac{1}{\sqrt6} & 0 & \frac{5}{\sqrt{30}}\end{bmatrix}$

and $P^T = P^{-1} = \begin{bmatrix}\frac{2}{\sqrt6} & \frac{-1}{\sqrt6} & \frac{1}{\sqrt6}\\ \frac{1}{\sqrt5} & \frac{2}{\sqrt5} & 0\\ \frac{-2}{\sqrt{30}} & \frac{1}{\sqrt{30}} & \frac{5}{\sqrt{30}}\end{bmatrix}$

$P^{-1}AP = P^TAP = \begin{bmatrix}8&0&0\\0&2&0\\0&0&2\end{bmatrix} = D.$

APPLICATIONS OF EIGEN VALUES AND EIGEN VECTORS

To find natural frequencies (Eigen values) and mode shapes (Eigen vectors)

Working procedure:

1. The matrices mass (m) and Stiffness (K) are given for the given system of Equations

$$[m]\ddot{x} + [K]x = 0$$

When $a = \ddot{x}$

2. write Eigen value matrix corresponding to the above system is $[K-\lambda m]X = 0$

Write the corresponding characteristic Equation $|K-\lambda m|=0$

4. Solve the above Equation for Eigen values λ_i (i = 1, 2, ... n) which are called Natural frequencies

5. Find the eigen vector corresponding to Each $\lambda_i (i=1,2,\dots n)$ which are called shapes.

Determine the natural frequencies and normal modes of vibrating system for which mass $m = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and Stiffness $K = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

Soln:- Free vibrating system Equation is $[m]\ddot{x} + [K]x = 0$

characteristic Equation for the $[m]\ddot{x} + [K]x = 0$

is $|K - \lambda m| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 1 & 3-2\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(3-2\lambda) - 1 = 0$$

$$\Rightarrow 6 - 4\lambda - 3\lambda + 2\lambda^2 - 1 = 0$$

$$\Rightarrow 2\lambda^2 - 7\lambda + 5 = 0$$

$$\Rightarrow 2\lambda^2 - 2\lambda - 5\lambda + 5 = 0$$

$$\Rightarrow 2\lambda(\lambda-1) - 5(\lambda-1) = 0$$

$$\Rightarrow (\lambda-1)(2\lambda-5) = 0$$

$\Rightarrow \lambda = 1, \frac{5}{2}$ are natural frequencies

Normal Modes

case i, $\lambda = 1$

consider $(K - \lambda m)x = 0$

$$\Rightarrow \begin{bmatrix} 2-1 & 1 \\ 1 & 3-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 : R_2 - R_1$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + x_2 = 0$ 43

Taking $x_2 = k$

$x_1 + k = 0 \Rightarrow x_1 = -k$

$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -k \\ k \end{bmatrix} = k\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is the Eigen vector corresponding to $\lambda = 1$.

Normal mode at $\lambda = 1$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Case ii, $\lambda = \frac{5}{2}$

$\begin{bmatrix} 2 - 5/2 & 1 \\ 1 & 3 - 2(\frac{5}{2}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -\frac{1}{2} & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$R_1 \leftrightarrow R_2$ $\begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$R_2 : R_2 + \frac{1}{2}R_1$ $\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 - 2x_2 = 0$

Let $x_2 = k$

$x_1 - 2k = 0 \Rightarrow x_1 = 2k$

$X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2k \\ k \end{bmatrix} = k\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is the Eigen vector corresponding to $\lambda = 5/2$

Normal mode at $\lambda = 5/2$ is $X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Normal modes are $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

QUADRATIC FORMS

Quadratic form: A Second degree homogeneous polynomial in 'n' variables $x_1, x_2 \cdots x_n$ is called a Quadratic form (Q.F)

Eg: 1. $2x_1^2 + 4x_1x_2 - 3x_2^2$ is a Quadratic form in two variables x_1 and x_2

2. $x^2+y^2+z^2-2xy+4yz-zx$ is a Quadratic form in three variables x, y and z.

General form of a Quadratic form in n variables is given by $Q = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j$

$$= [x_1\ x_2 \cdots x_n] \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & - & - \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= X^T A X$$

Where A is a real Symmetric matrix.

$\therefore$ Every real Quadratic form in n variables $x_1, x_2 \cdots x_n$ can be expressed in the form $x^T A x$.

MATRIX OF QUADRATIC FORM

A Quadratic form Q can be expressed as $Q = x^T A X$. The Symmetric matrix A is called the matrix Quadratic form.

Let the Quadratic form be $x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_3x_1$

Write the coefficients of

Write the coefficients of

	x_1	x_2	x_3
x_1	x_1^2	$\frac{x_1x_2}{2}$	$\frac{x_1x_3}{2}$
x_2	$\frac{x_1x_2}{2}$	x_2^2	$\frac{x_2x_3}{2}$
x_3	$\frac{x_1x_3}{2}$	$\frac{x_2x_3}{2}$	x_3^2

write the coefficients of square terms along the diagonal and divide the product terms x_1x_2, x_2x_3 and x_3x_1 by 2.

Find the symmetric matrix corresponding to the Quadratic form $x^2+2y^2+3z^2+4xy+5yz+6xz$

Soln

		coefficient of		
		x	y	z
	x	x^2	$\frac{xy}{2}$	$\frac{xz}{2}$
coefficient of	y	$\frac{xy}{2}$	y^2	$\frac{yz}{2}$
	z	$\frac{xz}{2}$	$\frac{yz}{2}$	z^2

Let A be the symmetric matrix of order 3 then

$$A = \begin{bmatrix} 1 & \frac{4}{2} & \frac{6}{2} \\ \frac{4}{2} & 2 & 5/2 \\ \frac{6}{2} & 5/2 & 3 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 5/2 \\ 3 & 5/2 & 3 \end{bmatrix}$$

Find the Quadratic form corresponding to the matrix

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & 4 \\ -3 & 4 & 2 \end{bmatrix}$$

Soln: Let $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & 4 \\ -3 & 4 & 2 \end{bmatrix}$ which is a symmetric matrix of order 3.

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then $X^T = [x_1 \ x_2 \ x_3]$

The Quadratic form of A is given by X^TAX

$$= [x_1 \ x_2 \ x_3] \begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & 4 \\ -3 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= [x_1+2x_2-3x_3 \quad 2x_1-x_2+4x_3 \quad -3x_1+4x_2+2x_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= (x_1+2x_2-3x_3)x_1 + (2x_1-x_2+4x_3)x_2 + (-3x_1+4x_2+2x_3)x_3$$

$\Rightarrow x_1^2 + 2x_1x_2 - 3x_1x_3 + 2x_1x_2 - x_2^2 + 4x_2x_3 - 3x_1x_3 + 4x_2x_3 + 2x_3^2$

$\Rightarrow x_1^2 - x_2^2 + 2x_3^2 + 4x_1x_2 - 6x_1x_3 + 8x_2x_3$

LINEAR TRANSFORMATION OF A QUADRATIC FORM

Let X^TAX be the Quadratic form in n variables $x_1, x_2 \cdots x_n$. Let $X = PY$ be the linear Transformation which transforms the variables $x_1, x_2, \cdots x_n$ to the variables $y_1, y_2 \cdots y_n$.

$$\text{Now } X^TAX = (PY)^T A (PY)$$
$$= Y^TP^TAPY$$
$$= Y^T(P^TAP)Y$$
$$= Y^TDY \quad \text{where } P^TAP = D$$

Y^TDY is a quadratic form in variables $y_1, y_2 \cdots y_n$.

$\therefore Y^TDY$ is the linear transformation of the quadratic form X^TAX.

CANONICAL FORM OF A QUADRATIC FORM

Let X^TAX be a quadratic form in n variables, then there exist a real non-singular linear transformation $X = PY$ which transforms X^TAX to another quadratic form of type $Y^TDY = \lambda_1y_1^2 + \lambda_2y_2^2 + \cdots + \lambda_ny_n^2$ then Y^TDY is called the canonical form (or) sum of squares form of X^TAX. where $D = dia[\lambda_1, \lambda_2 \cdots \lambda_n]$ and $\lambda_1, \lambda_2 \cdots \lambda_n$ are the eigen values of A

RANK, INDEX AND SIGNATURE OF QUADRATIC FORM

The given quadratic form can be reduced to the canonical form as $\lambda_1y_1^2 + \lambda_2y_2^2 + \cdots + \lambda_ny_n^2$

RANK: The number of non-zero terms in the canonical form of the quadratic form is called the rank of the quadratic form. It is denoted by r.

INDEX

The number of positive terms in canonical form is said to be Index of the Quadratic form. It is denoted by p

SIGNATURE

The difference between the positive and negative terms in the canonical form is said to be Signature of Quadratic form. It is denoted by s.

ie $s = 2p - r$

NATURE OF THE QUADRATIC FORMS

The Quadratic form x^TAx in n variables is said to be

i) positive definite : If all the Eigen values of A are positive.

ii) Negative definite : If all the Eigen values of A are negative.

iii) positive Semi definite : if all the Eigen values of A are positive and atleast one Eigen value is zero.

iv) Negative Semi definite : If all the Eigen values of A are negative and atleast one Eigen value is zero.

(v) Indefinite : If some of the Eigen value of A are positive and some of the Eigen values of A are negative.

... the nature of the Quadratic form $x^2+3y^2+3z^2-2yz$

Soln: Given Quadratic form is $x^2+3y^2+3z-2yz$

The matrix corresponding to Quadratic form is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

The characteristic Equation of A is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-1 & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[(3-\lambda)^2-1\right]=0$$

$$\Rightarrow (1-\lambda)\left[9+\lambda^2-6\lambda-1\right]=0$$

$$\Rightarrow (1-\lambda)\left[\lambda^2-6\lambda+8\right]=0$$

$$\Rightarrow \lambda = 1,2,4$$

Since all the Eigen values of A are positive.

The nature of Q.F is positive definite.

Discuss the nature of the Quadratic form $2x^2+2y^2+2z^2-2xy-2zx-2yz$

Soln: Given Quadratic form is $2x^2+2y^2+2z^2-2xy-2zx-2yz$

The matrix corresponding to Quadratic form is

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

The characteristic Equation of A is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\left[(2-\lambda)^2-1\right]+1\left[-2+\lambda-1\right]-1\left[1+2-\lambda\right]=0$$

$$\Rightarrow (2-\lambda)\left[4+\lambda^2-4\lambda-1\right]+1\left[\lambda-3\right]-1\left[3-\lambda\right]=0$$

$$\Rightarrow (2-\lambda)\left[\lambda^2-4\lambda+3\right]+\lambda-3-3+\lambda=0$$

$$\Rightarrow (2-\lambda)[(\lambda-1)(\lambda-3)] + 2\lambda - 6 = 0$$

$$\Rightarrow (2-\lambda)(\lambda-1)(\lambda-3) + 2(\lambda-3) = 0$$

$$\Rightarrow (\lambda-3)\left[(2-\lambda)(\lambda-1)+2\right] = 0$$

$$\Rightarrow (\lambda-3)\left[2\lambda - 2 + \lambda^2 + \lambda + 2\right] = 0$$

$$\Rightarrow (\lambda-3)\left[-\lambda^2 + 3\lambda\right] = 0$$

$$\Rightarrow (\lambda-3) = 0 \quad \& \quad -\lambda^2 + 3\lambda = 0$$

$$\Rightarrow \lambda = 3 \quad (\&) \quad -\lambda(\lambda-3) = 0$$

$$\Rightarrow \lambda = 0; 3$$

$$\therefore \quad \lambda = 0, 3, 3$$

Thus the Eigen values of A are $\lambda = 0, 3, 3$

Since one Eigen value of A is zero and remaining are positive.

$\therefore$ The nature of the Q.F is semi definite.

METHODS FOR REDUCING A QUADRATIC FORM TO CANONICAL FORM

The following methods are used for reducing a Q.F to canonical form

(1) Diagonalisation

(2) orthogonalisation

(3) Lagrange's reduction.

METHOD : I DIAGONALISATION

Working rule:

(1) write the matrix A of the given Q.F

(2) write $A_n = I_n A I_n$

(3) Apply row and column operation on Equation ① to reduce A_n to a diagonal matrix

(4) Apply the same row operations on pre-factor I on R.H.S and identical column operation on

...t factors in R.H.S of Equation ①

⑤ By these operations, $A = IAI$ will be reduced to the form $D = P^TAP$ where D is the diagonal matrix

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The canonical form is given by

$$Y^TDY = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \quad \text{where } D = \text{dig}[\lambda_1, \lambda_2 \cdots \lambda_n]$$

REDUCE the Quadratic form to the canonical form $x^2+y^2+2z^2-2xy+4zx+4yz$. Also find its rank, Index, Signature and nature.

Soln: Given Quadratic form is $x^2+y^2+2z^2-2xy+4zx+4yz$

The matrix form of the given Q.F is

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

write $A = I_3 A I_3$

$$\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 : R_2 + R_1$; $R_3 : R_3 - 2R_1$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 4 \\ 0 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 : C_2 + C_1$; $C_3 : C_3 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \rightleftarrows R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & -2 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \rightleftarrows C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 4 \\ 0 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$R_3 : R_3 + 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 4 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 1 \\ -3 & 1 & 2 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$C_3 : C_3 + 2C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 1 \\ -3 & 1 & 2 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

i.e. $D = P^T A P$

where $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

The canonical form to given Q.F is $Y^T D Y$.

$$= [y_1\ y_2\ y_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= y_1^2 - 2y_2^2 + 8y_3^2$$

Rank $(r) = 3$

Index $(p) = 2$

Signature $(s) = 2p - r = 2(2) - 3 = 4 - 3 = 1$

Nature of Q.F is indefinite.

METHOD 2 : ORTHOGONALISATION

Working rule :

(1) Write the matrix A of Q.F

(2) Find the Eigen values of A

(3) Find the Eigen vectors of A corresponding to the Eigen values such that the eigen vector must be pairwise orthogonal.

(4) Find the Normalized vectors $e_1 = \frac{X_1}{\|X_1\|}, e_2 = \frac{X_2}{\|X_2\|}$ - - - - - $e_n = \frac{X_n}{\|X_n\|}$

(5) Write the Modal matrix $P = [e_1\ e_2 \cdots e_n]$

(6) P being orthogonal matrix then $P^{-1} = P^T$

$\Rightarrow P^T A P = D$, where D is a diagonal matrix formed by Eigen values

(7) The canonical form is $Y^T D Y = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$

[illegible] quadratic form [illegible] to the canonical form by orthogonal reduction and find the nature, rank, index and signature

Soln: Given QF is $3x^2+5y^2+3z^2-2yz+2zx-2xy$

The matrix form of the given QF is

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic equation of A is $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)\left[(5-\lambda)(3-\lambda)-1\right]+1\left[-3+\lambda+1\right]+1\left[1-5+\lambda\right]=0$$

$$\Rightarrow (3-\lambda)\left[\lambda^2-8\lambda+14\right]+1(\lambda-2)+1\left[\lambda-4\right]=0$$

$$\Rightarrow (3-\lambda)\left[\lambda^2-8\lambda+14\right]+\lambda-2+\lambda-4=0$$

$$\Rightarrow (3-\lambda)\left[\lambda^2-8\lambda+14\right]+2\lambda-6=0$$

$$\Rightarrow (3-\lambda)\left[\lambda^2-8\lambda+14\right]-2\left[3-\lambda\right]=0$$

$$\Rightarrow (3-\lambda)\left[\lambda^2-8\lambda+14-2\right]=0$$

$$\Rightarrow (3-\lambda)\left[\lambda^2-8\lambda+12\right]=0$$

$$\Rightarrow (3-\lambda)(\lambda-6)(\lambda-2)=0$$

$$\Rightarrow \lambda = 2, 3, 6$$

The Eigen values of A are $\lambda = 2, 3, 6$

<u>To find the Eigen vectors of A corresponding to Eigen values</u>

Consider $(A-\lambda I)X = 0$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\text{i.e. } \begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- (1)}$$

Case i, $\lambda = 2$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 : R_2 + R_1$; $R_3 : R_3 - R_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x_2 = 0 \Rightarrow x_2 = 0$$

$$x_1 - x_2 + x_3 = 0$$

let $x_3 = K$

$$x_1 - 0 + K = 0$$

$$\Rightarrow x_1 = -K$$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -K \\ 0 \\ K \end{bmatrix} = K \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is the Eigen vector corresponding to Eigen value $\lambda = 2$

Case ii, $\lambda = 3$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \rightleftarrows R_3$
$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 : R_2 + R_1$
$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 : R_3 + R_2$
$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_2 - x_3 = 0$

$x_1 - x_2 = 0$

let $x_3 = K$

$x_2 - K = 0$

$\Rightarrow x_2 = K$

$x_1 - K = 0$

$\Rightarrow x_1 = K$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} K \\ K \\ K \end{bmatrix} = K \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is the Eigen vector corresponding to $\lambda = 3$

case iii $\lambda = 6$

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \rightleftarrows R_3$

$$\begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -1 \\ -3 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 : R_2 + R_1$; $R_3 : R_3 + 3R_1$

$$\begin{bmatrix} 1 & -1 & -3 \\ 0 & -2 & -4 \\ 0 & -4 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 : \frac{R_2}{-2}$

$$\begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 2 \\ 0 & -4 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 : R_3 + 4R_2$

$$\begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 - x_2 - 3x_3 = 0$

$x_2 + 2x_3 = 0$

let $x_3 = K$

$x_2 + 2K = 0$

$x_2 = -2K$

$x_1 + 2K - 3K = 0$

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$x_1 - K = 0$

$x_1 = K.$

$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} K \\ -2K \\ K \end{bmatrix} = K\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is the Eigen vector corresponding to $\lambda = 6$

The Eigen vectors of A are

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

We have $X_1^T X_2 = 0$; $X_2^T X_3 = 0$ and $X_3^T X_1 = 0$

$\therefore X_1, X_2$ and X_3 are pairwise orthogonal

Normalized Eigen vectors are

$$e_1 = \frac{X_1}{\|X_1\|} = \frac{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}{\sqrt{2}} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$e_2 = \frac{X_2}{\|X_2\|} = \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\sqrt{3}} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$e_3 = \frac{X_3}{\|X_3\|} = \frac{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}{\sqrt{6}} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

Modal Matrix $P = [e_1 \; e_2 \; e_3]$

$$P = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix} \Rightarrow P^T = \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Since P is orthogonal matrix then $P^{-1} = P^T$

$$D = P^{-1}AP = P^TAP = \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

The canonical form of given Q.F is $Y^T D Y = 2y_1^2 + 3y_2^2 + 6y_3^2$

Rank $(r) = 3$

Index $(P) = 3$

Signature $(s) = 2p - r = 2(3) - 3 = 3$

The Nature of Q.F is positive definite.

METHOD 3 LAGRANGE'S METHOD

Working rule:

Let $X^T A X$ be a Q.F of n variables

1. Take the common terms of given Q.F
2. Make perfect square suitable by regrouping the terms
3. Finally the resulting relation gives the required canonical form.

Using Lagrange's method reduce the Q.F $x^2 + 6y^2 + 18z^2 + 4xy + 8xz - 4yz$ to canonical form. Also find the nature, rank, index and signature of the Q.F.

Given Q.F is $x^2 + 6y^2 + 18z^2 + 4xy + 8xz - 4yz$

$$\Rightarrow x^2 + 4y^2 + 2y^2 + 16z^2 + 2z^2 + 4xy + 8xz - 4yz$$

$$\Rightarrow (x)^2 + (2y)^2 + (4z)^2 + 2 \cdot x \cdot 2y + 2 \cdot 2y \cdot 4z + 2 \cdot 4z \cdot x + 2y^2 + 2z^2 - 20yz$$

$$\Rightarrow (x + 2y + 4z)^2 + 2y^2 - 20yz + 2z^2$$

$$\Rightarrow (x + 2y + 4z)^2 + 2[y^2 - 10yz) + 2z^2$$

$$\Rightarrow (x + 2y + 4z)^2 + 2[y^2 - 2 \cdot y(5z) + (5z)^2 - (5z)^2] + 2z^2$$

$$\Rightarrow (x + 2y + 4z)^2 + 2[y^2 - 2 \cdot y \cdot (5z) + (5z)^2] - 2 \cdot (25) z^2 + 2z^2$$

$$\Rightarrow (x + 2y + 4z)^2 + 2[(y - 5z)^2] - 48z^2$$

$$\Rightarrow y_1^2 + 2y_2^2 - 48y_3^2$$ which is required canonical form

Where $y_1 = x+2y+4z$, $y_2 = y-5z$ and $y_3 = z$

Hence Rank $(r) = 3$

Index $(p) = 2$

Signature $(s) = 2p - r = 4 - 3 = 1$.

The Nature of Q.F is indefinite.

using Lagrange's method, reduce that the Q.F $x_1^2 + 4x_2^2 + x_3^2 - 4x_1x_2 + 2x_3x_1 - 4x_2x_3$. Also find its nature, index, rank and signature.

Soln: Given Q.F is $x_1^2 + 4x_2^2 + x_3^2 - 4x_1x_2 + 2x_3x_1 - 4x_2x_3$

$$\Rightarrow (x_1)^2 + (2x_2)^2 + (x_3)^2 - 2 \cdot x_1 \cdot 2x_2 - 2 \cdot 2x_2 \cdot x_3 + 2 \cdot x_3 x_1$$

$$\Rightarrow (x_1 - 2x_2 + x_3)^2$$

$\Rightarrow y_1^2$, which is required canonical form

where $y_1 = x_1 - 2x_2 + x_3$

$\therefore$ Rank $(r) = 1$

Index $(p) = 1$

Signature $(s) = 2p - r = 2 - 1 = 1$.

The nature of the Q.F is positive semi definite.