FirstRanker.com
Firstranker's choice
www.FirstRanker.com

VECTOR CALCULUS

Differentiation of vector:-

If $\bar{r} = \bar{f}(t)$ is a vector function, then differentiation of $\bar{r}$ w.r.t to 't' is $\frac{d\bar{f}}{dt} = \lim_{\delta t \to 0} \frac{\bar{f}(t+\delta t) - \bar{f}(t)}{\delta t} = \frac{d\bar{r}}{dt}$.

If $\lim_{\delta t \to 0} \frac{\delta \bar{r}}{\delta t} = \lim_{\delta t \to 0} \frac{\bar{f}(t+\delta t) - \bar{f}(t)}{\delta t}$ exists; then the value of this limit is denoted by $\frac{d\bar{r}}{dt}$ and is called the derivative of $\bar{r}$ w.r.t to 't'.

Let $\bar{r} = f(t)$ be a vector function of the scalar variable t. Let δt be a small increment in t and $\delta \bar{r}$ the corresponding increment in $\bar{r}$.

Then $\bar{r} + \delta \bar{r} = \bar{f}(t+\delta t)$ so that $\delta \bar{r} = \bar{f}(t+\delta t) - \bar{r}$

$\delta \bar{r} = \bar{f}(t+\delta t) - \bar{f}(t)$

$\frac{\delta \bar{r}}{\delta t} = \frac{\bar{f}(t+\delta t) - \bar{f}(t)}{\delta t}$.

Since $\frac{d\bar{r}}{dt}$ is itself a vector function of 't', it's derivative is denoted by $\frac{d^2\bar{r}}{dt^2}$ and is called the second derivative of $\bar{r}$ w.r.t to 't'. By we can define higher order derivatives of $\bar{r}$.

Important Formulae of Differentiation of vectors

(i) $\frac{d}{dt}(\bar{F} + \bar{G} - \bar{H}) = \frac{d\bar{F}}{dt} + \frac{d\bar{G}}{dt} - \frac{d\bar{H}}{dt}$

(ii) $\frac{d}{dt}[\bar{F}\phi] = \bar{F}\frac{d\phi}{dt} + \frac{d\bar{F}}{dt}\phi$

(iii) $\frac{d}{dt}[\bar{F}\cdot\bar{G}] = \bar{F}\cdot\frac{d\bar{G}}{dt} + \frac{d\bar{F}}{dt}\cdot\bar{G}$

(iv) $\frac{d}{dt}[\bar{F}\times\bar{G}] = \bar{F}\times\frac{d\bar{G}}{dt} + \frac{d\bar{F}}{dt}\times\bar{G}$

(v) $\frac{d}{dt}[\bar{F}\,\bar{G}\,\bar{H}] = \left[\frac{d\bar{F}}{dt}\,\bar{G}\,\bar{H}\right] + \left[\bar{F}\,\frac{d\bar{G}}{dt}\,\bar{H}\right] + \left[\bar{F}\,\bar{G}\,\frac{d\bar{H}}{dt}\right]$

(vi) $\frac{d}{dt}[(\bar{F}\times\bar{G})\times\bar{H}] = \left(\frac{d\bar{F}}{dt}\times\bar{G}\right)\times\bar{H} + \left(\bar{F}\times\frac{d\bar{G}}{dt}\right)\times\bar{H} + (\bar{F}\times\bar{G})\times\frac{d\bar{H}}{dt}$

where $\bar{F}, \bar{G}, \bar{H}$ are vectors and ϕ, t Scalars.

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Differentiation of constant vector

1) If $\bar{F}$ be a vector of constant magnitude and direction then $\bar{F} \cdot \frac{d\bar{F}}{dt} = 0$.

Proof:- $\bar{F}(t)$ has a constant magnitude $\Rightarrow |\bar{F}(t)| =$ constant

$$\bar{F}(t) \cdot \bar{F}(t) = |\bar{F}(t)|^2 = \text{constant}.$$

$$\Rightarrow \frac{d}{dt}(\bar{F} \cdot \bar{F}) = 0 \Rightarrow \bar{F} \cdot \frac{d\bar{F}}{dt} + \frac{d\bar{F}}{dt} \cdot \bar{F} = 0$$

$$\Rightarrow 2\bar{F} \cdot \frac{d\bar{F}}{dt} = 0 \Rightarrow \bar{F} \cdot \frac{d\bar{F}}{dt} = 0.$$

$$\Rightarrow \frac{d\bar{F}}{dt} \perp \bar{F}.$$

2) If $\bar{F}(t)$ has constant direction then $\bar{F} \times \frac{d\bar{F}}{dt} = \bar{0}$.

Proof:- Let $|\bar{F}(t)| = f(t)$

Let $\bar{G}(t)$ be a unit vector in the direction of $\bar{F}(t)$

so that $\bar{F}(t) = f(t)\bar{G}(t)$

$$\frac{d\bar{F}}{dt} = f\frac{d\bar{G}}{dt} + \frac{df}{dt}\bar{G} \quad \text{— (1)}$$

If $\bar{F}(t)$ has constant direction so has $\bar{G}(t)$

Thus $\bar{G}(t)$ is a constant vector and $\frac{d\bar{G}}{dt} = 0$

From (1) $\frac{d\bar{F}}{dt} = \frac{df}{dt} \cdot \bar{G}$

$$\bar{F} \times \frac{d\bar{F}}{dt} = f\bar{G} \times \left(\frac{df}{dt}\bar{G}\right)$$

$$= f\frac{df}{dt}(\bar{G} \times \bar{G}) = \bar{0} \quad [\because \bar{G} \times \bar{G} = \bar{0}]$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

1) If $\bar{A} = 5t^2\bar{i} + t\bar{j} - t^3\bar{k}$, $\bar{B} = \sin t\,\bar{i} - \cos t\,\bar{j}$ find (i) $\frac{d}{dt}(\bar{A}\cdot\bar{B})$

(ii) $\frac{d}{dt}(\bar{A}\times\bar{B})$

Sol:- (i) $\frac{d}{dt}(\bar{A}\cdot\bar{B}) = \bar{A}\cdot\frac{d\bar{B}}{dt} + \frac{d\bar{A}}{dt}\cdot\bar{B}$

$= (5t^2\bar{i} + t\bar{j} - t^3\bar{k})\cdot[\cos t\,\bar{i} - (-\sin t\,\bar{j})] + (10t\bar{i} + \bar{j} - 3t^2\bar{k})\cdot(\sin t\,\bar{i} - \cos t\,\bar{j})$

$= (5t^2\cos t + t\sin t) + 10t\sin t - \cos t$

$= 5t^2\cos t + 11t\sin t - \cos t$

(ii) $\frac{d}{dt}(\bar{A}\times\bar{B}) = \bar{A}\times\frac{d\bar{B}}{dt} + \frac{d\bar{A}}{dt}\times\bar{B}$

$= (5t^2\bar{i} + t\bar{j} - t^3\bar{k})\times(\cos t\,\bar{i} + \sin t\,\bar{j}) + (10t\bar{i} + \bar{j} - 3t^2\bar{k})\times(\sin t\,\bar{i} - \cos t\,\bar{j})$

$= 5t^2\sin t\,\bar{k} - t\cos t\,\bar{k} - t^3\cos t\,\bar{j} + 3t^3\sin t\,\bar{k} - 10t\cos t\,\bar{k} - \sin t\,\bar{k} - 3t^2\sin t\,\bar{j} - 3t^2\cos t\,\bar{i}$

$= (t^3\sin t - 3t^2\cos t)\bar{i} - t^2(t\cos t + 3\sin t)\bar{j} + [(5t^2 - 1)\sin t - 11t\cos t]\bar{k}$

2) Show that if $\bar{R} = A\sin\omega t + B\cos\omega t$ where A, B, ω are constants, then $\frac{d^2\bar{R}}{dt^2} = -\omega^2\bar{R}$ and $\bar{R}\times\frac{d\bar{R}}{dt} = -\omega\,\bar{A}\times\bar{B}$

Sol:- $\bar{R} = A\sin\omega t + B\cos\omega t$

$\frac{d\bar{R}}{dt} = A\omega\cos\omega t - B\omega\sin\omega t$

$\frac{d^2\bar{R}}{dt^2} = -A\omega^2\sin\omega t - B\omega^2\cos\omega t$

$= -\omega^2(A\sin\omega t + B\cos\omega t) = -\omega^2\bar{R}$

Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$$\bar{R} \times \frac{d\bar{R}}{dt} = (A\sin\omega t + B\cos\omega t) \times (A\omega\cos\omega t - B\omega\sin\omega t)$$

$$= -\omega \sin^2\omega t (\bar{A}\times\bar{B}) + \omega\cos^2\omega t (\bar{B}\times\bar{A}) \qquad [\bar{A}\times\bar{A} = \bar{0},\ \bar{B}\times\bar{B} = \bar{0}]$$

$$= -\omega \sin^2\omega t (\bar{A}\times\bar{B}) - \omega\cos^2\omega t (\bar{A}\times\bar{B}) \qquad \sin^2\omega t + \cos^2\omega t = 1$$

$$= -\omega(\bar{A}\times\bar{B})[\sin^2\omega t + \cos^2\omega t] \qquad \bar{A}\times\bar{B} = -(\bar{B}\times\bar{A})$$

$$= -\omega(\bar{A}\times\bar{B})$$

2) Given $\bar{R} = t^m\bar{A} + t^n\bar{B}$ where A, B are constant vectors, show that if $\bar{R}$ and $\frac{d^2\bar{R}}{dt^2}$ are parallel vectors, then $m+n = 1$ unless $m = n$.

Sol:- Given $\bar{R} = t^m\bar{A} + t^n\bar{B}$

$$\frac{d\bar{R}}{dt} = mt^{m-1}\bar{A} + nt^{n-1}\bar{B}$$

$$\frac{d^2\bar{R}}{dt^2} = m(m-1)t^{m-2}\bar{A} + n(n-1)t^{n-2}\bar{B}$$

For parallel vector $\bar{R} \times \frac{d^2\bar{R}}{dt^2} = \bar{0}$

$$\Rightarrow (t^m\bar{A} + t^n\bar{B}) \times [m(m-1)t^{m-2}\cdot\bar{A} + n(n-1)t^{n-2}\bar{B}] = \bar{0}$$

$$\Rightarrow n(n-1)t^m t^{n-2}(\bar{A}\times\bar{B}) + m(m-1)t^n t^{m-2}(\bar{B}\times\bar{A}) = \bar{0} \qquad [\bar{A}\times\bar{B} = -\bar{B}\times\bar{A}]$$

$$\Rightarrow n(n-1)t^{m+n-2}(\bar{A}\times\bar{B}) - m(m-1)t^{m+n-2}(\bar{A}\times\bar{B}) = \bar{0}$$

$$\Rightarrow t^{m+n-2}(\bar{A}\times\bar{B})[n^2 - n - m^2 + m] = \bar{0}$$

$$\Rightarrow t^{m+n-2}(\bar{A}\times\bar{B})[(n+m)(n-m) - (n-m)] = \bar{0}$$

$$\Rightarrow t^{m+n-2}(\bar{A}\times\bar{B})[(n-m)[m+n-1]] = \bar{0} \Rightarrow m \neq n \text{ either } m+n = 1$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$\bar{P} = t^2\hat{i} + t^3\hat{j} - t\hat{k}$ and $\bar{Q} = 2\sin t\,\hat{i} - \cos t\,\hat{j} + 5t\hat{k}$ (3)

find (i) $\frac{d}{dt}(\bar{P}\cdot\bar{Q})$ (ii) $\frac{d}{dt}(\bar{P}\times\bar{Q})$.

Sol:- $\bar{P}\cdot\bar{Q} = 10t^2\sin t - t^3\cos t - 5t^2$

$\frac{d}{dt}(\bar{P}\cdot\bar{Q}) = 10t^2\cos t + 20t\sin t - 3t^2\cos t + t^3\sin t - 10t$

$= t^3\sin t + 7t^2\cos t + 20t\sin t - 10t$

$\bar{P}\times\bar{Q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t^3 & -t \\ 2\sin t & -\cos t & 5t \end{vmatrix} = \hat{i}(5t^4 - t\cos t) + \hat{j}(-t^2\sin t + 25t^3) + \hat{k}(-5t^2\cos t - t^3 2\sin t)$

$= (5t^4 - t\cos t)\hat{i} - \hat{j}(2t\sin t + 25t^3) - \hat{k}(5t^2\cos t + 2t^3\sin t)$

(4) If $\frac{d\bar{v}}{dt} = \bar{\omega}\times\bar{v}$ and $\frac{d\bar{v}}{dt} = \bar{\omega}\times\bar{v}$

P.T. $\frac{d}{dt}(\bar{v}\times\bar{v}) = \bar{\omega}\times(\bar{v}\times\bar{v})$

Sol:- $\frac{d}{dt}(\bar{v}\times\bar{v}) = \bar{v}\times\frac{d\bar{v}}{dt} + \frac{d\bar{v}}{dt}\times\bar{v}$

$= \bar{v}\times(\bar{\omega}\times\bar{v}) + (\bar{\omega}\times\bar{v})\times\bar{v}$

$= (\bar{v}\cdot\bar{v})\bar{\omega} - (\bar{v}\cdot\bar{\omega})\bar{v} + (\bar{v}\cdot\bar{\omega})\bar{v} - (\bar{v}\cdot\bar{v})\bar{\omega}$

$= (\bar{\omega}\cdot\bar{v})\bar{v} - (\bar{\omega}\cdot\bar{v})\bar{v}$

$= \bar{\omega}\times(\bar{v}\times\bar{v})$

$(\bar{v}\cdot\bar{v} = \bar{v}\cdot\bar{\omega})$

$(\bar{v}\cdot\bar{\omega} = \bar{\omega}\cdot\bar{v})$

$(\bar{a}\times\bar{b})\times\bar{c} = (\bar{c}\cdot\bar{a})\bar{b} - (\bar{c}\cdot\bar{b})\bar{a}$

$\bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}$

(5) If $\bar{A} = x^2y\hat{i} - 2xz^3\hat{j} + xz^2\hat{k}$

and $\bar{B} = 2z\hat{i} + y\hat{j} - x^2\hat{k}$, find $\frac{\partial^2}{\partial x\partial y}(\bar{A}\times\bar{B})$ at $(1, 0, -2)$

Sol:- $\bar{A}\times\bar{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x^2y & -2xz^3 & xz^2 \\ 2z & y & -x^2 \end{vmatrix} = \hat{i}(2x^2z^3 - xyz^2) + \hat{j}(2xz^3 + x^4yz) + \hat{k}(x^2y^2 + 4xz^4)$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$$\frac{\partial^2}{\partial x \partial y}(\bar{a}\times\bar{b}) = (0-t^2)\bar{i} + (0+4x^3t)\bar{j} + (4xyz+0)\bar{k}$$

$$\frac{\partial^2}{\partial x \partial y}(\bar{a}\times\bar{b}) = -4(\bar{i}+2\bar{j}) \text{ at } (1,0,-2)$$

6) If $\bar{r} = a\cos t\,\bar{i} + a\sin t\,\bar{j} + at\tan\alpha\,\bar{k}$.

Find the value of $\left|\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2}\right|$ (ii) $\left[\frac{d\bar{r}}{dt}, \frac{d^2\bar{r}}{dt^2}, \frac{d^3\bar{r}}{dt^3}\right]$.

Sol:- $\bar{r} = a\cos t\,\bar{i} + a\sin t\,\bar{j} + at\tan\alpha\,\bar{k}$

$$\frac{d\bar{r}}{dt} = -a\sin t\,\bar{i} + a\cos t\,\bar{j} + a\tan\alpha\,\bar{k}$$

$$\frac{d^2\bar{r}}{dt^2} = -a\cos t\,\bar{i} - a\sin t\,\bar{j} \quad \text{and} \quad \frac{d^3\bar{r}}{dt^3} = a\sin t\,\bar{i} - a\cos t\,\bar{j}$$

$$\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -a\sin t & a\cos t & a\tan\alpha \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = a^2\sin t\tan\alpha\,\bar{i} - a^2\cos t\tan\alpha\,\bar{j} + \bar{k}(a^2\sin^2 t + a^2\cos^2 t)$$

$$= a^2\sin t\tan\alpha\,\bar{i} - a^2\cos t\tan\alpha\,\bar{j} + a^2\bar{k}$$

$$\left|\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2}\right| = \sqrt{a^4\sin^2 t\tan^2\alpha + a^4\cos^2 t\tan^2\alpha + a^4} = a^2\sqrt{\tan^2\alpha + 1} = a^2\sec^2\alpha$$

Also $\left[\frac{d\bar{r}}{dt}\ \frac{d^2\bar{r}}{dt^2}\ \frac{d^3\bar{r}}{dt^3}\right] = \begin{vmatrix} -a\sin t & a\cos t & a\tan\alpha \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix}$

$$= a\tan\alpha(a^2\cos^2 t + a^2\sin^2 t)$$

$\cos^2 t + \sin^2 t = 1$

$$= a^3\tan\alpha(\cos^2 t + \sin^2 t)$$

$$= a^3\tan\alpha$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Find the unit tangent vector at any point on the curve $x = t^2+2$, $y = 4t-5$, $z = 2t^2-6t$ where t is any variable. Also determine the unit tangent vector at the point $t=2$.

Sol:- If $\bar{r}$ is the position vector of any point (x,y,z) on the given curve then $\bar{r} = x\hat{i}+y\hat{j}+z\hat{k}$.

$$= (t^2+2)\hat{i} + (4t-5)\hat{j} + (2t^2-6t)\hat{k}.$$

and $\frac{d\bar{r}}{dt} = 2t\hat{i} + 4\hat{j} + (4t-6)\hat{k}$.

$$\left|\frac{d\bar{r}}{dt}\right| = \sqrt{(2t)^2+(4)^2+(4t-6)^2} = \sqrt{4t^2+16+16t^2+36-48t}$$

$$= \sqrt{20t^2-48t+52} = 2\sqrt{5t^2-12t+13}.$$

The unit tangent vector $\bar{T} = \frac{\frac{d\bar{r}}{dt}}{\left|\frac{d\bar{r}}{dt}\right|} = \frac{2[t\hat{i}+2\hat{j}+(2t-3)\hat{k}]}{2\sqrt{5t^2-12t+13}}$

$$= \frac{(t\hat{i}+2\hat{j}+2t-3\hat{k})}{\sqrt{5t^2-12t+13}}$$

Also the unit tangent vector at the point $t=2$ is $\frac{2\hat{i}+2\hat{j}+[2(2)-3]\hat{k}}{\sqrt{5(2)^2-12(2)+13}}$.

$$= \frac{1}{3}(2\hat{i}+2\hat{j}+\hat{k})$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Curves in a Space

(i) **Tangent** : If $\bar{R} = \bar{f}(t)$ be any space curve then the tangent at any point is represented by $\frac{d\bar{R}}{dt}$ at the point. If 's' is the parameter in place of 't' then the magnitude of the tangent vector, $\left|\frac{d\bar{R}}{ds}\right| = 1$, Thus the unit tangent vector $\boxed{\bar{T} = \frac{d\bar{R}}{ds}}$.

(ii) **principal Normal** :- Since 'T' is a unit vector.

$\boxed{\frac{d\bar{T}}{ds} \cdot \bar{T} = 0}$; i.e $\frac{d\bar{T}}{ds}$ is perpendicular to $\bar{T}$

Here $\frac{d\bar{T}}{ds}$ is unit vector along the vector $\bar{N}$ which known as principal normal at given point.

(iii) **Binormal** :- If $\bar{T}, \bar{N}$ represent unit tangent, unit (i.e principal normal) then Binormal $\boxed{\bar{B} = \bar{T} \times \bar{N}}$

(iv) **curvature** :- The arc rate of turning of the tan (i.e the magnitude of $\frac{d\bar{T}}{ds}$ is called the curvature curve and is denoted by 'k'.

Since $\frac{d\bar{T}}{ds}$ is in the direction of the principal normal $\bar{N}$

$$\boxed{\frac{d\bar{T}}{ds} = k\bar{N}}$$

(v) **Torsion** :- The arc rate of turning the binormal $\frac{d\bar{B}}{ds}$ is called torsion of the curve and is denoted by 'τ', we may, therefore, write $\boxed{\frac{d\bar{B}}{ds} = -\tau\bar{N}}$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

1) Find the curvature and torsion of the curve $x = a\cos t,\ y = a\sin t,\ z = bt$.

Ans:-

$$\bar{r} = a\cos t\,\bar{i} + a\sin t\,\bar{j} + bt\,\bar{k}$$

$$\frac{d\bar{r}}{dt} = -a\sin t\,\bar{i} + a\cos t\,\bar{j} + b\bar{k}$$

$$\frac{ds}{dt} = \left|\frac{d\bar{r}}{dt}\right| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2+b^2}$$

$$\bar{T} = \frac{d\bar{r}}{ds} = \frac{d\bar{r}/dt}{ds/dt} = \frac{-a\sin t\,\bar{i} + a\cos t\,\bar{j} + b\bar{k}}{\sqrt{a^2+b^2}}$$

$$\frac{d\bar{T}}{dt} = \frac{-a\cos t\,\bar{i} - a\sin t\,\bar{j}}{\sqrt{a^2+b^2}}$$

$$\frac{d\bar{T}}{ds} = \frac{d\bar{T}/dt}{ds/dt} = \frac{-a\cos t\,\bar{i} - a\sin t\,\bar{j}}{\sqrt{a^2+b^2}} \cdot \frac{1}{\sqrt{a^2+b^2}} = \frac{-a(\cos t\,\bar{i} + \sin t\,\bar{j})}{a^2+b^2}$$

$$k = \left|\frac{d\bar{T}}{ds}\right| = \frac{\sqrt{a^2\cos^2 t + a^2\sin^2 t}}{a^2+b^2} = \frac{a}{a^2+b^2}$$

$$\bar{N} = \frac{1}{k}\frac{d\bar{T}}{ds} = \frac{a^2+b^2}{a}\left(\frac{-a\cos t\,\bar{i} - a\sin t\,\bar{j}}{a^2+b^2}\right)$$

$$= -\cos t\,\bar{i} - \sin t\,\bar{j}$$

$$\bar{B} = \bar{T}\times\bar{N} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{-a\sin t}{\sqrt{a^2+b^2}} & \frac{a\cos t}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{b\sin t}{\sqrt{a^2+b^2}}\bar{i} - \frac{b\cos t}{\sqrt{a^2+b^2}}\bar{j} + a\bar{k}$$

$$\frac{d\bar{B}}{dt} = \frac{b\cos t\,\bar{i} + b\sin t\,\bar{j}}{\sqrt{a^2+b^2}}$$

$$\frac{d\bar{B}}{ds} = \frac{d\bar{B}}{dt}\Big/\frac{ds}{dt} = \frac{b\cos t\,\bar{i} + b\sin t\,\bar{j}}{\sqrt{a^2+b^2}} \cdot \frac{1}{\sqrt{a^2+b^2}} = \frac{b\cos t\,\bar{i} + b\sin t\,\bar{j}}{a^2+b^2}$$

But $\frac{d\bar{B}}{ds} = -\tau\bar{N}$

$$\frac{b\cos t\,\bar{i} + b\sin t\,\bar{j}}{a^2+b^2} = -\tau(-\cos t\,\bar{i} - \sin t\,\bar{j}) \Rightarrow \tau = \frac{b}{a^2+b^2}.$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$x = 3\cos t,\ y = 3\sin t,\ z = 4t$

$\bar{r} = 3\cos t\,\bar{i} + 3\sin t\,\bar{j} + 4t\,\bar{k} \Rightarrow \frac{d\bar{r}}{dt} = -3\sin t\,\bar{i} + 3\cos t\,\bar{j} + 4\bar{k}$

$\frac{d\bar{r}}{dt} = \bar{T} \Rightarrow \left|\frac{d\bar{r}}{ds}\right| = 1 \Rightarrow \left|\frac{d\bar{r}/dt}{ds/dt}\right| = 1 \Rightarrow \frac{ds}{dt} = \left|\frac{d\bar{r}}{dt}\right|$

$\frac{ds}{dt} = \left|\frac{d\bar{r}}{dt}\right| = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = \sqrt{9+16} = \sqrt{25} = 5$

$\bar{T} = \frac{d\bar{r}}{ds} = \frac{d\bar{r}/dt}{ds/dt} = \frac{-3\sin t\,\bar{i} + 3\cos t\,\bar{j} + 4\bar{k}}{5}$

$\frac{d\bar{T}}{dt} = \frac{1}{5}(-3\cos t\,\bar{i} - 3\sin t\,\bar{j})$

$\therefore \frac{d\bar{T}}{ds} = \frac{d\bar{T}/dt}{ds/dt} = \frac{1}{5}(-3\cos t\,\bar{i} - 3\sin t\,\bar{j})\cdot\frac{1}{5} = \frac{1}{25}(-3\cos t\,\bar{i} - 3\sin t\,\bar{j})$

$\bar{N} = \left|\frac{d\bar{T}}{ds}\right| = \frac{\sqrt{9\cos^2 t + 9\sin^2 t}}{25} = \frac{3}{25}$

$\bar{N} = \frac{1}{k}\frac{d\bar{T}}{ds} = \frac{25}{3}\cdot\frac{1}{25}(-3\cos t\,\bar{i} - 3\sin t\,\bar{j}) = -\cos t\,\bar{i} - \sin t\,\bar{j}$

3) $\bar{B} = \bar{T}\times\bar{N} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -\frac{3}{5}\sin t & \frac{3}{5}\cos t & \frac{4}{5} \\ -\cos t & -\sin t & 0 \end{vmatrix}$

$= \bar{i}\left(\frac{4}{5}\sin t\right) + \frac{4}{5}\cos t\,\bar{j} + \frac{3}{5}\bar{k}$

$= \frac{1}{5}(4\sin t\,\bar{i} - 4\cos t\,\bar{j} + 3\bar{k})$.

$\frac{d\bar{B}}{dt} = \frac{1}{5}(4\cos t\,\bar{i} + 4\sin t\,\bar{j})$.

$\frac{d\bar{B}}{ds} = \frac{d\bar{B}}{dt}\Big/\frac{ds}{dt} = \frac{1}{5}\cdot\frac{(4\cos t\,\bar{i} + 4\sin t\,\bar{j})}{5} = \frac{4\cos t\,\bar{i} + 4\sin t\,\bar{j}}{25}$

But $\frac{d\bar{B}}{ds} = -\tau\bar{N}$

$\frac{1}{25}(4\cos t\,\bar{i} + 4\sin t\,\bar{j}) = -\tau(-\cos t\,\bar{i} - \sin t\,\bar{j})$

$\tau = \frac{4}{25}$

$x = t - \frac{t^3}{3},\ y = t^2,\ z = t + \frac{t^3}{3}$

$k = \rho = \frac{1}{(1+t^2)^2}$

$x = t,\ y = t^2,\ z = \frac{2}{3}t^3$

$t = 1$ ⇒ k, τ.

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

unit tangent vector to the curve $x=t, y=t^2, z=t^3$ at $t=1$

$\bar{r} = x\hat{i}+y\hat{j}+z\hat{k} \Rightarrow \bar{r} = t\hat{i}+t^2\hat{j}+t^3\hat{k}$

$\frac{d\bar{r}}{dt} = \hat{i}+2t\hat{j}+3t^2\hat{k}$, At $t=1$, $\frac{d\bar{r}}{dt} = \hat{i}+2\hat{j}+3\hat{k}$.

But $\left|\frac{d\bar{r}}{dt}\right| = \sqrt{1+4+9} = \sqrt{14}$

unit tangent vector to the curve $= \frac{\frac{d\bar{r}}{dt}}{\left|\frac{d\bar{r}}{dt}\right|} = \frac{\hat{i}+2\hat{j}+3\hat{k}}{\sqrt{14}}$

3. A particle moves along the curve $x=2t^2$, $y=t^2-4t$, $z=3t-5$ where t is the time. Find the components of its velocity and acceleration at time $t=1$ in the direction $\hat{i}-3\hat{j}+2\hat{k}$.

Sol:- $\bar{r} = x\hat{i}+y\hat{j}+z\hat{k} \Rightarrow \bar{r} = 2t^2\hat{i}+(t^2-4t)\hat{j}+(3t-5)\hat{k}$

Velocity $= \bar{v} = \frac{d\bar{r}}{dt} = 4t\hat{i}+(2t-4)\hat{j}+3\hat{k}$

when $t=1$, $\frac{d\bar{r}}{dt} = 4\hat{i}-2\hat{j}+3\hat{k}$.

Acceleration $= \frac{d^2\bar{r}}{dt^2} = 4\hat{i}+2\hat{j}$.

when $t=1$; $\frac{d^2\bar{r}}{dt^2} = 4\hat{i}+2\hat{j}$.

Component of velocity in the direction of $\hat{i}-3\hat{j}+2\hat{k}$ is

$= \frac{(4\hat{i}-2\hat{j}+3\hat{k})(\hat{i}-3\hat{j}+2\hat{k})}{\sqrt{1+9+4}} = \frac{4+6+6}{\sqrt{14}} = \frac{16}{\sqrt{14}}$

component of acceleration in the direction of $(4\hat{i}+2\hat{j})$ $\hat{i}-3\hat{j}+2\hat{k}$ is

$= \frac{(4\hat{i}+2\hat{j})(\hat{i}-3\hat{j}+2\hat{k})}{\sqrt{1+9+4}} = \frac{4-6}{\sqrt{14}} = \frac{-2}{\sqrt{14}}$

4) A particle moves so that its position vector is given by $\bar{r} = \cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}$ where 'ω' is a constant. show that velocity $\bar{v}$ of the particle is perpendicular to $\bar{r}$ and $\bar{r}\times\bar{v}$ is a constant vector.

Sol:- The position vector of the particle is given by $\bar{r} = \cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}$.

The velocity is given by $\bar{v} = \frac{d\bar{r}}{dt} = -\omega\sin\omega t\,\hat{i} + \omega\cos\omega t\,\hat{j}$

consider $\bar{v}\cdot\bar{r} = -\omega\cos\omega t\sin\omega t + \omega\sin\omega t\cos\omega t = 0$ so $\bar{v}$ is $\perp$ to $\bar{r}$

now $\bar{r}\times\bar{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\omega t & \sin\omega t & 0 \\ -\omega\sin\omega t & \omega\cos\omega t & 0 \end{vmatrix} = \hat{k}(\omega\cos^2\omega t + \omega\sin^2\omega t)$

$= \omega\hat{k}$ = constant vector.

Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

① Find the angle between the tangents to the curve

$\bar{r} = t^2\bar{i} + 2t\bar{j} - t^3\bar{k}$ at the points $t=1$ and $t=-1$

Sol:- $\frac{d\bar{r}}{dt} = 2t\bar{i} + 2\bar{j} - 3t^2\bar{k}$ is a vector along the tangent at any point 't'

If $\bar{T}_1$ and $\bar{T}_2$ are the vectors along the tangents at $t=1$ and $t=-1$, respectively, then

$\bar{T}_1 = 2\bar{i} + 2\bar{j} - 3\bar{k}$ and $\bar{T}_2 = -2\bar{i} + 2\bar{j} + 3\bar{k}$

If θ is the angle between $\bar{T}_1$ and $\bar{T}_2$ then

$$\cos\theta = \frac{\bar{T}_1 \cdot \bar{T}_2}{|\bar{T}_1|\,|\bar{T}_2|} = \frac{2(-2) + 2(+2) + 3(-3)}{\sqrt{4+4+9}\ \sqrt{4+4+9}} = \frac{-4+4-9}{\sqrt{17}\ \sqrt{17}} = \frac{9}{17}$$

$$\theta = \cos^{-1}(9/17)$$

Velocity and Acceleration:

If the scalar variable 't' be the time and $\bar{r}$ be the position vector of moving particle with respect to the origin 'O', then $\delta\bar{r}$ is the displacement of the particle in time δt. If $\bar{v}$ represents the velocity of the particle at P in time δt,

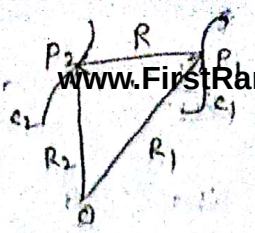
then $\bar{v} = \lim_{\delta t \to 0} \frac{\delta\bar{r}}{\delta t} = \frac{d\bar{r}}{dt}$

and acceleration is given by $\bar{a} = \lim_{\delta t \to 0} \frac{\delta\bar{v}}{dt} = \frac{d^2\bar{r}}{dt^2}$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com

Relative velocity and acceleration



Let two particles P_1 and P_2 moving along the curves C_1 and C_2 have position vectors R_1 and R_2 at time t, we get

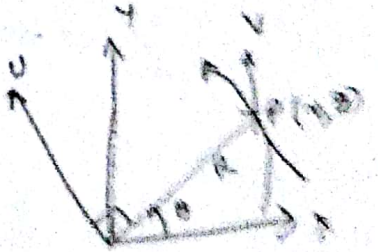
$$\frac{d\bar{R}}{dt} = \frac{d\bar{R}_2}{dt} - \frac{d\bar{R}_1}{dt} \quad -(1)$$

This defines the relative velocity of P_2 w.r.t P_1 and states that the velocity of P_2 relative to P_1 = velocity of P_2 − velocity of P_1.

differentiating (1), we have, $\frac{d^2\bar{R}}{dt^2} = \frac{d^2\bar{R}_2}{dt^2} - \frac{d^2\bar{R}_1}{dt^2}$

i.e acceleration of P_2 relative to P_1 = acceleration of P_2 − acceleration of P_1.

(2) Find the radial and transverse acceleration of a particle moving in a plane curve.



Sol:- At any time t, let the position vector of the moving particle $P(r,\theta)$ be $\bar{R}$

so that $\bar{R} = r\hat{R} = r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

∴ Its velocity $\bar{v} = \frac{d\bar{R}}{dt} = \frac{dr}{dt}\hat{R} + r\frac{d\hat{R}}{dt}$ —(1)

As $\hat{R} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

and $\frac{d\hat{R}}{dt} = (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})\frac{d\theta}{dt}$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice

$\frac{d\hat{R}}{dt} \perp \hat{R}$ and $\left|\frac{d\hat{R}}{dt}\right| = \frac{d\theta}{dt}$ i.e. $\hat{U}$ is a unit vector

www.FirstRanker.com www.FirstRanker.com

Then $\frac{d\hat{R}}{dt} = \frac{d\theta}{dt} U$

① becomes, $\bar{V} = \frac{dr}{dt}\hat{R} + r\frac{d\theta}{dt} U$

Thus the radial and transverse components of the velocity are $\frac{dr}{dt}$ and $r\frac{d\theta}{dt}$.

Also $\bar{A} = \frac{d\bar{V}}{dt} = \frac{d^2r}{dt^2}\hat{R} + \frac{dr}{dt}\frac{d\hat{R}}{dt} + \frac{dr}{dt}\frac{d\theta}{dt} U + r\frac{d^2\theta}{dt^2} U + r\frac{d\theta}{dt}$

$$= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{R} + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right) U$$

$$\left[\because U = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right.$$

$$\left. \text{gives } \frac{dU}{dt} = -\frac{d\theta}{dt}\hat{R}\right]$$

Thus the radial and transverse components of the acceleration are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \text{ and } 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

1) A particle moves along the curve $x = t^3+1$, $y = t^2$, $z = 2t+3$ where 't' is the time. Find the components of its velocity and acceleration at $t=1$ in the direction $\hat{i}+\hat{j}+3\hat{k}$.

Sol:- Velocity $\frac{d\bar{R}}{dt} = \frac{d}{dt}\left[(t^3+1)\hat{i} + t^2\hat{j} + (2t+3)\hat{k}\right]$

$= 3t^2\hat{i} + 2t\hat{j} + 2\hat{k} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ at $t=1$

and acceleration $= \frac{d^2\bar{R}}{dt^2} = 6t\hat{i} + 2\hat{j} + 0\hat{k} = 6\hat{i} + 2\hat{j}$ at $t=1$

Now unit vector in the direction of $\hat{i}+\hat{j}+3\hat{k}$ is

$$\frac{\hat{i}+\hat{j}+3\hat{k}}{\sqrt{1^2+1^2+3^2}} = \frac{1}{\sqrt{11}}(\hat{i}+\hat{j}+3\hat{k})$$

component of velocity at $t=1$ in the direction $\hat{i}+\hat{j}+3\hat{k}$

$$= \frac{(3\hat{i}+2\hat{j}+2\hat{k})(\hat{i}+\hat{j}+3\hat{k})}{\sqrt{11}} = \frac{3+2+6}{\sqrt{11}} = \frac{11}{\sqrt{11}} = \sqrt{11}$$

and component of acceleration at $t=1$ in the direction

$$\hat{i}+\hat{j}+3\hat{k} = (6\hat{i}+2\hat{j})\cdot\frac{(\hat{i}+\hat{j}+3\hat{k})}{\sqrt{11}} = \frac{6+2}{\sqrt{11}} = \frac{8}{\sqrt{11}}$$

2) A particle moves along the curve $\bar{R} = (t^2-4t)\hat{i} + (t^2+4t)\hat{j} + (2t^2-3t^3)\hat{k}$ where 't' denotes time. Find the magnitude of acceleration along the tangent and normal at time $t=2$.

Sol:- Velocity $\frac{d\bar{R}}{dt} = (3t^2-4)\hat{i} + (2t+4)\hat{j} + (16t-9t^2)\hat{k}$

and acceleration $\frac{d^2\bar{R}}{dt^2} = 6t\hat{i} + 2\hat{j} + (16-8t)\hat{k}$

At $t=2$, velocity $\bar{V} = 8\hat{i} + 8\hat{j} - 4\hat{k}$ and

acceleration $\bar{A} = 12\hat{i} + 2\hat{j} - 20\hat{k}$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Hence the velocity is along the tangent to the curve,

the component of A along the tangent $= A \cdot \frac{V}{|V|}$

$$= (12\hat{i} + 2\hat{j} - 20\hat{k}) \cdot \frac{8\hat{i} + 8\hat{j} - 4\hat{k}}{\sqrt{64+64+16}} = \frac{12\times 8 + 2\times 8 + (-20)\times(-4)}{12} = 12$$

Now the component of A along the normal

$= |A -$ Resolved part of A along the tangent $|$

$$= \left| 12\hat{i} + 2\hat{j} - 20\hat{k} - 16\,\frac{8\hat{i} + 8\hat{j} - 4\hat{k}}{12} \right| = \frac{1}{3}\left| 4\hat{i} - 26\hat{j} - 44\hat{k} \right|$$

$$= 2\sqrt{73}$$

3) The position vector of a particle at time 't' is

$$\bar{R} = \cos(t-1)\,\hat{i} + \sinh(t-1)\,\bar{j} + \alpha t^3\,\bar{k}$$

Find the condition imposed on α by requiring that at time $t=1$, the acceleration is normal to the position vector.

Sol: $\text{velocity} = \frac{d\bar{R}}{dt} = -\sin(t-1)\,\hat{i} + \cosh(t-1)\,\bar{j} + 3\alpha t^2\,\bar{k}$

$$\text{acceleration} = \frac{d^2\bar{R}}{dt^2} = -\cos(t-1)\,\hat{i} + \sinh(t-1)\,\bar{j} + 6\alpha t\,\bar{k}$$

$$= -\hat{i} + 6\alpha\,\bar{k} \quad \text{at } t=1$$

Also $\bar{R} = \hat{i} + \alpha\bar{k}$ at $t=1$.

If $\bar{R}$ and acceleration at $t=1$ are normal,

then their scalar product is zero.

$$(-\hat{i} + 6\alpha\bar{k}) \cdot (\hat{i} + \alpha\bar{k}) = 0$$

$$-1 + 6\alpha^2 = 0 \Rightarrow \alpha^2 = 1/6$$

$$\text{or } \alpha = \frac{1}{\sqrt{6}}$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Scalar point function :- A function $\phi(x,y,z)$ is called a scalar point function if it associates a scalar with every point in space. The temperature distribution in a heated body, density of body and potential due to gravity are the examples of a scalar point function.

Vector point function :- If a function $F(x,y,z)$ defines a vector at every point of a region, then $F(x,y,z)$ is called a vector point function. The velocity of moving fluid, gravitational force are the example of vector point function.

Vector operator Del or Nabla :-

The operator $\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$ is called the vector operator Del or Nabla.

The vector operator ∇ has properties analogus to those of ordinary vectors and can be applied to (i) scalar point functions (ii) vector point functions

Gradient :- The vector function ∇f is defined as the gradient of the scalar point function f and written as grad f

$$\text{grad } f = \nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

Directional Derivative :- The component of $\nabla\phi$ in the direction of $\bar{a}$ is given by $\nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$ and is called the directional derivative of ϕ in direction of $\bar{a}$.

The directional derivative of ϕ at a point P is maximum in the direction of normal at P and its maximum value is $|\nabla\phi|$.

Properties of gradient :- If ϕ and ψ are two scalar point functions then (i) $\nabla(c_1\phi + c_2\psi) = c_1\nabla\phi + c_2\nabla\psi$ where c_1, c_2 are constants

(ii) $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

(iii) $\nabla\left(\frac{\phi}{\psi}\right) = \frac{\psi\nabla\phi - \phi\nabla\psi}{\psi^2}$, $\psi \neq 0$.

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}|$

$r = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{1/2}$

$r^n = (x^2+y^2+z^2)^{n/2}$

$\nabla r^n = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{n/2}$

$= \frac{n}{2}(x^2+y^2+z^2)^{n/2-1} \cdot [2x\bar{i} + 2y\bar{j} + 2z\bar{k}]$

$= \frac{n}{2}(r^2)^{\frac{n-2}{2}}(2x\bar{i} + 2y\bar{j} + 2z\bar{k})$

$= n(r^2)^{\frac{n-2}{2}}(x\bar{i} + y\bar{j} + z\bar{k})$

$\nabla r^n = nr^{n-2}\bar{r}$.

4) $\nabla(\bar{a}\cdot\bar{r}) = \bar{a}$ where $\bar{a}$ is constant vector

Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$ where a_1, a_2, a_3 are constants

$\bar{a}\cdot\bar{r} = a_1x + a_2y + a_3z$

$\nabla(\bar{a}\cdot\bar{r}) = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(a_1x + a_2y + a_3z)$

$= a_1\bar{i} + a_2\bar{j} + a_3\bar{k} = \bar{a}$

5) Find the grad ϕ where ϕ is given by $\phi = 3x^2y - y^3z^2$ at the point $(1,-2,-1)$

Ans:- Grad $\phi = \nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(3x^2y - y^3z^2)$

$= \bar{i}\left(\frac{\partial}{\partial x}\right)(3x^2y - y^3z^2) + \bar{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \bar{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$

$= 6xy\bar{i} + \bar{j}(3x^2 - 3y^2z^2) + (-2y^3z)\bar{k}$

$= -12\bar{i} - 9\bar{j} - 16\bar{k}$ at the point $(1,-2,-1)$.

2)

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

p.T $\nabla r^n = nr^{n-2}\bar{r}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Sol:- $\nabla r^n = \bar{i}\frac{\partial}{\partial x}(r^n) + \bar{j}\frac{\partial}{\partial y}(r^n) + \bar{k}\frac{\partial}{\partial z}(r^n)$

$= nr^{n-1}\frac{\partial r}{\partial x}\bar{i} + nr^{n-1}\frac{\partial r}{\partial y}\bar{j} + nr^{n-1}\frac{\partial r}{\partial z}\bar{k}$

$= nr^{n-1}\frac{x}{r}\bar{i} + nr^{n-1}\frac{y}{r}\bar{j} + nr^{n-1}\frac{z}{r}\bar{k}$

$= nr^{n-2}(x\bar{i} + y\bar{j} + z\bar{k}) = nr^{n-2}\bar{r}$

$r = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

$2r\frac{\partial r}{\partial x} = 2x$

$\frac{\partial r}{\partial x} = \frac{x}{r}$

2) If $u = x+y+z$, $v = x^2+y^2+z^2$, $w = yz+zx+xy$, p.T grad u, grad v, grad w are coplanar.

Sol:- grad $u = \bar{i}+\bar{j}+\bar{k}$, grad $v = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$

grad $w = (y+z)\bar{i} + (z+x)\bar{j} + (x+y)\bar{k}$

we know that three vectors are coplanar if their scalar triple product is zero.

$$[\text{grad } u, \text{grad } v, \text{grad } w] = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix}$$ operate $R_2 + R_3$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix} = 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix} = 0.$$

Hence grad u, grad v, grad w are coplanar.

3) Find a unit vector normal to the surface $xy^3z^2 = 4$ at the point $(-1, -1, 2)$

Sol:- A vector normal to the given surface is

$\nabla(xy^3z^2) = \bar{i}\frac{\partial}{\partial x}(xy^3z^2) + \bar{j}\frac{\partial}{\partial y}(xy^3z^2) + \bar{k}\frac{\partial}{\partial z}(xy^3z^2)$

$= y^3z^2\bar{i} + 3xy^2z^2\bar{j} + 2xy^3z\bar{k}$

$= -4\bar{i} - 12\bar{j} + 4\bar{k}$ at the point $(-1, -1, 2)$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Hence the required unit normal to the surface

$$= \frac{i - 4\hat{i} - 12\hat{j} + 4\hat{k}}{\sqrt{(-4)^2 + (-12)^2 + 4^2}} = -\frac{1}{\sqrt{11}}(\hat{i} + 3\hat{j} - \hat{k})$$

4) Find the directional derivative of $f(x,y,z) = xy^2 + yz^3$ at the point $(2,-1,1)$ in the direction of vector $\hat{i} - 2\hat{j} - 2\hat{k}$.

Sol:- $\nabla f = \hat{i}\frac{\partial}{\partial x}(xy^2 + yz^3) + \hat{j}\frac{\partial}{\partial y}(xy^2 + yz^3) + \hat{k}\frac{\partial}{\partial z}(xy^2 + yz^3)$

$= y^2\hat{i} + (2xy + z^3)\hat{j} + 3yz^2\hat{k}$

$= \hat{i} - 3\hat{j} - 3\hat{k}$ at the point $(2,-1,1)$

∴ Directional derivative of f in the direction $\hat{i} + 2\hat{j} + 2\hat{k}$

$= (\hat{i} - 3\hat{j} - 3\hat{k}) \cdot \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{1^2 + 2^2 + 2^2}} = -\frac{11}{3}$

5) Find the directional derivative of $f = x^2 - y^2 + 2z^2$ at point $P(1,2,3)$ in the direction of the line PQ where Q is the point $(5,0,4)$. Also calculate the magnitude of maximum directional derivative.

Sol:- $\nabla f = \hat{i}\frac{\partial}{\partial x}(x^2 - y^2 + 2z^2) + \hat{j}\frac{\partial}{\partial y}(x^2 - y^2 + 2z^2) + \hat{k}\frac{\partial}{\partial z}(x^2 - y^2 + 2z^2)$

$= 2x\hat{i} - 2y\hat{j} + 4z\hat{k}$

$= 2\hat{i} - 4\hat{j} + 12\hat{k}$ at $P(1,2,3)$

Also $\vec{PQ} = \vec{OQ} - \vec{OP} = (5\hat{i} + 0\hat{j} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$

$= 4\hat{i} - 2\hat{j} + \hat{k} = \vec{b}$ say

∴ unit vector of $b = \hat{b} = \frac{\vec{b}}{|b|} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{16 + 4 + 1}} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}}$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Thus the directional derivative of f is the direction of $\vec{A}$

$$\nabla f \cdot \hat{A} = (2\vec{i} - 4\vec{j} + 12\vec{k}) \cdot \frac{(4\vec{i} - 2\vec{j} + \vec{k})}{\sqrt{21}} = \frac{8+8+12}{\sqrt{21}} = \frac{28}{\sqrt{21}}$$

∴ The directional derivative of is maximum in the direction of the normal to the surface ie in the direction of ∇f.

Hence maximum value of this directional derivative

$$= |\nabla f| = |2\vec{i} - 4\vec{j} + 12\vec{k}| = \sqrt{4+16+144} = \sqrt{164}$$

6) Find the directional derivative of $\phi = 5x^2y - 5y^2z + 2.5z^2x$ at the point $P(1,1,1)$ in the direction of the line

$$\frac{x-1}{2} = \frac{y-3}{-2} = z$$

Sol:- we have $\nabla\phi = \vec{i}\frac{d\phi}{dx} + \vec{j}\frac{d\phi}{dy} + \vec{k}\frac{d\phi}{dz}$

$$= (10xy + 2.5z^2)\vec{i} + (5x^2 - 10yz)\vec{j} + (-5y^2 + 5xz)\vec{k}$$

$$= 12.5\vec{i} - 5\vec{j} \text{ at } P(1,1,1)$$

Also direction of the given line is $\hat{A} = \frac{2\vec{i} - 2\vec{j} + \vec{k}}{3}$

Hence the required directional derivative $= \nabla\phi \cdot \hat{n}$

$$= (12.5\vec{i} - 5\vec{j}) \cdot \frac{2\vec{i} - 2\vec{j} + \vec{k}}{3}$$

$$= \frac{25+10}{3} = \frac{35}{3}$$

7) Find the angle between the surfaces $x^2+y^2+z^2 = 9$ and $z = x^2+y^2-3$ at the point $(2,-1,2)$

Sol:- Let $f_1 = x^2+y^2+z^2-9 = 0$ and $f_2 = x^2+y^2-z-3 = 0$.

Then $N_1 = \nabla f_1$ at $(2,-1,2)$

$$= (2x\vec{i} + 2y\vec{j} + 2z\vec{k}) \text{ at } (2,-1,2) = 4\vec{i} - 2\vec{j} + 4\vec{k}$$

and $N_2 = \nabla f_2$ at $(2,-1,2)$

$$= (2x\vec{i} + 2y\vec{j} - \vec{k}) \text{ at } (2,-1,2) = 4\vec{i} - 2\vec{j} - \vec{k}$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Since the angle θ between the two surfaces at a point is the angle between their normals at the point and N_1, N_2 are the normals at $(2,-1,2)$ to the given surfaces

$$\cos\theta = \frac{N_1 \cdot N_2}{|N_1||N_2|} = \frac{(4\bar{i}-2\bar{j}+4\bar{k})(4\bar{i}-2\bar{j}-\bar{k})}{\sqrt{16+4+16}\ \sqrt{16+4+1}} = \frac{16}{6\sqrt{21}}$$

Hence the required angle $\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$

8) Find the values a and b such that the surfaces $ax^2 - byz = (a+2)x$, and $4x^2y + z^3 = 4$ cut orthogonally

Sol:- Let $f_1 = ax^2 - byz - (a+2)x = 0$ — ①

$f_2 = 4x^2y + z^3 - 4 = 0$ — ②

$$\nabla f_1 = (2ax - a - 2)\bar{i} - bz\bar{j} - by\bar{k}$$
$$= (a-2)\bar{i} - 2b\bar{j} + b\bar{k} \text{ at } (1,-1,2)$$

$$\nabla f_2 = 8xy\,\bar{i} + 4x^2\bar{j} + 3z^2\bar{k}$$
$$= -8\bar{i} + 4\bar{j} + 12\bar{k} \text{ at } (1,-1,2)$$

The surfaces ① and ② will cut orthogonally if $\nabla f_1 \perp \nabla f_2$

i.e $-8(a-2) - 8b + 12b = 0$.

$-2a + b + 4 = 0$ — ③.

Also since the point $(1,-1,2)$ lies on (i) and (ii)

$a + 2b - (a+2) = 0$ or $b = 1$.

From ③ $-2a + 5 = 0$ or $a = 5/2$

Hence $a = 5/2$, and $b = 1$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice

Exercise - 8.3

www.FirstRanker.com www.FirstRanker.com

1) (a) find $\nabla\phi$, if $\phi = \log(x^2+y^2+z^2)$

(b) Show that $\text{grad}(1/r) = -R/r^3$.

Sol: (a) $\nabla\phi = \bar{i}\frac{\partial}{\partial x}\log(x^2+y^2+z^2) + \bar{j}\frac{\partial}{\partial y}\log(x^2+y^2+z^2) + \bar{k}\frac{\partial}{\partial z}\log(x^2+y^2+z^2)$

$= \bar{i}\frac{2x}{x^2+y^2+z^2} + \bar{j}\frac{2y}{x^2+y^2+z^2} + \bar{k}\frac{2z}{x^2+y^2+z^2}$

2)

$= \frac{2(x\bar{i}+y\bar{j}+z\bar{k})}{(x^2+y^2+z^2)} = \frac{2\bar{R}}{r^2}$

(b) $\text{grad}(1/r) = \bar{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \bar{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \bar{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$

$= \bar{i}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial x} + \bar{j}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial y} + \bar{k}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial z}$

$= \bar{i}\left(\frac{-1}{r^2}\right)\left(\frac{x}{r}\right) + \bar{j}\left(\frac{-1}{r^2}\right)\left(\frac{y}{r}\right) + \bar{k}\left(\frac{-1}{r^2}\right)\left(\frac{z}{r}\right)$

$= \frac{-1}{r^3}(x\bar{i}+y\bar{j}+z\bar{k})$

$= \frac{-\bar{r}}{r^3}$

2) Find a unit vector normal to the surface $x^3+y^3+3xyz = 3$ at the point $(1, 2, -1)$

Sol: Let $\phi = x^3+y^3+3xyz-3$

A vector normal to the above surface

$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

$= (3x^2+3yz)\bar{i} + (3y^2+3xz)\bar{j} + 3xy\bar{k}$

www.FirstRanker.com

Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

At the point $(1,-2,-1)$

$$\nabla\phi = [3+3(2)(-1)]\bar{i} + [3(0)+3(1)(-1)]\bar{j} + 3(1)(2)\bar{k}$$

$$= -3\bar{i} + 9\bar{j} + 6\bar{k}$$

$$= -3(\bar{i} - 3\bar{j} - 2\bar{k})$$

unit vector normal to the surface $= \frac{\nabla\phi}{|\nabla\phi|}$

$$= \frac{-3(\bar{i} - 3\bar{j} - 2\bar{k})}{3\sqrt{1+9+4}}$$

$$= \frac{-(\bar{i} - 3\bar{j} - 2\bar{k})}{\sqrt{14}}$$

(3) Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point $(1,-2,1)$ in the direction of the ve $2\bar{i} - \bar{j} - 2\bar{k}$

Sol: Here $\nabla\phi = (2xyz + 4z^2)\bar{i} + x^2z\bar{j} + (x^2y + 8xz)\bar{k}$

$= 8\bar{i} - \bar{j} - 10\bar{k}$ at the point $(1,-2,-1)$

∴ Directional derivative of ϕ in the direction $2\bar{i} - \bar{j} - 2\bar{k}$

$$= \frac{(8\bar{i} - \bar{j} - 10\bar{k}) \cdot (2\bar{i} - \bar{j} - 2\bar{k})}{\sqrt{4+1+4}}$$

$$= \frac{16+1+20}{3} = \frac{37}{3}$$

(4) What is the directional derivative of $\phi = xy^2 + yz^3$ at the point $(2,-1,1)$ in the direction of the normal the surface $xy^2z = 4$ at $(-1,2,1)$

Sol: A vector normal to the surface x

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$$\nabla(x\log z) - y^2 = \log z\,i - 2y\,j + \frac{x}{z}k$$

$$= -4j - k$$

Here $\nabla f = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(xy^2 + yz^3)$

$$= y^2 i + (2xy + z^3)j + 3yz^2 k \quad \text{at } (2,-1,1)$$

$$= i - 3j - 3k$$

Directional derivative of 'f' in the direction of $-4j - k$

$$= (i - 3j - 3k)\cdot\frac{(-4j-k)}{\sqrt{16+1}} = \frac{12+3}{\sqrt{17}} = \frac{15}{\sqrt{17}}$$

Find the values of constants a, b, c so that the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at $(1,2,-1)$ has a maximum magnitude 64 in a direction parallel to the z-axis.

$$\nabla\phi = \nabla(axy^2 + byz + cz^2x^3)$$

$$= (ay^2 + 3cz^2x^2)i + (2axy + bz)j + (by + 2czx^3)k$$

$$\nabla\phi_{(1,2,-1)} = (4a + 3c)i + (4a - b)j + (2b - 2c)k$$

Given that $\nabla\phi$ is parallel to z-axis and

$$\nabla\phi \cdot k = 64$$

$\therefore \nabla\phi \cdot i = 0$; $\nabla\phi \cdot j = 0$ and $\nabla\phi \cdot k = 64$

$4a + 3c = 0$; $4a - b = 0$; $2b - 2c = 64$

$\rightarrow (1)$ $\rightarrow (2)$ $\rightarrow (3)$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Solving (1), (2), (3) for a, b, c

$$a = 6,\ b = 24,\ c = -8$$

(8) If the directional derivative of $\phi = ax^2y + by^2z + cz^2x$ at the point $(1,1,1)$ has maximum magnitude 15 in the direction parallel to the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$. Find the values of a, b and c.

Sol: Given $\nabla\phi = ax^2y + by^2z + cz^2x$

$$\nabla\phi = (2axy + cz^2)\bar{i} + (ax^2 + 2byz)\bar{j} + (by^2 + 2czx)\bar{k}$$

At the point $(1,1,1)$

$$\nabla\phi = (2a + c)\bar{i} + (a + 2b)\bar{j} + (b + 2c)\bar{k}$$

Now the maximum value of the directional derivative is in the direction of ∇f

i.e, $|\nabla\phi| = 15$

$$\Rightarrow (2a+c)^2 + (a+2b)^2 + (b+2c)^2 = (15)^2$$

But the directional derivative is given to be maximum parallel to the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$

$$\Rightarrow \frac{2a+c}{2} = \frac{a+2b}{-2} = \frac{2c+b}{1} \Rightarrow 2a + c = -2b - a$$

$$3a + 2b + c = 0 \rightarrow$$

and $2b + a = -2(2c + b)$

$$a + 4b + 4c = 0 \rightarrow (2)$$

Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com

$\frac{a}{4} = \frac{b}{-11} = \frac{c}{10} = k$ (say)

$a = 4k$, $b = -11k$, $c = 10k$

Now, $(2a+c)^2 + (2b+a)^2 + (2c+b)^2 = (15)^2$

$(8k+10k)^2 + (-22k+4k)^2 + (20k-11k)^2 = (15)^2$

$(18k)^2 + (-18k)^2 + (8k)^2 = (15)^2$

$k = \pm 5/9$

$a = \pm 20/9$; $b = \pm \frac{55}{9}$; $c = \pm \frac{50}{9}$

4) In what direction from $(3,1,-2)$ is the directional derivative of $\phi = x^2y^2z^4$ maximum? Find also the magnitude of this maximum.

Or

$$\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(x^2y^2z^4)$$

$$= 2xy^2z^4\bar{i} + 2x^2yz^4\bar{j} + 4x^2y^2z^3\bar{k}$$

Directional derivative at $(3,1,-2)$

$$= 96\bar{i} + 288\bar{j} - 288\bar{k}$$

The directional derivative is maximum in the direction $96\bar{i} + 288\bar{j} - 288\bar{k}$

$$= 96(\bar{i} + 3\bar{j} - 3\bar{k})$$

In any other direction it will have its component which is less than its maximum value

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Maximum value $= 48\sqrt{4+9}$

$= 96\sqrt{19}$

(8) What is the greatest rate of increase of $u = x^2 + yz$ at the point $(1, -1, 3)$

Sol: Greatest rate $= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(x^2 + yz)$

$= 2x\bar{i} + z\bar{j} + y\bar{k}$

At $(1, -1, 3)$, Greatest rate $= 2\bar{i} + 3\bar{j} - \bar{k}$

(9) Calculate

(9) The temperature of points in space is given by $T(x, y, z) = x^2 + y^2 - z$. A mosquito located at (desires to fly in such a direction that it get warm as soon as possible. In what direction should it move?

Sol: Let $\phi = (x^2 + y^2 - z)$

$\nabla\phi = 2x\bar{i} + 2y\bar{j} - \bar{k}$

at $(1, 1, 2) = 2\bar{i} + 2\bar{j} - \bar{k}$

$\frac{\nabla\phi}{|\nabla\phi|} = \frac{2\bar{i} + 2\bar{j} - \bar{k}}{\sqrt{4 + 4 + 1}}$

$= \frac{2\bar{i} + 2\bar{j} - \bar{k}}{3}$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Calculate the angle b/w the normals to the surface $xy = z^2$ at the points $(4, 1, 2)$ and $(3, 3, -3)$

A vector normal to the surface is

$$\nabla(xy - z^2) = \bar{i}\frac{\partial}{\partial x}(xy - z^2) + \bar{j}\frac{\partial}{\partial y}(xy - z^2) + \bar{k}\frac{\partial}{\partial z}(xy - z^2)$$

$$= \bar{i}y + \bar{j}x - 2z\bar{k}$$

At $(4, 1, 2)$ and $(3, 3, -3)$

$$\bar{N}_1 = \bar{i} + 4\bar{j} - 4\bar{k} \quad ; \quad \bar{N}_2 = 3\bar{i} + 3\bar{j} - 6\bar{k}$$

$$\cos\theta = \frac{\bar{N}_1 \cdot \bar{N}_2}{|\bar{N}_1||\bar{N}_2|} = \frac{3 + 12 - 24}{\sqrt{1+16+16}\sqrt{9+9+36}}$$

$$= \frac{-9}{\sqrt{33}\sqrt{54}} = \frac{-9}{\sqrt{11 \times 3} \cdot \sqrt{2 \times 27}}$$

$$= \frac{-9}{\sqrt{81}\sqrt{22}} = \frac{-1}{\sqrt{22}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{22}}\right)$$

Find the values of λ and μ so that the surface $\lambda x^2 y + \mu z^3 = 4$ may cut the surface $5x^2 = yz + 9x$ orthogonally at $(1, -1, 2)$

Let $f_1 = \lambda x^2 y + \mu z^3 - 4 = 0 \quad \rightarrow (1)$

$$\nabla f_1 = 2\lambda xy\bar{i} + \lambda x^2\bar{j} + 3\mu z^2\bar{k}$$

and $f_2 = 5x^2 - yz - 9x = 0 \quad \rightarrow (2)$

$$\nabla f_2 = (10x - 9)\bar{i} - z\bar{j} - y\bar{k} = 0$$

At $(1, -1, 2) \Rightarrow \nabla f_1 = -2\lambda\bar{i} + \lambda\bar{j} + 12\mu\bar{k}$

At $(1, -1, 2) \Rightarrow \nabla f_2 = \bar{i} - 4\bar{j} + 2\bar{k}$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

The surfaces (1) & (2) will cut orthogonally if

$$\nabla f_1 \cdot \nabla f_2 = 0$$

i.e., $(-2\lambda)(1) + \lambda(-4) + 12(1)\mu = 0$

$\Rightarrow -2\lambda - 4\lambda + 24\mu = 0$

$\Rightarrow -6\lambda + 24\mu = 0 \quad \rightarrow (3)$

Also since the point $(1,-1,2)$ lies on (1) & (2)

(1) $\Rightarrow -\lambda + 8\mu - 4 = 0$

(2) $\Rightarrow 5 + 4 - 9 = 0$

From (3), $-6\lambda + 24\mu(1) = 0$

$-6\lambda = -24$

$\lambda = 4$

$\therefore \lambda = 4$ and $\mu = 1$.

$-6\lambda + 24\mu = 0$

$-6\lambda + 48\mu - 24 = 0$

$-24\mu + 24 = 0$

$\mu = 1$

(12) Find the angle b/w the tangent planes to the surface. $x\log z = y^2 - 1$, $x^2 y = 2 - z$ at the point $(1,$

Sol: The angle b/w the tangent planes to the surfaces ϕ_1 and ϕ_2 is $\cos^{-1}\left[\frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}\right]$

$\phi_1 = x\log z - y^2 + 1 \Rightarrow \nabla\phi_1 = \log z\,\bar{i} - 2y\,\bar{j} + \frac{x}{z}\,\bar{k}$

$\phi_2 = x^2 y + z - 2 \Rightarrow \nabla\phi_2 = 2xy\,\bar{i} + x^2\,\bar{j} + \bar{k}$

$\nabla\phi_1(1,1,1) = -2\bar{j} + \bar{k}$; $\nabla\phi_2(1,1,1) = 2\bar{i} + \bar{j} + \bar{k}$

$$\theta = \cos^{-1}\left[\frac{(-2\bar{j} + \bar{k})(2\bar{i} + \bar{j} + \bar{k})}{\sqrt{5}\,\sqrt{6}}\right] = \cos^{-1}\left(\frac{-1}{\sqrt{30}}\right)$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Divergence of a vector point function

The divergence of a differentiable vector point function $\bar{v}$ is denoted by div $\bar{v}$ and is defined as

$$\text{div } \bar{v} = \nabla \cdot \bar{v} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot \bar{v} = \bar{i} \cdot \frac{\partial \bar{v}}{\partial x} + \bar{j} \cdot \frac{\partial v}{\partial y} + \bar{k} \cdot \frac{\partial v}{\partial z}.$$

Obviously, the divergence of a vector point function is a scalar point function.

If $\bar{v} = v_1\bar{i} + v_2\bar{j} + v_3\bar{k}$ then $\text{div } \bar{v} = \nabla \cdot \bar{v} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot (v_1\bar{i} + v_2\bar{j} + v_3\bar{k})$

Since $\bar{i} \cdot \bar{i} = \bar{j} \cdot \bar{j} = \bar{k} \cdot \bar{k} = 1$

and $\bar{i} \cdot \bar{j} = \bar{j} \cdot \bar{k} = \bar{k} \cdot \bar{i} = 0$.

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}.$$

Curl of a vector point function :-

The curl (or rotation) of a differentiable vector point function $\bar{v}$ is denoted by curl $\bar{v}$ and is defined as

$$\text{curl } \bar{v} = \nabla \times \bar{v} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \times \bar{v} = \bar{i} \times \frac{\partial \bar{v}}{\partial x} + \bar{j} \times \frac{\partial \bar{v}}{\partial y} + \bar{k} \times \frac{\partial \bar{v}}{\partial z}.$$

Obviously, the curl of a vector point function is a vector point function.

If $\bar{v} = v_1\bar{i} + v_2\bar{j} + v_3\bar{k}$, then $\text{curl } \bar{v} = \nabla \times \bar{v} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \times (v_1\bar{i} + v_2\bar{j} + v_3\bar{k})$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 & v_2 & v_3 \end{vmatrix} = \bar{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \bar{j}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \bar{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right).$$

Ex:-① If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, show that (i) $\text{div } \bar{r} = 3$ (ii) $\text{curl } \bar{r} = \bar{0}$.

(i) $\text{div } \bar{r} = \nabla \cdot \bar{r} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot (x\bar{i} + y\bar{j} + z\bar{k}) = 1 + 1 + 1 = 3.$

(ii) $\text{curl } \bar{r} = \nabla \times \bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix}$

$$= \bar{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \bar{j}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + \bar{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right).$$

$$= \bar{i}(0) + \bar{j}(0) + \bar{k}(0) = \bar{0}.$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
FirstRanker's choice
www.FirstRanker.com www.FirstRanker.com

Ex 2:- Find the divergence and curl of the vector

$\bar{v} = (xyz)\,\bar{i} + (3x^2y)\,\bar{j} + (xz^2 - y^2z)\,\bar{k}$ at the point $(2,-1,1)$

Sol:- $\operatorname{div} \bar{v} = \frac{d}{dx}(xyz) + \frac{d}{dy}(3x^2y) + \frac{d}{dz}(xz^2 - y^2z)$.

$= yz + 3x^2 + 2xz - y^2$

$= -1 + 12 + 4 - 1 = 14$ at $(2,-1,1)$.

$$\operatorname{curl} \bar{v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix} = \bar{i}(-2yz) + \bar{j}(xy - z^2) + \bar{k}(6xy - xz)$$

$= 2\bar{i} - 8\bar{j} - 14\bar{k}$ at $(2,-1,1)$.

Ex 3:- Find $\operatorname{div} \bar{F}$ and $\operatorname{curl} \bar{F}$ where $\bar{F} = \operatorname{grad}(x^3 + y^3 + z^3 - 3xyz)$

Sol:- Let $\phi = x^3 + y^3 + z^3 - 3xyz$

$\bar{F} = \operatorname{grad} \phi = \left(\bar{i}\frac{d}{dx} + \bar{j}\frac{d}{dy} + \bar{k}\frac{d}{dz}\right)(x^3 + y^3 + z^3 - 3xyz)$.

$= (3x^2 - 3yz)\,\bar{i} + (3y^2 - 3xz)\,\bar{j} + (3z^2 - 3xy)\,\bar{k}$

$\therefore \operatorname{div} \bar{F} = \frac{d}{dx}(3x^2 - 3yz) + \frac{d}{dy}(3y^2 - 3xz) + \frac{d}{dz}(3z^2 - 3xy)$

$= 6x + 6y + 6z = 6(x + y + z)$

and $\operatorname{curl} \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$

$= \bar{i}(-3x + 3x) + \bar{j}(-3y + 3y) + \bar{k}(-3z + 3z)$

$= \bar{0}$

Note:- (1) If $\operatorname{div} \bar{v} = 0$, then $\bar{v}$ is called a solenoidal vector point function

(2) If $\operatorname{curl} \bar{v} = \bar{0}$, " " is said to be an irrotational vector

Otherwise rotational.

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

(1) If $\bar{f} = x^2y\,\bar{i} - 2xz\,\bar{j} + 2yz\,\bar{k}$, find (i) div $\bar{f}$ (ii) curl $\bar{f}$ (iii) curl curl $\bar{f}$

Soln: $\bar{f} = x^2y\,\bar{i} - 2xz\,\bar{j} + 2yz\,\bar{k}$

(i) $\nabla\cdot\bar{f} = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(-2xz) + \frac{\partial}{\partial z}(2yz) = 2xy + 0 + 2y = 2y(x+1)$

(ii) $\nabla\times\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & -2xz & 2yz \end{vmatrix} = \bar{i}(2z+2x) + \bar{j}(0) + \bar{k}(-2z - x^2)$

$= (2x+2z)\,\bar{i} - (x^2+2z)\,\bar{k}$

(iii) curl curl $\bar{f} = \nabla\times(\nabla\times\bar{f}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x+2z & 0 & -(x^2+2z) \end{vmatrix} = \bar{i}(0) + \bar{j}(2+2z) + \bar{k}(0)$

$= (2x+2)\,\bar{j}$.

(2) Show that the vector field given by $\bar{f} = (x+3y)\,\bar{i} + (y-3z)\,\bar{j} + (x-2z)\,\bar{k}$ is solenoidal.

Soln: $\bar{f} = (x+3y)\,\bar{i} + (y-3z)\,\bar{j} + (x-2z)\,\bar{k}$.

$\nabla\cdot\bar{f} = \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-3z) + \frac{\partial}{\partial z}(x-2z) = 1+1-2 = 0$.

$\therefore \bar{f}$ is solenoidal.

(3) Find the constant 'a' so that $\bar{f} = y(ax^2+z)\,\bar{i} + x(y^2-z^2)\,\bar{j} + 2xy(z-xy)\,\bar{k}$ is solenoidal.

Soln: If $\bar{f}$ is solenoidal, $\nabla\cdot\bar{f} = 0$

$\nabla\cdot\bar{f} = 0 \Rightarrow \frac{\partial}{\partial x}[y(ax^2+z)] + \frac{\partial}{\partial y}[x(y^2-z^2)] + \frac{\partial}{\partial z}[2xy(z-xy)]$

$\Rightarrow 2axy + 2xy + 2xy = 0$.

$\Rightarrow 2xy(a+2) = 0 \Rightarrow a = -2$.

(4) Show that $\bar{v} = (\sin y + z)\,\bar{i} + (x\cos y - z)\,\bar{j} + (x-y)\,\bar{k}$ is irrotational.

Soln: A vector $\bar{v}$ is irrotational if $\nabla\times\bar{v} = \bar{0}$.

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$$\begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+3y+az & bx+2y-z & 2x+cy+3z \end{vmatrix}$$

is irrotational.

3) Find the constants a, b, c if the vector

$\bar{f} = (2x+3y+az)\bar{i} + (bx+2y+3z)\bar{j} + (2x+cy+3z)\bar{k}$ is irrotational.

Sol:- If $\bar{f}$ is irrotational $\Rightarrow \nabla \times \bar{f} = \bar{0}$

$$\nabla \times \bar{f} = \bar{0} \Rightarrow \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+3y+az & bx+2y+z & 2x+cy+3z \end{vmatrix} = \bar{0}$$

$$\Rightarrow (c+3)\bar{i} + \bar{j}(a-2) + \bar{k}(b-3) = \bar{0}$$

$a-2=0$ i.e. $a=2$

$b-3=0$ i.e. $b=3$

$c-3=0$ i.e. $c=3$

4) If $\bar{f} = (x+y+z)\bar{i} + \bar{j} - (x+y)\bar{k}$ prove that $\bar{f} \cdot \text{curl}\, \bar{f} = 0$.

$$\text{curl}\, \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y+z & 1 & -(x+y) \end{vmatrix} = \bar{i}(-1) + \bar{j}(0+1) + \bar{k}(-0) = -\bar{i} + \bar{j} - \bar{k}$$

$$\bar{f} \cdot \text{curl}\, \bar{f} = (x+y+z)(-1) + 1 + x+y$$

$$= -x-y-z+x+y = \bar{0}$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Prove that $\text{div}(r^n \bar{r}) = (n+3) r^n$. (18)

Where $r = |\bar{r}|$

Hence show that $\bar{r}/r^3$ is solenoidal.

Sol:- Given that $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$. and $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{r} \quad \text{by} \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

$$\frac{\partial}{\partial x}(xr^n) = 1 \cdot r^n + x \cdot n r^{n-1} \frac{\partial r}{\partial x} = r^n + x \cdot n \cdot r^{n-1} \frac{x}{r} = r^n + nx^2 r^{n-2}.$$

By $\frac{\partial}{\partial y}(yr^n) = r^n + ny^2 r^{n-2}$ and $\frac{\partial}{\partial z}(zr^n) = r^n + nz^2 r^{n-2}$.

$$r^n \bar{r} = r^n \cdot (x\hat{i} + y\hat{j} + z\hat{k}) = xr^n\hat{i} + yr^n\hat{j} + zr^n\hat{k}.$$

$$\text{div}(r^n \bar{r}) = \frac{\partial}{\partial x}(xr^n) + \frac{\partial}{\partial y}(yr^n) + \frac{\partial}{\partial z}(zr^n)$$

$$= r^n + nx^2 r^{n-2} + r^n + ny^2 r^{n-2} + r^n + nz^2 r^{n-2}.$$

$$= 3r^n + nr^{n-2}(x^2 + y^2 + z^2)$$

$$= 3r^n + nr^{n-2} \cdot r^2 = 3r^n + nr^n = (3+n) r^n.$$

when $n = -3$, $\text{div}\left(\frac{\bar{r}}{r^3}\right) = 0$ i.e. $\frac{\bar{r}}{r^3}$ is solenoidal.

1) $\text{div grad} f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$.

Sol:- $\nabla^2 f = \nabla \cdot \nabla f = \nabla \cdot \left[\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right]$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f.$$

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the laplacian operator and $\nabla^2 f = 0$ is called the laplace's equation.

Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

2) If $\bar{r} = xi + yj + zk$, then prove that $\nabla^2(r^n) = n(n+1)r^{n-2}$ where $r = |\bar{r}|$

Sol:- Given that $\bar{r} = xi + yj + zk$ and $r = \sqrt{x^2+y^2+z^2}$

$$r^n = (x^2+y^2+z^2)^{n/2}.$$

To find $\nabla^2(r^n) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) r^n = \Sigma \frac{\partial^2}{\partial x^2} r^n$ — (1)

Take $\frac{\partial}{\partial x}(r^n) = n \cdot r^{n-1} \frac{\partial r}{\partial x} = n r^{n-1} \frac{x}{r} = n x \cdot r^{n-2}$

$$\frac{\partial^2}{\partial x^2}(r^n) = n \cdot r^{n-2} + x \cdot n \cdot (n-2) r^{n-3} \frac{\partial r}{\partial x}$$

$$= n\left[r^{n-2} + (n-2) r^{n-3} \frac{x^2}{r}\right] = n r^{n-2} + n(n-2) r^{n-4} x^2$$

Equation (1) we get

$$\nabla^2 r^n = \Sigma \frac{\partial^2}{\partial x^2} r^n$$

$$= \Sigma n \cdot r^{n-2} + n(n-2) r^{n-4} x^2$$

$$= 3n r^{n-2} + n(n-2) r^{n-4} (x^2+y^2+z^2)$$

$$= 3n r^{n-2} + (n^2-2n) r^{n-4} (r^2)$$

$$= 3n r^{n-2} + (n^2-2n) r^{n-2}$$

$$= r^{n-2}[3n + n^2 - 2n]$$

$$= r^{n-2}[n + n^2]$$

$$= n(n+1) r^{n-2}$$

$$\nabla^2 r^n = n(n+1) r^{n-2}$$

Treating ∇ as a vector, the ... represents application of ∇ ... easily ... three times

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Show that $r^\alpha \bar{R}$ is any irrotational vector for any value of α but is solenoidal if $\alpha+3=0$ where $\bar{R} = x\bar{i}+y\bar{j}+z\bar{k}$ and 'r' is the magnitude of $\bar{R}$.

$r = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{1/2} \Rightarrow r^\alpha = (x^2+y^2+z^2)^{\alpha/2}$

Sol:- Let $A = r^\alpha \bar{R} = (x^2+y^2+z^2)^{\alpha/2}(x\bar{i}+y\bar{j}+z\bar{k}) = \Sigma x(x^2+y^2+z^2)^{\alpha/2}\bar{i}$

$$\text{curl } \bar{A} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x(x^2+y^2+z^2)^{\alpha/2} & y(x^2+y^2+z^2)^{\alpha/2} & z(x^2+y^2+z^2)^{\alpha/2} \end{vmatrix}$$

$$= \Sigma \bar{i}\left[\frac{\partial}{\partial y}\left[z(x^2+y^2+z^2)^{\alpha/2}\right] - \frac{\partial}{\partial z}\left[y(x^2+y^2+z^2)^{\alpha/2}\right]\right.$$

$$= \Sigma \bar{i}\left[\frac{\alpha z}{2}(x^2+y^2+z^2)^{\frac{\alpha}{2}-1} 2y - \frac{\alpha y}{2}(x^2+y^2+z^2)^{\frac{\alpha}{2}-1}(2z)\right] = \bar{0}$$

Hence $\bar{A}$ is irrotational for any value of α.

But $\text{div } \bar{A} = \nabla\cdot(r^\alpha \bar{R}) = (\alpha+3)r^\alpha$ which is zero for $\alpha+3=0$

i.e A is solenoidal if $\alpha+3=0$.

---

2). curl grad f = $\nabla \times \nabla f = \bar{0}$

$$\nabla \times \nabla f = \nabla \times \left(\frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j} + \frac{\partial f}{\partial z}\bar{k}\right)$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial y \partial z}\right) + \bar{j}\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right) + \bar{k}\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)$$

$= \bar{0}$.

The above result can also be written as

grad(div $\bar{i}$) = curl(curl $\bar{i}$) + $\nabla^2 \bar{i}$

i) $\nabla\cdot\nabla\phi = \nabla^2\phi$

ii) $\nabla\times\nabla\phi = \bar{0}$

iii) $\nabla\times(\nabla\times\bar{i}) = \nabla(\nabla\cdot\bar{i}) - \nabla^2\bar{i}$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

(3) div curl $\bar{F} = 0 \Rightarrow \nabla \cdot (\nabla \times \bar{F}) = 0$

Let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

$$\text{curl } \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} = \bar{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \bar{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \bar{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$\text{div}(\text{curl } \bar{F}) = \nabla \cdot (\nabla \times \bar{F})$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= \frac{\partial^2 F_3}{\partial x\,\partial y} - \frac{\partial^2 F_2}{\partial x\,\partial z} + \frac{\partial^2 F_1}{\partial y\,\partial z} - \frac{\partial^2 F_3}{\partial y\,\partial x} + \frac{\partial^2 F_2}{\partial z\,\partial x} - \frac{\partial^2 F_1}{\partial z\,\partial y}$$

$$= 0$$

(4) curl curl $\bar{F}$ = grad div $\bar{F} - \nabla^2 \bar{F}$

i.e. $\nabla \times (\nabla \times \bar{F}) = \nabla(\nabla \cdot \bar{F}) - \nabla^2 \bar{F}$

Let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

then curl $\bar{F} = \nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$

$$= \bar{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \bar{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \bar{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

curl curl $\bar{F}$ = $\nabla \times (\nabla \times \bar{F})$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} & \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} & \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F_3}{\partial z} - \frac{\partial F_2}{\partial z}\right)\right]$$

$$+ \bar{j}\left[\frac{\partial}{\partial z}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x} - \frac{\partial F_3}{\partial x}\right)\right]$$

$$+ \bar{k}\left[\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x} - \frac{\partial F_2}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right]$$

$$= \bar{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) - \left(\frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}\right)\right]$$

$$+ \bar{j}\left[\frac{\partial}{\partial y}\left(\frac{\partial F_3}{\partial z} + \frac{\partial F_1}{\partial x}\right) - \left(\frac{\partial^2 F_2}{\partial z^2} + \frac{\partial^2 F_2}{\partial x^2}\right)\right]$$

$$+ \bar{k}\left[\frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) - \left(\frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_3}{\partial y^2}\right)\right]$$

$$= \bar{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) - \left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}\right)\right]$$

$$+ \bar{j}\left[\frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) - \left(\frac{\partial^2 F_2}{\partial x^2} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_2}{\partial z^2}\right)\right]$$

$$+ \bar{k}\left[\frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) - \left(\frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_3}{\partial z^2}\right)\right]$$

$$= \nabla(\nabla \cdot \bar{F}) - \nabla^2 \bar{F}$$

$$= \nabla(\nabla \cdot \bar{F}) - \nabla^2 \bar{F} = \text{grad}(\text{div } \bar{F})$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
www.FirstRanker.com

Properties of Divergence and Curl

1) For a constant vector $\bar{a}$, $\text{div}\,\bar{a} = 0$, $\text{curl}\,\bar{a} = \bar{0}$

2) $\text{div}(\bar{A}+\bar{B}) = \text{div}\,\bar{A} + \text{div}\,\bar{B}$ (or) $\nabla\cdot(\bar{A}+\bar{B}) = \nabla\cdot\bar{A} + \nabla\cdot\bar{B}$

Proof :- $\text{div}(\bar{A}+\bar{B}) = \nabla\cdot(\bar{A}+\bar{B}) = \left(\bar{i}\frac{d}{dx} + \bar{j}\frac{d}{dy} + \bar{k}\frac{d}{dz}\right)\cdot(\bar{A}+\bar{B})$

$$= \bar{i}\cdot\frac{d}{dx}(\bar{A}+\bar{B}) + \bar{j}\cdot\frac{d}{dy}(\bar{A}+\bar{B}) + \bar{k}\cdot\frac{d}{dz}(\bar{A}+\bar{B})$$

$$= \bar{i}\cdot\left(\frac{d\bar{A}}{dx}+\frac{d\bar{B}}{dx}\right) + \bar{j}\cdot\left(\frac{d\bar{A}}{dy}+\frac{d\bar{B}}{dy}\right) + \bar{k}\cdot\left(\frac{d\bar{A}}{dz}+\frac{d\bar{B}}{dz}\right)$$

$$= \left(\bar{i}\cdot\frac{d\bar{A}}{dx} + \bar{j}\cdot\frac{d\bar{A}}{dy} + \bar{k}\cdot\frac{d\bar{A}}{dz}\right) + \left(\bar{i}\cdot\frac{d\bar{B}}{dx} + \bar{j}\cdot\frac{d\bar{B}}{dy} + \bar{k}\cdot\frac{d\bar{B}}{dz}\right)$$

$$= \nabla\cdot\bar{A} + \nabla\cdot\bar{B} = \text{div}\,\bar{A} + \text{div}\,\bar{B}$$

3) $\text{curl}(\bar{A}+\bar{B}) = \text{curl}\,\bar{A} + \text{curl}\,\bar{B}$ (or) $\nabla\times(\bar{A}+\bar{B}) = \nabla\times\bar{A} + \nabla\times\bar{B}$

$$\text{curl}(\bar{A}+\bar{B}) = \nabla\times(\bar{A}+\bar{B}) = \left(\bar{i}\frac{d}{dx} + \bar{j}\frac{d}{dy} + \bar{k}\frac{d}{dz}\right)\times(\bar{A}+\bar{B})$$

$$= \Sigma\,\bar{i}\times\frac{d}{dx}(\bar{A}+\bar{B}) = \Sigma\,\bar{i}\times\left(\frac{d\bar{A}}{dx}+\frac{d\bar{B}}{dx}\right)$$

$$= \Sigma\,\bar{i}\times\frac{d\bar{A}}{dx} + \Sigma\,\bar{i}\times\frac{d\bar{B}}{dx}$$

$$= (\nabla\times\bar{A}) + (\nabla\times\bar{B}) = \text{curl}\,\bar{A} + \text{curl}\,\bar{B}$$

4) If $\bar{A}$ is a vector function and 'ϕ' is a scalar function

then $\text{div}(\phi\bar{A}) = \phi\,\text{div}\,\bar{A} + (\text{grad}\,\phi)\cdot\bar{A}$ (or) $\nabla\cdot(\phi\bar{A}) = \phi(\nabla\cdot\bar{A}) + (\nabla\phi)\cdot\bar{A}$

Sol:- $\text{div}(\phi\bar{A}) = \nabla\cdot(\phi\bar{A}) = \left(\bar{i}\frac{d}{dx} + \bar{j}\frac{d}{dy} + \bar{k}\frac{d}{dz}\right)\cdot(\phi\bar{A})$

$$= \bar{i}\frac{d}{dx}(\phi\bar{A}) + \bar{j}\frac{d}{dy}(\phi\bar{A}) + \bar{k}\frac{d}{dz}(\phi\bar{A})$$

$$= \Sigma\,\bar{i}\cdot\frac{d}{dx}(\phi\bar{A}) = \Sigma\,\bar{i}\cdot\left[\phi\frac{d\bar{A}}{dx} + \frac{d\phi}{dx}\bar{A}\right] = \phi\,\Sigma\left(\bar{i}\cdot\frac{d\bar{A}}{dx}\right) + \Sigma\left(\bar{i}\frac{d\phi}{dx}\right)\cdot\bar{A}$$

$$= \phi(\nabla\cdot\bar{A}) + \nabla\phi\cdot\bar{A}$$

$$= \phi\,\text{div}\,\bar{A} + (\text{grad}\,\phi)\cdot\bar{A}$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

If $\bar{A}$ is a vector function, and ϕ is scalar function, then

$$\text{curl}(\phi\bar{A}) = (\text{grad}\phi) \times \bar{A} + \phi \, \text{curl}\, \bar{A}$$

(or) $$\nabla \times (\phi\bar{A}) = (\nabla\phi) \times \bar{A} + \phi(\nabla \times \bar{A})$$

Sol:- $$\text{curl}(\phi\bar{A}) = \nabla \times (\phi\bar{A}) = \left(\bar{i}\frac{d}{dx} + \bar{j}\frac{d}{dy} + \bar{k}\frac{d}{dz}\right) \times (\phi\bar{A})$$

$$= \sum \bar{i} \times \frac{d}{dx}(\phi\bar{A}) = \sum \bar{i} \times \left(\frac{d\phi}{dx}\bar{A} + \phi\frac{d\bar{A}}{dx}\right)$$

$$= \sum \bar{i} \times \frac{d\phi}{dx}\bar{A} + \phi \sum \bar{i} \times \frac{d\bar{A}}{dx}$$

$\bar{a} \times (m\bar{b}) = (m\bar{a}) \times \bar{b}$
$= m(\bar{a} \times \bar{b})$

$$= \sum \frac{d\phi}{dx}\bar{i} \times \bar{A} + \phi \sum \bar{i} \times \frac{d\bar{A}}{dx}$$

$$= \nabla\phi \times \bar{A} + \phi(\nabla \times \bar{A})$$

$$= (\text{grad}\phi) \times \bar{A} + \phi \, \text{curl}\, \bar{A}$$

6) $$\nabla(\bar{A} \cdot \bar{B}) = (\bar{A} \cdot \nabla)\bar{B} + (\bar{B} \cdot \nabla)\bar{A} + \bar{A} \times (\nabla \times \bar{B}) + \bar{B} \times (\nabla \times \bar{A})$$

Sol:- $$\nabla(\bar{A} \cdot \bar{B}) = \sum \bar{i} \frac{d}{dx}(\bar{A} \cdot \bar{B}) = \sum \bar{i}\left(\bar{A} \cdot \frac{d\bar{B}}{dx} + \frac{d\bar{A}}{dx} \cdot \bar{B}\right)$$

$$= \sum \left(\bar{A} \cdot \frac{d\bar{B}}{dx}\right)\bar{i} + \sum \left(\bar{B} \cdot \frac{d\bar{A}}{dx}\right)\bar{i} \quad \text{— (1)}$$

We know that:
$\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c}$
$(\bar{a} \cdot \bar{b})\bar{c} = (\bar{a} \cdot \bar{c})\bar{b} - \bar{a} \times (\bar{b} \times \bar{c})$

$$\left(\bar{A} \cdot \frac{d\bar{B}}{dx}\right)\bar{i} = (\bar{A} \cdot \bar{i})\frac{d\bar{B}}{dx} - \bar{A} \times \left(\frac{d\bar{B}}{dx} \times \bar{i}\right)$$

$$= (\bar{A} \cdot \bar{i})\frac{d\bar{B}}{dx} + \bar{A} \times \left(\bar{i} \times \frac{d\bar{B}}{dx}\right)$$

$$\sum \left(\bar{A} \cdot \frac{d\bar{B}}{dx}\right)\bar{i} = \left(\bar{A} \cdot \sum \bar{i}\frac{d}{dx}\right)\bar{B} + \bar{A} \times \sum \left(\bar{i} \times \frac{d\bar{B}}{dx}\right)$$

$$= (\bar{A} \cdot \nabla)\bar{B} + \bar{A} \times (\nabla \times \bar{B}) \quad \text{— (2)}$$

Interchanging $\bar{A}$ and $\bar{B}$,

$$\sum \left(\bar{B} \cdot \frac{d\bar{A}}{dx}\right)\bar{i} = (\bar{B} \cdot \nabla)\bar{A} + \bar{B} \times (\nabla \times \bar{A}) \quad \text{— (3)}$$

Substituting values from (2) and (3) in (1), we get

$$\nabla(\bar{A} \cdot \bar{B}) = (\bar{A} \cdot \nabla)\bar{B} + (\bar{B} \cdot \nabla)\bar{A} + \bar{A} \times (\nabla \times \bar{B}) + \bar{B} \times (\nabla \times \bar{A})$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$\cdot(\bar{A}\times\bar{B}) = \bar{B}\cdot(\nabla\times\bar{A}) - \bar{A}\cdot(\nabla\times\bar{B})$ (21)

$$\nabla\cdot(\bar{A}\times\bar{B}) = \sum\bar{i}\cdot\frac{\partial}{\partial x}(\bar{A}\times\bar{B}) = \sum\bar{i}\cdot\left(\frac{\partial\bar{A}}{\partial x}\times\bar{B}+\bar{A}\times\frac{\partial\bar{B}}{\partial x}\right)$$

$$= \sum\bar{i}\cdot\left(\frac{\partial\bar{A}}{\partial x}\times\bar{B}\right)+\sum\bar{i}\cdot\left(\bar{A}\times\frac{\partial\bar{B}}{\partial x}\right)$$

Since $\bar{a}\cdot(\bar{b}\times\bar{c})=(\bar{a}\times\bar{b})\cdot\bar{c}$

$$= \sum\bar{i}\cdot\left(\frac{\partial\bar{A}}{\partial x}\times\bar{B}\right)-\sum\bar{i}\cdot\left(\frac{\partial\bar{B}}{\partial x}\times\bar{A}\right)$$

$$= \sum\left(\bar{i}\times\frac{\partial\bar{A}}{\partial x}\right)\cdot\bar{B}-\sum\left(\bar{i}\times\frac{\partial\bar{B}}{\partial x}\right)\cdot\bar{A}$$

$$= (\nabla\times\bar{A})\cdot\bar{B}-(\nabla\times\bar{B})\cdot\bar{A}$$

$$= \bar{B}\cdot(\nabla\times\bar{A})-\bar{A}\cdot(\nabla\times\bar{B})$$

ii) $\nabla\times(\bar{A}\times\bar{B}) = (\nabla\cdot\bar{B})\bar{A}-(\nabla\cdot\bar{A})\bar{B}+(\bar{B}\cdot\nabla)\bar{A}-(\bar{A}\cdot\nabla)\bar{B}$

$$\nabla\times(\bar{A}\times\bar{B}) = \sum\bar{i}\times\frac{\partial}{\partial x}(\bar{A}\times\bar{B}) = \sum\bar{i}\times\left(\frac{\partial\bar{A}}{\partial x}\times\bar{B}+\bar{A}\times\frac{\partial\bar{B}}{\partial x}\right)$$

$$= \sum\bar{i}\times\left(\frac{\partial\bar{A}}{\partial x}\times\bar{B}\right)+\sum\bar{i}\times\left(\bar{A}\times\frac{\partial\bar{B}}{\partial x}\right)$$

$$= \sum\left[(\bar{i}\cdot\bar{B})\frac{\partial\bar{A}}{\partial x}-\left(\bar{i}\cdot\frac{\partial\bar{A}}{\partial x}\right)\bar{B}\right]+\sum\left[\left(\bar{i}\cdot\frac{\partial\bar{B}}{\partial x}\right)\bar{A}-(\bar{i}\cdot\bar{A})\frac{\partial\bar{B}}{\partial x}\right]$$

$$= \sum(\bar{B}\cdot\bar{i})\frac{\partial\bar{A}}{\partial x}-\sum\left(\bar{i}\cdot\frac{\partial\bar{A}}{\partial x}\right)\bar{B}+\sum\left(\bar{i}\cdot\frac{\partial\bar{B}}{\partial x}\right)\bar{A}-\sum(\bar{A}\cdot\bar{i})\frac{\partial\bar{B}}{\partial x}$$

$$= \left(\bar{B}\cdot\sum\bar{i}\frac{\partial}{\partial x}\right)\bar{A}-(\nabla\cdot\bar{A})\bar{B}+(\nabla\cdot\bar{B})\bar{A}-\left(\bar{A}\cdot\sum\bar{i}\frac{\partial}{\partial x}\right)\bar{B}$$

$$= (\bar{B}\cdot\nabla)\bar{A}-(\nabla\cdot\bar{A})\bar{B}+(\nabla\cdot\bar{B})\bar{A}-(\bar{A}\cdot\nabla)\bar{B}$$

2.

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Physical's chainter
www.FirstRanker.com www.FirstRanker.com

Physical interpretation of the divergence of a vector :-

If $\bar{v}$ represents the velocity of the fluid, then $\nabla \cdot \bar{v}$ gives the rate at which the fluid is originating at a point per unit volume. Similarly if $\bar{v}$ represents an electric flux then $\operatorname{div} \bar{v}$ is the amount of flux which diverges per unit volume in unit time. If $\nabla \cdot \bar{F} = 0$ then the field is said to be a solenoidal field. Gravitational or electrostatic fields are examples of solenoidal fields.

Physical interpretation of the curl of a vector :-

The curl of a vector function $\bar{F}$ gives a measure of angular velocity at every point of the vector field.

If $\operatorname{curl} \bar{F} = \bar{0}$ the $\bar{F}$ is said to irrotational field.

Ex:- $\nabla^2(r^n) = n(n+1) r^{n-2}$

$$\nabla^2(r^n) = \frac{d^2}{dx^2}(r^n) + \frac{d^2}{dy^2}(r^n) + \frac{d^2}{dz^2}(r^n) \quad \text{---(1)}$$

$$\frac{d}{dx}(r^n) = n r^{n-1} \frac{dr}{dx} = n r^{n-1} \frac{x}{r} = n r^{n-2} x$$

$$\frac{d^2}{dx^2}(r^n) = n\left[r^{n-2} + (n-2) r^{n-3} \frac{dr}{dx} \cdot x\right] = n\left[r^{n-2} + (n-2) r^{n-3} \frac{x}{r} x\right]$$

$$= n\left[r^{n-2} + (n-2) r^{n-4} x^2\right]$$

$$\frac{d^2}{dy^2}(r^n) = n\left[r^{n-2} + (n-2) r^{n-4} y^2\right] \quad \text{---(3)} \quad \text{and} \quad \frac{d^2}{dz^2}(r^n) = n\left[r^{n-2} + (n-2)\right.$$

Adding (3), (2) and (4), (1) gives

$$\nabla^2(r^n) = n\left[3r^{n-2} + (n-2) r^{n-4}(x^2+y^2+z^2)\right.$$

$$= n\left[3r^{n-2} + (n-2) r^{n-4} r^2\right] = n(n+1) r^{n-2}$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

(5)

If ϕ and ψ are scalar fields and $\bar{F}$ is a vector field show that $\bar{F} \cdot \text{curl} \bar{F} = 0$

since $\bar{F} = \frac{1}{\phi} \nabla \psi \Rightarrow \text{curl} \bar{F} = \nabla \times \left(\frac{1}{\phi} \nabla \psi\right)$

$\text{curl} \bar{F} = \nabla \frac{1}{\phi} \times \nabla \psi + \frac{1}{\phi} \nabla \times (\nabla \psi)$ $\left[\nabla \times \phi \bar{A} = \nabla \phi \times \bar{A} + \phi \nabla \times \bar{A}\right]$.

$\nabla \times \nabla \psi = 0$

$= \nabla \frac{1}{\phi} \times \nabla \psi$

$\bar{F} \cdot \text{curl} \bar{F} = \frac{1}{\phi} \nabla \psi \left(\nabla \frac{1}{\phi} \times \nabla \psi\right) = 0$, for it scalar triple product in which two factors are equal.

If $\bar{r}$ and $\bar{R}$ have their usual meanings and $\bar{A}$ is a constant vector, prove that $\nabla \times \left(\frac{\bar{A} \times \bar{R}}{r^n}\right) = \frac{2-n}{r^n} \bar{A} + \frac{n(\bar{A} \cdot \bar{R})}{r^{n+2}} \bar{R}$

$\nabla \times \left[\bar{r}^{-n} (\bar{A} \times \bar{R})\right] = r^{-n} \left[\nabla \times (\bar{A} \times \bar{R})\right] + \nabla r^{-n} \times (\bar{A} \times \bar{R})$

$= r^{-n} \left[(\nabla \cdot \bar{R}) \bar{A} - (\nabla \cdot \bar{A}) \bar{R}\right] + \left[-n r^{-(n+1)} \frac{\bar{R}}{r}\right] \times (\bar{A} \times \bar{R})$

$= \bar{r}^{-n} (3\bar{A} - \bar{A}) - n r^{-(n+2)} \bar{R} \times (\bar{A} \times \bar{R})$ $\left[\because \nabla \cdot \bar{R} = 3, \ (\bar{A} \cdot \nabla) \bar{R} = \bar{A}\right]$.

$= 2\bar{A} \bar{r}^{-n} - n r^{-(n+2)} \left[(\bar{R} \cdot \bar{R}) \bar{A} - (\bar{R} \cdot \bar{A}) \bar{R}\right]$

$= \frac{2\bar{A}}{r^n} - \frac{n}{r^{n+2}} \left[r^2 \bar{A} - (\bar{A} \cdot \bar{R}) \bar{R}\right]$.

$= \frac{2-n}{r^n} \bar{A} + \frac{n(\bar{A} \cdot \bar{R})}{r^{n+2}} \bar{R}$.

If r is the distance of a point (x, y, z) from the origin, prove that $\text{curl}\left(\bar{k} \times \text{grad} \frac{1}{r}\right) + \text{grad}\left(\bar{k} \cdot \text{grad} \frac{1}{r}\right) = 0$ where $\bar{k}$ is the unit vector in the direction of z.

Given $\frac{1}{r} = \nabla \frac{1}{r} = \left(\bar{i} \frac{d}{dx} + \bar{j} \frac{d}{dy} + \bar{k} \frac{d}{dz}\right) (x^2 + y^2 + z^2)^{-1/2}$

$= -\frac{1}{2} (x^2 + y^2 + z^2)^{-3/2} (2x\bar{i} + 2y\bar{j} + 2z\bar{k})$

$= -(x^2 + y^2 + z^2)^{-3/2} (x\bar{i} + y\bar{j} + z\bar{k})$.

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$$\text{curl}\left(\bar{F}\times\text{grad}\frac{1}{r}\right) = \nabla\times\left(-\frac{x\bar{i}+y\bar{j}+z\bar{k}}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y/(x^2+y^2+z^2)^{3/2} & -x/(x^2+y^2+z^2)^{3/2} & 0 \end{vmatrix}$$

$$= \bar{i}\frac{\partial}{\partial z}\left[\frac{z}{(x^2+y^2+z^2)^{3/2}}\right] + \bar{j}\frac{\partial}{\partial z}\left[\frac{y}{(x^2+y^2+z^2)^{3/2}}\right] - \bar{k}\left[\frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right)\right]$$

$$= \frac{-3xz\bar{i} - 3yz\bar{j} + (x^2+y^2-2z^2)\bar{k}}{(x^2+y^2+z^2)^{5/2}} \quad \text{---(1)}$$

$$\text{grad}\left(\bar{F}\cdot\text{grad}\frac{1}{r}\right) = \nabla\left[-\bar{F}\cdot\frac{(x\bar{i}+y\bar{j}+z\bar{k})}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\left[\frac{-z}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= \frac{-3xz\bar{i}}{(x^2+y^2+z^2)^{5/2}} + \frac{3yz}{(x^2+y^2+z^2)^{5/2}}\bar{j} + \left(\frac{3z^2-x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}\right)\bar{k}$$

Adding (i) and (ii) we get

$$\text{curl}\left(\bar{F}\times\text{grad}\frac{1}{r}\right) + \text{grad}\left(\bar{F}\cdot\text{grad}\frac{1}{r}\right) = 0$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Exercise – 8.11

Divergence: The divergence of a continuously differentiable vector point function $\bar{f}$ is denoted by $\operatorname{div}\bar{f}$ and is defined by the equation

$$\operatorname{div}\bar{f} = \nabla\cdot\bar{f} = \bar{i}\cdot\frac{\partial\bar{f}}{\partial x} + \bar{j}\cdot\frac{\partial\bar{f}}{\partial y} + \bar{k}\cdot\frac{\partial\bar{f}}{\partial z}$$

If $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

then $\operatorname{div}\bar{f} = \nabla\cdot\bar{f} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot(f_1\bar{i} + f_2\bar{j} + f_3\bar{k})$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Curl: The curl of a continuously differentiable vector point function $\bar{f}$ is defined by the equation

$$\operatorname{Curl}\bar{f} = \nabla\times\bar{f} = \bar{i}\times\frac{\partial\bar{f}}{\partial x} + \bar{j}\times\frac{\partial\bar{f}}{\partial y} + \bar{k}\times\frac{\partial\bar{f}}{\partial z}$$

If $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$, then

$$\nabla\times\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix} = \bar{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \bar{j}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \bar{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

Ex: If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, show that

(i) $\nabla\cdot\bar{r} = 3$, (ii) $\nabla\times\bar{r} = \bar{0}$.

Sol: (i) $\nabla\cdot\bar{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$

(ii) $\nabla\times\bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = \bar{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \bar{j}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + \bar{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$$= \frac{\left(\sqrt{x^2+y^2+z^2}\cdot\frac{2x}{2\sqrt{x^2+y^2+z^2}}\right)+\sqrt{x^2+y^2+z^2}\cdot\frac{2x}{2\sqrt{x^2+y^2+z^2}}}{(x^2+y^2+z^2)}$$

$$+\frac{\sqrt{x^2+y^2+z^2}\cdot\frac{2z}{2\sqrt{x^2+y^2+z^2}}}{(x^2+y^2+z^2)}$$

$$= \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}} = \frac{2}{\sqrt{x^2+y^2+z^2}}$$

(11) Find the value of 'a' if the vector $(ax^2y+yz)\bar{i}+(xy^2-xz^2)\bar{j}+(2xyz-2x^2y^2)\bar{k}$ has a zero divergence. Find the curl of the above vector which has zero divergence.

Sol: If $\bar{f}$ is solenoidal $\Rightarrow \nabla\cdot\bar{f}=0$

$$\nabla\cdot\bar{f}=0 \Rightarrow \frac{\partial}{\partial x}(ax^2y+yz)+\frac{\partial}{\partial y}(xy^2-xz^2)+\frac{\partial}{\partial z}(2xyz-2x^2y^2)$$

$$\Rightarrow 2axy+2xy+2xy=0$$

$$\Rightarrow 2xy(a+2)=0 \Rightarrow a=-2$$

$$\nabla\times\bar{f}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ ax^2y+yz & xy^2-2z^2 & 2xyz-2x^2y^2\end{vmatrix}$$

$$=\bar{i}(2xz-4x^2y+2xz)+\bar{j}(y-2yz+4xy^2)+\bar{k}(y^2-z^2-ax^2-z)$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com

$= \bar{i}(4xz - 4x^2y) + \bar{j}(y - 2y^2z + 4xy^2) + \bar{k}(y^2 - x^2 - z)$ 25

show that each of the following vectors are solenoidal

(i) Let $\bar{F} = (-x^2 + yz)\bar{i} + (4y - z^2x)\bar{j} + (2xz - 4z)\bar{k}$

for solenoidal vector $\Rightarrow$ div $\bar{F} = 0$

$$\text{div } \bar{F} = \frac{\partial}{\partial x}(-x^2 + yz) + \frac{\partial}{\partial y}(4y - z^2x) + \frac{\partial}{\partial z}(2xz - 4z)$$

$$= -2x + 4 + 2x - 4$$

$$= 0$$

Hence the vector is solenoidal.

(ii) $\bar{F} = 3y^4z^2\bar{i} + 4x^3z^2\bar{j} + 3x^2y^2\bar{k}$

$$\text{div } \bar{F} = \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(3x^2y^2)$$

$$= 0 + 0 + 0$$

$$= 0$$

(iii) If ϕ and ψ are scalar functions, then prove that $\nabla\phi \times \nabla\psi$ is solenoidal.

Given that ϕ and ψ are scalar function

$$\nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

$$\nabla\psi = \frac{\partial\psi}{\partial x}\bar{i} + \frac{\partial\psi}{\partial y}\bar{j} + \frac{\partial\psi}{\partial z}\bar{k}$$

$$\nabla\phi \times \nabla\psi = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \\ \frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial y} & \partial\psi/\partial z \end{vmatrix}$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$$= \bar{i}\left(\frac{\partial\phi}{\partial y}\cdot\frac{\partial\psi}{\partial z} - \frac{\partial\phi}{\partial z}\cdot\frac{\partial\psi}{\partial y}\right) + \bar{j}\left(\frac{\partial\phi}{\partial z}\cdot\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial x}\cdot\frac{\partial\psi}{\partial z}\right) + \bar{k}\left(\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial x}\right)$$

A vector field $\bar{f}$ is solenoidal if $\operatorname{div}\bar{f} = 0$ if $\bar{p} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ then

$$\operatorname{div}\bar{p} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

To prove that $\nabla\cdot(\nabla\phi \times \nabla\psi) = 0$

so $\nabla\cdot(\nabla\phi\times\nabla\psi) = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\cdot\frac{\partial\psi}{\partial z} - \frac{\partial\psi}{\partial y}\cdot\frac{\partial\phi}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\cdot\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial x}\cdot\frac{\partial\psi}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial\phi}{\partial y}\cdot\frac{\partial\psi}{\partial x}\right)$

$$= \frac{\partial\phi}{\partial y}\cdot\frac{\partial^2\psi}{\partial x\partial z} + \frac{\partial\psi}{\partial z}\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial\psi}{\partial y}\cdot\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial\phi}{\partial z}\cdot\frac{\partial^2\psi}{\partial x\partial y} + \frac{\partial\phi}{\partial z}\cdot\frac{\partial^2\psi}{\partial y\partial x} + \frac{\partial\psi}{\partial x}\cdot\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\psi}{\partial y\partial z} - \frac{\partial\psi}{\partial z}\cdot\frac{\partial^2\phi}{\partial y\partial x}$$

$$+ \frac{\partial\phi}{\partial x}\frac{\partial^2\psi}{\partial z\partial y} + \frac{\partial\psi}{\partial y}\cdot\frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial\phi}{\partial y}\cdot\frac{\partial^2\psi}{\partial z\partial x} - \frac{\partial\psi}{\partial x}\cdot\frac{\partial^2\phi}{\partial z\partial y}$$

$$= 0 \qquad \left(\because \frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}\right)$$

$\nabla\cdot(\nabla\phi\times\nabla\psi) = 0$

$\nabla\phi \times \nabla\psi \neq 0$

are solenoidal.

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

If $\bar{A}$ and $\bar{B}$ are irrotational, prove that $\bar{A} \times \bar{B}$ is solenoidal

Sol: Given $\bar{A}$ and $\bar{B}$ are irrotational

$$\nabla \times \bar{A} = \bar{0}, \quad \nabla \times \bar{B} = \bar{0}$$

$$\nabla \cdot (\bar{A} \times \bar{B}) = \bar{B} \cdot (\nabla \times \bar{A}) - \bar{A} \cdot (\nabla \times \bar{B})$$

$$= \bar{B} \cdot \bar{0} - \bar{A} \cdot \bar{0}$$

$$= 0$$

$\nabla \cdot (\bar{A} \times \bar{B}) = 0 \Rightarrow \bar{A} \times \bar{B}$ is solenoidal.

(7) If $v = x^2 + y^2 + z^2$ and $\bar{V} = x\bar{i} + y\bar{j} + z\bar{k}$, show that div $(v\bar{V}) = 5v$.

$$v\bar{V} = (x^2 + y^2 + z^2)(x\bar{i} + y\bar{j} + z\bar{k})$$

$$= x(x^2 + y^2 + z^2)\bar{i} + y(x^2 + y^2 + z^2)\bar{j} + z(x^2 + y^2 + z^2)\bar{k}$$

$$\text{div}(v\bar{V}) = \frac{\partial}{\partial x}(x^3 + y^2 + z^2) + \frac{\partial}{\partial y}(x^2 + y^3 + z^2) + \frac{\partial}{\partial z}(x^2 + y^2 + z^3)$$

$$= 3x^2 + y^2 + z^2 + x^2 + 3y^2 + z^2 + x^2 + y^2 + 3z^2$$

$$= 5(x^2 + y^2 + z^2)$$

$$= 5v.$$

(8) If $\bar{R} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r \neq 0$, show that

(i) $\nabla\left(\frac{1}{r}\right) = -\bar{R}/r^4$

$$\nabla\left(\frac{1}{r}\right) = \bar{i}\frac{\partial}{\partial x}\left(\frac{1}{r^2}\right)$$

$$= \bar{i}\,\frac{r^2 \cdot \frac{\partial}{\partial x}(1) - 2r \cdot \frac{\partial r}{\partial x} \cdot 1}{(r^2)^2}$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$$= -\frac{\Sigma \bar{i} \cdot 2x \frac{x}{x}}{x^4} = -\frac{2\bar{R}}{x^4}$$

(ii) $\nabla \cdot \left(\frac{\bar{R}}{x^2}\right) = \frac{1}{x^2}$

$$\Sigma \bar{i} \cdot \frac{\partial}{\partial x}\left(\frac{\bar{R}}{x^2}\right) = \Sigma \bar{i} \cdot \frac{\left[x^2 \frac{\partial \bar{R}}{\partial x} - \bar{R} \cdot 2x \frac{\partial x}{\partial x}\right]}{(x^2)^2}$$

$$= \Sigma \bar{i} \cdot \frac{\left[x^2 \cdot \bar{i} - \bar{R} \cdot 2x \cdot \frac{x}{x}\right]}{x^4}$$

$$= \frac{3x^2 - 2\bar{R} \cdot \bar{R}}{x^4} = \frac{3x^2 - 2x^2}{x^4}$$

$$\Rightarrow \frac{x^2}{x^4} = 1/x^2$$

(iii) $div\,(x^n \bar{R}) = (n+3)x^n$; $curl\, x^n \bar{R} = \bar{0}$

$x^n = (x^2+y^2+z^2)^{n/2}$; $\bar{R} = x\bar{i} + y\bar{j} + z\bar{k}$

$$div\,(x^n \bar{r}) = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{n/2} x + \frac{\partial}{\partial y}(x^2+y^2+z^2)^{n/2} y + \frac{\partial}{\partial z}(x^2+y^2+z^2)^{n/2} \cdot z$$

$$= (x^2+y^2+z^2)^{n/2} + \frac{n}{2} \cdot 2x(x^2+y^2+z^2)^{\frac{n}{2}-1} \cdot x + (x^2+y^2+z^2)^{n/2} + \frac{n}{2} \cdot y(x^2+y^2+z^2)^{n/2-1} \cdot y + (x^2+y^2+z^2)^{n/2} + \frac{n}{2} \cdot 2z(x^2+y^2+z^2)^{\frac{n}{2}-1} \cdot z$$

$$= (x^2+y^2+z^2)^{n/2-1}\left[3(x^2+y^2+z^2) + n(x^2+y^2+z^2)\right]$$

$$= (x^2+y^2+z^2)^{n/2}[3+n]$$

$$= (n+3)x^n$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$\text{curl}(r^n \bar{R}) = \bar{0}$

$r^n = (x^2+y^2+z^2)^{n/2}(x\bar{i}+y\bar{j}+z\bar{k})$

$= \Sigma\, x(x^2+y^2+z^2)^{n/2}\,\bar{i}$

$$\text{curl}(r^n\bar{R}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x(x^2+y^2+z^2)^{n/2} & y(x^2+y^2+z^2)^{n/2} & -z(x^2+y^2+z^2)^{n/2} \end{vmatrix}$$

$$= \Sigma\,\bar{i}\left[\frac{nz}{2}(x^2+y^2+z^2)^{n/2-1}(2y) - \frac{ny}{2}(x^2+y^2+z^2)^{n/2-1}(2z)\right.$$

$= \bar{0}$

(v) $\nabla\left(\nabla\cdot\frac{\bar{R}}{r}\right) = \frac{-2\bar{R}}{r^3}$

$\bar{r} = (x\bar{i}+y\bar{j}+z\bar{k}) \Rightarrow r = \sqrt{x^2+y^2+z^2}$.

$$\text{div}\left(\frac{\bar{R}}{r}\right) = \nabla\cdot\frac{\bar{R}}{r} = \nabla\cdot\frac{(x\bar{i}+y\bar{j}+z\bar{k})}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \frac{\sqrt{x^2+y^2+z^2} - \frac{x^2}{\sqrt{x^2+y^2+z^2}}}{(x^2+y^2+z^2)} + \frac{\sqrt{x^2+y^2+z^2} - \frac{y^2}{\sqrt{x^2+y^2+z^2}}}{(x^2+y^2+z^2)}$$

$$+ \frac{\sqrt{x^2+y^2+z^2} - \frac{z^2}{\sqrt{x^2+y^2+z^2}}}{(x^2+y^2+z^2)}$$

$$\Rightarrow \frac{x^2+y^2+z^2-x^2+x^2+y^2+z^2-y^2+x^2+y^2+z^2-z^2}{(x^2+y^2+z^2)^{3/2}}$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$$\frac{\alpha(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}} = \alpha(x^2+y^2+z^2)^{-1/2}$$

$$\nabla\left(\nabla\cdot\frac{\bar{R}}{r}\right) = \alpha\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2}\cdot\bar{i} + \alpha\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2}\bar{j}$$
$$+ \alpha\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2}\cdot\bar{k}$$

$$= -\alpha\cdot\frac{1}{2}(x^2+y^2+z^2)^{\frac{-1}{2}-1}\cdot 2x\bar{i} + \alpha\left(\frac{-1}{2}\right)(x^2+y^2+z^2)^{-1/2}$$
$$+ \alpha\left(\frac{-1}{2}\right)(x^2+y^2+z^2)^{-1/2}$$

$$= \frac{-\alpha(x\bar{i}+y\bar{j}+z\bar{k})}{(x^2+y^2+z^2)^{3/2}} = \frac{-\alpha\bar{R}}{r^3}$$

(9) If $\bar{V}_1$ and $\bar{V}_2$ be the vectors joining the fixed points (x_1,y_1,z_1) and (x_2,y_2,z_2) respectively to a variable point (x,y,z). prove that (i) $\operatorname{div}(\bar{V}_1\times\bar{V}_2) = 0$
(ii) $\operatorname{grad}(V_1\cdot V_2) = V_1+V_2$ (iii) $\operatorname{curl}(V_1\times V_2) = 2(V_1-V_2)$

Here $\bar{V}_1 = (x-x_1)\bar{i}+(y-y_1)\bar{j}+(z-z_1)\bar{k} = \sum\bar{i}(x-x_1)$

$\bar{V}_2 = (x-x_2)\bar{i}+(y-y_2)\bar{j}+(z-z_2)\bar{k} = \sum\bar{i}(x-x_2)$

$\bar{V}_1\cdot\bar{V}_2 = (x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)$

$\operatorname{grad}(\bar{V}_1\cdot\bar{V}_2) = \nabla(\bar{V}_1\cdot\bar{V}_2) = \sum\bar{i}\frac{\partial}{\partial x}[(x-x_1)(x-x_2)+$
$(y-y_1)(y-y_2)+(z-z_1)$

$= \sum\bar{i}[(x-x_1)+(x-x_2)]$

$= \sum\bar{i}(x-x_1)+\sum\bar{i}(x-x_2)$

$= \bar{V}_1+\bar{V}_2$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
FirstRanker's choice

$\bar{V}_1 \times \bar{V}_2$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x-x_1 & y-y_1 & z-z_1 \\ x-x_2 & y-y_2 & z-z_2 \end{vmatrix}$$

www.FirstRanker.com www.FirstRanker.com

$$= \bar{i}\left[(y-y_1)(z-z_2)-(z-z_1)(y-y_2)\right] + \bar{j}\left[(z-z_1)(x-x_2) - (x-x_1)(z-z_2)\right] + \bar{k}\left[(x-x_1)(y-y_2)-(y-y_1)(x-x_2)\right]$$

$\operatorname{div}(\bar{V}_1 \times \bar{V}_2) = 0.$

$$\operatorname{curl}(\bar{V}_1 \times \bar{V}_2) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y_1z_2 - y_1z - yz_2 & -x_1z_2 + x_1z + xz_2 & x_1y_2 - x_1y - xy_2 \\ -y_2z_1 + y_2z + y_1z_1 & +x_2z_1 - x_2z - xz_1 & -x_2y_1 + x_2y + xy_1 \end{vmatrix}$$

$$= \bar{i}\left[-x_1+x_2-x_1+x_2\right] + \bar{j}\left[-y_1+y_2+y_2-y_1\right] + \bar{k}\left[z_2-z_1+z_2-z_1\right]$$

$$= 2(x_2-x_1)\bar{i} + 2(y_2-y_1)\bar{j} + 2(z_2-z_1)\bar{k}$$

$$= 2\left[\left[x_2\bar{i}+y_2\bar{j}+z_2\bar{k}\right] - \left[x_1\bar{i}+y_1\bar{j}+z_1\bar{k}\right]\right]$$

$$= 2\left[\bar{V}_2 - \bar{V}_1\right]$$

b) Show that (i) $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$

$$\nabla^2 = \nabla\cdot\left[\nabla f(r)\right] = \nabla\cdot\left[f'(r)\nabla r\right]$$

$$= \nabla\cdot\left[f'(r)\frac{\bar{r}}{r}\right] = \nabla\cdot\left[\frac{f'(r)}{r}\cdot\bar{r}\right]$$

$\nabla\cdot(\phi\bar{F}) = \nabla\phi\cdot\bar{F} + \phi(\nabla\cdot\bar{F})$: $\phi = \frac{f'(r)}{r}$, $\bar{F} = \bar{r}$

$$\nabla^2 f(r) = \nabla\cdot\left[\frac{f'(r)}{r}\cdot\bar{r}\right] = \nabla\frac{f'(r)}{r}\cdot\bar{r} + \frac{f'(r)}{r}(\nabla\cdot\bar{r})$$

$$= \bar{r}\cdot\left[\frac{-1}{r^2}f'(r) + \frac{1}{r}f''(r)\right]\frac{\bar{r}}{r} + \frac{3}{r}f'(r)$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$$= \left[\frac{-1}{x^3} f'(x) + \frac{1}{x^2} f''(x)\right] x^2 + \frac{2}{x} f'(x)$$

$$= \left[\frac{-1}{x^3} f'(x) + \frac{1}{x^2} f''(x)\right] x^2 + \frac{2}{x} f'(x)$$

$$= -\frac{1}{x} f'(x) + f''(x) + \frac{2}{x} f'(x)$$

$$= f''(x) + \frac{1}{x} f'(x)$$

$$\nabla^2 f(x) = f''(x) + \frac{2}{x} f'(x)$$

(OR)

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \quad \text{and} \quad |\vec{r}| = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2$$

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

Similarly: $\frac{\partial r}{\partial y} = \frac{y}{r}$; $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\frac{\partial}{\partial x}[g(r)] = g'(r)\frac{\partial r}{\partial x} = g'(r)\frac{x}{r}$$

$$\frac{\partial^2}{\partial x^2}[g(r)] = g''(r)\frac{\partial r}{\partial x}\left(\frac{x}{r}\right) + g'(r)\left[\frac{1}{r}\frac{\partial r}{\partial x} - \frac{x}{r^2}\right]$$

$$= g''(r)\left(\frac{x}{r}\right)^2 + g'(r)\left(\frac{1}{r}\right)\frac{x}{r} + g'(r)$$

$$= g''(r)\left(\frac{x}{r}\right)^2 + g'(r)\left(\frac{1}{r}\right)\frac{x}{r} + g'(r)\left(\frac{1}{r}\right)$$

$$+ g''(r)\left(\frac{x}{r}\right)^2 = g''(r)\frac{x^2}{r^2} + \frac{2}{r}g'(r) - \frac{x^2}{r^3}g'(r)$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Similarly,

$$\frac{\partial^2}{\partial y^2}[f(r)] = f''(r)\frac{y^2}{r^2} - \frac{y^2}{r^3}f'(r) + \frac{1}{r}f'(r)$$

$$\frac{\partial^2}{\partial z^2}[f(r)] = f''(r)\frac{z^2}{r^2} - \frac{z^2}{r^3}f'(r) + \frac{1}{r}f'(r)$$

adding the above values, we get

$$\frac{\partial^2}{\partial x^2}f(r) + \frac{\partial^2}{\partial y^2}f(r) + \frac{\partial^2}{\partial z^2}f(r)$$

$$= \frac{1}{r^2}f''(r)(x^2+y^2+z^2) - \frac{1}{r^3}f'(r)(x^2+y^2+z^2) + \frac{3}{r}f'(r)$$

$$= \frac{1}{r^2}f''(r)\,r^2 - \frac{1}{r^3}f'(r)\,r^2 + \frac{3}{r}f'(r)$$

$$= f''(r) - \frac{1}{r}f'(r) + \frac{3}{r}f'(r)$$

$$= f''(r) + \frac{2}{r}f'(r)$$

$$= \nabla^2 f(r)$$

$$\nabla^2 f(r) = \left[\frac{\partial^2}{\partial x^2}f(r) + \frac{\partial^2}{\partial y^2}f(r) + \frac{\partial^2}{\partial z^2}f(r)\right]$$

(ii) prove that $\nabla\cdot[\phi\nabla\psi - \psi\nabla\phi] = \phi\nabla^2\psi - \psi\nabla^2\phi$

$$\nabla\cdot(\phi\bar{F}) = \nabla\phi\cdot\bar{F} + \phi\nabla\cdot\bar{F}$$

$$\nabla\cdot(\phi\nabla\psi - \psi\nabla\phi) = \nabla\cdot(\phi\nabla\psi) - \nabla\cdot(\psi\nabla\phi) \longrightarrow (1)$$

$$\nabla\cdot(\phi\nabla\psi) = \nabla\phi\cdot\nabla\psi + \phi\nabla\cdot\nabla\psi \longrightarrow (2)$$

Here ϕ is scalar function, $\nabla\psi$ is a vector

$$\nabla\cdot(\psi\nabla\phi) = \nabla\psi\cdot\nabla\phi + \psi\nabla\cdot\nabla\phi \longrightarrow (3)$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Here ψ is a scalar, $\nabla\phi$ is a vector.

Substituting ② & ③ in ①

$$\nabla\cdot(\phi\nabla\psi - \psi\nabla\phi) = \nabla\phi\cdot\nabla\psi + \phi\nabla^2\psi - \nabla\psi\cdot\nabla\phi - \psi\nabla^2\phi$$

$$= \phi\nabla^2\psi - \psi\nabla^2\phi$$

(11) If $\bar{A}$ is a constant vector and $\bar{R} = x\bar{i} + y\bar{j} + z\bar{k}$, prove that

(i) grad $(\bar{A}\cdot\bar{R}) = \bar{A}$ (ii) div $(\bar{A}\times\bar{R}) = 0$

(iii) curl $(\bar{A}\times\bar{R}) = 2\bar{A}$ (iv) curl $[(\bar{A}\cdot\bar{R})\bar{R}] = \bar{A}\times\bar{R}$

Let $\bar{A} = A_1\bar{i} + A_2\bar{j} + A_3\bar{k}$

(i) grad $(\bar{A}\cdot\bar{R}) = \nabla[A_1\bar{i} + A_2\bar{j} + A_3\bar{k}]\cdot[x\bar{i} + y\bar{j} + z\bar{k}]$

$$= \nabla[A_1x + A_2y + A_3z]$$

$$= \Sigma i\frac{\partial}{\partial x}[A_1x + A_2y + A_3z]$$

$$= \bar{i}\frac{\partial}{\partial x}(A_1x) + \bar{j}\frac{\partial}{\partial y}(A_2y) + \bar{k}\frac{\partial}{\partial z}(A_3z)$$

$$= A_1\bar{i} + A_2\bar{j} + A_3\bar{k}$$

$$= \bar{A}$$

(ii) $\bar{A}\times\bar{R} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ A_1 & A_2 & A_3 \\ x & y & z \end{vmatrix}$

$$= \bar{i}[A_2z - A_3y] + \bar{j}[A_3x - A_1z] + \bar{k}[A_1y - A_2x]$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$\frac{\partial}{\partial x}(A_3x - A_1z) + \frac{\partial}{\partial y}(A_3x - A_1z) + \frac{\partial}{\partial z}(A_1y - A_2x) = 0$

$$\nabla \times (\bar{A} \times \bar{R}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_2z - A_3y & A_3x - A_1z & A_1y - A_2x \end{vmatrix}$$

$$= \bar{i}(A_1 + A_1) + \bar{j}(A_2 + A_2) + \bar{k}(A_3 + A_3)$$

$$= 2(A_1\bar{i} + A_2\bar{j} + A_3\bar{k})$$

$$= 2\bar{A}.$$

$$(\bar{A} \cdot \bar{R}) = (A_1\bar{i} + A_2\bar{j} + A_3\bar{k}) \cdot (x\bar{i} + y\bar{j} + z\bar{k})$$

$$= A_1x + A_2y + A_3z.$$

$$(\bar{A} \cdot \bar{R})\bar{R} = (A_1x + A_2y + A_3z)(x\bar{i} + y\bar{j} + z\bar{k})$$

$$= (A_1x^2 + A_2xy + A_3zx)\bar{i} + (A_1xy + A_2y^2 + A_3yz)\bar{j} + (A_1xz + A_2yz + A_3z^2)\bar{k}$$

$$\nabla \times (\bar{A} \cdot \bar{R})\bar{R} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1x^2 + A_2xy + A_3zx & A_1xy + A_2y^2 + A_3yz & A_1xz + A_2yz + A_3z^2 \end{vmatrix}$$

$$= \bar{i}(A_2z - A_3y) + \bar{j}(A_3x - A_1z) + \bar{k}(A_1y - A_2x)$$

$$= \bar{A} \times \bar{R}$$

Calculate (i) curl(grad f) given $f(x, y, z) = x^2 + y^2 - z$

grad $f = \nabla f = 2x\bar{i} + 2y\bar{j} - \bar{k}$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com

(ii) $\nabla \times \nabla\phi = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ax & ay & -1 \end{vmatrix} = 0$

www.FirstRanker.com

(iii) $\bar{A} = x^2y\bar{i} + y^2z\bar{j} + z^2y\bar{k}$

$$\nabla \times \bar{A} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & y^2z & z^2y \end{vmatrix}$$

$$= \bar{i}(z^2 - y^2) + \bar{j}(0-0) + \bar{k}(0 - x^2)$$

$$= (z^2 - y^2)\bar{i} - x^2\bar{k}$$

$$\nabla \times (\nabla \times \bar{A}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2 - y^2 & 0 & -x^2 \end{vmatrix}$$

$$= \bar{i}(0-0) + \bar{j}(2z + 2x) + \bar{k}(2y)$$

$$= 2(x+z)\bar{j} + 2y\bar{k}$$

(iv) If $\phi = (x^2+y^2+z^2)^{-n}$, find div (grad ϕ) and determine n if div grad $\phi = 0$

Given that $\phi = (x^2+y^2+z^2)^{-n}$

(i) $\nabla\phi$ = grad $\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$

$$\nabla \cdot (\nabla\phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\frac{\partial\phi}{\partial x} = -n(x^2+y^2+z^2)^{-n-1} \cdot 2x$$

$$\frac{\partial^2\phi}{\partial x^2} = -2n\left[(x^2+y^2+z^2)^{-(n+1)} - x(n+1)(x^2+y^2+z^2)^{-n-2} \cdot 2x\right]$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$= -2n(x^2+y^2+z^2)^{-(n+2)} [x^2+y^2+z^2-2x^2(n+1)]$

$= -2n(x^2+y^2+z^2)^{-(n+2)} [y^2+z^2-x^2(2n+1)]$

Similarly by symmetry,

$\frac{\partial^2\phi}{\partial y^2} = -2n(x^2+y^2+z^2)^{-(n+2)} [x^2+z^2-y^2(2n+1)]$

$\frac{\partial^2\phi}{\partial z^2} = -2n(x^2+y^2+z^2)^{-(n+2)} [x^2+y^2-z^2(2n+1)]$

$\nabla \cdot \nabla\phi = -2n(x^2+y^2+z^2)^{-(n+2)} \cdot [y^2+z^2-2nx^2-x^2+x^2+z^2-2ny^2-y^2 + x^2+y^2-2nz^2-z^2]$

$= -2n(x^2+y^2+z^2)^{-(n+2)} \cdot [2x^2+2y^2+2z^2-2n(x^2+y^2+z^2)]$

$= -2n(x^2+y^2+z^2)^{-(n+1)} (1-2n)$

$= \frac{2n(2n-1)}{(x^2+y^2+z^2)^{n+1}}$

ii) If div grad $\phi = 0$

$\frac{2n(2n-1)}{(x^2+y^2+z^2)^{n+1}} = 0 \quad ; \quad n = \frac{1}{2}$

since 'n' can't be zero

iii) S.T div (grad r^n) $= n(n+1)r^{n-2}$ where $r^2 = x^2+y^2+z^2$

By definition grad $(r^n) = \bar{i}\frac{\partial}{\partial x}(r^n) + \bar{j}\frac{\partial}{\partial y}(r^n) + \bar{k}\frac{\partial}{\partial z}(r^n)$

$= \bar{i}\, n r^{n-1}\frac{\partial r}{\partial x} + \bar{j}\, n r^{n-1}\frac{\partial r}{\partial y} + \bar{k}\, n r^{n-1}\frac{\partial r}{\partial z}$

$= n r^{n-1}\left[\bar{i}\frac{\partial r}{\partial x} + \bar{j}\frac{\partial r}{\partial y} + \bar{k}\frac{\partial r}{\partial z}\right]$

Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Since $r^2 = x^2+y^2+z^2$

$\Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

$\text{grad } r^n = nr^{n-1}\left[\bar{i}\frac{x}{r} + \bar{j}\frac{y}{r} + \bar{k}\frac{z}{r}\right]$

$= nr^{n-2}\left[x\bar{i} + y\bar{j} + z\bar{k}\right]$

$\text{div}(\text{grad } r^n) = \frac{\partial}{\partial x}(nr^{n-2}x) + \frac{\partial}{\partial y}(nr^{n-2}y) + \frac{\partial}{\partial z}(nr^{n-2}z)$

$= \left(nr^{n-2} + nx(n-2)r^{n-3}\frac{\partial r}{\partial x}\right) + \left(nr^{n-2} + ny(n-2)r^{n-3}\frac{\partial r}{\partial y}\right)$

$+ \left(nr^{n-2} + nz(n-2)r^{n-3}\frac{\partial r}{\partial z}\right)$

$= 3nr^{n-2} + n(n-2)r^{n-3}\left[x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right]$

$= 3nr^{n-2} + n(n-2)r^{n-3}\left[x\left(\frac{x}{r}\right) + y\left(\frac{y}{r}\right) + z\left(\frac{z}{r}\right)\right]$

$= 3nr^{n-2} + n(n-2)r^{n-4}\left[x^2+y^2+z^2\right]$

$= 3nr^{n-2} + n(n-2)r^{n-4}\cdot r^2$

$= 3nr^{n-2} + n(n-2)r^{n-2}$

$= r^{n-2}\left[n^2+3n-2n\right]$

$= r^{n-2}\left[n^2+n\right]$

$= r^{n-2}n(n+1)$

If we put $n=-1$ then

$\text{div}(\text{grad } r^n) = -1(-1+1)r^{-1-2}$

$\nabla^2\left(\frac{1}{r}\right) = 0$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

VI : VECTOR INTEGRALS

Integration of vector Functions:

Let $\bar{f}(t)$ and $\bar{F}(t)$ be two vector functions of a scalar variable t such that $\frac{d}{dt}[\bar{F}(t)] = \bar{f}(t)$, then $\bar{F}(t)$ is called an integral of $\bar{f}(t)$ w.r.t to t and we write $\int \bar{f}(t)\,dt = \bar{F}(t)$

If $\bar{c}$ is any arbitary constant vector independent of t,

then $\frac{d}{dt}[\bar{F}(t) + \bar{c}] = \bar{f}(t)$

This is equivalent to $\int \bar{f}(t)\,dt = \bar{F}(t) + \bar{c}$

$\bar{F}(t)$ is called the indefinite integral of $\bar{f}(t)$. The constant vector $\bar{c}$ is called constant of integration and can be determined if some initial conditions are given.

The definite integral of $\bar{f}(t)$ between limits $t=a$ and $t=b$ written as $\int_a^b \bar{f}(t)\,dt = [\bar{F}(t)]_a^b = \bar{F}(b) - \bar{F}(a)$.

Note 1:- If $\bar{f}(t) = f_1(t)\,\bar{i} + f_2(t)\,\bar{j} + f_3(t)\,\bar{k}$, then $\int \bar{f}(t)\,dt = \bar{i}\int f_1(t)\,dt + \bar{j}\int f_2(t)\,dt + \bar{k}\int f_3(t)\,dt$

Note 2:- We can obtain some standard results for integration of vector functions by considering the derivatives of suitable vector functions. For example,

(i) $\frac{d}{dt}(\bar{r}\cdot\bar{a}) = \frac{d\bar{r}}{dt}\cdot\bar{a} + \bar{r}\cdot\frac{d\bar{a}}{dt} \Rightarrow \int\left(\frac{d\bar{r}}{dt}\cdot\bar{a} + \bar{r}\cdot\frac{d\bar{a}}{dt}\right)dt = \bar{r}\cdot\bar{a} + c$

Where 'c' is scalar quantity since the integrand is a scalar

(ii) $\frac{d}{dt}(\bar{r}^2) = 2\bar{r}\cdot\frac{d\bar{r}}{dt} \Rightarrow \int\left(2\bar{r}\cdot\frac{d\bar{r}}{dt}\right)dt = \bar{r}^2 + c$ where 'c' is the scalar quantity

(iii) $\frac{d}{dt}\left(\bar{r}\times\frac{d\bar{r}}{dt}\right) = \frac{d\bar{r}}{dt}\times\frac{d\bar{r}}{dt} + \bar{r}\times\frac{d^2\bar{r}}{dt^2} = \bar{r}\times\frac{d^2\bar{r}}{dt^2} \Rightarrow \int\left(\bar{r}\times\frac{d^2\bar{r}}{dt^2}\right)dt = \bar{r}\times\frac{d\bar{r}}{dt} + \bar{c}$

Here $\bar{c}$ is vector quantity since the integrand is a vector

(iv) If $\bar{a}$ is constant vector, then $\frac{d}{dt}(\bar{a}\times\bar{r}) = \frac{d\bar{r}}{dt}\times\bar{r} + \bar{a}\times\frac{d\bar{r}}{dt} = \bar{a}\times\frac{d\bar{r}}{dt}$

$\Rightarrow \int\left(\bar{a}\times\frac{d\bar{r}}{dt}\right)dt = \bar{a}\times\bar{r} + \bar{c}$ where $\bar{c}$ is vector quantity

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

① If $\bar{F}(t) = (3t^2 - 2t)\bar{i} + (6t-4)\bar{j} + 4t\bar{k}$, evaluate $\int_2^3 \bar{F}(t)\,dt$.

$$\int_2^3 \bar{F}(t)\,dt = \int_2^3 \left[(3t^2-2t)\bar{i} + (6t-4)\bar{j} + 4t\bar{k}\right] dt$$

$$= \bar{i}\int_2^3 (3t^2-2t)\,dt + \bar{j}\int_2^3 (6t-4)\,dt + \bar{k}\int_2^3 4t\,dt$$

$$= \bar{i}\left[t^3 - t^2\right]_2^3 + \bar{j}\left[3t^2 - 4t\right]_2^3 + \bar{k}\left[2t^2\right]_2^3$$

$$= 14\bar{i} + 11\bar{j} + 10\bar{k}$$

④ If $\bar{r}(t) = \begin{cases} 2\bar{i} - \bar{j} + 2\bar{k} & \text{when } t=2 \\ 4\bar{i} - 2\bar{j} + 3\bar{k} & \text{when } t=3. \end{cases}$ Show that $\int_2^3 \left(\bar{r}\cdot\frac{d\bar{r}}{dt}\right) dt = 10$.

Sol:- Since $\frac{d}{dt}\left[\bar{r}^2\right] = 2\bar{r}\cdot\frac{d\bar{r}}{dt}$

$$\int_2^3 \left(\bar{r}\cdot\frac{d\bar{r}}{dt}\right) dt = \frac{1}{2}\left[\bar{r}^2\right]_2^3 \quad \text{—①}$$

when $t=3$, $\bar{r}^2 = (4\bar{i} - 2\bar{j} + 3\bar{k})\cdot(4\bar{i} - 2\bar{j} + 3\bar{k}) = 16 + 4 + 9 = 29$

when $t=2$, $\bar{r}^2 = (2\bar{i} - \bar{j} + 2\bar{k})\cdot(2\bar{i} - \bar{j} + 2\bar{k}) = 4 + 1 + 4 = 9$

From ①, we have $\int_2^3 \left(\bar{r}\cdot\frac{d\bar{r}}{dt}\right) dt = \frac{1}{2}\left[29 - 9\right] = 10$.

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

(3) Given $\bar{R}(t) = 5t^2\bar{i} + t\bar{j} - t^3\bar{k}$, Prove that $\int_1^2 \left(\bar{R} \times \frac{d^2\bar{R}}{dt^2}\right) dt = -14\bar{i} + 75\bar{j} - 15\bar{k}$.

Sol:- $\frac{d}{dt}\left[\bar{R} \times \frac{d\bar{R}}{dt}\right] = \frac{d\bar{R}}{dt} \times \frac{d\bar{R}}{dt} + \bar{R} \times \frac{d^2\bar{R}}{dt^2} = \bar{R} \times \frac{d^2\bar{R}}{dt^2}$

$$\int_1^2 \left(\bar{R} \times \frac{d^2\bar{R}}{dt^2}\right) dt = \left[\bar{R} \times \frac{d\bar{R}}{dt}\right]_1^2 \leftarrow (1)$$

Let us now find $\bar{R} \times \frac{d\bar{R}}{dt} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 5t^2 & t & -t^3 \\ 10t & 1 & -3t^2 \end{vmatrix}$.

$$= \bar{i}\left[-3t^3 + t^3\right] + \bar{j}\left[-10t^4 + 15t^4\right] + \bar{k}\left[5t^2 - 10t^2\right)$$

$$= -2t^3\bar{i} + 5t^4\bar{j} - 5t^2\bar{k}$$

From (1) we have $\int_1^2 \left(\bar{R} \times \frac{d^2\bar{R}}{dt^2}\right) dt = \left[-2t^3\bar{i} + 5t^4\bar{j} - 5t^2\bar{k}\right]_1^2$.

$$= \left[-2t^3\right]_1^2 \bar{i} + \left[5t^4\right]_1^2 \bar{j} - \left[5t^2\bar{k}\right]_1^2$$

$$= -14\bar{i} + 75\bar{j} - 15\bar{k}$$

) The acceleration of a particle at any time $t>0$ is given by $\bar{a} = \frac{d\bar{v}}{dt} = 12\cos 2t\,\bar{i} - 8\sin 2t\,\bar{j} + 16t\bar{k}$. If the velocity $\bar{v}$ and the displacement $\bar{r}$ are zero at $t=0$, find $\bar{v}$ and $\bar{r}$ at any time.

Sol:- $\bar{a} = \frac{d\bar{v}}{dt} = 12\cos 2t\,\bar{i} - 8\sin 2t\,\bar{j} + 16t\,\bar{k}$

$$\bar{v} = \int \frac{d\bar{v}}{dt}\,dt = \bar{i}\int 12\cos 2t\,dt - \bar{j}\int 8\sin 2t + \bar{k}\int 16t\,dt.$$

$$\bar{v} = 6\sin 2t\,\bar{i} + 4\cos 2t\,\bar{j} + 8t^2\bar{k} + \bar{c}_1$$

when $t=0$, $\bar{v}=0$ $\quad 0 = 4\bar{j} + \bar{c}_1 \Rightarrow \bar{c}_1 = -4\bar{j}$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Then $\bar{v} = 6\sin 2t\,\bar{i} + 4\cos 2t\,\bar{j} + 8t^2\bar{k} - 4\bar{j}$

$$\frac{d\bar{r}}{dt} = \bar{v} = 6\sin 2t\,\bar{i} + (4\cos 2t - 4)\bar{j} + 8t^2\bar{k}$$

~~So dr~~

Integration $\bar{r} = \int \frac{d\bar{r}}{dt}\,dt = \bar{i}\int 6\sin 2t\,dt + \bar{j}\int (4\cos 2t - 4)\,dt + \bar{k}\int 8t^2\,dt$

$$\bar{r} = -3\cos 2t\,\bar{i} + (2\sin 2t - 4t)\bar{j} + \frac{8}{3}t^3 + \bar{c}_2$$

when $t = 0$, $\bar{r} = 0$.

$$0 = -3\bar{i} + \bar{c}_2 \Rightarrow \bar{c}_2 = 3\bar{i}$$

$$\bar{r} = -3\cos 2t\,\bar{i} + (2\sin 2t - 4t)\bar{j} + \frac{8}{3}t^3 + 3\bar{i}$$

$$\bar{r} = (3 - 3\cos 2t)\,\bar{i} + (2\sin 2t - 4t)\bar{j} + \frac{8}{3}t^3\bar{k}$$

2

$\bar{a} = 18\cos 3t\,\bar{i} - 8\sin 2t\,\bar{j} + 6t\,\bar{k}$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com

In electromagnetic theory, we have $\nabla \cdot D = \rho$, $\nabla \times E = -\frac{1}{c}\frac{\partial H}{\partial t}$

$\nabla \times H = \frac{1}{c}\left(\rho v + \frac{\partial D}{\partial t}\right)$ Prove that.

(i) $\nabla^2 D - \frac{1}{c^2}\frac{\partial^2 D}{\partial t^2} = \nabla \rho + \frac{1}{c^2}\frac{\partial}{\partial t}(\rho v)$ (ii) $\nabla^2 H - \frac{1}{c^2}\frac{\partial^2 H}{\partial t^2} = -\frac{1}{c}\nabla \times \rho v$

Sol:- (i) we have $\frac{1}{c^2}\left[\frac{\partial^2 D}{\partial t^2} + \frac{\partial}{\partial t}(\rho v)\right] = \frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial D}{\partial t} + \rho v\right)$

$= \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times H) = \frac{1}{c}\left(\nabla \times \frac{\partial H}{\partial t}\right)$

$= -\nabla \times (\nabla \times D)$

$= -[(\nabla \cdot D)\nabla - (\nabla \cdot \nabla)D]$

$= -(\nabla \cdot D)\nabla + \nabla^2 D$

$= -\nabla \rho + \nabla^2 D$

Hence $\nabla^2 D - \frac{1}{c^2}\frac{\partial^2 D}{\partial t^2} = \nabla \rho + \frac{1}{c^2}\frac{\partial}{\partial t}(\rho v)$

(ii) $L.H.S = \nabla^2 H - \frac{1}{c^2}\frac{\partial^2 H}{\partial t^2} = \nabla^2 H + \frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial H}{\partial t}\right)$

$= \nabla^2 H + \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times D)$

$= \nabla^2 H + \frac{1}{c}\left(\nabla \times \frac{\partial D}{\partial t}\right)$

$= \nabla^2 H + \nabla \times \left(\nabla \times H - \frac{1}{c}\rho v\right)$

$= \nabla^2 H + \nabla \times (\nabla \times H) - \frac{1}{c}\nabla \times \rho v$

$= \nabla^2 H + (\nabla \cdot H)\nabla - (\nabla \cdot \nabla)H - \frac{1}{c}\nabla \times \rho v$

$= \nabla^2 H - 0 - \nabla^2 H - \frac{1}{c}\nabla \times \rho v$

$= -\frac{1}{c}\nabla \times \rho v = R.H.S.$

Hence $\nabla^2 H - \frac{1}{c^2}\frac{\partial^2 H}{\partial t^2} = -\frac{1}{c}\nabla \times \rho v$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

2) Prove that $\bar{A}\cdot\nabla(\bar{B}\cdot\nabla\frac{1}{r}) = \frac{3(\bar{A}\cdot\bar{r})(\bar{B}\cdot\bar{r})}{r^5} - \frac{\bar{A}\cdot\bar{B}}{r^3}$ where $\bar{A}$ and $\bar{B}$ are constant vectors.

Sol:- $\nabla\frac{1}{r} = \nabla(r^{-1}) = -r^{-2}\frac{\partial r}{\partial x}\hat{i} + \left(-r^{-2}\frac{\partial r}{\partial y}\right)\hat{j} + \left(-r^{-2}\frac{\partial r}{\partial z}\right)\hat{k}$

$= -\frac{x}{r^3}\hat{i} - \frac{y}{r^3}\hat{j} - \frac{z}{r^3}\hat{k}$

$= -r^{-3}(x\hat{i}+y\hat{j}+z\hat{k}) = -\frac{\bar{r}}{r^3}$

$r^2 = x^2+y^2+z^2$

$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

$\nabla(\bar{B}\cdot\nabla\frac{1}{r}) = \nabla\left(\bar{B}\cdot\left(-\frac{\bar{r}}{r^3}\right)\right)$

$= -\nabla\left[\frac{\bar{B}\cdot\bar{r}}{r^3}\right]$

$= -\nabla\left(\frac{1}{r^3}\right)(\bar{B}\cdot\bar{r}) - \frac{1}{r^3}\nabla(\bar{B}\cdot\bar{r})$

$\nabla(\bar{B}\cdot\bar{r}) = \bar{B}$

$\Rightarrow \frac{3}{r^5}\bar{r}(\bar{B}\cdot\bar{r}) - \frac{1}{r^3}\bar{B}$

$\bar{A}\cdot\nabla(\bar{B}\cdot\nabla\frac{1}{r}) = \frac{3}{r^5}(\bar{B}\cdot\bar{r})(\bar{A}\cdot\bar{r}) - \frac{1}{r^3}(\bar{A}\cdot\bar{B})$

Line integrals :- Any integral which is to be evaluate along a curve is called a line integral.

The line integral of $\bar{F}$ along the curve 'c' is defined as $\int_c \bar{F}\cdot d\bar{r}$

Note:- $\int_c \bar{F}\cdot d\bar{r} = \int_c (F_1\hat{i}+F_2\hat{j}+F_3\hat{k})\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k})$

$= \int_c F_1\,dx + F_2\,dy + F_3\,dz$

(2) $\int_a^b \bar{q}\cdot d\bar{r}$ is called line integral if along the curve 'c'.

circulation :- Let 'c' be a simple closed curve, the line integral of $\bar{F}$ along 'c' is called circulation $\bar{F}$ i.e. $\oint \bar{F}\cdot d\bar{r}$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

1) If $\bar{F} = 3xy\,\hat{i} - y^2\hat{j}$, evaluate $\int_c \bar{F}\cdot d\bar{r}$ where 'c' is the curve in the xy-plane $y = 2x^2$ from (0,0) to (1,2).

Sol:- Since the particle moves in the xy-plane ($z=0$) we take $\bar{r} = x\hat{i} + y\hat{j}$. Then $\int_c \bar{F}\cdot d\bar{r}$, where 'c' is the parabola $y = 2x^2$

$$\int_c \bar{F}\cdot d\bar{r} = \int_c (3xy\,\hat{i} - y^2\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j}) = \int_c (3xy\,dx - y^2 dy) \quad -(1)$$

Substituting $y = 2x^2$ where x goes from 0 to 1,

(1) becomes $\int_{x=0}^{1} 3x(2x^2)\,dx - (2x^2)^2\,d(2x^2) = \int_0^1 (6x^3 - 16x^5)\,dx = -7/6$

2) A vector field is given by $\bar{F} = \sin y\,\hat{i} + x(1+\cos y)\,\hat{j}$. Evaluate the line integral over a circular path given by $x^2 + y^2 = a^2$, $z = 0$.

Sol:- As the particles moves in xy-plane ($z=0$), let $\bar{r} = x\hat{i} + y\hat{j}$ so that $d\bar{r} = dx\,\hat{i} + dy\,\hat{j}$. Also the circular path is $x = a\cos t$, $y = a\sin t$, $z = 0$, where 't' varies from 0 to 2π.

$$\oint_c \bar{F}\cdot d\bar{r} = \oint_c (\sin y\,\hat{i} + x(1+\cos y)\,\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j})$$

$$= \oint_c \sin y\,dx + x(1+\cos y)\,dy = \oint_c (\sin y\,dx + x\cos y\,dy + x\,dy)$$

$$= \oint_c [d(x\sin y) + x\,dy] = \int_0^{2\pi} d(a\cos t\sin(a\sin t)) + a^2\cos^2 t\,dt$$

$$= [a\cos t\sin(a\sin t)]_0^{2\pi} + \frac{a^2}{2}\int_0^{2\pi} 1+\cos 2t\,dt$$

$$= \frac{a^2}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} = \pi a^2$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

3). Find the work done in moving a particle in the force field $\bar{F} = 3x^2\hat{i} + (2xz-y)\hat{j} + z\hat{k}$ along (a) the straight line from (0,0,0) to (2,1,3).

(b) the curve defined by $x^2 = 4y$, $3x^3 = 8z$ from $x=0$ to $x=2$.

Sol:- $\int_C \bar{F}\cdot d\bar{r} = \int_C (3x^2\hat{i} + (2xz-y)\hat{j} + z\hat{k})\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$

$= \int_C 3x^2dx + (2xz-y)dy + zdz.$

(a) The equations of the straight line from (0,0,0) to (2,1,3) are $\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$ say.

$\therefore x = 2t,\ y = t,\ z = 3t$ are it's parametric equations. The points (0,0,0) and (2,1,3) corresponding to $t=0$ and $t=1$ respectively.

$\therefore$ work done $= \int_C \bar{F}\cdot d\bar{r} = \int_0^1 3(2t)^2 2dt + ((4t)(3t) - t)dt + (3t)3dt.$

$= \int_0^1 (36t^2 + 8t)dt = 16.$

(b) Let $x = t$ in $x^2 = 4y$, $3x^3 = 8z$. Then the parametric equations of 'c' are $x = t,\ y = t^2/4,\ z = 3t^3/8$ and 't' varies from 0 to 2.

$\therefore$ work done $= \int_C \bar{F}\cdot d\bar{r} = \int_0^2 3t^2dt + \left\{2t\left(\frac{3t^3}{8}\right) - \frac{t^2}{4}\right\}d\left(\frac{t^2}{4}\right) + \frac{3t^3}{8}d\left(\frac{3t^3}{8}\right)$

$= \int_0^2 \left(3t^2 - \frac{t^3}{8} + \frac{51}{64}t^5\right)dt$

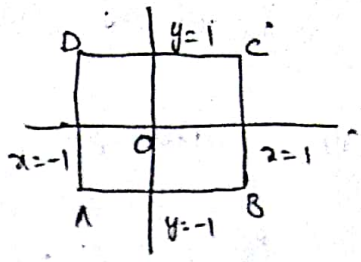
$= t^3 - \frac{t^4}{32} + \frac{17}{128}t^6\Big]_0^2 = 16.$

2.

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

① Evaluate the line integral $\int_C (x^2+xy)\,dx + (x^2+y^2)\,dy$ where C is the square formed by the lines $y = \pm 1$ and $x = \pm 1$.



Sol:- The given vector

$$\bar{F} = (x^2+xy)\,\bar{i} + (x^2+y^2)\,\bar{j}$$

To evaluate $\int_C (x^2+xy)\,dx + (x^2+y^2)\,dy$

C is the square formed by the lines $y = \pm 1$ and $x = \pm 1$.

The square is ABCD.

Along AB, x varies from -1 to 1.

$y = -1$ and $dy = 0$. $\Rightarrow \int_C (x^2-x)\,dx = \int_{-1}^{1} x^2\,dx - \int_{-1}^{1} x\,dx$

Along BC, $x = 1$, $dx = 0$ y varies -1 to 1. $= \left[\frac{x^3}{3}\right]_{-1}^{1} - \left[\frac{x^2}{2}\right]_{-1}^{1}$

$\int_{BC} \bar{F}\cdot d\bar{r} = \int_{-1}^{1} (1+y^2)\,dy = \left[y + \frac{y^3}{3}\right]_{-1}^{1}$ $= \left(\frac{1}{3}+\frac{1}{3}\right) - \left[\frac{1}{2}-\frac{1}{2}\right]$

$= [1+1] + \left[\frac{1}{3}+\frac{1}{3}\right] = 2+\frac{2}{3} = \frac{8}{3}$ $= \frac{2}{3}$

Along CD, $y = 1$, $dy = 0$, x varies from 1 to -1.

$\int_1^{-1} (x^2+x)\,dx = 0 \Rightarrow \int_{+1}^{-1} x^2\,dx = \left[\frac{x^3}{3}\right]_{+1}^{-1} = -\frac{1}{3} - \frac{1}{3} = -\frac{2}{3}$

Along DA, $x = -1$, $dx = 0$, y varies from 1 to -1

$\int_1^{-1} (1+y^2)\,dy = \left[y + \frac{y^3}{3}\right]_1^{-1} = \left[-1-\frac{1}{3}\right] + \left[\frac{1}{3}-\frac{1}{3}\right] = -\frac{8}{3}$

$\int_C (x^2+xy)\,dx + (x^2+y^2)\,dy = \int_{AB} (x^2+xy)\,dx + (x^2+y^2)\,dy + \int_{BC} (x^2+xy)\,dx + (x^2+y^2)\,dy + \int_{CD} (x^2+xy)\,dx + (x^2+y^2)\,dy$

$+ \int_{DA} (x^2+xy)\,dx + (x^2+y^2)\,dy$

$= \frac{2}{3} + \frac{8}{3} - \frac{2}{3} - \frac{8}{3} = 0$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

2) If $\bar{F} = (5xy - 6x^2)\,\hat{i} + (2y - 4x)\,\hat{j}$, evaluate $\int_C \bar{F}\cdot d\bar{r}$ along the curve 'C' in the xy plane $y = x^3$ from the point (1, 1) to (2, 8).

Sol:- The given function $\bar{F} = (5xy - 6x^2)\,\hat{i} + (2y - 4x)\,\hat{j}$

$y = x^3 \Rightarrow dy = 3x^2\,dx$

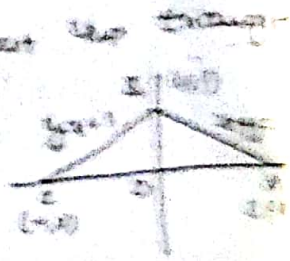
$$\int_C \bar{F}\cdot d\bar{r} = \int_C (5xy - 6x^2)\,dx + (2y - 4x)\,dy$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_1^2 (5x\cdot x^3 - 6x^2)\,dx + (2x^3 - 4x)\,3x^2\,dx$$

$$= \frac{5x^5}{5} - \frac{6x^3}{3} + \frac{6x^6}{6} - \frac{12x^4}{4}\Big]_1^2$$

$$= x^5 - 2x^3 + x^6 - 3x^4\Big]_1^2 = 32 - 16 + 64 - 48 - 1 + 2 - 1 + 3 = 35$$

3) Compute the line integral $\int_C y^2\,dx - x^2\,dy$ about the triangle whose vertices are (1,0) (0,1) and (-1,0)



Sol:- Let the points (1,0) (0,1) and (-1,0) be A, B and C respectively.

$$\int_C y^2\,dx - x^2\,dy = \int_{AB} y^2\,dx - x^2\,dy + \int_{BC} y^2\,dx - x^2\,dy + \int_{CA} y^2\,dx - x^2\,dy$$

$$= I_1 + I_2 + I_3.$$

$I_1 = \int_{AB} y^2\,dx - x^2\,dy$. The equation of the line AB is $x + y = 1$

$y = 1 - x \Rightarrow dy = -dx$

x varies from 1 to 0

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice

$$\int_1^0 (1+x)^2 dx + x^2 \int dx = \int_1^0 (1+2x+x^2)dx = \int_1^0 (1+2x+x^2)dx = -\frac{2}{3}$$

www.FirstRanker.com www.FirstRanker.com

$I_2 = \int_{Bc} y^2\,dx - x^2\,dy$, the equation of the line Bc is $y-x=1$

$y = 1+x$, $dy = dx$, x varies from 0 to -1.

$$I_2 = \int_0^{-1} [(1+x)^2 - x^2]\,dx = \int_0^{-1} (1+2x)\,dx = \left[x + x\frac{x^2}{2}\right]_0^{-1}$$

$$= -1+1 = 0.$$

$$I_3 = \int_{CA} y^2\,dx - x^2\,dy, \quad y=0,\ dy=0 \quad I_3 = 0.$$

$$\therefore \int_C y^2\,dx - x^2\,dy = -\frac{2}{3} \text{ along } ABCA.$$

4) If $\bar{F} = (3x^2+6y)\bar{i} - 14yz\bar{j} + 20xz^2\bar{k}$ evaluate $\int_C \bar{F}\cdot d\bar{r}$ from $(0,0,0)$ to $(1,1,1)$ along the path $x=t$, $y=t^2$, $z=t^3$.

Sol:- The given function $\bar{F} = (3x^2+6y)\bar{i} - 14yz\bar{j} + 20xz^2\bar{k}$.

The limits are $(0,0,0)$ to $(1,1,1)$, $x=0$ to 1, $y=0$ to 1, $z=0$ to 1

Given that $x=t$, $y=t^2$, $z=t^3$ therefore the limits t are 0 and 1

$dx = dt$, $dy = 2t\,dt$, $dz = 3t^2\,dt$

$$\int_C \bar{F}\,d\bar{r} = \int_0^1 (3x^2+6y)\,dt - (14yz)\,2t\,dt + (20xz^2)3t^2\,dt$$

$$= \int_0^1 (3t^2+6t^2)\,dt - (14t^5)2t\,dt + (20t^7)3t^2\,dt.$$

$$= \int_0^1 9t^2 - 28t^6\,dt + 60t^9\,dt = \int_0^1 (9t^2+32t^9)\,dt.$$

$$= 9\frac{t^3}{3} + 32\frac{t^7}{7}\Big]_0^1$$

$$= 3 + \frac{32}{7} = \frac{53}{7}$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

5) Using the line integral compute the work done by the force $\bar{F} = (2y+3)\bar{i} + xz\bar{j} + (yz-x)\bar{k}$ when it moves a particle from the point (0,0,0) to the the point (2,1,1) along the curve $x = 2t^2$, $y = t$, $z = t^3$.

Sol: $\bar{F} = (2y+3)\bar{i} + xz\bar{j} + (yz-x)\bar{k}$

$d\bar{R} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$$\int_C \bar{F}\cdot d\bar{R} = \int (2y+3)dx + \int xz\,dy + \int (yz-x)\,dz$$

$$= \int_0^1 (2t+3)\,4t\,dt + \int_0^1 (2t^3)\,dt + \int_0^1 (t^4 - 2t^2)\,3t^2\,dt$$

$$= \int_0^1 (8t^2+12t)\,dt + \int_0^1 2t^5\,dt + \int_0^1 (3t^6 - 6t^4)\,dt$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^1 (8t^2+12t)\,dt + \int_0^1 2t^5\,dt + \int_0^1 (3t^6 - 6t^4)\,dt$$

$$= \frac{8t^3}{3} + \frac{12t^2}{2} + \frac{2t^6}{6} + \frac{3t^7}{7} - \frac{6t^5}{5}\Big]_0^1$$

$$= \frac{8}{3} + 6 + \frac{1}{3} + \frac{3}{7} - \frac{6}{5} = 9 + \frac{3}{7} - \frac{6}{5} = \frac{315+15-42}{35}$$

$$= \frac{288}{35}$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

(6) If $\bar{F} = 2yi - zj + xk$, evaluate $\int_C \bar{F} \times d\bar{r}$ along the curve $x = \cos t$, $y = \sin t$, $z = 2\cos t$ from $t = 0$ to $t = \pi/2$

Sol: $\bar{F} = 2yi - zj + xk$, $d\bar{r} = dx\,i + dy\,j + dz\,k$

$$\bar{F} \times d\bar{r} = \begin{vmatrix} i & j & k \\ 2y & -z & x \\ dx & dy & dz \end{vmatrix}$$

$$= i(-z\,dz - x\,dy) + j(x\,dx - 2y\,dz) + k(2y\,dy + z\,dx)$$

$$\int \bar{F} \times d\bar{r} = \int_0^{\pi/2} \left[\cos t(\cos t\,dt + 2\cos t(-\sin t\,dt)\right] i + \int_0^{\pi/2} \left[\cos t(-\sin t\,dt) - 2\sin t(-2\sin t)\,dt\right] j$$

$$+ \int_0^{\pi/2} \left[2\sin t\cos t\,dt + 2\cos t(-\sin t\,dt)\right] k$$

$$= \int_0^{\pi/2} \left[-\cos^2 t\,dt + 2\sin 2t\,dt\right] i + \int_0^{\pi/2} \left(\frac{\sin 2t}{2} + 4\sin^2 t\right) dt\, j$$

$$+ \int_0^{\pi/2} (\sin 2t - \sin 2t)\,dt\, k$$

$$= i\int_0^{\pi/2} -\frac{1+\cos 2t}{2}\,dt + 2\sin 2t\,dt + j\int_0^{\pi/2} \left[\frac{-\sin 2t}{2} + 4\left(\frac{1-\cos 2t}{2}\right)\right] dt$$

$$= i\left[-\frac{1}{2}t - \frac{\sin 2t}{4} - \cos 2t\right]_0^{\pi/2} + j\left[\frac{\cos 2t}{4} + 2t - \frac{2\sin 2t}{2}\right]_0^{\pi/2}$$

$$= \left[-\frac{\pi}{4} + 0 + 1 - 1\right] i + \left[\frac{1}{4} + \pi + 0 - \frac{1}{4}\right] j$$

$$= \left[-\frac{\pi}{4}\right] i + \left[\pi\right] j$$

Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Surface Integrals :- Let $\bar{F}$ be a continuous vector point function defined over the smooth surface

Surface Integrals :- Any integral which is to be evaluated over a surface is called a surface Integral.

Then the surface integral of $\bar{F}$ over S is defined as $\int_S \bar{F} \cdot \bar{N}\, ds$

where $\bar{N}$ is the unit outward normal to S.

Cartesian form of surface Integrals :- Let $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$\bar{N} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$ where $\cos\alpha, \cos\beta, \cos\gamma$ are the d.c's of $\bar{N}$

$\bar{F}\cdot\bar{N} = F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma$.

Now $\int_S \bar{F}\cdot\bar{N}\, ds = \int_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\, ds$.

$= \int_S F_1\cos\alpha\, ds + F_2\cos\beta\, ds + F_3\cos\gamma\, ds$

$\int_S \bar{F}\cdot\bar{N}\, ds = \int_S F_1\, dy\,dz + F_2\, dz\,dx + F_3\, dx\,dy$.

Flux :- If 'S' be a closed surface then $\oint_S \bar{F}\cdot\bar{N}\, ds$ is called the flux over 'S'.

Note 1 :- (1) Let R_1 be the Projection of 'S' on xy-plane then $\int_S \bar{F}\cdot\bar{N}\, ds = \int_{R_1} \bar{F}\cdot\bar{N}\, \frac{dx\,dy}{|\cos\gamma|} = \int_{R_1} \bar{F}\cdot\bar{N}\, \frac{dx\,dy}{|\bar{N}\cdot\bar{k}|}$

(2) Let R_2 be the projection of 'S' on yz-plane then $\int_S \bar{F}\cdot\bar{N}\, ds = \int_{R_2} \bar{F}\cdot\bar{N}\, \frac{dy\,dz}{|\cos\alpha|} = \int_{R_2} \bar{F}\cdot\bar{N}\, \frac{dy\,dz}{|\bar{N}\cdot\bar{i}|}$

(3) Let R_3 be the Projection of zx-plane then $\int_S \bar{F}\cdot\bar{N}\, ds = \int_{R_3} \bar{F}\cdot\bar{N}\, \frac{dz\,dx}{|\cos\beta|} = \int_{R_3} \bar{F}\cdot\bar{N}\, \frac{dz\,dx}{|\bar{N}\cdot\bar{j}|}$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

ce Integrals : Any integral which is to be evaluated over a surface is called a surface integral.

Suppose 's' is a two sided surface, consider one side as positive. If 's' is a closed surface we choose the outside to be positive. Let $\bar{F}$ be a vector point function. Let ds be an element of area in the surface and $\bar{n}$ be the unit vector normal to the surface, drawn on the positive side. Let 'θ' be the angle between $\bar{F}$ and $\bar{n}$. The normal component of $\bar{F}$ is given by the dot product $\bar{F} = \bar{F}\cos\theta$. The integral of this normal component i.e of $\bar{F}\cdot\bar{n}$ through an element of area ds over the surface 's' is called the surface integral of $\bar{F}$ over S.

So the surface integral is $\int_S \bar{F}\cdot\bar{n}\,ds$.

If $d\bar{s}$ is a vector whose in magnitude is ds and whose direction is that of $\bar{n}$. then $d\bar{s} = \bar{n}\,ds$.

Hence surface integral is $\int_S \bar{F}\cdot d\bar{s}$.

Other surface integrals are $\int \phi\,ds$ or $\int_S \bar{F}\times d\bar{s}$

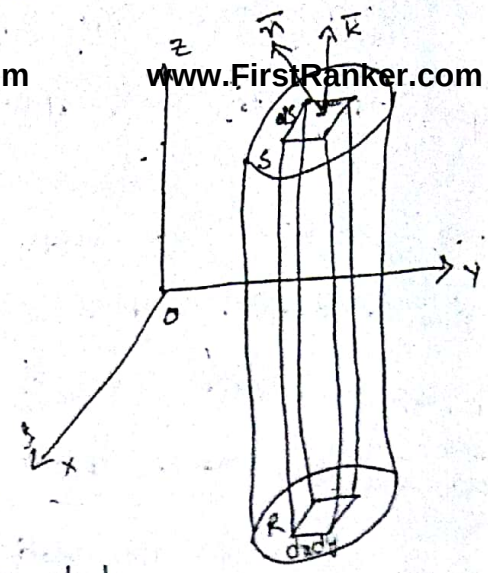
To evaluate surface integrals it is convenient to express them as double integrals taken over the projected area of the surfaces on one of the coordinate planes.

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Let $\bar{n} = \cos\alpha\,\bar{i} + \cos\beta\,\bar{j} + \cos\gamma\,\bar{k}$

where $\cos\alpha, \cos\beta, \cos\gamma$ are the d.c's of $\bar{n}$



$ds\cos\alpha$, $ds\cos\beta$, $ds\cos\gamma$ are the orthogonal projections of ds on the coordinate planes $x=0, y=0, z=0$ respectively.

$ds\cos\alpha = dy\,dz$, $ds\cos\beta = dz\,dx$, $ds\cos\gamma = dx\,dy$.

Considering subdivisions formed by planes parallel to the coordinate pla

Forming dot products $\bar{n}$ and unit vectors $\bar{i}, \bar{j}, \bar{k}$

$|\bar{i}\cdot\bar{n}| = \cos\alpha$, $|\bar{j}\cdot\bar{n}| = \cos\beta$, $|\bar{k}\cdot\bar{n}| = \cos\gamma$.

$$ds = \frac{dy\,dz}{\cos\alpha} = \frac{dy\,dz}{|\bar{i}\cdot\bar{n}|} \quad ; \quad ds = \frac{dz\,dx}{\cos\beta} = \frac{dz\,dx}{|\bar{j}\cdot\bar{n}|} \quad , \quad ds = \frac{dx\,dy}{|\bar{k}\cdot\bar{n}|}$$

Surface integral $\iint_S \bar{F}\cdot\bar{n}\,ds$.

$$\iint_S \bar{F}\bar{n}\,ds = \iint_{S_1} \bar{F}\cdot\bar{n}\,\frac{dy\,dz}{\bar{i}\cdot\bar{n}} = \iint_{S_2} \bar{F}\cdot\bar{n}\,\frac{dz\,dx}{\bar{j}\cdot\bar{n}} = \iint_{S_3} \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{\bar{k}\cdot\bar{n}}$$

where S_1, S_2, S_3 are the projections of S on the planes

$x=0, y=0, z=0$ respectively

2

cost

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

1) Evaluate $\int_S \bar{F}\cdot\bar{N}\,ds$ where $\bar{F} = z\bar{i} + x\bar{j} - 3y^2z\,\bar{k}$ and 's' is the surface $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$.

Sol:- $\bar{F} = z\bar{i} + x\bar{j} - 3y^2z\,\bar{k}$

Here 's' is the surface $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$.

$\bar{N}$ = unit outward normal to $S = \frac{\nabla\phi}{|\nabla\phi|}$ where $\phi = x^2+y^2=16$

$$\bar{N} = \frac{2x\bar{i}+2y\bar{j}}{\sqrt{4x^2+4y^2}} = \frac{x\bar{i}+y\bar{j}}{4}$$

$x^2+y^2=16$

$z = 0$ to 5.

$y^2 = 16-x^2$

$y = \sqrt{16-x^2}$

$y = \sqrt{16} = 4.$

$y = 0$ to 4.

Let 'R' be the projection of S on yz-plane

$$\int_S \bar{F}\cdot\bar{N}\,ds = \int_R \bar{F}\cdot\bar{N}\,\frac{dy\,dz}{|\bar{N}\cdot\bar{i}|}$$

$$= \int_R (z\bar{i}+x\bar{j}-3y^2z\,\bar{k})\cdot\left(\frac{x\bar{i}+y\bar{j}}{4}\right)\frac{dy\,dz}{\left|\frac{x\bar{i}+y\bar{j}}{4}\cdot\bar{i}\right|}$$

$$= \int_R (2x+xy)\,\frac{dy\,dz}{x}$$

$$= \int_{z=0}^{5}\int_{y=0}^{4} (2+y)\,dy\,dz = \int_{z=0}^{5}\left[2y+\frac{y^2}{2}\right]_0^4 dz$$

$$= \int_0^5 (8+8)\,dz = \left[\frac{16z}{2}+8z\right]_0^5$$

$$= 2\times25 + 8\times5$$

$$= 90$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

2) Evaluate $\int_S \bar{F}\cdot\bar{N}\,ds$ where $\bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$ and 'S' is the part of the plane $2x+3y+6z=12$ located in the first octant.

Sol:- $\bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$.

Here 'S' is the part of the plane $2x+3y+6z=12$ located in the first octant.

$\bar{N}$ = unit outward normal to $S = \frac{\nabla\phi}{|\nabla\phi|}$ where $\phi = 2x+3y+6z-12$

$$\bar{N} = \frac{2\bar{i}+3\bar{j}+6\bar{k}}{\sqrt{4+9+36}} = \frac{1}{7}(2\bar{i}+3\bar{j}+6\bar{k})$$

Let 'R' be the projection of on xy-plane then

$$\int_S \bar{F}\cdot\bar{N}\,ds = \int_R \bar{F}\cdot\bar{N}\,\frac{dx\,dy}{|\bar{N}\cdot\bar{k}|}$$

$2x+3y+6z=12$

$6z = 12-2x-3y$

$z = \frac{12-2x-3y}{6}$

$3y = 12-2x$

$y = \frac{12-2x}{3}$

$x = 0$ to 6

$$\int_S \bar{F}\cdot\bar{N}\,ds = \int_R (18z\bar{i}-12\bar{j}+3y\bar{k})\cdot\frac{1}{7}(2\bar{i}+3\bar{j}+6\bar{k})\cdot\frac{dx\,dy}{|\frac{1}{7}(2\bar{i}+3\bar{j}+6\bar{k})\cdot\bar{k}|}$$

$$= \int_R (36z-36+18y)\,\frac{dx\,dy}{6} = \int_R (6z-6+3y)\,dx\,dy$$

$$= \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} \left[6\left(\frac{12-2x-3y}{6}\right) - 6 + 3y\right] dy\,dx$$

$$= \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} [(6-2x)\,dy]\,dx = \int_{x=0}^{6} [6y - 2xy]_0^{\frac{12-2x}{3}}\,dx$$

$$= \int_{x=0}^{6} \left(6\left(\frac{12-2x}{3}\right) - 2x\left[\frac{12-2x}{3}\right]\right) dx$$

$$= \int_{x=0}^{6} \left(24 - 12x + \frac{4x^2}{3}\right) dx = \left[24x - \frac{12x^2}{2} + \frac{4x^3}{9}\right]_0^6$$

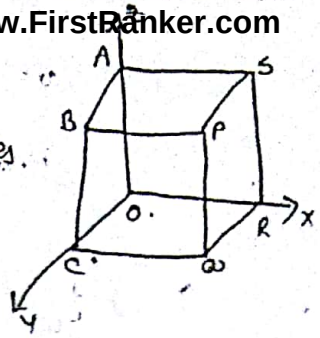
$$= 24\times6 - 6\times36 - \frac{4\times6^3}{9}$$

$$= 144 - 216 + 96 = 24$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$= 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$, evaluate $\int_S \bar{F}\cdot\bar{N}\,ds$, where 'S' is the surface of the bounded by $x=0,\ x=a,\ y=0,\ y=a,\ z=0,\ z=a$.



Sol:- consider the cube surrounded by the following faces.

(i) For the face PQRS, $\hat{i}$ is the outward normal

$\bar{N} = \hat{i}$; $x=a$, $ds = dy\,dz$

$$\int_S \bar{F}\cdot\bar{N}\,ds = \int_{y=0}^{a}\int_{z=0}^{a} (4xz\,\hat{i} - y^2\hat{j} + yz\,\hat{k})\cdot\hat{i}\,dy\,dz = \int_{y=0}^{a}\int_{z=0}^{a} 4xz\,dy\,dz$$

$$= \int_{y=0}^{a} 4a\left[\frac{z^2}{2}\right]_0^a dy = \int_{y=0}^{a} 4a\,\frac{a^2}{2}\,dy = \int_{y=0}^{a} 2a^3\,dy = 2a^4$$

$\because x = a$ on this face

$4a\,[y]_0^a \left[\frac{z^2}{2}\right]_0^a = 4a^2\cdot\frac{a^2}{2} = 2a^4$

(ii) For the face OABC, $-\hat{i}$ is the outward normal.

$\bar{N} = -\hat{i}$, $x=0$, and $ds = dy\,dz$. $\because x=0$ on this face

$$\int_{R_2} \bar{F}\cdot\bar{N}\,ds = \iint_{R_2} (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\cdot(-\hat{i})\,dy\,dz = -\iint_{R_2} -4xz\,dy\,dz = 0.$$

(iii) For the face BPQC, $\hat{j}$ is the outward normal.

$\bar{N} = \hat{j}$, $y=a$, and $ds = dx\,dz$

$$\int_{R_3} \bar{F}\cdot\bar{N}\,ds = \iint_{R_3} (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\cdot\hat{j}\,dx\,dz = -\iint_{R_3} y^2\,dx\,dz$$

$$= -a^2\int_0^a\int_0^a dx\,dz = -a^4.$$

(iv) For the face OPSA, $-\hat{j}$ is the outward normal.

$\bar{N} = -\hat{j}$, $y=0$, and $ds = dx\,dz$

$$\iint_{R_4} \bar{F}\cdot\bar{N}\,ds = \iint_{R_4} (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\cdot(-\hat{j})\,dx\,dz = \iint_{R_4} +y^2\,dx\,dz = 0.$$

(v) For the face ABPS, $\hat{k}$ is the outward normal

$\bar{N} = \hat{k}$, $z=a$, and $ds = dx\,dy$. $\bar{F}\cdot\bar{N} = (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\cdot\hat{k} = yz$

$$\iint_{R_5} \bar{F}\cdot\bar{N}\,ds = \iint_{R_5} yz\,dx\,dy = a\int_{x=0}^{a}\int_{y=0}^{a} y\,dx\,dy = a^4/4$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

(vi) For the face $BCEG$, outward normal

$\bar{n} = -\bar{i}$, $x = 0$, $ds = dy\,dz$.

$\bar{F}\cdot\bar{n} = (4xz\bar{i} - y^2\bar{j} + yz\bar{k})(-\bar{i}) = -yz$

$\iint_{R_6} \bar{F}\cdot\bar{n}\,ds = -\iint_{R_6} yz\,dy\,dz = 0$ [$\because x = 0$ on this face]

Hence adding all we get $\int_S \bar{F}\cdot\bar{n}\,ds = 2a^4 + 0 - a^4 + \frac{1}{2}a^4 + 0 = \frac{3}{2}a^4$.

4) $\bar{F} = 2y\bar{i} - 3\bar{j} + x^2\bar{k}$ and 'S' is surface $y^2 = 8x$ is first octant bounded by the planes $y = 4$ and $z = 6$, evaluate $\int_S \bar{F}\cdot\bar{n}\,ds$

Sol:- $\bar{F} = 2y\bar{i} - 3\bar{j} + x^2\bar{k}$.

Here 'S' is the surface $y^2 = 8x$ first octant bounded by the planes $y = 4$ and $z = 6$

$\phi = 8x - y^2$

$\bar{n}$ = unit outward normal to $S = \frac{\nabla\phi}{|\nabla\phi|}$.

$\bar{n} = \frac{8\bar{i} - 2y\bar{j}}{\sqrt{64 + 4y^2}} = \frac{4\bar{i} - y\bar{j}}{\sqrt{16 + y^2}}$.

Let 'R' be the projection of 'S' on the yz-plane then

$\int_S \bar{F}\cdot\bar{n}\,ds = \int_R \bar{F}\cdot\bar{n}\,\frac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$

For R: $x = y^2/8$, $y = 0$ to 4, $z = 0$ to 6.

$\int_S \bar{F}\cdot\bar{n}\,ds = \int_R \bar{F}\cdot\bar{n}\,\frac{dy\,dz}{|\bar{n}\cdot\bar{i}|} = \int_{z=0}^{6}\int_{y=0}^{4} (2y\bar{i} - 3\bar{j} + x^2\bar{k})\cdot\frac{(4\bar{i} - y\bar{j})}{\sqrt{16 + y^2}}$

$= \int_{z=0}^{6}\int_{y=0}^{4} (8y + 3y)\frac{dy\,dz}{4} = \frac{11}{4}\int_{z=0}^{6}\left[\int_{y=0}^{4} y\right]dz$

$= \frac{11}{4}\int_{z=0}^{6}\left[\frac{y^2}{2}\right]_0^4 dz = \frac{11}{4}\int_{z=0}^{6} 8\,dz = \frac{88}{4}[z]_0^6$

$= 22 \times 6 = 132$.

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
FirstRanker's Choice

Problem:- If velocity vector is $\bar{F} = y\bar{i} + 2\bar{j} + xz\bar{k}$ m/sec. Show that the flux of water through the parabolic cylinder $y = x^2$, $0 \le x \le 3$, $0 \le z \le 2$ is 69 m^3/sec.

www.FirstRanker.com www.FirstRanker.com

Sol:- $\bar{F} = y\bar{i} + 2\bar{j} + xz\bar{k}$

Here 'S' is the surface $y = x^2$ first octant bounded by the planes $x = 3$ and $z = 2$.

$\bar{N}$ = unit outward normal to $S = \frac{\nabla\phi}{|\nabla\phi|}$.

$\bar{N} = \frac{2x\bar{i} - \bar{j}}{\sqrt{4x^2+1}}$ Let 'R' be the Projection of 'S' on the xz-plane.

$$\iint_S \bar{F}\cdot\bar{N}\, ds = \iint_R \bar{F}\cdot\bar{N}\, \frac{dx\,dz}{|\bar{N}\cdot\bar{j}|}$$

$\bar{F}\cdot\bar{N} = (y\bar{i} + 2\bar{j} + xz\bar{k})\cdot\left(\frac{2x\bar{i} - \bar{j}}{\sqrt{4x^2+1}}\right)$

$\bar{N}\cdot\bar{j} = \frac{2x\bar{i} - \bar{j}}{\sqrt{4x^2+1}}\cdot\bar{j} = \frac{-1}{\sqrt{4x^2+1}}$

For R: $y = x^2$, $x = 0$ to 3, $z = 0$ to 2

$$\iint_S \bar{F}\cdot\bar{N}\, ds = \iint_R \bar{F}\cdot\bar{N}\, \frac{dx\,dz}{|\bar{N}\cdot\bar{j}|} = \int_{x=0}^{3}\int_{z=0}^{2} \frac{2xy - 2}{\sqrt{4x^2+1}} \Big/ \frac{-1}{\sqrt{4x^2+1}}\, dx\,dz$$

$$= +2\int_{x=0}^{3}\int_{z=0}^{2} (xy+1)\, dx\,dz = +2\int_{x=0}^{3}\int_{z=0}^{2} (x^3+1)\, dx\,dz \quad [\because y = x^2]$$

$$= +2\int_{z=0}^{2}\left[\frac{x^4}{4} + x\right]_0^3 dz = +2\int_{z=0}^{2}\left(\frac{81}{4} - 3\right) dz$$

$$= +2\left[\frac{81}{4}z - 3z\right]_0^2$$

$$= +2\left[\frac{81}{4}(2) - 3(2)\right]$$

$$= +2\cdot\frac{81}{2} - 2(3)(2)$$

$$= +81 - 12 = +69\ m^3/\text{sec}$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Evaluate $\int_S \bar{F}\cdot\bar{N}\,ds$ where $\bar{F} = yz\,\bar{i} + zx\,\bar{j} + xy\,\bar{k}$ and 'S' is the surface of $x^2+y^2+z^2=1$ in the first octant.

Sol:-

$\bar{F} = yz\,\bar{i} + zx\,\bar{j} + xy\,\bar{k}$

Here 'S' is the surface of $x^2+y^2+z^2=1$ in the first octant.

$\bar{N}$ = unit outward normal $= \frac{\nabla\phi}{|\nabla\phi|}$

where $\phi = x^2+y^2+z^2-1$

$$\bar{N} = \frac{2x\bar{i}+2y\bar{j}+2z\bar{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{2(x\bar{i}+y\bar{j}+z\bar{k})}{2\sqrt{x^2+y^2+z^2}} = x\bar{i}+y\bar{j}+z\bar{k}$$

Let 'R' be the Projection of xy-plane $\int_S \bar{F}\cdot\bar{N}\,ds = \int_R \bar{F}\cdot\bar{N}\,\frac{dx\,dy}{|\bar{N}\cdot\bar{k}|}$

$$\int_S \bar{F}\cdot\bar{N}\,ds = \int_R \bar{F}\cdot\bar{N}\,\frac{dx\,dy}{|\bar{N}\cdot\bar{k}|}$$

$$= \int_R (yz\,\bar{i}+zx\,\bar{j}+xy\,\bar{k})\cdot(x\bar{i}+y\bar{j}+z\bar{k})\cdot\frac{dx\,dy}{|(x\bar{i}+y\bar{j}+z\bar{k})\cdot\bar{k}|}$$

$$= \int_R (xyz+xyz+xyz)\,\frac{dx\,dy}{z} = \int_R 3xy\,\frac{dx\,dy}{z}$$

$$= \int_R 3xy\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} (3xy)\,dy\,dx = \int_{x=0}^{1} 3x\left(\frac{y^2}{2}\right)_0^{\sqrt{1-x^2}}\,dx$$

$$= \int_0^1 3x\,\frac{(1-x^2)}{2}\,dx = \frac{1}{2}\int_0^1 (3x-3x^3)\,dx$$

$$= \frac{1}{2}\left[\frac{3x^2}{2}-\frac{3x^4}{4}\right]_0^1 = \frac{1}{2}\left[\frac{3}{2}-\frac{3}{4}\right]$$

$$= \frac{1}{2}\left[\frac{3}{4}\right] = \frac{3}{8}$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice

Volume Integrals: $\int_V \bar{F}\,dv$ is called volume integral of $\bar{F}$ over the region 'V'.

www.FirstRanker.com www.FirstRanker.com

Another form of volume integrals are $\int \phi\,dv$ and $\iiint_V (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})\,dx\,dy\,dz$.

① If $\bar{F} = 2xz\,\hat{i} - x\hat{j} + y^2\hat{k}$. Evaluate $\int_V \bar{F}\,dv$ where 'V' is the region bounded by the surfaces $x=0$, $x=2$, $y=0$, $y=6$, $z=x^2$, $z=4$.

Sol:- $\bar{F} = 2xz\,\hat{i} - x\hat{j} + y^2\hat{k}$

Here 'V' is the region bounded by the surfaces $x=0$, $x=2$, $y=0$, $y=6$, $z=x^2$, $z=4$

For V: $x = 0$ to 2, $y = 0$ to 6, $z = x^2$ to 4.

$$\iiint_V \bar{F}\,dv = \iiint_V (2xz\,\hat{i} - x\hat{j} + y^2\hat{k})\,dx\,dy\,dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{6}\left[\int_{z=x^2}^{4}(2xz\,\hat{i} - x\hat{j} + y^2\hat{k})\,dz\right]dx\,dy$$

$$= \int_{x=0}^{2}\int_{y=0}^{6}\left[2x\left(\frac{z^2}{2}\right)_{x^2}^{4}\hat{i} - x[z]_{x^2}^{4}\hat{j} + y^2[z]_{x^2}^{4}\hat{k}\right]dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{6}\left(16x - x^5\right)\hat{i} - (4x - x^3)\hat{j} + (4y^2 - x^2y^2)\hat{k}\,dy\,dx$$

$$= \int_{x=0}^{2}\left[(16xy - x^5y)_0^6\,\hat{i} - (4xy - x^3y)_0^6\,\hat{j} + \left(\frac{4}{3}y^3 - x^2\frac{y^3}{3}\right)_0^8\hat{k}\right]dx$$

$$= \int_{x=0}^{2}\left[(96x - 6x^5)\hat{i} - (24x - 6x^3)\hat{j} + (288 - 72x^2)\hat{k}\right]dx$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com

$= \left[96\frac{x^2}{2} - 6\frac{x^6}{6}\right]_0^2 \bar{i} - \left[\frac{12}{2}x^2 - \frac{6}{5}\frac{x^4}{2}\right]_0^2 \bar{j} + \left[288x - \frac{24}{4}x^3\right]_0^2 \bar{k}$

$= \left(48x^2 - x^6\right)_0^2 \bar{i} - \left(12x^2 - \frac{3}{2}x^4\right)_0^2 \bar{j} + \left(288x - 24x^3\right)_0^2 \bar{k}$

$= (192 - 64)\bar{i} - (48 - 24)\bar{j} + (576 - 192)\bar{k}$

$= 128\bar{i} - 24\bar{j} + 384\bar{k}$

② Evaluate $\iiint_V (2x+y)\,dv$ where 'V' is closed region bounded the cylinder $z = 4-x^2$ and the plane $x=0, y=0, y=2$ and

Sol:- 'V' is the closed region bounded the cylinder $z =$ and the plane $x=0$, $y=0$, $y=2$, and $z=0$

$z =$

$z =$

$x =$

For V: $z = 0$ to $4-x^2$

$y = 0$ to 2

$x = 0$ to 2

$$\iiint_V (2x+y)\,dv = \int_{x=0}^{2}\int_{y=0}^{2}\left[\int_{z=0}^{4-x^2}(2x+y)\,dz\right]dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2}\left[(2x+y)z\right]_0^{4-x^2} dy\,dx = \int_{x=0}^{2}\int_{y=0}^{2}(2x+y)(4-x^2)\,dy$$

$$= \int_{x=0}^{2}\left[(8x - 2x^3 + 4y - x^2y)\,dy\right]dx$$

$$= \int_{x=0}^{2}\left[8x[y]_0^2 - 2x^3[y]_0^2 + 4\left[\frac{y^2}{2}\right]_0^2 - x^2\left[\frac{y^2}{2}\right]_0^2\right]dx = \int_{x=0}^{2}(16x - 4x^3 + 8 - 2x^2)$$

$$= 16\left[\frac{x^2}{2}\right]_0^2 - 4\left[\frac{x^4}{4}\right]_0^2 + 8[x]_0^2 - 2\left[\frac{x^3}{3}\right]_0^2$$

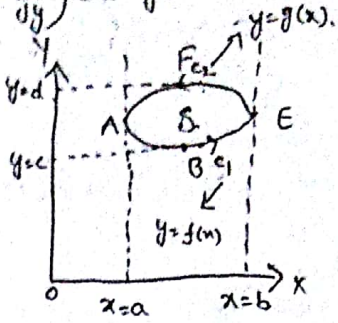
$$= 32 - 16 + 16 - \frac{16}{3}$$

$$= \frac{96-16}{3} = \frac{80}{3}$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Let 'S' be a closed region in xy-plane bounded by a simple closed curve 'C'. Let P and Q be continuous and differentiable scalar point functions of x and y in S. Then $\oint_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$.



Proof:- 'S' be a closed region in xy-plane enclosed by a closed curve "C".

P and Q are continuous and differentiable scalar point functions of x and y in S.

Let any line parallel to either co-ordinate axis cut 'C' in atmost two points.

Let 'S' lie between the lines $x=a$, $x=b$, $y=c$, $y=d$.

Let $y=f(x)$ be the curve "C_1" [i.e ABE] and

$y=g(x)$ " " " "C_2" [i.e EFA].

Generally $f(x) \le g(x)$.

Consider $\iint_S \frac{\partial P}{\partial y}\,dx\,dy = \int_{x=a}^{b} \int_{y=f(x)}^{y=g(x)} \left[\frac{\partial P}{\partial y}\,dy\right] dx = \int_{x=a}^{b} \left[P(x,y)\right]_{f(x)}^{g(x)} dx$.

$= \int_{x=a}^{b} [P(x,g) - P(x,f)]\,dx = \int_{x=a}^{b} P(x,g)\,dx - \int_{x=a}^{b} P(x,f)\,dx$.

$= -\int_{x=b}^{a} P(x,g)\,dx - \int_{x=a}^{b} P(x,f)\,dx$.

$= -\left[\int_{x=b}^{a} P(x,g)\,dx + \int_{x=a}^{b} P(x,f)\,dx\right]$

$= -\left[\int_{C_2} P(x,y)\,dx + \int_{C_1} P(x,y)\,dx\right]$.

$= -\left[\int_{C_1} P\,dx + \int_{C_2} P\,dx\right] = -\oint_C P\,dx$.

$\iint_S \frac{\partial P}{\partial y}\,dx\,dy = -\oint_C P\,dx \Rightarrow \oint_C P\,dx = -\iint_S \frac{\partial P}{\partial y}\,dx\,dy$. — (1)

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
FirstRanker's choice
www.FirstRanker.com www.FirstRanker.com

Hy prove that.

$$\oint_c Q\,dy = \iint_S \frac{\partial Q}{\partial x}\,dx\,dy \quad \text{— (II)}.$$

$$\oint_c P\,dx + \oint_c Q\,dy = -\iint_S \frac{\partial P}{\partial y}\,dx\,dy + \iint_S \frac{\partial Q}{\partial x}\,dx\,dy.$$

$$\oint_c P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Note ① :- Let "S" be closed region in yz-plane enclose by a curve 'c' Let P and Q continuous and differentiable scalar functions are y and z in S.

Then $\oint_c P\,dy + Q\,dz = \iint_S \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial z}\right) dy\,dz.$

Note ② :- Let 'S' be closed region zx-plane enclosed by a curve c. Let P and Q continuous and differentiable scalar functions are z and x in S.

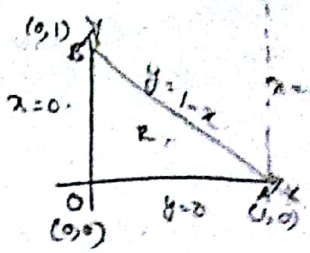
Then $\oint_c P\,dz + Q\,dx = \iint_S \left(\frac{\partial Q}{\partial z} - \frac{\partial P}{\partial x}\right) dz\,dx.$

z.

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com

1) Verify Green's theorem for $\oint_C (3x^2-8y^2)dx + (4y-6xy)dy$, where 'C' is the boundary of the region bounded by $x=0$, $y=0$, and $x+y=1$.



Sol:- $\oint_C (3x^2-8y^2)dx + (4y-6xy)dy$

By Green's theorem $\oint_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$P = 3x^2 - 8y^2$, $Q = 4y - 6xy \Rightarrow \frac{dP}{dy} = -16y$; $\frac{\partial Q}{\partial x} = -6y$.

$\frac{\partial Q}{\partial x} - \frac{dP}{dy} = -6y + 16y = 10y$.

and 'S' is the boundary of region bounded by $x=0$, $y=0$, $x+y=1$.

For S; $y = 0$ to $1-x$; $x = 0$ to 1.

$$\oint_C (3x-8y^2)dx + (4y-6xy)dy = \iint_S 10y\,dx\,dy = \int_{x=0}^{1} \int_{y=0}^{1-x} [10y\,dy]\,dx.$$

$$= \int_{x=0}^{1} [5y^2]_0^{1-x} = 5\int_{x=0}^{1} (1+x^2-2x)\,dx = 5\left[x + \frac{x^3}{3} - x^2\right]_0^1 = 5\left[1+\frac{1}{3}-1\right] = 5/3$$

The region R = OAB

Along OA, $y=0$, $dy=0$, x varies from 0 to 1.

$$\int_{OA} \bar{f}\cdot d\bar{r} = \int_0^1 3x\,dx = 3\frac{x^2}{2}\Big]_0^1 = \frac{3}{2}$$

Along AB, $x+y=1$, $y=1-x$, $dy=-dx$. x varies from 1 to 0.

$$\int_{AB} \bar{f}\cdot d\bar{r} = \int_1^0 [3x - 8(1-x)^2]\,dx - [4(1-x) - 6x(1-x)](-dx) = \int_1^0 (14x^2 + 2x - 12)\,dx$$

$$= \int_0^1 (14x^2 - 29x + 12)\,dx = 14\frac{x^3}{3} - 29\frac{x^2}{2} + 12x\Big]_0^1 = \frac{14}{3} - \frac{29}{2} + 12 = \frac{13}{6}$$

Along BO $x=0$, $y = 1$ to 0

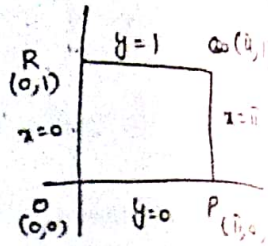
$$\int_{BO} \bar{f}\cdot d\bar{r} = \int_1^0 4y\,dy = 4\frac{y^2}{2}\Big]_1^0 = -\frac{4}{2} = -2$$

$$\int_C (3x-8y^2)dx + (4y-6xy)dy = \frac{3}{2} + \frac{13}{6} - 2 = \frac{5}{3}$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

7) Verify Green's theorem for $\oint_C (x^2 - \cosh y)\,dx + (y + \sin x)\,dy$ where C is the rectangle with vertices $(0,0)$ $(\pi,0)$ $(\pi,1)$ $(0,1)$.



Sol:- By Green's theorem $\oint_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$.

Here $P = x^2 - \cosh y$, $Q = y + \sin x$.

$\frac{\partial P}{\partial y} = -\sinh y$ $\quad \frac{\partial Q}{\partial x} = \cos x$ $\quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \cos x + \sinh y$.

and 'S' is the surface of rectangle with vertices $(0,0)$ $(\pi,0)$ $(\pi,1)$ and $(0,1)$. For S; $x = 0$ to π, $y = 0$ to 1.

$$\oint_C (x^2 - \cosh y)\,dx + (y + \sin x)\,dy = \iint_S (\cos x + \sinh y)\,dx\,dy$$

$$= \int_{y=0}^{1} \left[\int_{x=0}^{\pi} (\cos x + \sinh y)\,dx\right] dy = \int_{y=0}^{1} \left[(\sin x)_0^{\pi} + \sinh y\,[x]_0^{\pi}\right] dy$$

$$= \int_{y=0}^{1} \left[(0-0) + (\pi - 0)\sinh y\right] dy = \int_{y=0}^{1} \pi \sinh y\,dy = \pi \cosh y\Big]_0^1$$

$$= \pi\left[\cosh 1 - \cosh 0\right]$$

$$= \pi\left[\cosh 1 - 1\right]$$

8) Find the area of a circle of radius a using Green's Theo

Sol:- Area $= \frac{1}{2}\int_C (x\,dy - y\,dx)$

choosing $x = a\cos\theta$, $y = a\sin\theta$ we get

$$A = \frac{1}{2}\int_0^{2\pi} (a^2\cos^2\theta + a^2\sin^2\theta)\,d\theta$$

$$= \frac{1}{2}a^2\int_0^{2\pi} d\theta = \pi a^2$$

Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Problem: Apply Green's theorem to prove that the area (A) enclosed by a plane curve 'c' is $\frac{1}{2}\int_c x\,dy - y\,dx$. Hence find the area of an ellipse whose semi major and semi minor axes are of lengths a and b.

Sol:- By Green's theorem $\oint_c P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$.

Let $P = -y$, $Q = x$, $\frac{\partial P}{\partial y} = -1$, $\frac{\partial Q}{\partial x} = 1$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - (-1) = 2$

$\oint_c -y\,dx + x\,dy = \iint_S 2\,dx\,dy$.

$\int ds = \int dx\,dy$

$\int_S dx\,dy = S$.

$\frac{1}{2}\oint_c x\,dy - y\,dx = \iint_S dx\,dy$.

$S = \frac{1}{2}\oint_c x\,dy - y\,dx$

∴ The area bounded by a simple closed curve 'c' is

$\frac{1}{2}\oint_c x\,dy - y\,dx$.

Given that equation of ellipse is

$x = a\cos\theta$, $y = b\sin\theta$, $0 \le \theta \le 2\pi$

$dx = -a\sin\theta\,d\theta$, $dy = b\cos\theta\,d\theta$

$\theta : 0 \to 2\pi$

∴ The area of the given ellipse $= \frac{1}{2}\oint x\,dy - y\,dx$

$= \frac{1}{2}\int_0^{2\pi} a\cos\theta\,(b\cos\theta)\,d\theta - b\sin\theta\,(-a\sin\theta\,d\theta)$

$= \frac{1}{2}ab\int_0^{2\pi} (\cos^2\theta + \sin^2\theta)\,d\theta = \frac{1}{2}ab\,[\theta]_0^{2\pi}$

$= \frac{1}{2}ab\,[2\pi] = \pi ab$ sq. units.

Note: Area of the ellipse $= \pi ab$ sq. units.

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

of the region defined by $y^2 = 8x$ and $x = 2$.

Sol:- By Green's theorem $\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$P = x^2 - 2xy, \quad Q = x^2y + 2$



$\frac{\partial P}{\partial y} = -2x, \quad \frac{\partial Q}{\partial x} = 2xy$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2xy + 2x$

and 'R' is the surface of the region bounded $y^2 = 8x$ and x

for $y = -\sqrt{8x}$ to $\sqrt{8x}$ $x = 0$ to 2.

$$\oint (x^2 - 2xy)dx + (x^2y + 2)dy = \iint_R (2xy + 2x)\,dx\,dy$$

$$= \int_{x=0}^{2} \left[\int_{y=-\sqrt{8x}}^{\sqrt{8x}} (2xy + 2x)\,dy \right] dx = \int_{x=0}^{2} \left[xy^2 + 2xy \right]_{-\sqrt{8x}}^{\sqrt{8x}} dx$$

$$= \int_{x=0}^{2} x(8x - 8x) + 2x\left[\sqrt{8x} + \sqrt{8x}\right] dx$$

$$= \int_{x=0}^{2} 4\sqrt{8}\, x^{3/2}\, dx = 4\sqrt{8}\left[\frac{x^{5/2}}{5/2}\right]_0^2$$

$$= 4\sqrt{8}\,\frac{2}{5}\,x^{5/2}\Big|_0^2 = \frac{4\cdot 2\sqrt{8}\; x^{5/2}}{5}\Big|_0^2$$

$$= \frac{16\sqrt{2}\cdot 2^{5/2}}{5} = \frac{16\sqrt{2}\cdot 2^2\sqrt{2}}{5}$$

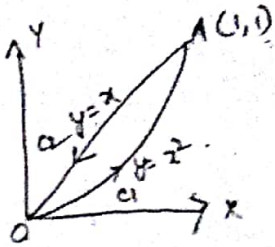
$$= \frac{128}{5}$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

1) Verify Green's theorem in the plane for $\oint_C (xy+y^2)dx + x^2dy$.

where 'c' is the closed curve of the region bounded by $y=x$ and $y=x^2$



Sol:- By Green's theorem, we have

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dx\,dy.$$

The two curves $y=x$ and $y=x^2$ intersect at $O(0,0)$ and $A(1,1)$

$$\oint_C (xy+y^2)\,dx + x^2\,dy \cdot \text{ where } P = xy+y^2,\ Q = x^2.$$

$$\frac{dQ}{dx} = 2x, \quad \frac{dP}{dy} = x+2y.$$

$$\oint_C P\,dx + Q\,dy = \iint_R (2x-x-2y)\,dx\,dy = \iint_R (x-2y)\,dx\,dy = \int_0^1 \int_{y=x^2}^{x} (x-2y)\,dy\,dx$$

$$= \int_0^1 \left(xy - 2\frac{y^2}{2}\right)_{y=x^2}^{x} dx = \int_0^1 (x^2-x^3)\,dx = \left(\frac{x^3}{3} - \frac{x^4}{4}\right)_0^1 = -\frac{1}{20} \quad -(1)$$

consider the integral.

Along c, $y = x^2 \Rightarrow dy = 2x\,dx$

$$\oint_C (xy+y^2)\,dx + x^2\,dy$$

$$\int_0^1 (x(x^2)+x^4)\,dx + x^2(2x\,dx) = \int_0^1 (3x^3+x^4)\,dx = \frac{19}{20}$$

$\frac{3x^4}{4} + \frac{x^5}{5}\Big]_0^1$

$\frac{3}{4} + \frac{1}{5} = \frac{15+4}{20}$

Along $y=x$, $dy = dx$.

$$\int_1^0 (x(x)+x^2)\,dx + x^2\,dx = \int_1^0 3x^2\,dx = 3\frac{x^3}{3}\Big]_1^0 = 0-1 = -1$$

so $\oint_C (xy+y^2)\,dx + x^2\,dy = \frac{19}{20} - 1 = -\frac{1}{20}$ $-(2)$.

Here the Green's theorem is verified from the equality of (1) and (2).

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

2) If 'c' is a simple closed curve in xy-plane not enclosing the origin, show that $\int_c \bar{F} \cdot d\bar{R} = 0$, where $\bar{F} = \frac{y\bar{i} - x\bar{j}}{x^2+y^2}$.

Soln:- $\int_c \bar{F} \cdot d\bar{R} = \int_c \frac{y\bar{i} - x\bar{j}}{x^2+y^2} \cdot (dx\bar{i} + dy\bar{j})$

$P dx + Q dy$

$\frac{dQ}{dy}$

$P = \frac{y}{x^2+y^2}$, $Q = \frac{-x}{x^2+y^2}$

$$\frac{dQ}{dx} = \frac{-(x^2+y^2) + x(2x)}{(x^2+y^2)^2}$$

$$\frac{dP}{dy} = \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2}$$

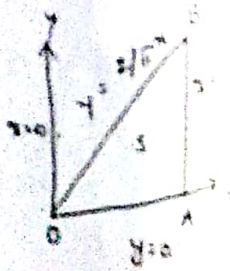
$$\frac{dQ}{dx} - \frac{dP}{dy} = -x^2 - y^2 + 2x^2 - \ldots$$

$$\frac{dQ}{dx} - \frac{dP}{dy} = 0.$$

$$\int_c \bar{F} \cdot d\bar{R} = \iint_S \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dx\, dy$$

$$= \iint_S \left[\frac{-(x^2+y^2) + x(2x)}{(x^2+y^2)^2} - \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2} \right] dx\, dy = 0.$$

3) Evaluate using Green's theorem, evaluate $\int_c (y - \sin x)\, dx + \cos x\, dy$ where 'c' is the plane triangle enclosed by the lines $y = 0$, $x = \pi/2$ and $y = \frac{2x}{\pi}$.



Sol:- By Green's theorem

$$\oint_c P\,dx + Q\,dy = \iint_S \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dx\, dy.$$

Here $P = y - \sin x$, $Q = \cos x$. $\frac{dP}{dy} = 1$, $\frac{dQ}{dx} = -\sin x$.

$$\frac{dQ}{dx} - \frac{dP}{dy} = -\sin x - 1.$$

and 'S' is the surface of a triangle bounded by the lines $y = 0$, $x = \pi/2$, $y = 2x/\pi$. For S: $y = 0$ to $\frac{2x}{\pi}$.

$x = 0$ to $\pi/2$.

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$$\oint_C (y+\sin x)\,dx + \cos x\,dy = \iint_R (-\sin x - 1)\,dx\,dy.$$

$$= \int_{x=0}^{\pi/2}\left[\int_{y=0}^{2x/\pi} (-\sin x - 1)\,dy\right]dx = \int_{x=0}^{\pi/2} (-\sin x - 1)\,y\Big]_0^{2x/\pi}\,dx.$$

$$= \int_{x=0}^{\pi/2} (-\sin x - 1)\frac{2x}{\pi}\,dx = \int_{x=0}^{\pi/2} -\frac{2x}{\pi}\sin x\,dx - \int_{x=0}^{\pi/2}\frac{2x}{\pi}\,dx.$$

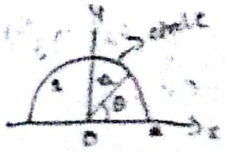
$$= \frac{2}{\pi}\int_{x=0}^{\pi/2} x\,d(\cos x) - \frac{1}{\pi}\left[x^2\right]_0^{\pi/2} \qquad \int u\,dv = uv - \int v\,du.$$

$$= \frac{2}{\pi}\left[(x\cos x)_0^{\pi/2} - \int_0^{\pi/2}\cos x\,dx\right] - \frac{1}{\pi}\frac{\pi^2}{4}.$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2}\cos\frac{\pi}{2} - [\sin x]_0^{\pi/2}\right] - \frac{\pi}{4}$$

$$= \frac{2}{\pi}\left[0 - (1-0)\right] - \frac{\pi}{4} = -\frac{2}{\pi} - \frac{\pi}{4}.$$

(4) Apply Green's theorem to evaluate $\int_C (2x^2-y^2)\,dx + (x^2+y^2)\,dy$, where 'c' is the boundary of the area enclosed by the x-axis and the upper-half of the circle $x^2+y^2=a^2$.



Sol:- $$\int_C (2x^2-y^2)\,dx + (x^2+y^2)\,dy = \iint_S \left[\frac{d}{dx}(x^2+y^2) - \frac{d}{dy}(2x^2-y^2)\right]dx\,dy.$$

$$= 2\iint_S (x+y)\,dx\,dy = 2\int_0^a\int_0^{\pi} r(\cos\theta+\sin\theta)\,r\,dr\,d\theta$$

changing the polar coordinates (r,θ), r varies from 0 to a and θ varies from 0 to π.

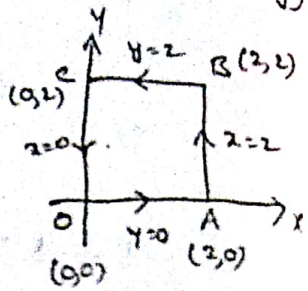
$$= 2\int_0^a r^2\,dr \int_0^{\pi}(\cos\theta+\sin\theta)\,d\theta = 2\frac{a^3}{3}(1+1) = \frac{4a^3}{3}.$$

$$\int_0^{\pi}(\cos\theta+\sin\theta)\,d\theta = [\sin\theta - \cos\theta]_0^{\pi}$$
$$= [\sin\pi - \cos\pi] - [\sin 0 - \cos 0]$$
$$= [0-(-1)] - [0-1]$$
$$= 1+1$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com

Verify Green's theorem in the plane for $\oint_C (x^2-xy^3)\,dx + (y^2-2xy)\,dy$



'C' is a square with vertices (0,0) (2,0) (2,2) (0,2)

$P = x^2-xy^3, \quad Q = y^2-2xy.$

$\frac{\partial Q}{\partial x} = -2y \quad \frac{\partial P}{\partial y} = -3xy^2.$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -2y+3xy^2.$

$\oint_C (x^2-xy^3)\,dx + (y^2-2xy)\,dy = \iint_R (-2y+3xy^2)\,dx\,dy.$

1) Along OA: $y=0$ hence $dy=0$, x varies from 0 to 2.

$$\int_{OA} (x^2-xy^3)dx + (y^2-2xy)dy = \int_{x=0}^{2} x^2\,dx = \frac{x^3}{3}\Big]_0^2 = 8/3$$

2) Along AB: $x=2$, $dx=0$, y varies from 0 to 2.

$$\int_{AB} (x^2-xy^3)\,dx + (y^2-2xy)\,dy = \int_{y=0}^{2} (y^2-4y)\,dy = \frac{y^3}{3} - 4\frac{y^2}{2}\Big]_0^2 = \frac{8}{3} - 8 = -\frac{16}{3}$$

3) Along BC: $y=2$, $dy=0$, x varies from 2 to 0.

$$\int_{BC} (x^2-xy^3)\,dx + (y^2-2xy)\,dy = \int_{x=2}^{0} (x^2-2x^3)\,dx = \frac{x^3}{3} - 2\frac{x^4}{4}\Big]_2^0$$

$$= -\frac{8}{3} + 16 = \frac{40}{3}$$

4) Along CO: $x=0$, $dx=0$, y varies from 2 to 0.

$$\int_{CO} (x^2-xy^3)\,dx + (y^2-2xy)\,dy = \int_{y=2}^{0} y^2\,dy = \frac{y^3}{3}\Big]_2^0 = -\frac{8}{3}.$$

$$\therefore \int_C (x^2-xy^3)\,dx + (y^2-2xy)\,dy = \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3} = 8.$$

Now RHS $\iint_R (-2y+3xy^2)\,dy\,dx = \int_{x=0}^{2}\int_{y=0}^{2} (-2y+3xy^2)\,dy\,dx$

$$= \int_{x=0}^{2} \left(-2\frac{y^2}{2} + 3x\frac{y^3}{3}\right)_0^2 dx = \int_{x=0}^{2} (-4+8x)\,dx$$

$$= -4x + 8\frac{x^2}{2}\Big]_0^2 = -8+16 = 8$$

www.FirstRanker.com
Scanned by CamScanner

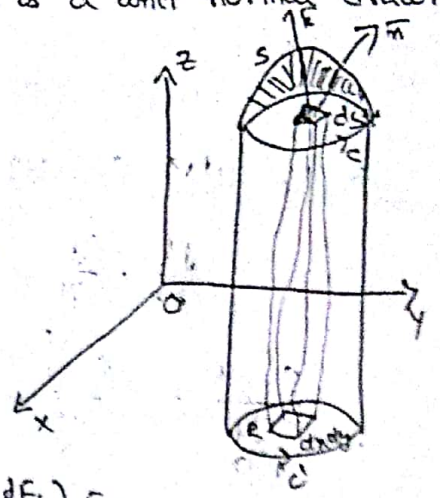
FirstRanker.com
www.FirstRanker.com

Stokes Theorem (Relation between Line and Surface Integrals)

Let $\bar{F}$ be any vector point function having continuous partial derivatives. Let 'S' be an open surface bounded by closed curve C, then

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds$$

where $\bar{n}$ is a unit normal drawn outward at any point to S.



Proof:- Let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

$$\text{curl}\,\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\bar{i} + \bar{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\bar{k}$$

Let the unit normal vector makes angles α, β, γ with positive direction of x, y, z axes, then $\bar{n} = \cos\alpha\,\bar{i} + \cos\beta\,\bar{j} + \cos\gamma\,\bar{k}$.

Also $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \Rightarrow d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$$(\nabla\times\bar{F})\cdot\bar{n} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma$$

and $\bar{F}\cdot d\bar{r} = (F_1\bar{i} + F_2\bar{j} + F_3\bar{k})\cdot(dx\bar{i} + dy\bar{j} + dz\bar{k}) = F_1\,dx + F_2\,dy + F_3\,dz$

Now Stokes theorem can also be written as,

$$\oint_C (F_1\,dx + F_2\,dy + F_3\,dz) = \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma\right] ds \quad \text{— (1)}$$

Let $z = f(x, y)$ be the equation of the surface S where projection on the xy plane is R. Then the projection of curve C on the xy-plane is the curve C' which is the boundary of the region R.

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$$\oint_C F_1(x,y,z)\,dx = \oint_{C_1} F_1(x,y,f)\,dx + \oint_{C_1} F_1(x,y,f)\,dy.$$

$$= -\iint_R \frac{\partial}{\partial y} F_1(x,y,f)\,dx\,dy \quad [\text{by Green's th.}]$$

$$= -\iint_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial f}{\partial y}\right) dx\,dy \quad -(2).$$

Now the d.r's of the normal $\bar{n}$ to the surface $z = f(x,y)$ are

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1.$$

$$\frac{\cos\alpha}{\frac{\partial f}{\partial x}} = \frac{\cos\beta}{\frac{\partial f}{\partial y}} = \frac{\cos\gamma}{-1} \Rightarrow \frac{\partial f}{\partial y} = -\frac{\cos\beta}{\cos\gamma}.$$

Since $dx\,dy$ is the projection of ds on the xy plane.

$$dx\,dy = ds\cos\gamma$$

By (2) $$\oint_C F_1(x,y,z)\,dx = -\iint_S \left(\frac{\partial F_1}{\partial y} - \frac{\partial F_1}{\partial z}\frac{\cos\beta}{\cos\gamma}\right) ds\cos\gamma$$

$$= \iint_S \left(\frac{\partial F_1}{\partial y}\cos\gamma - \frac{\partial F_1}{\partial z}\cos\beta\right) ds.$$

$$= \iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right) ds \quad -(3).$$

By we can prove that.

$$\oint_C F_2(x,y,z)\,dy = \iint_S \left(\frac{\partial F_2}{\partial x}\cos\gamma - \frac{\partial F_2}{\partial z}\cos\alpha\right) ds \quad -(4).$$

$$\oint_C F_3(x,y,z)\,dz = \iint_S \left(\frac{\partial F_3}{\partial y}\cos\alpha - \frac{\partial F_3}{\partial x}\cos\beta\right) ds \quad -(5).$$

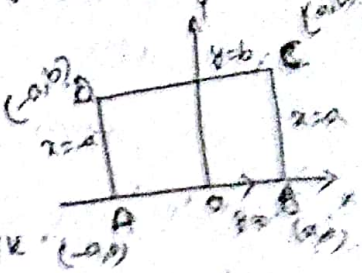
Adding (3) (4) and (5) and using (1) we get.

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds.$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice [30]
www.FirstRanker.com www.FirstRanker.com

1) Verify Stoke's theorem for $\bar{F} = (x^2+y^2)\bar{i} - 2xy\bar{j}$ taken around the rectangle bounded by the lines $x = \pm a$, $y = 0$, $y = b$.



Sol:- By Stoke's theorem $\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\bar{F}\cdot\bar{n}\,ds$.

$$\text{curl}\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = \bar{k}(-2y-2y) = -4y\bar{k}$$

vector $\perp$ to xy-plane $\bar{n} = \bar{k}$

$$\iint_S (\nabla\times\bar{F})\cdot\bar{n}\,ds = \int_{y=0}^{y=b}\int_{x=-a}^{x=a} (-4y\bar{k})\cdot\bar{k}\,dx\,dy = -4\int_{y=0}^{b}\int_{x=-a}^{a} y\,dx\,dy$$

$$= -4\int_{y=0}^{b}[xy]_{-a}^{a}\,dy = -4\int_{y=0}^{b} 2ay\,dy$$

$$= -4[ay^2]_0^b = -4ab^2$$

Along AB; $y=0$, $dy=0$.

x varies from $-a$ to a.

$$\oint_C \bar{F}\cdot d\bar{r} = \int (x^2+y^2)\,dx - 2xy\,dy = \int_{-a}^{a} x^2\,dx = \left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{2a^3}{3}$$

Along BC $x=a$, $dx=0$, y varies from 0 to b.

$$\oint_C \bar{F}\cdot d\bar{r} = \int_C (x^2+y^2)\,dx - 2xy\,dy = -\int_C 2ay\,dy = -2a\left[\frac{y^2}{2}\right]_0^b = -ab^2$$

Along CD $y=b$, $dy=0$, x varies from a to $-a$.

$$\oint_C \bar{F}\cdot d\bar{r} = \int_a^{-a} (x^2+b^2)\,dx = \left[\frac{x^3}{3} + b^2x\right]_a^{-a} = -\frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2$$

$$= -\frac{2a^3}{3} - 2ab^2$$

Along DA: $x=-a$, $dx=0$, y varies from b to 0.

$$\int_C \bar{F}\cdot d\bar{r} = \int_b^0 2ay\,dy = ay^2\Big]_b^0 = -ab^2$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r} + \int_{CD} \bar{F}\cdot d\bar{r} + \int_{DA} \bar{F}\cdot d\bar{r}$$

$$= \frac{2a^3}{3} - ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2$$

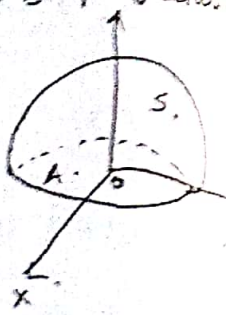
$$= -4ab^2$$

Stoke's theorem is verified.

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Verify Stoke's theorem for the vector field $\bar{F} = (2x-y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$ over the upper half surface $x^2+y^2+z^2=1$, bounded by c's Projection on xy-plane.



Sol:- The projection of the upper half of given sphere on the xy-plane ($z=0$) is the circle $c\,[x^2+y^2=1]$

$$\oint_c \bar{F}\cdot d\bar{r} = \oint_c (2x-y)\,dx - yz^2\,dy - y^2z\,dz.$$

Put $x=\cos\theta$, $y=\sin\theta$, $dx=-\sin\theta\,d\theta$

$$= \oint_c (2x-y)\,dx.$$

$$= \int_{\theta=0}^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta\,d\theta) = \int_{\theta=0}^{2\pi} (\sin 2\theta + \sin^2\theta)\,d\theta.$$

$$= 4\int_0^{\pi/2} \frac{1-\cos 2\theta}{2}\,d\theta = 4\left[\frac{1}{2}\theta - \frac{\sin 2\theta}{2}\right]$$

$$= 4\,\frac{1}{2}\,\frac{\pi}{2} = \pi. \quad -(1)$$

Now $\operatorname{curl}\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$

$$= (-2yz + 2yz)\bar{i} + 0\cdot\bar{j} + \bar{k}(0+1) = \bar{k}$$

$$\int \operatorname{curl}\bar{F}\cdot\bar{n}\,ds = \int_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{k}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

where A is projection of S on xy-plane and $ds = \frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$

$$= \int_R dx\,dy = \text{area of circle } c = \pi r^2 \quad [\because r=1]$$

$$= \pi. \quad -(2).$$

Hence the Stoke's theorem is verified from the equality of (1) and (2).

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

5) Apply Stoke's theorem for calculating $\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F} = y\bar{i} + z\bar{j} + x\bar{k}$ and 'C' is the curve of intersection $x^2+y^2+z^2 = a^2$, $x+z = a$.

Sol:- Given $\bar{F} = y\bar{i} + z\bar{j} + x\bar{k}$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = \bar{i}(0-1) + \bar{j}(0-1) + \bar{k}(0-1)$$

$$= -\bar{i} - \bar{j} - \bar{k}.$$

$\bar{N}$ = unit normal to the plane $(x+z=a) = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\bar{i}+\bar{k}}{\sqrt{2}}$.

By Stoke's theorem, $\int_C \bar{F}\cdot d\bar{r} = \int_S (\nabla\times\bar{F})\cdot\bar{n}\, ds$.

$$= \int_S (-\bar{i}-\bar{j}-\bar{k})\cdot\frac{\bar{i}+\bar{k}}{\sqrt{2}}\, ds.$$

$$= -\sqrt{2}\int_S ds = -\sqrt{2}\,(\text{area of } S)$$

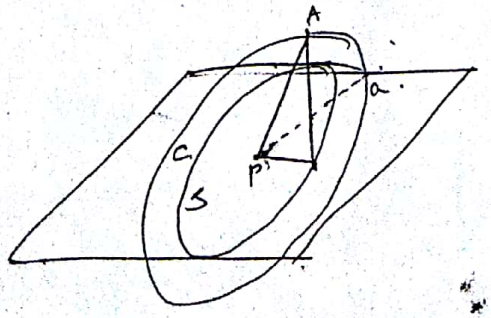
$$= -\sqrt{2}\,(\text{area of given circle})$$

$$= -\sqrt{2}\,\pi r^2 \text{ where } r \text{ is radius of given circle}$$

Now $r^2 = a^2 - p^2$ where 'p' is the perpendicular distance of centre (0,0,0) of the sphere from the plane $(x+z-a=0)$ $= \left|\frac{-a}{\sqrt{1+0+1}}\right| = \left(\frac{a}{\sqrt{2}}\right)$

$\therefore r^2 = a^2 - \frac{a^2}{2} = \frac{a^2}{2}$

The required value of given integral is $-\sqrt{2}\left(\pi\frac{a^2}{2}\right) = -\frac{\pi a^2}{\sqrt{2}}$.

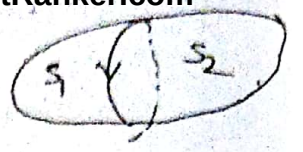


www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

6) If S be any closed surface, PT $\int \text{curl}\,\vec{f} \cdot \vec{n}\, dS = 0$.

Sol:- Cut open the surface 'S' by any plane and let S_1, S_2 denote it's upper and lower portions.



Let c be the common curve bounding both these portions.

$$\int_S \nabla \times \vec{F} \cdot \vec{n}\, dS = \int_{S_1} \nabla \times \vec{F} \cdot \vec{n}\, dS + \int_{S_2} (\nabla \times \vec{F}) \cdot \vec{n}\, dS$$

$$= \int_c \vec{F} \cdot d\vec{r} - \int_c \vec{F} \cdot d\vec{r} = 0.$$

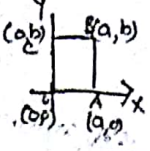
On applying Stoke's theorem. The second integral is negative because it is traversed in a direction opposite to that of

2.

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com

Stokes theorem for the vector field (i) $\bar{F} = (x^2-y^2)\bar{i} + 2xy\bar{j}$ over the box bounded by the planes $x=0, x=a, y=0, y=b, z=0, z=c$, if the face $z=0$ is cut.



Soln: $\bar{F} = (x^2-y^2)\bar{i} + 2xy\bar{j}$

Along OA; $y=0, dy=0$, x varies from 0 to a

$$\int_0^a \bar{f}\cdot d\bar{r} = \int_0^a x^2\,dx = \frac{a^3}{3}$$

Along AB, $x=a, dx=0$, y varies from 0 to b.

$$\int_0^b \bar{f}\cdot d\bar{r} = \int_0^b (a^2-y^2)(0) + 2ay\,dy = \int_0^b 2ay\,dy$$
$$= ab^2$$

Along BC, $y=b, dy=0$, x varies from a to 0.

$$\int_a^0 \bar{f}\cdot d\bar{r} = \int_a^0 (x^2-b^2)\,dx = -\frac{a^3}{3} + ab^2$$

Along CO, $x=0, dx=0$, y varies from b to 0

$$\int_b^0 \bar{f}\cdot d\bar{r} = \int_b^0 0\,dy = 0$$

$$\int_c \bar{f}\cdot d\bar{r} = \int_0^a \bar{f}\cdot d\bar{r} + \int_0^b \bar{f}\cdot d\bar{r} + \int_a^0 \bar{f}\cdot d\bar{r} + \int_b^0 \bar{f}\,d\bar{r}$$
$$= \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 + 0 = 2ab^2$$
$$= \text{L.H.S}$$

$$\iint_S (\nabla\times\bar{F})\cdot\bar{n}\,ds$$

$$\nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-y^2 & 2xy & 0 \end{vmatrix}$$

$$= \bar{i}(0) + \bar{j}(0-0) + \bar{k}(2y+2y)$$
$$= 4y\bar{k}$$

$$\int_0^a\int_0^b 4y\bar{k}\cdot\bar{k}\,dx\,dy = 4\int_0^a \left(\frac{y^2}{2}\right)_0^b dx$$

$$= 4\int_0^a \frac{b^2}{2}\,dx = 4\cdot\frac{b^2x}{2}\Big]_0^a$$
$$= 2ab^2 = \text{R.H.S}$$

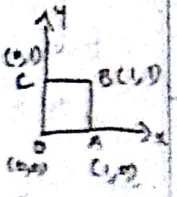
$$\therefore \int_c \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\bar{n}\,ds.$$

Stoke's theorem is verified.

(ii) $\bar{F} = (z^2, 5x, 0)$ and $S: 0\le x\le 1, 0\le y\le 1, z=1$

$$\bar{F} = z^2\bar{i} + 5x\bar{j} + 0\cdot\bar{k}$$

$z=1$



$$\bar{F} = \bar{i} + 5x\bar{j}$$

Along OA, $y=0, dy=0$, x varies from 0 to 1

$$\int_{OA}^{+} \bar{f}\cdot d\bar{r} = \int_0^1 1\,dx = 1$$

Along AB, $x=1, dx=0$, y varies from 0 to 1

$$\int_{AB}^{+} \bar{f}\cdot d\bar{r} = \int_0^1 5\,dy = 5y\Big]_0^1 = 5$$

Along BC, $y=1, dy=0$, x varies from 1 to 0

$$\int_{BC} \bar{f}\cdot d\bar{r} = \int_1^0 1\,dx = -1$$

Along CO, $x=0, dx=0$, y varies from 1 to 0

$$\int_{CO} \bar{F}\cdot d\bar{r} = \int_1^0 5(0)\,dy = 0$$

$$\int_c \bar{F}\cdot d\bar{r} = \int_{OA} \bar{F}\cdot d\bar{r} + \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r} + \int_{CO} \bar{F}\,d\bar{r}$$
$$= 1 + 5 - 1 + 0 = 5$$

$$\nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2 & 5x & 0 \end{vmatrix}$$

$$= \bar{i}(0) + \bar{j}(2z) + \bar{k}(5) = 2z\bar{j} + 5\bar{k}$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

along z-axis

$$\int_0^1\int_0^1 5\,dx\,dy = 5.$$

$$\therefore \int_c \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\bar{n}\,ds.$$

---

(2) Verify Stoke's theorem for a vector field defined by $\bar{F} = -y^3\bar{i} + x^3\bar{j}$ in the region $x^2+y^2\le 1$, $z=0$.

Sol:- By Stoke's theorem $\oint_c \bar{F}\cdot d\bar{r} = \int (\nabla\times\bar{F})\cdot\bar{n}\,ds$

Here $\bar{F} = -y^3\bar{i} + x^3\bar{j}$

$$\nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ d/dx & d/dy & d/dz \\ -y^3 & x^3 & 0 \end{vmatrix}$$

$$= \bar{i}(0-0) + \bar{j}(0-0) + \bar{k}(3x^2+3y^2).$$

and 'S' is the circular disc

$$x^2+y^2\le 1,\ z=0.$$

For S:- $\bar{n} = \bar{k}$, $x = r\cos\theta$, $y = r\sin\theta$, $z = 0$.

$ds = dx\,dy$.

$dx\,dy = r\,dr\,d\theta$

$r = 0$ to 1, $\theta = 0$ to 2π.

$$\oint_c \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\bar{n}\,ds.$$

$$\iint_S (\nabla\times\bar{F})\cdot\bar{n}\,ds = \iint_S (3x^2+3y^2)\bar{k}\cdot\bar{k}\,dx\,dy.$$

$$= \iint_S (3x^2+3y^2)\,dx\,dy.$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} (3r^2\cos^2\theta + 3r^2\sin^2\theta)\,r\,dr\,d\theta.$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} 3r^3\,dr\,d\theta = \int_0^{2\pi} \frac{3r^4}{4}\Big]_0^1 d\theta$$

$$= \frac{3}{4}\int_{\theta=0}^{2\pi} d\theta = \frac{3}{4}\theta\Big]_0^{2\pi} = \frac{3}{4}\cdot 2\pi = \frac{3\pi}{2}$$

Verification:- $\bar{F} = -y^3\bar{i} + x^3\bar{j}$

and 'c' is boundary of a circular disc $x^2+y^2\le 1$, $z=0$.

For c: $x = 1\cdot\cos\theta$, $y = 1\cdot\sin\theta$

$dx = -\sin\theta\,d\theta$, $dy = \cos\theta\,d\theta$

$z=0$, $dz=0$, $\theta = 0$ to 2π

$$\oint_c \bar{F}\cdot d\bar{r} = \oint_c (-y^3\bar{i} + x^3\bar{j})\cdot(dx\,\bar{i} + \dots$$

$$= \oint_c -y^3\,dx + x^3\,dy.$$

$$= \int_0^{2\pi} -\sin^3\theta(-\sin\theta\,d\theta) + \cos^3\theta\cos\dots$$

$$= \int_0^{2\pi} (\sin^4\theta + \cos^4\theta)\,d\theta = \int_0^{2\pi} (\sin^2\theta + \cos^2\theta)^2 - 2\times\frac{1}{2}\sin^2\dots$$

$$= \int_{\theta=0}^{2\pi} \left(1 - \frac{1}{2}\sin^2 2\theta\right) d\theta = \int_{\theta=0}^{2\pi} 1 - \frac{1}{2}\left(\frac{1-\dots}{2}\right)$$

$$= \theta - \frac{1}{4}\theta + \frac{1}{4}\frac{\sin 4\theta}{4}\Big]_{\theta=0}^{2\pi}$$

$$= 2\pi - \frac{1}{4}2\pi + \frac{1}{16}\sin 4(2\pi)$$

$$= \frac{3\pi}{2} + 0 = 3\pi/2$$

Stoke's theorem is verified

Scanned by CamScanner

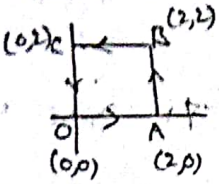
FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Stoke's theorem for

$\bar{F} = (y-z+2)\bar{i} + (yz+4)\bar{j} - xz\bar{k}$ where

'S' is the surface of the cube

$x=0, y=0, z=0, x=2, y=2, z=2$ above the xy-plane.



Soln By Stoke's theorem

$$\int_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\bar{n}\, ds.$$

Along $\vec{OA}$, $y=0$, $dy=0$, $z=2$, x varies from 0 to 2

$$\int_{OA} \bar{F}\cdot d\bar{r} = \int_0^2 (0-2+2)\,dx = 0.$$

Along $\vec{AB}$, $x=2$, $dx=0$, $z=2$, y varies from 0 to 2

$$\int_{AB} \bar{F}\cdot d\bar{r} = \int_0^2 (2y+4)\,dy = \left[y^2+4y\right]_0^2 = 4+8 = 12$$

Along $\vec{BC}$, $y=2$, $z=2$, $dy=0$, x varies from 2 to 0

$$\int_{BC} \bar{F}\cdot d\bar{r} = \int_2^0 2\,dx = 2x\Big]_2^0 = -4$$

Along $\vec{CO}$, $x=0$, $dx=0$, $z=2$, y varies from 2 to 0

$$\int_{CO} \bar{F}\cdot d\bar{r} = \int_2^0 (2y+4)\,dy = 2y^2+4y\Big]_2^0 = -12$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_{OA} \bar{F}\cdot d\bar{r} + \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r} + \int_{CO} \bar{F}\cdot d\bar{r}$$

$$= 0 + 12 - 4 - 12 = -4$$

Verification: $\iint_S (\nabla\times\bar{F})\cdot\bar{n}\, ds$

$$\nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y-z+2 & yz+4 & -xz \end{vmatrix}$$

$$= \bar{i}(-y) + \bar{j}(-1+z) + \bar{k}(0-1)$$

$$= -y\bar{i} + (z-1)\bar{j} - \bar{k}$$

$$\iint_S (\nabla\times\bar{F})\cdot\bar{n}\, ds = \int_0^2\int_0^2 (-y\bar{i}+(z-1)\bar{j}-\bar{k})\cdot\bar{k}\, dx\, dy$$

$$= \int_0^2\int_0^2 -dx\,dy = -\int_0^2 dx\, y\Big]_0^2$$

$$= -2\int_0^2 dx = -2x\Big]_0^2 = -4.$$

Stoke's theorem verified.

4) Evaluate $\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F} = y\bar{i} + xz^2\bar{j} - xy^2\bar{k}$

C is the circle $x^2+y^2=4$, $z=1.5$.

By Stoke's theorem $\oint_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\bar{n}\, ds$

$$\nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & xz^2 & -xy^2 \end{vmatrix} = \bar{i}(-2xy-2xz) + \bar{k}(z^2-1).$$

$$(\nabla\times\bar{F})\cdot\bar{n}\, ds = (\nabla\times\bar{F})\cdot\bar{k}\, ds = (z^2-1)\, ds.$$

Let $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$.

$$\iint_S (\nabla\times\bar{F})\cdot\bar{n}\, ds = \int_{r=0}^{2}\int_{\theta=0}^{2\pi} \left[\left(\frac{3}{2}\right)^2 - 1\right] r\,dr\,d\theta.$$

$$= \frac{19}{4}\int_0^2 r\,dr \cdot \pi = \frac{19}{4}\left(\frac{r^2}{2}\right)_0^2 \pi$$

$$= \frac{19}{4}\times\frac{4}{2}\times\pi$$

$$= \frac{19}{2}\pi$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

5) Evaluate by Stoke's theorem

$\oint_C (yz\,dx + zx\,dy + xy\,dz)$ where C is the circle $x^2+y^2=1$, $z=y^2$.

Sol:- By Stoke's theorem

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\bar{n}\,dS.$$

$\bar{F} = yz\bar{i} + zx\bar{j} + xy\bar{k}$

$$\nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & zx & xy \end{vmatrix}$$

$= \bar{i}(x-x) + \bar{j}(y-y) + \bar{k}(z-z)$

$= \bar{0}$

$\iint_S \bar{0}\cdot\bar{n}\,dS = 0$

6) If 'S' be the surface of the sphere $x^2+y^2+z^2=1$, prove that

$\iint_S (\nabla\times\bar{F})\cdot d\bar{S} = 0$

Sol:- Let S_1 and S_2 be the upper and lower portions of the sphere $x^2+y^2+z^2=1$ by any plane. Let 'c' be the boundary for S_1 and S_2.

$\iint_{S_1} (\nabla\times\bar{F})\cdot\bar{n}\,dS = \iint_{S_2} (\nabla\times\bar{F})\cdot\bar{n}\,dS$

Reversing the direction of S_2 in the direction of S_1

$\iint_S (\nabla\times\bar{F})\cdot\bar{n}\,dS - \iint_{S_2} (\nabla\times\bar{F})\cdot\bar{n}\,dS$

Applying Stoke's theorem

$$\iint_S (\nabla\times\bar{F})\cdot\bar{n}\,ds = \int_C \bar{F}\cdot d\bar{r} - \int_C \bar{F}\cdot d\bar{r}$$

$= 0$.

7) Prove that $\int_C (\bar{A}\times\bar{R})\cdot d\bar{R} = 2\bar{A}$, $\bar{A}$ being any constant vector, and deduce that $\oint_C \bar{R}\times d\bar{R}$ is twice the vector area of the surface enclosed

Sol:- By Stoke's theorem

$$\int_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\bar{n}\,ds$$

Here $\bar{F} = \bar{A}\times\bar{R}$, $\because \bar{A}$ is constant

$$\int_C (\bar{A}\times\bar{R})\cdot d\bar{R} = \iint_S \nabla\times(\bar{A}\times\bar{R})\cdot d\bar{s}$$

$\nabla\times(\bar{A}\times\bar{R}) = (\nabla\cdot\bar{R})\bar{A} - (\nabla\cdot\bar{A})\bar{R}$

$= 3\bar{A} - \bar{A} = 2\bar{A}$

Here $\bar{F} = \bar{A}\times\bar{R}$ where $\bar{A}$ is constant

$\oint_C (\bar{A}\times\bar{R})\cdot d\bar{R} = \int_S \nabla\times(\bar{A}\times\bar{R})\cdot\bar{n}\,ds$

$\oint_C \bar{A}\cdot(\bar{R}\times d\bar{R}) = \int_S (\nabla\cdot\bar{R})\bar{A} - (\nabla\cdot\bar{A})\bar{R}\cdot\bar{n}\,ds$

$\bar{A}\cdot\oint_C (\bar{R}\times d\bar{R}) = \int_S (3\bar{A} - \bar{A})\cdot\bar{n}\,ds$

$= \int_S 2\bar{A}\cdot\bar{n}\,ds$

$= \int_S \bar{A}\cdot 2\bar{n}\,ds$

$\oint_C (\bar{R}\times d\bar{R}) = \int_S 2\bar{n}\,ds$

$\oint_C (\bar{R}\times d\bar{R}) = 2\int_S d\bar{s}$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Scalar point function...

use stoke's theorem to prove that (i) curl (grad ϕ) = $\bar{0}$ (ii) div curl $\bar{F}$ = 0

(i) Let $\bar{F} = \nabla\phi$, so by stoke's theorem

$$\int_S (\nabla \times \bar{F}) \cdot \bar{n}\, ds = \int_S (\nabla \times \nabla\phi) \cdot \bar{n}\, ds.$$

$$= \oint \nabla\phi \cdot d\bar{r}$$

$$= \oint_c d\phi = 0.$$

$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) dx\,dy\,dz$

$\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = d\phi$

also circulation $\oint_c d\phi = 0$ along any closed curve being initial and end points are one and the same.

$\int_S (\nabla \times \nabla\phi) \cdot \bar{n}\, ds = 0$ over the surface S, which implies that

curl (grad ϕ) = $\bar{0}$

(ii) Let 'V' be any volume enclosed by a closed curve S. Then by Gauss's divergence theorem

$$\iiint_V \nabla \cdot (\text{curl } \bar{F})\, dv = \iint_S (\text{curl } \bar{F}) \cdot \bar{n}\, ds. \quad -(1)$$

Divide the surface S into two portions S_1 and S_2 by a closed curve C



$$\iint_S (\text{curl } \bar{F}) \cdot \bar{n}\, ds$$

$$\iint_{S_1} (\text{curl } \bar{F}) \cdot \bar{n}\, ds_1 + \iint_{S_2} (\text{curl } \bar{F}) \cdot \bar{n}\, ds_2$$

$$= \int_C \bar{F} \cdot d\bar{r} - \int_C \bar{F} \cdot d\bar{r} = 0$$

By stoke's theorem, since the integrals along the boundaries of S_1 and S_2 are opposite

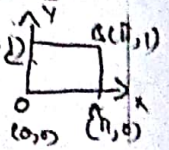
$$(1) \Rightarrow \iiint \nabla \cdot (\text{curl } \bar{F})\, dv = 0$$

Since this is true for all volume elements V, we have

$\nabla \cdot \text{curl } \bar{F} = 0 \Rightarrow dv\,(\text{curl } \bar{F}) = 0$.

2.

9) Evaluate $\oint_c (\sin z\, dx - \cos x\, dy + \sin y\, dz$ where 'c' is the boundary of the rectangle $0 \le x \le \pi$, $0 \le y \le 1$, $z = 3$.



Sol:- Along OA, $y = 0$, $dy = 0$

x varies from 0 to π.

$$\int_{OA} \bar{F} \cdot d\bar{r} = \int_0^{\pi} \sin 3\, dx = \sin 3 \cdot \pi$$

Along AB, $x = \pi$, $dx = 0$, y varies from 0 to 1

$$\int_{AB} \bar{F} \cdot d\bar{r} = \int_0^1 -\cos\pi\, dy = -\cos\pi = 1$$

Along BC, $y = 1$, $dy = 0$, y varies from π to

$$\int_{BC} \bar{F} \cdot d\bar{r} = \int_{\pi}^{0} -\sin 3\, dx = -\sin 3\,\pi$$

Along CO, $x = 0$, $dx = 0$, y varies from 1 to

$$\int_{CO} \bar{F} \cdot d\bar{r} = \int_1^0 -\cos 0\, dy = -y\Big]_1^0 = 1$$

$$\oint_c \bar{F} \cdot d\bar{r} = \int_{OA} \bar{F} \cdot d\bar{r} + \int_{AB} \bar{F} \cdot d\bar{r} + \int_{BC} \bar{F} \cdot d\bar{r} + \int_{CO} \bar{F} \cdot d\bar{r}$$

$$= \sin 3\pi + 1 - \sin 3\pi + 1$$

$$= 2.$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
FirstRanker Service
www.FirstRanker.com www.FirstRanker.com

Q) Stoke's theorem to evaluate

$\int (\nabla \times \bar{F}) \cdot \bar{n}\, ds$ where $\bar{F} = yi + (x - 2xz)j - xyk$

$= \nabla \times \bar{F}$

and 's' is the surface of the sphere $x^2+y^2+z^2 = a^2$ above the xy-plane.

Sol:- By stoke's theorem

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, ds$$

$$\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & x-2xz & -xy \end{vmatrix}$$

$= i(-x+2x) + j(0+y) + k(1-2z-x)$

$= xi + yj - 2zk$

above xy-plane of the sphere

$\bar{N}$ is unit outward normal along z-axis i.e. $x^2+y^2 = a^2$ [$\because z=0$]

$$\iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, ds = \int_0^a \int_0^{2\pi} (xi + yj - 2zk) \cdot k \, dx\, dy$$

$$= \int_0^a \int_0^{2\pi} -2(0)\, dx\, dy \quad [\because z=0]$$

$= 0$

b) Evaluate $\int (\nabla \times \bar{v}) \cdot ds$ over the surface of the paraboloid $z = 1 - x^2 - y^2$, $z \geq 0$ where $\bar{v} = yi + 2j + xk$

Sol:- By stoke's theorem

$$\oint_C \bar{v} \cdot d\bar{r} = \iint_S (\nabla \times \bar{v}) \cdot d\bar{s}$$

$$\nabla \times \bar{v} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix}$$

$= -i - j - k = -(i + j + k)$

$\bar{N} = \bar{k}$, for $z \geq 0$

$x^2+y^2 = 1$, $x = r\cos\theta$, $y = r\sin\theta$, $dx\, dy = r\, dr\, d\theta$

$dx = -r\sin\theta\, d\theta$, $dy = r\cos\theta\, d\theta$

$$\iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, ds = \int_0^{2\pi} \int_0^1 -(i+j+k) \cdot k \, dx\, dy$$

$$= -\int_0^1 \int_0^{2\pi} dx\, dy = -\int_{\theta=0}^{2\pi} \int_0^1 r\, dr\, d\theta$$

$$= -\int_0^1 2\pi \cdot r\, dr$$

$$= -2\pi \cdot \frac{r^2}{2}\Big]_0^1$$

$= -\pi$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com

Stoke's theorem for $\bar{F} = -y^3\bar{i} + x^3\bar{j}$ closed curve www.FirstRanker.com

'C' is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol:- $\bar{F} = -y^3\bar{i} + x^3\bar{j}$ Let $x = a\cos\theta$, $y = b\sin\theta$

$dx = -a\sin\theta\, d\theta$, $dy = b\cos\theta\, d\theta$.

$$\int_C \bar{F}\cdot d\bar{r} = \int_C (-y^3\bar{i} + x^3\bar{j})(dx\bar{i} + dy\bar{j}) = \int_C -y^3dx + x^3dy.$$

$$= \int_0^{2\pi} (-b^3\sin^3\theta)(-a\sin\theta\, d\theta) + (a^3\cos^3\theta)(b\cos\theta\, d\theta)$$

$$= ab^3\int_0^{2\pi} \sin^4\theta\, d\theta + a^3b\int_0^{2\pi} \cos^4\theta\, d\theta$$

$$= ab^3 \cdot 4\int_0^{\pi/2} \sin^4\theta\, d\theta + a^3b \cdot 4\int_0^{\pi/2} \cos^4\theta\, d\theta.$$

$$= ab^3 \cdot 4 \cdot \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + a^3b \cdot 4 \cdot \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi ab}{4}(a^2+b^2)$$

$$\text{curl } \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^3 & x^3 & 0 \end{vmatrix} = \bar{k}(3x^2+3y^2)$$

$$\int_S (\nabla\times\bar{F})\cdot\bar{n}\, ds = \int_S \bar{k}(3x^2+3y^2)\cdot\bar{k}\, ds = 3\int_S (x^2+y^2)\, ds$$

polar coordinates be $x = ar\cos\theta$, $y = br\sin\theta$

$dx\, dy = ab\, r\, dr\, d\theta$

'r' varies from 0 to 1

'θ' " " 0 to 2π

$$\int_S (\nabla\times\bar{F})\cdot\bar{n}\, ds = 3\int_0^{2\pi}\int_0^1 (a^2r^2\cos^2\theta + b^2r^2\sin^2\theta)\, r\, ab\, dr\, d\theta$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com

$$\int_0^{\pi}\int_0^{} a^3 b r^3 \cos^3\theta \, dr\, d\theta \qquad 3\int_0^{2\pi}\int_0^{1} ab^3 r^3 \sin^2\theta \, dr\, d\theta$$

$$= 3a^3b \left(\frac{r^4}{4}\right)_0^1 \int_0^{2\pi} \cos^2\theta \, d\theta + 3ab^3 \left(\frac{r^4}{4}\right)_0^1 \int_0^{2\pi} \sin^2\theta \, d\theta .$$

$$= \frac{3a^3b}{4} \times 4 \int_0^{\pi/2} \cos^2\theta \, d\theta + \frac{3ab^3}{4} \times 4 \int_0^{\pi/2} \sin^2\theta \, d\theta .$$

$$= 3a^3b \cdot \frac{1}{2}\cdot\frac{\pi}{2} + 3ab^3 \cdot \frac{1}{2}\cdot\frac{\pi}{2} = \frac{3ab\pi}{4}(a^2+b^2)$$

L.H.S = R.H.S Stoke's theorem verified.

2.

a) For a solenoidal vector $\bar{F}$, show that

curl curl curl curl $\bar{F} = \nabla^4 \bar{F}$

Sol:- A vector $\bar{F}$ is said to be solenoidal if $\nabla \cdot \bar{F} = 0$

we know that $\nabla \times (\nabla \times \bar{F}) = (\nabla \cdot \bar{F})\nabla - (\nabla \cdot \nabla)\bar{F}$

$= (\nabla \cdot \bar{F})\nabla - \nabla^2 \bar{F}$

$$\nabla \times (\nabla \times \bar{F}) = -\nabla^2 \bar{F} \quad [\because \nabla \cdot \bar{F} = 0].$$

$$\nabla \times \nabla \times [\nabla \times (\nabla \times \bar{F})] = \nabla \times [\nabla \times (-\nabla^2 \bar{F})]$$

$$= \nabla \cdot (-\nabla^2 \bar{F})\nabla - (\nabla \cdot \nabla)(-\nabla^2 \bar{F})$$

$$= \nabla [\nabla \cdot [\nabla \times (\nabla \times \bar{F})]] + \nabla^4 \bar{F}$$

$$= \nabla [\text{div}\,[\text{curl}\,(\nabla \times \bar{F})]] + \nabla^4 \bar{F}$$

$$= \nabla^4 \bar{F} \quad [\because \text{div}\,[\text{curl}\,(\nabla \times \bar{F})] = 0]$$

2.

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Gauss's Divergence theorem :- Let S be a closed surface enclosed a volume V. If $\bar{F}$ is a continuously differentiable vector point function, then $\int_V \text{div}\,\bar{F}\,dV = \int_S \bar{F}\cdot\bar{N}\,dS$ where $\bar{N}$ is the outward drawn unit normal vector at any point of S

Cartesian form :- Let $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ and $\bar{N} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$.

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the d.c's of $\bar{N}$.

$$\bar{F}\cdot\bar{N} = F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma.$$

Also $$\text{div}\,\bar{F} = \nabla\cdot\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Hence the divergence theorem can be written as.

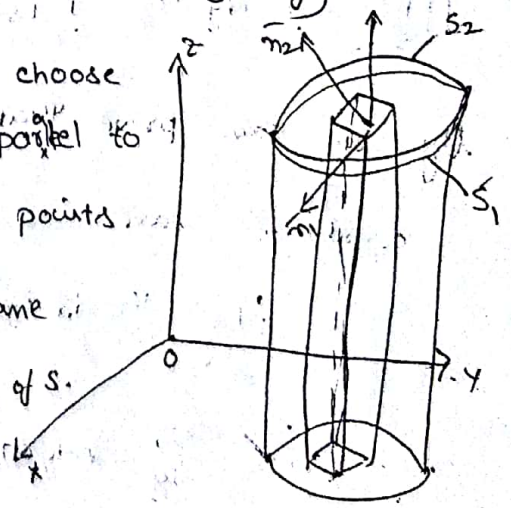
$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz = \int_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\,dS.$$

$$= \iint_S (F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy)$$

Proof :- Let S be a closed surface. Let us choose the coordinate axes so that any line parallel to the coordinate axes cuts S in at most two points.

Let R be the projection of S on xy-plane.

S_1 and S_2 be the lower and upper parts of S.



$z = f(x,y)$ and $z = g(x,y)$ be the equations of S_1 and S_2.

The surface can be put in the form $f(x,y) \le z \le g(x,y)$.

$$\iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz = \iint_R \left[\int_{z=f(x,y)}^{z=g(x,y)} \frac{\partial F_3}{\partial z}\,dz\right] dx\,dy.$$

$$= \iint_R \left[F_3(x,y,z)\right]_f^g\,dx\,dy$$

$$= \iint_R \left[F_3(x,y,g) - F_3(x,y,f)\right] dx\,dy.$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$$\iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz = \iint_R F_3(x,y,f)\,dx\,dy - \iint_R F_3(x,y,g)\,dx\,dy \quad -(1)$$

$ds = \frac{dx\,dy}{|\vec{N}\cdot\vec{k}|}$

For the upper part S_2, $\boxed{z = g(x,y)}$ $\quad dx\,dy = ds\cos\gamma$

$= (\vec{N}\cdot\vec{k})\,ds$

Since the normal to S_2 makes an acute angle γ with $\vec{k}$.

$$\iint_R F_3(x,y,g)\,dx\,dy = \int_{S_2} F_3\,\vec{N}\cdot\vec{k}\,ds \quad -(2)$$

For the lower portion S_1, $\boxed{z = f(x,y)}$ $\quad dx\,dy = -ds\cos\gamma$

$= -(\vec{N}\cdot\vec{k})\,ds$

Since the normal to S_1 makes an obtuse angle γ with $\vec{k}$

$$\iint_R F_3(x,y,f)\,dx\,dy = -\int_{S_1} F_3\,(\vec{n}\cdot\vec{k})\,ds \quad -(3)$$

from (1) (2) and (3).

$$\iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz = \int_{S_2} F_3(\vec{N}\cdot\vec{k})\,ds + \int_{S_1} F_3(\vec{n}\cdot\vec{k})\,ds = \int_S F_3(\vec{n}\cdot\vec{k})\,ds$$

By use can prove that, by Projecting S on the other coordinate planes

$$\iiint_V \frac{\partial F_1}{\partial x}\,dx\,dy\,dz = \int_S F_1\,(\vec{N}\cdot\vec{i})\,ds \quad -(ii)$$

$$\iiint_V \frac{\partial F_2}{\partial y}\,dx\,dy\,dz = \int_S F_2\,(\vec{N}\cdot\vec{j})\,ds \quad -(iii)$$

Adding (1) (ii) and (iii)

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz = \int_S (F_1\vec{i} + F_2\vec{j} + F_3\vec{k})\cdot\vec{N}\,ds$$

$$\iiint_V \operatorname{div}\vec{F}\,dv = \int_S \vec{F}\cdot\vec{N}\,ds$$

=,

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

1) Verify Divergence theorem, $\int \bar{F} = (x-yz)\bar{i} + (y-zx)\bar{j} + (z-xy)\bar{k}$ taken over a rectangular parallelepiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le b$

Sol:- $\nabla \cdot \bar{F} = \frac{\partial}{\partial x}(x-yz) + \frac{\partial}{\partial y}(y-zx) + \frac{\partial}{\partial z}(z-xy) = 3x+3y+3z$

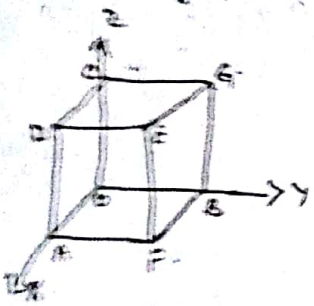
$$\iiint_V (\nabla \cdot \bar{F})\,dv = \iiint 3(x+y+z)\,dx\,dy\,dz = 3\int_{z=0}^{a}\int_{y=0}^{a}\int_{x=0}^{a} (x+y+z)\,dx\,dy\,dz.$$

$$= 3\int_{z=0}^{a}\int_{y=0}^{a}\left(\frac{x^2}{2}+xy+zx\right)_0^a dy\,dz = 3\int_{z=0}^{a}\int_{y=0}^{a}\left(\frac{a^2}{2}+ay+az\right)dy\,dz.$$

$$= 3\int_{z=0}^{c}\left[\frac{a^2}{2}y + a\frac{y^2}{2} + azy\right]_0^b dz = 3\int_{z=0}^{c}\left[\frac{a^2b}{2}+\frac{ab^2}{2}+abz\right]dz.$$

$$= 3\left[\frac{a^2b}{2}z + \frac{ab^2}{2}z + ab\frac{z^2}{2}\right]_0^c.$$

$$\frac{3}{2}a^2bc + \frac{3}{2}ab^2c + \frac{3}{2}abc^2 = \frac{3}{2}abc(a+b+c)$$



To calculate $\iint_S \bar{F}\cdot\bar{n}\,ds$ over the six faces of the rectangular parallelepiped over the face

(1) over the face DEFA, $\bar{n} = \bar{i}$, $x = a$, $ds = dy\,dz$.

$$\iint_{DEFA} \bar{F}\cdot\bar{n}\,ds = \int_{z=0}^{c}\int_{y=0}^{b}\left[(a^2-yz)\bar{i} + (y^2-za)\bar{j} + (z^2-ay)\bar{k}\right]\cdot\bar{i}\,dy\,dz.$$

$$= \int_{z=0}^{c}\int_{y=0}^{b}(a^2-yz)\,dy\,dz = \int_{z=0}^{c}\left[a^2y - z\frac{y^2}{2}\right]_0^b dz = \int_{z=0}^{c}\left(a^2b - z\frac{b^2}{2}\right)dz.$$

$$= a^2bz - b^2\frac{z^2}{4}\Big]_0^c = a^2bc - \frac{c^2b^2}{4}$$

(2) over the face BGCO, $\bar{n} = -\bar{i}$, $ds = dy\,dz$, $x = 0$.

$$\iint_{BGCO} \bar{F}\cdot\bar{n}\,ds = \int_{z=0}^{c}\int_{y=0}^{b} yz\,dy\,dz = \int_0^c \frac{y^2}{2}z\Big]_0^b dz = \int_0^c \frac{b^2}{2}z\,dz$$

$$= \frac{b^2}{2}\int_0^c z\,dz = \frac{b^2}{2}\frac{z^2}{2}\Big]_0^c = \frac{b^2c^2}{4}.$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

3) Over the face BGEF, $\hat{n} = \hat{j}$, $y = b$. $ds = dx\,dz$

$$\iint_{BGEF} \vec{F}\cdot\hat{n}\,ds = \int_{z=0}^{c}\int_{x=0}^{a} (b^2 - zx)\,dx\,dz = \int_{z=0}^{c}\left[b^2x - z\frac{x^2}{2}\right]_0^a dz = \int_{z=0}^{c}\left[b^2a - z\frac{a^2}{2}\right]dz$$

$$= b^2az - \frac{a^2}{2}\frac{z^2}{2}\Big]_0^c = ab^2c - \frac{a^2c^2}{4}.$$

4) over the face OADC, $\hat{n} = -\hat{j}$, $y = 0$

$$\iint_{OADC} \vec{F}\cdot\hat{n}\,ds = \int_{z=0}^{c}\int_{x=0}^{a} zx\,dx\,dz = \int_{z=0}^{c} z\frac{x^2}{2}\Big]_0^a dz = \int_{z=0}^{c} z\frac{a^2}{2}\,dz = \frac{a^2}{2}\frac{z^2}{2}\Big]_0^c$$

5) over the face GCDE, $\hat{n} = \hat{k}$, $z = c$, $ds = dx\,dy$.

$$\iint_{GCDE} \vec{F}\cdot\hat{n}\,ds = \int_{y=0}^{b}\int_{x=0}^{a} (c^2 - xy)\,dx\,dy = \int_{y=0}^{b} c^2x - \frac{x^2}{2}y\Big]_0^a dy = \int_{y=0}^{b}\left(c^2a - \frac{a^2}{2}y\right)$$

6) over the face AFBO, $\hat{n} = -\hat{k}$, $z = 0$, $ds = dx\,dy$. $= c^2ay - \frac{a^2}{2}\frac{y^2}{2}\Big]_0^b$

$$\iint_{AFBO} \vec{F}\cdot\hat{n}\,ds = \int_{y=0}^{b}\int_{x=0}^{a} xy\,dx\,dy = \int_{y=0}^{b} \frac{x^2}{2}y\Big]_0^a dy = \int_{y=0}^{b} \frac{a^2}{2}y\,dy = \frac{a^2}{2}\int_0^b$$

$$= \frac{a^2b^2}{4}$$

Adding the integrals over the six faces

$$\iint_S \vec{F}\cdot\hat{n}\,ds = a^2bc - \frac{c^2b^2}{4} + c^2b^2 + b^2ca - \frac{a^2b^2}{4} + c^2ab - \frac{a^2b^2}{4} + \frac{a^2b^2}{4}$$

$$= a^2bc + b^2ca + c^2ab = abc\,(a+b+c)$$

Hence $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V (\nabla\cdot\vec{F})\,dv$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com

Show that $\int_S \bar{r}\cdot\bar{N}\,ds = 3V$ where 'V' is the volume enclosed by the surface 'S'. www.FirstRanker.com

Sol:- By Gauss divergence theorem $\int_S \bar{R}\cdot\bar{N}\,ds = \iiint_V \text{div}\,\bar{R}\,dv$

$\nabla\cdot\bar{R} = \text{div}\,\bar{R} = 3$ where $\bar{R} = x\bar{i}+y\bar{j}+z\bar{k}$.

$$\int_S \bar{R}\cdot\bar{N}\,ds = \iiint_V 3\,dv = 3V.$$

3) For any closed surface 'S', P.T $\int_S (\nabla\times\bar{F})\cdot\bar{N}\,ds = 0$

Sol:- By divergence theorem $\int_S (\nabla\times\bar{F})\cdot\bar{N}\,ds = \iiint_V \text{div}(\text{curl}\,\bar{F})\,dv$

where 'V' is the volume enclosed by 'S'.

But we know that $\text{div}(\text{curl}\,\bar{F}) = 0$.

$$\therefore \int_S (\nabla\times\bar{F})\cdot\bar{N}\,ds = 0.$$

4) use divergence theorem to show that $\int_S \nabla r^2\cdot\bar{N}\,ds = 6V$

Sol:- $\nabla r^2 = \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right) r^2 = 2r\frac{\partial r}{\partial x}\bar{i} + 2r\frac{\partial r}{\partial y}\bar{j} + 2r\frac{\partial r}{\partial z}\bar{k}$.

$$= 2r\left(\frac{x}{r}\bar{i}+\frac{y}{r}\bar{j}+\frac{z}{r}\bar{k}\right)$$

$$= 2(x\bar{i}+y\bar{j}+z\bar{k}) = 2\bar{R}$$

$$\nabla\cdot\nabla r^2 = \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\cdot(2x\bar{i}+2y\bar{j}+2z\bar{k})$$

$$= 2+2+2 = 6.$$

$$\int_S \nabla r^2\cdot\bar{N}\,ds = \iiint_V \nabla\cdot\nabla r^2\,dv = \iiint_V 6\,dv = 6V.$$

5) If $\bar{F} = ax\bar{i}+by\bar{j}+cz\bar{k}$ a, b, c are constants, show that $\int_S \bar{F}\cdot\bar{N}\,ds = \frac{4}{3}\pi(a+b+c)$ where 'S' is the surface of a unit sphere

Sol:- By Gauss divergence theorem $\int_S \bar{F}\cdot\bar{N}\,ds = \iiint_V \text{div}\,\bar{F}\,dv$.

$$\text{div}\,\bar{F} = \frac{\partial}{\partial x}(ax)+\frac{\partial}{\partial y}(by)+\frac{\partial}{\partial z}(cz) = a+b+c$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$\int_S \bar{F}\cdot\bar{N}\,ds = a+b+c \iiint_V dv = (a+b+c)V$

where 'V' is the volume of the unit sphere $= \frac{4}{3}\pi r^3$

$= \frac{4}{3}\pi(1)^3$

$= \frac{4}{3}\pi$

$\therefore \int_S \bar{F}\cdot\bar{N}\,ds = \frac{4}{3}\pi(a+b+c)$

6) $\oint_S (a^2x^2+b^2y^2+c^2z^2)^{+1/2}\,ds$ over the ellipsoid $ax^2+by^2+cz^2=1$

Sol:- By G.D.T $\int_S \bar{F}\cdot\bar{N}\,ds = \iiint_V \text{div}\,\bar{F}\,dv$

Here $\bar{F}\cdot\bar{N} = (a^2x^2+b^2y^2+c^2z^2)^{+1/2}$

and 'S' is the surface of an ellipsoid $ax^2+by^2+cz^2=1$

$\bar{N}$ = unit outward normal $= \frac{\nabla\phi}{|\nabla\phi|}$ where $\phi = ax^2+by^2+cz^2-1$

$= \frac{2ax\bar{i}+2by\bar{j}+2cz\bar{k}}{\sqrt{4a^2x^2+4b^2y^2+4c^2z^2}}$

Given $\bar{F} = ax\bar{i}+by\bar{j}+cz\bar{k}$ $= \frac{ax\bar{i}+by\bar{j}+cz\bar{k}}{(a^2x^2+b^2y^2+c^2z^2)^{1/2}}$

$\text{div}\,\bar{F} = a+b+c$

and 'V' is the volume of an ellipsoid $ax^2+by^2+cz^2=1$

$\int_S (ax^2+by^2+cz^2)^{-1/2}\,ds = \iiint_V (a+b+c)\,dv = (a+b+c)V$

$= (a+b+c)\frac{4\pi}{3}\frac{1}{\sqrt{abc}}$

$\therefore \bar{F}\cdot\bar{N} = (ax\bar{i}+by\bar{j}+cz\bar{k})\cdot\left(\frac{ax\bar{i}+by\bar{j}+cz\bar{k}}{(a^2x^2+b^2y^2+c^2z^2)^{1/2}}\right)$

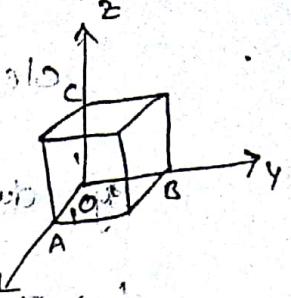
$= \frac{(a^2x^2+b^2y^2+c^2z^2)}{(a^2x^2+b^2y^2+c^2z^2)^{1/2}} = (a^2x^2+b^2y^2+c^2z^2)^{1/2}$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
www.FirstRanker.com www.FirstRanker.com

7) Evaluate $\int_S \bar{F}\cdot\bar{N}\,ds$ where 'S' is the surface of the cube bounded by $x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$

and $\bar{F} = 4xz\,\bar{i} - y^2\,\bar{j} + yz\,\bar{k}$



Sol: By Gauss divergence theorem $\int_S \bar{F}\cdot\bar{N}\,ds = \iiint_V \operatorname{div}\bar{F}\,dv$

$$\operatorname{div}\bar{F} = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)$$

$$= 4z - 2y + y = 4z - y$$

$$\int_S \bar{F}\cdot\bar{N}\,ds = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (4z-y)\,dz\,dy\,dx = \int_{x=0}^{1}\int_{y=0}^{1}\left[4\frac{z^2}{2} - yz\right]_0^1 dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} (2-y)\,dy\,dx$$

$$= \int_{x=0}^{1}\left(2y - \frac{y^2}{2}\right)_0^1 dx = \int_0^1 \left(2-\frac{1}{2}\right)dx$$

$$= \frac{3}{2}\int_0^1 dx = \frac{3}{2}$$

8) Evaluate $\int_S (a^2x^2 + b^2y^2 + c^2z^2)^{-1/2}\,ds$, where 'S' is the surface of the ellipsoid $ax^2 + by^2 + cz^2 = 1$.

Sol:- Taking $\phi = ax^2 + by^2 + cz^2 - 1 = 0$

$$\nabla\phi = 2ax\,\bar{i} + 2by\,\bar{j} + 2cz\,\bar{k}$$

$\therefore$ unit vector normal to the ellipsoid $= \bar{N} = \dfrac{\nabla\phi}{|\nabla\phi|}$

$$= \frac{ax\,\bar{i} + by\,\bar{j} + cz\,\bar{k}}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}}$$

since $\bar{F}\cdot\bar{N} = (a^2x^2 + b^2y^2 + c^2z^2)^{-1/2}$

$$\bar{F}\cdot\frac{(ax\,\bar{i} + by + cz\,\bar{k})}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}} = (a^2x^2 + b^2y^2 + c^2z^2)^{-1/2}$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$\therefore \vec{F}\cdot(ax\hat{i}+by\hat{j}+cz\hat{k}) = 1$ [$\because ax^2+by^2+cz^2=1$]

clearly $\vec{F} = x\hat{i}+y\hat{j}+z\hat{k}$

By divergence theorem

$$\int_S \vec{F}\cdot\vec{N}\,ds = \int_V \operatorname{div}\vec{F}\,dv = \int_V \left[\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)\right]dv = 3\int_V dv = 3V$$

$$= 3\cdot\frac{4\pi}{3}\frac{1}{\sqrt{abc}} = \frac{4\pi}{\sqrt{abc}} \quad [\because \text{vol of ellipsoid} = \tfrac{4\pi}{3}\tfrac{1}{\sqrt{abc}}]$$

9) Apply Divergence theorem to evaluate $\int(lx^2+my^2+nz^2)\,ds$ taken over the sphere $(x-a)^2+(y-b)^2+(z-c)^2=\rho^2$; l, m, n being the direction cosines of the external normal to the sphere

Sol:- The parametric equations of the sphere are

$x = a+\rho\sin\theta\cos\phi$, $y = b+\rho\sin\theta\sin\phi$, $z = c+\rho\cos\theta$

and to cover the whole sphere, ρ varies from 0 to ρ, θ varies from 0 to π, ϕ varies from 0 to 2π

$$\int_S (lx^2+my^2+nz^2)\,ds = \int_S (x^2\hat{i}+y^2\hat{j}+z^2\hat{k})\cdot\vec{N}\,ds$$

$$= \int_V \operatorname{div}(x^2\hat{i}+y^2\hat{j}+z^2\hat{k})\,dv = 2\int_V (x+y+z)\,dv$$

$$= 2\int_0^{2\pi}\int_0^{\pi}\int_0^{\rho}[(a+b+c)+\rho(\sin\theta\cos\phi+\sin\theta\sin\phi+\cos\theta)]\times\rho^2\sin\theta\,d\rho\,d\theta\,d\phi$$

$$= 2(a+b+c)\frac{\rho^3}{3}\int_0^{\pi}(1-\cos\theta)\,d\theta\cdot 2\pi$$

$$= \frac{8\pi}{3}(a+b+c)\rho^3.$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Evaluate $\int_S \bar{F}\cdot\bar{N}\,ds$ where $\bar{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ and 'S' is the surface bounding the region $x^2+y^2=4$, $z=0$ and $z=3$

Sol: By divergence theorem $\int_S \bar{F}\cdot\bar{N}\,ds = \int_V \text{div}\,\bar{F}\,dv$ ... (1)

$\nabla\cdot\bar{F} = \frac{d}{dx}(4x) + \frac{d}{dy}(-2y^2) + \frac{d}{dz}(z^2)$

$= 4 - 4y + 2z$

$\int_S \bar{F}\cdot\bar{N}\,ds = \iiint_V (4-4y+2z)\,dx\,dy\,dz$

$y^2 = 4-x^2$

$y = \pm\sqrt{4-x^2}$

$x^2 = 2$

$x = \pm 2$

$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(4-4y+2z)\,dx\,dy\,dz.$

$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[4z - 4yz + 2\frac{z^2}{2}\right]_0^3 dx\,dy$

$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[4z - 4yz + z^2\right]_0^3 dx\,dy = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}[12 - 12y + 9]\,dx\,dy.$

$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(21-12y)\,dx\,dy = \int_{-2}^{2}\left[21y - 12\frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$

$= 42\int_{-2}^{2}\sqrt{4-x^2}\,dx = 84\int_0^2\sqrt{4-x^2}\,dx$

$= 84\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$

$= 84\pi$

$\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

(4) Evaluate $\int_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = 2x^2y\,\bar{i} - y^2\,\bar{j} + 4xz^2\,\bar{k}$ and S is the closed surface of the region in the first octant bounded by the cylinder $y^2+z^2=9$ and the planes $x=0, x=2, y=0$ and $z=0$.

Sol:- By Gauss divergence theorem $\iint_S \bar{F}\cdot\bar{n}\,ds = \iiint_V \text{div}\,\bar{F}\,dv$.

$$= \int_{x=0}^{2}\int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} (4xy - 2y + 8xz)\,dx\,dy\,dz$$

$x = 0$ to 2, $z = \sqrt{9-y^2}$, $z = 0$ to $\sqrt{9-y^2}$, $y = 0$ to 3

$$= \int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} \left[4\frac{x^2}{2}y - 2xy + 8\frac{x^2}{2}z\right]_0^2 dy\,dz = \int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} (2x^2y - 2xy + 4x^2z)\Big|_0^2\,dy\,dz$$

$$= \int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} (8y - 4y + 16z)\,dy\,dz = \int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} (4y + 16z)\,dy\,dz.$$

$$= \int_{y=0}^{3}\left[4yz + 16\frac{z^2}{2}\right]_0^{\sqrt{9-y^2}} dy = \int_{y=0}^{3}\left[4y\sqrt{9-y^2} + 8(9-y^2)\right] dy.$$

$$= 4\int_{y=0}^{3}\left[y\sqrt{9-y^2}\,dy + 2(9-y^2)\right] dy.$$

$9-y^2 = t$
$-2y\,dy = dt$
$y\,dy = -\frac{dt}{2}$

$$= 4\left[-\frac{1}{2}(9-y^2)^{3/2}\cdot\frac{2}{3} + 18y - \frac{2}{3}y^3\right]_{y=0}^{3}$$

$$= 4\left[54 - 18\right] - 4\left[-\frac{1}{2}(9-y^2)^{3/2}\cdot\frac{2}{3}\right]$$

$$= 4\times 36 + 36$$

$$= 144 + 36$$

$$= 180$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Use divergence theorem to evaluate $\iint_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = x^3\bar{i} + y^3\bar{j} + z^3\bar{k}$ and 'S' is the surface of the sphere $x^2+y^2+z^2 = a^2$

According the Gauss divergence theorem

$$\iiint_V \text{div } \bar{F}\, dv = \iint_S \bar{F}\cdot\bar{n}\, ds$$

$$\text{div } \bar{F} = 3(x^2+y^2+z^2)$$

$$\iint_S \bar{F}\cdot\bar{n}\, ds = \iiint_V 3(x^2+y^2+z^2)\, dx\,dy\,dz.$$

using the Spherical coordinates

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

$$dv = r^2\sin\theta\, dr\, d\theta\, d\phi.$$

$$\iint_S \bar{F}\cdot\bar{n}\, ds = 3\int_0^a\int_0^{\pi}\int_0^{2\pi} r^4\sin\theta\, dr\, d\theta\, d\phi$$

$$= 3\left[\int_0^a r^4 dr \int_0^{\pi}\sin\theta\, d\theta \int_0^{2\pi} d\phi\right] = 3\left[\frac{r^5}{5}\right]_0^a (-\cos\theta)_0^{\pi} \cdot 2\pi$$

$$= 3\left[\frac{r^5}{5}\right]_0^a (1+1)\cdot 2\pi$$

$$= \frac{12\pi a^5}{5}.$$

$x^2+y^2+z^2 = r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta.$

$= r^2\sin^2\theta(\cos^2\phi + \sin^2\phi) + r^2\cos^2\theta.$

$= r^2(\sin^2\theta + \cos^2\theta) = r^2$

www.FirstRanker
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

2) Find $\iint_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = (2x+3z)\hat{i} - (xz+y)\hat{j} + (y+2z)\hat{k}$ and 'S' is the surface of the sphere having centre at $(3,-1,2)$ and radius 3.

Sol:- Let 'V' be the volume enclosed by the surface S.

Then Gauss divergence theorem, we have

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iiint_V \text{div}\,\bar{F}\,dv$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(2x+3z) + \frac{\partial}{\partial y}(-xz-y) + \frac{\partial}{\partial z}(y+2z)\right] dv.$$

$$= \iiint_V (2-1+2)\,dv = 3\iiint_V dv = 3V.$$

But 'V' is the volume of a sphere of radius is 3.

$$V = \frac{4}{3}\pi r^3 \quad [r=3]$$

$$V = \frac{4}{3}\pi(3)^3$$

$$= 36\pi$$

Hence $\iint_S \bar{F}\cdot\bar{n}\,ds = 3(36\pi) = 108\pi$

3) Evaluate $\iint (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$ over the surface of a sphere of radius 'a'.

Sol:- By Gauss divergence theorem $\int_S \bar{F}\cdot\bar{n}\,ds = \int_V \text{div}\,\bar{F}\,dv$

(or) $\int_S F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dv$

Here $F_1 = x$, $F_2 = y$, $F_3 = z$ $\Rightarrow \frac{\partial F_1}{\partial x} = 1$, $\frac{\partial F_2}{\partial y} = 1$, $\frac{\partial F_3}{\partial z} = 1$

$$\iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy = \iiint_V (1+1+1)\,dv = 3V$$

$$= 3\,\frac{4}{3}\pi a^3$$

$$= 4\pi a^3$$

[$\because$ 'V' is the volume of the sphere of radius is 'a']

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

If ϕ and ψ are scalar point functions which are uniform and continuous and possessing derivative in any direction within the volume V enclosed by the surface 's' then show that $\iiint_V (\phi \nabla^2\psi - \psi\nabla^2\phi)dv = \int_S (\phi\nabla\psi - \psi\nabla\phi)\cdot\bar{n}\,ds$

By Gauss divergence theorem we have $\int_S \bar{F}\cdot\bar{n}\,ds = \iiint_V \text{div}\,\bar{F}\,dv$ — (1).

taking $\bar{F} = \phi\nabla\psi$ then (1) becomes

$$\int_S (\phi\nabla\psi)\cdot\bar{n}\,ds = \iiint_V \nabla\cdot(\phi\nabla\psi)\,dv = \iiint_V (\nabla\phi\cdot\nabla\psi + \phi\nabla^2\psi)\,dv \quad -(2)$$

Interchanging ϕ and ψ in (2) we have

$$\int_S (\psi\nabla\phi)\cdot\bar{n}\,ds = \iiint_V [\nabla\psi\cdot\nabla\phi) + \psi\nabla^2\phi]\,dv \quad -(3)$$

$[\because \nabla\phi\cdot\nabla\psi = \nabla\psi\cdot\nabla\phi]$

on subtracting $\int_S (\phi\nabla\psi - \psi\nabla\phi)\cdot\bar{n}\,ds = \iiint_V [\phi\nabla^2\psi - \psi\nabla^2\phi]\,dv$ — (4)

which is known as Green's second identity.

2.

Irrotational fields :- An irrotational field $\bar{F}$ is characterised by one of the following conditions (i) $\nabla\times\bar{F} = \bar{0}$ (ii) circulation $\int \bar{F}\cdot d\bar{r}$ along every closed surface is zero.

(iii) $\bar{F} = \nabla\phi$ (for some scalar point function ϕ) if the domain is simply connected.

If $\nabla\bar{F} = 0$ by stokes theorem $\int_C \bar{F}\cdot d\bar{r} = \int_S (\nabla\times\bar{F})\cdot ds = 0$ (ie circulation along every closed surface is zero).

Again since $\text{curl}(\text{grad}\,\phi) = 0$ ie an irrotational field for which $\nabla\times\bar{F} = \bar{0}$; the vector $\bar{F}$ can always be expressed as the gradient of a scalar point function ϕ provided the domain is simply connected.

Thus $\bar{F} = \nabla\phi$ where a scalar function ϕ is called the potential. In a irrotational field $\bar{F}$ cannot be expressed as the gradient of a scalar potential.

Result :- In an irrotational field, the line integral of $\bar{F}$ between two points is independent of the path of integration and is equal to the potential difference between these points.

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com

Let $\bar{F} = \nabla\phi$ in region R. Let A and B be any two points in the region R and C be any path joining A and B in R then we have

$$\int_C \bar{F}\cdot d\bar{r} = \int_C \nabla\phi \cdot d\bar{r} = \int_C \left(\bar{i}\frac{d\phi}{dx} + \bar{j}\frac{d\phi}{dy} + \bar{k}\frac{d\phi}{dz}\right)\cdot(dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k})$$

$$= \int_C \left(\frac{d\phi}{dx}dx + \frac{d\phi}{dy}dy + \frac{d\phi}{dz}dz\right) = \int_C d\phi = \int_A^B d\phi = [\phi]_A^B = \phi_B - \phi_A$$

which shows that the line integral depends only on point A and B and not on path C.

Note 1:- If $\bar{F}$ is a vector force acting on a particle then $\oint \bar{F}\cdot d\bar{r}$ represents the work done in moving the particle around the closed path C. When $\oint_C \bar{F}\cdot d\bar{r} = 0$ the field is said to be conservative. i.e. no work is done in displacement from a point to another point in the field and back to that point and the mechanical energy is conserved. Thus every irrotational field is conservative.

Solenoidal Fields:- A solenoidal field $\bar{F}$ is characterised by any one of the following conditions:-

(i) $\nabla\cdot\bar{F} = 0$ (ii) Flux $\int \bar{F}\cdot\bar{N}\,ds$ across every closed surface is zero since $\nabla\cdot(\nabla\times v) = 0$.

i.e. div (curl v) = 0.

Therefore, in a solenoidal field for which $\nabla\cdot\bar{F} = 0$, then vector $\bar{F}$ can always be expressed as the curl of a vector function v thus $\bar{F} = \nabla\times v$.

2

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice

Cylindrical coordinate system is orthogonal

www.FirstRanker.com | www.FirstRanker.com

Any point P, we have $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$ so that

$$\bar{R} = \rho\cos\phi\,\bar{i} + \rho\sin\phi\,\bar{j} + z\bar{k}$$

If T_ρ, T_ϕ, T_z are unit vectors at P in the directions of the tangents to the ρ, ϕ, z curves then

$$T_\rho = \frac{\frac{\partial\bar{R}}{\partial\rho}}{|\partial\bar{R}/\partial\rho|} = \frac{\cos\phi\,\bar{i} + \sin\phi\,\bar{j}}{\sqrt{\cos^2\phi + \sin^2\phi}} = \cos\phi\,\bar{i} + \sin\phi\,\bar{j}$$

$$T_\phi = \frac{\frac{\partial\bar{R}}{\partial\phi}}{|\partial\bar{R}/\partial\phi|} = \frac{-\rho\sin\phi\,\bar{i} + \rho\cos\phi\,\bar{j}}{\sqrt{\rho^2\sin^2\phi + \rho^2\cos^2\phi}} = -\sin\phi\,\bar{i} + \cos\phi\,\bar{j}$$

$$T_z = \frac{\frac{\partial\bar{R}}{\partial z}}{|\partial\bar{R}/\partial z|} = \frac{\bar{k}}{1} = \bar{k}$$

Now $T_\rho \cdot T_\phi = (\cos\phi\,\bar{i} + \sin\phi\,\bar{j}) \cdot (-\sin\phi\,\bar{i} + \cos\phi\,\bar{j})$

$= 0$

$T_\phi \cdot T_z = (-\sin\phi\,\bar{i} + \cos\phi\,\bar{j}) \cdot \bar{k} = 0$

$T_\rho \cdot T_z = (\cos\phi\,\bar{i} + \sin\phi\,\bar{j}) \cdot \bar{k} = 0.$

∴ The cylindrical coordinate system is orthogonal.

Also $T_\rho \times T_\phi = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \end{vmatrix} = \bar{k}(\cos^2\phi + \sin^2\phi) = \bar{k} = T_z$

$T_\phi \times T_z = T_\rho$, $T_z \times T_\rho = T_\phi$

∴ The cylindrical coordinate system is a right handed orthogonal coordinate system

www.FirstRanker.com

Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com

Del applied to functions in cylindrical coordinates.

www.FirstRanker.com

① Expression for grad f

$$\nabla f = \frac{\partial f}{\partial \rho} T_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \phi} T_\phi + \frac{\partial f}{\partial z} T_z$$

② Expression for div $\bar{F}$ where $\bar{F} = f_1 T_u + f_2 T_v + f_3 T_w$

$$\nabla \cdot \bar{F} = \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho f_1) + \frac{\partial f_2}{\partial \phi} + \frac{\partial}{\partial z} (\rho f_3) \right]$$

③ Expression for curl $\bar{F}$

$$\nabla \times \bar{F} = \begin{vmatrix} T_\rho/\rho & T_\phi & T_z/\rho \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ f_1 & \rho f_2 & f_3 \end{vmatrix}$$

④ Expression for $\nabla^2 f$

$$\nabla^2 f = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial f}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Problem ① Express the vector $z\hat{i} - 2x\hat{j} + y\hat{k}$ in cylindrical coordinates.

Sol:- we have $x = \rho\cos\phi$; $y = \rho\sin\phi$, $z = z$

So that $\bar{R} = \rho\cos\phi\,\hat{i} + \rho\sin\phi\,\hat{j} + z\hat{k}$

If T_ρ, T_ϕ, T_z are unit vectors along the tangents to ρ, ϕ and z respectively

$$T_\rho = \frac{\partial \bar{R}}{\partial \rho} \Big/ \left| \frac{\partial \bar{R}}{\partial \rho} \right| = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$

$$T_\phi = \frac{\partial \bar{R}}{\partial \phi} \Big/ \left| \frac{\partial \bar{R}}{\partial \phi} \right| = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

$$T_z = \partial\bar{R}/\partial z \Big/ \left| \frac{\partial \bar{R}}{\partial z} \right| = \hat{k}$$

www.FirstRanker.com

Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Given $\bar{F} = z\bar{i} - 2x\bar{j} + y\bar{k}$

Let the expression for $\bar{F}$ in cylindrical coordinates be

$$\bar{F} = f_1 T_\rho + f_2 T_\phi + f_3 T_z$$

where $f_1 = \bar{F} \cdot T_\rho$, $f_2 = \bar{F} \cdot T_\phi$, $f_3 = \bar{F} \cdot T_z$

$$f_1 = (z\bar{i} - 2x\bar{j} + y\bar{k}) \cdot (\cos\phi\, \bar{i} + \sin\phi\, \bar{j})$$

$$= z\cos\phi - 2x\sin\phi$$

$$f_2 = (z\bar{i} - 2x\bar{j} + y\bar{k}) \cdot (-\sin\phi\, \bar{i} + \cos\phi\, \bar{j})$$

$$= -z\sin\phi - 2x\cos\phi$$

$$f_3 = (z\bar{i} - 2x\bar{j} + y\bar{k}) \cdot \bar{k} = y = \rho\sin\phi$$

required expression is $\bar{F} = f_1 T_\rho + f_2 T_\phi + f_3 T_z$

$$\bar{F} = (z\cos\phi - 2x\sin\phi) T_\rho - (z\sin\phi + 2x\cos\phi) T_\phi + y T_z$$

$$\bar{F} = (z\cos\phi - \rho\sin 2\phi) T_\rho - (z\sin\phi + 2\rho\cos^2\phi) T_\phi + \rho\sin\phi\, T_z$$

(2)

Express the vector $2y\bar{i} - z\bar{j} + 3x\bar{k}$ in cylindrical coordinates

Sol:- We have $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$

So that $\bar{R} = \rho\cos\phi\, \bar{i} + \rho\sin\phi\, \bar{j} + z\bar{k}$

If T_ρ, T_ϕ, T_z are unit vectors along the tangents to ρ, ϕ and z respectively

$$T_\rho = \frac{\partial \bar{R}}{\partial \rho} \Big/ \left|\frac{\partial \bar{R}}{\partial \rho}\right| = \cos\phi\, \bar{i} + \sin\phi\, \bar{j}$$

$$T_\phi = \frac{\partial \bar{R}}{\partial \phi} \Big/ \left|\frac{\partial \bar{R}}{\partial \phi}\right| = -\sin\phi\, \bar{i} + \cos\phi\, \bar{j}$$

$$T_z = \frac{\partial \bar{R}}{\partial z} \Big/ \left|\frac{\partial \bar{R}}{\partial z}\right| = \bar{k}$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Given $\bar{F} = 2yi - z\bar{j} + 3x\bar{k}$ Let the expression for $\bar{F}$ in cylindrical coordinates be

$$\bar{F} = f_1 T_\rho + f_2 T_\phi + f_3 T_z$$

where $f_1 = \bar{F} \cdot T_\rho$, $f_2 = \bar{F} \cdot T_\phi$, $f_3 = \bar{F} \cdot T_z$

$$f_1 = (2y\,\bar{i} - z\bar{j} + 3x\bar{k}) \cdot (\cos\phi\,\bar{i} + \sin\phi\,\bar{j})$$

$$= 2y\cos\phi - z\sin\phi$$

$$= 2(\rho\sin\phi)\cos\phi - z\sin\phi$$

$$= \rho\sin 2\phi - z\sin\phi$$

$$f_2 = (2y\,\bar{i} - z\bar{j} + 3x\bar{k}) \cdot (-\sin\phi\,\bar{i} + \cos\phi\,\bar{j})$$

$$= -2y\sin\phi - z\cos\phi$$

$$= -2(\rho\sin\phi)\sin\phi - z\cos\phi = -2\rho\sin^2\phi - z\cos\phi$$

$$f_3 = (2y\,\bar{i} - z\bar{j} + 3x\bar{k}) \cdot \bar{k} = 3x = 3\rho\cos\phi$$

Required expression is $\bar{F} = f_1 T_\rho + f_2 T_\phi + f_3 T_z$

$$\bar{F} = (\rho\sin 2\phi - z\sin\phi)\,T_\rho - (2\rho\sin^2\phi + z\cos\phi)\,T_\phi + 3\rho\cos\phi\,\bar{k}$$

③ Express the following vectors in cylindrical coordinates

$2x\bar{i} - 3y^2\bar{j} + 2x\bar{k}$

Sol:- We have $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$

So that $\bar{r} = \rho\cos\phi\,\bar{i} + \rho\sin\phi\,\bar{j} + z\bar{k}$

If T_ρ, T_ϕ, T_z are unit vectors along the tangents to ρ, ϕ and z respectively $T_\rho = \frac{\partial\bar{r}}{\partial\rho} \Big/ \left|\frac{\partial\bar{r}}{\partial\rho}\right| = \cos\phi\,\bar{i} + \sin\phi\,\bar{j}$

$$T_\phi = \frac{\partial\bar{r}}{\partial\phi} \Big/ \left|\frac{\partial\bar{r}}{\partial\phi}\right| = -\sin\phi\,\bar{i} + \cos\phi\,\bar{j}$$

$$T_z = \frac{\partial\bar{r}}{\partial z} \Big/ \left|\frac{\partial\bar{r}}{\partial z}\right| = \bar{k}$$

FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Given $\bar{F} = 2xz\hat{i} - 3y^2\hat{j} + 2z\hat{k}$

Find the expression for $\bar{F}$ in cylindrical coordinates

Sol: $\bar{F} = f_1 T_\rho + f_2 T_\phi + f_3 T_z$

where $f_1 = \bar{F} \cdot T_\rho$, $f_2 = \bar{F} \cdot T_\phi$, $f_3 = \bar{F} \cdot T_z$

$f_1 = (2x\hat{i} - 3y^2\hat{j} + 2z\hat{k}) \cdot (\cos\phi\,\hat{i} + \sin\phi\,\hat{j})$

$f_1 = 2x\cos\phi - 3y^2\sin\phi$

$= 2(\rho\cos\phi)\cos\phi - 3(\rho\sin\phi)^2\sin\phi$

$= 2\rho\cos^2\phi - 3\rho^2\sin^3\phi$

$f_2 = (2x\hat{i} - 3y^2\hat{j} + 2z\hat{k}) \cdot (-\sin\phi\,\hat{i} + \cos\phi\,\hat{j})$

$= -2x\sin\phi - 3y^2\cos\phi$

$= -2(\rho\cos\phi)\sin\phi - 3(\rho\sin\phi)^2\cos\phi$

$= -\rho\sin 2\phi - 3\rho^2\sin^2\phi\cos\phi$

$f_3 = (2x\hat{i} - 3y^2\hat{j} + 2z\hat{k}) \cdot \hat{k} = 2z = 2\rho\cos\phi$

Required expression is

$\bar{F} = f_1 T_\rho + f_2 T_\phi + f_3 T_z$

$\bar{F} = (2\rho\cos^2\phi - 3\rho^2\sin^3\phi)\,T_\rho - (\rho\sin 2\phi + 3\rho^2\sin^2\phi\cos\phi)\,T_\phi + 2\rho\cos\phi$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Problem (i) :-

S.T $\nabla(\log \rho)$ and $\nabla\phi$, $\rho \neq 0$, $\phi \neq 0$ are solenoidal vectors.

Sol :- $f = \log \rho$ is a function of ρ only

To P.T $\nabla \cdot (\nabla f) = 0 \Rightarrow \nabla^2 f$

$$\nabla^2 f = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \frac{\partial^2}{\partial \rho^2}(\log \rho) + \frac{1}{\rho}\frac{\partial}{\partial \rho}(\log \rho) + 0 + 0$$

$$= -\frac{1}{\rho^2} + \frac{1}{\rho^2} = 0.$$

$\therefore \nabla \log \rho$ is solenoidal.

(ii) $f = \nabla\phi$ is a function of ϕ only. we have to show that $\nabla \cdot (\nabla f)$ i.e $\nabla^2 f = 0$

$$\nabla^2 f = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= 0 + 0 + \frac{1}{\rho^2}\frac{\partial^2 \phi}{\partial \phi^2} + 0$$

$$= 0.$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice

herical coordinate system is orthogonal

www.FirstRanker.com www.FirstRanker.com

At any point P, we have $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$\Rightarrow \bar{R} = r\sin\theta\cos\phi\,\bar{i} + r\sin\theta\sin\phi\,\bar{j} + r\cos\theta\,\bar{k}$

If T_r, T_θ, T_ϕ be the unit vectors at P, in the directions of tangents to the r, θ, ϕ – curves respectively then:

$$T_r = \frac{\frac{\partial \bar{R}}{\partial r}}{\left|\frac{\partial \bar{R}}{\partial r}\right|} = \frac{\sin\theta\cos\phi\,\bar{i} + \sin\theta\sin\phi\,\bar{j} + \cos\theta\,\bar{k}}{\sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta}}$$

$\cos^2\phi + \sin^2\phi = 1$
$\sin^2\theta + \cos^2\theta = 1$

$$= \frac{\sin\theta\cos\phi\,\bar{i} + \sin\theta\sin\phi\,\bar{j} + \cos\theta\,\bar{k}}{\sqrt{\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta}}$$

$$= \sin\theta\cos\phi\,\bar{i} + \sin\theta\sin\phi\,\bar{j} + \cos\theta\,\bar{k}$$

$$T_\theta = \frac{\frac{\partial \bar{R}}{\partial \theta}}{\left|\frac{\partial \bar{R}}{\partial \theta}\right|} = \frac{r\cos\theta\cos\phi\,\bar{i} + r\cos\theta\sin\phi\,\bar{j} - r\sin\theta\,\bar{k}}{\sqrt{r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta}}$$

$$= \frac{r(\cos\theta\cos\phi\,\bar{i} + \cos\theta\sin\phi\,\bar{j} - \sin\theta\,\bar{k})}{r\sqrt{\cos^2\theta(\cos^2\phi + \sin^2\phi) + \sin^2\theta}}$$

$$= \cos\theta\cos\phi\,\bar{i} + \cos\theta\sin\phi\,\bar{j} - \sin\theta\,\bar{k}$$

$$T_\phi = \frac{\frac{\partial \bar{R}}{\partial \phi}}{\left|\frac{\partial \bar{R}}{\partial \phi}\right|} = \frac{-r\sin\theta\sin\phi\,\bar{i} + r\sin\theta\cos\phi\,\bar{j}}{\sqrt{r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi}}$$

$$= \frac{r\sin\theta(-\sin\phi\,\bar{i} + \cos\phi\,\bar{j})}{r\sin\theta\sqrt{\sin^2\phi + \cos^2\phi}}$$

$$= -\sin\phi\,\bar{i} + \cos\phi\,\bar{j}$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com

$= \sin\theta\cos\theta(\cos^2\phi + \sin^2\phi) - \sin\theta\cos\theta$

$= 0$

Similarly $T_\theta \cdot T_\phi = 0$ and $T_\phi \cdot T_r = 0$.

$$T_r \times T_\theta = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \end{vmatrix}$$

$$= \bar{i}\left[-\sin^2\theta\sin\phi - \cos^2\theta\sin\phi\right] + \bar{j}\left[\cos^2\theta\cos\phi + \sin^2\theta\cos\phi\right]$$
$$+ \bar{k}\left[\sin\theta\cos\phi\cos\theta\sin\phi - \sin\theta\cos\theta\sin\phi\cos\phi\right]$$

$$= -\sin\phi\,\bar{i} + \cos\phi\,\bar{j} = T_\phi$$

Similarly $T_\theta \times T_\phi = T_r$ and $T_\phi \times T_r = T_\theta$.

∴ The spherical polar coordinate system is a right handed orthogonal coordinate system.

Del applied to functions in spherical polar coordinates

Let T_r, T_θ, T_ϕ be the unit vectors in the directions of the tangents to the r, θ, ϕ – curves.

(i) Expression for grad f :-

$$\nabla f = \frac{\partial f}{\partial r} T_r + \frac{1}{r}\frac{\partial f}{\partial \theta} T_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi} T_\phi$$

(ii) Expression for div $\bar{F}$, where $\bar{F} = f_1 T_r + f_2 T_\theta + f_3 T_\phi$

$$\nabla \cdot \bar{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(f_1 r^2) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(f_2 \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial f_3}{\partial \phi}$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$$\nabla \times \bar{F} = \begin{vmatrix} \frac{T_r}{r^2\sin\theta} & \frac{T_\theta}{r\sin\theta} & \frac{T_\phi}{r} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ f_1 & rf_2 & r\sin\theta f_3 \end{vmatrix}$$

(iv) Expression for $\nabla^2 f$:-

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial f}{\partial\theta}$$

(i) Express the vector field $2y\,i - 2j + 3z\,k$ in spherical polar coordinate system.

Sol:- we have $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

So that $\bar{R} = r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}$

If T_r, T_θ, T_ϕ be the unit vectors along the tangents to r, θ, ϕ curves respectively, then

$$T_r = \frac{\frac{\partial\bar{R}}{\partial r}}{\left|\frac{\partial\bar{R}}{\partial r}\right|} = \frac{\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}}{\sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta}}$$

$$= \frac{\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}}{\sqrt{\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta}}$$

$$= \boxed{\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}}$$

$$T_\theta = \frac{\frac{\partial\bar{R}}{\partial\theta}}{\left|\frac{\partial\bar{R}}{\partial\theta}\right|} = \frac{r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,\hat{k}}{\sqrt{r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta}}$$

$$= \frac{r[\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}]}{r\sqrt{\cos^2\theta(\cos^2\phi + \sin^2\phi) + \sin^2\theta}}$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com
www.FirstRanker.com

$= \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$

$$T_\phi = \frac{\frac{\partial \bar{r}}{\partial \phi}}{\left|\frac{\partial \bar{r}}{\partial \phi}\right|} = \frac{-r\sin\theta\sin\phi\,\hat{i} + r\sin\theta\cos\phi\,\hat{j}}{\sqrt{r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi}}$$

$$T_\phi = \frac{r\sin\theta\left[-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}\right]}{r\sin\theta\left[\sqrt{\sin^2\phi+\cos^2\phi}\right]} = \boxed{-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}}$$

Let the expression for $\bar{F} = 2y\hat{i} - z\hat{j} + 3x\hat{k}$ in spherical polar coordinates be $\bar{F} = f_1 T_r + f_2 T_\theta + f_3 T_\phi$ — (1)

$f_1 = \bar{F}\cdot T_r$, $f_2 = \bar{F}\cdot T_\theta$ and $f_3 = \bar{F}\cdot T_\phi$

$f_1 = (2y\hat{i} - z\hat{j} + 3x\hat{k})\cdot(\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k})$

$f_1 = 2y\sin\theta\cos\phi - z\sin\theta\sin\phi + 3x\cos\theta$

$f_1 = 2(r\sin\theta\sin\phi)\sin\theta\cos\phi - r\cos\theta\sin\theta\sin\phi + 3r\sin\theta\cos\phi\cos\theta$

$f_1 = r\sin^2\theta\sin 2\phi - r\sin\theta\cos\theta\sin\phi + 3r\sin\theta\cos\theta\cos\phi$

$f_2 = \bar{F}\cdot T_\theta = \left[2y\hat{i} - z\hat{j} + 3x\hat{k}\right]\cdot\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$

$= 2y\cos\theta\cos\phi - z\cos\theta\sin\phi - 3x\sin\theta$

$= 2(r\sin\theta\sin\phi)\cos\theta\cos\phi - r\cos\theta\cos\theta\sin\phi - 3r\sin\theta\cos\phi\sin\theta$

$= r\sin\theta\cos\theta\sin 2\phi - r\cos^2\theta\sin\phi - 3r\sin^2\theta\cos\phi$

$f_3 = \bar{F}\cdot T_\phi = (2y\hat{i} - z\hat{j} + 3x\hat{k})\cdot(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j})$

$= -2y\sin\theta - z\cos\phi$

$= -2(r\sin\theta\sin\phi)\sin\phi - r\cos\theta\cos\phi$

$= -2r\sin\theta\sin^2\phi - r\cos\theta\cos\phi$

Substituting the values f_1, f_2, f_3 in (1) we get

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$\bar{F} = f_1 T_r + f_2 T_\theta + f_3 T_\phi$

$\bar{F} = [r\sin^2\theta \sin 2\phi - r\sin\theta\cos\theta\sin\phi + 3r\sin\theta\cos\theta\cos\phi]\, T_r$

$+ [r\sin\theta\cos\theta\sin 2\phi - r\cos^2\theta\sin\phi - 3r\sin^2\theta\cos\phi]\, T_\theta$

$+ [-2r\sin\theta\sin^2\phi - r\cos\theta\cos\phi]\, T_\phi$

$\bar{F} = [r\sin^2\theta\sin 2\phi - r\sin\theta\cos\theta\sin\phi + 3r\sin\theta\cos\theta\cos\phi]\, T_r$

$+ [r\sin\theta\cos\theta\sin 2\phi - r\cos^2\theta\sin\phi - 3r\sin^2\theta\cos\phi]\, T_\theta$

$- [2r\sin\theta\sin^2\phi + r\cos\theta\cos\phi]\, T_\phi$

2) P.T $\nabla(\cos\theta) \times \nabla\phi = \nabla\left(\frac{1}{r}\right)$, $r \neq 0$

Sol:- In spherical polar coordinates

$\nabla f = \frac{\partial f}{\partial r} T_r + \frac{1}{r}\frac{\partial f}{\partial \theta} T_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi} T_\phi$

$\nabla(\cos\theta) = 0 + \frac{1}{r}\frac{\partial}{\partial\theta}(\cos\theta)\, T_\theta + 0 = -\frac{1}{r}\sin\theta\, T_\theta$

$\nabla\phi = 0 + 0 + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\phi)\, T_\phi = \frac{1}{r\sin\theta} T_\phi$

$\nabla\left(\frac{1}{r}\right) = \frac{\partial}{\partial r}\left(\frac{1}{r}\right) T_r + 0 + 0 = -\frac{1}{r^2} T_r$

$\nabla(\cos\theta) \times \nabla\phi = \left(-\frac{1}{r}\sin\theta\, T_\theta\right) \times \left(\frac{1}{r\sin\theta} T_\phi\right)$

$= -\frac{1}{r^2}(T_\theta \times T_\phi) = -\frac{1}{r^2} T_r = \nabla\left(\frac{1}{r}\right)$

3) If $\bar{F} = r^3\cos\theta\, T_r - \frac{1}{r} T_\theta + \frac{1}{r\sin\theta} T_\phi$ find the value of $\bar{F} \times \text{curl}\, \bar{F}$

Sol:- In spherical polar coordinates

$$\text{curl}\, \bar{F} = \begin{vmatrix} T_r/r^2\sin\theta & T_\theta/r\sin\theta & T_\phi/r \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ f_1 & rf_2 & r\sin\theta f_3 \end{vmatrix}$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice

$f_1 = r\cos\theta$, $f_2 = -\frac{1}{r}$, $f_3 = \frac{1}{r\sin\theta}$.

www.FirstRanker.com www.FirstRanker.com

$$\text{curl } \bar{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} T_r & rT_\theta & r\sin\theta\, T_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ r^2\cos\theta & -1 & 1 \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left[r\sin\theta\, T_\phi\,(0 - r^2(-\sin\theta)\right] = \frac{r^3\sin^2\theta\, T_\phi}{r^2\sin\theta} = r\sin\theta\, T_\phi$$

$$\bar{F}\times\text{curl }\bar{F} = \left[r^2\cos\theta\, T_r - \frac{1}{r}T_\theta + \frac{1}{r\sin\theta}T_\phi\right]\times\left[r\sin\theta\, T_\phi\right]$$

$$= -r^3\sin\theta\cos\theta\, T_\theta - \sin\theta\, T_r$$

$$= -(r^3\sin\theta\cos\theta\, T_\theta + \sin\theta\, T_r)$$

2.

4) Express the following vectors in spherical polar coordinates

$x\bar{i} + 2y\,\bar{j} + yz\,\bar{k}$.

Sol:- we have $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

So that $\bar{R} = r\sin\theta\cos\phi\,\bar{i} + r\sin\theta\sin\phi\,\bar{j} + r\cos\theta\,\bar{k}$.

If T_r, T_θ, T_ϕ be the unit vectors along the tangents to r, θ, ϕ curves respectively, then

$$T_r = \frac{\partial\bar{R}}{\partial r}\Big/\left|\frac{\partial\bar{R}}{\partial r}\right| = \sin\theta\cos\phi\,\bar{i} + \sin\theta\sin\phi\,\bar{j} + \cos\theta\,\bar{k}.$$

$$T_\theta = \frac{\partial\bar{R}}{\partial\theta}\Big/\left|\frac{\partial\bar{R}}{\partial\theta}\right| = \cos\theta\cos\phi\,\bar{i} + \cos\theta\sin\phi\,\bar{j} - \sin\theta\,\bar{k}.$$

$$T_\phi = \frac{\partial\bar{R}}{\partial\phi}\Big/\left|\frac{\partial\bar{R}}{\partial\phi}\right| = -\sin\phi\,\bar{i} + \cos\phi\,\bar{j}$$

Let the expression for $\bar{F} = x\bar{i} + 2y\bar{j} + yz\bar{k}$ in spherical polar coordinates be $\bar{F} = f_1T_r + f_2T_\theta + f_3T_\phi$

Then $f_1 = \bar{F}\cdot T_r = (x\bar{i} + 2y\bar{j} + yz\bar{k})\cdot(\sin\theta\cos\phi\,\bar{i} + \sin\theta\sin\phi\,\bar{j} + \cos\theta\,\bar{k})$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
www.FirstRanker.com

$= r\sin\theta\cos\phi$ ... $+ 2y \sin\theta\sin\phi$ ... $+ yz\cos\theta$

$= r\sin\theta\cos\phi\,\sin\theta\cos\phi + 2(r\sin\theta\sin\phi)\sin\theta\sin\phi + (r\sin\theta\sin\phi)(r\cos\theta)\cos\theta$

$= r\sin^2\theta\cos^2\phi + 2r\sin^2\theta\sin^2\phi + r^2\sin\theta\sin\phi\cos^2\theta$.

$f_2 = \bar{F}\cdot T_\theta = (x\hat{i} + 2y\hat{j} + yz\hat{k})\cdot(\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k})$

$= x\cos\theta\cos\phi + 2y\cos\theta\sin\phi - yz\sin\theta$

$= (r\sin\theta\cos\phi)\cos\theta\cos\phi + 2(r\sin\theta\sin\phi)\cos\theta\sin\phi$
$\quad - (r\sin\theta\sin\phi)(r\cos\theta)\sin\theta$.

$= r\sin\theta\cos\theta\cos^2\phi + r\sin 2\theta\sin^2\phi - r^2\sin^2\theta\cos\theta\sin\phi$

$f_3 = \bar{F}\cdot T_\phi = (x\hat{i} + 2y\hat{j} + yz\hat{k})\cdot(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j})$

$= -x\sin\phi + 2y\cos\phi$

$= -r\sin\theta\cos\phi\sin\phi + 2(r\sin\theta\sin\phi)\cos\phi$

$= -r\sin\theta\sin\phi\cos\phi + r\sin\theta\sin 2\phi$

$\bar{F} = f_1 T_r + f_2 T_\theta + f_3 T_\phi$

$\bar{F} = [r\sin^2\theta\cos^2\phi + 2r\sin^2\theta\sin^2\phi + r^2\sin\theta\sin\phi\cos^2\theta]\,T_r$

$+ [r\sin\theta\cos\theta\cos^2\phi + r\sin 2\theta\sin^2\phi - r^2\sin^2\theta\cos\theta\sin\phi]\,T_\theta$

$+ [r\sin\theta\sin\phi\cos\phi - r\sin\theta\sin 2\phi]\,T_\phi$

$=$

5) $2y\,\hat{i} + yz\,\hat{j} + 2x\,\hat{k}$

$\bar{R} = r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}$

$T_r = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$

$T_\theta = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$

$T_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$\bar{F} = f_1 T_r + f_2 T_\theta + f_3 T_\phi$

Then $f_1 = \bar{F} \cdot T_r = xy \sin\theta\cos\phi + yz \sin\theta\sin\phi + zx\cos\theta$

$= (r\sin\theta\cos\phi)(r\sin\theta\sin\phi)\sin\theta\cos\phi$

$+ (r\sin\theta\sin\phi)(r\cos\theta)\sin\theta\sin\phi + (r\cos\theta)(r\sin\theta\cos\phi)\cos\theta$

$= r^2\sin^3\theta\sin\phi\cos^2\phi + r^2\sin^2\theta\cos\theta\sin^2\phi + r^2\sin\theta\cos^2\theta\cos\phi$

$f_2 = \bar{F} \cdot T_\theta = (xy\,\hat{i} + yz\,\hat{j} + zx\,\hat{k}) \cdot (\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k})$

$= xy\cos\theta\cos\phi + yz\cos\theta\sin\phi - zx\sin\theta$

$= (r\sin\theta\cos\phi)(r\sin\theta\sin\phi)\cos\theta\cos\phi + (r\sin\theta\sin\phi)(r\cos\theta)\cos\theta\sin\phi$

$- [r\cos\theta][r\sin\theta\cos\phi]\sin\theta$

$= r^2\sin^2\theta\cos\theta\sin\phi\cos^2\phi + r^2\sin\theta\cos^2\theta\sin^2\phi$

$- r^2\sin^2\theta\cos\theta\cos\phi$

$f_3 = \bar{F} \cdot T_\phi = (xy\,\hat{i} + yz\,\hat{j} + zx\,\hat{k}) \cdot (-\sin\phi\,\hat{i} + \cos\phi\,\hat{j})$

$= -xy\sin\phi + \cos\phi\, yz$

$= -(r\sin\theta\cos\phi)(r\sin\theta\sin\phi)\sin\phi + (r\sin\theta\sin\phi)(r\cos\theta)\cos\phi$

$= -r^2\sin^2\theta\sin^2\phi\cos\phi + r^2\sin\theta\cos\theta\sin\theta\cos\phi$

$\bar{F} = f_1 T_r + f_2 T_\theta + f_3 T_\phi$

$= [r^2\sin^3\theta\sin\phi\cos^2\phi + r^2\sin^2\theta\cos\theta\sin^2\phi + r^2\sin\theta\cos^2\theta\cos\phi]T_r$

$+ [r^2\sin^2\theta\cos\theta\sin\phi\cos^2\phi + r^2\sin\theta\cos^2\theta\sin^2\phi - r^2\sin^2\theta\cos\theta\cos\phi]T_\theta$

$- [r^2\sin^2\theta\sin^2\phi\cos\phi - r^2\sin\theta\cos\theta\sin\phi\cos\phi]$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice

www.FirstRanker.com www.FirstRanker.com

SPECIAL FUNCTIONS

BETA FUNCTION: The definite integral $\int_0^1 x^m (1-x)^{n-1} dx$, for $m>0, n>0$ is called the Beta function (or) The Eulerian integral of the first kind and is denoted by $B(m,n)$ and read as "Beta m, n"

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \; ; \; m>0, n>0$$

PROPERTIES OF BETA FUNCTION

① $B(m,n) = B(n,m)$

Proof:- By the definition

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

$$= \int_0^1 (1-x)^{m-1} (1-(1-x))^{n-1} dx$$

$$\boxed{\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx}$$

$$= \int_0^1 (1-x)^{m-1} x^{n-1} dx$$

$$= B(n,m)$$

$$\therefore \boxed{B(m,n) = B(n,m)}$$

2. $B(m,n) = 2 \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

Proof:- By the definition

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

put $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\, d\theta$

as $x \to 0, \theta \to 0$ and $x \to 1, \theta \to \pi/2$

$$B(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} (1-\sin^2\theta)^{n-1} 2\sin\theta\cos\theta\, d\theta$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta \cos^{2n-2}\theta \cdot 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

$$\therefore \beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

③ $\beta\left(\frac{1}{2},\frac{1}{2}\right) = \pi$

proof: We know that

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

put $m = n = \frac{1}{2}$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = 2\int_0^{\pi/2} \sin^{2(\frac{1}{2})-1}\theta \cos^{2(\frac{1}{2})-1}\theta\, d\theta$$

$$= 2\int_0^{\pi/2} d\theta$$

$$= 2[\theta]_0^{\pi/2} = 2\left[\frac{\pi}{2}\right] - 2(0)$$

$$\therefore \beta\left(\frac{1}{2},\frac{1}{2}\right) = \pi$$

DIFFERENT FORMS OF BETA FUNCTION

Form I: Show that $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$

Proof: By the definition $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$

put $x = \frac{y}{1+y} \Rightarrow 1 - \frac{1}{1+y}$.

$$\Rightarrow dx = \frac{1}{(1+y)^2}\, dy$$

As $x \to 0 : y \to 0$ and $x \to 1 : y \to \infty$

$$\beta(m,n) = \int_0^\infty \left[\frac{y}{(1+y)}\right]^{m-1} \left[1 - \frac{y}{1+y}\right]^{n-1} \frac{1}{(1+y)^2}\, dy$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m-1}} \cdot \frac{1}{(1+y)^{n-1}} \cdot \frac{1}{(1+y)^2}\, dy$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice

FORM: III Show that $\beta(m,n) = a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}}\,dx$ www.FirstRanker.com www.FirstRanker.com

Proof :- Consider $a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}}\,dx$

put $a\cdot x = bt$

$\Rightarrow x = \frac{b}{a}t \quad \Rightarrow dx = \frac{b}{a}dt$

As $x \to 0;\ t \to 0$ and as $x \to \infty;\ t \to \infty$

$$= a^m b^n \int_0^\infty \frac{\frac{b^{m-1}}{a^{m-1}}\,t^{m-1}}{(bt+b)^{m+n}} \cdot \frac{b}{a}\,dt$$

$$= \frac{a^m b^n}{a^m b^{m+n}}\, b^m \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\,dt$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m-1+n}}\,dx$$

$$= \beta(m,n)$$

Hence $\beta(m,n) = a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}}\,dx$

FORM: IV Show that $\int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx = (b-a)^{m+n-1}\beta(m,n)$

Proof :- Now $\int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx$

Put $x = a+(b-a)t \Rightarrow dx = (b-a)\,dt$

as $x \to a, t \to 0$ and as $x \to b;\ t \to 1$

$$= \int_0^1 [(b-a)t]^{m-1}[b-a-(b-a)t]^{n-1}(b-a)\,dt$$

$$= (b-a)^{m+n-1}\int_0^1 t^{m-1}(1-t)^{n-1}\,dt$$

$$= (b-a)^{m+n-1}\beta(m,n)$$

Hence $\int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx = (b-a)^{m+n-1}\beta(m,n)$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$\therefore \int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

By property $B(m,n) = B(n,m)$

$$\therefore B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

Form : II Show that $B(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$

proof :- We know that

$$B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx \quad \text{——— (1)}$$

Consider $\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

put $x = \frac{1}{y} \Rightarrow dx = \frac{-1}{y^2}\,dy$

as $x \to 1 ; y \to 1$ and $x \to \infty ; y \to 0$

$$= \int_1^0 \frac{(\frac{1}{y})^{m-1}}{(1+\frac{1}{y})^{m+n}} \left(\frac{-1}{y^2}\right) dy$$

$$= \int_0^1 \frac{y^{m+n}}{y^{m-1}(1+y)^{m+n}} \cdot \frac{1}{y^2}\,dy$$

$$= \int_0^1 \frac{y^{m+n-m+1-2}}{(1+y)^{m+n}}\,dy$$

$$= \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}\,dy = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

From (1), we have $B(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$

Hence $B(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
www.FirstRanker.com www.FirstRanker.com

Evaluate $B(m,n)$

Soln: By the definition

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$B(1,1) = \int_0^1 x^{1-1}(1-x)^{1-1}dx = \int_0^1 dx = [x]_0^1 = 1-0 = 1$$

$$\therefore B(1,1) = 1$$

② Evaluate $\int_0^1 \frac{x^2}{\sqrt{1-x^5}}dx$ interms of Beta function.

Soln: Consider $\int_0^1 \frac{x^2}{\sqrt{1-x^5}}dx$

put $x^5 = t \Rightarrow x = t^{1/5}$

$$dx = \frac{1}{5}t^{-4/5}dt$$

as $x \to 0; t \to 0$ and $x \to 1; t \to 1$

$$= \int_0^1 \frac{t^{2/5}}{\sqrt{1-t}} \frac{1}{5}t^{-4/5}dt$$

$$= \frac{1}{5}\int_0^1 t^{-2/5}(1-t)^{-1/2}dt$$

$$= \frac{1}{5}\int_0^1 t^{3/5-1}(1-t)^{\frac{1}{2}-1}dt$$

$$= \frac{1}{5}B\left(\frac{3}{5},\frac{1}{5}\right)$$

③ Evaluate $\int_0^1 x^5(1-x^3)^{10}dx$

Soln: Consider $\int_0^1 x^5(1-x^3)^{10}dx$

put $x^3 = t \Rightarrow x = t^{1/3}$

$$dx = \frac{1}{3}t^{-2/3}dt$$

as $x \to 0; t \to 0$ and $x \to 1, t \to 1$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$$\int_0^1 t^{1/3}(1-t)^{10}\frac{1}{3}t^{-2/3}dt$$

$$= \frac{1}{3}\int_0^1 t\,(1-t)^{10}dt$$

$$= \frac{1}{3}\int_0^1 t^{2-1}(1-t)^{11-1}dt$$

$$= \frac{1}{3}B(2,11)$$

(4) Evaluate $\int_0^1 \frac{1}{(1-x^3)^{\frac{1}{3}}}dx$ using Beta function.

Soln: Consider $\int_0^1 \frac{1}{(1-x^3)^{\frac{1}{3}}}dx$

put $x^3 = t \Rightarrow x = t^{1/3}$

$\Rightarrow dx = \frac{1}{3}t^{-2/3}dt$

as $x \to 0$; $t \to 0$ and as $x \to 1$; $t \to 1$

$$= \int_0^1 \frac{1}{(1-t)^{\frac{1}{3}}}\frac{1}{3}t^{-2/3}dt$$

$$= \frac{1}{3}\int_0^1 t^{-2/3}(1-t)^{-1/3}dt$$

$$= \frac{1}{3}\int_0^1 t^{\frac{1}{3}-1}(1-t)^{\frac{2}{3}-1}dt$$

$$= \frac{1}{3}B\left(\frac{1}{3},\frac{2}{3}\right)$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice

www.FirstRanker.com www.FirstRanker.com

Def:- The definite integral $\int_0^\infty e^{-x} x^{n-1} dx$; $n>0$ is called the Gamma function (or) Eulerian integral of the second kind and is denoted by $\Gamma(n)$ and read as "Gamma n"

Thus $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$; $n>0$

PROPERTIES OF GAMMA FUNCTION

1. Show that $\Gamma(1) = 1$

Proof: By the definition of Gamma function

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

$$\Gamma(1) = \int_0^\infty e^{-x} dx = \left[\frac{e^{-x}}{-1}\right]_0^\infty = -[e^{-\infty} - e^{-0}]$$

$$= -[0-1] = 1$$

$$\therefore \boxed{\Gamma(1) = 1}$$

② Show that $\Gamma(n+1) = n\Gamma(n)$ for $n>0$

proof:-

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^{n} dx$$

$$= \left[-x^n e^{-x}\right]_0^\infty + n\int_0^\infty e^{-x} x^{n-1} dx$$

$$= 0 + n\Gamma(n)$$

$$\boxed{\Gamma(n+1) = n\Gamma(n)}$$

③ Show that $\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1} dx$

proof: By the definition,

we have $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$

put $x = t^2$

$dx = 2t\,dt$

as $x \to 0$, $t \to 0$ and as $x \to \infty$, $t \to \infty$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$$\Gamma(n) = \int_0^\infty e^{-t^2} t^{2n-2}\, 2t\, dt$$

$$= 2\int_0^\infty e^{-t^2} t^{2n-1}\, dt$$

Hence $\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1}\, dx$

(4) Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof:- By the def; we have $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$

We have $\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1}\, dx$

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2} dx \quad \text{—— (1)}$$

Also, we have $\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-y^2} dy$ —— (2)

(1) × (2) $\Rightarrow$ $\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$

Changing into polar coordinates we have

$x = r\cos\theta$; $y = r\sin\theta$; $dx\,dy = r\,dr\,d\theta$

θ varies from 0 to $\pi/2$ and r varies from 0 to ∞

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2} r\, dr\, d\theta$$

put $r^2 = t \Rightarrow 2r\,dr = dt$

$\Rightarrow r\,dr = \frac{dt}{2}$

$$= 2\int_0^{\pi/2}\int_0^\infty e^{-t}\, dt\, d\theta$$

$$= 2\int_0^{\pi/2} \left[\frac{e^{-t}}{-1}\right]_0^\infty d\theta$$

$$= 2\int_0^{\pi/2} -[e^{-\infty} - e^{0}]\, d\theta$$

$$= 2\int_0^{\pi/2} d\theta$$

$$= 2[\theta]_0^{\pi/2} \; ; \; 2\left(\frac{\pi}{2}\right) - 0 = \pi$$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi$$

$$\therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Evaluate $\int_0^\infty x^\nu e^{-x^\nu} dx$

Soln: consider $\int_0^\infty x^\nu e^{-x^\nu} dx$

put $x^\nu = t \Rightarrow x = t^{1/\nu}$

$dx = \frac{1}{\nu} t^{-1/\nu} dt$

as $x \to 0; t \to 0$ and as $x \to \infty, t \to \infty$

$= \int_0^\infty t e^{-t} \frac{1}{\nu} t^{-1/\nu} dt$

$= \frac{1}{\nu} \int_0^\infty e^{-t} t^{1/\nu} dt$

$= \frac{1}{\nu} \int_0^\infty e^{-t} t^{\frac{3}{\nu}-1} dt$

$= \frac{1}{\nu} \Gamma(\frac{3}{\nu})$

$= \frac{1}{\nu} (\frac{3}{\nu}-1) \Gamma(\frac{3}{\nu}-1)$

$= \frac{1}{\nu} (\frac{1}{\nu}) \Gamma(\frac{1}{\nu})$

$= \frac{\sqrt{\pi}}{4}$

② Show that $\Gamma(n) = \int_0^1 (\log \frac{1}{x})^{n-1} dx, \quad n > 0$

Soln: R.H.S $\int_0^1 (\log \frac{1}{x})^{n-1} dx$

put $t = \log \frac{1}{x} = -\log x$

$= \int_\infty^0 t^{n-1} (-e^{-t}) dt$

$= \int_0^\infty e^{-t} t^{n-1} dt$

$= \Gamma(n)$

$= L.H.S$

$\Rightarrow -t = \log x$

$\Rightarrow e^{-t} = x$

$\therefore x = e^{-t}$

$dx = -e^{-t} dt$

as $x \to 0; t \to \infty, dx = -1; t = 0$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Show that $\int_0^1 x^3\left(\log\frac{1}{x}\right)^4 dx = \frac{3}{128}$

Soln:-

L.H.S $\int_0^1 x^3\left(\log\frac{1}{x}\right)^4 dx$

put $\log\frac{1}{x} = t$

$\Rightarrow \frac{1}{x} = e^t$

$\Rightarrow x = e^{-t}$

$dx = -e^{-t}dt$

as $x \to 0; t \to \infty$ and

as $x \to 1; t \to 0$

$= \int_\infty^0 e^{-3t} t^4 (-e^{-t})\, dt$

$= \int_0^\infty e^{-4t} t^4\, dt$

again put $4t = u$

$4dt = du$

$dt = \frac{1}{4}du$

$= \int_0^\infty e^{-u}\left(\frac{u}{4}\right)^4 \frac{1}{4}\, du$

$= \int_0^\infty e^{-u}\frac{u^4}{4^5}\, du$

$= \frac{1}{4^5}\int_0^\infty e^{-u} u^{5-1}\, du$

$= \frac{1}{4^5}\Gamma(5)$

$= \frac{4!}{4^5} = \frac{4\cdot 3\cdot 2\cdot 1}{4\times 4\times 64} = \frac{3}{128}$

$= R.H.S$

$\int_0^1 \left(x^3\log\frac{1}{x}\right)^4 dx = \frac{3}{128}$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Evaluate $\int_0^\infty 3^{-4x^2}\,dx$

Soln:- Consider $\int_0^\infty 3^{-4x^2}\,dx$

$= \int_0^\infty e^{\log e^{3^{-4x^2}}}\,dx$

$= \int_0^\infty e^{-4x^2 \log 3}\,dx$

$e^{\log_e f(x)} = f(x)$

$4x^2 \log 3 = t$

$\Rightarrow x^2 = \frac{1}{4\log 3}\,t$

$\Rightarrow x = \frac{1}{2}\frac{1}{\sqrt{\log 3}}\,t^{\frac{1}{2}}$

$\Rightarrow dx = \frac{1}{2}\frac{1}{\sqrt{\log 3}}\,\frac{1}{2}\,t^{-\frac{1}{2}}\,dt$

$= dx = \frac{1}{4}\frac{1}{\sqrt{\log 3}}\,t^{-\frac{1}{2}}\,dt$

$= \int_0^\infty e^{-t}\,\frac{1}{4}\frac{1}{\sqrt{\log 3}}\,t^{-\frac{1}{2}}\,dt$

$= \frac{1}{4\sqrt{\log 3}} \int_0^\infty e^{-t}\,t^{\frac{1}{2}-1}\,dt$

$= \frac{1}{4\sqrt{\log 3}}\,\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{4\sqrt{\log 3}}$

$\therefore \int_0^\infty 3^{-4x^2}\,dx = \frac{\sqrt{\pi}}{4\sqrt{\log 3}}$

(5) Show that $\int_0^\infty \frac{x^a}{a^x}\,dx = \frac{\Gamma(a+1)}{(\log a)^{a+1}}$ $(a>1)$

Soln: LHS $\int_0^\infty \frac{x^a}{a^x}\,dx$

Put $a^x = e^t$

$\Rightarrow x \log a = t$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$x = \frac{1}{\log a} t$

$dx = \frac{1}{\log a} dt$

As $x \to 0$; $t \to 0$ and $x \to \infty$, $t \to \infty$

$$= \int_0^\infty \left(\frac{t}{\log a}\right)^a e^{-t} \frac{1}{\log a} dt$$

$$= \frac{1}{(\log a)^{a+1}} \int_0^\infty e^{-t} dt \cdot t^a$$

$$= \frac{1}{(\log a)^{a+1}} \int_0^\infty e^{-t} t^{(a+1)-1} dt$$

$$= \frac{1}{(\log a)^{a+1}} \Gamma(a+1)$$

$$= \frac{1}{(\log a)^{a+1}}$$

$$= R.H.S$$

$$\therefore \int_0^a \frac{x^a}{a^x} dx = \frac{\Gamma(a+1)}{(\log a)^{a+1}}$$

(5) Show that $\int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma(n)}{k^n}$

Soln:

L.H.S $\int_0^\infty e^{-kx} x^{n-1} dx$

put $kx = t \Rightarrow x = \frac{1}{k} t$

$dx = \frac{1}{k} dt$

$$= \int_0^\infty e^{-t} \left(\frac{t}{k}\right)^{n-1} \frac{1}{k} dt$$ as $x \to 0$; $t \to 0$ and $x \to \infty$, $t \to \infty$

$$= \int_0^\infty e^{-t} \frac{t^{n-1}}{k^{n-1}} \cdot \frac{1}{k} dt$$

$$= \frac{1}{k^n} \int_0^\infty e^{-t} t^{n-1} dt$$

$$= \frac{\Gamma(n)}{k^n}$$

$$= R.H.S$$

$$\therefore \int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma(n)}{k^n}.$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice

www.FirstRanker.com www.FirstRanker.com

Prove that $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ Where $m>0, n>0$

Proof :- We know that

$$\Gamma(m) = 2\int_0^\infty e^{-x^2} x^{2m-1}\,dx \quad \text{——} (1)$$

Ily $$\Gamma(n) = 2\int_0^\infty e^{-y^2} y^{2n-1}\,dy \quad \text{——} (2)$$

$(1)\times(2) \Rightarrow$ $$\Gamma m\,\Gamma n = 4\int_0^\infty e^{-x^2} x^{2m-1}\,dx \int_0^\infty e^{-y^2} y^{2n-1}\,dy$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1}\,dy \quad \text{——} (3)$$

Changing to polar co-ordinates by writing

$x = r\cos\theta$; $y = r\sin\theta$; $dx\,dy = r\,dr\,d\theta$.

r varies from 0 to ∞ and θ varies from 0 to $\pi/2$

$$\Gamma(m)\Gamma n = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2} (r\cos\theta)^{2m-1} (r\sin\theta)^{2n-1} r\,dr\,d\theta$$

$$= 2\int_{\theta=0}^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\,d\theta \; 2\int_{r=0}^{\infty} e^{-r^2} r^{2(m+n)-1}\,dr$$

$$= \beta(m,n)\,\Gamma(m+n)$$

$$\frac{\Gamma m\,\Gamma n}{\Gamma(m+n)} = \beta(m,n)$$

$$\therefore \beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$$

www.FirstRanker.com

Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com

Prove that $\Gamma n \, \Gamma(1-n) = \frac{\pi}{\sin n\pi}$ if n is not an integer

www.FirstRanker.com www.FirstRanker.com

Soln: we know that

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)} \quad \text{— (1)}$$

and also $\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$ — (2)

$$\therefore \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

put $m = 1-n$

$\Rightarrow m+n = 1$

$$\frac{\Gamma(1-n)\,\Gamma n}{\Gamma 1} = \int_0^\infty \frac{x^{n-1}}{(1+x)} dx$$

$$\Rightarrow \Gamma n \, \Gamma(1-n) = \int_0^\infty \frac{x^{n-1}}{(1+x)} dx$$

$\Rightarrow$ But we have $\int_0^\infty \frac{x^{n-1}}{(1+x)} dx = \frac{\pi}{\sin n\pi}$ for non integer n

$$\therefore \Gamma n \, \Gamma(1-n) = \frac{\pi}{\sin n\pi}$$ for non integer n.

(2) Prove that $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}} dx = 0$ using β and Γ functions

Soln:

LHS $\int_0^\infty \frac{x^8}{(1+x)^{24}} dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}$

$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}} dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}} dx$

$= \beta(9,15) - \beta(15,9)$

$= \beta(9,15) - \beta(9,15)$ $(\because \beta(m,n) = \beta(n,m)]$

$= 0$

$=$ R.H.S

www.FirstRanker.com

Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

(3) Evaluate $\int_0^2 (8-x^3)^{-1/3}\,dx$ using β and Γ functions

Soln: Consider $\int_0^2 (8-x^3)^{-1/3}\,dx$

Put $x^3 = 8t$

$\Rightarrow x = 2t^{1/3}$

$dx = \frac{2}{3}t^{-2/3}\,dt$

As $x \to 0$, $t \to 0$ and as $x \to 2$, $t \to 1$

$$= \int_0^1 (8-8t)^{-1/3}\,\frac{2}{3}t^{-2/3}\,dt$$

$$= \frac{2}{3}(8)^{-1/3}\int_0^1 t^{-2/3}(1-t)^{-1/3}\,dt$$

$$= \frac{1}{3}\int_0^1 t^{\frac{1}{3}-1}(1-t)^{\frac{2}{3}-1}\,dt$$

$$= \frac{1}{3}\beta\left(\frac{1}{3},\frac{2}{3}\right)$$

$$= \frac{1}{3}\frac{\Gamma 1/3\,\Gamma 2/3}{\Gamma \frac{1}{3}+\frac{2}{3}}$$

$$= \frac{1}{3}\Gamma 1/3\,\Gamma 1-1/3$$

$$= \frac{1}{3}\frac{\pi}{\sin \pi/3}$$

$$= \frac{\pi}{3}\frac{2}{\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$

(4) Express $\int_0^1 x^m(1-x^n)^p\,dx$ interms of Γ function.

Soln: Consider $\int_0^1 x^m(1-x^n)^p\,dx$

Put $x^n = t$

$\Rightarrow x = t^{1/n}$

$dx = \frac{1}{n}t^{\frac{1}{n}-1}\,dt$

$$= \int_0^1 t^{m/n}(1-t)^p\,\frac{1}{n}t^{\frac{1}{n}-1}\,dt$$

$$= \frac{1}{n}\int_0^1 t^{\left(\frac{m}{n}+\frac{1}{n}-1\right)}(1-t)^p\,dt$$

As $x \to 0$, $t \to 0$ and $x \to 1$, $t \to 1$

$$= \frac{1}{n}\int_0^1 t^{\left(\frac{m+1}{n}\right)-1}(1-t)^{(p+1)-1}\,dt$$

$$= \frac{1}{n}\beta\left(\frac{m+1}{n},p+1\right)$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$$\frac{\frac{1}{n}\,\Gamma\frac{m+1}{n}\,\Gamma(p+1)}{\Gamma\left(\frac{m+1}{n}+p+1\right)}$$

(5) Evaluate $\int_0^1 x^5(1-x^3)^{10}\,dx$

Soln: We know that

$$\int_0^1 x^m(1-x^n)^p\,dx = \frac{1}{n}\,\frac{\Gamma\frac{m+1}{n}\,\Gamma(p+1)}{\Gamma\left(\frac{m+1}{n}+p+1\right)}$$

Here $m=5;\ n=3;\ p=10$

$$\int_0^1 x^5(1-x^3)^{10}\,dx = \frac{1}{3}\,\frac{\Gamma\frac{6}{3}\,\Gamma 11}{\Gamma\left(\frac{6}{3}+11\right)} = \frac{1}{3}\,\frac{\Gamma 2\,\Gamma 11}{\Gamma 13}$$

$$= \frac{1}{396}$$

(6) Show that $\Gamma x\,\Gamma(-x) = \frac{-\pi}{x\sin\pi x}$

Soln: We know that

$$\Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

Now $\Gamma x\,\Gamma(-x) = \Gamma x\left[\frac{-1}{x}\,\Gamma(-x+1)\right]$ $\quad \left[\Gamma n = \frac{1}{n}\Gamma(n+1)\right]$

$$= \frac{-1}{x}\,\Gamma x\,\Gamma(1-x)$$

$$= \frac{-1}{x}\,\frac{\pi}{\sin\pi x}$$

$$= \frac{-\pi}{x\sin\pi x}$$

(7) Show that $\Gamma\left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}-x\right) = \pi\sec\pi x$

Soln: We know that $\Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

$$\Gamma\left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}-x\right) = \Gamma\left(\frac{1}{2}+x\right)\Gamma\left(1-\left(\frac{1}{2}+x\right)\right)$$

$$= \frac{\pi}{\sin\left[\left(\frac{1}{2}+x\right)\pi\right]}$$

$$= \frac{\pi}{\sin\left(\frac{\pi}{2}+\pi x\right)} = \frac{\pi}{\cos\pi x} = \pi\sec\pi x$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

Evaluate $\int_0^\infty \frac{x^2}{1+x^4}\,dx$ using β and Γ functions (8)

Soln:-

Consider $4\int_0^\infty \frac{x^2}{1+x^4}\,dx$

put $x^4 = t$

$\Rightarrow x = t^{1/4}$

$dx = \frac{1}{4}t^{-3/4}dt$

as $x \to 0; t \to 0$ and as $x \to \infty, t \to \infty$

$$= 4\int_0^\infty \frac{t^{2/4}\,t^{-3/4}}{4(1+t)}\,dt$$

$$= \int_0^\infty \frac{t^{-1/4}}{(1+t)}\,dt$$

$$= \int_0^\infty \frac{t^{3/4-1}}{(1+t)^{3/4+1/4}}\,dt$$

$$= \beta\left(\frac{3}{4}, \frac{1}{4}\right)$$

$$= \frac{\lceil 3/4 \lceil 1/4}{\lceil 3/4+1/4}$$

$$= \lceil 3/4 \lceil 1/4$$

$$= \lceil 1/4 \lceil 1-\frac{1}{4} = \frac{\pi}{\sin\frac{\pi}{4}} = \sqrt{2}\pi$$

(9) Evaluate $\int_0^{\pi/2} \sin^4\theta \cos^2\theta\,d\theta$

Soln:

We know that

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta\,d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

put $m = 4$; $n = 2$

$$\int_0^{\pi/2} \sin^4\theta \cos^2\theta\,d\theta = \frac{1}{2}\beta\left(\frac{4+1}{2}, \frac{2+1}{2}\right)$$

$$= \frac{1}{2}\beta\left(\frac{5}{2}, \frac{3}{2}\right)$$

$$= \frac{1}{2}\frac{\lceil 5/2 \lceil 3/2}{\lceil 5/2+3/2}$$

www.FirstRanker.com
Scanned by CamScanner

FirstRanker.com
Firstranker's choice
www.FirstRanker.com www.FirstRanker.com

$$= \frac{1}{2}\left(\frac{5}{2}-1\right)\left(\frac{3}{2}-1\right)\sqrt{\frac{3}{2}-1}\left(\frac{3}{2}-1\right)\sqrt{\frac{3}{2}-1}$$

$$= \frac{1}{6}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\ \frac{1}{2}\sqrt{\pi} = \frac{\pi}{16}.$$

(10) Show that $\int_0^\infty \sin x^2\,dx = \int_0^\infty \cos x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$

Soln: Let $A = \int_0^\infty \cos x^2\,dx$ and $B = \int_0^\infty \sin x^2\,dx$

Consider $A - iB = \int_0^\infty [\cos x^2 - i\sin x^2]\,dx = \int_0^\infty e^{-ix^2}\,dx$

put $ix^2 = t$

$\Rightarrow e^{i\pi/2}x^2 = t$

$\Rightarrow e^{i\pi/2}x = \sqrt{t}$

$\Rightarrow x = e^{-i\pi/2}\sqrt{t}$

$\Rightarrow dx = e^{-i\pi/2}\frac{1}{2\sqrt{t}}\,dt$

as $x \to 0; t \to 0$ and as $x \to \infty; t \to \infty$

$$A - iB = \int_0^\infty e^{-t}\,e^{-i\pi/2}\frac{1}{2\sqrt{t}}\,dt$$

$$= \frac{e^{-i\pi/2}}{2}\int_0^\infty e^{-t}t^{\frac{1}{2}-1}\,dt = \frac{e^{-i\pi/4}}{2}\Gamma\frac{1}{2}$$

$$= \frac{\sqrt{\pi}}{2}\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right]$$

$$= \frac{\sqrt{\pi}}{2}\left[\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right]$$

$$A - iB = \frac{1}{2}\sqrt{\frac{\pi}{2}} - i\frac{1}{2}\sqrt{\frac{\pi}{2}}$$

Comparing real and imaginary parts we have

$$A = \frac{1}{2}\sqrt{\frac{\pi}{2}} \text{ and } B = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

Hence $\int_0^\infty \cos x^2\,dx = \int_0^\infty \sin x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$

www.FirstRanker.com
Scanned by CamScanner