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VECTOR CALCULUS

Differentiation of vector:-

If $\bar{r} = \bar{f}(t)$ is a vector function, then differentiation of $\bar{r}$ w.r.t to 't' is $\frac{d\bar{f}}{dt} = \lim_{\delta t \to 0} \frac{\bar{f}(t+\delta t) - \bar{f}(t)}{\delta t} = \frac{d\bar{r}}{dt}$.

If $\lim_{\delta t \to 0} \frac{\delta \bar{r}}{\delta t} = \lim_{\delta t \to 0} \frac{\bar{f}(t+\delta t) - \bar{f}(t)}{\delta t}$ exists; then the value of this limit is denoted by $\frac{d\bar{r}}{dt}$ and is called the derivative of $\bar{r}$ w.r.t to 't'.

Let $\bar{r} = f(t)$ be a vector function of the scalar variable t. Let δt be a small increment in t and $\delta \bar{r}$ the corresponding increment in $\bar{r}$.

Then $\bar{r} + \delta \bar{r} = \bar{f}(t+\delta t)$ so that $\delta \bar{r} = \bar{f}(t+\delta t) - \bar{r}$

$\delta \bar{r} = \bar{f}(t+\delta t) - \bar{f}(t)$

$\frac{\delta \bar{r}}{\delta t} = \frac{\bar{f}(t+\delta t) - \bar{f}(t)}{\delta t}$.

Since $\frac{d\bar{r}}{dt}$ is itself a vector function of 't', it's derivative is denoted by $\frac{d^2\bar{r}}{dt^2}$ and is called the second derivative of $\bar{r}$ w.r.t to 't'. By we can define higher order derivatives of $\bar{r}$.

Important Formulae of Differentiation of vectors

(i) $\frac{d}{dt}(\bar{F} + \bar{G} - \bar{H}) = \frac{d\bar{F}}{dt} + \frac{d\bar{G}}{dt} - \frac{d\bar{H}}{dt}$

(ii) $\frac{d}{dt}[\bar{F}\phi] = \bar{F}\frac{d\phi}{dt} + \frac{d\bar{F}}{dt}\phi$

(iii) $\frac{d}{dt}[\bar{F}\cdot\bar{G}] = \bar{F}\cdot\frac{d\bar{G}}{dt} + \frac{d\bar{F}}{dt}\cdot\bar{G}$

(iv) $\frac{d}{dt}[\bar{F}\times\bar{G}] = \bar{F}\times\frac{d\bar{G}}{dt} + \frac{d\bar{F}}{dt}\times\bar{G}$

(v) $\frac{d}{dt}[\bar{F}\,\bar{G}\,\bar{H}] = \left[\frac{d\bar{F}}{dt}\,\bar{G}\,\bar{H}\right] + \left[\bar{F}\,\frac{d\bar{G}}{dt}\,\bar{H}\right] + \left[\bar{F}\,\bar{G}\,\frac{d\bar{H}}{dt}\right]$

(vi) $\frac{d}{dt}[(\bar{F}\times\bar{G})\times\bar{H}] = \left(\frac{d\bar{F}}{dt}\times\bar{G}\right)\times\bar{H} + \left(\bar{F}\times\frac{d\bar{G}}{dt}\right)\times\bar{H} + (\bar{F}\times\bar{G})\times\frac{d\bar{H}}{dt}$

where $\bar{F}, \bar{G}, \bar{H}$ are vectors and ϕ, t Scalars.

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Differentiation of constant vector

1) If $\bar{F}$ be a vector of constant magnitude and direction then $\bar{F} \cdot \frac{d\bar{F}}{dt} = 0$.

Proof:- $\bar{F}(t)$ has a constant magnitude $\Rightarrow |\bar{F}(t)| =$ constant

$$\bar{F}(t) \cdot \bar{F}(t) = |\bar{F}(t)|^2 = \text{constant}.$$

$$\Rightarrow \frac{d}{dt}(\bar{F} \cdot \bar{F}) = 0 \Rightarrow \bar{F} \cdot \frac{d\bar{F}}{dt} + \frac{d\bar{F}}{dt} \cdot \bar{F} = 0$$

$$\Rightarrow 2\bar{F} \cdot \frac{d\bar{F}}{dt} = 0 \Rightarrow \bar{F} \cdot \frac{d\bar{F}}{dt} = 0.$$

$$\Rightarrow \frac{d\bar{F}}{dt} \perp \bar{F}.$$

2) If $\bar{F}(t)$ has constant direction then $\bar{F} \times \frac{d\bar{F}}{dt} = \bar{0}$.

Proof:- Let $|\bar{F}(t)| = f(t)$

Let $\bar{G}(t)$ be a unit vector in the direction of $\bar{F}(t)$

so that $\bar{F}(t) = f(t)\,\bar{G}(t)$

$$\frac{d\bar{F}}{dt} = f\frac{d\bar{G}}{dt} + \frac{df}{dt}\bar{G} \quad \text{— (1)}$$

If $\bar{F}(t)$ has constant direction so has $\bar{G}(t)$

Thus $\bar{G}(t)$ is a constant vector and $\frac{d\bar{G}}{dt} = 0$

From (1) $\frac{d\bar{F}}{dt} = \frac{df}{dt} \cdot \bar{G}$

$$\bar{F} \times \frac{d\bar{F}}{dt} = f\bar{G} \times \left(\frac{df}{dt}\bar{G}\right)$$

$$= f\frac{df}{dt}(\bar{G} \times \bar{G}) = \bar{0} \quad [\because \bar{G} \times \bar{G} = \bar{0}]$$

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1) If $\bar{A} = 5t^2\bar{i} + t\bar{j} - t^3\bar{k}$, $\bar{B} = \sin t\,\bar{i} - \cos t\,\bar{j}$ find (i) $\frac{d}{dt}(\bar{A}\cdot\bar{B})$

(ii) $\frac{d}{dt}(\bar{A}\times\bar{B})$

Sol:- (i) $\frac{d}{dt}(\bar{A}\cdot\bar{B}) = \bar{A}\cdot\frac{d\bar{B}}{dt} + \frac{d\bar{A}}{dt}\cdot\bar{B}$

$= (5t^2\bar{i} + t\bar{j} - t^3\bar{k})\cdot[\cos t\,\bar{i} - (-\sin t\,\bar{j})] + (10t\bar{i} + \bar{j} - 3t^2\bar{k})\cdot(\sin t\,\bar{i} - \cos t\,\bar{j})$

$= (5t^2\cos t + t\sin t) + 10t\sin t - \cos t$

$= 5t^2\cos t + 11t\sin t - \cos t$

(ii) $\frac{d}{dt}(\bar{A}\times\bar{B}) = \bar{A}\times\frac{d\bar{B}}{dt} + \frac{d\bar{A}}{dt}\times\bar{B}$

$= (5t^2\bar{i} + t\bar{j} - t^3\bar{k})\times(\cos t\,\bar{i} + \sin t\,\bar{j}) + (10t\bar{i} + \bar{j} - 3t^2\bar{k})\times(\sin t\,\bar{i} - \cos t\,\bar{j})$

$= 5t^2\sin t\,\bar{k} - t\cos t\,\bar{k} - t^3\cos t\,\bar{j} + 3t^3\sin t\,\bar{k} - 10t\cos t\,\bar{k} - \sin t\,\bar{k} - 3t^2\sin t\,\bar{j} - 3t^2\cos t\,\bar{i}$

$= (t^3\sin t - 3t^2\cos t)\bar{i} - t^2(t\cos t + 3\sin t)\bar{j} + [(5t^2 - 1)\sin t - 11t\cos t]\bar{k}$

2) Show that if $\bar{R} = A\sin\omega t + B\cos\omega t$ where A, B, ω are constants, then $\frac{d^2\bar{R}}{dt^2} = -\omega^2\bar{R}$ and $\bar{R}\times\frac{d\bar{R}}{dt} = -\omega\,\bar{A}\times\bar{B}$

Sol:- $\bar{R} = A\sin\omega t + B\cos\omega t$

$\frac{d\bar{R}}{dt} = A\omega\cos\omega t - B\omega\sin\omega t$

$\frac{d^2\bar{R}}{dt^2} = -A\omega^2\sin\omega t - B\omega^2\cos\omega t$

$= -\omega^2(A\sin\omega t + B\cos\omega t) = -\omega^2\bar{R}$

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$$\bar{R} \times \frac{d\bar{R}}{dt} = (A\sin\omega t + B\cos\omega t) \times (A\omega\cos\omega t - B\omega\sin\omega t)$$

$$= -\omega \sin^2\omega t (\bar{A}\times\bar{B}) + \omega\cos^2\omega t (\bar{B}\times\bar{A}) \qquad [\bar{A}\times\bar{A} = \bar{0},\ \bar{B}\times\bar{B} = \bar{0}]$$

$$= -\omega \sin^2\omega t (\bar{A}\times\bar{B}) - \omega\cos^2\omega t (\bar{A}\times\bar{B}) \qquad \sin^2\omega t + \cos^2\omega t = 1$$

$$= -\omega(\bar{A}\times\bar{B})[\sin^2\omega t + \cos^2\omega t] \qquad \bar{A}\times\bar{B} = -(\bar{B}\times\bar{A})$$

$$= -\omega(\bar{A}\times\bar{B})$$

2) Given $\bar{R} = t^m\bar{A} + t^n\bar{B}$ where A, B are constant vectors, show that if $\bar{R}$ and $\frac{d^2\bar{R}}{dt^2}$ are parallel vectors, then $m+n = 1$ unless $m = n$.

Sol:- Given $\bar{R} = t^m\bar{A} + t^n\bar{B}$

$$\frac{d\bar{R}}{dt} = mt^{m-1}\bar{A} + nt^{n-1}\bar{B}$$

$$\frac{d^2\bar{R}}{dt^2} = m(m-1)t^{m-2}\bar{A} + n(n-1)t^{n-2}\bar{B}$$

For parallel vector $\bar{R} \times \frac{d^2\bar{R}}{dt^2} = \bar{0}$

$$\Rightarrow (t^m\bar{A} + t^n\bar{B}) \times [m(m-1)t^{m-2}\cdot\bar{A} + n(n-1)t^{n-2}\bar{B}] = \bar{0}$$

$$\Rightarrow n(n-1)t^m t^{n-2}(\bar{A}\times\bar{B}) + m(m-1)t^n t^{m-2}(\bar{B}\times\bar{A}) = \bar{0} \qquad [\bar{A}\times\bar{B} = -\bar{B}\times\bar{A}]$$

$$\Rightarrow n(n-1)t^{m+n-2}(\bar{A}\times\bar{B}) - m(m-1)t^{m+n-2}(\bar{A}\times\bar{B}) = \bar{0}$$

$$\Rightarrow t^{m+n-2}(\bar{A}\times\bar{B})[n^2 - n - m^2 + m] = \bar{0}$$

$$\Rightarrow t^{m+n-2}(\bar{A}\times\bar{B})[(n+m)(n-m) - (n-m)] = \bar{0}$$

$$\Rightarrow t^{m+n-2}(\bar{A}\times\bar{B})[(n-m)[m+n-1]] = \bar{0} \Rightarrow m \neq n \text{ either } m+n = 1$$

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$\bar{P} = t^2\hat{i} + t^3\hat{j} - t\hat{k}$ and $\bar{Q} = 2\sin t\,\hat{i} - \cos t\,\hat{j} + 5t\hat{k}$ (3)

find (i) $\frac{d}{dt}(\bar{P}\cdot\bar{Q})$ (ii) $\frac{d}{dt}(\bar{P}\times\bar{Q})$.

Sol:- $\bar{P}\cdot\bar{Q} = 10t^2\sin t - t^3\cos t - 5t^2$

$\frac{d}{dt}(\bar{P}\cdot\bar{Q}) = 10t^2\cos t + 20t\sin t - 3t^2\cos t + t^3\sin t - 10t$

$= t^3\sin t + 7t^2\cos t + 20t\sin t - 10t$

$\bar{P}\times\bar{Q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t^3 & -t \\ 2\sin t & -\cos t & 5t \end{vmatrix} = \hat{i}(5t^4 - t\cos t) + \hat{j}(-t^2\sin t + 25t^3) + \hat{k}(-5t^2\cos t - t^3 2\sin t)$

$= (5t^4 - t\cos t)\hat{i} - \hat{j}(2t\sin t + 25t^3) - \hat{k}(5t^2\cos t + 2t^3\sin t)$

(4) If $\frac{d\bar{v}}{dt} = \bar{\omega}\times\bar{v}$ and $\frac{d\bar{v}}{dt} = \bar{\omega}\times\bar{v}$

P.T. $\frac{d}{dt}(\bar{v}\times\bar{v}) = \bar{\omega}\times(\bar{v}\times\bar{v})$

Sol:- $\frac{d}{dt}(\bar{v}\times\bar{v}) = \bar{v}\times\frac{d\bar{v}}{dt} + \frac{d\bar{v}}{dt}\times\bar{v}$

$= \bar{v}\times(\bar{\omega}\times\bar{v}) + (\bar{\omega}\times\bar{v})\times\bar{v}$

$= (\bar{v}\cdot\bar{v})\bar{\omega} - (\bar{v}\cdot\bar{\omega})\bar{v} + (\bar{v}\cdot\bar{\omega})\bar{v} - (\bar{v}\cdot\bar{v})\bar{\omega}$

$= (\bar{\omega}\cdot\bar{v})\bar{v} - (\bar{\omega}\cdot\bar{v})\bar{v}$

$= \bar{\omega}\times(\bar{v}\times\bar{v})$

$(\bar{v}\cdot\bar{v} = \bar{v}\cdot\bar{\omega})$

$(\bar{v}\cdot\bar{\omega} = \bar{\omega}\cdot\bar{v})$

$(\bar{a}\times\bar{b})\times\bar{c} = (\bar{c}\cdot\bar{a})\bar{b} - (\bar{c}\cdot\bar{b})\bar{a}$

$\bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}$

(5) If $\bar{A} = x^2y\hat{i} - 2xz^3\hat{j} + xz^2\hat{k}$

and $\bar{B} = 2z\hat{i} + y\hat{j} - x^2\hat{k}$, find $\frac{\partial^2}{\partial x\partial y}(\bar{A}\times\bar{B})$ at $(1, 0, -2)$

Sol:- $\bar{A}\times\bar{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x^2y & -2xz^3 & xz^2 \\ 2z & y & -x^2 \end{vmatrix} = \hat{i}(2x^2z^3 - xyz^2) + \hat{j}(2xz^3 + x^4yz) + \hat{k}(x^2y^2 + 4xz^4)$

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$$\frac{\partial^2}{\partial x \partial y}(\bar{a}\times\bar{b}) = (0-t^2)\bar{i} + (0+4x^3t)\bar{j} + (4xyz+0)\bar{k}$$

$$\frac{\partial^2}{\partial x \partial y}(\bar{a}\times\bar{b}) = -4(\bar{i}+2\bar{j}) \text{ at } (1,0,-2)$$

6) If $\bar{r} = a\cos t\,\bar{i} + a\sin t\,\bar{j} + at\tan\alpha\,\bar{k}$.

Find the value of $\left|\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2}\right|$ (ii) $\left[\frac{d\bar{r}}{dt}, \frac{d^2\bar{r}}{dt^2}, \frac{d^3\bar{r}}{dt^3}\right]$.

Sol:- $\bar{r} = a\cos t\,\bar{i} + a\sin t\,\bar{j} + at\tan\alpha\,\bar{k}$

$$\frac{d\bar{r}}{dt} = -a\sin t\,\bar{i} + a\cos t\,\bar{j} + a\tan\alpha\,\bar{k}$$

$$\frac{d^2\bar{r}}{dt^2} = -a\cos t\,\bar{i} - a\sin t\,\bar{j} \quad \text{and} \quad \frac{d^3\bar{r}}{dt^3} = a\sin t\,\bar{i} - a\cos t\,\bar{j}$$

$$\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -a\sin t & a\cos t & a\tan\alpha \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = a^2\sin t\tan\alpha\,\bar{i} - a^2\cos t\tan\alpha\,\bar{j} + \bar{k}(a^2\sin^2 t + a^2\cos^2 t)$$

$$= a^2\sin t\tan\alpha\,\bar{i} - a^2\cos t\tan\alpha\,\bar{j} + a^2\bar{k}$$

$$\left|\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2}\right| = \sqrt{a^4\sin^2 t\tan^2\alpha + a^4\cos^2 t\tan^2\alpha + a^4} = a^2\sqrt{\tan^2\alpha + 1} = a^2\sec^2\alpha$$

Also $\left[\frac{d\bar{r}}{dt}\ \frac{d^2\bar{r}}{dt^2}\ \frac{d^3\bar{r}}{dt^3}\right] = \begin{vmatrix} -a\sin t & a\cos t & a\tan\alpha \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix}$

$$= a\tan\alpha(a^2\cos^2 t + a^2\sin^2 t)$$

$\cos^2 t + \sin^2 t = 1$

$$= a^3\tan\alpha(\cos^2 t + \sin^2 t)$$

$$= a^3\tan\alpha$$

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Find the unit tangent vector at any point on the curve $x = t^2+2$, $y = 4t-5$, $z = 2t^2-6t$ where t is any variable. Also determine the unit tangent vector at the point $t=2$.

Sol:- If $\bar{r}$ is the position vector of any point (x,y,z) on the given curve then $\bar{r} = x\hat{i}+y\hat{j}+z\hat{k}$.

$$= (t^2+2)\hat{i} + (4t-5)\hat{j} + (2t^2-6t)\hat{k}.$$

and $\frac{d\bar{r}}{dt} = 2t\hat{i} + 4\hat{j} + (4t-6)\hat{k}$.

$$\left|\frac{d\bar{r}}{dt}\right| = \sqrt{(2t)^2+(4)^2+(4t-6)^2} = \sqrt{4t^2+16+16t^2+36-48t}$$

$$= \sqrt{20t^2-48t+52} = 2\sqrt{5t^2-12t+13}.$$

The unit tangent vector $\bar{T} = \frac{\frac{d\bar{r}}{dt}}{\left|\frac{d\bar{r}}{dt}\right|} = \frac{2[t\hat{i}+2\hat{j}+(2t-3)\hat{k}]}{2\sqrt{5t^2-12t+13}}$

$$= \frac{(t\hat{i}+2\hat{j}+2t-3\hat{k})}{\sqrt{5t^2-12t+13}}$$

Also the unit tangent vector at the point $t=2$ is $\frac{2\hat{i}+2\hat{j}+[2(2)-3]\hat{k}}{\sqrt{5(2)^2-12(2)+13}}$.

$$= \frac{1}{3}(2\hat{i}+2\hat{j}+\hat{k})$$

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Curves in a Space

(i) **Tangent** : If $\bar{R} = \bar{f}(t)$ be any space curve then the tangent at any point is represented by $\frac{d\bar{R}}{dt}$ at the point. If 's' is the parameter in place of 't' then the magnitude of the tangent vector, $\left|\frac{d\bar{R}}{ds}\right| = 1$, Thus the unit tangent vector $\boxed{\bar{T} = \frac{d\bar{R}}{ds}}$.

(ii) **principal Normal** :- Since 'T' is a unit vector.

$\boxed{\frac{d\bar{T}}{ds} \cdot \bar{T} = 0}$; i.e $\frac{d\bar{T}}{ds}$ is perpendicular to $\bar{T}$

Here $\frac{d\bar{T}}{ds}$ is unit vector along the vector $\bar{N}$ which known as principal normal at given point.

(iii) **Binormal** :- If $\bar{T}, \bar{N}$ represent unit tangent, unit (i.e principal normal) then Binormal $\boxed{\bar{B} = \bar{T} \times \bar{N}}$

(iv) **curvature** :- The arc rate of turning of the tan (i.e the magnitude of $\frac{d\bar{T}}{ds}$ is called the curvature curve and is denoted by 'k'.

Since $\frac{d\bar{T}}{ds}$ is in the direction of the principal normal $\bar{N}$

$$\boxed{\frac{d\bar{T}}{ds} = k\bar{N}}$$

(v) **Torsion** :- The arc rate of turning the binormal $\frac{d\bar{B}}{ds}$ is called torsion of the curve and is denoted by 'τ', we may, therefore, write $\boxed{\frac{d\bar{B}}{ds} = -\tau\bar{N}}$

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1) Find the curvature and torsion of the curve $x = a\cos t,\ y = a\sin t,\ z = bt$.

Ans:-

$$\bar{r} = a\cos t\,\bar{i} + a\sin t\,\bar{j} + bt\,\bar{k}$$

$$\frac{d\bar{r}}{dt} = -a\sin t\,\bar{i} + a\cos t\,\bar{j} + b\bar{k}$$

$$\frac{ds}{dt} = \left|\frac{d\bar{r}}{dt}\right| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2+b^2}$$

$$\bar{T} = \frac{d\bar{r}}{ds} = \frac{d\bar{r}/dt}{ds/dt} = \frac{-a\sin t\,\bar{i} + a\cos t\,\bar{j} + b\bar{k}}{\sqrt{a^2+b^2}}$$

$$\frac{d\bar{T}}{dt} = \frac{-a\cos t\,\bar{i} - a\sin t\,\bar{j}}{\sqrt{a^2+b^2}}$$

$$\frac{d\bar{T}}{ds} = \frac{d\bar{T}/dt}{ds/dt} = \frac{-a\cos t\,\bar{i} - a\sin t\,\bar{j}}{\sqrt{a^2+b^2}} \cdot \frac{1}{\sqrt{a^2+b^2}} = \frac{-a(\cos t\,\bar{i} + \sin t\,\bar{j})}{a^2+b^2}$$

$$k = \left|\frac{d\bar{T}}{ds}\right| = \frac{\sqrt{a^2\cos^2 t + a^2\sin^2 t}}{a^2+b^2} = \frac{a}{a^2+b^2}$$

$$\bar{N} = \frac{1}{k}\frac{d\bar{T}}{ds} = \frac{a^2+b^2}{a}\left(\frac{-a\cos t\,\bar{i} - a\sin t\,\bar{j}}{a^2+b^2}\right)$$

$$= -\cos t\,\bar{i} - \sin t\,\bar{j}$$

$$\bar{B} = \bar{T}\times\bar{N} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{-a\sin t}{\sqrt{a^2+b^2}} & \frac{a\cos t}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{b\sin t}{\sqrt{a^2+b^2}}\bar{i} - \frac{b\cos t}{\sqrt{a^2+b^2}}\bar{j} + a\bar{k}$$

$$\frac{d\bar{B}}{dt} = \frac{b\cos t\,\bar{i} + b\sin t\,\bar{j}}{\sqrt{a^2+b^2}}$$

$$\frac{d\bar{B}}{ds} = \frac{d\bar{B}}{dt}\Big/\frac{ds}{dt} = \frac{b\cos t\,\bar{i} + b\sin t\,\bar{j}}{\sqrt{a^2+b^2}} \cdot \frac{1}{\sqrt{a^2+b^2}} = \frac{b\cos t\,\bar{i} + b\sin t\,\bar{j}}{a^2+b^2}$$

But $\frac{d\bar{B}}{ds} = -\tau\bar{N}$

$$\frac{b\cos t\,\bar{i} + b\sin t\,\bar{j}}{a^2+b^2} = -\tau(-\cos t\,\bar{i} - \sin t\,\bar{j}) \Rightarrow \tau = \frac{b}{a^2+b^2}.$$

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$x = 3\cos t,\ y = 3\sin t,\ z = 4t$

$\bar{r} = 3\cos t\,\bar{i} + 3\sin t\,\bar{j} + 4t\,\bar{k} \Rightarrow \frac{d\bar{r}}{dt} = -3\sin t\,\bar{i} + 3\cos t\,\bar{j} + 4\bar{k}$

$\frac{d\bar{r}}{dt} = \bar{T} \Rightarrow \left|\frac{d\bar{r}}{ds}\right| = 1 \Rightarrow \left|\frac{d\bar{r}/dt}{ds/dt}\right| = 1 \Rightarrow \frac{ds}{dt} = \left|\frac{d\bar{r}}{dt}\right|$

$\frac{ds}{dt} = \left|\frac{d\bar{r}}{dt}\right| = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = \sqrt{9+16} = \sqrt{25} = 5$

$\bar{T} = \frac{d\bar{r}}{ds} = \frac{d\bar{r}/dt}{ds/dt} = \frac{-3\sin t\,\bar{i} + 3\cos t\,\bar{j} + 4\bar{k}}{5}$

$\frac{d\bar{T}}{dt} = \frac{1}{5}(-3\cos t\,\bar{i} - 3\sin t\,\bar{j})$

$\therefore \frac{d\bar{T}}{ds} = \frac{d\bar{T}/dt}{ds/dt} = \frac{1}{5}(-3\cos t\,\bar{i} - 3\sin t\,\bar{j})\cdot\frac{1}{5} = \frac{1}{25}(-3\cos t\,\bar{i} - 3\sin t\,\bar{j})$

$\bar{N} = \left|\frac{d\bar{T}}{ds}\right| = \frac{\sqrt{9\cos^2 t + 9\sin^2 t}}{25} = \frac{3}{25}$

$\bar{N} = \frac{1}{k}\frac{d\bar{T}}{ds} = \frac{25}{3}\cdot\frac{1}{25}(-3\cos t\,\bar{i} - 3\sin t\,\bar{j}) = -\cos t\,\bar{i} - \sin t\,\bar{j}$

3) $\bar{B} = \bar{T}\times\bar{N} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -\frac{3}{5}\sin t & \frac{3}{5}\cos t & \frac{4}{5} \\ -\cos t & -\sin t & 0 \end{vmatrix}$

$= \bar{i}\left(\frac{4}{5}\sin t\right) + \frac{4}{5}\cos t\,\bar{j} + \frac{3}{5}\bar{k}$

$= \frac{1}{5}(4\sin t\,\bar{i} - 4\cos t\,\bar{j} + 3\bar{k})$.

$\frac{d\bar{B}}{dt} = \frac{1}{5}(4\cos t\,\bar{i} + 4\sin t\,\bar{j})$.

$\frac{d\bar{B}}{ds} = \frac{d\bar{B}}{dt}\Big/\frac{ds}{dt} = \frac{1}{5}\cdot\frac{(4\cos t\,\bar{i} + 4\sin t\,\bar{j})}{5} = \frac{4\cos t\,\bar{i} + 4\sin t\,\bar{j}}{25}$

But $\frac{d\bar{B}}{ds} = -\tau\bar{N}$

$\frac{1}{25}(4\cos t\,\bar{i} + 4\sin t\,\bar{j}) = -\tau(-\cos t\,\bar{i} - \sin t\,\bar{j})$

$\tau = \frac{4}{25}$

$x = t - \frac{t^3}{3},\ y = t^2,\ z = t + \frac{t^3}{3}$

$k = \rho = \frac{1}{(1+t^2)^2}$

$x = t,\ y = t^2,\ z = \frac{2}{3}t^3$

$t = 1$ ⇒ k, τ.

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unit tangent vector to the curve $x=t, y=t^2, z=t^3$ at $t=1$

$\bar{r} = x\hat{i}+y\hat{j}+z\hat{k} \Rightarrow \bar{r} = t\hat{i}+t^2\hat{j}+t^3\hat{k}$

$\frac{d\bar{r}}{dt} = \hat{i}+2t\hat{j}+3t^2\hat{k}$, At $t=1$, $\frac{d\bar{r}}{dt} = \hat{i}+2\hat{j}+3\hat{k}$.

But $\left|\frac{d\bar{r}}{dt}\right| = \sqrt{1+4+9} = \sqrt{14}$

unit tangent vector to the curve $= \frac{\frac{d\bar{r}}{dt}}{\left|\frac{d\bar{r}}{dt}\right|} = \frac{\hat{i}+2\hat{j}+3\hat{k}}{\sqrt{14}}$

3. A particle moves along the curve $x=2t^2$, $y=t^2-4t$, $z=3t-5$ where t is the time. Find the components of its velocity and acceleration at time $t=1$ in the direction $\hat{i}-3\hat{j}+2\hat{k}$.

Sol:- $\bar{r} = x\hat{i}+y\hat{j}+z\hat{k} \Rightarrow \bar{r} = 2t^2\hat{i}+(t^2-4t)\hat{j}+(3t-5)\hat{k}$

Velocity $= \bar{v} = \frac{d\bar{r}}{dt} = 4t\hat{i}+(2t-4)\hat{j}+3\hat{k}$

when $t=1$, $\frac{d\bar{r}}{dt} = 4\hat{i}-2\hat{j}+3\hat{k}$.

Acceleration $= \frac{d^2\bar{r}}{dt^2} = 4\hat{i}+2\hat{j}$.

when $t=1$; $\frac{d^2\bar{r}}{dt^2} = 4\hat{i}+2\hat{j}$.

Component of velocity in the direction of $\hat{i}-3\hat{j}+2\hat{k}$ is

$= \frac{(4\hat{i}-2\hat{j}+3\hat{k})(\hat{i}-3\hat{j}+2\hat{k})}{\sqrt{1+9+4}} = \frac{4+6+6}{\sqrt{14}} = \frac{16}{\sqrt{14}}$

component of acceleration in the direction of $(4\hat{i}+2\hat{j})$ $\hat{i}-3\hat{j}+2\hat{k}$ is

$= \frac{(4\hat{i}+2\hat{j})(\hat{i}-3\hat{j}+2\hat{k})}{\sqrt{1+9+4}} = \frac{4-6}{\sqrt{14}} = \frac{-2}{\sqrt{14}}$

4) A particle moves so that its position vector is given by $\bar{r} = \cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}$ where 'ω' is a constant. show that velocity $\bar{v}$ of the particle is perpendicular to $\bar{r}$ and $\bar{r}\times\bar{v}$ is a constant vector.

Sol:- The position vector of the particle is given by $\bar{r} = \cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}$.

The velocity is given by $\bar{v} = \frac{d\bar{r}}{dt} = -\omega\sin\omega t\,\hat{i} + \omega\cos\omega t\,\hat{j}$

consider $\bar{v}\cdot\bar{r} = -\omega\cos\omega t\sin\omega t + \omega\sin\omega t\cos\omega t = 0$ so $\bar{v}$ is $\perp$ to $\bar{r}$

now $\bar{r}\times\bar{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\omega t & \sin\omega t & 0 \\ -\omega\sin\omega t & \omega\cos\omega t & 0 \end{vmatrix} = \hat{k}(\omega\cos^2\omega t + \omega\sin^2\omega t)$

$= \omega\hat{k}$ = constant vector.

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① Find the angle between the tangents to the curve

$\bar{r} = t^2\bar{i} + 2t\bar{j} - t^3\bar{k}$ at the points $t=1$ and $t=-1$

Sol:- $\frac{d\bar{r}}{dt} = 2t\bar{i} + 2\bar{j} - 3t^2\bar{k}$ is a vector along the tangent at any point 't'

If $\bar{T}_1$ and $\bar{T}_2$ are the vectors along the tangents at $t=1$ and $t=-1$, respectively, then

$\bar{T}_1 = 2\bar{i} + 2\bar{j} - 3\bar{k}$ and $\bar{T}_2 = -2\bar{i} + 2\bar{j} + 3\bar{k}$

If θ is the angle between $\bar{T}_1$ and $\bar{T}_2$ then

$$\cos\theta = \frac{\bar{T}_1 \cdot \bar{T}_2}{|\bar{T}_1|\,|\bar{T}_2|} = \frac{2(-2) + 2(+2) + 3(-3)}{\sqrt{4+4+9}\ \sqrt{4+4+9}} = \frac{-4+4-9}{\sqrt{17}\ \sqrt{17}} = \frac{9}{17}$$

$$\theta = \cos^{-1}(9/17)$$

Velocity and Acceleration:

If the scalar variable 't' be the time and $\bar{r}$ be the position vector of moving particle with respect to the origin 'O', then $\delta\bar{r}$ is the displacement of the particle in time δt. If $\bar{v}$ represents the velocity of the particle at P in time δt,

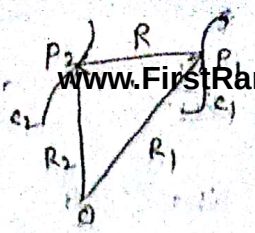
then $\bar{v} = \lim_{\delta t \to 0} \frac{\delta\bar{r}}{\delta t} = \frac{d\bar{r}}{dt}$

and acceleration is given by $\bar{a} = \lim_{\delta t \to 0} \frac{\delta\bar{v}}{dt} = \frac{d^2\bar{r}}{dt^2}$

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Relative velocity and acceleration



Let two particles P_1 and P_2 moving along the curves C_1 and C_2 have position vectors R_1 and R_2 at time t, we get

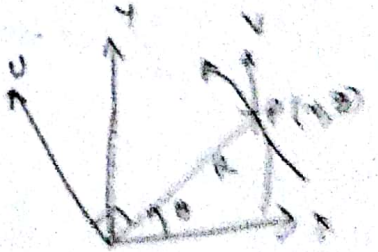
$$\frac{d\bar{R}}{dt} = \frac{d\bar{R}_2}{dt} - \frac{d\bar{R}_1}{dt} \quad -(1)$$

This defines the relative velocity of P_2 w.r.t P_1 and states that the velocity of P_2 relative to P_1 = velocity of P_2 − velocity of P_1.

differentiating (1), we have, $\frac{d^2\bar{R}}{dt^2} = \frac{d^2\bar{R}_2}{dt^2} - \frac{d^2\bar{R}_1}{dt^2}$

i.e acceleration of P_2 relative to P_1 = acceleration of P_2 − acceleration of P_1.

(2) Find the radial and transverse acceleration of a particle moving in a plane curve.



Sol:- At any time t, let the position vector of the moving particle $P(r,\theta)$ be $\bar{R}$

so that $\bar{R} = r\hat{R} = r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

∴ Its velocity $\bar{v} = \frac{d\bar{R}}{dt} = \frac{dr}{dt}\hat{R} + r\frac{d\hat{R}}{dt}$ —(1)

As $\hat{R} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

and $\frac{d\hat{R}}{dt} = (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})\frac{d\theta}{dt}$

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$\frac{d\hat{R}}{dt} \perp \hat{R}$ and $\left|\frac{d\hat{R}}{dt}\right| = \frac{d\theta}{dt}$ i.e. $\hat{U}$ is a unit vector

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Then $\frac{d\hat{R}}{dt} = \frac{d\theta}{dt} U$

① becomes, $\bar{V} = \frac{dr}{dt}\hat{R} + r\frac{d\theta}{dt} U$

Thus the radial and transverse components of the velocity are $\frac{dr}{dt}$ and $r\frac{d\theta}{dt}$.

Also $\bar{A} = \frac{d\bar{V}}{dt} = \frac{d^2r}{dt^2}\hat{R} + \frac{dr}{dt}\frac{d\hat{R}}{dt} + \frac{dr}{dt}\frac{d\theta}{dt} U + r\frac{d^2\theta}{dt^2} U + r\frac{d\theta}{dt}$

$$= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{R} + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right) U$$

$$\left[\because U = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right.$$

$$\left. \text{gives } \frac{dU}{dt} = -\frac{d\theta}{dt}\hat{R}\right]$$

Thus the radial and transverse components of the acceleration are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \text{ and } 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$$

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1) A particle moves along the curve $x = t^3+1$, $y = t^2$, $z = 2t+3$ where 't' is the time. Find the components of its velocity and acceleration at $t=1$ in the direction $\hat{i}+\hat{j}+3\hat{k}$.

Sol:- Velocity $\frac{d\bar{R}}{dt} = \frac{d}{dt}\left[(t^3+1)\hat{i} + t^2\hat{j} + (2t+3)\hat{k}\right]$

$= 3t^2\hat{i} + 2t\hat{j} + 2\hat{k} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ at $t=1$

and acceleration $= \frac{d^2\bar{R}}{dt^2} = 6t\hat{i} + 2\hat{j} + 0\hat{k} = 6\hat{i} + 2\hat{j}$ at $t=1$

Now unit vector in the direction of $\hat{i}+\hat{j}+3\hat{k}$ is

$$\frac{\hat{i}+\hat{j}+3\hat{k}}{\sqrt{1^2+1^2+3^2}} = \frac{1}{\sqrt{11}}(\hat{i}+\hat{j}+3\hat{k})$$

component of velocity at $t=1$ in the direction $\hat{i}+\hat{j}+3\hat{k}$

$$= \frac{(3\hat{i}+2\hat{j}+2\hat{k})(\hat{i}+\hat{j}+3\hat{k})}{\sqrt{11}} = \frac{3+2+6}{\sqrt{11}} = \frac{11}{\sqrt{11}} = \sqrt{11}$$

and component of acceleration at $t=1$ in the direction

$$\hat{i}+\hat{j}+3\hat{k} = (6\hat{i}+2\hat{j})\cdot\frac{(\hat{i}+\hat{j}+3\hat{k})}{\sqrt{11}} = \frac{6+2}{\sqrt{11}} = \frac{8}{\sqrt{11}}$$

2) A particle moves along the curve $\bar{R} = (t^2-4t)\hat{i} + (t^2+4t)\hat{j} + (2t^2-3t^3)\hat{k}$ where 't' denotes time. Find the magnitude of acceleration along the tangent and normal at time $t=2$.

Sol:- Velocity $\frac{d\bar{R}}{dt} = (3t^2-4)\hat{i} + (2t+4)\hat{j} + (16t-9t^2)\hat{k}$

and acceleration $\frac{d^2\bar{R}}{dt^2} = 6t\hat{i} + 2\hat{j} + (16-8t)\hat{k}$

At $t=2$, velocity $\bar{V} = 8\hat{i} + 8\hat{j} - 4\hat{k}$ and

acceleration $\bar{A} = 12\hat{i} + 2\hat{j} - 20\hat{k}$

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Hence the velocity is along the tangent to the curve,

the component of A along the tangent $= A \cdot \frac{V}{|V|}$

$$= (12\hat{i} + 2\hat{j} - 20\hat{k}) \cdot \frac{8\hat{i} + 8\hat{j} - 4\hat{k}}{\sqrt{64+64+16}} = \frac{12\times 8 + 2\times 8 + (-20)\times(-4)}{12} = 12$$

Now the component of A along the normal

$= |A -$ Resolved part of A along the tangent $|$

$$= \left| 12\hat{i} + 2\hat{j} - 20\hat{k} - 16\,\frac{8\hat{i} + 8\hat{j} - 4\hat{k}}{12} \right| = \frac{1}{3}\left| 4\hat{i} - 26\hat{j} - 44\hat{k} \right|$$

$$= 2\sqrt{73}$$

3) The position vector of a particle at time 't' is

$$\bar{R} = \cos(t-1)\,\hat{i} + \sinh(t-1)\,\bar{j} + \alpha t^3\,\bar{k}$$

Find the condition imposed on α by requiring that at time $t=1$, the acceleration is normal to the position vector.

Sol: $\text{velocity} = \frac{d\bar{R}}{dt} = -\sin(t-1)\,\hat{i} + \cosh(t-1)\,\bar{j} + 3\alpha t^2\,\bar{k}$

$$\text{acceleration} = \frac{d^2\bar{R}}{dt^2} = -\cos(t-1)\,\hat{i} + \sinh(t-1)\,\bar{j} + 6\alpha t\,\bar{k}$$

$$= -\hat{i} + 6\alpha\,\bar{k} \quad \text{at } t=1$$

Also $\bar{R} = \hat{i} + \alpha\bar{k}$ at $t=1$.

If $\bar{R}$ and acceleration at $t=1$ are normal,

then their scalar product is zero.

$$(-\hat{i} + 6\alpha\bar{k}) \cdot (\hat{i} + \alpha\bar{k}) = 0$$

$$-1 + 6\alpha^2 = 0 \Rightarrow \alpha^2 = 1/6$$

$$\text{or } \alpha = \frac{1}{\sqrt{6}}$$

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Scalar point function :- A function $\phi(x,y,z)$ is called a scalar point function if it associates a scalar with every point in space. The temperature distribution in a heated body, density of body and potential due to gravity are the examples of a scalar point function.

Vector point function :- If a function $F(x,y,z)$ defines a vector at every point of a region, then $F(x,y,z)$ is called a vector point function. The velocity of moving fluid, gravitational force are the example of vector point function.

Vector operator Del or Nabla :-

The operator $\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$ is called the vector operator Del or Nabla.

The vector operator ∇ has properties analogus to those of ordinary vectors and can be applied to (i) scalar point functions (ii) vector point functions

Gradient :- The vector function ∇f is defined as the gradient of the scalar point function f and written as grad f

$$\text{grad } f = \nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

Directional Derivative :- The component of $\nabla\phi$ in the direction of $\bar{a}$ is given by $\nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$ and is called the directional derivative of ϕ in direction of $\bar{a}$.

The directional derivative of ϕ at a point P is maximum in the direction of normal at P and its maximum value is $|\nabla\phi|$.

Properties of gradient :- If ϕ and ψ are two scalar point functions then (i) $\nabla(c_1\phi + c_2\psi) = c_1\nabla\phi + c_2\nabla\psi$ where c_1, c_2 are constants

(ii) $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

(iii) $\nabla\left(\frac{\phi}{\psi}\right) = \frac{\psi\nabla\phi - \phi\nabla\psi}{\psi^2}$, $\psi \neq 0$.

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where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}|$

$r = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{1/2}$

$r^n = (x^2+y^2+z^2)^{n/2}$

$\nabla r^n = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{n/2}$

$= \frac{n}{2}(x^2+y^2+z^2)^{n/2-1} \cdot [2x\bar{i} + 2y\bar{j} + 2z\bar{k}]$

$= \frac{n}{2}(r^2)^{\frac{n-2}{2}}(2x\bar{i} + 2y\bar{j} + 2z\bar{k})$

$= n(r^2)^{\frac{n-2}{2}}(x\bar{i} + y\bar{j} + z\bar{k})$

$\nabla r^n = nr^{n-2}\bar{r}$.

4) $\nabla(\bar{a}\cdot\bar{r}) = \bar{a}$ where $\bar{a}$ is constant vector

Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$ where a_1, a_2, a_3 are constants

$\bar{a}\cdot\bar{r} = a_1x + a_2y + a_3z$

$\nabla(\bar{a}\cdot\bar{r}) = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(a_1x + a_2y + a_3z)$

$= a_1\bar{i} + a_2\bar{j} + a_3\bar{k} = \bar{a}$

5) Find the grad ϕ where ϕ is given by $\phi = 3x^2y - y^3z^2$ at the point $(1,-2,-1)$

Ans:- Grad $\phi = \nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(3x^2y - y^3z^2)$

$= \bar{i}\left(\frac{\partial}{\partial x}\right)(3x^2y - y^3z^2) + \bar{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \bar{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$

$= 6xy\bar{i} + \bar{j}(3x^2 - 3y^2z^2) + (-2y^3z)\bar{k}$

$= -12\bar{i} - 9\bar{j} - 16\bar{k}$ at the point $(1,-2,-1)$.

2)

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p.T $\nabla r^n = nr^{n-2}\bar{r}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Sol:- $\nabla r^n = \bar{i}\frac{\partial}{\partial x}(r^n) + \bar{j}\frac{\partial}{\partial y}(r^n) + \bar{k}\frac{\partial}{\partial z}(r^n)$

$= nr^{n-1}\frac{\partial r}{\partial x}\bar{i} + nr^{n-1}\frac{\partial r}{\partial y}\bar{j} + nr^{n-1}\frac{\partial r}{\partial z}\bar{k}$

$= nr^{n-1}\frac{x}{r}\bar{i} + nr^{n-1}\frac{y}{r}\bar{j} + nr^{n-1}\frac{z}{r}\bar{k}$

$= nr^{n-2}(x\bar{i} + y\bar{j} + z\bar{k}) = nr^{n-2}\bar{r}$

$r = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

$2r\frac{\partial r}{\partial x} = 2x$

$\frac{\partial r}{\partial x} = \frac{x}{r}$

2) If $u = x+y+z$, $v = x^2+y^2+z^2$, $w = yz+zx+xy$, p.T grad u, grad v, grad w are coplanar.

Sol:- grad $u = \bar{i}+\bar{j}+\bar{k}$, grad $v = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$

grad $w = (y+z)\bar{i} + (z+x)\bar{j} + (x+y)\bar{k}$

we know that three vectors are coplanar if their scalar triple product is zero.

$$[\text{grad } u, \text{grad } v, \text{grad } w] = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix}$$ operate $R_2 + R_3$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix} = 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix} = 0.$$

Hence grad u, grad v, grad w are coplanar.

3) Find a unit vector normal to the surface $xy^3z^2 = 4$ at the point $(-1, -1, 2)$

Sol:- A vector normal to the given surface is

$\nabla(xy^3z^2) = \bar{i}\frac{\partial}{\partial x}(xy^3z^2) + \bar{j}\frac{\partial}{\partial y}(xy^3z^2) + \bar{k}\frac{\partial}{\partial z}(xy^3z^2)$

$= y^3z^2\bar{i} + 3xy^2z^2\bar{j} + 2xy^3z\bar{k}$

$= -4\bar{i} - 12\bar{j} + 4\bar{k}$ at the point $(-1, -1, 2)$

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Hence the required unit normal to the surface

$$= \frac{i - 4\hat{i} - 12\hat{j} + 4\hat{k}}{\sqrt{(-4)^2 + (-12)^2 + 4^2}} = -\frac{1}{\sqrt{11}}(\hat{i} + 3\hat{j} - \hat{k})$$

4) Find the directional derivative of $f(x,y,z) = xy^2 + yz^3$ at the point $(2,-1,1)$ in the direction of vector $\hat{i} - 2\hat{j} - 2\hat{k}$.

Sol:- $\nabla f = \hat{i}\frac{\partial}{\partial x}(xy^2 + yz^3) + \hat{j}\frac{\partial}{\partial y}(xy^2 + yz^3) + \hat{k}\frac{\partial}{\partial z}(xy^2 + yz^3)$

$= y^2\hat{i} + (2xy + z^3)\hat{j} + 3yz^2\hat{k}$

$= \hat{i} - 3\hat{j} - 3\hat{k}$ at the point $(2,-1,1)$

∴ Directional derivative of f in the direction $\hat{i} + 2\hat{j} + 2\hat{k}$

$= (\hat{i} - 3\hat{j} - 3\hat{k}) \cdot \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{1^2 + 2^2 + 2^2}} = -\frac{11}{3}$

5) Find the directional derivative of $f = x^2 - y^2 + 2z^2$ at point $P(1,2,3)$ in the direction of the line PQ where Q is the point $(5,0,4)$. Also calculate the magnitude of maximum directional derivative.

Sol:- $\nabla f = \hat{i}\frac{\partial}{\partial x}(x^2 - y^2 + 2z^2) + \hat{j}\frac{\partial}{\partial y}(x^2 - y^2 + 2z^2) + \hat{k}\frac{\partial}{\partial z}(x^2 - y^2 + 2z^2)$

$= 2x\hat{i} - 2y\hat{j} + 4z\hat{k}$

$= 2\hat{i} - 4\hat{j} + 12\hat{k}$ at $P(1,2,3)$

Also $\vec{PQ} = \vec{OQ} - \vec{OP} = (5\hat{i} + 0\hat{j} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$

$= 4\hat{i} - 2\hat{j} + \hat{k} = \vec{b}$ say

∴ unit vector of $b = \hat{b} = \frac{\vec{b}}{|b|} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{16 + 4 + 1}} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}}$

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Thus the directional derivative of f is the direction of $\vec{A}$

$$\nabla f \cdot \hat{A} = (2\vec{i} - 4\vec{j} + 12\vec{k}) \cdot \frac{(4\vec{i} - 2\vec{j} + \vec{k})}{\sqrt{21}} = \frac{8+8+12}{\sqrt{21}} = \frac{28}{\sqrt{21}}$$

∴ The directional derivative of is maximum in the direction of the normal to the surface ie in the direction of ∇f.

Hence maximum value of this directional derivative

$$= |\nabla f| = |2\vec{i} - 4\vec{j} + 12\vec{k}| = \sqrt{4+16+144} = \sqrt{164}$$

6) Find the directional derivative of $\phi = 5x^2y - 5y^2z + 2.5z^2x$ at the point $P(1,1,1)$ in the direction of the line

$$\frac{x-1}{2} = \frac{y-3}{-2} = z$$

Sol:- we have $\nabla\phi = \vec{i}\frac{d\phi}{dx} + \vec{j}\frac{d\phi}{dy} + \vec{k}\frac{d\phi}{dz}$

$$= (10xy + 2.5z^2)\vec{i} + (5x^2 - 10yz)\vec{j} + (-5y^2 + 5xz)\vec{k}$$

$$= 12.5\vec{i} - 5\vec{j} \text{ at } P(1,1,1)$$

Also direction of the given line is $\hat{A} = \frac{2\vec{i} - 2\vec{j} + \vec{k}}{3}$

Hence the required directional derivative $= \nabla\phi \cdot \hat{n}$

$$= (12.5\vec{i} - 5\vec{j}) \cdot \frac{2\vec{i} - 2\vec{j} + \vec{k}}{3}$$

$$= \frac{25+10}{3} = \frac{35}{3}$$

7) Find the angle between the surfaces $x^2+y^2+z^2 = 9$ and $z = x^2+y^2-3$ at the point $(2,-1,2)$

Sol:- Let $f_1 = x^2+y^2+z^2-9 = 0$ and $f_2 = x^2+y^2-z-3 = 0$.

Then $N_1 = \nabla f_1$ at $(2,-1,2)$

$$= (2x\vec{i} + 2y\vec{j} + 2z\vec{k}) \text{ at } (2,-1,2) = 4\vec{i} - 2\vec{j} + 4\vec{k}$$

and $N_2 = \nabla f_2$ at $(2,-1,2)$

$$= (2x\vec{i} + 2y\vec{j} - \vec{k}) \text{ at } (2,-1,2) = 4\vec{i} - 2\vec{j} - \vec{k}$$

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Since the angle θ between the two surfaces at a point is the angle between their normals at the point and N_1, N_2 are the normals at $(2,-1,2)$ to the given surfaces

$$\cos\theta = \frac{N_1 \cdot N_2}{|N_1||N_2|} = \frac{(4\bar{i}-2\bar{j}+4\bar{k})(4\bar{i}-2\bar{j}-\bar{k})}{\sqrt{16+4+16}\ \sqrt{16+4+1}} = \frac{16}{6\sqrt{21}}$$

Hence the required angle $\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$

8) Find the values a and b such that the surfaces $ax^2 - byz = (a+2)x$, and $4x^2y + z^3 = 4$ cut orthogonally

Sol:- Let $f_1 = ax^2 - byz - (a+2)x = 0$ — ①

$f_2 = 4x^2y + z^3 - 4 = 0$ — ②

$$\nabla f_1 = (2ax - a - 2)\bar{i} - bz\bar{j} - by\bar{k}$$
$$= (a-2)\bar{i} - 2b\bar{j} + b\bar{k} \text{ at } (1,-1,2)$$

$$\nabla f_2 = 8xy\,\bar{i} + 4x^2\bar{j} + 3z^2\bar{k}$$
$$= -8\bar{i} + 4\bar{j} + 12\bar{k} \text{ at } (1,-1,2)$$

The surfaces ① and ② will cut orthogonally if $\nabla f_1 \perp \nabla f_2$

i.e $-8(a-2) - 8b + 12b = 0$.

$-2a + b + 4 = 0$ — ③.

Also since the point $(1,-1,2)$ lies on (i) and (ii)

$a + 2b - (a+2) = 0$ or $b = 1$.

From ③ $-2a + 5 = 0$ or $a = 5/2$

Hence $a = 5/2$, and $b = 1$

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Exercise - 8.3

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1) (a) find $\nabla\phi$, if $\phi = \log(x^2+y^2+z^2)$

(b) Show that $\text{grad}(1/r) = -R/r^3$.

Sol: (a) $\nabla\phi = \bar{i}\frac{\partial}{\partial x}\log(x^2+y^2+z^2) + \bar{j}\frac{\partial}{\partial y}\log(x^2+y^2+z^2) + \bar{k}\frac{\partial}{\partial z}\log(x^2+y^2+z^2)$

$= \bar{i}\frac{2x}{x^2+y^2+z^2} + \bar{j}\frac{2y}{x^2+y^2+z^2} + \bar{k}\frac{2z}{x^2+y^2+z^2}$

2)

$= \frac{2(x\bar{i}+y\bar{j}+z\bar{k})}{(x^2+y^2+z^2)} = \frac{2\bar{R}}{r^2}$

(b) $\text{grad}(1/r) = \bar{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \bar{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \bar{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$

$= \bar{i}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial x} + \bar{j}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial y} + \bar{k}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial z}$

$= \bar{i}\left(\frac{-1}{r^2}\right)\left(\frac{x}{r}\right) + \bar{j}\left(\frac{-1}{r^2}\right)\left(\frac{y}{r}\right) + \bar{k}\left(\frac{-1}{r^2}\right)\left(\frac{z}{r}\right)$

$= \frac{-1}{r^3}(x\bar{i}+y\bar{j}+z\bar{k})$

$= \frac{-\bar{r}}{r^3}$

2) Find a unit vector normal to the surface $x^3+y^3+3xyz = 3$ at the point $(1, 2, -1)$

Sol: Let $\phi = x^3+y^3+3xyz-3$

A vector normal to the above surface

$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

$= (3x^2+3yz)\bar{i} + (3y^2+3xz)\bar{j} + 3xy\bar{k}$

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At the point $(1,-2,-1)$

$$\nabla\phi = [3+3(2)(-1)]\bar{i} + [3(0)+3(1)(-1)]\bar{j} + 3(1)(2)\bar{k}$$

$$= -3\bar{i} + 9\bar{j} + 6\bar{k}$$

$$= -3(\bar{i} - 3\bar{j} - 2\bar{k})$$

unit vector normal to the surface $= \frac{\nabla\phi}{|\nabla\phi|}$

$$= \frac{-3(\bar{i} - 3\bar{j} - 2\bar{k})}{3\sqrt{1+9+4}}$$

$$= \frac{-(\bar{i} - 3\bar{j} - 2\bar{k})}{\sqrt{14}}$$

(3) Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point $(1,-2,1)$ in the direction of the ve $2\bar{i} - \bar{j} - 2\bar{k}$

Sol: Here $\nabla\phi = (2xyz + 4z^2)\bar{i} + x^2z\bar{j} + (x^2y + 8xz)\bar{k}$

$= 8\bar{i} - \bar{j} - 10\bar{k}$ at the point $(1,-2,-1)$

∴ Directional derivative of ϕ in the direction $2\bar{i} - \bar{j} - 2\bar{k}$

$$= \frac{(8\bar{i} - \bar{j} - 10\bar{k}) \cdot (2\bar{i} - \bar{j} - 2\bar{k})}{\sqrt{4+1+4}}$$

$$= \frac{16+1+20}{3} = \frac{37}{3}$$

(4) What is the directional derivative of $\phi = xy^2 + yz^3$ at the point $(2,-1,1)$ in the direction of the normal to the surface $xy^2z = 4$ at $(-1,-2,1)$

Sol: A vector normal to the surface x

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$$\nabla(x\log z) - y^2 = \log z\,i - 2y\,j + \frac{x}{z}k$$

$$= -4j - k$$

Here $\nabla f = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(xy^2 + yz^3)$

$$= y^2 i + (2xy + z^3)j + 3yz^2 k \quad \text{at } (2,-1,1)$$

$$= i - 3j - 3k$$

Directional derivative of 'f' in the direction of $-4j - k$

$$= (i - 3j - 3k)\cdot\frac{(-4j-k)}{\sqrt{16+1}} = \frac{12+3}{\sqrt{17}} = \frac{15}{\sqrt{17}}$$

Find the values of constants a, b, c so that the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at $(1,2,-1)$ has a maximum magnitude 64 in a direction parallel to the z-axis.

$$\nabla\phi = \nabla(axy^2 + byz + cz^2x^3)$$

$$= (ay^2 + 3cz^2x^2)i + (2axy + bz)j + (by + 2czx^3)k$$

$$\nabla\phi_{(1,2,-1)} = (4a + 3c)i + (4a - b)j + (2b - 2c)k$$

Given that $\nabla\phi$ is parallel to z-axis and

$$\nabla\phi \cdot k = 64$$

$\therefore \nabla\phi \cdot i = 0$; $\nabla\phi \cdot j = 0$ and $\nabla\phi \cdot k = 64$

$4a + 3c = 0$; $4a - b = 0$; $2b - 2c = 64$

$\rightarrow (1)$ $\rightarrow (2)$ $\rightarrow (3)$

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Solving (1), (2), (3) for a, b, c

$$a = 6,\ b = 24,\ c = -8$$

(8) If the directional derivative of $\phi = ax^2y + by^2z + cz^2x$ at the point $(1,1,1)$ has maximum magnitude 15 in the direction parallel to the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$. Find the values of a, b and c.

Sol: Given $\nabla\phi = ax^2y + by^2z + cz^2x$

$$\nabla\phi = (2axy + cz^2)\bar{i} + (ax^2 + 2byz)\bar{j} + (by^2 + 2czx)\bar{k}$$

At the point $(1,1,1)$

$$\nabla\phi = (2a + c)\bar{i} + (a + 2b)\bar{j} + (b + 2c)\bar{k}$$

Now the maximum value of the directional derivative is in the direction of ∇f

i.e, $|\nabla\phi| = 15$

$$\Rightarrow (2a+c)^2 + (a+2b)^2 + (b+2c)^2 = (15)^2$$

But the directional derivative is given to be maximum parallel to the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$

$$\Rightarrow \frac{2a+c}{2} = \frac{a+2b}{-2} = \frac{2c+b}{1} \Rightarrow 2a + c = -2b - a$$

$$3a + 2b + c = 0 \rightarrow$$

and $2b + a = -2(2c + b)$

$$a + 4b + 4c = 0 \rightarrow (2)$$

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$\frac{a}{4} = \frac{b}{-11} = \frac{c}{10} = k$ (say)

$a = 4k$, $b = -11k$, $c = 10k$

Now, $(2a+c)^2 + (2b+a)^2 + (2c+b)^2 = (15)^2$

$(8k+10k)^2 + (-22k+4k)^2 + (20k-11k)^2 = (15)^2$

$(18k)^2 + (-18k)^2 + (8k)^2 = (15)^2$

$k = \pm 5/9$

$a = \pm 20/9$; $b = \pm \frac{55}{9}$; $c = \pm \frac{50}{9}$

4) In what direction from $(3,1,-2)$ is the directional derivative of $\phi = x^2y^2z^4$ maximum? Find also the magnitude of this maximum.

Or

$$\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(x^2y^2z^4)$$

$$= 2xy^2z^4\bar{i} + 2x^2yz^4\bar{j} + 4x^2y^2z^3\bar{k}$$

Directional derivative at $(3,1,-2)$

$$= 96\bar{i} + 288\bar{j} - 288\bar{k}$$

The directional derivative is maximum in the direction $96\bar{i} + 288\bar{j} - 288\bar{k}$

$$= 96(\bar{i} + 3\bar{j} - 3\bar{k})$$

In any other direction it will have its component which is less than its maximum value

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Maximum value $= 48\sqrt{4+9}$

$= 96\sqrt{19}$

(8) What is the greatest rate of increase of $u = x^2 + yz$ at the point $(1, -1, 3)$

Sol: Greatest rate $= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(x^2 + yz)$

$= 2x\bar{i} + z\bar{j} + y\bar{k}$

At $(1, -1, 3)$, Greatest rate $= 2\bar{i} + 3\bar{j} - \bar{k}$

(9) Calculate

(9) The temperature of points in space is given by $T(x, y, z) = x^2 + y^2 - z$. A mosquito located at (desires to fly in such a direction that it get warm as soon as possible. In what direction should it move?

Sol: Let $\phi = (x^2 + y^2 - z)$

$\nabla\phi = 2x\bar{i} + 2y\bar{j} - \bar{k}$

at $(1, 1, 2) = 2\bar{i} + 2\bar{j} - \bar{k}$

$\frac{\nabla\phi}{|\nabla\phi|} = \frac{2\bar{i} + 2\bar{j} - \bar{k}}{\sqrt{4 + 4 + 1}}$

$= \frac{2\bar{i} + 2\bar{j} - \bar{k}}{3}$

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Calculate the angle b/w the normals to the surface $xy = z^2$ at the points $(4, 1, 2)$ and $(3, 3, -3)$

A vector normal to the surface is

$$\nabla(xy - z^2) = \bar{i}\frac{\partial}{\partial x}(xy - z^2) + \bar{j}\frac{\partial}{\partial y}(xy - z^2) + \bar{k}\frac{\partial}{\partial z}(xy - z^2)$$

$$= \bar{i}y + \bar{j}x - 2z\bar{k}$$

At $(4, 1, 2)$ and $(3, 3, -3)$

$$\bar{N}_1 = \bar{i} + 4\bar{j} - 4\bar{k} \quad ; \quad \bar{N}_2 = 3\bar{i} + 3\bar{j} - 6\bar{k}$$

$$\cos\theta = \frac{\bar{N}_1 \cdot \bar{N}_2}{|\bar{N}_1||\bar{N}_2|} = \frac{3 + 12 - 24}{\sqrt{1+16+16}\sqrt{9+9+36}}$$

$$= \frac{-9}{\sqrt{33}\sqrt{54}} = \frac{-9}{\sqrt{11 \times 3} \cdot \sqrt{2 \times 27}}$$

$$= \frac{-9}{\sqrt{81}\sqrt{22}} = \frac{-1}{\sqrt{22}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{22}}\right)$$

Find the values of λ and μ so that the surface $\lambda x^2 y + \mu z^3 = 4$ may cut the surface $5x^2 = yz + 9x$ orthogonally at $(1, -1, 2)$

Let $f_1 = \lambda x^2 y + \mu z^3 - 4 = 0 \quad \rightarrow (1)$

$$\nabla f_1 = 2\lambda xy\bar{i} + \lambda x^2\bar{j} + 3\mu z^2\bar{k}$$

and $f_2 = 5x^2 - yz - 9x = 0 \quad \rightarrow (2)$

$$\nabla f_2 = (10x - 9)\bar{i} - z\bar{j} - y\bar{k} = 0$$

At $(1, -1, 2) \Rightarrow \nabla f_1 = -2\lambda\bar{i} + \lambda\bar{j} + 12\mu\bar{k}$

At $(1, -1, 2) \Rightarrow \nabla f_2 = \bar{i} - 4\bar{j} + 2\bar{k}$

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The surfaces (1) & (2) will cut orthogonally if

$$\nabla f_1 \cdot \nabla f_2 = 0$$

i.e., $(-2\lambda)(1) + \lambda(-4) + 12(1)\mu = 0$

$\Rightarrow -2\lambda - 4\lambda + 24\mu = 0$

$\Rightarrow -6\lambda + 24\mu = 0 \quad \rightarrow (3)$

Also since the point $(1,-1,2)$ lies on (1) & (2)

(1) $\Rightarrow -\lambda + 8\mu - 4 = 0$

(2) $\Rightarrow 5 + 4 - 9 = 0$

From (3), $-6\lambda + 24\mu(1) = 0$

$-6\lambda = -24$

$\lambda = 4$

$\therefore \lambda = 4$ and $\mu = 1$.

$-6\lambda + 24\mu = 0$

$-6\lambda + 48\mu - 24 = 0$

$-24\mu + 24 = 0$

$\mu = 1$

(12) Find the angle b/w the tangent planes to the surface. $x\log z = y^2 - 1$, $x^2 y = 2 - z$ at the point $(1,$

Sol: The angle b/w the tangent planes to the surfaces ϕ_1 and ϕ_2 is $\cos^{-1}\left[\frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}\right]$

$\phi_1 = x\log z - y^2 + 1 \Rightarrow \nabla\phi_1 = \log z\,\bar{i} - 2y\,\bar{j} + \frac{x}{z}\,\bar{k}$

$\phi_2 = x^2 y + z - 2 \Rightarrow \nabla\phi_2 = 2xy\,\bar{i} + x^2\,\bar{j} + \bar{k}$

$\nabla\phi_1(1,1,1) = -2\bar{j} + \bar{k}$; $\nabla\phi_2(1,1,1) = 2\bar{i} + \bar{j} + \bar{k}$

$$\theta = \cos^{-1}\left[\frac{(-2\bar{j} + \bar{k})(2\bar{i} + \bar{j} + \bar{k})}{\sqrt{5}\,\sqrt{6}}\right] = \cos^{-1}\left(\frac{-1}{\sqrt{30}}\right)$$

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Divergence of a vector point function

The divergence of a differentiable vector point function $\bar{v}$ is denoted by div $\bar{v}$ and is defined as

$$\text{div } \bar{v} = \nabla \cdot \bar{v} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot \bar{v} = \bar{i} \cdot \frac{\partial \bar{v}}{\partial x} + \bar{j} \cdot \frac{\partial v}{\partial y} + \bar{k} \cdot \frac{\partial v}{\partial z}.$$

Obviously, the divergence of a vector point function is a scalar point function.

If $\bar{v} = v_1\bar{i} + v_2\bar{j} + v_3\bar{k}$ then $\text{div } \bar{v} = \nabla \cdot \bar{v} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot (v_1\bar{i} + v_2\bar{j} + v_3\bar{k})$

Since $\bar{i} \cdot \bar{i} = \bar{j} \cdot \bar{j} = \bar{k} \cdot \bar{k} = 1$

and $\bar{i} \cdot \bar{j} = \bar{j} \cdot \bar{k} = \bar{k} \cdot \bar{i} = 0$.

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}.$$

Curl of a vector point function :-

The curl (or rotation) of a differentiable vector point function $\bar{v}$ is denoted by curl $\bar{v}$ and is defined as

$$\text{curl } \bar{v} = \nabla \times \bar{v} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \times \bar{v} = \bar{i} \times \frac{\partial \bar{v}}{\partial x} + \bar{j} \times \frac{\partial \bar{v}}{\partial y} + \bar{k} \times \frac{\partial \bar{v}}{\partial z}.$$

Obviously, the curl of a vector point function is a vector point function.

If $\bar{v} = v_1\bar{i} + v_2\bar{j} + v_3\bar{k}$, then $\text{curl } \bar{v} = \nabla \times \bar{v} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \times (v_1\bar{i} + v_2\bar{j} + v_3\bar{k})$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 & v_2 & v_3 \end{vmatrix} = \bar{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \bar{j}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \bar{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right).$$

Ex:-① If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, show that (i) $\text{div } \bar{r} = 3$ (ii) $\text{curl } \bar{r} = \bar{0}$.

(i) $\text{div } \bar{r} = \nabla \cdot \bar{r} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot (x\bar{i} + y\bar{j} + z\bar{k}) = 1 + 1 + 1 = 3.$

(ii) $\text{curl } \bar{r} = \nabla \times \bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix}$

$$= \bar{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \bar{j}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + \bar{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right).$$

$$= \bar{i}(0) + \bar{j}(0) + \bar{k}(0) = \bar{0}.$$

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Ex 2:- Find the divergence and curl of the vector

$\bar{v} = (xyz)\,\bar{i} + (3x^2y)\,\bar{j} + (xz^2 - y^2z)\,\bar{k}$ at the point $(2,-1,1)$

Sol:- $\operatorname{div} \bar{v} = \frac{d}{dx}(xyz) + \frac{d}{dy}(3x^2y) + \frac{d}{dz}(xz^2 - y^2z)$.

$= yz + 3x^2 + 2xz - y^2$

$= -1 + 12 + 4 - 1 = 14$ at $(2,-1,1)$.

$$\operatorname{curl} \bar{v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix} = \bar{i}(-2yz) + \bar{j}(xy - z^2) + \bar{k}(6xy - xz)$$

$= 2\bar{i} - 8\bar{j} - 14\bar{k}$ at $(2,-1,1)$.

Ex 3:- Find $\operatorname{div} \bar{F}$ and $\operatorname{curl} \bar{F}$ where $\bar{F} = \operatorname{grad}(x^3 + y^3 + z^3 - 3xyz)$

Sol:- Let $\phi = x^3 + y^3 + z^3 - 3xyz$

$\bar{F} = \operatorname{grad} \phi = \left(\bar{i}\frac{d}{dx} + \bar{j}\frac{d}{dy} + \bar{k}\frac{d}{dz}\right)(x^3 + y^3 + z^3 - 3xyz)$.

$= (3x^2 - 3yz)\,\bar{i} + (3y^2 - 3xz)\,\bar{j} + (3z^2 - 3xy)\,\bar{k}$

$\therefore \operatorname{div} \bar{F} = \frac{d}{dx}(3x^2 - 3yz) + \frac{d}{dy}(3y^2 - 3xz) + \frac{d}{dz}(3z^2 - 3xy)$

$= 6x + 6y + 6z = 6(x + y + z)$

and $\operatorname{curl} \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$

$= \bar{i}(-3x + 3x) + \bar{j}(-3y + 3y) + \bar{k}(-3z + 3z)$

$= \bar{0}$

Note:- (1) If $\operatorname{div} \bar{v} = 0$, then $\bar{v}$ is called a solenoidal vector point function

(2) If $\operatorname{curl} \bar{v} = \bar{0}$, " " is said to be an irrotational vector

Otherwise rotational.

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(1) If $\bar{f} = x^2y\,\bar{i} - 2xz\,\bar{j} + 2yz\,\bar{k}$, find (i) div $\bar{f}$ (ii) curl $\bar{f}$ (iii) curl curl $\bar{f}$

Soln: $\bar{f} = x^2y\,\bar{i} - 2xz\,\bar{j} + 2yz\,\bar{k}$

(i) $\nabla\cdot\bar{f} = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(-2xz) + \frac{\partial}{\partial z}(2yz) = 2xy + 0 + 2y = 2y(x+1)$

(ii) $\nabla\times\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & -2xz & 2yz \end{vmatrix} = \bar{i}(2z+2x) + \bar{j}(0) + \bar{k}(-2z - x^2)$

$= (2x+2z)\,\bar{i} - (x^2+2z)\,\bar{k}$

(iii) curl curl $\bar{f} = \nabla\times(\nabla\times\bar{f}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x+2z & 0 & -(x^2+2z) \end{vmatrix} = \bar{i}(0) + \bar{j}(2+2z) + \bar{k}(0)$

$= (2x+2)\,\bar{j}$.

(2) Show that the vector field given by $\bar{f} = (x+3y)\,\bar{i} + (y-3z)\,\bar{j} + (x-2z)\,\bar{k}$ is solenoidal.

Soln: $\bar{f} = (x+3y)\,\bar{i} + (y-3z)\,\bar{j} + (x-2z)\,\bar{k}$.

$\nabla\cdot\bar{f} = \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-3z) + \frac{\partial}{\partial z}(x-2z) = 1+1-2 = 0$.

$\therefore \bar{f}$ is solenoidal.

(3) Find the constant 'a' so that $\bar{f} = y(ax^2+z)\,\bar{i} + x(y^2-z^2)\,\bar{j} + 2xy(z-xy)\,\bar{k}$ is solenoidal.

Soln: If $\bar{f}$ is solenoidal, $\nabla\cdot\bar{f} = 0$

$\nabla\cdot\bar{f} = 0 \Rightarrow \frac{\partial}{\partial x}[y(ax^2+z)] + \frac{\partial}{\partial y}[x(y^2-z^2)] + \frac{\partial}{\partial z}[2xy(z-xy)]$

$\Rightarrow 2axy + 2xy + 2xy = 0$.

$\Rightarrow 2xy(a+2) = 0 \Rightarrow a = -2$.

(4) Show that $\bar{v} = (\sin y + z)\,\bar{i} + (x\cos y - z)\,\bar{j} + (x-y)\,\bar{k}$ is irrotational.

Soln: A vector $\bar{v}$ is irrotational if $\nabla\times\bar{v} = \bar{0}$.

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$$\begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+3y+az & bx+2y-z & 2x+cy+3z \end{vmatrix}$$

is irrotational.

3) Find the constants a, b, c if the vector

$\bar{f} = (2x+3y+az)\bar{i} + (bx+2y+3z)\bar{j} + (2x+cy+3z)\bar{k}$ is irrotational.

Sol:- If $\bar{f}$ is irrotational $\Rightarrow \nabla \times \bar{f} = \bar{0}$

$$\nabla \times \bar{f} = \bar{0} \Rightarrow \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+3y+az & bx+2y+z & 2x+cy+3z \end{vmatrix} = \bar{0}$$

$$\Rightarrow (c+3)\bar{i} + \bar{j}(a-2) + \bar{k}(b-3) = \bar{0}$$

$a-2=0$ i.e. $a=2$

$b-3=0$ i.e. $b=3$

$c-3=0$ i.e. $c=3$

4) If $\bar{f} = (x+y+z)\bar{i} + \bar{j} - (x+y)\bar{k}$ prove that $\bar{f} \cdot \text{curl}\, \bar{f} = 0$.

$$\text{curl}\, \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y+z & 1 & -(x+y) \end{vmatrix} = \bar{i}(-1) + \bar{j}(0+1) + \bar{k}(-0) = -\bar{i} + \bar{j} - \bar{k}$$

$$\bar{f} \cdot \text{curl}\, \bar{f} = (x+y+z)(-1) + 1 + x+y$$

$$= -x-y-z+x+y = \bar{0}$$

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Prove that $\text{div}(r^n \bar{r}) = (n+3) r^n$. (18)

Where $r = |\bar{r}|$

Hence show that $\bar{r}/r^3$ is solenoidal.

Sol:- Given that $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$. and $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{r} \quad \text{by} \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

$$\frac{\partial}{\partial x}(xr^n) = 1 \cdot r^n + x \cdot n r^{n-1} \frac{\partial r}{\partial x} = r^n + x \cdot n \cdot r^{n-1} \frac{x}{r} = r^n + nx^2 r^{n-2}.$$

By $\frac{\partial}{\partial y}(yr^n) = r^n + ny^2 r^{n-2}$ and $\frac{\partial}{\partial z}(zr^n) = r^n + nz^2 r^{n-2}$.

$$r^n \bar{r} = r^n \cdot (x\hat{i} + y\hat{j} + z\hat{k}) = xr^n\hat{i} + yr^n\hat{j} + zr^n\hat{k}.$$

$$\text{div}(r^n \bar{r}) = \frac{\partial}{\partial x}(xr^n) + \frac{\partial}{\partial y}(yr^n) + \frac{\partial}{\partial z}(zr^n)$$

$$= r^n + nx^2 r^{n-2} + r^n + ny^2 r^{n-2} + r^n + nz^2 r^{n-2}.$$

$$= 3r^n + nr^{n-2}(x^2 + y^2 + z^2)$$

$$= 3r^n + nr^{n-2} \cdot r^2 = 3r^n + nr^n = (3+n) r^n.$$

when $n = -3$, $\text{div}\left(\frac{\bar{r}}{r^3}\right) = 0$ i.e. $\frac{\bar{r}}{r^3}$ is solenoidal.

1) $\text{div grad} f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$.

Sol:- $\nabla^2 f = \nabla \cdot \nabla f = \nabla \cdot \left[\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right]$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f.$$

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the laplacian operator and $\nabla^2 f = 0$ is called the laplace's equation.

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2) If $\bar{r} = xi + yj + zk$, then prove that $\nabla^2(r^n) = n(n+1)r^{n-2}$ where $r = |\bar{r}|$

Sol:- Given that $\bar{r} = xi + yj + zk$ and $r = \sqrt{x^2+y^2+z^2}$

$$r^n = (x^2+y^2+z^2)^{n/2}.$$

To find $\nabla^2(r^n) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) r^n = \Sigma \frac{\partial^2}{\partial x^2} r^n$ — (1)

Take $\frac{\partial}{\partial x}(r^n) = n \cdot r^{n-1} \frac{\partial r}{\partial x} = n r^{n-1} \frac{x}{r} = n x \cdot r^{n-2}$

$$\frac{\partial^2}{\partial x^2}(r^n) = n \cdot r^{n-2} + x \cdot n \cdot (n-2) r^{n-3} \frac{\partial r}{\partial x}$$

$$= n\left[r^{n-2} + (n-2) r^{n-3} \frac{x^2}{r}\right] = n r^{n-2} + n(n-2) r^{n-4} x^2$$

Equation (1) we get

$$\nabla^2 r^n = \Sigma \frac{\partial^2}{\partial x^2} r^n$$

$$= \Sigma n \cdot r^{n-2} + n(n-2) r^{n-4} x^2$$

$$= 3n r^{n-2} + n(n-2) r^{n-4} (x^2+y^2+z^2)$$

$$= 3n r^{n-2} + (n^2-2n) r^{n-4} (r^2)$$

$$= 3n r^{n-2} + (n^2-2n) r^{n-2}$$

$$= r^{n-2}[3n + n^2 - 2n]$$

$$= r^{n-2}[n + n^2]$$

$$= n(n+1) r^{n-2}$$

$$\nabla^2 r^n = n(n+1) r^{n-2}$$

Treating ∇ as a vector, the ... represents application of ∇ ... easily ... three times

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Show that $r^\alpha \bar{R}$ is any irrotational vector for any value of α but is solenoidal if $\alpha+3=0$ where $\bar{R} = x\bar{i}+y\bar{j}+z\bar{k}$ and 'r' is the magnitude of $\bar{R}$.

$r = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{1/2} \Rightarrow r^\alpha = (x^2+y^2+z^2)^{\alpha/2}$

Sol:- Let $A = r^\alpha \bar{R} = (x^2+y^2+z^2)^{\alpha/2}(x\bar{i}+y\bar{j}+z\bar{k}) = \Sigma x(x^2+y^2+z^2)^{\alpha/2}\bar{i}$

$$\text{curl } \bar{A} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x(x^2+y^2+z^2)^{\alpha/2} & y(x^2+y^2+z^2)^{\alpha/2} & z(x^2+y^2+z^2)^{\alpha/2} \end{vmatrix}$$

$$= \Sigma \bar{i}\left[\frac{\partial}{\partial y}\left[z(x^2+y^2+z^2)^{\alpha/2}\right] - \frac{\partial}{\partial z}\left[y(x^2+y^2+z^2)^{\alpha/2}\right]\right.$$

$$= \Sigma \bar{i}\left[\frac{\alpha z}{2}(x^2+y^2+z^2)^{\frac{\alpha}{2}-1} 2y - \frac{\alpha y}{2}(x^2+y^2+z^2)^{\frac{\alpha}{2}-1}(2z)\right] = \bar{0}$$

Hence $\bar{A}$ is irrotational for any value of α.

But $\text{div } \bar{A} = \nabla\cdot(r^\alpha \bar{R}) = (\alpha+3)r^\alpha$ which is zero for $\alpha+3=0$

i.e A is solenoidal if $\alpha+3=0$.

---

2). curl grad f = $\nabla \times \nabla f = \bar{0}$

$$\nabla \times \nabla f = \nabla \times \left(\frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j} + \frac{\partial f}{\partial z}\bar{k}\right)$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial y \partial z}\right) + \bar{j}\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right) + \bar{k}\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)$$

$= \bar{0}$.

The above result can also be written as

grad(div $\bar{i}$) = curl(curl $\bar{i}$) + $\nabla^2 \bar{i}$

i) $\nabla\cdot\nabla\phi = \nabla^2\phi$

ii) $\nabla\times\nabla\phi = \bar{0}$

iii) $\nabla\times(\nabla\times\bar{i}) = \nabla(\nabla\cdot\bar{i}) - \nabla^2\bar{i}$

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(3) div curl $\bar{F} = 0 \Rightarrow \nabla \cdot (\nabla \times \bar{F}) = 0$

Let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

$$\text{curl } \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} = \bar{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \bar{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \bar{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$\text{div}(\text{curl } \bar{F}) = \nabla \cdot (\nabla \times \bar{F})$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= \frac{\partial^2 F_3}{\partial x\,\partial y} - \frac{\partial^2 F_2}{\partial x\,\partial z} + \frac{\partial^2 F_1}{\partial y\,\partial z} - \frac{\partial^2 F_3}{\partial y\,\partial x} + \frac{\partial^2 F_2}{\partial z\,\partial x} - \frac{\partial^2 F_1}{\partial z\,\partial y}$$

$$= 0$$

(4) curl curl $\bar{F}$ = grad div $\bar{F} - \nabla^2 \bar{F}$

i.e. $\nabla \times (\nabla \times \bar{F}) = \nabla(\nabla \cdot \bar{F}) - \nabla^2 \bar{F}$

Let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

then curl $\bar{F} = \nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$

$$= \bar{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \bar{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \bar{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

curl curl $\bar{F}$ = $\nabla \times (\nabla \times \bar{F})$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} & \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} & \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F_3}{\partial z} - \frac{\partial F_2}{\partial z}\right)\right]$$

$$+ \bar{j}\left[\frac{\partial}{\partial z}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x} - \frac{\partial F_3}{\partial x}\right)\right]$$

$$+ \bar{k}\left[\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x} - \frac{\partial F_2}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right]$$

$$= \bar{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) - \left(\frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}\right)\right]$$

$$+ \bar{j}\left[\frac{\partial}{\partial y}\left(\frac{\partial F_3}{\partial z} + \frac{\partial F_1}{\partial x}\right) - \left(\frac{\partial^2 F_2}{\partial z^2} + \frac{\partial^2 F_2}{\partial x^2}\right)\right]$$

$$+ \bar{k}\left[\frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) - \left(\frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_3}{\partial y^2}\right)\right]$$

$$= \bar{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) - \left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}\right)\right]$$

$$+ \bar{j}\left[\frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) - \left(\frac{\partial^2 F_2}{\partial x^2} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_2}{\partial z^2}\right)\right]$$

$$+ \bar{k}\left[\frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) - \left(\frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_3}{\partial z^2}\right)\right]$$

$$= \nabla(\nabla \cdot \bar{F}) - \nabla^2 \bar{F}$$

$$= \nabla(\nabla \cdot \bar{F}) - \nabla^2 \bar{F} = \text{grad}(\text{div } \bar{F})$$

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Properties of Divergence and Curl

1) For a constant vector $\bar{a}$, $\text{div}\,\bar{a} = 0$, $\text{curl}\,\bar{a} = \bar{0}$

2) $\text{div}(\bar{A}+\bar{B}) = \text{div}\,\bar{A} + \text{div}\,\bar{B}$ (or) $\nabla\cdot(\bar{A}+\bar{B}) = \nabla\cdot\bar{A} + \nabla\cdot\bar{B}$

Proof :- $\text{div}(\bar{A}+\bar{B}) = \nabla\cdot(\bar{A}+\bar{B}) = \left(\bar{i}\frac{d}{dx} + \bar{j}\frac{d}{dy} + \bar{k}\frac{d}{dz}\right)\cdot(\bar{A}+\bar{B})$

$$= \bar{i}\cdot\frac{d}{dx}(\bar{A}+\bar{B}) + \bar{j}\cdot\frac{d}{dy}(\bar{A}+\bar{B}) + \bar{k}\cdot\frac{d}{dz}(\bar{A}+\bar{B})$$

$$= \bar{i}\cdot\left(\frac{d\bar{A}}{dx}+\frac{d\bar{B}}{dx}\right) + \bar{j}\cdot\left(\frac{d\bar{A}}{dy}+\frac{d\bar{B}}{dy}\right) + \bar{k}\cdot\left(\frac{d\bar{A}}{dz}+\frac{d\bar{B}}{dz}\right)$$

$$= \left(\bar{i}\cdot\frac{d\bar{A}}{dx} + \bar{j}\cdot\frac{d\bar{A}}{dy} + \bar{k}\cdot\frac{d\bar{A}}{dz}\right) + \left(\bar{i}\cdot\frac{d\bar{B}}{dx} + \bar{j}\cdot\frac{d\bar{B}}{dy} + \bar{k}\cdot\frac{d\bar{B}}{dz}\right)$$

$$= \nabla\cdot\bar{A} + \nabla\cdot\bar{B} = \text{div}\,\bar{A} + \text{div}\,\bar{B}$$

3) $\text{curl}(\bar{A}+\bar{B}) = \text{curl}\,\bar{A} + \text{curl}\,\bar{B}$ (or) $\nabla\times(\bar{A}+\bar{B}) = \nabla\times\bar{A} + \nabla\times\bar{B}$

$$\text{curl}(\bar{A}+\bar{B}) = \nabla\times(\bar{A}+\bar{B}) = \left(\bar{i}\frac{d}{dx} + \bar{j}\frac{d}{dy} + \bar{k}\frac{d}{dz}\right)\times(\bar{A}+\bar{B})$$

$$= \Sigma\,\bar{i}\times\frac{d}{dx}(\bar{A}+\bar{B}) = \Sigma\,\bar{i}\times\left(\frac{d\bar{A}}{dx}+\frac{d\bar{B}}{dx}\right)$$

$$= \Sigma\,\bar{i}\times\frac{d\bar{A}}{dx} + \Sigma\,\bar{i}\times\frac{d\bar{B}}{dx}$$

$$= (\nabla\times\bar{A}) + (\nabla\times\bar{B}) = \text{curl}\,\bar{A} + \text{curl}\,\bar{B}$$

4) If $\bar{A}$ is a vector function and 'ϕ' is a scalar function

then $\text{div}(\phi\bar{A}) = \phi\,\text{div}\,\bar{A} + (\text{grad}\,\phi)\cdot\bar{A}$ (or) $\nabla\cdot(\phi\bar{A}) = \phi(\nabla\cdot\bar{A}) + (\nabla\phi)\cdot\bar{A}$

Sol:- $\text{div}(\phi\bar{A}) = \nabla\cdot(\phi\bar{A}) = \left(\bar{i}\frac{d}{dx} + \bar{j}\frac{d}{dy} + \bar{k}\frac{d}{dz}\right)\cdot(\phi\bar{A})$

$$= \bar{i}\frac{d}{dx}(\phi\bar{A}) + \bar{j}\frac{d}{dy}(\phi\bar{A}) + \bar{k}\frac{d}{dz}(\phi\bar{A})$$

$$= \Sigma\,\bar{i}\cdot\frac{d}{dx}(\phi\bar{A}) = \Sigma\,\bar{i}\cdot\left[\phi\frac{d\bar{A}}{dx} + \frac{d\phi}{dx}\bar{A}\right] = \phi\,\Sigma\left(\bar{i}\cdot\frac{d\bar{A}}{dx}\right) + \Sigma\left(\bar{i}\frac{d\phi}{dx}\right)\cdot\bar{A}$$

$$= \phi(\nabla\cdot\bar{A}) + \nabla\phi\cdot\bar{A}$$

$$= \phi\,\text{div}\,\bar{A} + (\text{grad}\,\phi)\cdot\bar{A}$$

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If $\bar{A}$ is a vector function, and ϕ is scalar function, then

$$\text{curl}(\phi\bar{A}) = (\text{grad}\phi) \times \bar{A} + \phi \, \text{curl}\, \bar{A}$$

(or) $$\nabla \times (\phi\bar{A}) = (\nabla\phi) \times \bar{A} + \phi(\nabla \times \bar{A})$$

Sol:- $$\text{curl}(\phi\bar{A}) = \nabla \times (\phi\bar{A}) = \left(\bar{i}\frac{d}{dx} + \bar{j}\frac{d}{dy} + \bar{k}\frac{d}{dz}\right) \times (\phi\bar{A})$$

$$= \sum \bar{i} \times \frac{d}{dx}(\phi\bar{A}) = \sum \bar{i} \times \left(\frac{d\phi}{dx}\bar{A} + \phi\frac{d\bar{A}}{dx}\right)$$

$$= \sum \bar{i} \times \frac{d\phi}{dx}\bar{A} + \phi \sum \bar{i} \times \frac{d\bar{A}}{dx}$$

$\bar{a} \times (m\bar{b}) = (m\bar{a}) \times \bar{b}$
$= m(\bar{a} \times \bar{b})$

$$= \sum \frac{d\phi}{dx}\bar{i} \times \bar{A} + \phi \sum \bar{i} \times \frac{d\bar{A}}{dx}$$

$$= \nabla\phi \times \bar{A} + \phi(\nabla \times \bar{A})$$

$$= (\text{grad}\phi) \times \bar{A} + \phi \, \text{curl}\, \bar{A}$$

6) $$\nabla(\bar{A} \cdot \bar{B}) = (\bar{A} \cdot \nabla)\bar{B} + (\bar{B} \cdot \nabla)\bar{A} + \bar{A} \times (\nabla \times \bar{B}) + \bar{B} \times (\nabla \times \bar{A})$$

Sol:- $$\nabla(\bar{A} \cdot \bar{B}) = \sum \bar{i} \frac{d}{dx}(\bar{A} \cdot \bar{B}) = \sum \bar{i}\left(\bar{A} \cdot \frac{d\bar{B}}{dx} + \frac{d\bar{A}}{dx} \cdot \bar{B}\right)$$

$$= \sum \left(\bar{A} \cdot \frac{d\bar{B}}{dx}\right)\bar{i} + \sum \left(\bar{B} \cdot \frac{d\bar{A}}{dx}\right)\bar{i} \quad \text{— (1)}$$

We know that:
$\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c}$
$(\bar{a} \cdot \bar{b})\bar{c} = (\bar{a} \cdot \bar{c})\bar{b} - \bar{a} \times (\bar{b} \times \bar{c})$

$$\left(\bar{A} \cdot \frac{d\bar{B}}{dx}\right)\bar{i} = (\bar{A} \cdot \bar{i})\frac{d\bar{B}}{dx} - \bar{A} \times \left(\frac{d\bar{B}}{dx} \times \bar{i}\right)$$

$$= (\bar{A} \cdot \bar{i})\frac{d\bar{B}}{dx} + \bar{A} \times \left(\bar{i} \times \frac{d\bar{B}}{dx}\right)$$

$$\sum \left(\bar{A} \cdot \frac{d\bar{B}}{dx}\right)\bar{i} = \left(\bar{A} \cdot \sum \bar{i}\frac{d}{dx}\right)\bar{B} + \bar{A} \times \sum \left(\bar{i} \times \frac{d\bar{B}}{dx}\right)$$

$$= (\bar{A} \cdot \nabla)\bar{B} + \bar{A} \times (\nabla \times \bar{B}) \quad \text{— (2)}$$

Interchanging $\bar{A}$ and $\bar{B}$,

$$\sum \left(\bar{B} \cdot \frac{d\bar{A}}{dx}\right)\bar{i} = (\bar{B} \cdot \nabla)\bar{A} + \bar{B} \times (\nabla \times \bar{A}) \quad \text{— (3)}$$

Substituting values from (2) and (3) in (1), we get

$$\nabla(\bar{A} \cdot \bar{B}) = (\bar{A} \cdot \nabla)\bar{B} + (\bar{B} \cdot \nabla)\bar{A} + \bar{A} \times (\nabla \times \bar{B}) + \bar{B} \times (\nabla \times \bar{A})$$

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$\cdot(\bar{A}\times\bar{B}) = \bar{B}\cdot(\nabla\times\bar{A}) - \bar{A}\cdot(\nabla\times\bar{B})$ (21)

$$\nabla\cdot(\bar{A}\times\bar{B}) = \sum\bar{i}\cdot\frac{\partial}{\partial x}(\bar{A}\times\bar{B}) = \sum\bar{i}\cdot\left(\frac{\partial\bar{A}}{\partial x}\times\bar{B}+\bar{A}\times\frac{\partial\bar{B}}{\partial x}\right)$$

$$= \sum\bar{i}\cdot\left(\frac{\partial\bar{A}}{\partial x}\times\bar{B}\right)+\sum\bar{i}\cdot\left(\bar{A}\times\frac{\partial\bar{B}}{\partial x}\right)$$

since $\bar{a}\cdot(\bar{b}\times\bar{c}) = (\bar{a}\times\bar{b})\cdot\bar{c}$

$$= \sum\bar{i}\cdot\left(\frac{\partial\bar{A}}{\partial x}\times\bar{B}\right)-\sum\bar{i}\cdot\left(\frac{\partial\bar{B}}{\partial x}\times\bar{A}\right)$$

$$= \sum\left(\bar{i}\times\frac{\partial\bar{A}}{\partial x}\right)\cdot\bar{B}-\sum\left(\bar{i}\times\frac{\partial\bar{B}}{\partial x}\right)\cdot\bar{A}$$

$$= (\nabla\times\bar{A})\cdot\bar{B}-(\nabla\times\bar{B})\cdot\bar{A}$$

$$= \bar{B}\cdot(\nabla\times\bar{A})-\bar{A}\cdot(\nabla\times\bar{B})$$

ii) $\nabla\times(\bar{A}\times\bar{B}) = (\nabla\cdot\bar{B})\bar{A}-(\nabla\cdot\bar{A})\bar{B}+(\bar{B}\cdot\nabla)\bar{A}-(\bar{A}\cdot\nabla)\bar{B}$

$$\nabla\times(\bar{A}\times\bar{B}) = \sum\bar{i}\times\frac{\partial}{\partial x}(\bar{A}\times\bar{B}) = \sum\bar{i}\times\left(\frac{\partial\bar{A}}{\partial x}\times\bar{B}+\bar{A}\times\frac{\partial\bar{B}}{\partial x}\right)$$

$$= \sum\bar{i}\times\left(\frac{\partial\bar{A}}{\partial x}\times\bar{B}\right)+\sum\bar{i}\times\left(\bar{A}\times\frac{\partial\bar{B}}{\partial x}\right)$$

$$= \sum\left[(\bar{i}\cdot\bar{B})\frac{\partial\bar{A}}{\partial x}-\left(\bar{i}\cdot\frac{\partial\bar{A}}{\partial x}\right)\bar{B}\right]+\sum\left[\left(\bar{i}\cdot\frac{\partial\bar{B}}{\partial x}\right)\bar{A}-(\bar{i}\cdot\bar{A})\frac{\partial\bar{B}}{\partial x}\right]$$

$$= \sum(\bar{B}\cdot\bar{i})\frac{\partial\bar{A}}{\partial x}-\sum\left(\bar{i}\cdot\frac{\partial\bar{A}}{\partial x}\right)\bar{B}+\sum\left(\bar{i}\cdot\frac{\partial\bar{B}}{\partial x}\right)\bar{A}-\sum(\bar{A}\cdot\bar{i})\frac{\partial\bar{B}}{\partial x}$$

$$= \left(\bar{B}\cdot\sum\bar{i}\frac{\partial}{\partial x}\right)\bar{A}-(\nabla\cdot\bar{A})\bar{B}+(\nabla\cdot\bar{B})\bar{A}-\left(\bar{A}\cdot\sum\bar{i}\frac{\partial}{\partial x}\right)\bar{B}$$

$$= (\bar{B}\cdot\nabla)\bar{A}-(\nabla\cdot\bar{A})\bar{B}+(\nabla\cdot\bar{B})\bar{A}-(\bar{A}\cdot\nabla)\bar{B}$$

2.

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Physical interpretation of the divergence of a vector :-

If $\bar{v}$ represents the velocity of the fluid, then $\nabla \cdot \bar{v}$ gives the rate at which the fluid is originating at a point per unit volume. Similarly if $\bar{v}$ represents an electric flux then $\operatorname{div} \bar{v}$ is the amount of flux which diverges per unit volume in unit time. If $\nabla \cdot \bar{F} = 0$ then the field is said to be a solenoidal field. Gravitational or electrostatic fields are examples of solenoidal fields.

Physical interpretation of the curl of a vector :-

The curl of a vector function $\bar{F}$ gives a measure of angular velocity at every point of the vector field.

If $\operatorname{curl} \bar{F} = \bar{0}$ the $\bar{F}$ is said to irrotational field.

Ex:- $\nabla^2(r^n) = n(n+1) r^{n-2}$

$$\nabla^2(r^n) = \frac{d^2}{dx^2}(r^n) + \frac{d^2}{dy^2}(r^n) + \frac{d^2}{dz^2}(r^n) \quad \text{---(1)}$$

$$\frac{d}{dx}(r^n) = n r^{n-1} \frac{dr}{dx} = n r^{n-1} \frac{x}{r} = n r^{n-2} x$$

$$\frac{d^2}{dx^2}(r^n) = n\left[r^{n-2} + (n-2) r^{n-3} \frac{dr}{dx} \cdot x\right] = n\left[r^{n-2} + (n-2) r^{n-3} \frac{x}{r} x\right]$$

$$= n\left[r^{n-2} + (n-2) r^{n-4} x^2\right]$$

$$\frac{d^2}{dy^2}(r^n) = n\left[r^{n-2} + (n-2) r^{n-4} y^2\right] \quad \text{---(3)} \quad \text{and} \quad \frac{d^2}{dz^2}(r^n) = n\left[r^{n-2} + (n-2)\right.$$

Adding (3), (2) and (4), (1) gives

$$\nabla^2(r^n) = n\left[3r^{n-2} + (n-2) r^{n-4}(x^2+y^2+z^2)\right.$$

$$= n\left[3r^{n-2} + (n-2) r^{n-4} r^2\right] = n(n+1) r^{n-2}$$

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(5)

If ϕ and ψ are scalar fields and $\bar{F}$ is a vector field show that $\bar{F} \cdot \text{curl} \bar{F} = 0$

since $\bar{F} = \frac{1}{\phi} \nabla \psi \Rightarrow \text{curl} \bar{F} = \nabla \times \left(\frac{1}{\phi} \nabla \psi\right)$

$\text{curl} \bar{F} = \nabla \frac{1}{\phi} \times \nabla \psi + \frac{1}{\phi} \nabla \times (\nabla \psi)$ $\left[\nabla \times \phi \bar{A} = \nabla \phi \times \bar{A} + \phi \nabla \times \bar{A}\right]$.

$\nabla \times \nabla \psi = 0$

$= \nabla \frac{1}{\phi} \times \nabla \psi$

$\bar{F} \cdot \text{curl} \bar{F} = \frac{1}{\phi} \nabla \psi \left(\nabla \frac{1}{\phi} \times \nabla \psi\right) = 0$, for it scalar triple product in which two factors are equal.

If $\bar{r}$ and $\bar{R}$ have their usual meanings and $\bar{A}$ is a constant vector, prove that $\nabla \times \left(\frac{\bar{A} \times \bar{R}}{r^n}\right) = \frac{2-n}{r^n} \bar{A} + \frac{n(\bar{A} \cdot \bar{R})}{r^{n+2}} \bar{R}$

$\nabla \times \left[\bar{r}^{-n} (\bar{A} \times \bar{R})\right] = r^{-n} \left[\nabla \times (\bar{A} \times \bar{R})\right] + \nabla r^{-n} \times (\bar{A} \times \bar{R})$

$= r^{-n} \left[(\nabla \cdot \bar{R}) \bar{A} - (\nabla \cdot \bar{A}) \bar{R}\right] + \left[-n r^{-(n+1)} \frac{\bar{R}}{r}\right] \times (\bar{A} \times \bar{R})$

$= \bar{r}^{-n} (3\bar{A} - \bar{A}) - n r^{-(n+2)} \bar{R} \times (\bar{A} \times \bar{R})$ $\left[\because \nabla \cdot \bar{R} = 3, \ (\bar{A} \cdot \nabla) \bar{R} = \bar{A}\right]$.

$= 2\bar{A} \bar{r}^{-n} - n r^{-(n+2)} \left[(\bar{R} \cdot \bar{R}) \bar{A} - (\bar{R} \cdot \bar{A}) \bar{R}\right]$

$= \frac{2\bar{A}}{r^n} - \frac{n}{r^{n+2}} \left[r^2 \bar{A} - (\bar{A} \cdot \bar{R}) \bar{R}\right]$.

$= \frac{2-n}{r^n} \bar{A} + \frac{n(\bar{A} \cdot \bar{R})}{r^{n+2}} \bar{R}$.

If r is the distance of a point (x, y, z) from the origin, prove that $\text{curl}\left(\bar{k} \times \text{grad} \frac{1}{r}\right) + \text{grad}\left(\bar{k} \cdot \text{grad} \frac{1}{r}\right) = 0$ where $\bar{k}$ is the unit vector in the direction of z.

Given $\frac{1}{r} = \nabla \frac{1}{r} = \left(\bar{i} \frac{d}{dx} + \bar{j} \frac{d}{dy} + \bar{k} \frac{d}{dz}\right) (x^2 + y^2 + z^2)^{-1/2}$

$= -\frac{1}{2} (x^2 + y^2 + z^2)^{-3/2} (2x\bar{i} + 2y\bar{j} + 2z\bar{k})$

$= -(x^2 + y^2 + z^2)^{-3/2} (x\bar{i} + y\bar{j} + z\bar{k})$.

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$$\text{curl}\left(\bar{F}\times\text{grad}\frac{1}{r}\right) = \nabla\times\left(-\frac{x\bar{i}+y\bar{j}+z\bar{k}}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y/(x^2+y^2+z^2)^{3/2} & -x/(x^2+y^2+z^2)^{3/2} & 0 \end{vmatrix}$$

$$= \bar{i}\frac{\partial}{\partial z}\left[\frac{z}{(x^2+y^2+z^2)^{3/2}}\right] + \bar{j}\frac{\partial}{\partial z}\left[\frac{y}{(x^2+y^2+z^2)^{3/2}}\right] - \bar{k}\left[\frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right)\right]$$

$$= \frac{-3xz\bar{i} - 3yz\bar{j} + (x^2+y^2-2z^2)\bar{k}}{(x^2+y^2+z^2)^{5/2}} \quad \text{---(1)}$$

$$\text{grad}\left(\bar{F}\cdot\text{grad}\frac{1}{r}\right) = \nabla\left[-\bar{F}\cdot\frac{(x\bar{i}+y\bar{j}+z\bar{k})}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\left[\frac{-z}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= \frac{-3xz\bar{i}}{(x^2+y^2+z^2)^{5/2}} + \frac{3yz}{(x^2+y^2+z^2)^{5/2}}\bar{j} + \left(\frac{3z^2-x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}\right)\bar{k}$$

Adding (i) and (ii) we get

$$\text{curl}\left(\bar{F}\times\text{grad}\frac{1}{r}\right) + \text{grad}\left(\bar{F}\cdot\text{grad}\frac{1}{r}\right) = 0$$

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Exercise – 8.11

Divergence: The divergence of a continuously differentiable vector point function $\bar{f}$ is denoted by $\operatorname{div}\bar{f}$ and is defined by the equation

$$\operatorname{div}\bar{f} = \nabla\cdot\bar{f} = \bar{i}\cdot\frac{\partial\bar{f}}{\partial x} + \bar{j}\cdot\frac{\partial\bar{f}}{\partial y} + \bar{k}\cdot\frac{\partial\bar{f}}{\partial z}$$

If $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

then $\operatorname{div}\bar{f} = \nabla\cdot\bar{f} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot(f_1\bar{i} + f_2\bar{j} + f_3\bar{k})$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Curl: The curl of a continuously differentiable vector point function $\bar{f}$ is defined by the equation

$$\operatorname{Curl}\bar{f} = \nabla\times\bar{f} = \bar{i}\times\frac{\partial\bar{f}}{\partial x} + \bar{j}\times\frac{\partial\bar{f}}{\partial y} + \bar{k}\times\frac{\partial\bar{f}}{\partial z}$$

If $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$, then

$$\nabla\times\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix} = \bar{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \bar{j}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \bar{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

Ex: If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, show that

(i) $\nabla\cdot\bar{r} = 3$, (ii) $\nabla\times\bar{r} = \bar{0}$.

Sol: (i) $\nabla\cdot\bar{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$

(ii) $\nabla\times\bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = \bar{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \bar{j}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + \bar{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$

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$$= \frac{\left(\sqrt{x^2+y^2+z^2}\cdot\frac{2x}{2\sqrt{x^2+y^2+z^2}}\right)+\sqrt{x^2+y^2+z^2}\cdot\frac{2x}{2\sqrt{x^2+y^2+z^2}}}{(x^2+y^2+z^2)}$$

$$+\frac{\sqrt{x^2+y^2+z^2}\cdot\frac{2z}{2\sqrt{x^2+y^2+z^2}}}{(x^2+y^2+z^2)}$$

$$= \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}} = \frac{2}{\sqrt{x^2+y^2+z^2}}$$

(11) Find the value of 'a' if the vector $(ax^2y+yz)\bar{i}+(xy^2-xz^2)\bar{j}+(2xyz-2x^2y^2)\bar{k}$ has a zero divergence. Find the curl of the above vector which has zero divergence.

Sol: If $\bar{f}$ is solenoidal $\Rightarrow \nabla\cdot\bar{f}=0$

$$\nabla\cdot\bar{f}=0 \Rightarrow \frac{\partial}{\partial x}(ax^2y+yz)+\frac{\partial}{\partial y}(xy^2-xz^2)+\frac{\partial}{\partial z}(2xyz-2x^2y^2)$$

$$\Rightarrow 2axy+2xy+2xy=0$$

$$\Rightarrow 2xy(a+2)=0 \Rightarrow a=-2$$

$$\nabla\times\bar{f}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ ax^2y+yz & xy^2-2z^2 & 2xyz-2x^2y^2\end{vmatrix}$$

$$=\bar{i}(2xz-4x^2y+2xz)+\bar{j}(y-2yz+4xy^2)+\bar{k}(y^2-z^2-ax^2-z)$$

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$= \bar{i}(4xz - 4x^2y) + \bar{j}(y - 2y^2z + 4xy^2) + \bar{k}(y^2 - x^2 - z)$ 25

show that each of the following vectors are solenoidal

(i) Let $\bar{F} = (-x^2 + yz)\bar{i} + (4y - z^2x)\bar{j} + (2xz - 4z)\bar{k}$

for solenoidal vector $\Rightarrow$ div $\bar{F} = 0$

$$\text{div } \bar{F} = \frac{\partial}{\partial x}(-x^2 + yz) + \frac{\partial}{\partial y}(4y - z^2x) + \frac{\partial}{\partial z}(2xz - 4z)$$

$$= -2x + 4 + 2x - 4$$

$$= 0$$

Hence the vector is solenoidal.

(ii) $\bar{F} = 3y^4z^2\bar{i} + 4x^3z^2\bar{j} + 3x^2y^2\bar{k}$

$$\text{div } \bar{F} = \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(3x^2y^2)$$

$$= 0 + 0 + 0$$

$$= 0$$

(iii) If ϕ and ψ are scalar functions, then prove that $\nabla\phi \times \nabla\psi$ is solenoidal.

Given that ϕ and ψ are scalar function

$$\nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

$$\nabla\psi = \frac{\partial\psi}{\partial x}\bar{i} + \frac{\partial\psi}{\partial y}\bar{j} + \frac{\partial\psi}{\partial z}\bar{k}$$

$$\nabla\phi \times \nabla\psi = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \\ \frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial y} & \partial\psi/\partial z \end{vmatrix}$$

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$$= \bar{i}\left(\frac{\partial\phi}{\partial y}\cdot\frac{\partial\psi}{\partial z} - \frac{\partial\phi}{\partial z}\cdot\frac{\partial\psi}{\partial y}\right) + \bar{j}\left(\frac{\partial\phi}{\partial z}\cdot\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial x}\cdot\frac{\partial\psi}{\partial z}\right) + \bar{k}\left(\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial x}\right)$$

A vector field $\bar{f}$ is solenoidal if $\operatorname{div}\bar{f} = 0$ if $\bar{p} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ then

$$\operatorname{div}\bar{p} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

To prove that $\nabla\cdot(\nabla\phi \times \nabla\psi) = 0$

so $\nabla\cdot(\nabla\phi\times\nabla\psi) = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\cdot\frac{\partial\psi}{\partial z} - \frac{\partial\psi}{\partial y}\cdot\frac{\partial\phi}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\cdot\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial x}\cdot\frac{\partial\psi}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial\phi}{\partial y}\cdot\frac{\partial\psi}{\partial x}\right)$

$$= \frac{\partial\phi}{\partial y}\cdot\frac{\partial^2\psi}{\partial x\partial z} + \frac{\partial\psi}{\partial z}\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial\psi}{\partial y}\cdot\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial\phi}{\partial z}\cdot\frac{\partial^2\psi}{\partial x\partial y} + \frac{\partial\phi}{\partial z}\cdot\frac{\partial^2\psi}{\partial y\partial x} + \frac{\partial\psi}{\partial x}\cdot\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\psi}{\partial y\partial z} - \frac{\partial\psi}{\partial z}\cdot\frac{\partial^2\phi}{\partial y\partial x}$$

$$+ \frac{\partial\phi}{\partial x}\frac{\partial^2\psi}{\partial z\partial y} + \frac{\partial\psi}{\partial y}\cdot\frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial\phi}{\partial y}\cdot\frac{\partial^2\psi}{\partial z\partial x} - \frac{\partial\psi}{\partial x}\cdot\frac{\partial^2\phi}{\partial z\partial y}$$

$$= 0 \qquad \left(\because \frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}\right)$$

$\nabla\cdot(\nabla\phi\times\nabla\psi) = 0$

$\nabla\phi \times \nabla\psi \neq 0$

are solenoidal.

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If $\bar{A}$ and $\bar{B}$ are irrotational, prove that $\bar{A} \times \bar{B}$ is solenoidal

Sol: Given $\bar{A}$ and $\bar{B}$ are irrotational

$$\nabla \times \bar{A} = \bar{0}, \quad \nabla \times \bar{B} = \bar{0}$$

$$\nabla \cdot (\bar{A} \times \bar{B}) = \bar{B} \cdot (\nabla \times \bar{A}) - \bar{A} \cdot (\nabla \times \bar{B})$$

$$= \bar{B} \cdot \bar{0} - \bar{A} \cdot \bar{0}$$

$$= 0$$

$\nabla \cdot (\bar{A} \times \bar{B}) = 0 \Rightarrow \bar{A} \times \bar{B}$ is solenoidal.

(7) If $v = x^2 + y^2 + z^2$ and $\bar{V} = x\bar{i} + y\bar{j} + z\bar{k}$, show that div $(v\bar{V}) = 5v$.

$$v\bar{V} = (x^2 + y^2 + z^2)(x\bar{i} + y\bar{j} + z\bar{k})$$

$$= x(x^2 + y^2 + z^2)\bar{i} + y(x^2 + y^2 + z^2)\bar{j} + z(x^2 + y^2 + z^2)\bar{k}$$

$$\text{div}(v\bar{V}) = \frac{\partial}{\partial x}(x^3 + y^2 + z^2) + \frac{\partial}{\partial y}(x^2 + y^3 + z^2) + \frac{\partial}{\partial z}(x^2 + y^2 + z^3)$$

$$= 3x^2 + y^2 + z^2 + x^2 + 3y^2 + z^2 + x^2 + y^2 + 3z^2$$

$$= 5(x^2 + y^2 + z^2)$$

$$= 5v.$$

(8) If $\bar{R} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r \neq 0$, show that

(i) $\nabla\left(\frac{1}{r}\right) = -\bar{R}/r^4$

$$\nabla\left(\frac{1}{r}\right) = \bar{i}\frac{\partial}{\partial x}\left(\frac{1}{r^2}\right)$$

$$= \bar{i}\,\frac{r^2 \cdot \frac{\partial}{\partial x}(1) - 2r \cdot \frac{\partial r}{\partial x} \cdot 1}{(r^2)^2}$$

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$$= -\frac{\Sigma \bar{i} \cdot 2x \frac{x}{x}}{x^4} = -\frac{2\bar{R}}{x^4}$$

(ii) $\nabla \cdot \left(\frac{\bar{R}}{x^2}\right) = \frac{1}{x^2}$

$$\Sigma \bar{i} \cdot \frac{\partial}{\partial x}\left(\frac{\bar{R}}{x^2}\right) = \Sigma \bar{i} \cdot \frac{\left[x^2 \frac{\partial \bar{R}}{\partial x} - \bar{R} \cdot 2x \frac{\partial x}{\partial x}\right]}{(x^2)^2}$$

$$= \Sigma \bar{i} \cdot \frac{\left[x^2 \cdot \bar{i} - \bar{R} \cdot 2x \cdot \frac{x}{x}\right]}{x^4}$$

$$= \frac{3x^2 - 2\bar{R} \cdot \bar{R}}{x^4} = \frac{3x^2 - 2x^2}{x^4}$$

$$\Rightarrow \frac{x^2}{x^4} = 1/x^2$$

(iii) $div\,(x^n \bar{R}) = (n+3)x^n$; $curl\, x^n \bar{R} = \bar{0}$

$x^n = (x^2+y^2+z^2)^{n/2}$; $\bar{R} = x\bar{i} + y\bar{j} + z\bar{k}$

$$div\,(x^n \bar{r}) = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{n/2} x + \frac{\partial}{\partial y}(x^2+y^2+z^2)^{n/2} y + \frac{\partial}{\partial z}(x^2+y^2+z^2)^{n/2} \cdot z$$

$$= (x^2+y^2+z^2)^{n/2} + \frac{n}{2} \cdot 2x(x^2+y^2+z^2)^{\frac{n}{2}-1} \cdot x + (x^2+y^2+z^2)^{n/2} + \frac{n}{2} \cdot y(x^2+y^2+z^2)^{n/2-1} \cdot y + (x^2+y^2+z^2)^{n/2} + \frac{n}{2} \cdot 2z(x^2+y^2+z^2)^{\frac{n}{2}-1} \cdot z$$

$$= (x^2+y^2+z^2)^{n/2-1}\left[3(x^2+y^2+z^2) + n(x^2+y^2+z^2)\right]$$

$$= (x^2+y^2+z^2)^{n/2}[3+n]$$

$$= (n+3)x^n$$

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$\text{curl}(r^n \bar{R}) = \bar{0}$

$r^n = (x^2+y^2+z^2)^{n/2}(x\bar{i}+y\bar{j}+z\bar{k})$

$= \Sigma\, x(x^2+y^2+z^2)^{n/2}\,\bar{i}$

$$\text{curl}(r^n\bar{R}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x(x^2+y^2+z^2)^{n/2} & y(x^2+y^2+z^2)^{n/2} & -z(x^2+y^2+z^2)^{n/2} \end{vmatrix}$$

$$= \Sigma\,\bar{i}\left[\frac{nz}{2}(x^2+y^2+z^2)^{n/2-1}(2y) - \frac{ny}{2}(x^2+y^2+z^2)^{n/2-1}(2z)\right.$$

$= \bar{0}$

(v) $\nabla\left(\nabla\cdot\frac{\bar{R}}{r}\right) = \frac{-2\bar{R}}{r^3}$

$\bar{r} = (x\bar{i}+y\bar{j}+z\bar{k}) \Rightarrow r = \sqrt{x^2+y^2+z^2}$.

$$\text{div}\left(\frac{\bar{R}}{r}\right) = \nabla\cdot\frac{\bar{R}}{r} = \nabla\cdot\frac{(x\bar{i}+y\bar{j}+z\bar{k})}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \frac{\sqrt{x^2+y^2+z^2} - \frac{x^2}{\sqrt{x^2+y^2+z^2}}}{(x^2+y^2+z^2)} + \frac{\sqrt{x^2+y^2+z^2} - \frac{y^2}{\sqrt{x^2+y^2+z^2}}}{(x^2+y^2+z^2)}$$

$$+ \frac{\sqrt{x^2+y^2+z^2} - \frac{z^2}{\sqrt{x^2+y^2+z^2}}}{(x^2+y^2+z^2)}$$

$$\Rightarrow \frac{x^2+y^2+z^2-x^2+x^2+y^2+z^2-y^2+x^2+y^2+z^2-z^2}{(x^2+y^2+z^2)^{3/2}}$$

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$$\frac{\alpha(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}} = \alpha(x^2+y^2+z^2)^{-1/2}$$

$$\nabla\left(\nabla\cdot\frac{\bar{R}}{r}\right) = \alpha\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2}\cdot\bar{i} + \alpha\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2}\bar{j}$$
$$+ \alpha\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2}\cdot\bar{k}$$

$$= -\alpha\cdot\frac{1}{2}(x^2+y^2+z^2)^{\frac{-1}{2}-1}\cdot 2x\bar{i} + \alpha\left(\frac{-1}{2}\right)(x^2+y^2+z^2)^{-1/2}$$
$$+ \alpha\left(\frac{-1}{2}\right)(x^2+y^2+z^2)^{-1/2}$$

$$= \frac{-\alpha(x\bar{i}+y\bar{j}+z\bar{k})}{(x^2+y^2+z^2)^{3/2}} = \frac{-\alpha\bar{R}}{r^3}$$

(9) If $\bar{V}_1$ and $\bar{V}_2$ be the vectors joining the fixed points (x_1,y_1,z_1) and (x_2,y_2,z_2) respectively to a variable point (x,y,z). prove that (i) $\operatorname{div}(\bar{V}_1\times\bar{V}_2) = 0$
(ii) $\operatorname{grad}(V_1\cdot V_2) = V_1+V_2$ (iii) $\operatorname{curl}(V_1\times V_2) = 2(V_1-V_2)$

Here $\bar{V}_1 = (x-x_1)\bar{i}+(y-y_1)\bar{j}+(z-z_1)\bar{k} = \sum\bar{i}(x-x_1)$

$\bar{V}_2 = (x-x_2)\bar{i}+(y-y_2)\bar{j}+(z-z_2)\bar{k} = \sum\bar{i}(x-x_2)$

$\bar{V}_1\cdot\bar{V}_2 = (x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)$

$\operatorname{grad}(\bar{V}_1\cdot\bar{V}_2) = \nabla(\bar{V}_1\cdot\bar{V}_2) = \sum\bar{i}\frac{\partial}{\partial x}[(x-x_1)(x-x_2)+$
$(y-y_1)(y-y_2)+(z-z_1)$

$= \sum\bar{i}[(x-x_1)+(x-x_2)]$

$= \sum\bar{i}(x-x_1)+\sum\bar{i}(x-x_2)$

$= \bar{V}_1+\bar{V}_2$

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$\bar{V}_1 \times \bar{V}_2$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x-x_1 & y-y_1 & z-z_1 \\ x-x_2 & y-y_2 & z-z_2 \end{vmatrix}$$

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$$= \bar{i}\left[(y-y_1)(z-z_2)-(z-z_1)(y-y_2)\right] + \bar{j}\left[(z-z_1)(x-x_2) - (x-x_1)(z-z_2)\right] + \bar{k}\left[(x-x_1)(y-y_2)-(y-y_1)(x-x_2)\right]$$

$\operatorname{div}(\bar{V}_1 \times \bar{V}_2) = 0.$

$$\operatorname{curl}(\bar{V}_1 \times \bar{V}_2) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y_1z_2 - y_1z - yz_2 & -x_1z_2 + x_1z + xz_2 & x_1y_2 - x_1y - xy_2 \\ -y_2z_1 + y_2z + y_1z_1 & +x_2z_1 - x_2z - xz_1 & -x_2y_1 + x_2y + xy_1 \end{vmatrix}$$

$$= \bar{i}\left[-x_1+x_2-x_1+x_2\right] + \bar{j}\left[-y_1+y_2+y_2-y_1\right] + \bar{k}\left[z_2-z_1+z_2-z_1\right]$$

$$= 2(x_2-x_1)\bar{i} + 2(y_2-y_1)\bar{j} + 2(z_2-z_1)\bar{k}$$

$$= 2\left[\left[x_2\bar{i}+y_2\bar{j}+z_2\bar{k}\right] - \left[x_1\bar{i}+y_1\bar{j}+z_1\bar{k}\right]\right]$$

$$= 2\left[\bar{V}_2 - \bar{V}_1\right]$$

b) Show that (i) $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$

$$\nabla^2 = \nabla\cdot\left[\nabla f(r)\right] = \nabla\cdot\left[f'(r)\nabla r\right]$$

$$= \nabla\cdot\left[f'(r)\frac{\bar{r}}{r}\right] = \nabla\cdot\left[\frac{f'(r)}{r}\cdot\bar{r}\right]$$

$\nabla\cdot(\phi\bar{F}) = \nabla\phi\cdot\bar{F} + \phi(\nabla\cdot\bar{F})$: $\phi = \frac{f'(r)}{r}$, $\bar{F} = \bar{r}$

$$\nabla^2 f(r) = \nabla\cdot\left[\frac{f'(r)}{r}\cdot\bar{r}\right] = \nabla\frac{f'(r)}{r}\cdot\bar{r} + \frac{f'(r)}{r}(\nabla\cdot\bar{r})$$

$$= \bar{r}\cdot\left[\frac{-1}{r^2}f'(r) + \frac{1}{r}f''(r)\right]\frac{\bar{r}}{r} + \frac{3}{r}f'(r)$$

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$$= \left[\frac{-1}{x^3} f'(x) + \frac{1}{x^2} f''(x)\right] x^2 + \frac{2}{x} f'(x)$$

$$= \left[\frac{-1}{x^3} f'(x) + \frac{1}{x^2} f''(x)\right] x^2 + \frac{2}{x} f'(x)$$

$$= -\frac{1}{x} f'(x) + f''(x) + \frac{2}{x} f'(x)$$

$$= f''(x) + \frac{1}{x} f'(x)$$

$$\nabla^2 f(x) = f''(x) + \frac{2}{x} f'(x)$$

(OR)

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \quad \text{and} \quad |\vec{r}| = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2$$

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

Similarly: $\frac{\partial r}{\partial y} = \frac{y}{r}$; $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\frac{\partial}{\partial x}[g(r)] = g'(r)\frac{\partial r}{\partial x} = g'(r)\frac{x}{r}$$

$$\frac{\partial^2}{\partial x^2}[g(r)] = g''(r)\frac{\partial r}{\partial x}\left(\frac{x}{r}\right) + g'(r)\left[\frac{1}{r}\frac{\partial r}{\partial x} - \frac{x}{r^2}\right]$$

$$= g''(r)\left(\frac{x}{r}\right)^2 + g'(r)\left(\frac{1}{r}\right)\frac{x}{r} + g'(r)$$

$$= g''(r)\left(\frac{x}{r}\right)^2 + g'(r)\left(\frac{1}{r}\right)\frac{x}{r} + g'(r)\left(\frac{1}{r}\right)$$

$$+ g''(r)\left(\frac{x}{r}\right)^2 = g''(r)\frac{x^2}{r^2} + \frac{2}{r}g'(r) - \frac{x^2}{r^3}g'(r)$$

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Similarly,

$$\frac{\partial^2}{\partial y^2}[f(r)] = f''(r)\frac{y^2}{r^2} - \frac{y^2}{r^3}f'(r) + \frac{1}{r}f'(r)$$

$$\frac{\partial^2}{\partial z^2}[f(r)] = f''(r)\frac{z^2}{r^2} - \frac{z^2}{r^3}f'(r) + \frac{1}{r}f'(r)$$

adding the above values, we get

$$\frac{\partial^2}{\partial x^2}f(r) + \frac{\partial^2}{\partial y^2}f(r) + \frac{\partial^2}{\partial z^2}f(r)$$

$$= \frac{1}{r^2}f''(r)(x^2+y^2+z^2) - \frac{1}{r^3}f'(r)(x^2+y^2+z^2) + \frac{3}{r}f'(r)$$

$$= \frac{1}{r^2}f''(r)\,r^2 - \frac{1}{r^3}f'(r)\,r^2 + \frac{3}{r}f'(r)$$

$$= f''(r) - \frac{1}{r}f'(r) + \frac{3}{r}f'(r)$$

$$= f''(r) + \frac{2}{r}f'(r)$$

$$= \nabla^2 f(r)$$

$$\nabla^2 f(r) = \left[\frac{\partial^2}{\partial x^2}f(r) + \frac{\partial^2}{\partial y^2}f(r) + \frac{\partial^2}{\partial z^2}f(r)\right]$$

(ii) prove that $\nabla\cdot[\phi\nabla\psi - \psi\nabla\phi] = \phi\nabla^2\psi - \psi\nabla^2\phi$

$$\nabla\cdot(\phi\bar{F}) = \nabla\phi\cdot\bar{F} + \phi\nabla\cdot\bar{F}$$

$$\nabla\cdot(\phi\nabla\psi - \psi\nabla\phi) = \nabla\cdot(\phi\nabla\psi) - \nabla\cdot(\psi\nabla\phi) \longrightarrow (1)$$

$$\nabla\cdot(\phi\nabla\psi) = \nabla\phi\cdot\nabla\psi + \phi\nabla\cdot\nabla\psi \longrightarrow (2)$$

Here ϕ is scalar function, $\nabla\psi$ is a vector

$$\nabla\cdot(\psi\nabla\phi) = \nabla\psi\cdot\nabla\phi + \psi\nabla\cdot\nabla\phi \longrightarrow (3)$$

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Here ψ is a scalar, $\nabla\phi$ is a vector.

Substituting ② & ③ in ①

$$\nabla\cdot(\phi\nabla\psi - \psi\nabla\phi) = \nabla\phi\cdot\nabla\psi + \phi\nabla^2\psi - \nabla\psi\cdot\nabla\phi - \psi\nabla^2\phi$$

$$= \phi\nabla^2\psi - \psi\nabla^2\phi$$

(11) If $\bar{A}$ is a constant vector and $\bar{R} = x\bar{i} + y\bar{j} + z\bar{k}$, prove that

(i) grad $(\bar{A}\cdot\bar{R}) = \bar{A}$ (ii) div $(\bar{A}\times\bar{R}) = 0$

(iii) curl $(\bar{A}\times\bar{R}) = 2\bar{A}$ (iv) curl $[(\bar{A}\cdot\bar{R})\bar{R}] = \bar{A}\times\bar{R}$

Let $\bar{A} = A_1\bar{i} + A_2\bar{j} + A_3\bar{k}$

(i) grad $(\bar{A}\cdot\bar{R}) = \nabla[A_1\bar{i} + A_2\bar{j} + A_3\bar{k}]\cdot[x\bar{i} + y\bar{j} + z\bar{k}]$

$$= \nabla[A_1x + A_2y + A_3z]$$

$$= \Sigma i\frac{\partial}{\partial x}[A_1x + A_2y + A_3z]$$

$$= \bar{i}\frac{\partial}{\partial x}(A_1x) + \bar{j}\frac{\partial}{\partial y}(A_2y) + \bar{k}\frac{\partial}{\partial z}(A_3z)$$

$$= A_1\bar{i} + A_2\bar{j} + A_3\bar{k}$$

$$= \bar{A}$$

(ii) $\bar{A}\times\bar{R} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ A_1 & A_2 & A_3 \\ x & y & z \end{vmatrix}$

$$= \bar{i}[A_2z - A_3y] + \bar{j}[A_3x - A_1z] + \bar{k}[A_1y - A_2x]$$

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$\frac{\partial}{\partial x}(A_3x - A_1z) + \frac{\partial}{\partial y}(A_3x - A_1z) + \frac{\partial}{\partial z}(A_1y - A_2x) = 0$

$$\nabla \times (\bar{A} \times \bar{R}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_2z - A_3y & A_3x - A_1z & A_1y - A_2x \end{vmatrix}$$

$$= \bar{i}(A_1 + A_1) + \bar{j}(A_2 + A_2) + \bar{k}(A_3 + A_3)$$

$$= 2(A_1\bar{i} + A_2\bar{j} + A_3\bar{k})$$

$$= 2\bar{A}.$$

$$(\bar{A} \cdot \bar{R}) = (A_1\bar{i} + A_2\bar{j} + A_3\bar{k}) \cdot (x\bar{i} + y\bar{j} + z\bar{k})$$

$$= A_1x + A_2y + A_3z.$$

$$(\bar{A} \cdot \bar{R})\bar{R} = (A_1x + A_2y + A_3z)(x\bar{i} + y\bar{j} + z\bar{k})$$

$$= (A_1x^2 + A_2xy + A_3zx)\bar{i} + (A_1xy + A_2y^2 + A_3yz)\bar{j} + (A_1xz + A_2yz + A_3z^2)\bar{k}$$

$$\nabla \times (\bar{A} \cdot \bar{R})\bar{R} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1x^2 + A_2xy + A_3zx & A_1xy + A_2y^2 + A_3yz & A_1xz + A_2yz + A_3z^2 \end{vmatrix}$$

$$= \bar{i}(A_2z - A_3y) + \bar{j}(A_3x - A_1z) + \bar{k}(A_1y - A_2x)$$

$$= \bar{A} \times \bar{R}$$

Calculate (i) curl(grad f) given $f(x, y, z) = x^2 + y^2 - z$

grad $f = \nabla f = 2x\bar{i} + 2y\bar{j} - \bar{k}$

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(ii) $\nabla \times \nabla\phi = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ax & ay & -1 \end{vmatrix} = 0$

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(iii) $\bar{A} = x^2y\bar{i} + y^2z\bar{j} + z^2y\bar{k}$

$$\nabla \times \bar{A} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & y^2z & z^2y \end{vmatrix}$$

$$= \bar{i}(z^2 - y^2) + \bar{j}(0-0) + \bar{k}(0 - x^2)$$

$$= (z^2 - y^2)\bar{i} - x^2\bar{k}$$

$$\nabla \times (\nabla \times \bar{A}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2 - y^2 & 0 & -x^2 \end{vmatrix}$$

$$= \bar{i}(0-0) + \bar{j}(2z + 2x) + \bar{k}(2y)$$

$$= 2(x+z)\bar{j} + 2y\bar{k}$$

(iv) If $\phi = (x^2+y^2+z^2)^{-n}$, find div (grad ϕ) and determine n if div grad $\phi = 0$

Given that $\phi = (x^2+y^2+z^2)^{-n}$

(i) $\nabla\phi$ = grad $\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$

$$\nabla \cdot (\nabla\phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\frac{\partial\phi}{\partial x} = -n(x^2+y^2+z^2)^{-n-1} \cdot 2x$$

$$\frac{\partial^2\phi}{\partial x^2} = -2n\left[(x^2+y^2+z^2)^{-(n+1)} - x(n+1)(x^2+y^2+z^2)^{-n-2} \cdot 2x\right]$$

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$= -2n(x^2+y^2+z^2)^{-(n+2)} [x^2+y^2+z^2-2x^2(n+1)]$

$= -2n(x^2+y^2+z^2)^{-(n+2)} [y^2+z^2-x^2(2n+1)]$

Similarly by symmetry,

$\frac{\partial^2\phi}{\partial y^2} = -2n(x^2+y^2+z^2)^{-(n+2)} [x^2+z^2-y^2(2n+1)]$

$\frac{\partial^2\phi}{\partial z^2} = -2n(x^2+y^2+z^2)^{-(n+2)} [x^2+y^2-z^2(2n+1)]$

$\nabla \cdot \nabla\phi = -2n(x^2+y^2+z^2)^{-(n+2)} \cdot [y^2+z^2-2nx^2-x^2+x^2+z^2-2ny^2-y^2 + x^2+y^2-2nz^2-z^2]$

$= -2n(x^2+y^2+z^2)^{-(n+2)} \cdot [2x^2+2y^2+2z^2-2n(x^2+y^2+z^2)]$

$= -2n(x^2+y^2+z^2)^{-(n+1)} (1-2n)$

$= \frac{2n(2n-1)}{(x^2+y^2+z^2)^{n+1}}$

ii) If div grad $\phi = 0$

$\frac{2n(2n-1)}{(x^2+y^2+z^2)^{n+1}} = 0 \quad ; \quad n = \frac{1}{2}$

since 'n' can't be zero

iii) S.T div (grad r^n) $= n(n+1)r^{n-2}$ where $r^2 = x^2+y^2+z^2$

By definition grad $(r^n) = \bar{i}\frac{\partial}{\partial x}(r^n) + \bar{j}\frac{\partial}{\partial y}(r^n) + \bar{k}\frac{\partial}{\partial z}(r^n)$

$= \bar{i}\, n r^{n-1}\frac{\partial r}{\partial x} + \bar{j}\, n r^{n-1}\frac{\partial r}{\partial y} + \bar{k}\, n r^{n-1}\frac{\partial r}{\partial z}$

$= n r^{n-1}\left[\bar{i}\frac{\partial r}{\partial x} + \bar{j}\frac{\partial r}{\partial y} + \bar{k}\frac{\partial r}{\partial z}\right]$

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Since $r^2 = x^2+y^2+z^2$

$\Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

$\text{grad } r^n = nr^{n-1}\left[\bar{i}\frac{x}{r} + \bar{j}\frac{y}{r} + \bar{k}\frac{z}{r}\right]$

$= nr^{n-2}\left[x\bar{i} + y\bar{j} + z\bar{k}\right]$

$\text{div}(\text{grad } r^n) = \frac{\partial}{\partial x}(nr^{n-2}x) + \frac{\partial}{\partial y}(nr^{n-2}y) + \frac{\partial}{\partial z}(nr^{n-2}z)$

$= \left(nr^{n-2} + nx(n-2)r^{n-3}\frac{\partial r}{\partial x}\right) + \left(nr^{n-2} + ny(n-2)r^{n-3}\frac{\partial r}{\partial y}\right)$

$+ \left(nr^{n-2} + nz(n-2)r^{n-3}\frac{\partial r}{\partial z}\right)$

$= 3nr^{n-2} + n(n-2)r^{n-3}\left[x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right]$

$= 3nr^{n-2} + n(n-2)r^{n-3}\left[x\left(\frac{x}{r}\right) + y\left(\frac{y}{r}\right) + z\left(\frac{z}{r}\right)\right]$

$= 3nr^{n-2} + n(n-2)r^{n-4}\left[x^2+y^2+z^2\right]$

$= 3nr^{n-2} + n(n-2)r^{n-4}\cdot r^2$

$= 3nr^{n-2} + n(n-2)r^{n-2}$

$= r^{n-2}\left[n^2+3n-2n\right]$

$= r^{n-2}\left[n^2+n\right]$

$= r^{n-2}n(n+1)$

If we put $n=-1$ then

$\text{div}(\text{grad } r^n) = -1(-1+1)r^{-1-2}$

$\nabla^2\left(\frac{1}{r}\right) = 0$

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VI : VECTOR INTEGRALS

Integration of vector Functions:

Let $\bar{f}(t)$ and $\bar{F}(t)$ be two vector functions of a scalar variable t such that $\frac{d}{dt}[\bar{F}(t)] = \bar{f}(t)$, then $\bar{F}(t)$ is called an integral of $\bar{f}(t)$ w.r.t to t and we write $\int \bar{f}(t)\,dt = \bar{F}(t)$

If $\bar{c}$ is any arbitary constant vector independent of t,

then $\frac{d}{dt}[\bar{F}(t) + \bar{c}] = \bar{f}(t)$

This is equivalent to $\int \bar{f}(t)\,dt = \bar{F}(t) + \bar{c}$

$\bar{F}(t)$ is called the indefinite integral of $\bar{f}(t)$. The constant vector $\bar{c}$ is called constant of integration and can be determined if some initial conditions are given.

The definite integral of $\bar{f}(t)$ between limits $t=a$ and $t=b$ written as $\int_a^b \bar{f}(t)\,dt = [\bar{F}(t)]_a^b = \bar{F}(b) - \bar{F}(a)$.

Note 1:- If $\bar{f}(t) = f_1(t)\,\bar{i} + f_2(t)\,\bar{j} + f_3(t)\,\bar{k}$, then $\int \bar{f}(t)\,dt = \bar{i}\int f_1(t)\,dt + \bar{j}\int f_2(t)\,dt + \bar{k}\int f_3(t)\,dt$

Note 2:- We can obtain some standard results for integration of vector functions by considering the derivatives of suitable vector functions. For example,

(i) $\frac{d}{dt}(\bar{r}\cdot\bar{a}) = \frac{d\bar{r}}{dt}\cdot\bar{a} + \bar{r}\cdot\frac{d\bar{a}}{dt} \Rightarrow \int\left(\frac{d\bar{r}}{dt}\cdot\bar{a} + \bar{r}\cdot\frac{d\bar{a}}{dt}\right)dt = \bar{r}\cdot\bar{a} + c$

Where 'c' is scalar quantity since the integrand is a scalar

(ii) $\frac{d}{dt}(\bar{r}^2) = 2\bar{r}\cdot\frac{d\bar{r}}{dt} \Rightarrow \int\left(2\bar{r}\cdot\frac{d\bar{r}}{dt}\right)dt = \bar{r}^2 + c$ where 'c' is the scalar quantity

(iii) $\frac{d}{dt}\left(\bar{r}\times\frac{d\bar{r}}{dt}\right) = \frac{d\bar{r}}{dt}\times\frac{d\bar{r}}{dt} + \bar{r}\times\frac{d^2\bar{r}}{dt^2} = \bar{r}\times\frac{d^2\bar{r}}{dt^2} \Rightarrow \int\left(\bar{r}\times\frac{d^2\bar{r}}{dt^2}\right)dt = \bar{r}\times\frac{d\bar{r}}{dt} + \bar{c}$

Here $\bar{c}$ is vector quantity since the integrand is a vector

(iv) If $\bar{a}$ is constant vector, then $\frac{d}{dt}(\bar{a}\times\bar{r}) = \frac{d\bar{r}}{dt}\times\bar{r} + \bar{a}\times\frac{d\bar{r}}{dt} = \bar{a}\times\frac{d\bar{r}}{dt}$

$\Rightarrow \int\left(\bar{a}\times\frac{d\bar{r}}{dt}\right)dt = \bar{a}\times\bar{r} + \bar{c}$ where $\bar{c}$ is vector quantity

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① If $\bar{F}(t) = (3t^2 - 2t)\bar{i} + (6t-4)\bar{j} + 4t\bar{k}$, evaluate $\int_2^3 \bar{F}(t)\,dt$.

$$\int_2^3 \bar{F}(t)\,dt = \int_2^3 \left[(3t^2-2t)\bar{i} + (6t-4)\bar{j} + 4t\bar{k}\right] dt$$

$$= \bar{i}\int_2^3 (3t^2-2t)\,dt + \bar{j}\int_2^3 (6t-4)\,dt + \bar{k}\int_2^3 4t\,dt$$

$$= \bar{i}\left[t^3 - t^2\right]_2^3 + \bar{j}\left[3t^2 - 4t\right]_2^3 + \bar{k}\left[2t^2\right]_2^3$$

$$= 14\bar{i} + 11\bar{j} + 10\bar{k}$$

④ If $\bar{r}(t) = \begin{cases} 2\bar{i} - \bar{j} + 2\bar{k} & \text{when } t=2 \\ 4\bar{i} - 2\bar{j} + 3\bar{k} & \text{when } t=3. \end{cases}$ Show that $\int_2^3 \left(\bar{r}\cdot\frac{d\bar{r}}{dt}\right) dt = 10$.

Sol:- Since $\frac{d}{dt}\left[\bar{r}^2\right] = 2\bar{r}\cdot\frac{d\bar{r}}{dt}$

$$\int_2^3 \left(\bar{r}\cdot\frac{d\bar{r}}{dt}\right) dt = \frac{1}{2}\left[\bar{r}^2\right]_2^3 \quad \text{—①}$$

when $t=3$, $\bar{r}^2 = (4\bar{i} - 2\bar{j} + 3\bar{k})\cdot(4\bar{i} - 2\bar{j} + 3\bar{k}) = 16 + 4 + 9 = 29$

when $t=2$, $\bar{r}^2 = (2\bar{i} - \bar{j} + 2\bar{k})\cdot(2\bar{i} - \bar{j} + 2\bar{k}) = 4 + 1 + 4 = 9$

From ①, we have $\int_2^3 \left(\bar{r}\cdot\frac{d\bar{r}}{dt}\right) dt = \frac{1}{2}\left[29 - 9\right] = 10$.

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(3) Given $\bar{R}(t) = 5t^2\bar{i} + t\bar{j} - t^3\bar{k}$, Prove that $\int_1^2 \left(\bar{R} \times \frac{d^2\bar{R}}{dt^2}\right) dt = -14\bar{i} + 75\bar{j} - 15\bar{k}$.

Sol:- $\frac{d}{dt}\left[\bar{R} \times \frac{d\bar{R}}{dt}\right] = \frac{d\bar{R}}{dt} \times \frac{d\bar{R}}{dt} + \bar{R} \times \frac{d^2\bar{R}}{dt^2} = \bar{R} \times \frac{d^2\bar{R}}{dt^2}$

$$\int_1^2 \left(\bar{R} \times \frac{d^2\bar{R}}{dt^2}\right) dt = \left[\bar{R} \times \frac{d\bar{R}}{dt}\right]_1^2 \leftarrow (1)$$

Let us now find $\bar{R} \times \frac{d\bar{R}}{dt} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 5t^2 & t & -t^3 \\ 10t & 1 & -3t^2 \end{vmatrix}$.

$$= \bar{i}\left[-3t^3 + t^3\right] + \bar{j}\left[-10t^4 + 15t^4\right] + \bar{k}\left[5t^2 - 10t^2\right)$$

$$= -2t^3\bar{i} + 5t^4\bar{j} - 5t^2\bar{k}$$

From (1) we have $\int_1^2 \left(\bar{R} \times \frac{d^2\bar{R}}{dt^2}\right) dt = \left[-2t^3\bar{i} + 5t^4\bar{j} - 5t^2\bar{k}\right]_1^2$.

$$= \left[-2t^3\right]_1^2 \bar{i} + \left[5t^4\right]_1^2 \bar{j} - \left[5t^2\bar{k}\right]_1^2$$

$$= -14\bar{i} + 75\bar{j} - 15\bar{k}$$

) The acceleration of a particle at any time $t>0$ is given by $\bar{a} = \frac{d\bar{v}}{dt} = 12\cos 2t\,\bar{i} - 8\sin 2t\,\bar{j} + 16t\bar{k}$. If the velocity $\bar{v}$ and the displacement $\bar{r}$ are zero at $t=0$, find $\bar{v}$ and $\bar{r}$ at any time.

Sol:- $\bar{a} = \frac{d\bar{v}}{dt} = 12\cos 2t\,\bar{i} - 8\sin 2t\,\bar{j} + 16t\,\bar{k}$

$$\bar{v} = \int \frac{d\bar{v}}{dt}\,dt = \bar{i}\int 12\cos 2t\,dt - \bar{j}\int 8\sin 2t + \bar{k}\int 16t\,dt.$$

$$\bar{v} = 6\sin 2t\,\bar{i} + 4\cos 2t\,\bar{j} + 8t^2\bar{k} + \bar{c}_1$$

when $t=0$, $\bar{v}=0$ $\quad 0 = 4\bar{j} + \bar{c}_1 \Rightarrow \bar{c}_1 = -4\bar{j}$

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Then $\bar{v} = 6\sin 2t\,\bar{i} + 4\cos 2t\,\bar{j} + 8t^2\bar{k} - 4\bar{j}$

$$\frac{d\bar{r}}{dt} = \bar{v} = 6\sin 2t\,\bar{i} + (4\cos 2t - 4)\bar{j} + 8t^2\bar{k}$$

~~So dr~~

Integration $\bar{r} = \int \frac{d\bar{r}}{dt}\,dt = \bar{i}\int 6\sin 2t\,dt + \bar{j}\int (4\cos 2t - 4)\,dt + \bar{k}\int 8t^2\,dt$

$$\bar{r} = -3\cos 2t\,\bar{i} + (2\sin 2t - 4t)\bar{j} + \frac{8}{3}t^3 + \bar{c}_2$$

when $t = 0$, $\bar{r} = 0$.

$$0 = -3\bar{i} + \bar{c}_2 \Rightarrow \bar{c}_2 = 3\bar{i}$$

$$\bar{r} = -3\cos 2t\,\bar{i} + (2\sin 2t - 4t)\bar{j} + \frac{8}{3}t^3 + 3\bar{i}$$

$$\bar{r} = (3 - 3\cos 2t)\,\bar{i} + (2\sin 2t - 4t)\bar{j} + \frac{8}{3}t^3\bar{k}$$

2

$\bar{a} = 18\cos 3t\,\bar{i} - 8\sin 2t\,\bar{j} + 6t\,\bar{k}$

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In electromagnetic theory, we have $\nabla \cdot D = \rho$, $\nabla \times E = -\frac{1}{c}\frac{\partial H}{\partial t}$

$\nabla \times H = \frac{1}{c}\left(\rho v + \frac{\partial D}{\partial t}\right)$ Prove that.

(i) $\nabla^2 D - \frac{1}{c^2}\frac{\partial^2 D}{\partial t^2} = \nabla \rho + \frac{1}{c^2}\frac{\partial}{\partial t}(\rho v)$ (ii) $\nabla^2 H - \frac{1}{c^2}\frac{\partial^2 H}{\partial t^2} = -\frac{1}{c}\nabla \times \rho v$

Sol:- (i) we have $\frac{1}{c^2}\left[\frac{\partial^2 D}{\partial t^2} + \frac{\partial}{\partial t}(\rho v)\right] = \frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial D}{\partial t} + \rho v\right)$

$= \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times H) = \frac{1}{c}\left(\nabla \times \frac{\partial H}{\partial t}\right)$

$= -\nabla \times (\nabla \times D)$

$= -[(\nabla \cdot D)\nabla - (\nabla \cdot \nabla)D]$

$= -(\nabla \cdot D)\nabla + \nabla^2 D$

$= -\nabla \rho + \nabla^2 D$

Hence $\nabla^2 D - \frac{1}{c^2}\frac{\partial^2 D}{\partial t^2} = \nabla \rho + \frac{1}{c^2}\frac{\partial}{\partial t}(\rho v)$

(ii) $L.H.S = \nabla^2 H - \frac{1}{c^2}\frac{\partial^2 H}{\partial t^2} = \nabla^2 H + \frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial H}{\partial t}\right)$

$= \nabla^2 H + \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times D)$

$= \nabla^2 H + \frac{1}{c}\left(\nabla \times \frac{\partial D}{\partial t}\right)$

$= \nabla^2 H + \nabla \times \left(\nabla \times H - \frac{1}{c}\rho v\right)$

$= \nabla^2 H + \nabla \times (\nabla \times H) - \frac{1}{c}\nabla \times \rho v$

$= \nabla^2 H + (\nabla \cdot H)\nabla - (\nabla \cdot \nabla)H - \frac{1}{c}\nabla \times \rho v$

$= \nabla^2 H - 0 - \nabla^2 H - \frac{1}{c}\nabla \times \rho v$

$= -\frac{1}{c}\nabla \times \rho v = R.H.S.$

Hence $\nabla^2 H - \frac{1}{c^2}\frac{\partial^2 H}{\partial t^2} = -\frac{1}{c}\nabla \times \rho v$

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2) Prove that $\bar{A}\cdot\nabla(\bar{B}\cdot\nabla\frac{1}{r}) = \frac{3(\bar{A}\cdot\bar{r})(\bar{B}\cdot\bar{r})}{r^5} - \frac{\bar{A}\cdot\bar{B}}{r^3}$ where $\bar{A}$ and $\bar{B}$ are constant vectors.

Sol:- $\nabla\frac{1}{r} = \nabla(r^{-1}) = -r^{-2}\frac{\partial r}{\partial x}\hat{i} + \left(-r^{-2}\frac{\partial r}{\partial y}\right)\hat{j} + \left(-r^{-2}\frac{\partial r}{\partial z}\right)\hat{k}$

$= -\frac{x}{r^3}\hat{i} - \frac{y}{r^3}\hat{j} - \frac{z}{r^3}\hat{k}$

$= -r^{-3}(x\hat{i}+y\hat{j}+z\hat{k}) = -\frac{\bar{r}}{r^3}$

$r^2 = x^2+y^2+z^2$

$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

$\nabla(\bar{B}\cdot\nabla\frac{1}{r}) = \nabla\left(\bar{B}\cdot\left(-\frac{\bar{r}}{r^3}\right)\right)$

$= -\nabla\left[\frac{\bar{B}\cdot\bar{r}}{r^3}\right]$

$= -\nabla\left(\frac{1}{r^3}\right)(\bar{B}\cdot\bar{r}) - \frac{1}{r^3}\nabla(\bar{B}\cdot\bar{r})$

$\nabla(\bar{B}\cdot\bar{r}) = \bar{B}$

$\Rightarrow \frac{3}{r^5}\bar{r}(\bar{B}\cdot\bar{r}) - \frac{1}{r^3}\bar{B}$

$\bar{A}\cdot\nabla(\bar{B}\cdot\nabla\frac{1}{r}) = \frac{3}{r^5}(\bar{B}\cdot\bar{r})(\bar{A}\cdot\bar{r}) - \frac{1}{r^3}(\bar{A}\cdot\bar{B})$

Line integrals :- Any integral which is to be evaluate along a curve is called a line integral.

The line integral of $\bar{F}$ along the curve 'c' is defined as $\int_c \bar{F}\cdot d\bar{r}$

Note:- $\int_c \bar{F}\cdot d\bar{r} = \int_c (F_1\hat{i}+F_2\hat{j}+F_3\hat{k})\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k})$

$= \int_c F_1\,dx + F_2\,dy + F_3\,dz$

(2) $\int_a^b \bar{q}\cdot d\bar{r}$ is called line integral if along the curve 'c'.

circulation :- Let 'c' be a simple closed curve, the line integral of $\bar{F}$ along 'c' is called circulation $\bar{F}$ i.e. $\oint \bar{F}\cdot d\bar{r}$

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1) If $\bar{F} = 3xy\,\hat{i} - y^2\hat{j}$, evaluate $\int_c \bar{F}\cdot d\bar{r}$ where 'c' is the curve in the xy-plane $y = 2x^2$ from (0,0) to (1,2).

Sol:- Since the particle moves in the xy-plane ($z=0$) we take $\bar{r} = x\hat{i} + y\hat{j}$. Then $\int_c \bar{F}\cdot d\bar{r}$, where 'c' is the parabola $y = 2x^2$

$$\int_c \bar{F}\cdot d\bar{r} = \int_c (3xy\,\hat{i} - y^2\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j}) = \int_c (3xy\,dx - y^2 dy) \quad -(1)$$

Substituting $y = 2x^2$ where x goes from 0 to 1,

(1) becomes $\int_{x=0}^{1} 3x(2x^2)\,dx - (2x^2)^2\,d(2x^2) = \int_0^1 (6x^3 - 16x^5)\,dx = -7/6$

2) A vector field is given by $\bar{F} = \sin y\,\hat{i} + x(1+\cos y)\,\hat{j}$. Evaluate the line integral over a circular path given by $x^2 + y^2 = a^2$, $z = 0$.

Sol:- As the particles moves in xy-plane ($z=0$), let $\bar{r} = x\hat{i} + y\hat{j}$ so that $d\bar{r} = dx\,\hat{i} + dy\,\hat{j}$. Also the circular path is $x = a\cos t$, $y = a\sin t$, $z = 0$, where 't' varies from 0 to 2π.

$$\oint_c \bar{F}\cdot d\bar{r} = \oint_c (\sin y\,\hat{i} + x(1+\cos y)\,\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j})$$

$$= \oint_c \sin y\,dx + x(1+\cos y)\,dy = \oint_c (\sin y\,dx + x\cos y\,dy + x\,dy)$$

$$= \oint_c [d(x\sin y) + x\,dy] = \int_0^{2\pi} d(a\cos t\sin(a\sin t)) + a^2\cos^2 t\,dt$$

$$= [a\cos t\sin(a\sin t)]_0^{2\pi} + \frac{a^2}{2}\int_0^{2\pi} 1+\cos 2t\,dt$$

$$= \frac{a^2}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} = \pi a^2$$

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3). Find the work done in moving a particle in the force field $\bar{F} = 3x^2\hat{i} + (2xz-y)\hat{j} + z\hat{k}$ along (a) the straight line from (0,0,0) to (2,1,3).

(b) the curve defined by $x^2 = 4y$, $3x^3 = 8z$ from $x=0$ to $x=2$.

Sol:- $\int_C \bar{F}\cdot d\bar{r} = \int_C (3x^2\hat{i} + (2xz-y)\hat{j} + z\hat{k})\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$

$= \int_C 3x^2dx + (2xz-y)dy + zdz.$

(a) The equations of the straight line from (0,0,0) to (2,1,3) are $\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$ say.

$\therefore x = 2t,\ y = t,\ z = 3t$ are it's parametric equations. The points (0,0,0) and (2,1,3) corresponding to $t=0$ and $t=1$ respectively.

$\therefore$ work done $= \int_C \bar{F}\cdot d\bar{r} = \int_0^1 3(2t)^2 2dt + ((4t)(3t) - t)dt + (3t)3dt.$

$= \int_0^1 (36t^2 + 8t)dt = 16.$

(b) Let $x = t$ in $x^2 = 4y$, $3x^3 = 8z$. Then the parametric equations of 'c' are $x = t,\ y = t^2/4,\ z = 3t^3/8$ and 't' varies from 0 to 2.

$\therefore$ work done $= \int_C \bar{F}\cdot d\bar{r} = \int_0^2 3t^2dt + \left\{2t\left(\frac{3t^3}{8}\right) - \frac{t^2}{4}\right\}d\left(\frac{t^2}{4}\right) + \frac{3t^3}{8}d\left(\frac{3t^3}{8}\right)$

$= \int_0^2 \left(3t^2 - \frac{t^3}{8} + \frac{51}{64}t^5\right)dt$

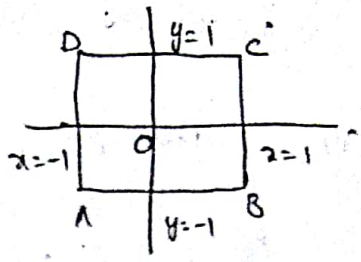
$= t^3 - \frac{t^4}{32} + \frac{17}{128}t^6\Big]_0^2 = 16.$

2.

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① Evaluate the line integral $\int_C (x^2+xy)\,dx + (x^2+y^2)\,dy$ where C is the square formed by the lines $y = \pm 1$ and $x = \pm 1$.



Sol:- The given vector

$$\bar{F} = (x^2+xy)\,\bar{i} + (x^2+y^2)\,\bar{j}$$

To evaluate $\int_C (x^2+xy)\,dx + (x^2+y^2)\,dy$

C is the square formed by the lines $y = \pm 1$ and $x = \pm 1$.

The square is ABCD.

Along AB, x varies from -1 to 1.

$y = -1$ and $dy = 0$. $\Rightarrow \int_C (x^2-x)\,dx = \int_{-1}^{1} x^2\,dx - \int_{-1}^{1} x\,dx$

Along BC, $x = 1$, $dx = 0$ y varies -1 to 1. $= \left[\frac{x^3}{3}\right]_{-1}^{1} - \left[\frac{x^2}{2}\right]_{-1}^{1}$

$\int_{BC} \bar{F}\cdot d\bar{r} = \int_{-1}^{1} (1+y^2)\,dy = \left[y + \frac{y^3}{3}\right]_{-1}^{1}$ $= \left(\frac{1}{3}+\frac{1}{3}\right) - \left[\frac{1}{2}-\frac{1}{2}\right]$

$= [1+1] + \left[\frac{1}{3}+\frac{1}{3}\right] = 2+\frac{2}{3} = \frac{8}{3}$ $= \frac{2}{3}$

Along CD, $y = 1$, $dy = 0$, x varies from 1 to -1.

$\int_1^{-1} (x^2+x)\,dx = 0 \Rightarrow \int_{+1}^{-1} x^2\,dx = \left[\frac{x^3}{3}\right]_{+1}^{-1} = -\frac{1}{3} - \frac{1}{3} = -\frac{2}{3}$

Along DA, $x = -1$, $dx = 0$, y varies from 1 to -1

$\int_1^{-1} (1+y^2)\,dy = \left[y + \frac{y^3}{3}\right]_1^{-1} = \left[-1-\frac{1}{3}\right] + \left[\frac{1}{3}-\frac{1}{3}\right] = -\frac{8}{3}$

$\int_C (x^2+xy)\,dx + (x^2+y^2)\,dy = \int_{AB} (x^2+xy)\,dx + (x^2+y^2)\,dy + \int_{BC} (x^2+xy)\,dx + (x^2+y^2)\,dy + \int_{CD} (x^2+xy)\,dx + (x^2+y^2)\,dy$

$+ \int_{DA} (x^2+xy)\,dx + (x^2+y^2)\,dy$

$= \frac{2}{3} + \frac{8}{3} - \frac{2}{3} - \frac{8}{3} = 0$

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2) If $\bar{F} = (5xy - 6x^2)\,\hat{i} + (2y - 4x)\,\hat{j}$, evaluate $\int_C \bar{F}\cdot d\bar{r}$ along the curve 'C' in the xy plane $y = x^3$ from the point (1, 1) to (2, 8).

Sol:- The given function $\bar{F} = (5xy - 6x^2)\,\hat{i} + (2y - 4x)\,\hat{j}$

$y = x^3 \Rightarrow dy = 3x^2\,dx$

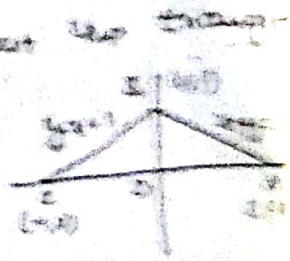
$$\int_C \bar{F}\cdot d\bar{r} = \int_C (5xy - 6x^2)\,dx + (2y - 4x)\,dy$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_1^2 (5x\cdot x^3 - 6x^2)\,dx + (2x^3 - 4x)\,3x^2\,dx$$

$$= \frac{5x^5}{5} - \frac{6x^3}{3} + \frac{6x^6}{6} - \frac{12x^4}{4}\Big]_1^2$$

$$= x^5 - 2x^3 + x^6 - 3x^4\Big]_1^2 = 32 - 16 + 64 - 48 - 1 + 2 - 1 + 3 = 35$$

3) Compute the line integral $\int_C y^2\,dx - x^2\,dy$ about the triangle whose vertices are (1,0) (0,1) and (-1,0)



Sol:- Let the points (1,0) (0,1) and (-1,0) be A, B and C respectively.

$$\int_C y^2\,dx - x^2\,dy = \int_{AB} y^2\,dx - x^2\,dy + \int_{BC} y^2\,dx - x^2\,dy + \int_{CA} y^2\,dx - x^2\,dy$$

$$= I_1 + I_2 + I_3.$$

$I_1 = \int_{AB} y^2\,dx - x^2\,dy$. The equation of the line AB is $x + y = 1$

$y = 1 - x \Rightarrow dy = -dx$

x varies from 1 to 0

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$$\int_1^0 (1+x)^2 dx + x^2 \int dx = \int_1^0 (1+2x+x^2)dx = \int_1^0 (1+2x+x^2)dx = -\frac{2}{3}$$

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$I_2 = \int_{Bc} y^2\,dx - x^2\,dy$, the equation of the line Bc is $y-x=1$

$y = 1+x$, $dy = dx$, x varies from 0 to -1.

$$I_2 = \int_0^{-1} [(1+x)^2 - x^2]\,dx = \int_0^{-1} (1+2x)\,dx = \left[x + x\frac{x^2}{2}\right]_0^{-1}$$

$$= -1+1 = 0.$$

$$I_3 = \int_{CA} y^2\,dx - x^2\,dy, \quad y=0,\ dy=0 \quad I_3 = 0.$$

$$\therefore \int_C y^2\,dx - x^2\,dy = -\frac{2}{3} \text{ along } ABCA.$$

4) If $\bar{F} = (3x^2+6y)\bar{i} - 14yz\bar{j} + 20xz^2\bar{k}$ evaluate $\int_C \bar{F}\cdot d\bar{r}$ from $(0,0,0)$ to $(1,1,1)$ along the path $x=t$, $y=t^2$, $z=t^3$.

Sol:- The given function $\bar{F} = (3x^2+6y)\bar{i} - 14yz\bar{j} + 20xz^2\bar{k}$.

The limits are $(0,0,0)$ to $(1,1,1)$, $x=0$ to 1, $y=0$ to 1, $z=0$ to 1

Given that $x=t$, $y=t^2$, $z=t^3$ therefore the limits t are 0 and 1

$dx = dt$, $dy = 2t\,dt$, $dz = 3t^2\,dt$

$$\int_C \bar{F}\,d\bar{r} = \int_0^1 (3x^2+6y)\,dt - (14yz)\,2t\,dt + (20xz^2)3t^2\,dt$$

$$= \int_0^1 (3t^2+6t^2)\,dt - (14t^5)2t\,dt + (20t^7)3t^2\,dt.$$

$$= \int_0^1 9t^2 - 28t^6\,dt + 60t^9\,dt = \int_0^1 (9t^2+32t^9)\,dt.$$

$$= 9\frac{t^3}{3} + 32\frac{t^7}{7}\Big]_0^1$$

$$= 3 + \frac{32}{7} = \frac{53}{7}$$

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5) Using the line integral compute the work done by the force $\bar{F} = (2y+3)\bar{i} + xz\bar{j} + (yz-x)\bar{k}$ when it moves a particle from the point (0,0,0) to the the point (2,1,1) along the curve $x = 2t^2$, $y = t$, $z = t^3$.

Sol: $\bar{F} = (2y+3)\bar{i} + xz\bar{j} + (yz-x)\bar{k}$

$d\bar{R} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$$\int_C \bar{F}\cdot d\bar{R} = \int (2y+3)dx + \int xz\,dy + \int (yz-x)\,dz$$

$$= \int_0^1 (2t+3)\,4t\,dt + \int_0^1 (2t^3)\,dt + \int_0^1 (t^4 - 2t^2)\,3t^2\,dt$$

$$= \int_0^1 (8t^2+12t)\,dt + \int_0^1 2t^5\,dt + \int_0^1 (3t^6 - 6t^4)\,dt$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^1 (8t^2+12t)\,dt + \int_0^1 2t^5\,dt + \int_0^1 (3t^6 - 6t^4)\,dt$$

$$= \frac{8t^3}{3} + \frac{12t^2}{2} + \frac{2t^6}{6} + \frac{3t^7}{7} - \frac{6t^5}{5}\Big]_0^1$$

$$= \frac{8}{3} + 6 + \frac{1}{3} + \frac{3}{7} - \frac{6}{5} = 9 + \frac{3}{7} - \frac{6}{5} = \frac{315+15-42}{35}$$

$$= \frac{288}{35}$$

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(6) If $\bar{F} = 2yi - zj + xk$, evaluate $\int_C \bar{F} \times d\bar{r}$ along the curve $x = \cos t$, $y = \sin t$, $z = 2\cos t$ from $t = 0$ to $t = \pi/2$

Sol: $\bar{F} = 2yi - zj + xk$, $d\bar{r} = dx\,i + dy\,j + dz\,k$

$$\bar{F} \times d\bar{r} = \begin{vmatrix} i & j & k \\ 2y & -z & x \\ dx & dy & dz \end{vmatrix}$$

$$= i(-z\,dz - x\,dy) + j(x\,dx - 2y\,dz) + k(2y\,dy + z\,dx)$$

$$\int \bar{F} \times d\bar{r} = \int_0^{\pi/2} \left[\cos t(\cos t\,dt + 2\cos t(-\sin t\,dt)\right] i + \int_0^{\pi/2} \left[\cos t(-\sin t\,dt) - 2\sin t(-2\sin t)\,dt\right] j$$

$$+ \int_0^{\pi/2} \left[2\sin t\cos t\,dt + 2\cos t(-\sin t\,dt)\right] k$$

$$= \int_0^{\pi/2} \left[-\cos^2 t\,dt + 2\sin 2t\,dt\right] i + \int_0^{\pi/2} \left(\frac{\sin 2t}{2} + 4\sin^2 t\right) dt\, j$$

$$+ \int_0^{\pi/2} (\sin 2t - \sin 2t)\,dt\, k$$

$$= i\int_0^{\pi/2} -\frac{1+\cos 2t}{2}\,dt + 2\sin 2t\,dt + j\int_0^{\pi/2} \left[\frac{-\sin 2t}{2} + 4\left(\frac{1-\cos 2t}{2}\right)\right] dt$$

$$= i\left[-\frac{1}{2}t - \frac{\sin 2t}{4} - \cos 2t\right]_0^{\pi/2} + j\left[\frac{\cos 2t}{4} + 2t - \frac{2\sin 2t}{2}\right]_0^{\pi/2}$$

$$= \left[-\frac{\pi}{4} + 0 + 1 - 1\right] i + \left[\frac{1}{4} + \pi + 0 - \frac{1}{4}\right] j$$

$$= \left[-\frac{\pi}{4}\right] i + \left[\pi\right] j$$

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Surface Integrals :- Let $\bar{F}$ be a continuous vector point function defined over the smooth surface

Surface Integrals :- Any integral which is to be evaluated over a surface is called a surface Integral.

Then the surface integral of $\bar{F}$ over S is defined as $\int_S \bar{F} \cdot \bar{N}\, ds$

where $\bar{N}$ is the unit outward normal to S.

Cartesian form of surface Integrals :- Let $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$\bar{N} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$ where $\cos\alpha, \cos\beta, \cos\gamma$ are the d.c's of $\bar{N}$

$\bar{F}\cdot\bar{N} = F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma$.

Now $\int_S \bar{F}\cdot\bar{N}\, ds = \int_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\, ds$.

$= \int_S F_1\cos\alpha\, ds + F_2\cos\beta\, ds + F_3\cos\gamma\, ds$

$\int_S \bar{F}\cdot\bar{N}\, ds = \int_S F_1\, dy\,dz + F_2\, dz\,dx + F_3\, dx\,dy$.

Flux :- If 'S' be a closed surface then $\oint_S \bar{F}\cdot\bar{N}\, ds$ is called the flux over 'S'.

Note 1 :- (1) Let R_1 be the Projection of 'S' on xy-plane then $\int_S \bar{F}\cdot\bar{N}\, ds = \int_{R_1} \bar{F}\cdot\bar{N}\, \frac{dx\,dy}{|\cos\gamma|} = \int_{R_1} \bar{F}\cdot\bar{N}\, \frac{dx\,dy}{|\bar{N}\cdot\bar{k}|}$

(2) Let R_2 be the projection of 'S' on yz-plane then $\int_S \bar{F}\cdot\bar{N}\, ds = \int_{R_2} \bar{F}\cdot\bar{N}\, \frac{dy\,dz}{|\cos\alpha|} = \int_{R_2} \bar{F}\cdot\bar{N}\, \frac{dy\,dz}{|\bar{N}\cdot\bar{i}|}$

(3) Let R_3 be the Projection of zx-plane then $\int_S \bar{F}\cdot\bar{N}\, ds = \int_{R_3} \bar{F}\cdot\bar{N}\, \frac{dz\,dx}{|\cos\beta|} = \int_{R_3} \bar{F}\cdot\bar{N}\, \frac{dz\,dx}{|\bar{N}\cdot\bar{j}|}$

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ce Integrals : Any integral which is to be evaluated over a surface is called a surface integral.

Suppose 's' is a two sided surface, consider one side as positive. If 's' is a closed surface we choose the outside to be positive. Let $\bar{F}$ be a vector point function. Let ds be an element of area in the surface and $\bar{n}$ be the unit vector normal to the surface, drawn on the positive side. Let 'θ' be the angle between $\bar{F}$ and $\bar{n}$. The normal component of $\bar{F}$ is given by the dot product $\bar{F} = \bar{F}\cos\theta$. The integral of this normal component i.e of $\bar{F}\cdot\bar{n}$ through an element of area ds over the surface 's' is called the surface integral of $\bar{F}$ over S.

So the surface integral is $\int_S \bar{F}\cdot\bar{n}\,ds$.

If $d\bar{s}$ is a vector whose in magnitude is ds and whose direction is that of $\bar{n}$. then $d\bar{s} = \bar{n}\,ds$.

Hence surface integral is $\int_S \bar{F}\cdot d\bar{s}$.

Other surface integrals are $\int \phi\,ds$ or $\int_S \bar{F}\times d\bar{s}$

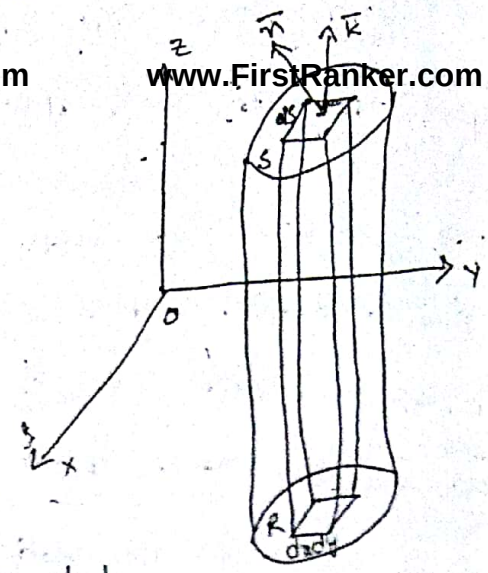
To evaluate surface integrals it is convenient to express them as double integrals taken over the projected area of the surfaces on one of the coordinate planes.

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Let $\bar{n} = \cos\alpha\,\bar{i} + \cos\beta\,\bar{j} + \cos\gamma\,\bar{k}$

where $\cos\alpha, \cos\beta, \cos\gamma$ are the d.c's of $\bar{n}$



$ds\cos\alpha$, $ds\cos\beta$, $ds\cos\gamma$ are the orthogonal projections of ds on the coordinate planes $x=0, y=0, z=0$ respectively.

$ds\cos\alpha = dy\,dz$, $ds\cos\beta = dz\,dx$, $ds\cos\gamma = dx\,dy$.

Considering subdivisions formed by planes parallel to the coordinate pla

Forming dot products $\bar{n}$ and unit vectors $\bar{i}, \bar{j}, \bar{k}$

$|\bar{i}\cdot\bar{n}| = \cos\alpha$, $|\bar{j}\cdot\bar{n}| = \cos\beta$, $|\bar{k}\cdot\bar{n}| = \cos\gamma$.

$$ds = \frac{dy\,dz}{\cos\alpha} = \frac{dy\,dz}{|\bar{i}\cdot\bar{n}|} \quad ; \quad ds = \frac{dz\,dx}{\cos\beta} = \frac{dz\,dx}{|\bar{j}\cdot\bar{n}|} \quad , \quad ds = \frac{dx\,dy}{|\bar{k}\cdot\bar{n}|}$$

Surface integral $\iint_S \bar{F}\cdot\bar{n}\,ds$.

$$\iint_S \bar{F}\bar{n}\,ds = \iint_{S_1} \bar{F}\cdot\bar{n}\,\frac{dy\,dz}{\bar{i}\cdot\bar{n}} = \iint_{S_2} \bar{F}\cdot\bar{n}\,\frac{dz\,dx}{\bar{j}\cdot\bar{n}} = \iint_{S_3} \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{\bar{k}\cdot\bar{n}}$$

where S_1, S_2, S_3 are the projections of S on the planes

$x=0, y=0, z=0$ respectively

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1) Evaluate $\int_S \bar{F}\cdot\bar{N}\,ds$ where $\bar{F} = z\bar{i} + x\bar{j} - 3y^2z\,\bar{k}$ and 's' is the surface $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$

Sol:- $\bar{F} = z\bar{i} + x\bar{j} - 3y^2z\,\bar{k}$

Here 's' is the surface $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$.

$\bar{N}$ = unit outward normal to $S = \frac{\nabla\phi}{|\nabla\phi|}$ where $\phi = x^2+y^2=16$

$$\bar{N} = \frac{2x\bar{i}+2y\bar{j}}{\sqrt{4x^2+4y^2}} = \frac{x\bar{i}+y\bar{j}}{4}$$

$x^2+y^2=16$

$z = 0$ to 5.

$y^2 = 16-x^2$

$y = \sqrt{16-x^2}$

$y = \sqrt{16} = 4.$

$y = 0$ to 4.

Let 'R' be the projection of S on yz-plane

$$\int_S \bar{F}\cdot\bar{N}\,ds = \int_R \bar{F}\cdot\bar{N}\,\frac{dy\,dz}{|\bar{N}\cdot\bar{i}|}$$

$$= \int_R (z\bar{i}+x\bar{j}-3y^2z\,\bar{k})\cdot\left(\frac{x\bar{i}+y\bar{j}}{4}\right)\frac{dy\,dz}{\left|\frac{x\bar{i}+y\bar{j}}{4}\cdot\bar{i}\right|}$$

$$= \int_R (2x+xy)\,\frac{dy\,dz}{x}$$

$$= \int_{z=0}^{5}\int_{y=0}^{4} (2+y)\,dy\,dz = \int_{z=0}^{5}\left[2y+\frac{y^2}{2}\right]_0^4 dz$$

$$= \int_0^5 (8+8)\,dz = \left[\frac{16z^2}{2}+8z\right]_0^5$$

$$= 2\times 25 + 8\times 5$$

$$= 90$$

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2) Evaluate $\int_S \bar{F}\cdot\bar{N}\,ds$ where $\bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$ and 'S' is the part of the plane $2x+3y+6z=12$ located in the first octant.

Sol:- $\bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$.

Here 'S' is the part of the plane $2x+3y+6z=12$ located in the first octant.

$\bar{N}$ = unit outward normal to $S = \frac{\nabla\phi}{|\nabla\phi|}$ where $\phi = 2x+3y+6z-12$

$$\bar{N} = \frac{2\bar{i}+3\bar{j}+6\bar{k}}{\sqrt{4+9+36}} = \frac{1}{7}(2\bar{i}+3\bar{j}+6\bar{k})$$

Let 'R' be the projection of on xy-plane then

$$\int_S \bar{F}\cdot\bar{N}\,ds = \int_R \bar{F}\cdot\bar{N}\,\frac{dx\,dy}{|\bar{N}\cdot\bar{k}|}$$

$2x+3y+6z=12$

$6z = 12-2x-3y$

$z = \frac{12-2x-3y}{6}$

$3y = 12-2x$

$y = \frac{12-2x}{3}$

$x = 0$ to 6

$$\int_S \bar{F}\cdot\bar{N}\,ds = \int_R (18z\bar{i}-12\bar{j}+3y\bar{k})\cdot\frac{1}{7}(2\bar{i}+3\bar{j}+6\bar{k})\cdot\frac{dx\,dy}{|\frac{1}{7}(2\bar{i}+3\bar{j}+6\bar{k})\cdot\bar{k}|}$$

$$= \int_R (36z-36+18y)\,\frac{dx\,dy}{6} = \int_R (6z-6+3y)\,dx\,dy$$

$$= \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} \left[6\left(\frac{12-2x-3y}{6}\right) - 6 + 3y\right] dy\,dx$$

$$= \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} [(6-2x)\,dy]\,dx = \int_{x=0}^{6} [6y - 2xy]_0^{\frac{12-2x}{3}}\,dx$$

$$= \int_{x=0}^{6} \left(6\left(\frac{12-2x}{3}\right) - 2x\left[\frac{12-2x}{3}\right]\right) dx$$

$$= \int_{x=0}^{6} \left(24 - 12x + \frac{4x^2}{3}\right) dx = \left[24x - \frac{12x^2}{2} + \frac{4x^3}{9}\right]_0^6$$

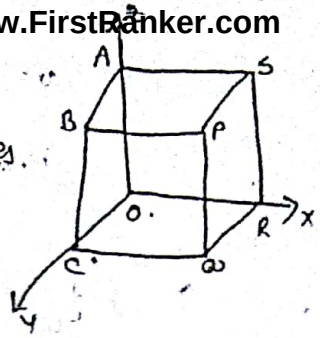
$$= 24\times6 - 6\times36 - \frac{4\times6^3}{9}$$

$$= 144 - 216 + 96 = 24$$

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$= 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$, evaluate $\int_S \bar{F}\cdot\bar{N}\,ds$, where 'S' is the surface of the bounded by $x=0,\ x=a,\ y=0,\ y=a,\ z=0,\ z=a$.



Sol:- consider the cube surrounded by the following faces.

(i) For the face PQRS, $\hat{i}$ is the outward normal

$\bar{N} = \hat{i}$; $x=a$, $ds = dy\,dz$

$$\int_S \bar{F}\cdot\bar{N}\,ds = \int_{y=0}^{a}\int_{z=0}^{a} (4xz\,\hat{i} - y^2\hat{j} + yz\,\hat{k})\cdot\hat{i}\,dy\,dz = \int_{y=0}^{a}\int_{z=0}^{a} 4xz\,dy\,dz$$

$$= \int_{y=0}^{a} 4a\left[\frac{z^2}{2}\right]_0^a dy = \int_{y=0}^{a} 4a\,\frac{a^2}{2}\,dy = \int_{y=0}^{a} 2a^3\,dy = 2a^4$$

$\because x = a$ on this face

$4a\,[y]_0^a \left[\frac{z^2}{2}\right]_0^a = 4a^2\cdot\frac{a^2}{2} = 2a^4$

(ii) For the face OABC, $-\hat{i}$ is the outward normal.

$\bar{N} = -\hat{i}$, $x=0$, and $ds = dy\,dz$. $\because x=0$ on this face

$$\int_{R_2} \bar{F}\cdot\bar{N}\,ds = \iint_{R_2} (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\cdot(-\hat{i})\,dy\,dz = -\iint_{R_2} -4xz\,dy\,dz = 0.$$

(iii) For the face BPQC, $\hat{j}$ is the outward normal.

$\bar{N} = \hat{j}$, $y=a$, and $ds = dx\,dz$

$$\int_{R_3} \bar{F}\cdot\bar{N}\,ds = \iint_{R_3} (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\cdot\hat{j}\,dx\,dz = -\iint_{R_3} y^2\,dx\,dz$$

$$= -a^2\int_0^a\int_0^a dx\,dz = -a^4.$$

(iv) For the face OPSA, $-\hat{j}$ is the outward normal.

$\bar{N} = -\hat{j}$, $y=0$, and $ds = dx\,dz$

$$\iint_{R_4} \bar{F}\cdot\bar{N}\,ds = \iint_{R_4} (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\cdot(-\hat{j})\,dx\,dz = \iint_{R_4} +y^2\,dx\,dz = 0.$$

(v) For the face ABPS, $\hat{k}$ is the outward normal

$\bar{N} = \hat{k}$, $z=a$, and $ds = dx\,dy$. $\bar{F}\cdot\bar{N} = (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\cdot\hat{k} = yz$

$$\iint_{R_5} \bar{F}\cdot\bar{N}\,ds = \iint_{R_5} yz\,dx\,dy = a\int_{x=0}^{a}\int_{y=0}^{a} y\,dx\,dy = a^4/4$$

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(vi) For the face $BCEG$, outward normal

$\bar{n} = -\bar{i}$, $x = 0$, $ds = dy\,dz$.

$\bar{F}\cdot\bar{n} = (4xz\bar{i} - y^2\bar{j} + yz\bar{k})(-\bar{i}) = -yz$

$\iint_{R_6} \bar{F}\cdot\bar{n}\,ds = -\iint_{R_6} yz\,dy\,dz = 0$ [$\because x = 0$ on this face]

Hence adding all we get $\int_S \bar{F}\cdot\bar{n}\,ds = 2a^4 + 0 - a^4 + \frac{1}{2}a^4 + 0 = \frac{3}{2}a^4$.

4) $\bar{F} = 2y\bar{i} - 3\bar{j} + x^2\bar{k}$ and 'S' is surface $y^2 = 8x$ is first octant bounded by the planes $y = 4$ and $z = 6$, evaluate $\int_S \bar{F}\cdot\bar{n}\,ds$

Sol:- $\bar{F} = 2y\bar{i} - 3\bar{j} + x^2\bar{k}$.

Here 'S' is the surface $y^2 = 8x$ first octant bounded by the planes $y = 4$ and $z = 6$

$\phi = 8x - y^2$

$\bar{n}$ = unit outward normal to $S = \frac{\nabla\phi}{|\nabla\phi|}$.

$\bar{n} = \frac{8\bar{i} - 2y\bar{j}}{\sqrt{64 + 4y^2}} = \frac{4\bar{i} - y\bar{j}}{\sqrt{16 + y^2}}$.

Let 'R' be the projection of 'S' on the yz-plane then

$\int_S \bar{F}\cdot\bar{n}\,ds = \int_R \bar{F}\cdot\bar{n}\,\frac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$

For R: $x = y^2/8$, $y = 0$ to 4, $z = 0$ to 6.

$\int_S \bar{F}\cdot\bar{n}\,ds = \int_R \bar{F}\cdot\bar{n}\,\frac{dy\,dz}{|\bar{n}\cdot\bar{i}|} = \int_{z=0}^{6}\int_{y=0}^{4} (2y\bar{i} - 3\bar{j} + x^2\bar{k})\cdot\frac{(4\bar{i} - y\bar{j})}{\sqrt{16 + y^2}}$

$= \int_{z=0}^{6}\int_{y=0}^{4} (8y + 3y)\frac{dy\,dz}{4} = \frac{11}{4}\int_{z=0}^{6}\left[\int_{y=0}^{4} y\right]dz$

$= \frac{11}{4}\int_{z=0}^{6}\left[\frac{y^2}{2}\right]_0^4 dz = \frac{11}{4}\int_{z=0}^{6} 8\,dz = \frac{88}{4}[z]_0^6$

$= 22 \times 6 = 132$.

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Problem:- If velocity vector is $\bar{F} = y\bar{i} + 2\bar{j} + xz\bar{k}$ m/sec. Show that the flux of water through the parabolic cylinder $y = x^2$, $0 \le x \le 3$, $0 \le z \le 2$ is 69 m^3/sec.

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Sol:- $\bar{F} = y\bar{i} + 2\bar{j} + xz\bar{k}$

Here 'S' is the surface $y = x^2$ first octant bounded by the planes $x = 3$ and $z = 2$.

$\bar{N}$ = unit outward normal to $S = \frac{\nabla\phi}{|\nabla\phi|}$.

$\bar{N} = \frac{2x\bar{i} - \bar{j}}{\sqrt{4x^2+1}}$ Let 'R' be the Projection of 'S' on the xz-plane.

$$\iint_S \bar{F}\cdot\bar{N}\, ds = \iint_R \bar{F}\cdot\bar{N}\, \frac{dx\,dz}{|\bar{N}\cdot\bar{j}|}$$

$\bar{F}\cdot\bar{N} = (y\bar{i} + 2\bar{j} + xz\bar{k})\cdot\left(\frac{2x\bar{i} - \bar{j}}{\sqrt{4x^2+1}}\right)$

$\bar{N}\cdot\bar{j} = \frac{2x\bar{i} - \bar{j}}{\sqrt{4x^2+1}}\cdot\bar{j} = \frac{-1}{\sqrt{4x^2+1}}$

For R: $y = x^2$, $x = 0$ to 3, $z = 0$ to 2

$$\iint_S \bar{F}\cdot\bar{N}\, ds = \iint_R \bar{F}\cdot\bar{N}\, \frac{dx\,dz}{|\bar{N}\cdot\bar{j}|} = \int_{x=0}^{3}\int_{z=0}^{2} \frac{2xy - 2}{\sqrt{4x^2+1}} \Big/ \frac{-1}{\sqrt{4x^2+1}}\, dx\,dz$$

$$= +2\int_{x=0}^{3}\int_{z=0}^{2} (xy+1)\, dx\,dz = +2\int_{x=0}^{3}\int_{z=0}^{2} (x^3+1)\, dx\,dz \quad [\because y = x^2]$$

$$= +2\int_{z=0}^{2}\left[\frac{x^4}{4} + x\right]_0^3 dz = +2\int_{z=0}^{2}\left(\frac{81}{4} - 3\right) dz$$

$$= +2\left[\frac{81}{4}z - 3z\right]_0^2$$

$$= +2\left[\frac{81}{4}(2) - 3(2)\right]$$

$$= +2\cdot\frac{81}{2} - 2(3)(2)$$

$$= +81 - 12 = +69\ m^3/\text{sec}$$

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Evaluate $\int_S \bar{F}\cdot\bar{N}\,ds$ where $\bar{F} = yz\,\bar{i} + zx\,\bar{j} + xy\,\bar{k}$ and 'S' is the surface of $x^2+y^2+z^2=1$ in the first octant.

Sol:-

$\bar{F} = yz\,\bar{i} + zx\,\bar{j} + xy\,\bar{k}$

Here 'S' is the surface of $x^2+y^2+z^2=1$ in the first octant.

$\bar{N}$ = unit outward normal $= \frac{\nabla\phi}{|\nabla\phi|}$

where $\phi = x^2+y^2+z^2-1$

$$\bar{N} = \frac{2x\bar{i}+2y\bar{j}+2z\bar{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{2(x\bar{i}+y\bar{j}+z\bar{k})}{2\sqrt{x^2+y^2+z^2}} = x\bar{i}+y\bar{j}+z\bar{k}$$

Let 'R' be the Projection of xy-plane $\int_S \bar{F}\cdot\bar{N}\,ds = \int_R \bar{F}\cdot\bar{N}\,\frac{dx\,dy}{|\bar{N}\cdot\bar{k}|}$

$$\int_S \bar{F}\cdot\bar{N}\,ds = \int_R \bar{F}\cdot\bar{N}\,\frac{dx\,dy}{|\bar{N}\cdot\bar{k}|}$$

$$= \int_R (yz\,\bar{i}+zx\,\bar{j}+xy\,\bar{k})\cdot(x\bar{i}+y\bar{j}+z\bar{k})\cdot\frac{dx\,dy}{|(x\bar{i}+y\bar{j}+z\bar{k})\cdot\bar{k}|}$$

$$= \int_R (xyz+xyz+xyz)\,\frac{dx\,dy}{z} = \int_R 3xy\,\frac{dx\,dy}{z}$$

$$= \int_R 3xy\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} (3xy)\,dy\,dx = \int_{x=0}^{1} 3x\left(\frac{y^2}{2}\right)_0^{\sqrt{1-x^2}}\,dx$$

$$= \int_0^1 3x\,\frac{(1-x^2)}{2}\,dx = \frac{1}{2}\int_0^1 (3x-3x^3)\,dx$$

$$= \frac{1}{2}\left[\frac{3x^2}{2}-\frac{3x^4}{4}\right]_0^1 = \frac{1}{2}\left[\frac{3}{2}-\frac{3}{4}\right]$$

$$= \frac{1}{2}\left[\frac{3}{4}\right] = \frac{3}{8}$$

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Volume Integrals: $\int_V \bar{F}\,dv$ is called volume integral of $\bar{F}$ over the region 'V'.

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Another form of volume integrals are $\int \phi\,dv$ and $\iiint_V (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})\,dx\,dy\,dz$.

① If $\bar{F} = 2xz\,\hat{i} - x\hat{j} + y^2\hat{k}$. Evaluate $\int_V \bar{F}\,dv$ where 'V' is the region bounded by the surfaces $x=0$, $x=2$, $y=0$, $y=6$, $z=x^2$, $z=4$.

Sol:- $\bar{F} = 2xz\,\hat{i} - x\hat{j} + y^2\hat{k}$

Here 'V' is the region bounded by the surfaces $x=0$, $x=2$, $y=0$, $y=6$, $z=x^2$, $z=4$

For V: $x = 0$ to 2, $y = 0$ to 6, $z = x^2$ to 4.

$$\iiint_V \bar{F}\,dv = \iiint_V (2xz\,\hat{i} - x\hat{j} + y^2\hat{k})\,dx\,dy\,dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{6}\left[\int_{z=x^2}^{4}(2xz\,\hat{i} - x\hat{j} + y^2\hat{k})\,dz\right]dx\,dy$$

$$= \int_{x=0}^{2}\int_{y=0}^{6}\left[2x\left(\frac{z^2}{2}\right)_{x^2}^{4}\hat{i} - x[z]_{x^2}^{4}\hat{j} + y^2[z]_{x^2}^{4}\hat{k}\right]dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{6}\left(16x - x^5\right)\hat{i} - (4x - x^3)\hat{j} + (4y^2 - x^2y^2)\hat{k}\,dy\,dx$$

$$= \int_{x=0}^{2}\left[(16xy - x^5y)_0^6\,\hat{i} - (4xy - x^3y)_0^6\,\hat{j} + \left(\frac{4}{3}y^3 - x^2\frac{y^3}{3}\right)_0^8\hat{k}\right]dx$$

$$= \int_{x=0}^{2}\left[(96x - 6x^5)\hat{i} - (24x - 6x^3)\hat{j} + (288 - 72x^2)\hat{k}\right]dx$$

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$= \left[96\frac{x^2}{2} - 6\frac{x^6}{6}\right]_0^2 \bar{i} - \left[\frac{12}{2}x^2 - \frac{6}{5}\frac{x^4}{2}\right]_0^2 \bar{j} + \left[288x - \frac{24}{4}x^3\right]_0^2 \bar{k}$

$= \left(48x^2 - x^6\right)_0^2 \bar{i} - \left(12x^2 - \frac{3}{2}x^4\right)_0^2 \bar{j} + \left(288x - 24x^3\right)_0^2 \bar{k}$

$= (192 - 64)\bar{i} - (48 - 24)\bar{j} + (576 - 192)\bar{k}$

$= 128\bar{i} - 24\bar{j} + 384\bar{k}$

② Evaluate $\iiint_V (2x+y)\,dv$ where 'V' is closed region bounded the cylinder $z = 4-x^2$ and the plane $x=0, y=0, y=2$ and

Sol:- 'V' is the closed region bounded the cylinder $z =$ and the plane $x=0$, $y=0$, $y=2$, and $z=0$

$z =$

$z =$

$x =$

For V: $z = 0$ to $4-x^2$

$y = 0$ to 2

$x = 0$ to 2

$$\iiint_V (2x+y)\,dv = \int_{x=0}^{2}\int_{y=0}^{2}\left[\int_{z=0}^{4-x^2}(2x+y)\,dz\right]dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2}\left[(2x+y)z\right]_0^{4-x^2} dy\,dx = \int_{x=0}^{2}\int_{y=0}^{2}(2x+y)(4-x^2)\,dy$$

$$= \int_{x=0}^{2}\left[(8x - 2x^3 + 4y - x^2y)\,dy\right]dx$$

$$= \int_{x=0}^{2}\left[8x[y]_0^2 - 2x^3[y]_0^2 + 4\left[\frac{y^2}{2}\right]_0^2 - x^2\left[\frac{y^2}{2}\right]_0^2\right]dx = \int_{x=0}^{2}(16x - 4x^3 + 8 - 2x^2)$$

$$= 16\left[\frac{x^2}{2}\right]_0^2 - 4\left[\frac{x^4}{4}\right]_0^2 + 8[x]_0^2 - 2\left[\frac{x^3}{3}\right]_0^2$$

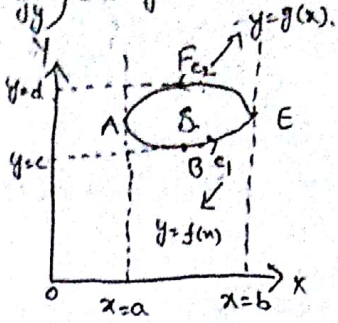
$$= 32 - 16 + 16 - \frac{16}{3}$$

$$= \frac{96-16}{3} = \frac{80}{3}$$

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Let 'S' be a closed region in xy-plane bounded by a simple closed curve 'C'. Let P and Q be continuous and differentiable scalar point functions of x and y in S. Then $\oint_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$.



Proof:- 'S' be a closed region in xy-plane enclosed by a closed curve "C".

P and Q are continuous and differentiable scalar point functions of x and y in S.

Let any line parallel to either co-ordinate axis cut 'C' in atmost two points.

Let 'S' lie between the lines $x=a$, $x=b$, $y=c$, $y=d$.

Let $y=f(x)$ be the curve "C_1" [i.e ABE] and

$y=g(x)$ " " " "C_2" [i.e EFA].

Generally $f(x) \le g(x)$.

Consider $\iint_S \frac{\partial P}{\partial y}\,dx\,dy = \int_{x=a}^{b} \int_{y=f(x)}^{y=g(x)} \left[\frac{\partial P}{\partial y}\,dy\right] dx = \int_{x=a}^{b} \left[P(x,y)\right]_{f(x)}^{g(x)} dx$.

$= \int_{x=a}^{b} [P(x,g) - P(x,f)]\,dx = \int_{x=a}^{b} P(x,g)\,dx - \int_{x=a}^{b} P(x,f)\,dx$.

$= -\int_{x=b}^{a} P(x,g)\,dx - \int_{x=a}^{b} P(x,f)\,dx$.

$= -\left[\int_{x=b}^{a} P(x,g)\,dx + \int_{x=a}^{b} P(x,f)\,dx\right]$

$= -\left[\int_{C_2} P(x,y)\,dx + \int_{C_1} P(x,y)\,dx\right]$.

$= -\left[\int_{C_1} P\,dx + \int_{C_2} P\,dx\right] = -\oint_C P\,dx$.

$\iint_S \frac{\partial P}{\partial y}\,dx\,dy = -\oint_C P\,dx \Rightarrow \oint_C P\,dx = -\iint_S \frac{\partial P}{\partial y}\,dx\,dy$. — (1)

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Hy prove that.

$$\oint_c Q\,dy = \iint_S \frac{\partial Q}{\partial x}\,dx\,dy \quad \text{— (II)}.$$

$$\oint_c P\,dx + \oint_c Q\,dy = -\iint_S \frac{\partial P}{\partial y}\,dx\,dy + \iint_S \frac{\partial Q}{\partial x}\,dx\,dy.$$

$$\oint_c P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Note ① :- Let "S" be closed region in yz-plane enclose by a curve 'c' Let P and Q continuous and differentiable scalar functions are y and z in S.

Then $\oint_c P\,dy + Q\,dz = \iint_S \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial z}\right) dy\,dz.$

Note ② :- Let 'S' be closed region zx-plane enclosed by a curve c. Let P and Q continuous and differentiable scalar functions are z and x in S.

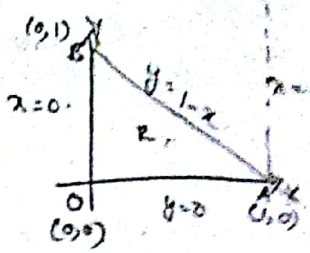
Then $\oint_c P\,dz + Q\,dx = \iint_S \left(\frac{\partial Q}{\partial z} - \frac{\partial P}{\partial x}\right) dz\,dx.$

z.

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1) Verify Green's theorem for $\oint_C (3x^2-8y^2)dx + (4y-6xy)dy$, where 'C' is the boundary of the region bounded by $x=0$, $y=0$, and $x+y=1$.



Sol:- $\oint_C (3x^2-8y^2)dx + (4y-6xy)dy$

By Green's theorem $\oint_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$P = 3x^2 - 8y^2$, $Q = 4y - 6xy \Rightarrow \frac{dP}{dy} = -16y$; $\frac{\partial Q}{\partial x} = -6y$.

$\frac{\partial Q}{\partial x} - \frac{dP}{dy} = -6y + 16y = 10y$.

and 'S' is the boundary of region bounded by $x=0$, $y=0$, $x+y=1$.

For S; $y = 0$ to $1-x$; $x = 0$ to 1.

$$\oint_C (3x-8y^2)dx + (4y-6xy)dy = \iint_S 10y\,dx\,dy = \int_{x=0}^{1} \int_{y=0}^{1-x} [10y\,dy]\,dx.$$

$$= \int_{x=0}^{1} [5y^2]_0^{1-x} = 5\int_{x=0}^{1} (1+x^2-2x)\,dx = 5\left[x + \frac{x^3}{3} - x^2\right]_0^1 = 5\left[1+\frac{1}{3}-1\right] = 5/3$$

The region R = OAB

Along OA, $y=0$, $dy=0$, x varies from 0 to 1.

$$\int_{OA} \bar{f}\cdot d\bar{r} = \int_0^1 3x\,dx = 3\frac{x^2}{2}\Big]_0^1 = \frac{3}{2}$$

Along AB, $x+y=1$, $y=1-x$, $dy=-dx$. x varies from 1 to 0.

$$\int_{AB} \bar{f}\cdot d\bar{r} = \int_1^0 [3x - 8(1-x)^2]\,dx - [4(1-x) - 6x(1-x)](-dx) = \int_1^0 (14x^2 + 2x - 12)\,dx$$

$$= \int_0^1 (14x^2 - 29x + 12)\,dx = 14\frac{x^3}{3} - 29\frac{x^2}{2} + 12x\Big]_0^1 = \frac{14}{3} - \frac{29}{2} + 12 = \frac{13}{6}$$

Along BO $x=0$, $y = 1$ to 0

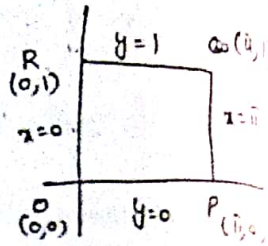
$$\int_{BO} \bar{f}\cdot d\bar{r} = \int_1^0 4y\,dy = 4\frac{y^2}{2}\Big]_1^0 = -\frac{4}{2} = -2$$

$$\int_C (3x-8y^2)dx + (4y-6xy)dy = \frac{3}{2} + \frac{13}{6} - 2 = \frac{5}{3}$$

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7) Verify Green's theorem for $\oint_C (x^2 - \cosh y)\,dx + (y + \sin x)\,dy$ where C is the rectangle with vertices $(0,0)$ $(\pi,0)$ $(\pi,1)$ $(0,1)$.



Sol:- By Green's theorem $\oint_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$.

Here $P = x^2 - \cosh y$, $Q = y + \sin x$.

$\frac{\partial P}{\partial y} = -\sinh y$ $\quad \frac{\partial Q}{\partial x} = \cos x$ $\quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \cos x + \sinh y$.

and 'S' is the surface of rectangle with vertices $(0,0)$ $(\pi,0)$ $(\pi,1)$ and $(0,1)$. For S; $x = 0$ to π, $y = 0$ to 1.

$$\oint_C (x^2 - \cosh y)\,dx + (y + \sin x)\,dy = \iint_S (\cos x + \sinh y)\,dx\,dy$$

$$= \int_{y=0}^{1} \left[\int_{x=0}^{\pi} (\cos x + \sinh y)\,dx\right] dy = \int_{y=0}^{1} \left[(\sin x)_0^{\pi} + \sinh y\,[x]_0^{\pi}\right] dy$$

$$= \int_{y=0}^{1} \left[(0-0) + (\pi - 0)\sinh y\right] dy = \int_{y=0}^{1} \pi \sinh y\,dy = \pi \cosh y\Big]_0^1$$

$$= \pi\left[\cosh 1 - \cosh 0\right]$$

$$= \pi\left[\cosh 1 - 1\right]$$

8) Find the area of a circle of radius a using Green's Theo

Sol:- Area $= \frac{1}{2}\int_C (x\,dy - y\,dx)$

choosing $x = a\cos\theta$, $y = a\sin\theta$ we get

$$A = \frac{1}{2}\int_0^{2\pi} (a^2\cos^2\theta + a^2\sin^2\theta)\,d\theta$$

$$= \frac{1}{2}a^2\int_0^{2\pi} d\theta = \pi a^2$$

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Problem: Apply Green's theorem to prove that the area (A) enclosed by a plane curve 'c' is $\frac{1}{2}\int_c x\,dy - y\,dx$. Hence find the area of an ellipse whose semi major and semi minor axes are of lengths a and b.

Sol:- By Green's theorem $\oint_c P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$.

Let $P = -y$, $Q = x$, $\frac{\partial P}{\partial y} = -1$, $\frac{\partial Q}{\partial x} = 1$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - (-1) = 2$

$\oint_c -y\,dx + x\,dy = \iint_S 2\,dx\,dy$.

$\int ds = \int dx\,dy$

$\int_S dx\,dy = S$.

$\frac{1}{2}\oint_c x\,dy - y\,dx = \iint_S dx\,dy$.

$S = \frac{1}{2}\oint_c x\,dy - y\,dx$

∴ The area bounded by a simple closed curve 'c' is

$\frac{1}{2}\oint_c x\,dy - y\,dx$.

Given that equation of ellipse is

$x = a\cos\theta$, $y = b\sin\theta$, $0 \le \theta \le 2\pi$

$dx = -a\sin\theta\,d\theta$, $dy = b\cos\theta\,d\theta$

$\theta : 0 \to 2\pi$

∴ The area of the given ellipse $= \frac{1}{2}\oint x\,dy - y\,dx$

$= \frac{1}{2}\int_0^{2\pi} a\cos\theta\,(b\cos\theta)\,d\theta - b\sin\theta\,(-a\sin\theta\,d\theta)$

$= \frac{1}{2}ab\int_0^{2\pi} (\cos^2\theta + \sin^2\theta)\,d\theta = \frac{1}{2}ab\,[\theta]_0^{2\pi}$

$= \frac{1}{2}ab\,[2\pi] = \pi ab$ sq. units.

Note: Area of the ellipse $= \pi ab$ sq. units.

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of the region defined by $y^2 = 8x$ and $x = 2$.

Sol:- By Green's theorem $\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$P = x^2 - 2xy, \quad Q = x^2y + 2$



$\frac{\partial P}{\partial y} = -2x, \quad \frac{\partial Q}{\partial x} = 2xy$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2xy + 2x$

and 'R' is the surface of the region bounded $y^2 = 8x$ and x

for $y = -\sqrt{8x}$ to $\sqrt{8x}$ $x = 0$ to 2.

$$\oint (x^2 - 2xy)dx + (x^2y + 2)dy = \iint_R (2xy + 2x)\,dx\,dy$$

$$= \int_{x=0}^{2} \left[\int_{y=-\sqrt{8x}}^{\sqrt{8x}} (2xy + 2x)\,dy \right] dx = \int_{x=0}^{2} \left[xy^2 + 2xy \right]_{-\sqrt{8x}}^{\sqrt{8x}} dx$$

$$= \int_{x=0}^{2} x(8x - 8x) + 2x\left[\sqrt{8x} + \sqrt{8x}\right] dx$$

$$= \int_{x=0}^{2} 4\sqrt{8}\, x^{3/2}\, dx = 4\sqrt{8}\left[\frac{x^{5/2}}{5/2}\right]_0^2$$

$$= 4\sqrt{8}\,\frac{2}{5}\,x^{5/2}\Big|_0^2 = \frac{4\cdot 2\sqrt{8}\; x^{5/2}}{5}\Big|_0^2$$

$$= \frac{16\sqrt{2}\cdot 2^{5/2}}{5} = \frac{16\sqrt{2}\cdot 2^2\sqrt{2}}{5}$$

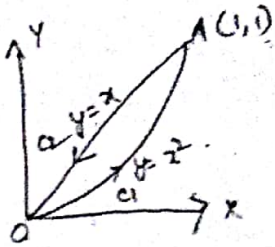
$$= \frac{128}{5}$$

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1) Verify Green's theorem in the plane for $\oint_C (xy+y^2)dx + x^2dy$.

where 'c' is the closed curve of the region bounded by $y=x$ and $y=x^2$



Sol:- By Green's theorem, we have

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dx\,dy.$$

The two curves $y=x$ and $y=x^2$ intersect at $O(0,0)$ and $A(1,1)$

$$\oint_C (xy+y^2)\,dx + x^2\,dy \cdot \text{ where } P = xy+y^2,\ Q = x^2.$$

$$\frac{dQ}{dx} = 2x, \quad \frac{dP}{dy} = x+2y.$$

$$\oint_C P\,dx + Q\,dy = \iint_R (2x-x-2y)\,dx\,dy = \iint_R (x-2y)\,dx\,dy = \int_0^1 \int_{y=x^2}^{x} (x-2y)\,dy\,dx$$

$$= \int_0^1 \left(xy - 2\frac{y^2}{2}\right)_{y=x^2}^{x} dx = \int_0^1 (x^2-x^3)\,dx = \left(\frac{x^3}{3} - \frac{x^4}{4}\right)_0^1 = -\frac{1}{20} \quad -(1)$$

consider the integral.

Along c, $y = x^2 \Rightarrow dy = 2x\,dx$

$$\oint_C (xy+y^2)\,dx + x^2\,dy$$

$$\int_0^1 (x(x^2)+x^4)\,dx + x^2(2x\,dx) = \int_0^1 (3x^3+x^4)\,dx = \frac{19}{20}$$

$\frac{3x^4}{4} + \frac{x^5}{5}\Big]_0^1$

$\frac{3}{4} + \frac{1}{5} = \frac{15+4}{20}$

Along $y=x$, $dy = dx$.

$$\int_1^0 (x(x)+x^2)\,dx + x^2\,dx = \int_1^0 3x^2\,dx = 3\frac{x^3}{3}\Big]_1^0 = 0-1 = -1$$

so $\oint_C (xy+y^2)\,dx + x^2\,dy = \frac{19}{20} - 1 = -\frac{1}{20}$ $-(2)$.

Here the Green's theorem is verified from the equality of (1) and (2).

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2) If 'c' is a simple closed curve in xy-plane not enclosing the origin, show that $\int_c \bar{F} \cdot d\bar{R} = 0$, where $\bar{F} = \frac{y\bar{i} - x\bar{j}}{x^2+y^2}$.

Soln:- $\int_c \bar{F} \cdot d\bar{R} = \int_c \frac{y\bar{i} - x\bar{j}}{x^2+y^2} \cdot (dx\bar{i} + dy\bar{j})$

$P dx + Q dy$

$\frac{dQ}{dy}$

$P = \frac{y}{x^2+y^2}$, $Q = \frac{-x}{x^2+y^2}$

$$\frac{dQ}{dx} = \frac{-(x^2+y^2) + x(2x)}{(x^2+y^2)^2}$$

$$\frac{dP}{dy} = \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2}$$

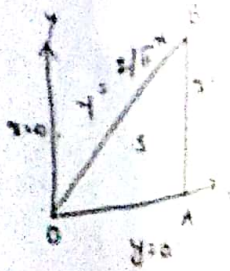
$$\frac{dQ}{dx} - \frac{dP}{dy} = -x^2 - y^2 + 2x^2 - \ldots$$

$$\frac{dQ}{dx} - \frac{dP}{dy} = 0.$$

$$\int_c \bar{F} \cdot d\bar{R} = \iint_S \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dx\, dy$$

$$= \iint_S \left[\frac{-(x^2+y^2) + x(2x)}{(x^2+y^2)^2} - \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2} \right] dx\, dy = 0.$$

3) Evaluate using Green's theorem, evaluate $\int_c (y - \sin x)\, dx + \cos x\, dy$ where 'c' is the plane triangle enclosed by the lines $y = 0$, $x = \pi/2$ and $y = \frac{2x}{\pi}$.



Sol:- By Green's theorem

$$\oint_c P\,dx + Q\,dy = \iint_S \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dx\, dy.$$

Here $P = y - \sin x$, $Q = \cos x$. $\frac{dP}{dy} = 1$, $\frac{dQ}{dx} = -\sin x$.

$$\frac{dQ}{dx} - \frac{dP}{dy} = -\sin x - 1.$$

and 'S' is the surface of a triangle bounded by the lines $y = 0$, $x = \pi/2$, $y = 2x/\pi$. For S: $y = 0$ to $\frac{2x}{\pi}$.

$x = 0$ to $\pi/2$.

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$$\oint_C (y+\sin x)\,dx + \cos x\,dy = \iint_R (-\sin x - 1)\,dx\,dy.$$

$$= \int_{x=0}^{\pi/2}\left[\int_{y=0}^{2x/\pi} (-\sin x - 1)\,dy\right]dx = \int_{x=0}^{\pi/2} (-\sin x - 1)\,y\Big]_0^{2x/\pi}\,dx.$$

$$= \int_{x=0}^{\pi/2} (-\sin x - 1)\frac{2x}{\pi}\,dx = \int_{x=0}^{\pi/2} -\frac{2x}{\pi}\sin x\,dx - \int_{x=0}^{\pi/2}\frac{2x}{\pi}\,dx.$$

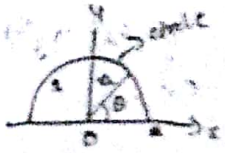
$$= \frac{2}{\pi}\int_{x=0}^{\pi/2} x\,d(\cos x) - \frac{1}{\pi}\left[x^2\right]_0^{\pi/2} \qquad \int u\,dv = uv - \int v\,du.$$

$$= \frac{2}{\pi}\left[(x\cos x)_0^{\pi/2} - \int_0^{\pi/2}\cos x\,dx\right] - \frac{1}{\pi}\frac{\pi^2}{4}.$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2}\cos\frac{\pi}{2} - [\sin x]_0^{\pi/2}\right] - \frac{\pi}{4}$$

$$= \frac{2}{\pi}\left[0 - (1-0)\right] - \frac{\pi}{4} = -\frac{2}{\pi} - \frac{\pi}{4}.$$

(4) Apply Green's theorem to evaluate $\int_C (2x^2-y^2)\,dx + (x^2+y^2)\,dy$, where 'c' is the boundary of the area enclosed by the x-axis and the upper-half of the circle $x^2+y^2=a^2$.



Sol:- $$\int_C (2x^2-y^2)\,dx + (x^2+y^2)\,dy = \iint_S \left[\frac{d}{dx}(x^2+y^2) - \frac{d}{dy}(2x^2-y^2)\right]dx\,dy.$$

$$= 2\iint_S (x+y)\,dx\,dy = 2\int_0^a\int_0^{\pi} r(\cos\theta+\sin\theta)\,r\,dr\,d\theta$$

changing the polar coordinates (r,θ), r varies from 0 to a and θ varies from 0 to π.

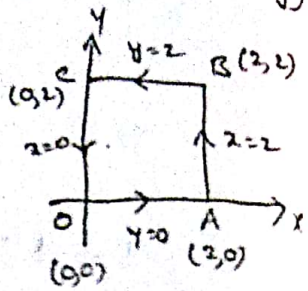
$$= 2\int_0^a r^2\,dr \int_0^{\pi}(\cos\theta+\sin\theta)\,d\theta = 2\frac{a^3}{3}(1+1) = \frac{4a^3}{3}.$$

$$\int_0^{\pi}(\cos\theta+\sin\theta)\,d\theta = [\sin\theta - \cos\theta]_0^{\pi}$$
$$= [\sin\pi - \cos\pi] - [\sin 0 - \cos 0]$$
$$= [0-(-1)] - [0-1]$$
$$= 1+1$$

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Verify Green's theorem in the plane for $\oint_C (x^2-xy^3)\,dx + (y^2-2xy)\,dy$



'C' is a square with vertices (0,0) (2,0) (2,2) (0,2)

$P = x^2-xy^3, \quad Q = y^2-2xy.$

$\frac{\partial Q}{\partial x} = -2y \quad \frac{\partial P}{\partial y} = -3xy^2.$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -2y+3xy^2.$

$\oint_C (x^2-xy^3)\,dx + (y^2-2xy)\,dy = \iint_R (-2y+3xy^2)\,dx\,dy.$

1) Along OA: $y=0$ hence $dy=0$, x varies from 0 to 2.

$$\int_{OA} (x^2-xy^3)dx + (y^2-2xy)dy = \int_{x=0}^{2} x^2\,dx = \frac{x^3}{3}\Big]_0^2 = 8/3$$

2) Along AB: $x=2$, $dx=0$, y varies from 0 to 2.

$$\int_{AB} (x^2-xy^3)\,dx + (y^2-2xy)\,dy = \int_{y=0}^{2} (y^2-4y)\,dy = \frac{y^3}{3} - 4\frac{y^2}{2}\Big]_0^2 = \frac{8}{3} - 8 = -\frac{16}{3}$$

3) Along BC: $y=2$, $dy=0$, x varies from 2 to 0.

$$\int_{BC} (x^2-xy^3)\,dx + (y^2-2xy)\,dy = \int_{x=2}^{0} (x^2-2x^3)\,dx = \frac{x^3}{3} - 2\frac{x^4}{4}\Big]_2^0$$

$$= -\frac{8}{3} + 16 = \frac{40}{3}$$

4) Along CO: $x=0$, $dx=0$, y varies from 2 to 0.

$$\int_{CO} (x^2-xy^3)\,dx + (y^2-2xy)\,dy = \int_{y=2}^{0} y^2\,dy = \frac{y^3}{3}\Big]_2^0 = -\frac{8}{3}.$$

$$\therefore \int_C (x^2-xy^3)\,dx + (y^2-2xy)\,dy = \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3} = 8.$$

Now RHS $\iint_R (-2y+3xy^2)\,dy\,dx = \int_{x=0}^{2}\int_{y=0}^{2} (-2y+3xy^2)\,dy\,dx$

$$= \int_{x=0}^{2} \left(-2\frac{y^2}{2} + 3x\frac{y^3}{3}\right)_0^2 dx = \int_{x=0}^{2} (-4+8x)\,dx$$

$$= -4x + 8\frac{x^2}{2}\Big]_0^2 = -8+16 = 8$$

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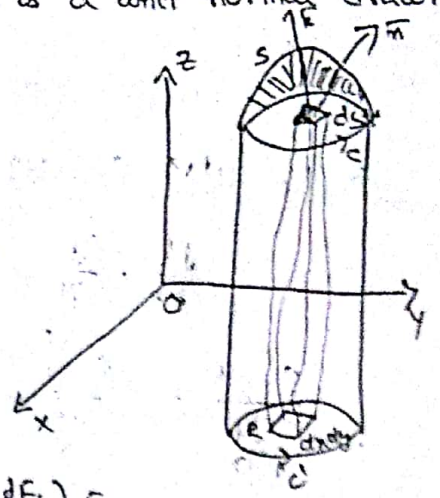
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Stokes Theorem (Relation between Line and Surface Integrals)

Let $\bar{F}$ be any vector point function having continuous partial derivatives. Let 'S' be an open surface bounded by closed curve C, then

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds$$

where $\bar{n}$ is a unit normal drawn outward at any point to S.



Proof:- Let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

$$\text{curl}\,\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\bar{i} + \bar{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\bar{k}$$

Let the unit normal vector makes angles α, β, γ with positive direction of x, y, z axes, then $\bar{n} = \cos\alpha\,\bar{i} + \cos\beta\,\bar{j} + \cos\gamma\,\bar{k}$.

Also $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \Rightarrow d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$$(\nabla\times\bar{F})\cdot\bar{n} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma$$

and $\bar{F}\cdot d\bar{r} = (F_1\bar{i} + F_2\bar{j} + F_3\bar{k})\cdot(dx\bar{i} + dy\bar{j} + dz\bar{k}) = F_1\,dx + F_2\,dy + F_3\,dz$

Now Stokes theorem can also be written as,

$$\oint_C (F_1\,dx + F_2\,dy + F_3\,dz) = \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma\right] ds \quad \text{— (1)}$$

Let $z = f(x, y)$ be the equation of the surface S where projection on the xy plane is R. Then the projection of curve C on the xy-plane is the curve C' which is the boundary of the region R.

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$$\oint_C F_1(x,y,z)\,dx = \oint_{C_1} F_1(x,y,f)\,dx + \oint_{C_1} F_1(x,y,f)\,dy.$$

$$= -\iint_R \frac{\partial}{\partial y} F_1(x,y,f)\,dx\,dy \quad [\text{by Green's th.}]$$

$$= -\iint_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial f}{\partial y}\right) dx\,dy \quad -(2).$$

Now the d.r's of the normal $\bar{n}$ to the surface $z = f(x,y)$ are

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1.$$

$$\frac{\cos\alpha}{\frac{\partial f}{\partial x}} = \frac{\cos\beta}{\frac{\partial f}{\partial y}} = \frac{\cos\gamma}{-1} \Rightarrow \frac{\partial f}{\partial y} = -\frac{\cos\beta}{\cos\gamma}.$$

Since $dx\,dy$ is the projection of ds on the xy plane.

$$dx\,dy = ds\cos\gamma$$

By (2) $$\oint_C F_1(x,y,z)\,dx = -\iint_S \left(\frac{\partial F_1}{\partial y} - \frac{\partial F_1}{\partial z}\frac{\cos\beta}{\cos\gamma}\right) ds\cos\gamma$$

$$= \iint_S \left(\frac{\partial F_1}{\partial y}\cos\gamma - \frac{\partial F_1}{\partial z}\cos\beta\right) ds.$$

$$= \iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right) ds \quad -(3).$$

By we can prove that.

$$\oint_C F_2(x,y,z)\,dy = \iint_S \left(\frac{\partial F_2}{\partial x}\cos\gamma - \frac{\partial F_2}{\partial z}\cos\alpha\right) ds \quad -(4).$$

$$\oint_C F_3(x,y,z)\,dz = \iint_S \left(\frac{\partial F_3}{\partial y}\cos\alpha - \frac{\partial F_3}{\partial x}\cos\beta\right) ds \quad -(5).$$

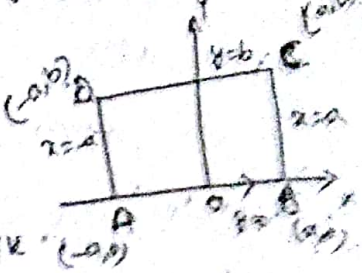
Adding (3) (4) and (5) and using (1) we get.

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds.$$

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1) Verify Stoke's theorem for $\bar{F} = (x^2+y^2)\bar{i} - 2xy\bar{j}$ taken around the rectangle bounded by the lines $x = \pm a$, $y = 0$, $y = b$.



Sol:- By Stoke's theorem $\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\bar{F}\cdot\bar{n}\,ds$.

$$\text{curl}\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = \bar{k}(-2y-2y) = -4y\bar{k}$$

vector $\perp$ to xy-plane $\bar{n} = \bar{k}$

$$\iint_S (\nabla\times\bar{F})\cdot\bar{n}\,ds = \int_{y=0}^{y=b}\int_{x=-a}^{x=a} (-4y\bar{k})\cdot\bar{k}\,dx\,dy = -4\int_{y=0}^{b}\int_{x=-a}^{a} y\,dx\,dy$$

$$= -4\int_{y=0}^{b}[xy]_{-a}^{a}\,dy = -4\int_{y=0}^{b} 2ay\,dy$$

$$= -4[ay^2]_0^b = -4ab^2$$

Along AB; $y=0$, $dy=0$.

x varies from $-a$ to a.

$$\oint_C \bar{F}\cdot d\bar{r} = \int (x^2+y^2)\,dx - 2xy\,dy = \int_{-a}^{a} x^2\,dx = \left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{2a^3}{3}$$

Along BC $x=a$, $dx=0$, y varies from 0 to b.

$$\oint_C \bar{F}\cdot d\bar{r} = \int_C (x^2+y^2)\,dx - 2xy\,dy = -\int_C 2ay\,dy = -2a\left[\frac{y^2}{2}\right]_0^b = -ab^2$$

Along CD $y=b$, $dy=0$, x varies from a to $-a$.

$$\oint_C \bar{F}\cdot d\bar{r} = \int_a^{-a} (x^2+b^2)\,dx = \left[\frac{x^3}{3} + b^2x\right]_a^{-a} = -\frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2$$

$$= -\frac{2a^3}{3} - 2ab^2$$

Along DA: $x=-a$, $dx=0$, y varies from b to 0.

$$\int_C \bar{F}\cdot d\bar{r} = \int_b^0 2ay\,dy = ay^2\Big]_b^0 = -ab^2$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r} + \int_{CD} \bar{F}\cdot d\bar{r} + \int_{DA} \bar{F}\cdot d\bar{r}$$

$$= \frac{2a^3}{3} - ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2$$

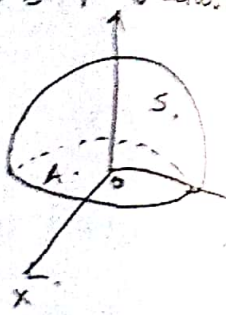
$$= -4ab^2$$

Stoke's theorem is verified.

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Verify Stoke's theorem for the vector field $\bar{F} = (2x-y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$ over the upper half surface $x^2+y^2+z^2=1$, bounded by c's Projection on xy-plane.



Sol:- The projection of the upper half of given sphere on the xy-plane ($z=0$) is the circle $c\,[x^2+y^2=1]$

$$\oint_c \bar{F}\cdot d\bar{r} = \oint_c (2x-y)\,dx - yz^2\,dy - y^2z\,dz.$$

Put $x=\cos\theta$, $y=\sin\theta$, $dx=-\sin\theta\,d\theta$

$$= \oint_c (2x-y)\,dx.$$

$$= \int_{\theta=0}^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta\,d\theta) = \int_{\theta=0}^{2\pi} (\sin 2\theta + \sin^2\theta)\,d\theta.$$

$$= 4\int_0^{\pi/2} \frac{1-\cos 2\theta}{2}\,d\theta = 4\left[\frac{1}{2}\theta - \frac{\sin 2\theta}{2}\right]$$

$$= 4\,\frac{1}{2}\,\frac{\pi}{2} = \pi. \quad -(1)$$

Now $\operatorname{curl}\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$

$$= (-2yz + 2yz)\bar{i} + 0\cdot\bar{j} + \bar{k}(0+1) = \bar{k}$$

$$\int \operatorname{curl}\bar{F}\cdot\bar{n}\,ds = \int_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{k}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

where A is projection of S on xy-plane and $ds = \frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$

$$= \int_R dx\,dy = \text{area of circle } c = \pi r^2 \quad [\because r=1]$$

$$= \pi. \quad -(2).$$

Hence the Stoke's theorem is verified from the equality of (1) and (2).

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5) Apply Stoke's theorem for calculating $\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F} = y\bar{i} + z\bar{j} + x\bar{k}$ and 'C' is the curve of intersection $x^2+y^2+z^2 = a^2$, $x+z = a$.

Sol:- Given $\bar{F} = y\bar{i} + z\bar{j} + x\bar{k}$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = \bar{i}(0-1) + \bar{j}(0-1) + \bar{k}(0-1)$$

$$= -\bar{i} - \bar{j} - \bar{k}.$$

$\bar{N}$ = unit normal to the plane $(x+z=a) = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\bar{i}+\bar{k}}{\sqrt{2}}$.

By Stoke's theorem, $\int_C \bar{F}\cdot d\bar{r} = \int_S (\nabla\times\bar{F})\cdot\bar{n}\, ds$.

$$= \int_S (-\bar{i}-\bar{j}-\bar{k})\cdot\frac{\bar{i}+\bar{k}}{\sqrt{2}}\, ds.$$

$$= -\sqrt{2}\int_S ds = -\sqrt{2}\,(\text{area of } S)$$

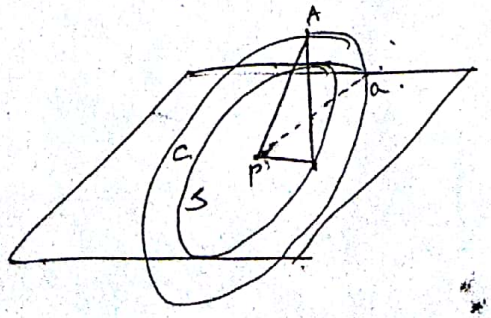
$$= -\sqrt{2}\,(\text{area of given circle})$$

$$= -\sqrt{2}\,\pi r^2 \text{ where } r \text{ is radius of given circle}$$

Now $r^2 = a^2 - p^2$ where 'p' is the perpendicular distance of centre (0,0,0) of the sphere from the plane $(x+z-a=0)$ $= \left|\frac{-a}{\sqrt{1+0+1}}\right| = \left(\frac{a}{\sqrt{2}}\right)$

$\therefore r^2 = a^2 - \frac{a^2}{2} = \frac{a^2}{2}$

The required value of given integral is $-\sqrt{2}\left(\pi\frac{a^2}{2}\right) = -\frac{\pi a^2}{\sqrt{2}}$.

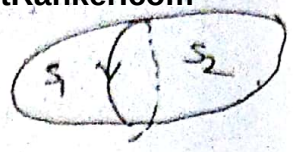


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6) If S be any closed surface, PT $\int \text{curl}\,\vec{f} \cdot \vec{n}\, dS = 0$.

Sol:- Cut open the surface 'S' by any plane and let S_1, S_2 denote it's upper and lower portions.



Let c be the common curve bounding both these portions.

$$\int_S \nabla \times \vec{F} \cdot \vec{n}\, dS = \int_{S_1} \nabla \times \vec{F} \cdot \vec{n}\, dS + \int_{S_2} (\nabla \times \vec{F}) \cdot \vec{n}\, dS$$

$$= \int_c \vec{F} \cdot d\vec{r} - \int_c \vec{F} \cdot d\vec{r} = 0.$$

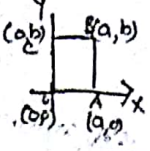
On applying Stoke's theorem. The second integral is negative because it is traversed in a direction opposite to that of

2.

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Stokes theorem for the vector field (i) $\bar{F} = (x^2-y^2)\bar{i} + 2xy\bar{j}$ over the box bounded by the planes $x=0, x=a, y=0, y=b, z=0, z=c$, if the face $z=0$ is cut.



Soln: $\bar{F} = (x^2-y^2)\bar{i} + 2xy\bar{j}$

Along OA; $y=0, dy=0$, x varies from 0 to a

$$\int_0^a \bar{f}\cdot d\bar{r} = \int_0^a x^2\,dx = \frac{a^3}{3}$$

Along AB, $x=a, dx=0$, y varies from 0 to b.

$$\int_0^b \bar{f}\cdot d\bar{r} = \int_0^b (a^2-y^2)(0) + 2ay\,dy = \int_0^b 2ay\,dy$$
$$= ab^2$$

Along BC, $y=b, dy=0$, x varies from a to 0.

$$\int_a^0 \bar{f}\cdot d\bar{r} = \int_a^0 (x^2-b^2)\,dx = -\frac{a^3}{3} + ab^2$$

Along CO, $x=0, dx=0$, y varies from b to 0

$$\int_b^0 \bar{f}\cdot d\bar{r} = \int_b^0 0\,dy = 0$$

$$\int_c \bar{f}\cdot d\bar{r} = \int_0^a \bar{f}\cdot d\bar{r} + \int_0^b \bar{f}\cdot d\bar{r} + \int_a^0 \bar{f}\cdot d\bar{r} + \int_b^0 \bar{f}\,d\bar{r}$$
$$= \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 + 0 = 2ab^2$$
$$= \text{L.H.S}$$

$$\iint_S (\nabla\times\bar{F})\cdot\bar{n}\,ds$$

$$\nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-y^2 & 2xy & 0 \end{vmatrix}$$

$$= \bar{i}(0) + \bar{j}(0-0) + \bar{k}(2y+2y)$$
$$= 4y\bar{k}$$

$$\int_0^a\int_0^b 4y\bar{k}\cdot\bar{k}\,dx\,dy = 4\int_0^a \left(\frac{y^2}{2}\right)_0^b dx$$

$$= 4\int_0^a \frac{b^2}{2}\,dx = 4\cdot\frac{b^2x}{2}\Big]_0^a$$
$$= 2ab^2 = \text{R.H.S}$$

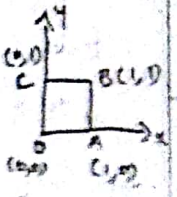
$$\therefore \int_c \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\bar{n}\,ds.$$

Stoke's theorem is verified.

(ii) $\bar{F} = (z^2, 5x, 0)$ and $S: 0\le x\le 1, 0\le y\le 1, z=1$

$$\bar{F} = z^2\bar{i} + 5x\bar{j} + 0\cdot\bar{k}$$

$z=1$



$$\bar{F} = \bar{i} + 5x\bar{j}$$

Along OA, $y=0, dy=0$, x varies from 0 to 1

$$\int_{OA}^{+} \bar{f}\cdot d\bar{r} = \int_0^1 1\,dx = 1$$

Along AB, $x=1, dx=0$, y varies from 0 to 1

$$\int_{AB}^{+} \bar{f}\cdot d\bar{r} = \int_0^1 5\,dy = 5y\Big]_0^1 = 5$$

Along BC, $y=1, dy=0$, x varies from 1 to 0

$$\int_{BC} \bar{f}\cdot d\bar{r} = \int_1^0 1\,dx = -1$$

Along CO, $x=0, dx=0$, y varies from 1 to 0

$$\int_{CO} \bar{F}\cdot d\bar{r} = \int_1^0 5(0)\,dy = 0$$

$$\int_c \bar{F}\cdot d\bar{r} = \int_{OA} \bar{F}\cdot d\bar{r} + \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r} + \int_{CO} \bar{F}\,d\bar{r}$$
$$= 1 + 5 - 1 + 0 = 5$$

$$\nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2 & 5x & 0 \end{vmatrix}$$

$$= \bar{i}(0) + \bar{j}(2z) + \bar{k}(5) = 2z\bar{j} + 5\bar{k}$$

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along z-axis

$$\int_0^1\int_0^1 5\,dx\,dy = 5.$$

$$\therefore \int_c \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\bar{n}\,ds.$$

---

(2) Verify Stoke's theorem for a vector field defined by $\bar{F} = -y^3\bar{i} + x^3\bar{j}$ in the region $x^2+y^2\le 1$, $z=0$.

Sol:- By Stoke's theorem $\oint_c \bar{F}\cdot d\bar{r} = \int (\nabla\times\bar{F})\cdot\bar{n}\,ds$

Here $\bar{F} = -y^3\bar{i} + x^3\bar{j}$

$$\nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ d/dx & d/dy & d/dz \\ -y^3 & x^3 & 0 \end{vmatrix}$$

$$= \bar{i}(0-0) + \bar{j}(0-0) + \bar{k}(3x^2+3y^2).$$

and 'S' is the circular disc

$$x^2+y^2\le 1,\ z=0.$$

For S:- $\bar{n} = \bar{k}$, $x = r\cos\theta$, $y = r\sin\theta$, $z = 0$.

$ds = dx\,dy$.

$dx\,dy = r\,dr\,d\theta$

$r = 0$ to 1, $\theta = 0$ to 2π.

$$\oint_c \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\bar{n}\,ds.$$

$$\iint_S (\nabla\times\bar{F})\cdot\bar{n}\,ds = \iint_S (3x^2+3y^2)\bar{k}\cdot\bar{k}\,dx\,dy.$$

$$= \iint_S (3x^2+3y^2)\,dx\,dy.$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} (3r^2\cos^2\theta + 3r^2\sin^2\theta)\,r\,dr\,d\theta.$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} 3r^3\,dr\,d\theta = \int_0^{2\pi} \frac{3r^4}{4}\Big]_0^1 d\theta$$

$$= \frac{3}{4}\int_{\theta=0}^{2\pi} d\theta = \frac{3}{4}\theta\Big]_0^{2\pi} = \frac{3}{4}\cdot 2\pi = \frac{3\pi}{2}$$

Verification:- $\bar{F} = -y^3\bar{i} + x^3\bar{j}$

and 'c' is boundary of a circular disc $x^2+y^2\le 1$, $z=0$.

For c: $x = 1\cdot\cos\theta$, $y = 1\cdot\sin\theta$

$dx = -\sin\theta\,d\theta$, $dy = \cos\theta\,d\theta$

$z=0$, $dz=0$, $\theta = 0$ to 2π

$$\oint_c \bar{F}\cdot d\bar{r} = \oint_c (-y^3\bar{i} + x^3\bar{j})\cdot(dx\,\bar{i} + \dots$$

$$= \oint_c -y^3\,dx + x^3\,dy.$$

$$= \int_0^{2\pi} -\sin^3\theta(-\sin\theta\,d\theta) + \cos^3\theta\cos\dots$$

$$= \int_0^{2\pi} (\sin^4\theta + \cos^4\theta)\,d\theta = \int_0^{2\pi} (\sin^2\theta + \cos^2\theta)^2 - 2\times\frac{1}{2}\sin^2\dots$$

$$= \int_{\theta=0}^{2\pi} \left(1 - \frac{1}{2}\sin^2 2\theta\right) d\theta = \int_{\theta=0}^{2\pi} 1 - \frac{1}{2}\left(\frac{1-\dots}{2}\right)$$

$$= \theta - \frac{1}{4}\theta + \frac{1}{4}\frac{\sin 4\theta}{4}\Big]_{\theta=0}^{2\pi}$$

$$= 2\pi - \frac{1}{4}2\pi + \frac{1}{16}\sin 4(2\pi)$$

$$= \frac{3\pi}{2} + 0 = 3\pi/2$$

Stoke's theorem is verified

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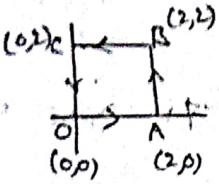
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Stoke's theorem for

$\bar{F} = (y-z+2)\bar{i} + (yz+4)\bar{j} - xz\bar{k}$ where

'S' is the surface of the cube

$x=0, y=0, z=0, x=2, y=2, z=2$ above the xy-plane.



Soln By Stoke's theorem

$$\int_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\bar{n}\, ds.$$

Along $\vec{OA}$, $y=0$, $dy=0$, $z=2$, x varies from 0 to 2

$$\int_{OA} \bar{F}\cdot d\bar{r} = \int_0^2 (0-2+2)\,dx = 0.$$

Along $\vec{AB}$, $x=2$, $dx=0$, $z=2$, y varies from 0 to 2

$$\int_{AB} \bar{F}\cdot d\bar{r} = \int_0^2 (2y+4)\,dy = \left[y^2+4y\right]_0^2 = 4+8 = 12$$

Along $\vec{BC}$, $y=2$, $z=2$, $dy=0$, x varies from 2 to 0

$$\int_{BC} \bar{F}\cdot d\bar{r} = \int_2^0 2\,dx = 2x\Big]_2^0 = -4$$

Along $\vec{CO}$, $x=0$, $dx=0$, $z=2$, y varies from 2 to 0

$$\int_{CO} \bar{F}\cdot d\bar{r} = \int_2^0 (2y+4)\,dy = 2y^2+4y\Big]_2^0 = -12$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_{OA} \bar{F}\cdot d\bar{r} + \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r} + \int_{CO} \bar{F}\cdot d\bar{r}$$

$$= 0 + 12 - 4 - 12 = -4$$

Verification: $\iint_S (\nabla\times\bar{F})\cdot\bar{n}\, ds$

$$\nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y-z+2 & yz+4 & -xz \end{vmatrix}$$

$$= \bar{i}(-y) + \bar{j}(-1+z) + \bar{k}(0-1)$$

$$= -y\bar{i} + (z-1)\bar{j} - \bar{k}$$

$$\iint_S (\nabla\times\bar{F})\cdot\bar{n}\, ds = \int_0^2\int_0^2 (-y\bar{i}+(z-1)\bar{j}-\bar{k})\cdot\bar{k}\, dx\, dy$$

$$= \int_0^2\int_0^2 -dx\,dy = -\int_0^2 dx\, y\Big]_0^2$$

$$= -2\int_0^2 dx = -2x\Big]_0^2 = -4.$$

Stoke's theorem verified.

4) Evaluate $\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F} = y\bar{i} + xz^2\bar{j} - xy^2\bar{k}$

C is the circle $x^2+y^2=4$, $z=1.5$.

By Stoke's theorem $\oint_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\bar{n}\, ds$

$$\nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & xz^2 & -xy^2 \end{vmatrix} = \bar{i}(-2xy-2xz) + \bar{k}(z^2-1).$$

$$(\nabla\times\bar{F})\cdot\bar{n}\, ds = (\nabla\times\bar{F})\cdot\bar{k}\, ds = (z^2-1)\, ds.$$

Let $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$.

$$\iint_S (\nabla\times\bar{F})\cdot\bar{n}\, ds = \int_{r=0}^{2}\int_{\theta=0}^{2\pi} \left[\left(\frac{3}{2}\right)^2 - 1\right] r\,dr\,d\theta.$$

$$= \frac{19}{4}\int_0^2 r\,dr \cdot \pi = \frac{19}{4}\left(\frac{r^2}{2}\right)_0^2 \pi$$

$$= \frac{19}{4}\times\frac{4}{2}\times\pi$$

$$= \frac{19}{2}\pi$$

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5) Evaluate by Stoke's theorem

$\oint_C (yz\,dx + zx\,dy + xy\,dz)$ where C is the circle $x^2+y^2=1$, $z=y^2$.

Sol:- By Stoke's theorem

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\bar{n}\,dS.$$

$\bar{F} = yz\bar{i} + zx\bar{j} + xy\bar{k}$

$$\nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & zx & xy \end{vmatrix}$$

$= \bar{i}(x-x) + \bar{j}(y-y) + \bar{k}(z-z)$

$= \bar{0}$

$\iint_S \bar{0}\cdot\bar{n}\,dS = 0$

6) If 'S' be the surface of the sphere $x^2+y^2+z^2=1$, prove that

$\iint_S (\nabla\times\bar{F})\cdot d\bar{S} = 0$

Sol:- Let S_1 and S_2 be the upper and lower portions of the sphere $x^2+y^2+z^2=1$ by any plane. Let 'c' be the boundary for S_1 and S_2.

$\iint_{S_1} (\nabla\times\bar{F})\cdot\bar{n}\,dS = \iint_{S_2} (\nabla\times\bar{F})\cdot\bar{n}\,dS$

Reversing the direction of S_2 in the direction of S_1

$\iint_S (\nabla\times\bar{F})\cdot\bar{n}\,dS - \iint_{S_2} (\nabla\times\bar{F})\cdot\bar{n}\,dS$

Applying Stoke's theorem

$$\iint_S (\nabla\times\bar{F})\cdot\bar{n}\,ds = \int_C \bar{F}\cdot d\bar{r} - \int_C \bar{F}\cdot d\bar{r}$$

$= 0$.

7) Prove that $\int_C (\bar{A}\times\bar{R})\cdot d\bar{R} = 2\bar{A}$, $\bar{A}$ being any constant vector, and deduce that $\oint_C \bar{R}\times d\bar{R}$ is twice the vector area of the surface enclosed

Sol:- By Stoke's theorem

$$\int_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\bar{n}\,ds$$

Here $\bar{F} = \bar{A}\times\bar{R}$, $\because \bar{A}$ is constant

$$\int_C (\bar{A}\times\bar{R})\cdot d\bar{R} = \iint_S \nabla\times(\bar{A}\times\bar{R})\cdot d\bar{s}$$

$\nabla\times(\bar{A}\times\bar{R}) = (\nabla\cdot\bar{R})\bar{A} - (\nabla\cdot\bar{A})\bar{R}$

$= 3\bar{A} - \bar{A} = 2\bar{A}$

Here $\bar{F} = \bar{A}\times\bar{R}$ where $\bar{A}$ is constant

$\oint_C (\bar{A}\times\bar{R})\cdot d\bar{R} = \int_S \nabla\times(\bar{A}\times\bar{R})\cdot\bar{n}\,ds$

$\oint_C \bar{A}\cdot(\bar{R}\times d\bar{R}) = \int_S (\nabla\cdot\bar{R})\bar{A} - (\nabla\cdot\bar{A})\bar{R}\cdot\bar{n}\,ds$

$\bar{A}\cdot\oint_C (\bar{R}\times d\bar{R}) = \int_S (3\bar{A} - \bar{A})\cdot\bar{n}\,ds$

$= \int_S 2\bar{A}\cdot\bar{n}\,ds$

$= \int_S \bar{A}\cdot 2\bar{n}\,ds$

$\oint_C (\bar{R}\times d\bar{R}) = \int_S 2\bar{n}\,ds$

$\oint_C (\bar{R}\times d\bar{R}) = 2\int_S d\bar{s}$

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Scalar point function...

use stoke's theorem to prove that (i) curl (grad ϕ) = $\bar{0}$ (ii) div curl $\bar{F}$ = 0

(i) Let $\bar{F} = \nabla\phi$, so by stoke's theorem

$$\int_S (\nabla \times \bar{F}) \cdot \bar{n}\, ds = \int_S (\nabla \times \nabla\phi) \cdot \bar{n}\, ds.$$

$$= \oint \nabla\phi \cdot d\bar{r}$$

$$= \oint_c d\phi = 0.$$

$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) dx\,dy\,dz$

$\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = d\phi$

also circulation $\oint_c d\phi = 0$ along any closed curve being initial and end points are one and the same.

$\int_S (\nabla \times \nabla\phi) \cdot \bar{n}\, ds = 0$ over the surface S, which implies that

curl (grad ϕ) = $\bar{0}$

(ii) Let 'V' be any volume enclosed by a closed curve S. Then by Gauss's divergence theorem

$$\iiint_V \nabla \cdot (\text{curl } \bar{F})\, dv = \iint_S (\text{curl } \bar{F}) \cdot \bar{n}\, ds. \quad -(1)$$

Divide the surface S into two portions S_1 and S_2 by a closed curve C



$$\iint_S (\text{curl } \bar{F}) \cdot \bar{n}\, ds$$

$$\iint_{S_1} (\text{curl } \bar{F}) \cdot \bar{n}\, ds_1 + \iint_{S_2} (\text{curl } \bar{F}) \cdot \bar{n}\, ds_2$$

$$= \int_C \bar{F} \cdot d\bar{r} - \int_C \bar{F} \cdot d\bar{r} = 0$$

By stoke's theorem, since the integrals along the boundaries of S_1 and S_2 are opposite

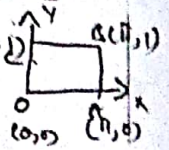
$$(1) \Rightarrow \iiint \nabla \cdot (\text{curl } \bar{F})\, dv = 0$$

Since this is true for all volume elements V, we have

$\nabla \cdot \text{curl } \bar{F} = 0 \Rightarrow dv\,(\text{curl } \bar{F}) = 0$.

2.

9) Evaluate $\oint_c (\sin z\, dx - \cos x\, dy + \sin y\, dz$ where 'c' is the boundary of the rectangle $0 \le x \le \pi$, $0 \le y \le 1$, $z = 3$.



Sol:- Along OA, $y = 0$, $dy = 0$

x varies from 0 to π.

$$\int_{OA} \bar{F} \cdot d\bar{r} = \int_0^{\pi} \sin 3\, dx = \sin 3 \cdot \pi$$

Along AB, $x = \pi$, $dx = 0$, y varies from 0 to 1

$$\int_{AB} \bar{F} \cdot d\bar{r} = \int_0^1 -\cos\pi\, dy = -\cos\pi = 1$$

Along BC, $y = 1$, $dy = 0$, y varies from π to

$$\int_{BC} \bar{F} \cdot d\bar{r} = \int_{\pi}^{0} -\sin 3\, dx = -\sin 3\,\pi$$

Along CO, $x = 0$, $dx = 0$, y varies from 1 to

$$\int_{CO} \bar{F} \cdot d\bar{r} = \int_1^0 -\cos 0\, dy = -y\Big]_1^0 = 1$$

$$\oint_c \bar{F} \cdot d\bar{r} = \int_{OA} \bar{F} \cdot d\bar{r} + \int_{AB} \bar{F} \cdot d\bar{r} + \int_{BC} \bar{F} \cdot d\bar{r} + \int_{CO} \bar{F} \cdot d\bar{r}$$

$$= \sin 3\pi + 1 - \sin 3\pi + 1$$

$$= 2.$$

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Q) Stoke's theorem to evaluate

$\int (\nabla \times \bar{F}) \cdot \bar{n}\, ds$ where $\bar{F} = yi + (x - 2xz)j - xyk$

$= \nabla \times \bar{F}$

and 's' is the surface of the sphere $x^2+y^2+z^2 = a^2$ above the xy-plane.

Sol:- By stoke's theorem

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, ds$$

$$\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & x-2xz & -xy \end{vmatrix}$$

$= i(-x+2x) + j(0+y) + k(1-2z-x)$

$= xi + yj - 2zk$

above xy-plane of the sphere

$\bar{N}$ is unit outward normal along z-axis i.e. $x^2+y^2 = a^2$ [$\because z=0$]

$$\iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, ds = \int_0^a \int_0^{2\pi} (xi + yj - 2zk) \cdot k \, dx\, dy$$

$$= \int_0^a \int_0^{2\pi} -2(0)\, dx\, dy \quad [\because z=0]$$

$= 0$

b) Evaluate $\int (\nabla \times \bar{v}) \cdot ds$ over the surface of the paraboloid $z = 1 - x^2 - y^2$, $z \geq 0$ where $\bar{v} = yi + 2j + xk$

Sol:- By stoke's theorem

$$\oint_C \bar{v} \cdot d\bar{r} = \iint_S (\nabla \times \bar{v}) \cdot d\bar{s}$$

$$\nabla \times \bar{v} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix}$$

$= -i - j - k = -(i + j + k)$

$\bar{N} = \bar{k}$, for $z \geq 0$

$x^2+y^2 = 1$, $x = r\cos\theta$, $y = r\sin\theta$, $dx\, dy = r\, dr\, d\theta$

$dx = -r\sin\theta\, d\theta$, $dy = r\cos\theta\, d\theta$

$$\iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, ds = \int_0^{2\pi} \int_0^1 -(i+j+k) \cdot k \, dx\, dy$$

$$= -\int_0^1 \int_0^{2\pi} dx\, dy = -\int_{\theta=0}^{2\pi} \int_0^1 r\, dr\, d\theta$$

$$= -\int_0^1 2\pi \cdot r\, dr$$

$$= -2\pi \cdot \frac{r^2}{2}\Big]_0^1$$

$= -\pi$

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Stoke's theorem for $\bar{F} = -y^3\bar{i} + x^3\bar{j}$ closed curve www.FirstRanker.com

'C' is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol:- $\bar{F} = -y^3\bar{i} + x^3\bar{j}$ Let $x = a\cos\theta$, $y = b\sin\theta$

$dx = -a\sin\theta\, d\theta$, $dy = b\cos\theta\, d\theta$.

$$\int_C \bar{F}\cdot d\bar{r} = \int_C (-y^3\bar{i} + x^3\bar{j})(dx\bar{i} + dy\bar{j}) = \int_C -y^3dx + x^3dy.$$

$$= \int_0^{2\pi} (-b^3\sin^3\theta)(-a\sin\theta\, d\theta) + (a^3\cos^3\theta)(b\cos\theta\, d\theta)$$

$$= ab^3\int_0^{2\pi} \sin^4\theta\, d\theta + a^3b\int_0^{2\pi} \cos^4\theta\, d\theta$$

$$= ab^3 \cdot 4\int_0^{\pi/2} \sin^4\theta\, d\theta + a^3b \cdot 4\int_0^{\pi/2} \cos^4\theta\, d\theta.$$

$$= ab^3 \cdot 4 \cdot \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + a^3b \cdot 4 \cdot \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi ab}{4}(a^2+b^2)$$

$$\text{curl } \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^3 & x^3 & 0 \end{vmatrix} = \bar{k}(3x^2+3y^2)$$

$$\int_S (\nabla\times\bar{F})\cdot\bar{n}\, ds = \int_S \bar{k}(3x^2+3y^2)\cdot\bar{k}\, ds = 3\int_S (x^2+y^2)\, ds$$

polar coordinates be $x = ar\cos\theta$, $y = br\sin\theta$

$dx\, dy = ab\, r\, dr\, d\theta$

'r' varies from 0 to 1

'θ' " " 0 to 2π

$$\int_S (\nabla\times\bar{F})\cdot\bar{n}\, ds = 3\int_0^{2\pi}\int_0^1 (a^2r^2\cos^2\theta + b^2r^2\sin^2\theta)\, r\, ab\, dr\, d\theta$$

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$$\int_0^{\pi}\int_0^{} a^3 b r^3 \cos^3\theta \, dr\, d\theta \qquad 3\int_0^{2\pi}\int_0^{1} ab^3 r^3 \sin^2\theta \, dr\, d\theta$$

$$= 3a^3b \left(\frac{r^4}{4}\right)_0^1 \int_0^{2\pi} \cos^2\theta \, d\theta + 3ab^3 \left(\frac{r^4}{4}\right)_0^1 \int_0^{2\pi} \sin^2\theta \, d\theta .$$

$$= \frac{3a^3b}{4} \times 4 \int_0^{\pi/2} \cos^2\theta \, d\theta + \frac{3ab^3}{4} \times 4 \int_0^{\pi/2} \sin^2\theta \, d\theta .$$

$$= 3a^3b \cdot \frac{1}{2}\cdot\frac{\pi}{2} + 3ab^3 \cdot \frac{1}{2}\cdot\frac{\pi}{2} = \frac{3ab\pi}{4}(a^2+b^2)$$

L.H.S = R.H.S Stoke's theorem verified.

2.

a) For a solenoidal vector $\bar{F}$, show that

curl curl curl curl $\bar{F} = \nabla^4 \bar{F}$

Sol:- A vector $\bar{F}$ is said to be solenoidal if $\nabla \cdot \bar{F} = 0$

we know that $\nabla \times (\nabla \times \bar{F}) = (\nabla \cdot \bar{F})\nabla - (\nabla \cdot \nabla)\bar{F}$

$= (\nabla \cdot \bar{F})\nabla - \nabla^2 \bar{F}$

$$\nabla \times (\nabla \times \bar{F}) = -\nabla^2 \bar{F} \quad [\because \nabla \cdot \bar{F} = 0].$$

$$\nabla \times \nabla \times [\nabla \times (\nabla \times \bar{F})] = \nabla \times [\nabla \times (-\nabla^2 \bar{F})]$$

$$= \nabla \cdot (-\nabla^2 \bar{F})\nabla - (\nabla \cdot \nabla)(-\nabla^2 \bar{F})$$

$$= \nabla [\nabla \cdot [\nabla \times (\nabla \times \bar{F})]] + \nabla^4 \bar{F}$$

$$= \nabla [\text{div}\,[\text{curl}\,(\nabla \times \bar{F})]] + \nabla^4 \bar{F}$$

$$= \nabla^4 \bar{F} \quad [\because \text{div}\,[\text{curl}\,(\nabla \times \bar{F})] = 0]$$

2.

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Gauss's Divergence theorem :- Let S be a closed surface enclosed a volume V. If $\bar{F}$ is a continuously differentiable vector point function, then $\int_V \text{div}\,\bar{F}\,dV = \int_S \bar{F}\cdot\bar{N}\,dS$ where $\bar{N}$ is the outward drawn unit normal vector at any point of S

Cartesian form :- Let $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ and $\bar{N} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$.

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the d.c's of $\bar{N}$.

$$\bar{F}\cdot\bar{N} = F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma.$$

Also $$\text{div}\,\bar{F} = \nabla\cdot\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Hence the divergence theorem can be written as.

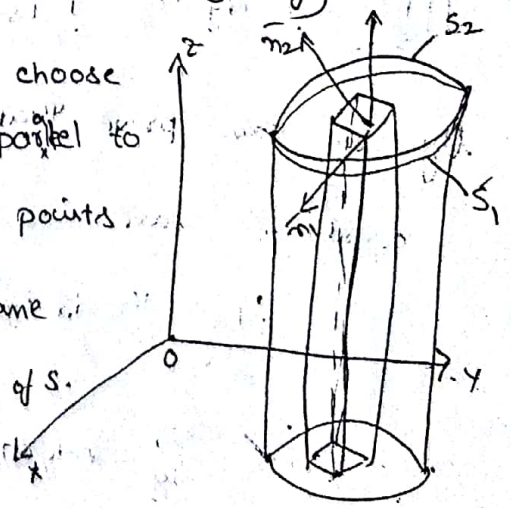
$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz = \int_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\,dS.$$

$$= \iint_S (F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy)$$

Proof :- Let S be a closed surface. Let us choose the coordinate axes so that any line parallel to the coordinate axes cuts S in at most two points.

Let R be the projection of S on xy-plane.

S_1 and S_2 be the lower and upper parts of S.



$z = f(x,y)$ and $z = g(x,y)$ be the equations of S_1 and S_2.

The surface can be put in the form $f(x,y) \le z \le g(x,y)$.

$$\iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz = \iint_R \left[\int_{z=f(x,y)}^{z=g(x,y)} \frac{\partial F_3}{\partial z}\,dz\right] dx\,dy.$$

$$= \iint_R \left[F_3(x,y,z)\right]_f^g\,dx\,dy$$

$$= \iint_R \left[F_3(x,y,g) - F_3(x,y,f)\right] dx\,dy.$$

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$$\iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz = \iint_R F_3(x,y,f)\,dx\,dy - \iint_R F_3(x,y,g)\,dx\,dy \quad -(1)$$

$ds = \frac{dx\,dy}{|\vec{N}\cdot\vec{k}|}$

For the upper part S_2, $\boxed{z = g(x,y)}$ $\quad dx\,dy = ds\cos\gamma$

$= (\vec{N}\cdot\vec{k})\,ds$

Since the normal to S_2 makes an acute angle γ with $\vec{k}$.

$$\iint_R F_3(x,y,g)\,dx\,dy = \int_{S_2} F_3\,\vec{N}\cdot\vec{k}\,ds \quad -(2)$$

For the lower portion S_1, $\boxed{z = f(x,y)}$ $\quad dx\,dy = -ds\cos\gamma$

$= -(\vec{N}\cdot\vec{k})\,ds$

Since the normal to S_1 makes an obtuse angle γ with $\vec{k}$

$$\iint_R F_3(x,y,f)\,dx\,dy = -\int_{S_1} F_3\,(\vec{n}\cdot\vec{k})\,ds \quad -(3)$$

from (1) (2) and (3).

$$\iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz = \int_{S_2} F_3(\vec{N}\cdot\vec{k})\,ds + \int_{S_1} F_3(\vec{n}\cdot\vec{k})\,ds = \int_S F_3(\vec{n}\cdot\vec{k})\,ds$$

By use can prove that, by Projecting S on the other coordinate planes

$$\iiint_V \frac{\partial F_1}{\partial x}\,dx\,dy\,dz = \int_S F_1\,(\vec{N}\cdot\vec{i})\,ds \quad -(ii)$$

$$\iiint_V \frac{\partial F_2}{\partial y}\,dx\,dy\,dz = \int_S F_2\,(\vec{N}\cdot\vec{j})\,ds \quad -(iii)$$

Adding (1) (ii) and (iii)

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz = \int_S (F_1\vec{i} + F_2\vec{j} + F_3\vec{k})\cdot\vec{N}\,ds$$

$$\iiint_V \operatorname{div}\vec{F}\,dv = \int_S \vec{F}\cdot\vec{N}\,ds$$

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1) Verify Divergence theorem, $\int \bar{F} = (x-yz)\bar{i} + (y-zx)\bar{j} + (z-xy)\bar{k}$ taken over a rectangular parallelepiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le b$

Sol:- $\nabla \cdot \bar{F} = \frac{\partial}{\partial x}(x-yz) + \frac{\partial}{\partial y}(y-zx) + \frac{\partial}{\partial z}(z-xy) = 3x+3y+3z$

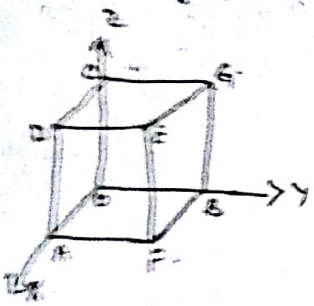
$$\iiint_V (\nabla \cdot \bar{F})\,dv = \iiint 3(x+y+z)\,dx\,dy\,dz = 3\int_{z=0}^{a}\int_{y=0}^{a}\int_{x=0}^{a} (x+y+z)\,dx\,dy\,dz.$$

$$= 3\int_{z=0}^{a}\int_{y=0}^{a}\left(\frac{x^2}{2}+xy+zx\right)_0^a dy\,dz = 3\int_{z=0}^{a}\int_{y=0}^{a}\left(\frac{a^2}{2}+ay+az\right)dy\,dz.$$

$$= 3\int_{z=0}^{c}\left[\frac{a^2}{2}y + a\frac{y^2}{2} + azy\right]_0^b dz = 3\int_{z=0}^{c}\left[\frac{a^2b}{2}+\frac{ab^2}{2}+abz\right]dz.$$

$$= 3\left[\frac{a^2b}{2}z + \frac{ab^2}{2}z + ab\frac{z^2}{2}\right]_0^c.$$

$$\frac{3}{2}a^2bc + \frac{3}{2}ab^2c + \frac{3}{2}abc^2 = \frac{3}{2}abc(a+b+c)$$



To calculate $\iint_S \bar{F}\cdot\bar{n}\,ds$ over the six faces of the rectangular parallelepiped over the face

(1) over the face DEFA, $\bar{n} = \bar{i}$, $x = a$, $ds = dy\,dz$.

$$\iint_{DEFA} \bar{F}\cdot\bar{n}\,ds = \int_{z=0}^{c}\int_{y=0}^{b}\left[(a^2-yz)\bar{i} + (y^2-za)\bar{j} + (z^2-ay)\bar{k}\right]\cdot\bar{i}\,dy\,dz.$$

$$= \int_{z=0}^{c}\int_{y=0}^{b}(a^2-yz)\,dy\,dz = \int_{z=0}^{c}\left[a^2y - z\frac{y^2}{2}\right]_0^b dz = \int_{z=0}^{c}\left(a^2b - z\frac{b^2}{2}\right)dz.$$

$$= a^2bz - b^2\frac{z^2}{4}\Big]_0^c = a^2bc - \frac{c^2b^2}{4}$$

(2) over the face BGCO, $\bar{n} = -\bar{i}$, $ds = dy\,dz$, $x = 0$.

$$\iint_{BGCO} \bar{F}\cdot\bar{n}\,ds = \int_{z=0}^{c}\int_{y=0}^{b} yz\,dy\,dz = \int_0^c \frac{y^2}{2}z\Big]_0^b dz = \int_0^c \frac{b^2}{2}z\,dz$$

$$= \frac{b^2}{2}\int_0^c z\,dz = \frac{b^2}{2}\frac{z^2}{2}\Big]_0^c = \frac{b^2c^2}{4}.$$

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3) Over the face BGEF, $\hat{n} = \hat{j}$, $y = b$. $ds = dx\,dz$

$$\iint_{BGEF} \vec{F}\cdot\hat{n}\,ds = \int_{z=0}^{c}\int_{x=0}^{a} (b^2 - zx)\,dx\,dz = \int_{z=0}^{c}\left[b^2x - z\frac{x^2}{2}\right]_0^a dz = \int_{z=0}^{c}\left[b^2a - z\frac{a^2}{2}\right]dz$$

$$= b^2az - \frac{a^2}{2}\frac{z^2}{2}\Big]_0^c = ab^2c - \frac{a^2c^2}{4}.$$

4) over the face OADC, $\hat{n} = -\hat{j}$, $y = 0$

$$\iint_{OADC} \vec{F}\cdot\hat{n}\,ds = \int_{z=0}^{c}\int_{x=0}^{a} zx\,dx\,dz = \int_{z=0}^{c} z\frac{x^2}{2}\Big]_0^a dz = \int_{z=0}^{c} z\frac{a^2}{2}\,dz = \frac{a^2}{2}\frac{z^2}{2}\Big]_0^c$$

5) over the face GCDE, $\hat{n} = \hat{k}$, $z = c$, $ds = dx\,dy$.

$$\iint_{GCDE} \vec{F}\cdot\hat{n}\,ds = \int_{y=0}^{b}\int_{x=0}^{a} (c^2 - xy)\,dx\,dy = \int_{y=0}^{b} c^2x - \frac{x^2}{2}y\Big]_0^a dy = \int_{y=0}^{b}\left(c^2a - \frac{a^2}{2}y\right)$$

6) over the face AFBO, $\hat{n} = -\hat{k}$, $z = 0$, $ds = dx\,dy$. $= c^2ay - \frac{a^2}{2}\frac{y^2}{2}\Big]_0^b$

$$\iint_{AFBO} \vec{F}\cdot\hat{n}\,ds = \int_{y=0}^{b}\int_{x=0}^{a} xy\,dx\,dy = \int_{y=0}^{b} \frac{x^2}{2}y\Big]_0^a dy = \int_{y=0}^{b} \frac{a^2}{2}y\,dy = \frac{a^2}{2}\int_0^b$$

$$= \frac{a^2b^2}{4}$$

Adding the integrals over the six faces

$$\iint_S \vec{F}\cdot\hat{n}\,ds = a^2bc - \frac{c^2b^2}{4} + c^2b^2 + b^2ca - \frac{a^2b^2}{4} + c^2ab - \frac{a^2b^2}{4} + \frac{a^2b^2}{4}$$

$$= a^2bc + b^2ca + c^2ab = abc\,(a+b+c)$$

Hence $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V (\nabla\cdot\vec{F})\,dv$

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Show that $\int_S \bar{r}\cdot\bar{N}\,ds = 3V$ where 'V' is the volume enclosed by the surface 'S'. www.FirstRanker.com

Sol:- By Gauss divergence theorem $\int_S \bar{R}\cdot\bar{N}\,ds = \iiint_V \text{div}\,\bar{R}\,dv$

$\nabla\cdot\bar{R} = \text{div}\,\bar{R} = 3$ where $\bar{R} = x\bar{i}+y\bar{j}+z\bar{k}$.

$$\int_S \bar{R}\cdot\bar{N}\,ds = \iiint_V 3\,dv = 3V.$$

3) For any closed surface 'S', P.T $\int_S (\nabla\times\bar{F})\cdot\bar{N}\,ds = 0$

Sol:- By divergence theorem $\int_S (\nabla\times\bar{F})\cdot\bar{N}\,ds = \iiint_V \text{div}(\text{curl}\,\bar{F})\,dv$

where 'V' is the volume enclosed by 'S'.

But we know that $\text{div}(\text{curl}\,\bar{F}) = 0$.

$$\therefore \int_S (\nabla\times\bar{F})\cdot\bar{N}\,ds = 0.$$

4) use divergence theorem to show that $\int_S \nabla r^2\cdot\bar{N}\,ds = 6V$

Sol:- $\nabla r^2 = \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right) r^2 = 2r\frac{\partial r}{\partial x}\bar{i} + 2r\frac{\partial r}{\partial y}\bar{j} + 2r\frac{\partial r}{\partial z}\bar{k}$.

$$= 2r\left(\frac{x}{r}\bar{i}+\frac{y}{r}\bar{j}+\frac{z}{r}\bar{k}\right)$$

$$= 2(x\bar{i}+y\bar{j}+z\bar{k}) = 2\bar{R}$$

$$\nabla\cdot\nabla r^2 = \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\cdot(2x\bar{i}+2y\bar{j}+2z\bar{k})$$

$$= 2+2+2 = 6.$$

$$\int_S \nabla r^2\cdot\bar{N}\,ds = \iiint_V \nabla\cdot\nabla r^2\,dv = \iiint_V 6\,dv = 6V.$$

5) If $\bar{F} = ax\bar{i}+by\bar{j}+cz\bar{k}$ a, b, c are constants, show that $\int_S \bar{F}\cdot\bar{N}\,ds = \frac{4}{3}\pi(a+b+c)$ where 'S' is the surface of a unit sphere

Sol:- By Gauss divergence theorem $\int_S \bar{F}\cdot\bar{N}\,ds = \iiint_V \text{div}\,\bar{F}\,dv$.

$$\text{div}\,\bar{F} = \frac{\partial}{\partial x}(ax)+\frac{\partial}{\partial y}(by)+\frac{\partial}{\partial z}(cz) = a+b+c$$

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$\int_S \bar{F}\cdot\bar{N}\,ds = a+b+c \iiint_V dv = (a+b+c)V$

where 'V' is the volume of the unit sphere $= \frac{4}{3}\pi r^3$

$= \frac{4}{3}\pi(1)^3$

$= \frac{4}{3}\pi$

$\therefore \int_S \bar{F}\cdot\bar{N}\,ds = \frac{4}{3}\pi(a+b+c)$

6) $\oint_S (a^2x^2+b^2y^2+c^2z^2)^{+1/2}\,ds$ over the ellipsoid $ax^2+by^2+cz^2=1$

Sol:- By G.D.T $\int_S \bar{F}\cdot\bar{N}\,ds = \iiint_V \text{div}\,\bar{F}\,dv$

Here $\bar{F}\cdot\bar{N} = (a^2x^2+b^2y^2+c^2z^2)^{+1/2}$

and 'S' is the surface of an ellipsoid $ax^2+by^2+cz^2=1$

$\bar{N}$ = unit outward normal $= \frac{\nabla\phi}{|\nabla\phi|}$ where $\phi = ax^2+by^2+cz^2-1$

$= \frac{2ax\bar{i}+2by\bar{j}+2cz\bar{k}}{\sqrt{4a^2x^2+4b^2y^2+4c^2z^2}}$

Given $\bar{F} = ax\bar{i}+by\bar{j}+cz\bar{k}$ $= \frac{ax\bar{i}+by\bar{j}+cz\bar{k}}{(a^2x^2+b^2y^2+c^2z^2)^{1/2}}$

$\text{div}\,\bar{F} = a+b+c$

and 'V' is the volume of an ellipsoid $ax^2+by^2+cz^2=1$

$\int_S (ax^2+by^2+cz^2)^{-1/2}\,ds = \iiint_V (a+b+c)\,dv = (a+b+c)V$

$= (a+b+c)\frac{4\pi}{3}\frac{1}{\sqrt{abc}}$

$\therefore \bar{F}\cdot\bar{N} = (ax\bar{i}+by\bar{j}+cz\bar{k})\cdot\left(\frac{ax\bar{i}+by\bar{j}+cz\bar{k}}{(a^2x^2+b^2y^2+c^2z^2)^{1/2}}\right)$

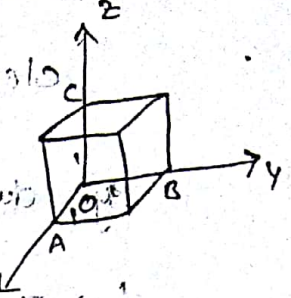
$= \frac{(a^2x^2+b^2y^2+c^2z^2)}{(a^2x^2+b^2y^2+c^2z^2)^{1/2}} = (a^2x^2+b^2y^2+c^2z^2)^{1/2}$

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7) Evaluate $\int_S \bar{F}\cdot\bar{N}\,ds$ where 'S' is the surface of the cube bounded by $x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$

and $\bar{F} = 4xz\,\bar{i} - y^2\,\bar{j} + yz\,\bar{k}$



Sol: By Gauss divergence theorem $\int_S \bar{F}\cdot\bar{N}\,ds = \iiint_V \operatorname{div}\bar{F}\,dv$

$$\operatorname{div}\bar{F} = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)$$

$$= 4z - 2y + y = 4z - y$$

$$\int_S \bar{F}\cdot\bar{N}\,ds = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (4z-y)\,dz\,dy\,dx = \int_{x=0}^{1}\int_{y=0}^{1}\left[4\frac{z^2}{2} - yz\right]_0^1 dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} (2-y)\,dy\,dx$$

$$= \int_{x=0}^{1}\left(2y - \frac{y^2}{2}\right)_0^1 dx = \int_0^1 \left(2-\frac{1}{2}\right)dx$$

$$= \frac{3}{2}\int_0^1 dx = \frac{3}{2}$$

8) Evaluate $\int_S (a^2x^2 + b^2y^2 + c^2z^2)^{-1/2}\,ds$, where 'S' is the surface of the ellipsoid $ax^2 + by^2 + cz^2 = 1$.

Sol:- Taking $\phi = ax^2 + by^2 + cz^2 - 1 = 0$

$$\nabla\phi = 2ax\,\bar{i} + 2by\,\bar{j} + 2cz\,\bar{k}$$

$\therefore$ unit vector normal to the ellipsoid $= \bar{N} = \dfrac{\nabla\phi}{|\nabla\phi|}$

$$= \frac{ax\,\bar{i} + by\,\bar{j} + cz\,\bar{k}}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}}$$

since $\bar{F}\cdot\bar{N} = (a^2x^2 + b^2y^2 + c^2z^2)^{-1/2}$

$$\bar{F}\cdot\frac{(ax\,\bar{i} + by + cz\,\bar{k})}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}} = (a^2x^2 + b^2y^2 + c^2z^2)^{-1/2}$$

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$\therefore \vec{F}\cdot(ax\hat{i}+by\hat{j}+cz\hat{k}) = 1$ [$\because ax^2+by^2+cz^2=1$]

clearly $\vec{F} = x\hat{i}+y\hat{j}+z\hat{k}$

By divergence theorem

$$\int_S \vec{F}\cdot\vec{N}\,ds = \int_V \operatorname{div}\vec{F}\,dv = \int_V \left[\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)\right]dv = 3\int_V dv = 3V$$

$$= 3\cdot\frac{4\pi}{3}\frac{1}{\sqrt{abc}} = \frac{4\pi}{\sqrt{abc}} \quad [\because \text{vol of ellipsoid} = \tfrac{4\pi}{3}\tfrac{1}{\sqrt{abc}}]$$

9) Apply Divergence theorem to evaluate $\int(lx^2+my^2+nz^2)\,ds$ taken over the sphere $(x-a)^2+(y-b)^2+(z-c)^2=\rho^2$; l, m, n being the direction cosines of the external normal to the sphere

Sol:- The parametric equations of the sphere are

$x = a+\rho\sin\theta\cos\phi$, $y = b+\rho\sin\theta\sin\phi$, $z = c+\rho\cos\theta$

and to cover the whole sphere, ρ varies from 0 to ρ, θ varies from 0 to π, ϕ varies from 0 to 2π

$$\int_S (lx^2+my^2+nz^2)\,ds = \int_S (x^2\hat{i}+y^2\hat{j}+z^2\hat{k})\cdot\vec{N}\,ds$$

$$= \int_V \operatorname{div}(x^2\hat{i}+y^2\hat{j}+z^2\hat{k})\,dv = 2\int_V (x+y+z)\,dv$$

$$= 2\int_0^{2\pi}\int_0^{\pi}\int_0^{\rho}[(a+b+c)+\rho(\sin\theta\cos\phi+\sin\theta\sin\phi+\cos\theta)]\times\rho^2\sin\theta\,d\rho\,d\theta\,d\phi$$

$$= 2(a+b+c)\frac{\rho^3}{3}\int_0^{\pi}(1-\cos\theta)\,d\theta\cdot 2\pi$$

$$= \frac{8\pi}{3}(a+b+c)\rho^3.$$

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Evaluate $\int_S \bar{F}\cdot\bar{N}\,ds$ where $\bar{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ and 'S' is the surface bounding the region $x^2+y^2=4$, $z=0$ and $z=3$

Sol: By divergence theorem $\int_S \bar{F}\cdot\bar{N}\,ds = \int_V \text{div}\,\bar{F}\,dv$ ... (1)

$\nabla\cdot\bar{F} = \frac{d}{dx}(4x) + \frac{d}{dy}(-2y^2) + \frac{d}{dz}(z^2)$

$= 4 - 4y + 2z$

$\int_S \bar{F}\cdot\bar{N}\,ds = \iiint_V (4-4y+2z)\,dx\,dy\,dz$

$y^2 = 4-x^2$

$y = \pm\sqrt{4-x^2}$

$x^2 = 2$

$x = \pm 2$

$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(4-4y+2z)\,dx\,dy\,dz.$

$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[4z - 4yz + 2\frac{z^2}{2}\right]_0^3 dx\,dy$

$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[4z - 4yz + z^2\right]_0^3 dx\,dy = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}[12 - 12y + 9]\,dx\,dy.$

$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(21-12y)\,dx\,dy = \int_{-2}^{2}\left[21y - 12\frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$

$= 42\int_{-2}^{2}\sqrt{4-x^2}\,dx = 84\int_0^2\sqrt{4-x^2}\,dx$

$= 84\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$

$= 84\pi$

$\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

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(4) Evaluate $\int_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = 2x^2y\,\bar{i} - y^2\,\bar{j} + 4xz^2\,\bar{k}$ and S is the closed surface of the region in the first octant bounded by the cylinder $y^2+z^2=9$ and the planes $x=0, x=2, y=0$ and $z=0$.

Sol:- By Gauss divergence theorem $\iint_S \bar{F}\cdot\bar{n}\,ds = \iiint_V \text{div}\,\bar{F}\,dv$.

$$= \int_{x=0}^{2}\int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} (4xy - 2y + 8xz)\,dx\,dy\,dz$$

$x = 0$ to 2, $z = \sqrt{9-y^2}$, $z = 0$ to $\sqrt{9-y^2}$, $y = 0$ to 3

$$= \int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} \left[4\frac{x^2}{2}y - 2xy + 8\frac{x^2}{2}z\right]_0^2 dy\,dz = \int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} (2x^2y - 2xy + 4x^2z)\Big|_0^2\,dy\,dz$$

$$= \int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} (8y - 4y + 16z)\,dy\,dz = \int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} (4y + 16z)\,dy\,dz.$$

$$= \int_{y=0}^{3}\left[4yz + 16\frac{z^2}{2}\right]_0^{\sqrt{9-y^2}} dy = \int_{y=0}^{3}\left[4y\sqrt{9-y^2} + 8(9-y^2)\right] dy.$$

$$= 4\int_{y=0}^{3}\left[y\sqrt{9-y^2}\,dy + 2(9-y^2)\right] dy.$$

$9-y^2 = t$
$-2y\,dy = dt$
$y\,dy = -\frac{dt}{2}$

$$= 4\left[-\frac{1}{2}(9-y^2)^{3/2}\cdot\frac{2}{3} + 18y - \frac{2}{3}y^3\right]_{y=0}^{3}$$

$$= 4\left[54 - 18\right] - 4\left[-\frac{1}{2}(9-y^2)^{3/2}\cdot\frac{2}{3}\right]$$

$$= 4\times 36 + 36$$

$$= 144 + 36$$

$$= 180$$

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Use divergence theorem to evaluate $\iint_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = x^3\bar{i} + y^3\bar{j} + z^3\bar{k}$ and 'S' is the surface of the sphere $x^2+y^2+z^2 = a^2$

According the Gauss divergence theorem

$$\iiint_V \text{div } \bar{F}\, dv = \iint_S \bar{F}\cdot\bar{n}\, ds$$

$$\text{div } \bar{F} = 3(x^2+y^2+z^2)$$

$$\iint_S \bar{F}\cdot\bar{n}\, ds = \iiint_V 3(x^2+y^2+z^2)\, dx\,dy\,dz.$$

using the Spherical coordinates

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

$$dv = r^2\sin\theta\, dr\, d\theta\, d\phi.$$

$$\iint_S \bar{F}\cdot\bar{n}\, ds = 3\int_0^a\int_0^{\pi}\int_0^{2\pi} r^4\sin\theta\, dr\, d\theta\, d\phi$$

$$= 3\left[\int_0^a r^4 dr \int_0^{\pi}\sin\theta\, d\theta \int_0^{2\pi} d\phi\right] = 3\left[\frac{r^5}{5}\right]_0^a (-\cos\theta)_0^{\pi} \cdot 2\pi$$

$$= 3\left[\frac{r^5}{5}\right]_0^a (1+1)\cdot 2\pi$$

$$= \frac{12\pi a^5}{5}.$$

$x^2+y^2+z^2 = r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta.$

$= r^2\sin^2\theta(\cos^2\phi + \sin^2\phi) + r^2\cos^2\theta.$

$= r^2(\sin^2\theta + \cos^2\theta) = r^2$

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2) Find $\iint_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = (2x+3z)\hat{i} - (xz+y)\hat{j} + (y+2z)\hat{k}$ and 'S' is the surface of the sphere having centre at $(3,-1,2)$ and radius 3.

Sol:- Let 'V' be the volume enclosed by the surface S.

Then Gauss divergence theorem, we have

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iiint_V \text{div}\,\bar{F}\,dv$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(2x+3z) + \frac{\partial}{\partial y}(-xz-y) + \frac{\partial}{\partial z}(y+2z)\right] dv.$$

$$= \iiint_V (2-1+2)\,dv = 3\iiint_V dv = 3V.$$

But 'V' is the volume of a sphere of radius is 3.

$$V = \frac{4}{3}\pi r^3 \quad [r=3]$$

$$V = \frac{4}{3}\pi(3)^3$$

$$= 36\pi$$

Hence $\iint_S \bar{F}\cdot\bar{n}\,ds = 3(36\pi) = 108\pi$

3) Evaluate $\iint (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$ over the surface of a sphere of radius 'a'.

Sol:- By Gauss divergence theorem $\int_S \bar{F}\cdot\bar{n}\,ds = \int_V \text{div}\,\bar{F}\,dv$

(or) $\int_S F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dv$

Here $F_1 = x$, $F_2 = y$, $F_3 = z$ $\Rightarrow \frac{\partial F_1}{\partial x} = 1$, $\frac{\partial F_2}{\partial y} = 1$, $\frac{\partial F_3}{\partial z} = 1$

$$\iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy = \iiint_V (1+1+1)\,dv = 3V$$

$$= 3\,\frac{4}{3}\pi a^3$$

$$= 4\pi a^3$$

[$\because$ 'V' is the volume of the sphere of radius is 'a']

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If ϕ and ψ are scalar point functions which are uniform and continuous and possessing derivative in any direction within the volume V enclosed by the surface 's' then show that $\iiint_V (\phi \nabla^2\psi - \psi\nabla^2\phi)dv = \int_S (\phi\nabla\psi - \psi\nabla\phi)\cdot\bar{n}\,ds$

By Gauss divergence theorem we have $\int_S \bar{F}\cdot\bar{n}\,ds = \iiint_V \text{div}\,\bar{F}\,dv$ — (1).

taking $\bar{F} = \phi\nabla\psi$ then (1) becomes

$$\int_S (\phi\nabla\psi)\cdot\bar{n}\,ds = \iiint_V \nabla\cdot(\phi\nabla\psi)\,dv = \iiint_V (\nabla\phi\cdot\nabla\psi + \phi\nabla^2\psi)\,dv \quad -(2)$$

Interchanging ϕ and ψ in (2) we have

$$\int_S (\psi\nabla\phi)\cdot\bar{n}\,ds = \iiint_V [\nabla\psi\cdot\nabla\phi) + \psi\nabla^2\phi]\,dv \quad -(3)$$

$[\because \nabla\phi\cdot\nabla\psi = \nabla\psi\cdot\nabla\phi]$

on subtracting $\int_S (\phi\nabla\psi - \psi\nabla\phi)\cdot\bar{n}\,ds = \iiint_V [\phi\nabla^2\psi - \psi\nabla^2\phi]\,dv$ — (4)

which is known as Green's second identity.

2.

Irrotational fields :- An irrotational field $\bar{F}$ is characterised by one of the following conditions (i) $\nabla\times\bar{F} = \bar{0}$ (ii) circulation $\int \bar{F}\cdot d\bar{r}$ along every closed surface is zero.

(iii) $\bar{F} = \nabla\phi$ (for some scalar point function ϕ) if the domain is simply connected.

If $\nabla\bar{F} = 0$ by stokes theorem $\int_C \bar{F}\cdot d\bar{r} = \int_S (\nabla\times\bar{F})\cdot ds = 0$ (ie circulation along every closed surface is zero).

Again since $\text{curl}(\text{grad}\,\phi) = 0$ ie an irrotational field for which $\nabla\times\bar{F} = \bar{0}$; the vector $\bar{F}$ can always be expressed as the gradient of a scalar point function ϕ provided the domain is simply connected.

Thus $\bar{F} = \nabla\phi$ where a scalar function ϕ is called the potential. In a irrotational field $\bar{F}$ cannot be expressed as the gradient of a scalar potential.

Result :- In an irrotational field, the line integral of $\bar{F}$ between two points is independent of the path of integration and is equal to the potential difference between these points.

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Let $\bar{F} = \nabla\phi$ in region R. Let A and B be any two points in the region R and C be any path joining A and B in R then we have

$$\int_C \bar{F}\cdot d\bar{r} = \int_C \nabla\phi \cdot d\bar{r} = \int_C \left(\bar{i}\frac{d\phi}{dx} + \bar{j}\frac{d\phi}{dy} + \bar{k}\frac{d\phi}{dz}\right)\cdot(dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k})$$

$$= \int_C \left(\frac{d\phi}{dx}dx + \frac{d\phi}{dy}dy + \frac{d\phi}{dz}dz\right) = \int_C d\phi = \int_A^B d\phi = [\phi]_A^B = \phi_B - \phi_A$$

which shows that the line integral depends only on point A and B and not on path C.

Note 1:- If $\bar{F}$ is a vector force acting on a particle then $\oint \bar{F}\cdot d\bar{r}$ represents the work done in moving the particle around the closed path C. When $\oint_C \bar{F}\cdot d\bar{r} = 0$ the field is said to be conservative. i.e. no work is done in displacement from a point to another point in the field and back to that point and the mechanical energy is conserved. Thus every irrotational field is conservative.

Solenoidal Fields:- A solenoidal field $\bar{F}$ is characterised by any one of the following conditions:-

(i) $\nabla\cdot\bar{F} = 0$ (ii) Flux $\int \bar{F}\cdot\bar{N}\,ds$ across every closed surface is zero since $\nabla\cdot(\nabla\times v) = 0$.

i.e. div (curl v) = 0.

Therefore, in a solenoidal field for which $\nabla\cdot\bar{F} = 0$, then vector $\bar{F}$ can always be expressed as the curl of a vector function v thus $\bar{F} = \nabla\times v$.

2

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Cylindrical coordinate system is orthogonal

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Any point P, we have $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$ so that

$$\bar{R} = \rho\cos\phi\,\bar{i} + \rho\sin\phi\,\bar{j} + z\bar{k}$$

If T_ρ, T_ϕ, T_z are unit vectors at P in the directions of the tangents to the ρ, ϕ, z curves then

$$T_\rho = \frac{\frac{\partial\bar{R}}{\partial\rho}}{|\partial\bar{R}/\partial\rho|} = \frac{\cos\phi\,\bar{i} + \sin\phi\,\bar{j}}{\sqrt{\cos^2\phi + \sin^2\phi}} = \cos\phi\,\bar{i} + \sin\phi\,\bar{j}$$

$$T_\phi = \frac{\frac{\partial\bar{R}}{\partial\phi}}{|\partial\bar{R}/\partial\phi|} = \frac{-\rho\sin\phi\,\bar{i} + \rho\cos\phi\,\bar{j}}{\sqrt{\rho^2\sin^2\phi + \rho^2\cos^2\phi}} = -\sin\phi\,\bar{i} + \cos\phi\,\bar{j}$$

$$T_z = \frac{\frac{\partial\bar{R}}{\partial z}}{|\partial\bar{R}/\partial z|} = \frac{\bar{k}}{1} = \bar{k}$$

Now $T_\rho \cdot T_\phi = (\cos\phi\,\bar{i} + \sin\phi\,\bar{j}) \cdot (-\sin\phi\,\bar{i} + \cos\phi\,\bar{j})$

$= 0$

$T_\phi \cdot T_z = (-\sin\phi\,\bar{i} + \cos\phi\,\bar{j}) \cdot \bar{k} = 0$

$T_\rho \cdot T_z = (\cos\phi\,\bar{i} + \sin\phi\,\bar{j}) \cdot \bar{k} = 0.$

∴ The cylindrical coordinate system is orthogonal.

Also $T_\rho \times T_\phi = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \end{vmatrix} = \bar{k}(\cos^2\phi + \sin^2\phi) = \bar{k} = T_z$

$T_\phi \times T_z = T_\rho$, $T_z \times T_\rho = T_\phi$

∴ The cylindrical coordinate system is a right handed orthogonal coordinate system

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Del applied to functions in cylindrical coordinates.

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① Expression for grad f

$$\nabla f = \frac{\partial f}{\partial \rho} T_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \phi} T_\phi + \frac{\partial f}{\partial z} T_z$$

② Expression for div $\bar{F}$ where $\bar{F} = f_1 T_u + f_2 T_v + f_3 T_w$

$$\nabla \cdot \bar{F} = \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho f_1) + \frac{\partial f_2}{\partial \phi} + \frac{\partial}{\partial z} (\rho f_3) \right]$$

③ Expression for curl $\bar{F}$

$$\nabla \times \bar{F} = \begin{vmatrix} T_\rho/\rho & T_\phi & T_z/\rho \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ f_1 & \rho f_2 & f_3 \end{vmatrix}$$

④ Expression for $\nabla^2 f$

$$\nabla^2 f = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial f}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Problem ① Express the vector $z\hat{i} - 2x\hat{j} + y\hat{k}$ in cylindrical coordinates.

Sol:- we have $x = \rho\cos\phi$; $y = \rho\sin\phi$, $z = z$

So that $\bar{R} = \rho\cos\phi\,\hat{i} + \rho\sin\phi\,\hat{j} + z\hat{k}$

If T_ρ, T_ϕ, T_z are unit vectors along the tangents to ρ, ϕ and z respectively

$$T_\rho = \frac{\partial \bar{R}}{\partial \rho} \Big/ \left| \frac{\partial \bar{R}}{\partial \rho} \right| = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$

$$T_\phi = \frac{\partial \bar{R}}{\partial \phi} \Big/ \left| \frac{\partial \bar{R}}{\partial \phi} \right| = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

$$T_z = \partial\bar{R}/\partial z \Big/ \left| \frac{\partial \bar{R}}{\partial z} \right| = \hat{k}$$

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Given $\bar{F} = z\bar{i} - 2x\bar{j} + y\bar{k}$

Let the expression for $\bar{F}$ in cylindrical coordinates be

$$\bar{F} = f_1 T_\rho + f_2 T_\phi + f_3 T_z$$

where $f_1 = \bar{F} \cdot T_\rho$, $f_2 = \bar{F} \cdot T_\phi$, $f_3 = \bar{F} \cdot T_z$

$$f_1 = (z\bar{i} - 2x\bar{j} + y\bar{k}) \cdot (\cos\phi\, \bar{i} + \sin\phi\, \bar{j})$$

$$= z\cos\phi - 2x\sin\phi$$

$$f_2 = (z\bar{i} - 2x\bar{j} + y\bar{k}) \cdot (-\sin\phi\, \bar{i} + \cos\phi\, \bar{j})$$

$$= -z\sin\phi - 2x\cos\phi$$

$$f_3 = (z\bar{i} - 2x\bar{j} + y\bar{k}) \cdot \bar{k} = y = \rho\sin\phi$$

required expression is $\bar{F} = f_1 T_\rho + f_2 T_\phi + f_3 T_z$

$$\bar{F} = (z\cos\phi - 2x\sin\phi) T_\rho - (z\sin\phi + 2x\cos\phi) T_\phi + y T_z$$

$$\bar{F} = (z\cos\phi - \rho\sin 2\phi) T_\rho - (z\sin\phi + 2\rho\cos^2\phi) T_\phi + \rho\sin\phi\, T_z$$

(2)

Express the vector $2y\bar{i} - z\bar{j} + 3x\bar{k}$ in cylindrical coordinates

Sol:- We have $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$

So that $\bar{R} = \rho\cos\phi\, \bar{i} + \rho\sin\phi\, \bar{j} + z\bar{k}$

If T_ρ, T_ϕ, T_z are unit vectors along the tangents to ρ, ϕ and z respectively

$$T_\rho = \frac{\partial \bar{R}}{\partial \rho} \Big/ \left|\frac{\partial \bar{R}}{\partial \rho}\right| = \cos\phi\, \bar{i} + \sin\phi\, \bar{j}$$

$$T_\phi = \frac{\partial \bar{R}}{\partial \phi} \Big/ \left|\frac{\partial \bar{R}}{\partial \phi}\right| = -\sin\phi\, \bar{i} + \cos\phi\, \bar{j}$$

$$T_z = \frac{\partial \bar{R}}{\partial z} \Big/ \left|\frac{\partial \bar{R}}{\partial z}\right| = \bar{k}$$

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Given $\bar{F} = 2yi - z\bar{j} + 3x\bar{k}$ Let the expression for $\bar{F}$ in cylindrical coordinates be

$$\bar{F} = f_1 T_\rho + f_2 T_\phi + f_3 T_z$$

where $f_1 = \bar{F} \cdot T_\rho$, $f_2 = \bar{F} \cdot T_\phi$, $f_3 = \bar{F} \cdot T_z$

$$f_1 = (2y\,\bar{i} - z\bar{j} + 3x\bar{k}) \cdot (\cos\phi\,\bar{i} + \sin\phi\,\bar{j})$$

$$= 2y\cos\phi - z\sin\phi$$

$$= 2(\rho\sin\phi)\cos\phi - z\sin\phi$$

$$= \rho\sin 2\phi - z\sin\phi$$

$$f_2 = (2y\,\bar{i} - z\bar{j} + 3x\bar{k}) \cdot (-\sin\phi\,\bar{i} + \cos\phi\,\bar{j})$$

$$= -2y\sin\phi - z\cos\phi$$

$$= -2(\rho\sin\phi)\sin\phi - z\cos\phi = -2\rho\sin^2\phi - z\cos\phi$$

$$f_3 = (2y\,\bar{i} - z\bar{j} + 3x\bar{k}) \cdot \bar{k} = 3x = 3\rho\cos\phi$$

Required expression is $\bar{F} = f_1 T_\rho + f_2 T_\phi + f_3 T_z$

$$\bar{F} = (\rho\sin 2\phi - z\sin\phi)\,T_\rho - (2\rho\sin^2\phi + z\cos\phi)\,T_\phi + 3\rho\cos\phi\,\bar{k}$$

③ Express the following vectors in cylindrical coordinates

$2x\bar{i} - 3y^2\bar{j} + 2x\bar{k}$

Sol:- We have $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$

So that $\bar{r} = \rho\cos\phi\,\bar{i} + \rho\sin\phi\,\bar{j} + z\bar{k}$

If T_ρ, T_ϕ, T_z are unit vectors along the tangents to ρ, ϕ and z respectively $T_\rho = \frac{\partial\bar{r}}{\partial\rho} \Big/ \left|\frac{\partial\bar{r}}{\partial\rho}\right| = \cos\phi\,\bar{i} + \sin\phi\,\bar{j}$

$$T_\phi = \frac{\partial\bar{r}}{\partial\phi} \Big/ \left|\frac{\partial\bar{r}}{\partial\phi}\right| = -\sin\phi\,\bar{i} + \cos\phi\,\bar{j}$$

$$T_z = \frac{\partial\bar{r}}{\partial z} \Big/ \left|\frac{\partial\bar{r}}{\partial z}\right| = \bar{k}$$

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Given $\bar{F} = 2xz\hat{i} - 3y^2\hat{j} + 2z\hat{k}$

Find the expression for $\bar{F}$ in cylindrical coordinates

Sol: $\bar{F} = f_1 T_\rho + f_2 T_\phi + f_3 T_z$

where $f_1 = \bar{F} \cdot T_\rho$, $f_2 = \bar{F} \cdot T_\phi$, $f_3 = \bar{F} \cdot T_z$

$f_1 = (2x\hat{i} - 3y^2\hat{j} + 2z\hat{k}) \cdot (\cos\phi\,\hat{i} + \sin\phi\,\hat{j})$

$f_1 = 2x\cos\phi - 3y^2\sin\phi$

$= 2(\rho\cos\phi)\cos\phi - 3(\rho\sin\phi)^2\sin\phi$

$= 2\rho\cos^2\phi - 3\rho^2\sin^3\phi$

$f_2 = (2x\hat{i} - 3y^2\hat{j} + 2z\hat{k}) \cdot (-\sin\phi\,\hat{i} + \cos\phi\,\hat{j})$

$= -2x\sin\phi - 3y^2\cos\phi$

$= -2(\rho\cos\phi)\sin\phi - 3(\rho\sin\phi)^2\cos\phi$

$= -\rho\sin 2\phi - 3\rho^2\sin^2\phi\cos\phi$

$f_3 = (2x\hat{i} - 3y^2\hat{j} + 2z\hat{k}) \cdot \hat{k} = 2z = 2\rho\cos\phi$

Required expression is

$\bar{F} = f_1 T_\rho + f_2 T_\phi + f_3 T_z$

$\bar{F} = (2\rho\cos^2\phi - 3\rho^2\sin^3\phi)\,T_\rho - (\rho\sin 2\phi + 3\rho^2\sin^2\phi\cos\phi)\,T_\phi + 2\rho\cos\phi$

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Problem (i) :-

S.T $\nabla(\log \rho)$ and $\nabla\phi$, $\rho \neq 0$, $\phi \neq 0$ are solenoidal vectors.

Sol :- $f = \log \rho$ is a function of ρ only

To P.T $\nabla \cdot (\nabla f) = 0 \Rightarrow \nabla^2 f$

$$\nabla^2 f = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \frac{\partial^2}{\partial \rho^2}(\log \rho) + \frac{1}{\rho}\frac{\partial}{\partial \rho}(\log \rho) + 0 + 0$$

$$= -\frac{1}{\rho^2} + \frac{1}{\rho^2} = 0.$$

$\therefore \nabla \log \rho$ is solenoidal.

(ii) $f = \nabla\phi$ is a function of ϕ only. we have to show that $\nabla \cdot (\nabla f)$ i.e $\nabla^2 f = 0$

$$\nabla^2 f = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= 0 + 0 + \frac{1}{\rho^2}\frac{\partial^2 \phi}{\partial \phi^2} + 0$$

$$= 0.$$

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herical coordinate system is orthogonal

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At any point P, we have $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$\Rightarrow \bar{R} = r\sin\theta\cos\phi\,\bar{i} + r\sin\theta\sin\phi\,\bar{j} + r\cos\theta\,\bar{k}$

If T_r, T_θ, T_ϕ be the unit vectors at P, in the directions of tangents to the r, θ, ϕ – curves respectively then:

$$T_r = \frac{\frac{\partial \bar{R}}{\partial r}}{\left|\frac{\partial \bar{R}}{\partial r}\right|} = \frac{\sin\theta\cos\phi\,\bar{i} + \sin\theta\sin\phi\,\bar{j} + \cos\theta\,\bar{k}}{\sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta}}$$

$\cos^2\phi + \sin^2\phi = 1$
$\sin^2\theta + \cos^2\theta = 1$

$$= \frac{\sin\theta\cos\phi\,\bar{i} + \sin\theta\sin\phi\,\bar{j} + \cos\theta\,\bar{k}}{\sqrt{\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta}}$$

$$= \sin\theta\cos\phi\,\bar{i} + \sin\theta\sin\phi\,\bar{j} + \cos\theta\,\bar{k}$$

$$T_\theta = \frac{\frac{\partial \bar{R}}{\partial \theta}}{\left|\frac{\partial \bar{R}}{\partial \theta}\right|} = \frac{r\cos\theta\cos\phi\,\bar{i} + r\cos\theta\sin\phi\,\bar{j} - r\sin\theta\,\bar{k}}{\sqrt{r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta}}$$

$$= \frac{r(\cos\theta\cos\phi\,\bar{i} + \cos\theta\sin\phi\,\bar{j} - \sin\theta\,\bar{k})}{r\sqrt{\cos^2\theta(\cos^2\phi + \sin^2\phi) + \sin^2\theta}}$$

$$= \cos\theta\cos\phi\,\bar{i} + \cos\theta\sin\phi\,\bar{j} - \sin\theta\,\bar{k}$$

$$T_\phi = \frac{\frac{\partial \bar{R}}{\partial \phi}}{\left|\frac{\partial \bar{R}}{\partial \phi}\right|} = \frac{-r\sin\theta\sin\phi\,\bar{i} + r\sin\theta\cos\phi\,\bar{j}}{\sqrt{r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi}}$$

$$= \frac{r\sin\theta(-\sin\phi\,\bar{i} + \cos\phi\,\bar{j})}{r\sin\theta\sqrt{\sin^2\phi + \cos^2\phi}}$$

$$= -\sin\phi\,\bar{i} + \cos\phi\,\bar{j}$$

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$= \sin\theta\cos\theta(\cos^2\phi + \sin^2\phi) - \sin\theta\cos\theta$

$= 0$

Similarly $T_\theta \cdot T_\phi = 0$ and $T_\phi \cdot T_r = 0$.

$$T_r \times T_\theta = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \end{vmatrix}$$

$$= \bar{i}\left[-\sin^2\theta\sin\phi - \cos^2\theta\sin\phi\right] + \bar{j}\left[\cos^2\theta\cos\phi + \sin^2\theta\cos\phi\right]$$
$$+ \bar{k}\left[\sin\theta\cos\phi\cos\theta\sin\phi - \sin\theta\cos\theta\sin\phi\cos\phi\right]$$

$$= -\sin\phi\,\bar{i} + \cos\phi\,\bar{j} = T_\phi$$

Similarly $T_\theta \times T_\phi = T_r$ and $T_\phi \times T_r = T_\theta$.

∴ The spherical polar coordinate system is a right handed orthogonal coordinate system.

Del applied to functions in spherical polar coordinates

Let T_r, T_θ, T_ϕ be the unit vectors in the directions of the tangents to the r, θ, ϕ – curves.

(i) Expression for grad f :-

$$\nabla f = \frac{\partial f}{\partial r} T_r + \frac{1}{r}\frac{\partial f}{\partial \theta} T_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi} T_\phi$$

(ii) Expression for div $\bar{F}$, where $\bar{F} = f_1 T_r + f_2 T_\theta + f_3 T_\phi$

$$\nabla \cdot \bar{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(f_1 r^2) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(f_2 \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial f_3}{\partial \phi}$$

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$$\nabla \times \bar{F} = \begin{vmatrix} \frac{T_r}{r^2\sin\theta} & \frac{T_\theta}{r\sin\theta} & \frac{T_\phi}{r} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ f_1 & rf_2 & r\sin\theta f_3 \end{vmatrix}$$

(iv) Expression for $\nabla^2 f$:-

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial f}{\partial\theta}$$

(i) Express the vector field $2y\,i - 2j + 3z\,k$ in spherical polar coordinate system.

Sol:- we have $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

So that $\bar{R} = r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}$

If T_r, T_θ, T_ϕ be the unit vectors along the tangents to r, θ, ϕ curves respectively, then

$$T_r = \frac{\frac{\partial\bar{R}}{\partial r}}{\left|\frac{\partial\bar{R}}{\partial r}\right|} = \frac{\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}}{\sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta}}$$

$$= \frac{\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}}{\sqrt{\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta}}$$

$$= \boxed{\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}}$$

$$T_\theta = \frac{\frac{\partial\bar{R}}{\partial\theta}}{\left|\frac{\partial\bar{R}}{\partial\theta}\right|} = \frac{r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,\hat{k}}{\sqrt{r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta}}$$

$$= \frac{r[\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}]}{r\sqrt{\cos^2\theta(\cos^2\phi + \sin^2\phi) + \sin^2\theta}}$$

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$= \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$

$$T_\phi = \frac{\frac{\partial \bar{r}}{\partial \phi}}{\left|\frac{\partial \bar{r}}{\partial \phi}\right|} = \frac{-r\sin\theta\sin\phi\,\hat{i} + r\sin\theta\cos\phi\,\hat{j}}{\sqrt{r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi}}$$

$$T_\phi = \frac{r\sin\theta\left[-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}\right]}{r\sin\theta\left[\sqrt{\sin^2\phi+\cos^2\phi}\right]} = \boxed{-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}}$$

Let the expression for $\bar{F} = 2y\hat{i} - z\hat{j} + 3x\hat{k}$ in spherical polar coordinates be $\bar{F} = f_1 T_r + f_2 T_\theta + f_3 T_\phi$ — (1)

$f_1 = \bar{F}\cdot T_r$, $f_2 = \bar{F}\cdot T_\theta$ and $f_3 = \bar{F}\cdot T_\phi$

$f_1 = (2y\hat{i} - z\hat{j} + 3x\hat{k})\cdot(\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k})$

$f_1 = 2y\sin\theta\cos\phi - z\sin\theta\sin\phi + 3x\cos\theta$

$f_1 = 2(r\sin\theta\sin\phi)\sin\theta\cos\phi - r\cos\theta\sin\theta\sin\phi + 3r\sin\theta\cos\phi\cos\theta$

$f_1 = r\sin^2\theta\sin 2\phi - r\sin\theta\cos\theta\sin\phi + 3r\sin\theta\cos\theta\cos\phi$

$f_2 = \bar{F}\cdot T_\theta = \left[2y\hat{i} - z\hat{j} + 3x\hat{k}\right]\cdot\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$

$= 2y\cos\theta\cos\phi - z\cos\theta\sin\phi - 3x\sin\theta$

$= 2(r\sin\theta\sin\phi)\cos\theta\cos\phi - r\cos\theta\cos\theta\sin\phi - 3r\sin\theta\cos\phi\sin\theta$

$= r\sin\theta\cos\theta\sin 2\phi - r\cos^2\theta\sin\phi - 3r\sin^2\theta\cos\phi$

$f_3 = \bar{F}\cdot T_\phi = (2y\hat{i} - z\hat{j} + 3x\hat{k})\cdot(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j})$

$= -2y\sin\theta - z\cos\phi$

$= -2(r\sin\theta\sin\phi)\sin\phi - r\cos\theta\cos\phi$

$= -2r\sin\theta\sin^2\phi - r\cos\theta\cos\phi$

Substituting the values f_1, f_2, f_3 in (1) we get

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$\bar{F} = f_1 T_r + f_2 T_\theta + f_3 T_\phi$

$\bar{F} = [r\sin^2\theta \sin 2\phi - r\sin\theta\cos\theta\sin\phi + 3r\sin\theta\cos\theta\cos\phi]\, T_r$

$+ [r\sin\theta\cos\theta\sin 2\phi - r\cos^2\theta\sin\phi - 3r\sin^2\theta\cos\phi]\, T_\theta$

$+ [-2r\sin\theta\sin^2\phi - r\cos\theta\cos\phi]\, T_\phi$

$\bar{F} = [r\sin^2\theta\sin 2\phi - r\sin\theta\cos\theta\sin\phi + 3r\sin\theta\cos\theta\cos\phi]\, T_r$

$+ [r\sin\theta\cos\theta\sin 2\phi - r\cos^2\theta\sin\phi - 3r\sin^2\theta\cos\phi]\, T_\theta$

$- [2r\sin\theta\sin^2\phi + r\cos\theta\cos\phi]\, T_\phi$

2) P.T $\nabla(\cos\theta) \times \nabla\phi = \nabla\left(\frac{1}{r}\right)$, $r \neq 0$

Sol:- In spherical polar coordinates

$\nabla f = \frac{\partial f}{\partial r} T_r + \frac{1}{r}\frac{\partial f}{\partial \theta} T_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi} T_\phi$

$\nabla(\cos\theta) = 0 + \frac{1}{r}\frac{\partial}{\partial\theta}(\cos\theta)\, T_\theta + 0 = -\frac{1}{r}\sin\theta\, T_\theta$

$\nabla\phi = 0 + 0 + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\phi)\, T_\phi = \frac{1}{r\sin\theta} T_\phi$

$\nabla\left(\frac{1}{r}\right) = \frac{\partial}{\partial r}\left(\frac{1}{r}\right) T_r + 0 + 0 = -\frac{1}{r^2} T_r$

$\nabla(\cos\theta) \times \nabla\phi = \left(-\frac{1}{r}\sin\theta\, T_\theta\right) \times \left(\frac{1}{r\sin\theta} T_\phi\right)$

$= -\frac{1}{r^2}(T_\theta \times T_\phi) = -\frac{1}{r^2} T_r = \nabla\left(\frac{1}{r}\right)$

3) If $\bar{F} = r^3\cos\theta\, T_r - \frac{1}{r} T_\theta + \frac{1}{r\sin\theta} T_\phi$ find the value of $\bar{F} \times \text{curl}\, \bar{F}$

Sol:- In spherical polar coordinates

$$\text{curl}\, \bar{F} = \begin{vmatrix} T_r/r^2\sin\theta & T_\theta/r\sin\theta & T_\phi/r \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ f_1 & rf_2 & r\sin\theta f_3 \end{vmatrix}$$

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$f_1 = r\cos\theta$, $f_2 = -\frac{1}{r}$, $f_3 = \frac{1}{r\sin\theta}$.

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$$\text{curl } \bar{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} T_r & rT_\theta & r\sin\theta\, T_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ r^2\cos\theta & -1 & 1 \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left[r\sin\theta\, T_\phi\,(0 - r^2(-\sin\theta)\right] = \frac{r^3\sin^2\theta\, T_\phi}{r^2\sin\theta} = r\sin\theta\, T_\phi$$

$$\bar{F}\times\text{curl }\bar{F} = \left[r^2\cos\theta\, T_r - \frac{1}{r}T_\theta + \frac{1}{r\sin\theta}T_\phi\right]\times\left[r\sin\theta\, T_\phi\right]$$

$$= -r^3\sin\theta\cos\theta\, T_\theta - \sin\theta\, T_r$$

$$= -(r^3\sin\theta\cos\theta\, T_\theta + \sin\theta\, T_r)$$

2.

4) Express the following vectors in spherical polar coordinates

$x\bar{i} + 2y\,\bar{j} + yz\,\bar{k}$.

Sol:- we have $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

So that $\bar{R} = r\sin\theta\cos\phi\,\bar{i} + r\sin\theta\sin\phi\,\bar{j} + r\cos\theta\,\bar{k}$.

If T_r, T_θ, T_ϕ be the unit vectors along the tangents to r, θ, ϕ curves respectively, then

$$T_r = \frac{\partial\bar{R}}{\partial r}\Big/\left|\frac{\partial\bar{R}}{\partial r}\right| = \sin\theta\cos\phi\,\bar{i} + \sin\theta\sin\phi\,\bar{j} + \cos\theta\,\bar{k}.$$

$$T_\theta = \frac{\partial\bar{R}}{\partial\theta}\Big/\left|\frac{\partial\bar{R}}{\partial\theta}\right| = \cos\theta\cos\phi\,\bar{i} + \cos\theta\sin\phi\,\bar{j} - \sin\theta\,\bar{k}.$$

$$T_\phi = \frac{\partial\bar{R}}{\partial\phi}\Big/\left|\frac{\partial\bar{R}}{\partial\phi}\right| = -\sin\phi\,\bar{i} + \cos\phi\,\bar{j}$$

Let the expression for $\bar{F} = x\bar{i} + 2y\bar{j} + yz\bar{k}$ in spherical polar coordinates be $\bar{F} = f_1T_r + f_2T_\theta + f_3T_\phi$

Then $f_1 = \bar{F}\cdot T_r = (x\bar{i} + 2y\bar{j} + yz\bar{k})\cdot(\sin\theta\cos\phi\,\bar{i} + \sin\theta\sin\phi\,\bar{j} + \cos\theta\,\bar{k})$

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$= r\sin\theta\cos\phi$ ... $+ 2y \sin\theta\sin\phi$ ... $+ yz\cos\theta$

$= r\sin\theta\cos\phi\,\sin\theta\cos\phi + 2(r\sin\theta\sin\phi)\sin\theta\sin\phi + (r\sin\theta\sin\phi)(r\cos\theta)\cos\theta$

$= r\sin^2\theta\cos^2\phi + 2r\sin^2\theta\sin^2\phi + r^2\sin\theta\sin\phi\cos^2\theta$.

$f_2 = \bar{F}\cdot T_\theta = (x\hat{i} + 2y\hat{j} + yz\hat{k})\cdot(\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k})$

$= x\cos\theta\cos\phi + 2y\cos\theta\sin\phi - yz\sin\theta$

$= (r\sin\theta\cos\phi)\cos\theta\cos\phi + 2(r\sin\theta\sin\phi)\cos\theta\sin\phi$
$\quad - (r\sin\theta\sin\phi)(r\cos\theta)\sin\theta$.

$= r\sin\theta\cos\theta\cos^2\phi + r\sin 2\theta\sin^2\phi - r^2\sin^2\theta\cos\theta\sin\phi$

$f_3 = \bar{F}\cdot T_\phi = (x\hat{i} + 2y\hat{j} + yz\hat{k})\cdot(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j})$

$= -x\sin\phi + 2y\cos\phi$

$= -r\sin\theta\cos\phi\sin\phi + 2(r\sin\theta\sin\phi)\cos\phi$

$= -r\sin\theta\sin\phi\cos\phi + r\sin\theta\sin 2\phi$

$\bar{F} = f_1 T_r + f_2 T_\theta + f_3 T_\phi$

$\bar{F} = [r\sin^2\theta\cos^2\phi + 2r\sin^2\theta\sin^2\phi + r^2\sin\theta\sin\phi\cos^2\theta]\,T_r$

$+ [r\sin\theta\cos\theta\cos^2\phi + r\sin 2\theta\sin^2\phi - r^2\sin^2\theta\cos\theta\sin\phi]\,T_\theta$

$+ [r\sin\theta\sin\phi\cos\phi - r\sin\theta\sin 2\phi]\,T_\phi$

$=$

5) $2y\,\hat{i} + yz\,\hat{j} + 2x\,\hat{k}$

$\bar{R} = r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}$

$T_r = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$

$T_\theta = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$

$T_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$

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$\bar{F} = f_1 T_r + f_2 T_\theta + f_3 T_\phi$

Then $f_1 = \bar{F} \cdot T_r = xy \sin\theta\cos\phi + yz \sin\theta\sin\phi + zx\cos\theta$

$= (r\sin\theta\cos\phi)(r\sin\theta\sin\phi)\sin\theta\cos\phi$

$+ (r\sin\theta\sin\phi)(r\cos\theta)\sin\theta\sin\phi + (r\cos\theta)(r\sin\theta\cos\phi)\cos\theta$

$= r^2\sin^3\theta\sin\phi\cos^2\phi + r^2\sin^2\theta\cos\theta\sin^2\phi + r^2\sin\theta\cos^2\theta\cos\phi$

$f_2 = \bar{F} \cdot T_\theta = (xy\,\hat{i} + yz\,\hat{j} + zx\,\hat{k}) \cdot (\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k})$

$= xy\cos\theta\cos\phi + yz\cos\theta\sin\phi - zx\sin\theta$

$= (r\sin\theta\cos\phi)(r\sin\theta\sin\phi)\cos\theta\cos\phi + (r\sin\theta\sin\phi)(r\cos\theta)\cos\theta\sin\phi$

$- [r\cos\theta][r\sin\theta\cos\phi]\sin\theta$

$= r^2\sin^2\theta\cos\theta\sin\phi\cos^2\phi + r^2\sin\theta\cos^2\theta\sin^2\phi$

$- r^2\sin^2\theta\cos\theta\cos\phi$

$f_3 = \bar{F} \cdot T_\phi = (xy\,\hat{i} + yz\,\hat{j} + zx\,\hat{k}) \cdot (-\sin\phi\,\hat{i} + \cos\phi\,\hat{j})$

$= -xy\sin\phi + \cos\phi\, yz$

$= -(r\sin\theta\cos\phi)(r\sin\theta\sin\phi)\sin\phi + (r\sin\theta\sin\phi)(r\cos\theta)\cos\phi$

$= -r^2\sin^2\theta\sin^2\phi\cos\phi + r^2\sin\theta\cos\theta\sin\theta\cos\phi$

$\bar{F} = f_1 T_r + f_2 T_\theta + f_3 T_\phi$

$= [r^2\sin^3\theta\sin\phi\cos^2\phi + r^2\sin^2\theta\cos\theta\sin^2\phi + r^2\sin\theta\cos^2\theta\cos\phi]T_r$

$+ [r^2\sin^2\theta\cos\theta\sin\phi\cos^2\phi + r^2\sin\theta\cos^2\theta\sin^2\phi - r^2\sin^2\theta\cos\theta\cos\phi]T_\theta$

$- [r^2\sin^2\theta\sin^2\phi\cos\phi - r^2\sin\theta\cos\theta\sin\phi\cos\phi]$

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SPECIAL FUNCTIONS

BETA FUNCTION: The definite integral $\int_0^1 x^m (1-x)^{n-1} dx$, for $m>0, n>0$ is called the Beta function (or) The Eulerian integral of the first kind and is denoted by $B(m,n)$ and read as "Beta m, n"

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \; ; \; m>0, n>0$$

PROPERTIES OF BETA FUNCTION

① $B(m,n) = B(n,m)$

Proof:- By the definition

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

$$= \int_0^1 (1-x)^{m-1} (1-(1-x))^{n-1} dx$$

$$\boxed{\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx}$$

$$= \int_0^1 (1-x)^{m-1} x^{n-1} dx$$

$$= B(n,m)$$

$$\therefore \boxed{B(m,n) = B(n,m)}$$

2. $B(m,n) = 2 \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

Proof:- By the definition

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

put $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\, d\theta$

as $x \to 0, \theta \to 0$ and $x \to 1, \theta \to \pi/2$

$$B(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} (1-\sin^2\theta)^{n-1} 2\sin\theta\cos\theta\, d\theta$$

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$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta \cos^{2n-2}\theta \cdot 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

$$\therefore \beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

③ $\beta\left(\frac{1}{2},\frac{1}{2}\right) = \pi$

proof: We know that

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

put $m = n = \frac{1}{2}$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = 2\int_0^{\pi/2} \sin^{2(\frac{1}{2})-1}\theta \cos^{2(\frac{1}{2})-1}\theta\, d\theta$$

$$= 2\int_0^{\pi/2} d\theta$$

$$= 2[\theta]_0^{\pi/2} = 2\left[\frac{\pi}{2}\right] - 2(0)$$

$$\therefore \beta\left(\frac{1}{2},\frac{1}{2}\right) = \pi$$

DIFFERENT FORMS OF BETA FUNCTION

Form I: Show that $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$

Proof: By the definition $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$

put $x = \frac{y}{1+y} \Rightarrow 1 - \frac{1}{1+y}$.

$$\Rightarrow dx = \frac{1}{(1+y)^2}\, dy$$

As $x \to 0 : y \to 0$ and $x \to 1 : y \to \infty$

$$\beta(m,n) = \int_0^\infty \left[\frac{y}{(1+y)}\right]^{m-1} \left[1 - \frac{y}{1+y}\right]^{n-1} \frac{1}{(1+y)^2}\, dy$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m-1}} \cdot \frac{1}{(1+y)^{n-1}} \cdot \frac{1}{(1+y)^2}\, dy$$

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FORM: III Show that $\beta(m,n) = a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}}\,dx$ www.FirstRanker.com www.FirstRanker.com

Proof :- Consider $a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}}\,dx$

put $a\cdot x = bt$

$\Rightarrow x = \frac{b}{a}t \quad \Rightarrow dx = \frac{b}{a}dt$

As $x \to 0;\ t \to 0$ and as $x \to \infty;\ t \to \infty$

$$= a^m b^n \int_0^\infty \frac{\frac{b^{m-1}}{a^{m-1}}\,t^{m-1}}{(bt+b)^{m+n}} \cdot \frac{b}{a}\,dt$$

$$= \frac{a^m b^n}{a^m b^{m+n}}\, b^m \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\,dt$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m-1+n}}\,dx$$

$$= \beta(m,n)$$

Hence $\beta(m,n) = a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}}\,dx$

FORM: IV Show that $\int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx = (b-a)^{m+n-1}\beta(m,n)$

Proof :- Now $\int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx$

Put $x = a+(b-a)t \Rightarrow dx = (b-a)\,dt$

as $x \to a, t \to 0$ and as $x \to b;\ t \to 1$

$$= \int_0^1 [(b-a)t]^{m-1}[b-a-(b-a)t]^{n-1}(b-a)\,dt$$

$$= (b-a)^{m+n-1}\int_0^1 t^{m-1}(1-t)^{n-1}\,dt$$

$$= (b-a)^{m+n-1}\beta(m,n)$$

Hence $\int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx = (b-a)^{m+n-1}\beta(m,n)$

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$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$\therefore \int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

By property $B(m,n) = B(n,m)$

$$\therefore B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

Form : II Show that $B(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$

proof :- We know that

$$B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx \quad \text{——— (1)}$$

Consider $\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

put $x = \frac{1}{y} \Rightarrow dx = \frac{-1}{y^2}\,dy$

as $x \to 1 ; y \to 1$ and $x \to \infty ; y \to 0$

$$= \int_1^0 \frac{(\frac{1}{y})^{m-1}}{(1+\frac{1}{y})^{m+n}} \left(\frac{-1}{y^2}\right) dy$$

$$= \int_0^1 \frac{y^{m+n}}{y^{m-1}(1+y)^{m+n}} \cdot \frac{1}{y^2}\,dy$$

$$= \int_0^1 \frac{y^{m+n-m+1-2}}{(1+y)^{m+n}}\,dy$$

$$= \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}\,dy = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

From (1), we have $B(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$

Hence $B(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$

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Evaluate $B(m,n)$

Soln: By the definition

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$B(1,1) = \int_0^1 x^{1-1}(1-x)^{1-1}dx = \int_0^1 dx = [x]_0^1 = 1-0 = 1$$

$$\therefore B(1,1) = 1$$

② Evaluate $\int_0^1 \frac{x^2}{\sqrt{1-x^5}}dx$ interms of Beta function.

Soln: Consider $\int_0^1 \frac{x^2}{\sqrt{1-x^5}}dx$

put $x^5 = t \Rightarrow x = t^{1/5}$

$$dx = \frac{1}{5}t^{-4/5}dt$$

as $x \to 0; t \to 0$ and $x \to 1; t \to 1$

$$= \int_0^1 \frac{t^{2/5}}{\sqrt{1-t}} \frac{1}{5}t^{-4/5}dt$$

$$= \frac{1}{5}\int_0^1 t^{-2/5}(1-t)^{-1/2}dt$$

$$= \frac{1}{5}\int_0^1 t^{3/5-1}(1-t)^{\frac{1}{2}-1}dt$$

$$= \frac{1}{5}B\left(\frac{3}{5},\frac{1}{5}\right)$$

③ Evaluate $\int_0^1 x^5(1-x^3)^{10}dx$

Soln: Consider $\int_0^1 x^5(1-x^3)^{10}dx$

put $x^3 = t \Rightarrow x = t^{1/3}$

$$dx = \frac{1}{3}t^{-2/3}dt$$

as $x \to 0; t \to 0$ and $x \to 1, t \to 1$

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$$\int_0^1 t^{1/3}(1-t)^{10}\frac{1}{3}t^{-2/3}dt$$

$$= \frac{1}{3}\int_0^1 t\,(1-t)^{10}dt$$

$$= \frac{1}{3}\int_0^1 t^{2-1}(1-t)^{11-1}dt$$

$$= \frac{1}{3}B(2,11)$$

(4) Evaluate $\int_0^1 \frac{1}{(1-x^3)^{\frac{1}{3}}}dx$ using Beta function.

Soln: Consider $\int_0^1 \frac{1}{(1-x^3)^{\frac{1}{3}}}dx$

put $x^3 = t \Rightarrow x = t^{1/3}$

$\Rightarrow dx = \frac{1}{3}t^{-2/3}dt$

as $x \to 0$; $t \to 0$ and as $x \to 1$; $t \to 1$

$$= \int_0^1 \frac{1}{(1-t)^{\frac{1}{3}}}\frac{1}{3}t^{-2/3}dt$$

$$= \frac{1}{3}\int_0^1 t^{-2/3}(1-t)^{-1/3}dt$$

$$= \frac{1}{3}\int_0^1 t^{\frac{1}{3}-1}(1-t)^{\frac{2}{3}-1}dt$$

$$= \frac{1}{3}B\left(\frac{1}{3},\frac{2}{3}\right)$$

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Def:- The definite integral $\int_0^\infty e^{-x} x^{n-1} dx$; $n>0$ is called the Gamma function (or) Eulerian integral of the second kind and is denoted by $\Gamma(n)$ and read as "Gamma n"

Thus $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$; $n>0$

PROPERTIES OF GAMMA FUNCTION

1. Show that $\Gamma(1) = 1$

Proof: By the definition of Gamma function

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

$$\Gamma(1) = \int_0^\infty e^{-x} dx = \left[\frac{e^{-x}}{-1}\right]_0^\infty = -[e^{-\infty} - e^{-0}]$$

$$= -[0-1] = 1$$

$$\therefore \boxed{\Gamma(1) = 1}$$

② Show that $\Gamma(n+1) = n\Gamma(n)$ for $n>0$

proof:-

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^{n} dx$$

$$= \left[-x^n e^{-x}\right]_0^\infty + n\int_0^\infty e^{-x} x^{n-1} dx$$

$$= 0 + n\Gamma(n)$$

$$\boxed{\Gamma(n+1) = n\Gamma(n)}$$

③ Show that $\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1} dx$

proof: By the definition,

we have $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$

put $x = t^2$

$dx = 2t\,dt$

as $x \to 0$, $t \to 0$ and as $x \to \infty$, $t \to \infty$

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$$\Gamma(n) = \int_0^\infty e^{-t^2} t^{2n-2}\, 2t\, dt$$

$$= 2\int_0^\infty e^{-t^2} t^{2n-1}\, dt$$

Hence $\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1}\, dx$

(4) Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof:- By the def; we have $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$

We have $\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1}\, dx$

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2} dx \quad \text{—— (1)}$$

Also, we have $\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-y^2} dy$ —— (2)

(1) × (2) $\Rightarrow$ $\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$

Changing into polar coordinates we have

$x = r\cos\theta$; $y = r\sin\theta$; $dx\,dy = r\,dr\,d\theta$

θ varies from 0 to $\pi/2$ and r varies from 0 to ∞

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2} r\, dr\, d\theta$$

put $r^2 = t \Rightarrow 2r\,dr = dt$

$\Rightarrow r\,dr = \frac{dt}{2}$

$$= 2\int_0^{\pi/2}\int_0^\infty e^{-t}\, dt\, d\theta$$

$$= 2\int_0^{\pi/2} \left[\frac{e^{-t}}{-1}\right]_0^\infty d\theta$$

$$= 2\int_0^{\pi/2} -[e^{-\infty} - e^{0}]\, d\theta$$

$$= 2\int_0^{\pi/2} d\theta$$

$$= 2[\theta]_0^{\pi/2} \; ; \; 2\left(\frac{\pi}{2}\right) - 0 = \pi$$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi$$

$$\therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

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Evaluate $\int_0^\infty x^\nu e^{-x^\nu} dx$

Soln: consider $\int_0^\infty x^\nu e^{-x^\nu} dx$

put $x^\nu = t \Rightarrow x = t^{1/\nu}$

$dx = \frac{1}{\nu} t^{-1/\nu} dt$

as $x \to 0; t \to 0$ and as $x \to \infty, t \to \infty$

$= \int_0^\infty t e^{-t} \frac{1}{\nu} t^{-1/\nu} dt$

$= \frac{1}{\nu} \int_0^\infty e^{-t} t^{1/\nu} dt$

$= \frac{1}{\nu} \int_0^\infty e^{-t} t^{\frac{3}{\nu}-1} dt$

$= \frac{1}{\nu} \Gamma(\frac{3}{\nu})$

$= \frac{1}{\nu} (\frac{3}{\nu}-1) \Gamma(\frac{3}{\nu}-1)$

$= \frac{1}{\nu} (\frac{1}{\nu}) \Gamma(\frac{1}{\nu})$

$= \frac{\sqrt{\pi}}{4}$

② Show that $\Gamma(n) = \int_0^1 (\log \frac{1}{x})^{n-1} dx, \quad n > 0$

Soln: R.H.S $\int_0^1 (\log \frac{1}{x})^{n-1} dx$

put $t = \log \frac{1}{x} = -\log x$

$= \int_\infty^0 t^{n-1} (-e^{-t}) dt$

$= \int_0^\infty e^{-t} t^{n-1} dt$

$= \Gamma(n)$

$= L.H.S$

$\Rightarrow -t = \log x$

$\Rightarrow e^{-t} = x$

$\therefore x = e^{-t}$

$dx = -e^{-t} dt$

as $x \to 0; t \to \infty, dx = -1; t = 0$

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Show that $\int_0^1 x^3\left(\log\frac{1}{x}\right)^4 dx = \frac{3}{128}$

Soln:-

L.H.S $\int_0^1 x^3\left(\log\frac{1}{x}\right)^4 dx$

put $\log\frac{1}{x} = t$

$\Rightarrow \frac{1}{x} = e^t$

$\Rightarrow x = e^{-t}$

$dx = -e^{-t}dt$

as $x \to 0; t \to \infty$ and

as $x \to 1; t \to 0$

$= \int_\infty^0 e^{-3t} t^4 (-e^{-t})\, dt$

$= \int_0^\infty e^{-4t} t^4\, dt$

again put $4t = u$

$4dt = du$

$dt = \frac{1}{4}du$

$= \int_0^\infty e^{-u}\left(\frac{u}{4}\right)^4 \frac{1}{4}\, du$

$= \int_0^\infty e^{-u}\frac{u^4}{4^5}\, du$

$= \frac{1}{4^5}\int_0^\infty e^{-u} u^{5-1}\, du$

$= \frac{1}{4^5}\Gamma(5)$

$= \frac{4!}{4^5} = \frac{4\cdot 3\cdot 2\cdot 1}{4\times 4\times 64} = \frac{3}{128}$

$= R.H.S$

$\int_0^1 \left(x^3\log\frac{1}{x}\right)^4 dx = \frac{3}{128}$

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Evaluate $\int_0^\infty 3^{-4x^2}\,dx$

Soln:- Consider $\int_0^\infty 3^{-4x^2}\,dx$

$= \int_0^\infty e^{\log e^{3^{-4x^2}}}\,dx$

$= \int_0^\infty e^{-4x^2 \log 3}\,dx$

$e^{\log_e f(x)} = f(x)$

$4x^2 \log 3 = t$

$\Rightarrow x^2 = \frac{1}{4\log 3}\,t$

$\Rightarrow x = \frac{1}{2}\frac{1}{\sqrt{\log 3}}\,t^{\frac{1}{2}}$

$\Rightarrow dx = \frac{1}{2}\frac{1}{\sqrt{\log 3}}\,\frac{1}{2}\,t^{-\frac{1}{2}}\,dt$

$= dx = \frac{1}{4}\frac{1}{\sqrt{\log 3}}\,t^{-\frac{1}{2}}\,dt$

$= \int_0^\infty e^{-t}\,\frac{1}{4}\frac{1}{\sqrt{\log 3}}\,t^{-\frac{1}{2}}\,dt$

$= \frac{1}{4\sqrt{\log 3}} \int_0^\infty e^{-t}\,t^{\frac{1}{2}-1}\,dt$

$= \frac{1}{4\sqrt{\log 3}}\,\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{4\sqrt{\log 3}}$

$\therefore \int_0^\infty 3^{-4x^2}\,dx = \frac{\sqrt{\pi}}{4\sqrt{\log 3}}$

(5) Show that $\int_0^\infty \frac{x^a}{a^x}\,dx = \frac{\Gamma(a+1)}{(\log a)^{a+1}}$ $(a>1)$

Soln: LHS $\int_0^\infty \frac{x^a}{a^x}\,dx$

Put $a^x = e^t$

$\Rightarrow x \log a = t$

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$x = \frac{1}{\log a} t$

$dx = \frac{1}{\log a} dt$

As $x \to 0$; $t \to 0$ and $x \to \infty$, $t \to \infty$

$$= \int_0^\infty \left(\frac{t}{\log a}\right)^a e^{-t} \frac{1}{\log a} dt$$

$$= \frac{1}{(\log a)^{a+1}} \int_0^\infty e^{-t} dt \cdot t^a$$

$$= \frac{1}{(\log a)^{a+1}} \int_0^\infty e^{-t} t^{(a+1)-1} dt$$

$$= \frac{1}{(\log a)^{a+1}} \Gamma(a+1)$$

$$= \frac{1}{(\log a)^{a+1}}$$

$$= R.H.S$$

$$\therefore \int_0^a \frac{x^a}{a^x} dx = \frac{\Gamma(a+1)}{(\log a)^{a+1}}$$

(5) Show that $\int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma(n)}{k^n}$

Soln:

L.H.S $\int_0^\infty e^{-kx} x^{n-1} dx$

put $kx = t \Rightarrow x = \frac{1}{k} t$

$dx = \frac{1}{k} dt$

$$= \int_0^\infty e^{-t} \left(\frac{t}{k}\right)^{n-1} \frac{1}{k} dt$$ as $x \to 0$; $t \to 0$ and $x \to \infty$, $t \to \infty$

$$= \int_0^\infty e^{-t} \frac{t^{n-1}}{k^{n-1}} \cdot \frac{1}{k} dt$$

$$= \frac{1}{k^n} \int_0^\infty e^{-t} t^{n-1} dt$$

$$= \frac{\Gamma(n)}{k^n}$$

$$= R.H.S$$

$$\therefore \int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma(n)}{k^n}.$$

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Prove that $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ Where $m>0, n>0$

Proof :- We know that

$$\Gamma(m) = 2\int_0^\infty e^{-x^2} x^{2m-1}\,dx \quad \text{——} (1)$$

Ily $$\Gamma(n) = 2\int_0^\infty e^{-y^2} y^{2n-1}\,dy \quad \text{——} (2)$$

$(1)\times(2) \Rightarrow$ $$\Gamma m\,\Gamma n = 4\int_0^\infty e^{-x^2} x^{2m-1}\,dx \int_0^\infty e^{-y^2} y^{2n-1}\,dy$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1}\,dy \quad \text{——} (3)$$

Changing to polar co-ordinates by writing

$x = r\cos\theta$; $y = r\sin\theta$; $dx\,dy = r\,dr\,d\theta$.

r varies from 0 to ∞ and θ varies from 0 to $\pi/2$

$$\Gamma(m)\Gamma n = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2} (r\cos\theta)^{2m-1} (r\sin\theta)^{2n-1} r\,dr\,d\theta$$

$$= 2\int_{\theta=0}^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\,d\theta \; 2\int_{r=0}^{\infty} e^{-r^2} r^{2(m+n)-1}\,dr$$

$$= \beta(m,n)\,\Gamma(m+n)$$

$$\frac{\Gamma m\,\Gamma n}{\Gamma(m+n)} = \beta(m,n)$$

$$\therefore \beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$$

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Prove that $\Gamma n \, \Gamma(1-n) = \frac{\pi}{\sin n\pi}$ if n is not an integer

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Soln: we know that

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)} \quad \text{— (1)}$$

and also $\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$ — (2)

$$\therefore \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

put $m = 1-n$

$\Rightarrow m+n = 1$

$$\frac{\Gamma(1-n)\,\Gamma n}{\Gamma 1} = \int_0^\infty \frac{x^{n-1}}{(1+x)} dx$$

$$\Rightarrow \Gamma n \, \Gamma(1-n) = \int_0^\infty \frac{x^{n-1}}{(1+x)} dx$$

$\Rightarrow$ But we have $\int_0^\infty \frac{x^{n-1}}{(1+x)} dx = \frac{\pi}{\sin n\pi}$ for non integer n

$$\therefore \Gamma n \, \Gamma(1-n) = \frac{\pi}{\sin n\pi}$$ for non integer n.

(2) Prove that $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}} dx = 0$ using β and Γ functions

Soln:

LHS $\int_0^\infty \frac{x^8}{(1+x)^{24}} dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}$

$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}} dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}} dx$

$= \beta(9,15) - \beta(15,9)$

$= \beta(9,15) - \beta(9,15)$ $(\because \beta(m,n) = \beta(n,m)]$

$= 0$

$=$ R.H.S

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(3) Evaluate $\int_0^2 (8-x^3)^{-1/3}\,dx$ using β and Γ functions

Soln: Consider $\int_0^2 (8-x^3)^{-1/3}\,dx$

Put $x^3 = 8t$

$\Rightarrow x = 2t^{1/3}$

$dx = \frac{2}{3}t^{-2/3}\,dt$

As $x \to 0$, $t \to 0$ and as $x \to 2$, $t \to 1$

$$= \int_0^1 (8-8t)^{-1/3}\,\frac{2}{3}t^{-2/3}\,dt$$

$$= \frac{2}{3}(8)^{-1/3}\int_0^1 t^{-2/3}(1-t)^{-1/3}\,dt$$

$$= \frac{1}{3}\int_0^1 t^{\frac{1}{3}-1}(1-t)^{\frac{2}{3}-1}\,dt$$

$$= \frac{1}{3}\beta\left(\frac{1}{3},\frac{2}{3}\right)$$

$$= \frac{1}{3}\frac{\Gamma 1/3\,\Gamma 2/3}{\Gamma \frac{1}{3}+\frac{2}{3}}$$

$$= \frac{1}{3}\Gamma 1/3\,\Gamma 1-1/3$$

$$= \frac{1}{3}\frac{\pi}{\sin \pi/3}$$

$$= \frac{\pi}{3}\frac{2}{\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$

(4) Express $\int_0^1 x^m(1-x^n)^p\,dx$ interms of Γ function.

Soln: Consider $\int_0^1 x^m(1-x^n)^p\,dx$

Put $x^n = t$

$\Rightarrow x = t^{1/n}$

$dx = \frac{1}{n}t^{\frac{1}{n}-1}\,dt$

$$= \int_0^1 t^{m/n}(1-t)^p\,\frac{1}{n}t^{\frac{1}{n}-1}\,dt$$

$$= \frac{1}{n}\int_0^1 t^{\left(\frac{m}{n}+\frac{1}{n}-1\right)}(1-t)^p\,dt$$

As $x \to 0$, $t \to 0$ and $x \to 1$, $t \to 1$

$$= \frac{1}{n}\int_0^1 t^{\left(\frac{m+1}{n}\right)-1}(1-t)^{(p+1)-1}\,dt$$

$$= \frac{1}{n}\beta\left(\frac{m+1}{n},p+1\right)$$

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$$\frac{\frac{1}{n}\,\Gamma\frac{m+1}{n}\,\Gamma(p+1)}{\Gamma\left(\frac{m+1}{n}+p+1\right)}$$

(5) Evaluate $\int_0^1 x^5(1-x^3)^{10}\,dx$

Soln: We know that

$$\int_0^1 x^m(1-x^n)^p\,dx = \frac{1}{n}\,\frac{\Gamma\frac{m+1}{n}\,\Gamma(p+1)}{\Gamma\left(\frac{m+1}{n}+p+1\right)}$$

Here $m=5;\ n=3;\ p=10$

$$\int_0^1 x^5(1-x^3)^{10}\,dx = \frac{1}{3}\,\frac{\Gamma\frac{6}{3}\,\Gamma 11}{\Gamma\left(\frac{6}{3}+11\right)} = \frac{1}{3}\,\frac{\Gamma 2\,\Gamma 11}{\Gamma 13}$$

$$= \frac{1}{396}$$

(6) Show that $\Gamma x\,\Gamma(-x) = \frac{-\pi}{x\sin\pi x}$

Soln: We know that

$$\Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

Now $\Gamma x\,\Gamma(-x) = \Gamma x\left[\frac{-1}{x}\,\Gamma(-x+1)\right]$ $\quad \left[\Gamma n = \frac{1}{n}\Gamma(n+1)\right]$

$$= \frac{-1}{x}\,\Gamma x\,\Gamma(1-x)$$

$$= \frac{-1}{x}\,\frac{\pi}{\sin\pi x}$$

$$= \frac{-\pi}{x\sin\pi x}$$

(7) Show that $\Gamma\left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}-x\right) = \pi\sec\pi x$

Soln: We know that $\Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

$$\Gamma\left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}-x\right) = \Gamma\left(\frac{1}{2}+x\right)\Gamma\left(1-\left(\frac{1}{2}+x\right)\right)$$

$$= \frac{\pi}{\sin\left[\left(\frac{1}{2}+x\right)\pi\right]}$$

$$= \frac{\pi}{\sin\left(\frac{\pi}{2}+\pi x\right)} = \frac{\pi}{\cos\pi x} = \pi\sec\pi x$$

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Evaluate $\int_0^\infty \frac{x^2}{1+x^4}\,dx$ using β and Γ functions (8)

Soln:-

Consider $4\int_0^\infty \frac{x^2}{1+x^4}\,dx$

put $x^4 = t$

$\Rightarrow x = t^{1/4}$

$dx = \frac{1}{4}t^{-3/4}dt$

as $x \to 0; t \to 0$ and as $x \to \infty, t \to \infty$

$$= 4\int_0^\infty \frac{t^{2/4}\,t^{-3/4}}{4(1+t)}\,dt$$

$$= \int_0^\infty \frac{t^{-1/4}}{(1+t)}\,dt$$

$$= \int_0^\infty \frac{t^{3/4-1}}{(1+t)^{3/4+1/4}}\,dt$$

$$= \beta\left(\frac{3}{4}, \frac{1}{4}\right)$$

$$= \frac{\lceil 3/4 \lceil 1/4}{\lceil 3/4+1/4}$$

$$= \lceil 3/4 \lceil 1/4$$

$$= \lceil 1/4 \lceil 1-\frac{1}{4} = \frac{\pi}{\sin\frac{\pi}{4}} = \sqrt{2}\pi$$

(9) Evaluate $\int_0^{\pi/2} \sin^4\theta \cos^2\theta\,d\theta$

Soln:

We know that

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta\,d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

put $m = 4$; $n = 2$

$$\int_0^{\pi/2} \sin^4\theta \cos^2\theta\,d\theta = \frac{1}{2}\beta\left(\frac{4+1}{2}, \frac{2+1}{2}\right)$$

$$= \frac{1}{2}\beta\left(\frac{5}{2}, \frac{3}{2}\right)$$

$$= \frac{1}{2}\frac{\lceil 5/2 \lceil 3/2}{\lceil 5/2+3/2}$$

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$$= \frac{1}{2}\left(\frac{5}{2}-1\right)\left(\frac{3}{2}-1\right)\sqrt{\frac{3}{2}-1}\left(\frac{3}{2}-1\right)\sqrt{\frac{3}{2}-1}$$

$$= \frac{1}{6}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\ \frac{1}{2}\sqrt{\pi} = \frac{\pi}{16}.$$

(10) Show that $\int_0^\infty \sin x^2\,dx = \int_0^\infty \cos x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$

Soln: Let $A = \int_0^\infty \cos x^2\,dx$ and $B = \int_0^\infty \sin x^2\,dx$

Consider $A - iB = \int_0^\infty [\cos x^2 - i\sin x^2]\,dx = \int_0^\infty e^{-ix^2}\,dx$

put $ix^2 = t$

$\Rightarrow e^{i\pi/2}x^2 = t$

$\Rightarrow e^{i\pi/2}x = \sqrt{t}$

$\Rightarrow x = e^{-i\pi/2}\sqrt{t}$

$\Rightarrow dx = e^{-i\pi/2}\frac{1}{2\sqrt{t}}\,dt$

as $x \to 0; t \to 0$ and as $x \to \infty; t \to \infty$

$$A - iB = \int_0^\infty e^{-t}\,e^{-i\pi/2}\frac{1}{2\sqrt{t}}\,dt$$

$$= \frac{e^{-i\pi/2}}{2}\int_0^\infty e^{-t}t^{\frac{1}{2}-1}\,dt = \frac{e^{-i\pi/4}}{2}\Gamma\frac{1}{2}$$

$$= \frac{\sqrt{\pi}}{2}\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right]$$

$$= \frac{\sqrt{\pi}}{2}\left[\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right]$$

$$A - iB = \frac{1}{2}\sqrt{\frac{\pi}{2}} - i\frac{1}{2}\sqrt{\frac{\pi}{2}}$$

Comparing real and imaginary parts we have

$$A = \frac{1}{2}\sqrt{\frac{\pi}{2}} \text{ and } B = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

Hence $\int_0^\infty \cos x^2\,dx = \int_0^\infty \sin x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$

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