

Differentiation of vector:-

If $\vec{r} = \vec{f}(t)$ is a vector function, then differentiation of $\vec{r}$ w.r.t to t

$$\text{is } \frac{d\vec{f}}{dt} = \lim_{\delta t \rightarrow 0} \frac{\vec{f}(t+\delta t) - \vec{f}(t)}{\delta t} = \frac{d\vec{r}}{dt}$$

$$\text{If } \lim_{\delta t \rightarrow 0} \frac{\delta \vec{r}}{\delta t} = \lim_{\delta t \rightarrow 0} \frac{\vec{f}(t+\delta t) - \vec{f}(t)}{\delta t} \text{ exists;}$$

then the value of this limit is denoted by $\frac{d\vec{r}}{dt}$ and is called the derivative of $\vec{r}$ w.r.t to t .

Since $\frac{d\vec{r}}{dt}$ is itself a vector function of t , its derivative is denoted by $\frac{d^2\vec{r}}{dt^2}$ and is called the second derivative of $\vec{r}$ w.r.t to t . By we can define higher order derivatives of $\vec{r}$.

Important Formulae of Differentiation of vectors

$$(i) \quad \frac{d}{dt} (\vec{F} + \vec{G} - \vec{H}) = \frac{d\vec{F}}{dt} + \frac{d\vec{G}}{dt} - \frac{d\vec{H}}{dt}$$

$$(ii) \quad \frac{d}{dt} [\vec{F} \phi] = \vec{F} \frac{d\phi}{dt} + \frac{d\vec{F}}{dt} \phi$$

$$(iii) \quad \frac{d}{dt} [\vec{F} \cdot \vec{G}] = \vec{F} \cdot \frac{d\vec{G}}{dt} + \frac{d\vec{F}}{dt} \cdot \vec{G}$$

$$(iv) \quad \frac{d}{dt} [\vec{F} \times \vec{G}] = \vec{F} \times \frac{d\vec{G}}{dt} + \frac{d\vec{F}}{dt} \times \vec{G}$$

$$\& \quad \frac{d}{dt} [\vec{F} \vec{G} \vec{H}] = \left[\frac{d\vec{F}}{dt} \vec{G} \vec{H} \right] + \left[\vec{F} \frac{d\vec{G}}{dt} \vec{H} \right] + \left[\vec{F} \vec{G} \frac{d\vec{H}}{dt} \right]$$

$$(vi) \quad \frac{d}{dt} [(\vec{F} \times \vec{G}) \times \vec{H}] = \left(\frac{d\vec{F}}{dt} \times \vec{G} \right) \times \vec{H} + (\vec{F} \times \frac{d\vec{G}}{dt}) \times \vec{H} + (\vec{F} \times \vec{G}) \times \frac{d\vec{H}}{dt}$$

where $\vec{F}, \vec{G}, \vec{H}$ are vectors and ϕ, t scalars.

1) If $\vec{F}$ be a vector of constant magnitude and direction then $\vec{F} \cdot \frac{d\vec{F}}{dt} = 0$.

Proof:- $\vec{F}(t)$ has a constant magnitude $\Rightarrow |\vec{F}(t)| = \text{constant}$

$$\vec{F}(t) \cdot \vec{F}(t) = |\vec{F}(t)|^2 = \text{constant}$$

$$\Rightarrow \frac{d}{dt} (\vec{F} \cdot \vec{F}) = 0 \Rightarrow \vec{F} \cdot \frac{d\vec{F}}{dt} + \frac{d\vec{F}}{dt} \cdot \vec{F} = 0$$

$$\Rightarrow 2\vec{F} \cdot \frac{d\vec{F}}{dt} = 0 \Rightarrow \vec{F} \cdot \frac{d\vec{F}}{dt} = 0$$

$$\Rightarrow \frac{d\vec{F}}{dt} \perp \vec{F}$$

2) If $\vec{F}(t)$ has constant direction then $\vec{F} \times \frac{d\vec{F}}{dt} = \vec{0}$

Proof:- Let $|\vec{F}(t)| = f(t)$

Let $\vec{a}(t)$ be a unit vector in the direction of $\vec{F}(t)$

$$\text{So that } \vec{F}(t) = f(t) \vec{a}(t)$$

$$\frac{d\vec{F}}{dt} = f \frac{d\vec{a}}{dt} + \frac{df}{dt} \vec{a} \quad \text{--- (1)}$$

If $\vec{F}(t)$ has constant direction so has $\vec{a}(t)$

Thus $\vec{a}(t)$ is a constant vector and $\frac{d\vec{a}}{dt} = \vec{0}$

$$\text{From (1)} \quad \frac{d\vec{F}}{dt} = \frac{df}{dt} \vec{a}$$

$$\vec{F} \times \frac{d\vec{F}}{dt} = f \vec{a} \times \left(\frac{df}{dt} \vec{a} \right)$$

$$= f \frac{df}{dt} (\vec{a} \times \vec{a}) = \vec{0} \quad [\because \vec{a} \times \vec{a} = \vec{0}]$$

(ii) $\frac{d}{dt}(\vec{A} \times \vec{B})$

Sol: (i) $\frac{d}{dt}(\vec{A} \cdot \vec{B}) = \vec{A} \cdot \frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt} \cdot \vec{B}$

$= (5t^2\vec{i} + t\vec{j} - t^3\vec{k}) \cdot [\cos t\vec{i} - (-\sin t\vec{j})] + (10t\vec{i} + \vec{j} - 3t^2\vec{k}) \cdot (\sin t\vec{i} - \cos t\vec{j})$

$= (5t^2 \cos t + t \sin t) + 10t \sin t - \cos t$

$= 5t^2 \cos t + 11t \sin t - \cos t$

(ii) $\frac{d}{dt}(\vec{A} \times \vec{B}) = \vec{A} \times \frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt} \times \vec{B}$

$= (5t^2\vec{i} + t\vec{j} - t^3\vec{k}) \times (\cos t\vec{i} + \sin t\vec{j}) + (10t\vec{i} + \vec{j} - 3t^2\vec{k}) \times (\sin t\vec{i} - \cos t\vec{j})$

$= 5t^2 \sin t \vec{k} - t \cos t \vec{k} - t^3 \cos t \vec{j} + 3t^2 \sin t \vec{i}$
 $- 10t \cos t \vec{k} - \sin t \vec{k} - 3t^2 \sin t \vec{j} - 3t^2 \cos t \vec{i}$

$= (t^3 \sin t - 3t^2 \cos t) \vec{i} - t^2 (t \cos t + 3 \sin t) \vec{j}$
 $+ [(5t^2 - 1) \sin t - 11t \cos t] \vec{k}$

2) Show that if $\vec{R} = A \sin \omega t + B \cos \omega t$ where A, B, ω

are constants, then $\frac{d^2 \vec{R}}{dt^2} = -\omega^2 \vec{R}$ and $\vec{R} \cdot \frac{d\vec{R}}{dt} = -\omega^2 \vec{R} \cdot \vec{R}$

Sol: $\vec{R} = A \sin \omega t + B \cos \omega t$

$\frac{d\vec{R}}{dt} = A\omega \cos \omega t - B\omega \sin \omega t$

$\frac{d^2 \vec{R}}{dt^2} = -A\omega^2 \sin \omega t - B\omega^2 \cos \omega t$

$= -\omega^2 (A \sin \omega t + B \cos \omega t) = -\omega^2 \vec{R}$

$$\begin{aligned}
 \vec{R} \times \frac{d\vec{R}}{dt} &= (A \sin \omega t + B \cos \omega t) \times (A \omega \cos \omega t - B \omega \sin \omega t) \\
 &= -\omega \sin^2 \omega t (\vec{A} \times \vec{B}) + \omega \cos^2 \omega t (\vec{B} \times \vec{A}) \\
 &= -\omega \sin^2 \omega t (\vec{A} \times \vec{B}) - \omega \cos^2 \omega t (\vec{A} \times \vec{B}) \quad \left[\begin{array}{l} \vec{A} \times \vec{A} = \vec{0} \\ \vec{B} \times \vec{B} = \vec{0} \end{array} \right] \\
 &= -\omega (\vec{A} \times \vec{B}) [\sin^2 \omega t + \cos^2 \omega t] \quad \sin^2 \omega t + \cos^2 \omega t = 1 \\
 &= -\omega (\vec{A} \times \vec{B}) \quad \vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})
 \end{aligned}$$

2) Given $\vec{R} = t^m \vec{A} + t^n \vec{B}$ where $\vec{A}, \vec{B}$ are constant vectors.
Show that if $\vec{R}$ and $\frac{d^2\vec{R}}{dt^2}$ are parallel vectors, then
 $m+n=1$ unless $m=n$.

Sol: Given $\vec{R} = t^m \vec{A} + t^n \vec{B}$

$$\frac{d\vec{R}}{dt} = m t^{m-1} \vec{A} + n t^{n-1} \vec{B}$$

$$\frac{d^2\vec{R}}{dt^2} = m(m-1) t^{m-2} \vec{A} + n(n-1) t^{n-2} \vec{B}$$

For parallel vector $\vec{R} \times \frac{d^2\vec{R}}{dt^2} = \vec{0}$

$$\Rightarrow (t^m \vec{A} + t^n \vec{B}) \times [m(m-1) t^{m-2} \vec{A} + n(n-1) t^{n-2} \vec{B}] = \vec{0}$$

$$\Rightarrow n(n-1) t^{m+n-2} (\vec{A} \times \vec{B}) + m(m-1) t^{n+m-2} (\vec{B} \times \vec{A}) = \vec{0} \quad \left[\begin{array}{l} \vec{A} \times \vec{B} \\ = -\vec{B} \times \vec{A} \end{array} \right]$$

$$\Rightarrow n(n-1) t^{m+n-2} (\vec{A} \times \vec{B}) - m(m-1) t^{m+n-2} (\vec{A} \times \vec{B}) = \vec{0}$$

$$\Rightarrow t^{m+n-2} (\vec{A} \times \vec{B}) [n^2 - n - m^2 + m] = \vec{0}$$

$$\Rightarrow t^{m+n-2} (\vec{A} \times \vec{B}) [(n+m)(n-m) - (n-m)] = \vec{0}$$

$$\Rightarrow t^{m+n-2} (\vec{A} \times \vec{B}) [(n-m)(m+n-1)] = \vec{0} \Rightarrow \begin{array}{l} m=n \text{ either} \\ m+n=1 \end{array}$$

find (i) $\frac{d}{dt}(\vec{p} \cdot \vec{a})$ (ii) $\frac{d}{dt}(\vec{p} \times \vec{a})$

Sol: $\vec{p} \cdot \vec{a} = 10t^2 \sin t - t^3 \cos t - 5t^2$

$$\frac{d}{dt}(\vec{p} \cdot \vec{a}) = 10t^2 \cos t + 20t \sin t - 3t^2 \cos t + t^3 \sin t - 10t$$

$$= t^3 \sin t + 7t^2 \cos t + 20t \sin t - 10t$$

$$\vec{p} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t^3 & -t \\ 2 \sin t & -\cos t & 5t \end{vmatrix} = \hat{i}(5t^4 - t \cos t) + \hat{j}(-t^2 2 \sin t + 25t^3) + \hat{k}(-5t^2 \cos t - t^3 2 \sin t)$$

$$= (5t^4 - t \cos t) \hat{i} - \hat{j}(2t^2 \sin t + 25t^3) - \hat{k}(5t^2 \cos t + 2t^3 \sin t)$$

4) If $\frac{d\vec{u}}{dt} = \vec{\omega} \times \vec{u}$ and $\frac{d\vec{v}}{dt} = \vec{\omega} \times \vec{v}$

P.T: $\frac{d}{dt}(\vec{u} \times \vec{v}) = \vec{\omega} \times (\vec{u} \times \vec{v})$

Sol: $\frac{d}{dt}(\vec{u} \times \vec{v}) = \vec{u} \times \frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt} \times \vec{v}$

$$= \vec{u} \times (\vec{\omega} \times \vec{v}) + (\vec{\omega} \times \vec{u}) \times \vec{v}$$

$$= (\vec{u} \cdot \vec{v})\vec{\omega} - (\vec{u} \cdot \vec{\omega})\vec{v} + (\vec{v} \cdot \vec{\omega})\vec{u} - (\vec{v} \cdot \vec{u})\vec{\omega}$$

$$= (\vec{\omega} \cdot \vec{v})\vec{u} - (\vec{\omega} \cdot \vec{u})\vec{v}$$

$$= \vec{\omega} \times (\vec{u} \times \vec{v})$$

$$(\vec{u} \cdot \vec{v}) = (\vec{v} \cdot \vec{u})$$

$$(\vec{v} \cdot \vec{\omega}) = (\vec{\omega} \cdot \vec{v})$$

$$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{c} \cdot \vec{a})\vec{b} - (\vec{c} \cdot \vec{b})\vec{a}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

5) If $\vec{A} = x^2 y \hat{i} - x y^2 \hat{j} + x z^2 \hat{k}$

and $\vec{B} = xz \hat{i} + yz \hat{j} - x^2 \hat{k}$, find $\frac{d}{dx}(\vec{A} \times \vec{B})$ at $(1, 2, -3)$

Sol: $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x^2 y & -x y^2 & x z^2 \\ xz & yz & -x^2 \end{vmatrix} = \hat{i}(x^2 z^3 - x y^2 z^2) + \hat{j}(x^3 z^2 + x^2 y z^2) + \hat{k}(x^2 y^2 z + 4 x^2 z^3)$

$$\frac{d^2}{dx dy} (\vec{A} \times \vec{B}) = (0 - z^2) \hat{i} + (0 + 4xz^2) \hat{j} + (4xy^2 + 0) \hat{k}$$

$$\frac{d^2}{dx dy} (\vec{A} \times \vec{B}) = 4(\hat{i} + 2\hat{j}) \text{ at } (1, 0, -2)$$

Q) If $\vec{r} = a \cos t \hat{i} + a \sin t \hat{j} + at \tan \alpha \hat{k}$

Find the value of $\left| \frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} \right|$ (ii) $\left[\frac{d\vec{r}}{dt}, \frac{d^2\vec{r}}{dt^2}, \frac{d^3\vec{r}}{dt^3} \right]$

Sol: $\vec{r} = a \cos t \hat{i} + a \sin t \hat{j} + at \tan \alpha \hat{k}$

$$\frac{d\vec{r}}{dt} = -a \sin t \hat{i} + a \cos t \hat{j} + a \tan \alpha \hat{k}$$

$$\frac{d^2\vec{r}}{dt^2} = -a \cos t \hat{i} - a \sin t \hat{j} \quad \text{and} \quad \frac{d^3\vec{r}}{dt^3} = a \sin t \hat{i} - a \cos t \hat{j}$$

$$\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a \sin t & a \cos t & a \tan \alpha \\ -a \cos t & -a \sin t & 0 \end{vmatrix} = \hat{i} (a^2 \sin t \tan \alpha) - \hat{j} (a^2 \cos t \tan \alpha) + \hat{k} (a^2 \sin^2 t + a^2 \cos^2 t)$$

$$= a^2 \sin t \tan \alpha \hat{i} - a^2 \cos t \tan \alpha \hat{j} + a^2 \hat{k}$$

$$\left| \frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} \right| = \sqrt{a^4 \sin^2 t \tan^2 \alpha + a^4 \cos^2 t \tan^2 \alpha + a^4} = a^2 \sqrt{\tan^2 \alpha + 1} = a^2 \sec \alpha$$

Ans $\left[\frac{d\vec{r}}{dt}, \frac{d^2\vec{r}}{dt^2}, \frac{d^3\vec{r}}{dt^3} \right] = \begin{vmatrix} -a \sin t & a \cos t & a \tan \alpha \\ -a \cos t & -a \sin t & 0 \\ a \sin t & -a \cos t & 0 \end{vmatrix}$

$$= a \tan \alpha (a^2 \cos^2 t + a^2 \sin^2 t)$$

$$= a^3 \tan \alpha (\cos^2 t + \sin^2 t)$$

$$= a^3 \tan \alpha$$

$$\boxed{\cos^2 t + \sin^2 t = 1}$$

$$x = t^2 + 2, y = 4t - 5, z = 2t - 6 \text{ where } t \text{ is any variable}$$

Also determine the unit tangent vector at the point $t = 2$

Sol: If $\vec{r}$ is the position vector of any point (x, y, z) on the

given curve then $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$,

$$= (t^2 + 2)\hat{i} + (4t - 5)\hat{j} + (2t - 6)\hat{k}$$

$$\text{and } \frac{d\vec{r}}{dt} = 2t\hat{i} + 4\hat{j} + (2t - 6)\hat{k}$$

$$\left| \frac{d\vec{r}}{dt} \right| = \sqrt{(2t)^2 + (4)^2 + (2t - 6)^2} = \sqrt{4t^2 + 16 + 16t^2 + 36 - 48t}$$

$$= \sqrt{20t^2 - 48t + 52} = 2\sqrt{5t^2 - 12t + 13}$$

$$\text{The unit tangent vector } \vec{T} = \frac{\frac{d\vec{r}}{dt}}{\left| \frac{d\vec{r}}{dt} \right|} = \frac{2[t\hat{i} + 2\hat{j} + (2t - 3)\hat{k}]}{2\sqrt{5t^2 - 12t + 13}}$$

$$= \frac{(t\hat{i} + 2\hat{j} + 2t - 3)\hat{k}}{\sqrt{5t^2 - 12t + 13}}$$

Also the unit tangent vector at the point

$$t = 2 \text{ is } \frac{2\hat{i} + 2\hat{j} + [2(2) - 3]\hat{k}}{\sqrt{5(2)^2 - 12(2) + 13}}$$

$$= \frac{1}{3} (2\hat{i} + 2\hat{j} + \hat{k})$$

2

Curves in a Space

(i) Tangent :- If $\vec{r} = \vec{r}(t)$ be any space curve then the tangent at any point is represented by $\frac{d\vec{r}}{dt}$ at the point. If 's' is the parameter in place of 't' then the magnitude of the tangent vector,

$$\left| \frac{d\vec{r}}{ds} \right| = 1, \text{ Thus the unit tangent vector } \boxed{\vec{T} = \frac{d\vec{r}}{ds}}$$

(ii) Principal Normal :- Since 'T' is a unit vector.

$$\boxed{\frac{d\vec{T}}{ds} \cdot \vec{T} = 0}; \text{ i.e. } \frac{d\vec{T}}{ds} \text{ is perpendicular to } \vec{T}$$

Here $\frac{d\vec{T}}{ds}$ is unit vector along the vector $\vec{N}$ which is known as principal normal at given point.

(iii) Binormal :- If $\vec{T}, \vec{N}$ represent unit tangent, unit normal (i.e. principal normal) then Binormal $\boxed{\vec{B} = \vec{T} \times \vec{N}}$

(iv) Curvature :- The rate of turning of the tangent vector is called the curvature of the curve and is denoted by 'k'.

Since $\frac{d\vec{T}}{ds}$ is in the direction of the principal normal $\vec{N}$

$$\boxed{\frac{d\vec{T}}{ds} = k\vec{N}}$$

(v) Torsion :- The rate of turning of the binormal vector $\frac{d\vec{B}}{ds}$ is called torsion of the curve and is denoted by 'τ'. we may, therefore, write $\boxed{\frac{d\vec{B}}{ds} = -\tau\vec{N}}$

Ans:- $\vec{r} = acost \hat{i} + asint \hat{j} + bt \hat{k}$

$$\frac{d\vec{r}}{dt} = -asint \hat{i} + acost \hat{j} + b \hat{k}$$

$$\frac{ds}{dt} = \left| \frac{d\vec{r}}{dt} \right| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2} = \sqrt{a^2 + b^2}$$

$$\hat{T} = \frac{d\vec{r}}{ds} = \frac{d\vec{r}/dt}{ds/dt} = \frac{-asint \hat{i} + acost \hat{j} + b \hat{k}}{\sqrt{a^2 + b^2}}$$

$$\frac{d\hat{T}}{dt} = \frac{-acost \hat{i} - asint \hat{j}}{\sqrt{a^2 + b^2}}$$

$$\frac{d\hat{T}}{ds} = \frac{d\hat{T}/dt}{ds/dt} = \frac{-acost \hat{i} - asint \hat{j}}{\sqrt{a^2 + b^2}} \cdot \frac{1}{\sqrt{a^2 + b^2}} = \frac{-a(cost \hat{i} + sint \hat{j})}{a^2 + b^2}$$

$$K = \left| \frac{d\hat{T}}{ds} \right| = \frac{\sqrt{a^2 \cos^2 t + a^2 \sin^2 t}}{a^2 + b^2} = \frac{a}{a^2 + b^2}$$

$$\hat{N} = \frac{1}{K} \frac{d\hat{T}}{ds} = \frac{a^2 + b^2}{a} \left(\frac{-acost \hat{i} - asint \hat{j}}{a^2 + b^2} \right) = -cost \hat{i} - sint \hat{j}$$

$$\hat{B} = \hat{T} \times \hat{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{-asint}{\sqrt{a^2 + b^2}} & \frac{acost}{\sqrt{a^2 + b^2}} & \frac{b}{\sqrt{a^2 + b^2}} \\ -cost & -sint & 0 \end{vmatrix} = \frac{bsint}{\sqrt{a^2 + b^2}} \hat{i} - \frac{bcost}{\sqrt{a^2 + b^2}} \hat{j} + a \hat{k}$$

$$\frac{d\hat{B}}{dt} = \frac{b(cost \hat{i} - bsint \hat{j})}{\sqrt{a^2 + b^2}}$$

$$\frac{d\hat{B}}{ds} = \frac{d\hat{B}/dt}{ds/dt} = \frac{b(cost \hat{i} - bsint \hat{j})}{\sqrt{a^2 + b^2}} \cdot \frac{1}{\sqrt{a^2 + b^2}} = \frac{b(cost \hat{i} - bsint \hat{j})}{a^2 + b^2}$$

But $\frac{d\hat{B}}{ds} = -\tau \hat{N}$

$$\frac{b(cost \hat{i} - bsint \hat{j})}{a^2 + b^2} = -\tau (-cost \hat{i} - sint \hat{j}) \Rightarrow \tau = \frac{b}{a^2 + b^2}$$

$$\vec{r} = 3\cos t \hat{i} + 3\sin t \hat{j} + 4t \hat{k} \Rightarrow \frac{d\vec{r}}{dt} = -3\sin t \hat{i} + 3\cos t \hat{j} + 4\hat{k}$$

$$\frac{d\vec{r}}{dt} = \vec{T} \Rightarrow \left| \frac{d\vec{r}}{dt} \right| = 5 \Rightarrow \left| \frac{d\vec{r}}{ds} \right| = 1 \Rightarrow \frac{ds}{dt} = \left| \frac{d\vec{r}}{dt} \right|$$

$$\frac{ds}{dt} = \left| \frac{d\vec{r}}{dt} \right| = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = \sqrt{9 + 16} = \sqrt{25} = 5$$

$$\vec{T} = \frac{d\vec{r}}{ds} = \frac{d\vec{r}/dt}{ds/dt} = \frac{-3\sin t \hat{i} + 3\cos t \hat{j} + 4\hat{k}}{5}$$

$$\frac{d\vec{T}}{dt} = \frac{1}{5} (-3\cos t \hat{i} - 3\sin t \hat{j})$$

$$\therefore \frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{ds/dt} = \frac{1}{5} (-3\cos t \hat{i} - 3\sin t \hat{j}) \cdot \frac{1}{5} = \frac{1}{25} (-3\cos t \hat{i} - 3\sin t \hat{j})$$

$$\vec{N} = \left| \frac{d\vec{T}}{ds} \right| = \frac{\sqrt{9\cos^2 t + 9\sin^2 t}}{25} = \frac{3}{25}$$

$$\vec{N} = \frac{1}{K} \frac{d\vec{T}}{ds} = \frac{25}{3} \cdot \frac{1}{25} (-3\cos t \hat{i} - 3\sin t \hat{j}) = -\cos t \hat{i} - \sin t \hat{j}$$

$$3) \quad \vec{B} = \vec{T} \times \vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{-3\sin t}{5} & \frac{3\cos t}{5} & \frac{4}{5} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \hat{i} \left(\frac{4}{5} \sin t \right) + \frac{4}{5} \cos t \hat{j} + \frac{3}{5} \hat{k}$$

$$= \frac{1}{5} (4\sin t \hat{i} + 4\cos t \hat{j} + 3\hat{k})$$

$$\frac{d\vec{B}}{dt} = \frac{1}{5} (4\cos t \hat{i} + 4\sin t \hat{j})$$

$$\frac{d\vec{B}}{ds} = \frac{d\vec{B}/dt}{ds/dt} = \frac{1}{5} \frac{(4\cos t \hat{i} + 4\sin t \hat{j})}{5} = \frac{4\cos t \hat{i} + 4\sin t \hat{j}}{25}$$

$$\text{But } \frac{d\vec{B}}{ds} = -\gamma \vec{N}$$

$$\frac{1}{25} (4\cos t \hat{i} + 4\sin t \hat{j}) = -\gamma (-\cos t \hat{i} - \sin t \hat{j})$$

$$x = t - \frac{t^3}{3}, y = t^2$$

$$z = t + \frac{t^3}{2}$$

$$k = \gamma = \frac{1}{(1+t^2)^2}$$

$$\gamma = \frac{4}{25}$$

$$z$$

$$x = t, y = t^2, z' = \frac{2}{3} t^3$$

$$t = 1 \text{ ಎಲ್ಲ } k, \gamma$$

Find the unit tangent vector to the curve $x=t, y=t^2, z=t^3$ at $t=1$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \Rightarrow \vec{r} = t\vec{i} + t^2\vec{j} + t^3\vec{k}$$

$$\frac{d\vec{r}}{dt} = \vec{i} + 2t\vec{j} + 3t^2\vec{k}, \text{ At } t=1, \frac{d\vec{r}}{dt} = \vec{i} + 2\vec{j} + 3\vec{k}$$

$$\text{But } \left| \frac{d\vec{r}}{dt} \right| = \sqrt{1+4+9} = \sqrt{14}$$

$$\text{unit tangent vector to the curve} = \frac{\frac{d\vec{r}}{dt}}{\left| \frac{d\vec{r}}{dt} \right|} = \frac{\vec{i} + 2\vec{j} + 3\vec{k}}{\sqrt{14}}$$

2. A particle moves along the curve $x=2t^2, y=t^3-4t, z=3t-5$ where t is the time. Find the components of its velocity and acceleration at time $t=1$ in the direction $\vec{i}-3\vec{j}+2\vec{k}$.

$$\text{Sol:- } \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \Rightarrow \vec{r} = 2t^2\vec{i} + (t^3-4t)\vec{j} + (3t-5)\vec{k}$$

$$\text{velocity} = \vec{v} = \frac{d\vec{r}}{dt} = 4t\vec{i} + (3t^2-4)\vec{j} + 3\vec{k}$$

$$\text{when } t=1, \frac{d\vec{r}}{dt} = 4\vec{i} + 2\vec{j} + 3\vec{k}$$

$$\text{Acceleration} = \frac{d^2\vec{r}}{dt^2} = 4\vec{i} + 2\vec{j}$$

$$\text{when } t=1, \frac{d^2\vec{r}}{dt^2} = 4\vec{i} + 2\vec{j}$$

Component of velocity in the direction of $\vec{i}-3\vec{j}+2\vec{k}$ is

$$= \frac{(4\vec{i} + 2\vec{j} + 3\vec{k})(\vec{i} - 3\vec{j} + 2\vec{k})}{\sqrt{1+9+4}} = \frac{4+6+6}{\sqrt{14}} = \frac{16}{\sqrt{14}}$$

Component of acceleration in the direction of $(4\vec{i} + 2\vec{j})(\vec{i} - 3\vec{j} + 2\vec{k})$ is

$$= \frac{(4\vec{i} + 2\vec{j})(\vec{i} - 3\vec{j} + 2\vec{k})}{\sqrt{1+9+4}} = \frac{4-6}{\sqrt{14}} = \frac{-2}{\sqrt{14}}$$

4) A particle moves so that its position vector is given by $\vec{r} = \cos \omega t \vec{i} + \sin \omega t \vec{j}$ where ω is a constant. Show that velocity $\vec{v}$ of the particle is perpendicular to $\vec{r}$ and $\vec{r} \times \vec{v}$ is a constant vector.

Sol:- The position vector of the particle is given by $\vec{r} = \cos \omega t \vec{i} + \sin \omega t \vec{j}$

The velocity is given by $\vec{v} = \frac{d\vec{r}}{dt} = -\omega \sin \omega t \vec{i} + \omega \cos \omega t \vec{j}$

consider $\vec{r} \cdot \vec{v} = -\omega \cos \omega t \sin \omega t + \omega \sin \omega t \cos \omega t = 0$ so $\vec{v}$ is $\perp$ to $\vec{r}$

$$\text{Now } \vec{r} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos \omega t & \sin \omega t & 0 \\ -\omega \sin \omega t & \omega \cos \omega t & 0 \end{vmatrix} = \vec{k} (\omega \cos^2 \omega t + \omega \sin^2 \omega t) = \omega \vec{k} = \text{constant vector}$$

① Find the angle between the tangents to the curve

$$\vec{r} = t^2 \hat{i} + 2t \hat{j} + 3 \hat{k} \text{ at the points } t=1$$

Sol:-

$$\frac{d\vec{r}}{dt} = 2t\hat{i} + 2\hat{j} = 2t\hat{i} + 2\hat{j}$$

is a vector along the tangent at any point 't'.

If $\vec{T}_1$ and $\vec{T}_2$ are the vectors along the tangents at $t=1$ and $t=-1$, respectively, then

$$\vec{T}_1 = 2\hat{i} + 2\hat{j} = 2\hat{i} + 2\hat{j} \text{ and } \vec{T}_2 = -2\hat{i} + 2\hat{j} + 3\hat{k}$$

Let θ be the angle between $\vec{T}_1$ and $\vec{T}_2$ then

$$\cos \theta = \frac{\vec{T}_1 \cdot \vec{T}_2}{|\vec{T}_1| |\vec{T}_2|} = \frac{2(-2) + 2(2) + 3(3)}{\sqrt{4+4+9} \sqrt{4+4+9}} = \frac{-4+4+9}{\sqrt{17} \sqrt{17}} = \frac{9}{17}$$

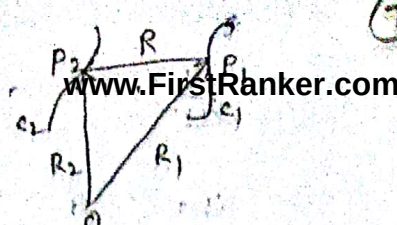
$$\theta = \cos^{-1} (9/17)$$

Velocity and Acceleration:

If the scalar variable 't' be the time and $\vec{r}$ be the position vector of moving particle with respect to the origin 'O', then $\delta \vec{r}$ is the displacement of the particle in time δt . If $\vec{v}$ represents the velocity of the particle at P in time δt ,

$$\vec{v} = \lim_{\delta t \rightarrow 0} \frac{\delta \vec{r}}{\delta t} = \frac{d\vec{r}}{dt}$$

and acceleration is given by $\vec{a} = \lim_{\delta t \rightarrow 0} \frac{\delta \vec{v}}{\delta t} = \frac{d^2 \vec{r}}{dt^2}$



Let two particles P_1 and P_2 moving along the curves c_1 and c_2 have position vectors R_1 and R_2 at time t , we get

$$\frac{d\vec{R}}{dt} = \frac{d\vec{R}_2}{dt} - \frac{d\vec{R}_1}{dt} \quad \text{--- (1)}$$

This defines the relative velocity of P_2 w.r.t P_1 and states that the velocity of P_2 relative to $P_1 =$

velocity of P_2 - velocity of P_1 .

differentiating (1), we have:
$$\frac{d^2\vec{R}}{dt^2} = \frac{d^2\vec{R}_2}{dt^2} - \frac{d^2\vec{R}_1}{dt^2}$$

i.e. acceleration of P_2 relative to $P_1 =$ acceleration of P_2 - acceleration of P_1 .

Find the radial and transverse acceleration of a particle moving in a plane curve.



Sol:- At any time t , let the position vector of the moving particle $P(r, \theta)$ be ' $\vec{R}$ ' so that $\vec{R} = r\hat{r} = r(\cos\theta\hat{i} + \sin\theta\hat{j})$

Its velocity $\vec{V} = \frac{d\vec{R}}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}$ --- (1)

As $\hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j}$

and $\frac{d\hat{r}}{dt} = (-\sin\theta\hat{i} + \cos\theta\hat{j})\frac{d\theta}{dt}$

then $\frac{d\hat{r}}{dt} = \frac{d\theta}{dt} \hat{u}$

① becomes, $\vec{v} = \frac{dr}{dt} \hat{r} + r \frac{d\theta}{dt} \hat{u}$

Thus the radial and transverse components of

the velocity are $\frac{dr}{dt}$ and $r \frac{d\theta}{dt}$.

Also $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2r}{dt^2} \hat{r} + \frac{dr}{dt} \frac{d\hat{r}}{dt} + \frac{dr}{dt} \frac{d\theta}{dt} \hat{u} + r \frac{d^2\theta}{dt^2} \hat{u} + r \frac{d\theta}{dt} \frac{d\hat{u}}{dt}$

$$= \left[\frac{d^2r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 \right] \hat{r} + \left(2 \frac{dr}{dt} \frac{d\theta}{dt} + r \frac{d^2\theta}{dt^2} \right) \hat{u}$$

$$[\because \hat{u} = -\sin\theta \hat{r} + \cos\theta \hat{\theta}$$

$$\text{gives } \frac{d\hat{u}}{dt} = -\frac{d\theta}{dt} \hat{r}$$

Thus the radial and transverse components of

acceleration are

$$\frac{d^2r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 \quad \text{and} \quad 2 \frac{dr}{dt} \frac{d\theta}{dt} + r \frac{d^2\theta}{dt^2}$$

2

- 1) A particle moves along the curve $\vec{r} = (t^3+1)\vec{i} + t^2\vec{j} + 2t\vec{k}$ where 't' is the time. Find the velocity and acceleration at $t=1$ in the direction $\vec{i} + \vec{j} + 3\vec{k}$.

Sol:- velocity $\frac{d\vec{r}}{dt} = \frac{d}{dt}[(t^3+1)\vec{i} + t^2\vec{j} + 2t\vec{k}]$

$$= 3t^2\vec{i} + 2t\vec{j} + 2\vec{k} = 3\vec{i} + 2\vec{j} + 2\vec{k} \text{ at } t=1$$

and acceleration $= \frac{d^2\vec{r}}{dt^2} = 6t\vec{i} + 2\vec{j} + 0\vec{k} = 6\vec{i} + 2\vec{j} \text{ at } t=1$

Now unit vector in the direction of $\vec{i} + \vec{j} + 3\vec{k}$ is

$$\frac{\vec{i} + \vec{j} + 3\vec{k}}{\sqrt{1^2 + 1^2 + 3^2}} = \frac{1}{\sqrt{11}}(\vec{i} + \vec{j} + 3\vec{k})$$

component of velocity at $t=1$ in the direction $\vec{i} + \vec{j} + 3\vec{k}$

$$= \frac{(3\vec{i} + 2\vec{j} + 2\vec{k}) \cdot (\vec{i} + \vec{j} + 3\vec{k})}{\sqrt{11}} = \frac{3+2+6}{\sqrt{11}} = \frac{11}{\sqrt{11}} = \sqrt{11}$$

and component of acceleration at $t=1$ in the direction

$$\vec{i} + \vec{j} + 3\vec{k} = \frac{(6\vec{i} + 2\vec{j}) \cdot (\vec{i} + \vec{j} + 3\vec{k})}{\sqrt{11}} = \frac{6+2}{\sqrt{11}} = \frac{8}{\sqrt{11}}$$

- 2) A particle moves along the curve $\vec{r} = (t^2-4t)\vec{i} + (t^2+4t)\vec{j} + (2t^2-3t^3)\vec{k}$

where 't' denotes time. Find the magnitude of

acceleration along the tangent and normal at time $t=2$.

Sol:- velocity $\frac{d\vec{r}}{dt} = (2t-4)\vec{i} + (2t+4)\vec{j} + (4t-9t^2)\vec{k}$

and acceleration $\frac{d^2\vec{r}}{dt^2} = 2\vec{i} + 2\vec{j} + (4-18t)\vec{k}$

At $t=2$, velocity $\vec{v} = 0\vec{i} + 8\vec{j} - 20\vec{k}$ and

acceleration $\vec{a} = 2\vec{i} + 2\vec{j} - 32\vec{k}$

the component of A along the tangent $= A \cdot \frac{V}{|V|}$

$$= (12\hat{i} + 2\hat{j} - 20\hat{k}) \cdot \frac{8\hat{i} + 8\hat{j} - 4\hat{k}}{\sqrt{64+64+16}} = \frac{12 \times 8 + 2 \times 8 + (-20) \times (-4)}{12} = 16$$

Now the component of A along the normal

$= |A - \text{Resolved part of } A \text{ along the tangent}|$

$$= \left| 12\hat{i} + 2\hat{j} - 20\hat{k} - 16 \frac{8\hat{i} + 8\hat{j} - 4\hat{k}}{12} \right| = \frac{1}{3} |4\hat{i} - 26\hat{j} - 44\hat{k}|$$

$$= 2\sqrt{73}$$

3) The position vector of a particle at time 't' is

$$\vec{R} = \cos(t-1)\hat{i} + \sinh(t-1)\hat{j} + \alpha t^3 \hat{k}$$

Find the condition imposed on α by requiring that at time $t=1$, the acceleration is normal to the position vector.

Sol: velocity $= \frac{d\vec{R}}{dt} = -\sin(t-1)\hat{i} + \cosh(t-1)\hat{j} + 3\alpha t^2 \hat{k}$

acceleration $= \frac{d^2\vec{R}}{dt^2} = -\cos(t-1)\hat{i} + \sinh(t-1)\hat{j} + 6\alpha t \hat{k}$

$$= -\hat{i} + 6\alpha \hat{k} \text{ at } t=1$$

Also $\vec{R} = \hat{i} + \alpha \hat{k}$ at $t=1$.

If $\vec{R}$ and acceleration at $t=1$ are normal,

then their scalar product is zero.

$$(-\hat{i} + 6\alpha \hat{k}) \cdot (\hat{i} + \alpha \hat{k}) = 0$$

$$-1 + 6\alpha^2 = 0 \Rightarrow \alpha^2 = \frac{1}{6}$$

$$\text{or } \alpha = \frac{1}{\sqrt{6}}$$

2

Ans:

Scalar point function:- A function $\phi(x, y, z)$ is called a scalar point function if it associates a scalar with every point in space.

The temperature distribution in a heated body, density of body and potential due to gravity are the examples of a scalar point function.

Vector point function:- If a function $F(x, y, z)$ defines a vector at every point of a region, then $F(x, y, z)$ is called a vector point function. The velocity of moving fluid, gravitational force are the example of vector point function.

Vector operator Del or Nabla:-

The operator $\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$ is called the vector operator

Del or Nabla.

The vector operator ∇ has properties analogous to those of ordinary vectors and can be applied to (i) scalar point functions
(ii) vector point functions

Gradient:- The vector function $\nabla \phi$ is defined as the gradient of the scalar point function ϕ and written as grad ϕ

$$\text{grad } \phi = \nabla \phi = \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z}$$

Directional Derivative:- The component of $\nabla \phi$ in the direction of $\vec{a}$ is given by $\nabla \phi \cdot \frac{\vec{a}}{|\vec{a}|}$ and is called the directional derivative of ϕ in direction of $\vec{a}$. The directional derivative of ϕ at a point P is maximum in the direction of normal at P and its maximum value is $|\nabla \phi|$.

Properties of gradient:- If ϕ and ψ are two scalar point functions,

then (i) $\nabla(c_1\phi + c_2\psi) = c_1\nabla\phi + c_2\nabla\psi$ where c_1, c_2 are constants

$$(ii) \nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$$

$$(iii) \nabla\left(\frac{\phi}{\psi}\right) = \frac{\psi\nabla\phi - \phi\nabla\psi}{\psi^2}, \psi \neq 0$$

$$r = \sqrt{x^2 + y^2 + z^2} = (x^2 + y^2 + z^2)^{1/2}$$

$$r^n = (x^2 + y^2 + z^2)^{n/2}$$

$$\nabla r^n = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2 + y^2 + z^2)^{n/2}$$

$$= \frac{n}{2} (x^2 + y^2 + z^2)^{n/2 - 1} \cdot [2x\hat{i} + 2y\hat{j} + 2z\hat{k}]$$

$$= \frac{n}{2} (r^2)^{\frac{n-2}{2}} (2x\hat{i} + 2y\hat{j} + 2z\hat{k})$$

$$= n(r^2)^{\frac{n-2}{2}} (x\hat{i} + y\hat{j} + z\hat{k})$$

$$\nabla r^n = nr^{n-2} \vec{r}$$

4) $\nabla(\vec{a} \cdot \vec{r}) = \vec{a}$ where $\vec{a}$ is constant vec.

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ where a_1, a_2, a_3 are const.

$$\vec{a} \cdot \vec{r} = a_1x + a_2y + a_3z$$

$$\nabla(\vec{a} \cdot \vec{r}) = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (a_1x + a_2y + a_3z)$$

$$= a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \vec{a}$$

5) Find the grad ϕ where ϕ is given by $\phi = 3xy - y^3z^2$ at the point $(1, -2, -1)$

Ans:- $\text{Grad } \phi = \nabla \phi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (3xy - y^3z^2)$

$$= \hat{i} \left(\frac{\partial}{\partial x} \right) (3xy - y^3z^2) + \hat{j} \left(\frac{\partial}{\partial y} \right) (3xy - y^3z^2) + \hat{k} \left(\frac{\partial}{\partial z} \right) (3xy - y^3z^2)$$

$$= 3y\hat{i} + \hat{j} (3x - 3y^3z^2) + \hat{k} (-2y^3z)$$

$$= -12\hat{i} - 9\hat{j} - 16\hat{k} \text{ at the point } (1, -2, -1)$$

Sol:- $\nabla r^n = \hat{i} \frac{d}{dx} (r^n) + \hat{j} \frac{d}{dy} (r^n) + \hat{k} \frac{d}{dz} (r^n)$

$$= n r^{n-1} \frac{dr}{dx} \hat{i} + n r^{n-1} \frac{dr}{dy} \hat{j} + n r^{n-1} \frac{dr}{dz} \hat{k}$$

$$= n r^{n-1} \frac{1}{r} \hat{i} + n r^{n-1} \frac{1}{r} \hat{j} + n r^{n-1} \frac{1}{r} \hat{k}$$

$$= n r^{n-2} (x \hat{i} + y \hat{j} + z \hat{k}) = n r^{n-2} \vec{r}$$

1) $r = \sqrt{x^2 + y^2 + z^2}$

$$r^2 = x^2 + y^2 + z^2$$

$$2r \frac{dr}{dx} = 2x$$

$$\frac{dr}{dx} = \frac{x}{r}$$

2) If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = yz + zx + xy$,
p.t. grad u , grad v , grad w are coplanar.

Sol:- grad $u = \hat{i} + \hat{j} + \hat{k}$, grad $v = 2x \hat{i} + 2y \hat{j} + 2z \hat{k}$
grad $w = (y+z) \hat{i} + (z+x) \hat{j} + (x+y) \hat{k}$

We know that three vectors are coplanar if their scalar triple product is zero.

$$[\text{grad } u, \text{grad } v, \text{grad } w] = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix}$$

operate $R_2 + R_3$

$$= 2 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix} = 2(x+y+z) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix} = 0$$

Hence grad u , grad v , grad w are coplanar.

3) Find a unit vector normal to the surface

$xy^3z^2 = 4$ at the point $(-1, -1, 2)$

Sol:- A vector normal to the given surface is

$$\nabla(xy^3z^2) = \hat{i} \frac{d}{dx} (xy^3z^2) + \hat{j} \frac{d}{dy} (xy^3z^2) + \hat{k} \frac{d}{dz} (xy^3z^2)$$

$$= y^3z^2 \hat{i} + 3xy^2z^2 \hat{j} + 2xy^3z \hat{k}$$

$$= -4\hat{i} - 12\hat{j} + 4\hat{k} \text{ at the point } (-1, -1, 2)$$

$$= \frac{-4\hat{i} - 12\hat{j} + 4\hat{k}}{\sqrt{(-4)^2 + (-12)^2 + 4^2}} = \frac{-1}{\sqrt{17}} (\hat{i} + 3\hat{j} - \hat{k})$$

4) Find the directional derivative of $f(x, y, z) = xy^2 + yz^2$ at the point $(2, -1, 1)$ in the direction of vector $\hat{i} + 2\hat{j} + 2\hat{k}$.

$$\begin{aligned} \text{Sol: } \nabla f &= \hat{i} \frac{\partial}{\partial x} (xy^2 + yz^2) + \hat{j} \frac{\partial}{\partial y} (xy^2 + yz^2) + \hat{k} \frac{\partial}{\partial z} (xy^2 + yz^2) \\ &= y^2 \hat{i} + (2xy + z^2) \hat{j} + 2yz \hat{k} \\ &= \hat{i} - 3\hat{j} - 3\hat{k} \text{ at the point } (2, -1, 1) \end{aligned}$$

$$\begin{aligned} \therefore \text{Directional derivative of } f \text{ in the direction } \hat{i} + 2\hat{j} + 2\hat{k} \\ = (\hat{i} - 3\hat{j} - 3\hat{k}) \cdot \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{1^2 + 2^2 + 2^2}} = -\frac{11}{3} \end{aligned}$$

5) Find the directional derivative of $f = x^2 - y^2 + 2z^2$ at point $P(1, 2, 3)$ in the direction of the line $\vec{r} = \vec{a} + t\vec{b}$ where $\vec{a}$ is the point $(5, 0, 4)$. Also calculate the magnitude of maximum directional derivative.

$$\begin{aligned} \text{Sol: } \nabla f &= \hat{i} \frac{\partial}{\partial x} (x^2 - y^2 + 2z^2) + \hat{j} \frac{\partial}{\partial y} (x^2 - y^2 + 2z^2) + \hat{k} \frac{\partial}{\partial z} (x^2 - y^2 + 2z^2) \\ &= 2x\hat{i} - 2y\hat{j} + 4z\hat{k} \\ &= 2\hat{i} - 4\hat{j} + 12\hat{k} \text{ at } P(1, 2, 3) \end{aligned}$$

$$\begin{aligned} \text{Also } \vec{PQ} &= \vec{a} - \vec{OP} = (5\hat{i} + 0\hat{j} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) \\ &= 4\hat{i} - 2\hat{j} + \hat{k} \end{aligned}$$

$$\therefore \text{unit vector of } \vec{A} = \frac{\vec{A}}{|\vec{A}|} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{16 + 4 + 1}} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}}$$

Thus the directional derivative of ϕ in the direction of $\hat{A}$ is

$$\nabla \phi \cdot \hat{A} = (2\hat{i} - 4\hat{j} + 12\hat{k}) \cdot \frac{(4\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{21}} = \frac{8+8+12}{\sqrt{21}} = \frac{28}{\sqrt{21}}$$

$\therefore$ The directional derivative of ϕ is maximum in the direction of the normal to the surface i.e. in the direction of $\nabla \phi$.

Hence maximum value of this directional derivative

$$= |\nabla \phi| = |2\hat{i} - 4\hat{j} + 12\hat{k}| = \sqrt{4+16+144} = \sqrt{164}$$

(6) Find the directional derivative of $\phi = 5x^2y - 5y^2z + 2.5z^2x$ at the point $P(1,1,1)$ in the direction of the line

$$\frac{x-1}{2} = \frac{y-3}{-2} = z$$

Sol:- we have $\nabla \phi = \hat{i} \frac{d\phi}{dx} + \hat{j} \frac{d\phi}{dy} + \hat{k} \frac{d\phi}{dz}$
 $= (10xy + 2.5z^2)\hat{i} + (5x^2 - 10yz)\hat{j} + (-5y^2 + 5xz)\hat{k}$
 $= 12.5\hat{i} - 5\hat{j}$ at $P(1,1,1)$

Also direction of the given line is $\hat{A} = \frac{2\hat{i} - 2\hat{j} + \hat{k}}{3}$

Hence the required directional derivative $= \nabla \phi \cdot \hat{A}$
 $= (12.5\hat{i} - 5\hat{j}) \cdot \frac{2\hat{i} - 2\hat{j} + \hat{k}}{3}$
 $= \frac{25+10}{3} = \frac{35}{3}$

7) Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at the point $(2,-1,2)$

Sol:- Let $f_1 = x^2+y^2+z^2-9=0$ and $f_2 = x^2+y^2-z-3=0$

Then $N_1 = \nabla f_1$ at $(2,-1,2)$
 $= (2x\hat{i} + 2y\hat{j} + 2z\hat{k})$ at $(2,-1,2) = 4\hat{i} - 2\hat{j} + 4\hat{k}$

and $N_2 = \nabla f_2$ at $(2,-1,2)$
 $= (2x\hat{i} + 2y\hat{j} - \hat{k})$ at $(2,-1,2) = 4\hat{i} - 2\hat{j} - \hat{k}$

Since the angle θ between the two surfaces at a point is the angle between their normals at the point and N_1, N_2 are the normals at $(2, -1, 2)$ to the given surfaces,

$$\cos \theta = \frac{N_1 \cdot N_2}{|N_1| |N_2|} = \frac{(4\hat{i} - 2\hat{j} + 4\hat{k})(4\hat{i} - 2\hat{j} - \hat{k})}{\sqrt{16+4+16} \sqrt{16+4+1}} = \frac{16}{6\sqrt{21}}$$

Hence the required angle $\theta = \cos^{-1} \left(\frac{8}{3\sqrt{21}} \right)$

8) Find the values a and b such that the surfaces $ax^2 - byz = (a+2)x$, and $4x^2y + z^3 = 4$ cut orthogonally

Sol:- Let $f_1 = ax^2 - byz - (a+2)x = 0$ — (1)

$$f_2 = 4x^2y + z^3 - 4 = 0. \text{ — (2)}$$

$$\begin{aligned} \nabla f_1 &= (2ax - a - 2)\hat{i} - bz\hat{j} - by\hat{k} \\ &= (a-2)\hat{i} - 2b\hat{j} + b\hat{k} \text{ at } (1, -1, 2) \end{aligned}$$

$$\begin{aligned} \nabla f_2 &= 8xy\hat{i} + 4x^2\hat{j} + 3z^2\hat{k} \\ &= -8\hat{i} + 4\hat{j} + 12\hat{k} \text{ at } (1, -1, 2) \end{aligned}$$

The surfaces (1) and (2) will cut orthogonally if $\nabla f_1 \cdot \nabla f_2 = 0$

$$\text{i.e. } -8(a-2) - 8b + 12b = 0.$$

$$-2a + b + 4 = 0 \text{ — (3)}$$

Also since the point $(1, -1, 2)$ lies on (1) and (2)

$$a + 2b - (a+2) = 0 \text{ or } b = 1.$$

$$\text{From (3) } -2a + 5 = 0 \text{ or } a = 5/2$$

$$\text{Hence } a = 5/2, \text{ and } b = 1$$

1) (a) find $\nabla\phi$, if $\phi = \log(x^2 + y^2 + z^2)$

(b) show that $\text{grad}(1/r) = -R/r^3$.

$$\begin{aligned} \text{Sol: (a)} \quad \nabla\phi &= \bar{i} \frac{\partial}{\partial x} \log(x^2 + y^2 + z^2) + \bar{j} \frac{\partial}{\partial y} \log(x^2 + y^2 + z^2) \\ &\quad + \bar{k} \frac{\partial}{\partial z} \log(x^2 + y^2 + z^2) \\ &= \bar{i} \frac{2x}{x^2 + y^2 + z^2} + \bar{j} \frac{2y}{x^2 + y^2 + z^2} + \bar{k} \frac{2z}{x^2 + y^2 + z^2} \\ &= \frac{2(\bar{i}x + \bar{j}y + \bar{k}z)}{(x^2 + y^2 + z^2)} = \frac{2\bar{r}}{r^2} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \text{grad}(1/r) &= \bar{i} \frac{\partial}{\partial x} \left(\frac{1}{r}\right) + \bar{j} \frac{\partial}{\partial y} \left(\frac{1}{r}\right) + \bar{k} \frac{\partial}{\partial z} \left(\frac{1}{r}\right) \\ &= \bar{i} \left(-\frac{1}{r^2}\right) \frac{\partial r}{\partial x} + \bar{j} \left(-\frac{1}{r^2}\right) \frac{\partial r}{\partial y} + \bar{k} \left(-\frac{1}{r^2}\right) \frac{\partial r}{\partial z} \\ &= \bar{i} \left(-\frac{1}{r^2}\right) \left(\frac{x}{r}\right) + \bar{j} \left(-\frac{1}{r^2}\right) \left(\frac{y}{r}\right) + \bar{k} \left(-\frac{1}{r^2}\right) \left(\frac{z}{r}\right) \\ &= -\frac{1}{r^3} (\bar{i}x + \bar{j}y + \bar{k}z) \\ &= -\frac{\bar{r}}{r^3} \end{aligned}$$

2) Find a unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1, 2, -1)$

Sol: Let $\phi = x^3 + y^3 + 3xyz - 3$

A vector normal to the above surface

$$\begin{aligned} \nabla\phi &= \bar{i} \frac{\partial\phi}{\partial x} + \bar{j} \frac{\partial\phi}{\partial y} + \bar{k} \frac{\partial\phi}{\partial z} \\ &= (3x^2 + 3yz)\bar{i} + (3y^2 + 3xz)\bar{j} + 3xy\bar{k} \end{aligned}$$

$$\begin{aligned}\nabla \phi &= [3 + 3(2)(-1)]\bar{i} + [3(2)^4 + 3(1)(-1)]\bar{j} + 3(1)(2)\bar{k} \\ &= -3\bar{i} + 9\bar{j} + 6\bar{k} \\ &= -3(\bar{i} - 3\bar{j} - 2\bar{k})\end{aligned}$$

$$\begin{aligned}\text{unit vector normal to the surface} &= \frac{\nabla \phi}{|\nabla \phi|} \\ &= \frac{-3(\bar{i} - 3\bar{j} - 2\bar{k})}{3\sqrt{1+9+4}} \\ &= \frac{-(\bar{i} - 3\bar{j} - 2\bar{k})}{\sqrt{14}}\end{aligned}$$

(3) Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point $(1, -2, 1)$ in the direction of the vector $2\bar{i} - \bar{j} - 2\bar{k}$

Sol: Here $\nabla \phi = (2xyz + 4z^2)\bar{i} + x^2z\bar{j} + (x^2y + 8xz)\bar{k}$
 $= 8\bar{i} - \bar{j} - 10\bar{k}$ at the point $(1, -2, 1)$

$\therefore$ Directional derivative of ϕ in the direction $2\bar{i} - \bar{j} - 2\bar{k}$

$$\begin{aligned}&= \frac{(8\bar{i} - \bar{j} - 10\bar{k}) \cdot (2\bar{i} - \bar{j} - 2\bar{k})}{\sqrt{4+1+4}} \\ &= \frac{16+1+20}{3} = \frac{37}{3}\end{aligned}$$

(4) What is the directional derivative of $\phi = xyz^2 + yz^3$ at the point $(2, 1, 1)$ in the direction of the normal to the surface $x \log z - yz = -4$ at $(-1, 2, 1)$

Sol: A vector normal to the surface is

$$\nabla(x \log z) - y^2 = \log z \bar{i} - 2y \bar{j} + \frac{x}{z} \bar{k}$$

$$= -4\bar{j} - \bar{k}$$

Here $\nabla f = \left(\bar{i} \frac{\partial}{\partial x} + \bar{j} \frac{\partial}{\partial y} + \bar{k} \frac{\partial}{\partial z}\right) (xy^2 + yz^3)$

$$= y^2 \bar{i} + (2xy + z^3) \bar{j} + 3yz^2 \bar{k} \text{ at } (2, -1, 1)$$

$$= \bar{i} - 3\bar{j} - 3\bar{k}$$

Directional derivative of 'f' in the direction of $-4\bar{j} - \bar{k}$

$$= \frac{(\bar{i} - 3\bar{j} - 3\bar{k}) \cdot (-4\bar{j} - \bar{k})}{\sqrt{16+1}} = \frac{12+3}{\sqrt{17}} = \frac{15}{\sqrt{17}}$$

9) Find the values of constants a, b, c so that the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has a maximum magnitude 64 in a direction parallel to the z-axis.

$$\nabla \phi = \nabla (axy^2 + byz + cz^2x^3)$$

$$= (ay^2 + 3cz^2x^2) \bar{i} + (2axy + bz) \bar{j} + (by + 2czx^3) \bar{k}$$

$$\nabla \phi_{(1,2,-1)} = (4a+3c) \bar{i} + (4a-b) \bar{j} + (2b-2c) \bar{k}$$

Given that $\nabla \phi$ is parallel to z-axis and

$$\nabla \phi \cdot \bar{i} = 0$$

$$\therefore \nabla \phi \cdot \bar{i} = 0 ; \nabla \phi \cdot \bar{j} = 0 \text{ and } \nabla \phi \cdot \bar{k} = 64$$

$$4a+3c=0 ; 4a-b=0 ; 2b-2c=64$$

$$\rightarrow (1)$$

$$\rightarrow (2)$$

$$\rightarrow (3)$$

Solving (1), (2), (3) for a, b, c

$$a = 6, b = 24, c = -8$$

(6) If the directional derivative of $\phi = ax^2y + by^2z + cz^2x$ at the point $(1, 1, 1)$ has maximum magnitude 15 in the direction parallel to the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z-1}{1}$ find the values of a, b and c .

Sol Given $\phi = ax^2y + by^2z + cz^2x$

$$\nabla\phi = (2axy + cz^2)\bar{i} + (ax^2 + 2byz)\bar{j} + (by^2 + 2cxz)\bar{k}$$

At the point $(1, 1, 1)$

$$\nabla\phi = (2a+c)\bar{i} + (a+2b)\bar{j} + (b+2c)\bar{k}$$

Now the maximum value of the directional derivative is in the direction of $\nabla\phi$

i.e., $|\nabla\phi| = 15$

$$\Rightarrow (2a+c)^2 + (a+2b)^2 + (b+2c)^2 = (15)^2$$

But the directional derivative is given to be maximum parallel to the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z-1}{1}$

$$\Rightarrow \frac{2a+c}{2} = \frac{a+2b}{-2} = \frac{b+2c}{1} \Rightarrow 2a+c = -2b-a$$

$$3a+2b+c=0 \quad \text{--- (1)}$$

and $2b+a = -2(2c+b)$

$$a+4b+4c=0 \rightarrow \text{--- (2)}$$

$$\frac{a}{4} = \frac{b}{-11} = \frac{c}{10} = k$$

$$a = 4k, b = -11k, c = 10k$$

Now,

$$(2a+c)^2 + (2b+a)^2 + (2c+b)^2 = (15)^2$$

$$(8k+10k)^2 + (-22k+4k)^2 + (20k-11k)^2 = (15)^2$$

$$(18k)^2 + (-18k)^2 + (8k)^2 = (15)^2$$

$$k = \pm 5/9$$

$$a = \pm 20/9 ; b = \pm 55/9 ; c = \pm 50/9$$

4) In what direction from $(3, 1, -2)$ is the directional derivative of $\phi = x^2 y^2 z^4$ maximum? Find also the magnitude of this maximum.

$$\nabla \phi = \left(\bar{i} \frac{\partial}{\partial x} + \bar{j} \frac{\partial}{\partial y} + \bar{k} \frac{\partial}{\partial z} \right) (x^2 y^2 z^4)$$

$$= 2xy^2 z^4 \bar{i} + 2x^2 y z^4 \bar{j} + 4x^2 y^2 z^3 \bar{k}$$

Directional derivative at $(3, 1, -2)$

$$= 96\bar{i} + 288\bar{j} - 288\bar{k}$$

The directional derivative is maximum in the direction $96\bar{i} + 288\bar{j} - 288\bar{k}$

$$= 96(\bar{i} + 3\bar{j} - 3\bar{k})$$

In any other direction it will have its component which is less than its maximum value

$$\begin{aligned}\text{Maximum value} &= 96\sqrt{19} + 9 \\ &= 96\sqrt{19}\end{aligned}$$

(8) What is the greatest rate of increase of $u = x^2 + y^2 + z^2$ at the point $(1, -1, 3)$

Sol:

$$\begin{aligned}\text{greatest rate} &= \left(\bar{i} \frac{\partial}{\partial x} + \bar{j} \frac{\partial}{\partial y} + \bar{k} \frac{\partial}{\partial z} \right) (x^2 + y^2 + z^2) \\ &= 2x\bar{i} + 2y\bar{j} + 2z\bar{k}\end{aligned}$$

$$\text{At } (1, -1, 3), \text{ greatest rate} = 2\bar{i} + 9\bar{j} - 6\bar{k}$$

(9) Calculate

(9) The temperature of points in space is given by $T(x, y, z) = x^2 + y^2 - z$. A mosquito located at $(1, 1, 2)$ desires to fly in such a direction that it get warm as soon as possible. In what direction should it move?

Sol:

$$\begin{aligned}\text{Let } \phi &= (x^2 + y^2 - z) \\ \nabla \phi &= 2x\bar{i} + 2y\bar{j} - \bar{k} \\ \text{at } (1, 1, 2) &= 2\bar{i} + 2\bar{j} - \bar{k}\end{aligned}$$

$$\begin{aligned}\frac{\nabla \phi}{|\nabla \phi|} &= \frac{2\bar{i} + 2\bar{j} - \bar{k}}{\sqrt{4 + 4 + 1}} \\ &= \frac{2\bar{i} + 2\bar{j} - \bar{k}}{3}\end{aligned}$$

calculate the angle b/w the normals to the surface $xy = z^2$ at the points $(4, 1, 2)$ and $(3, 3, -3)$

A vector normal to the surface is

$$\nabla (xy - z^2) = \bar{i} \frac{\partial}{\partial x} (xy - z^2) + \bar{j} \frac{\partial}{\partial y} (xy - z^2) + \bar{k} \frac{\partial}{\partial z} (xy - z^2)$$

$$= \bar{i}y + \bar{j}x - 2z\bar{k}$$

At $(4, 1, 2)$ and $(3, 3, -3)$

$$\bar{N}_1 = \bar{i} + 4\bar{j} - 4\bar{k} ; \bar{N}_2 = 3\bar{i} + 3\bar{j} - 6\bar{k}$$

$$\cos \theta = \frac{\bar{N}_1 \cdot \bar{N}_2}{|\bar{N}_1| |\bar{N}_2|} = \frac{3 + 12 - 24}{\sqrt{1+16+16} \sqrt{9+9+36}}$$

$$= \frac{-9}{\sqrt{33} \sqrt{54}} = \frac{-9}{\sqrt{11} \times \sqrt{3} \cdot \sqrt{2} \cdot \sqrt{3}}$$

$$= \frac{-9}{\sqrt{81} \sqrt{22}} = \frac{-1}{\sqrt{22}}$$

$$\theta = \cos^{-1} \left(\frac{1}{\sqrt{22}} \right)$$

Find the values of λ and μ so that the surfaces $\lambda xy + \mu z^3 = 4$ may cut the surface $xy = z^2$ orthogonally at $(1, -1, 2)$

$$\text{Let } f_1 = \lambda xy + \mu z^3 - 4 = 0 \rightarrow (1)$$

$$\nabla f_1 = \lambda y \bar{i} + \lambda x \bar{j} + 3\mu z^2 \bar{k}$$

$$\text{and } f_2 = 5x^2 - 2yz - 9x = 0 \rightarrow (2)$$

$$\nabla f_2 = (10x - 9) \bar{i} - 2z \bar{j} - 2y \bar{k} = 0$$

$$\text{At } (1, -1, 2) \Rightarrow \nabla f_1 = -\lambda \bar{i} + \lambda \bar{j} + 12\mu \bar{k}$$

$$\text{At } (1, -1, 2) \Rightarrow \nabla f_2 = \bar{i} - 4\bar{j} + 2\bar{k}$$

$$\nabla f_1 \cdot \nabla f_2 = 0$$

$$\text{i.e., } (-2\lambda)(1) + \lambda(-4) + 12(2)\mu = 0$$

$$\Rightarrow -2\lambda - 4\lambda + 24\mu = 0$$

$$\Rightarrow -6\lambda + 24\mu = 0 \quad \text{--- (3)}$$

Also since the point (1, -1, 2) lies on (1) & (2)

$$(1) \Rightarrow -\lambda + 8\mu - 4 = 0$$

$$(2) \Rightarrow 5 + 4 - 9 = 0$$

$$\text{From (3), } -6\lambda + 24\mu(1) = 0$$

$$-6\lambda = -24$$

$$\lambda = 4$$

$$\therefore \lambda = 4 \text{ and } \mu = 1.$$

(12) Find the angle b/w the tangent planes to the surfaces. $x \log z = y^2 - 1$, $x^2 y = 2 - z$ at the point (1,

sol: The angle b/w the tangent planes to the surfaces

$$\phi_1 \text{ and } \phi_2 = \cos^{-1} \left[\frac{\nabla \phi_1 \cdot \nabla \phi_2}{|\nabla \phi_1| |\nabla \phi_2|} \right]$$

$$\phi_1 = x \log z - y^2 + 1 \quad \nabla \phi_1 = \log z \bar{i} - 2y \bar{j} + \frac{x}{z} \bar{k}$$

$$\phi_2 = x^2 y + z - 2 \quad \nabla \phi_2 = 2xy \bar{i} + x^2 \bar{j} + \bar{k}$$

$$\nabla \phi_1(1, 1, 1) = -2\bar{j} + \bar{k} \quad \nabla \phi_2(1, 1, 1) = 2\bar{i} + \bar{j} + \bar{k}$$

$$0 = \cos^{-1} \left[\frac{(-2\bar{j} + \bar{k}) \cdot (2\bar{i} + \bar{j} + \bar{k})}{\sqrt{5} \sqrt{6}} \right] = \cos^{-1} \left(\frac{-1}{\sqrt{30}} \right)$$

Divergence of a Vector Point Function

the divergence of a differentiable vector point function $\vec{V}$ is denoted by $\text{div } \vec{V}$ and is defined as

$$\text{div } \vec{V} = \nabla \cdot \vec{V} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot \vec{V} = \hat{i} \cdot \frac{\partial V_x}{\partial x} + \hat{j} \cdot \frac{\partial V_y}{\partial y} + \hat{k} \cdot \frac{\partial V_z}{\partial z}$$

obviously, the divergence of a vector point function is a scalar point function.

If $\vec{V} = V_1 \hat{i} + V_2 \hat{j} + V_3 \hat{k}$, then $\text{div } \vec{V} = \nabla \cdot \vec{V} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (V_1 \hat{i} + V_2 \hat{j} + V_3 \hat{k})$

Since $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$

and $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

Curl of a Vector Point Function

The curl (or rotation) of a differentiable vector point function $\vec{V}$ is denoted by $\text{curl } \vec{V}$ and is defined as

$$\text{curl } \vec{V} = \nabla \times \vec{V} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \times \vec{V} = \hat{i} \times \frac{\partial \vec{V}}{\partial x} + \hat{j} \times \frac{\partial \vec{V}}{\partial y} + \hat{k} \times \frac{\partial \vec{V}}{\partial z}$$

obviously, the curl of a vector point function is a vector point function.

If $\vec{V} = V_1 \hat{i} + V_2 \hat{j} + V_3 \hat{k}$, then $\text{curl } \vec{V} = \nabla \times \vec{V} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \times (V_1 \hat{i} + V_2 \hat{j} + V_3 \hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \hat{i} \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \right) + \hat{j} \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} \right) + \hat{k} \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right)$$

Ex: - ① If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, show that (i) $\text{div } \vec{r} = 3$ (ii) $\text{curl } \vec{r} = \vec{0}$

(i) $\text{div } \vec{r} = \nabla \cdot \vec{r} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (x\hat{i} + y\hat{j} + z\hat{k}) = 1 + 1 + 1 = 3$

(ii) $\text{curl } \vec{r} = \nabla \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$

$$= \hat{i} \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z} \right) + \hat{j} \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x} \right) + \hat{k} \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right)$$

$$= \hat{i}(0) + \hat{j}(0) + \hat{k}(0) = \vec{0}$$

Ex 2:- Find the divergence and curl of the vector
 $\vec{v} = (xy^2)\vec{i} + (3x^2y)\vec{j} + (xz^2 - y^2z)\vec{k}$ at the point $(2, -1, 1)$

Sol:- $\text{div } \vec{v} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z)$
 $= y^2 + 3x^2 + 2xz - y^2$
 $= -1 + 12 + 4 - 1 = 14$ at $(2, -1, 1)$

$\text{curl } \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 3x^2y & xz^2 - y^2z \end{vmatrix} = \vec{i}(-2yz) + \vec{j}(xz^2 - y^2) + \vec{k}(6xy - 2z)$
 $= 2\vec{i} - 3\vec{j} - 14\vec{k}$ at $(2, -1, 1)$

Ex 3:- Find $\text{div } \vec{F}$ and $\text{curl } \vec{F}$ where $\vec{F} = \text{grad}(x^3 + y^3 + z^3 - 3xyz)$

Sol:- Let $\phi = x^3 + y^3 + z^3 - 3xyz$

$\vec{F} = \text{grad } \phi = \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) (x^3 + y^3 + z^3 - 3xyz)$
 $= (3x^2 - 3yz)\vec{i} + (3y^2 - 3xz)\vec{j} + (3z^2 - 3xy)\vec{k}$

$\text{div } \vec{F} = \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy)$
 $= 6x + 6y + 6z = 6(x + y + z)$

and, $\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$

$= \vec{i}(-3x + 3x) + \vec{j}(-3y + 3y) + \vec{k}(-3z + 3z)$
 $= \vec{0}$

Note:- ① If $\text{div } \vec{v} = 0$, then $\vec{v}$ is said to be a solenoidal vector field.
 ② If $\text{curl } \vec{v} = \vec{0}$, then $\vec{v}$ is said to be an irrotational vector field.

Q. 1. $\vec{f} = (x^2y)\vec{i} - 2xz\vec{j} + 2yz\vec{k}$, find (i)

Sol: $\vec{f} = x^2y\vec{i} - 2xz\vec{j} + 2yz\vec{k}$

(i) $\nabla \cdot \vec{f} = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(-2xz) + \frac{\partial}{\partial z}(2yz) = 2xy + 0 + 2y = 2y(2+1)$

(ii) $\nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix} = \vec{i}(2z+2z) + \vec{j}(0) + \vec{k}(-2z-x^2)$
 $= (2z+2z)\vec{i} - (x^2+2z)\vec{k}$

(iii) $\text{curl curl } \vec{f} = \nabla \times (\nabla \times \vec{f}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z+2z & 0 & -(x^2+2z) \end{vmatrix} = \vec{i}(0) + \vec{j}(2+2z) + \vec{k}(0)$
 $= (2z+2z)\vec{j}$

Q. 2. Show that the vector field given by $\vec{f} = (x+3y)\vec{i} + (y-3z)\vec{j} + (x-2z)\vec{k}$ is solenoidal.

Sol: $\vec{f} = (x+3y)\vec{i} + (y-3z)\vec{j} + (x-2z)\vec{k}$

$\nabla \cdot \vec{f} = \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-3z) + \frac{\partial}{\partial z}(x-2z) = 1+1-2 = 0$

$\therefore \vec{f}$ is solenoidal.

Q. 3. Find the constant 'a' so that $\vec{f} = y(ax^2+z)\vec{i} + x(y^2-z^2)\vec{j} + 2xy(z-xy)\vec{k}$ is solenoidal.

Sol: If $\vec{f}$ is solenoidal, $\nabla \cdot \vec{f} = 0$

$\nabla \cdot \vec{f} = 0 \Rightarrow \frac{\partial}{\partial x}[y(ax^2+z)] + \frac{\partial}{\partial y}[x(y^2-z^2)] + \frac{\partial}{\partial z}[2xy(z-xy)]$

$\Rightarrow 2axy + 2xy + 2xy = 0$

$\Rightarrow 2xy(a+2) = 0 \Rightarrow a = -2$

Q. 4. Show that $\vec{v} = (\sin y + z)\vec{i} + (x \cos y - z)\vec{j} + (x-y)\vec{k}$ is irrotational.

Sol: A vector $\vec{v}$ is irrotational if $\nabla \times \vec{v} = \vec{0}$.

Q) Find the constants a, b, c if $\vec{F}$ is irrotational.

$$\vec{F} = (2x+3y+az)\vec{i} + (bx+2y+3z)\vec{j} + (2x+cy+2z)\vec{k}$$

Sol: If $\vec{F}$ is irrotational $\Rightarrow \nabla \times \vec{F} = \vec{0}$

$$\nabla \times \vec{F} = \vec{0} \Rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+3y+az & bx+2y+3z & 2x+cy+2z \end{vmatrix} = \vec{0}$$

$$\Rightarrow (c+3)\vec{i} + (a-2)\vec{j} + (b-2)\vec{k} = \vec{0}$$

$$a-2=0 \quad \text{ie } a=2$$

$$b-3=0 \quad \text{ie } b=3$$

$$c-3=0 \quad \text{ie } c=3$$

Q) If $\vec{F} = (x+y+z)\vec{i} + (x+y+z)\vec{j} + (x+y+z)\vec{k}$ prove that $\vec{F} \cdot \text{curl } \vec{F} = 0$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+z & x+y+z & x+y+z \end{vmatrix} = \vec{i}(-1) + \vec{j}(0+1) + \vec{k}(-1) = -\vec{i} + \vec{j} - \vec{k}$$

$$\vec{F} \cdot \text{curl } \vec{F} = (x+y+z)(-1) + (x+y+z)(1) + (x+y+z)(-1) = -x-y-z+x+y+z = 0$$

Please show that $\vec{r}/r^3$ is solenoidal.

Sol: Given that $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial r}{\partial x} = \frac{x}{r} \quad \text{by} \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\frac{\partial}{\partial x} (x r^n) = 1 \cdot r^n + x \cdot n r^{n-1} \frac{\partial r}{\partial x} = r^n + x \cdot n r^{n-1} \frac{x}{r} = r^n + n x^2 r^{n-2}$$

$$= r^n + n x^2 r^{n-2}$$

$$\text{by } \frac{\partial}{\partial y} (y r^n) = r^n + n y^2 r^{n-2} \quad \text{and} \quad \frac{\partial}{\partial z} (z r^n) = r^n + n z^2 r^{n-2}$$

$$r^n \vec{r} = r^n (x\hat{i} + y\hat{j} + z\hat{k}) = x r^n \hat{i} + y r^n \hat{j} + z r^n \hat{k}$$

$$\text{div} (r^n \vec{r}) = \frac{\partial}{\partial x} (x r^n) + \frac{\partial}{\partial y} (y r^n) + \frac{\partial}{\partial z} (z r^n)$$

$$= r^n + n x^2 r^{n-2} + r^n + n y^2 r^{n-2} + r^n + n z^2 r^{n-2}$$

$$= 3r^n + n r^{n-2} (x^2 + y^2 + z^2)$$

$$= 3r^n + n r^{n-2} \cdot r^2 = 3r^n + n r^n = (3+n) r^n$$

when $n = -3$, $\text{div} \left(\frac{\vec{r}}{r^3} \right) = 0$ i.e. $\frac{\vec{r}}{r^3}$ is solenoidal.

$$\text{div grad } f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\text{Sol: } \nabla^2 f = \nabla \cdot \nabla f = \nabla \cdot \left[\frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} \right]$$

$$= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial z} \right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) f$$

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the Laplacian operator and $\nabla^2 f = 0$

is called the Laplace's equation.

Sol:- Given that $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = \sqrt{x^2 + y^2 + z^2}$

$$r^n = (x^2 + y^2 + z^2)^{n/2}$$

$$\text{To find } \nabla^2(r^n) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) r^n = \Sigma \frac{\partial^2}{\partial x^2} r^n \quad \text{--- (1)}$$

$$\text{Take } \frac{\partial}{\partial x} (r^n) = n \cdot r^{n-1} \frac{\partial r}{\partial x} = n r^{n-1} \frac{x}{r} = n x \cdot r^{n-2}$$

$$\frac{\partial^2}{\partial x^2} (r^n) = n \cdot r^{n-2} + x \cdot n \cdot (n-2) r^{n-3} \frac{\partial r}{\partial x}$$

$$= n \left[r^{n-2} + (n-2) r^{n-3} \frac{x^2}{r} \right] = n r^{n-2} + n(n-2) r^{n-4} x^2$$

Equation (1) we get

$$\nabla^2 r^n = \Sigma \frac{\partial^2}{\partial x^2} r^n$$

$$= \Sigma n \cdot r^{n-2} + n(n-2) r^{n-4} x^2$$

$$= 3n r^{n-2} + n(n-2) r^{n-4} (x^2 + y^2 + z^2)$$

$$= 3n r^{n-2} + (n^2 - 2n) r^{n-4} (r^2)$$

$$= 3n r^{n-2} + (n^2 - 2n) r^{n-2}$$

$$= r^{n-2} [3n + n^2 - 2n]$$

$$= r^{n-2} [n + n^2]$$

$$= n(n+1) r^{n-2}$$

$$\nabla^2 r^n = n(n+1) r^{n-2}$$

2

Treating ∇ as a vector, the repeated application of ∇ on a scalar gives a vector. Thus

show that $r^\alpha \vec{R}$ is any irrotational vector for any value of α but is solenoidal if $\alpha+3=0$ where $\vec{R} = x\vec{i} + y\vec{j} + z\vec{k}$ and r is the magnitude of $\vec{R}$.

$$r = \sqrt{x^2 + y^2 + z^2} = (x^2 + y^2 + z^2)^{1/2} \Rightarrow r^\alpha = (x^2 + y^2 + z^2)^{\alpha/2}$$

Soln: Let $\vec{A} = r^\alpha \vec{R} = (x^2 + y^2 + z^2)^{\alpha/2} (x\vec{i} + y\vec{j} + z\vec{k}) = \Sigma x(x^2 + y^2 + z^2)^{\alpha/2} \vec{i}$

$$\text{curl } \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x(x^2 + y^2 + z^2)^{\alpha/2} & y(x^2 + y^2 + z^2)^{\alpha/2} & z(x^2 + y^2 + z^2)^{\alpha/2} \end{vmatrix}$$

$$= \Sigma \vec{i} \left[\frac{\partial}{\partial y} [z(x^2 + y^2 + z^2)^{\alpha/2}] - \frac{\partial}{\partial z} [y(x^2 + y^2 + z^2)^{\alpha/2}] \right]$$

$$= \Sigma \vec{i} \left[\frac{\alpha z}{2} (x^2 + y^2 + z^2)^{\frac{\alpha}{2}-1} 2y - \frac{\alpha y}{2} (x^2 + y^2 + z^2)^{\frac{\alpha}{2}-1} (2z) \right] = \vec{0}$$

Hence $\vec{A}$ is irrotational for any value of α .

But $\text{div } \vec{A} = \nabla \cdot (r^\alpha \vec{R}) = (\alpha+3)r^\alpha$ which is zero for $\alpha+3=0$

i.e. $\vec{A}$ is solenoidal if $\alpha+3=0$.

2) $\text{curl grad } \phi = \nabla \times \nabla \phi = \vec{0}$

$$\nabla \times \nabla \phi = \nabla \times \left(\frac{\partial \phi}{\partial x} \vec{i} + \frac{\partial \phi}{\partial y} \vec{j} + \frac{\partial \phi}{\partial z} \vec{k} \right)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \vec{i} \left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial y \partial z} \right) + \vec{j} \left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial x \partial z} \right) + \vec{k} \left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial x \partial y} \right)$$

$$= \vec{0}$$

The above result can also be written as

$$\text{grad}(\text{div } \phi) = \text{curl}(\text{curl } \phi) + \nabla^2 \phi$$

$$1) \nabla \cdot \nabla \phi = \nabla^2 \phi$$

$$2) \nabla \times \nabla \phi = \vec{0}$$

$$3) \nabla \times (\nabla \times \phi) = \nabla(\nabla \cdot \phi) - \nabla^2 \phi$$

Let $\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$

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$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \hat{i} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) + \hat{j} \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) + \hat{k} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$\text{div}(\text{curl } \vec{F}) = \nabla \cdot (\nabla \times \vec{F})$$

$$\begin{aligned} &= \frac{\partial}{\partial x} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \\ &= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} \\ &= 0 \end{aligned}$$

$$(4) \quad \text{curl curl } \vec{F} = \text{grad div } \vec{F} - \nabla^2 \vec{F}$$

$$\text{i.e. } \nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$$

$$\text{Let } \vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$$

$$\text{then curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \hat{i} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) + \hat{j} \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) + \hat{k} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$\text{curl curl } \vec{F} = \nabla \times (\nabla \times \vec{F})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} & \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} & \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{vmatrix}$$

$$\begin{aligned} &= \hat{i} \left[\frac{\partial}{\partial x} \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) - \frac{\partial}{\partial z} \left(\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x} \right) \right] \\ &+ \hat{j} \left[\frac{\partial}{\partial y} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) - \frac{\partial}{\partial x} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \right] \\ &+ \hat{k} \left[\frac{\partial}{\partial z} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \right] \end{aligned}$$

$$= \hat{i} \left[\frac{\partial}{\partial x} \left(\frac{\partial F_1}{\partial z} + \frac{\partial F_3}{\partial x} \right) - \left(\frac{\partial^2 F_1}{\partial y \partial z} + \frac{\partial^2 F_3}{\partial x \partial z} \right) \right]$$

$$+ \hat{j} \left[\frac{\partial}{\partial y} \left(\frac{\partial F_3}{\partial z} + \frac{\partial F_1}{\partial x} \right) - \left(\frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial x} \right) \right]$$

$$+ \hat{k} \left[\frac{\partial}{\partial z} \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \right) - \left(\frac{\partial^2 F_3}{\partial x \partial z} + \frac{\partial^2 F_2}{\partial y \partial z} \right) \right]$$

$$= \nabla \cdot \left(\frac{\partial \vec{F}}{\partial x} + \frac{\partial \vec{F}}{\partial y} + \frac{\partial \vec{F}}{\partial z} \right) - \left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_3}{\partial z^2} \right)$$

$$+ \hat{j} \left[\frac{\partial}{\partial y} \left(\frac{\partial F_3}{\partial z} + \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \right) - \left(\frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial y^2} \right) \right]$$

$$+ \hat{k} \left[\frac{\partial}{\partial z} \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) - \left(\frac{\partial^2 F_3}{\partial x \partial z} + \frac{\partial^2 F_2}{\partial y \partial z} + \frac{\partial^2 F_3}{\partial z^2} \right) \right]$$

$$= \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$$

$$= \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F} = \text{grad}(\text{div } \vec{F})$$

1) For a constant vector $\vec{a}$, $\text{div } \vec{a} = 0$, $\text{curl } \vec{a} = \vec{0}$

2) $\text{div}(\vec{A} + \vec{B}) = \text{div } \vec{A} + \text{div } \vec{B}$ (or) $\nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$

Proof :- $\text{div}(\vec{A} + \vec{B}) = \nabla \cdot (\vec{A} + \vec{B}) = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (\vec{A} + \vec{B})$

$$= \hat{i} \cdot \frac{\partial}{\partial x} (\vec{A} + \vec{B}) + \hat{j} \cdot \frac{\partial}{\partial y} (\vec{A} + \vec{B}) + \hat{k} \cdot \frac{\partial}{\partial z} (\vec{A} + \vec{B})$$

$$= \hat{i} \cdot \left(\frac{\partial \vec{A}}{\partial x} + \frac{\partial \vec{B}}{\partial x} \right) + \hat{j} \cdot \left(\frac{\partial \vec{A}}{\partial y} + \frac{\partial \vec{B}}{\partial y} \right) + \hat{k} \cdot \left(\frac{\partial \vec{A}}{\partial z} + \frac{\partial \vec{B}}{\partial z} \right)$$

$$= \left(\hat{i} \cdot \frac{\partial \vec{A}}{\partial x} + \hat{j} \cdot \frac{\partial \vec{A}}{\partial y} + \hat{k} \cdot \frac{\partial \vec{A}}{\partial z} \right) + \left(\hat{i} \cdot \frac{\partial \vec{B}}{\partial x} + \hat{j} \cdot \frac{\partial \vec{B}}{\partial y} + \hat{k} \cdot \frac{\partial \vec{B}}{\partial z} \right)$$

$$= \nabla \cdot \vec{A} + \nabla \cdot \vec{B} = \text{div } \vec{A} + \text{div } \vec{B}$$

3) $\text{curl}(\vec{A} + \vec{B}) = \text{curl } \vec{A} + \text{curl } \vec{B}$ (or) $\nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$

$$\text{curl}(\vec{A} + \vec{B}) = \nabla \times (\vec{A} + \vec{B}) = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \times (\vec{A} + \vec{B})$$

$$= \sum \hat{i} \times \frac{\partial}{\partial x} (\vec{A} + \vec{B}) = \sum \hat{i} \times \left(\frac{\partial \vec{A}}{\partial x} + \frac{\partial \vec{B}}{\partial x} \right)$$

$$= \sum \hat{i} \times \frac{\partial \vec{A}}{\partial x} + \sum \hat{i} \times \frac{\partial \vec{B}}{\partial x}$$

$$= (\nabla \times \vec{A}) + (\nabla \times \vec{B}) = \text{curl } \vec{A} + \text{curl } \vec{B}$$

4) If $\vec{A}$ is a vector function and ϕ is a scalar function

$$\text{div}(\phi \vec{A}) = \phi \text{div } \vec{A} + (\text{grad } \phi) \cdot \vec{A} \quad (\text{or}) \quad \nabla \cdot (\phi \vec{A}) = \phi (\nabla \cdot \vec{A}) + (\nabla \phi) \cdot \vec{A}$$

$$\text{Sol:-} \quad \text{div}(\phi \vec{A}) = \nabla \cdot (\phi \vec{A}) = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (\phi \vec{A})$$

$$= \hat{i} \cdot \frac{\partial}{\partial x} (\phi \vec{A}) + \hat{j} \cdot \frac{\partial}{\partial y} (\phi \vec{A}) + \hat{k} \cdot \frac{\partial}{\partial z} (\phi \vec{A})$$

$$= \sum \hat{i} \cdot \frac{\partial}{\partial x} (\phi \vec{A}) = \sum \hat{i} \cdot \left[\phi \frac{\partial \vec{A}}{\partial x} + \frac{\partial \phi}{\partial x} \vec{A} \right] = \phi \sum \left(\hat{i} \cdot \frac{\partial \vec{A}}{\partial x} \right) + \sum \left(\hat{i} \cdot \frac{\partial \phi}{\partial x} \right) \cdot \vec{A}$$

$$= \phi (\nabla \cdot \vec{A}) + \nabla \phi \cdot \vec{A}$$

$$\text{curl}(\phi \bar{A}) = (\text{grad} \phi) \times \bar{A} + \phi \text{curl} \bar{A}$$

$$(\text{or}) \nabla \times (\phi \bar{A}) = (\nabla \phi) \times \bar{A} + \phi (\nabla \times \bar{A})$$

$$\text{Soln: } \text{curl}(\phi \bar{A}) = \nabla \times (\phi \bar{A}) = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \times (\phi \bar{A})$$

$$= \sum \hat{i} \times \frac{\partial}{\partial x} (\phi \bar{A}) = \sum \hat{i} \times \left(\frac{\partial \phi}{\partial x} \bar{A} + \phi \frac{\partial \bar{A}}{\partial x} \right)$$

$$= \sum \hat{i} \times \frac{\partial \phi}{\partial x} \bar{A} + \phi \sum \hat{i} \times \frac{\partial \bar{A}}{\partial x}$$

$$\boxed{\bar{a} \times (m \bar{b}) = (m \bar{a}) \times \bar{b} = m(\bar{a} \times \bar{b})}$$

$$= \sum \frac{\partial \phi}{\partial x} \hat{i} \times \bar{A} + \phi \sum \hat{i} \times \frac{\partial \bar{A}}{\partial x}$$

$$= \nabla \phi \times \bar{A} + \phi (\nabla \times \bar{A})$$

$$= (\text{grad} \phi) \times \bar{A} + \phi \text{curl} \bar{A}$$

$$6) \nabla (\bar{A} \cdot \bar{B}) = (\bar{A} \cdot \nabla) \bar{B} + (\bar{B} \cdot \nabla) \bar{A} + \bar{A} \times (\nabla \times \bar{B}) + \bar{B} \times (\nabla \times \bar{A})$$

$$\text{Soln: } \nabla (\bar{A} \cdot \bar{B}) = \sum \hat{i} \frac{\partial}{\partial x} (\bar{A} \cdot \bar{B}) = \sum \hat{i} \left(\bar{A} \cdot \frac{\partial \bar{B}}{\partial x} + \frac{\partial \bar{A}}{\partial x} \cdot \bar{B} \right)$$

$$= \sum \left(\bar{A} \cdot \frac{\partial \bar{B}}{\partial x} \right) \hat{i} + \sum \left(\bar{B} \cdot \frac{\partial \bar{A}}{\partial x} \right) \hat{i} \quad \text{--- (1)}$$

we know that:

$$\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c}) \bar{b} - (\bar{a} \cdot \bar{b}) \bar{c}$$

$$(\bar{a} \cdot \bar{b}) \bar{c} = (\bar{a} \cdot \bar{c}) \bar{b} - \bar{a} \times (\bar{b} \times \bar{c})$$

$$\left(\bar{A} \cdot \frac{\partial \bar{B}}{\partial x} \right) \hat{i} = \left(\bar{A} \cdot \hat{i} \right) \frac{\partial \bar{B}}{\partial x} - \bar{A} \times \left(\frac{\partial \bar{B}}{\partial x} \times \hat{i} \right)$$

$$= \left(\bar{A} \cdot \hat{i} \right) \frac{\partial \bar{B}}{\partial x} + \bar{A} \times \left(\hat{i} \times \frac{\partial \bar{B}}{\partial x} \right)$$

$$\sum \left(\bar{A} \cdot \frac{\partial \bar{B}}{\partial x} \right) \hat{i} = \left(\bar{A} \cdot \sum \hat{i} \frac{\partial}{\partial x} \right) \bar{B} + \bar{A} \times \sum \left(\hat{i} \times \frac{\partial \bar{B}}{\partial x} \right)$$

$$= (\bar{A} \cdot \nabla) \bar{B} + \bar{A} \times (\nabla \times \bar{B}) \quad \text{--- (2)}$$

Interchanging $\bar{A}$ and $\bar{B}$

$$\sum \left(\bar{B} \cdot \frac{\partial \bar{A}}{\partial x} \right) \hat{i} = (\bar{B} \cdot \nabla) \bar{A} + \bar{B} \times (\nabla \times \bar{A}) \quad \text{--- (3)}$$

Substituting values from (2) and (3) in (1), we get.

$$\begin{aligned}\nabla \cdot (\vec{A} \times \vec{B}) &= \sum \vec{i} \cdot \frac{\partial}{\partial x} (\vec{A} \times \vec{B}) = \sum \vec{i} \cdot \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B} + \vec{A} \times \frac{\partial \vec{B}}{\partial x} \right) \\ &= \sum \vec{i} \cdot \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B} \right) + \sum \vec{i} \cdot \left(\vec{A} \times \frac{\partial \vec{B}}{\partial x} \right) \\ &= \sum \vec{i} \cdot \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B} \right) - \sum \vec{i} \cdot \left(\frac{\partial \vec{B}}{\partial x} \times \vec{A} \right) \\ &= \sum \left(\vec{i} \times \frac{\partial \vec{A}}{\partial x} \right) \cdot \vec{B} - \sum \left(\vec{i} \times \frac{\partial \vec{B}}{\partial x} \right) \cdot \vec{A} \\ &= (\nabla \times \vec{A}) \cdot \vec{B} - (\nabla \times \vec{B}) \cdot \vec{A} \\ &= \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})\end{aligned}$$

Since $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

$$\nabla \times (\vec{A} \times \vec{B}) = (\nabla \cdot \vec{B}) \vec{A} - (\nabla \cdot \vec{A}) \vec{B} + (\vec{B} \cdot \nabla) \vec{A} - (\vec{A} \cdot \nabla) \vec{B}$$

$$\begin{aligned}\nabla \times (\vec{A} \times \vec{B}) &= \sum \vec{i} \times \frac{\partial}{\partial x} (\vec{A} \times \vec{B}) = \sum \vec{i} \times \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B} + \vec{A} \times \frac{\partial \vec{B}}{\partial x} \right) \\ &= \sum \vec{i} \times \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B} \right) + \sum \vec{i} \times \left(\vec{A} \times \frac{\partial \vec{B}}{\partial x} \right) \\ &= \sum \left[(\vec{i} \cdot \vec{B}) \frac{\partial \vec{A}}{\partial x} - \left(\vec{i} \cdot \frac{\partial \vec{A}}{\partial x} \right) \vec{B} \right] + \sum \left[\left(\vec{i} \cdot \frac{\partial \vec{B}}{\partial x} \right) \vec{A} - (\vec{i} \cdot \vec{A}) \frac{\partial \vec{B}}{\partial x} \right] \\ &= \sum (\vec{B} \cdot \vec{i}) \frac{\partial \vec{A}}{\partial x} - \sum \left(\vec{i} \cdot \frac{\partial \vec{A}}{\partial x} \right) \vec{B} + \sum \left(\vec{i} \cdot \frac{\partial \vec{B}}{\partial x} \right) \vec{A} - \sum (\vec{A} \cdot \vec{i}) \frac{\partial \vec{B}}{\partial x} \\ &= \left(\vec{B} \cdot \sum \vec{i} \frac{\partial}{\partial x} \right) \vec{A} - (\nabla \cdot \vec{A}) \vec{B} + (\nabla \cdot \vec{B}) \vec{A} - \left(\vec{A} \cdot \sum \vec{i} \frac{\partial}{\partial x} \right) \vec{B} \\ &= (\vec{B} \cdot \nabla) \vec{A} - (\nabla \cdot \vec{A}) \vec{B} + (\nabla \cdot \vec{B}) \vec{A} - (\vec{A} \cdot \nabla) \vec{B}\end{aligned}$$

2.

If $\vec{v}$ represents the velocity of the fluid, then $\nabla \cdot \vec{v}$ gives the rate at which the fluid is originating at a point per unit volume. Similarly if $\vec{v}$ represents an electric flux then $\text{div } \vec{v}$ is the amount of flux which diverges per unit volume in unit time. If $\nabla \cdot \vec{F} = 0$ then the field is said to be a solenoidal field. Gravitational or electrostatic field are examples of solenoidal fields.

Physical interpretation of the curl of a vector :-

The curl of a vector function $\vec{F}$ gives a measure of angular velocity at every point of the vector field.

If $\text{curl } \vec{F} = \vec{0}$ the $\vec{F}$ is said to be irrotational field.

Ex 1 - $\nabla^2(r^n) = n(n+1)r^{n-2}$

$$\nabla^2(r^n) = \frac{d^2}{dx^2}(r^n) + \frac{d^2}{dy^2}(r^n) + \frac{d^2}{dz^2}(r^n) \quad \text{--- (1)}$$

$$\frac{d}{dx}(r^n) = nr^{n-1} \frac{dr}{dx} = nr^{n-1} \frac{x}{r} = nr^{n-2}x$$

$$\frac{d^2}{dx^2}(r^n) = n \left[r^{n-2} + (n-2)r^{n-3} \frac{dr}{dx} \cdot x \right] = n \left[r^{n-2} + (n-2)r^{n-3} \frac{x}{r} x \right]$$

$$= n \left[r^{n-2} + (n-2)r^{n-4}x^2 \right]$$

$$\frac{d^2}{dy^2}(r^n) = n \left[r^{n-2} + (n-2)r^{n-4}y^2 \right] \quad \text{and} \quad \frac{d^2}{dz^2}(r^n) = n \left[r^{n-2} + (n-2)r^{n-4}z^2 \right] \quad \text{--- (2)}$$

Adding (2), (3) and (1), (1) gives

$$\nabla^2(r^n) = n \left[3r^{n-2} + (n-2)r^{n-4}(x^2 + y^2 + z^2) \right]$$

$$= n \left[3r^{n-2} + (n-2)r^{n-4}r^2 \right] = n(n+1)r^{n-2}$$

show that $\vec{r} \cdot \text{curl} \vec{F} = 0$

since $\vec{F} = \frac{1}{r} \nabla V \Rightarrow \text{curl} \vec{F} = \text{curl} \left(\frac{1}{r} \nabla V \right)$

$$\text{curl} \vec{F} = \nabla \frac{1}{r} \times \nabla V + \frac{1}{r} \nabla (\text{curl} V)$$

$$[\nabla \times \nabla V = \nabla V \times \vec{r} + \nabla (\nabla \cdot \vec{r})]$$

$$\nabla \times \nabla V = 0$$

$$= \nabla \frac{1}{r} \times \nabla V$$

$$\vec{r} \cdot \text{curl} \vec{F} = \frac{1}{r} \nabla V \cdot (\nabla \frac{1}{r} \times \nabla V) = 0, \text{ for scalar triple product}$$

in which two vectors are equal.

If r and $\vec{r}$ have their usual meanings and $\vec{A}$ is a constant vector, prove that $\nabla \times \left(\frac{\vec{A} \times \vec{r}}{r^n} \right) = \frac{2-n}{r^n} \vec{A} + \frac{n(\vec{A} \cdot \vec{r})}{r^{n+2}} \vec{r}$

$$\nabla \times \left[r^{-n} (\vec{A} \times \vec{r}) \right] = r^{-n} [\nabla \times (\vec{A} \times \vec{r})] + \nabla r^{-n} \times (\vec{A} \times \vec{r})$$

$$= r^{-n} [(\nabla \cdot \vec{r}) \vec{A} - (\nabla \cdot \vec{A}) \vec{r}] + \left[-n r^{-(n+1)} \frac{\vec{r}}{r} \right] \times (\vec{A} \times \vec{r})$$

$$= r^{-n} (3\vec{A} - \vec{A}) - n r^{-(n+2)} \vec{r} \times (\vec{A} \times \vec{r}) \quad [\because \nabla \cdot \vec{r} = 3, (\vec{A} \cdot \nabla) \vec{r} = \vec{A}]$$

$$= 2\vec{A} r^{-n} - n r^{-(n+2)} [(\vec{r} \cdot \vec{r}) \vec{A} - (\vec{r} \cdot \vec{A}) \vec{r}]$$

$$= \frac{2\vec{A}}{r^n} - \frac{n}{r^{n+2}} [r^2 \vec{A} - (\vec{A} \cdot \vec{r}) \vec{r}]$$

$$= \frac{2-n}{r^n} \vec{A} + \frac{n(\vec{A} \cdot \vec{r})}{r^{n+2}} \vec{r}$$

If r is the distance of a point (x, y, z) from the origin, prove that $\text{curl} (\vec{r} \cdot \text{grad} \frac{1}{r}) + \text{grad} (\vec{r} \cdot \text{grad} \frac{1}{r}) = 0$ where $\vec{r}$ is the vector in the direction of z .

$$\begin{aligned} \text{grad} \frac{1}{r} &= \nabla \frac{1}{r} = \left(\hat{i} \frac{d}{dx} + \hat{j} \frac{d}{dy} + \hat{k} \frac{d}{dz} \right) (x^2 + y^2 + z^2)^{-1/2} \\ &= \frac{1}{2} (x^2 + y^2 + z^2)^{-3/2} (2x\hat{i} + 2y\hat{j} + 2z\hat{k}) \\ &= -(x^2 + y^2 + z^2)^{-3/2} (x\hat{i} + y\hat{j} + z\hat{k}) \end{aligned}$$

$$\text{curl} (\vec{E} \times \text{grad } \frac{1}{r}) = \nabla \times (-x^2 + y^2 + z^2) - \nabla \times (2x - y^2)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y/(x^2+y^2+z^2)^{3/2} & -x/(x^2+y^2+z^2)^{3/2} & 0 \end{vmatrix}$$

$$= \hat{i} \frac{\partial}{\partial z} \left[\frac{2}{(x^2+y^2+z^2)^{3/2}} \right] + \hat{j} \frac{\partial}{\partial x} \left[\frac{y}{(x^2+y^2+z^2)^{3/2}} \right] - \hat{k} \left[\frac{\partial}{\partial x} \left(\frac{x}{(x^2+y^2+z^2)^{3/2}} \right) - \frac{\partial}{\partial y} \left(\frac{y}{(x^2+y^2+z^2)^{3/2}} \right) \right]$$

$$= \frac{-3xz\hat{i} - 3yz\hat{j} + (x^2+y^2-2z^2)\hat{k}}{(x^2+y^2+z^2)^{5/2}} \quad \text{--- (1)}$$

$$\text{grad} (\vec{E} \cdot \text{grad } \frac{1}{r}) = \nabla \left[-\vec{E} \cdot \frac{(x\hat{i} + y\hat{j} + z\hat{k})}{(x^2+y^2+z^2)^{3/2}} \right]$$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \left[\frac{-z}{(x^2+y^2+z^2)^{3/2}} \right]$$

$$= \frac{-3xz\hat{i}}{(x^2+y^2+z^2)^{5/2}} + \frac{3yz\hat{j}}{(x^2+y^2+z^2)^{5/2}} + \left(\frac{3x^2 - x^2 - y^2 - z^2}{(x^2+y^2+z^2)^{5/2}} \right) \hat{k}$$

Adding (1) and (ii) we get

$$\text{curl} (\vec{E} \times \text{grad } \frac{1}{r}) + \text{grad} (\vec{E} \cdot \text{grad } \frac{1}{r}) = 0$$

Divergence: The divergence of a continuously differentiable vector point function $\vec{F}$ is denoted by $\text{div } \vec{F}$ and is defined by the equation

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \vec{i} \cdot \frac{\partial \vec{F}}{\partial x} + \vec{j} \cdot \frac{\partial \vec{F}}{\partial y} + \vec{k} \cdot \frac{\partial \vec{F}}{\partial z}$$

$$\text{If } \vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$$

$$\begin{aligned} \text{then } \text{div } \vec{F} = \nabla \cdot \vec{F} &= \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot (F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}) \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \end{aligned}$$

Curl:- The curl of a continuously differentiable vector point function $\vec{F}$ is defined by the equation

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \vec{i} \times \frac{\partial \vec{F}}{\partial x} + \vec{j} \times \frac{\partial \vec{F}}{\partial y} + \vec{k} \times \frac{\partial \vec{F}}{\partial z}$$

$$\text{If } \vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}, \text{ then}$$

$$\begin{aligned} \nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \vec{i} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) + \vec{j} \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \\ &\quad + \vec{k} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \end{aligned}$$

Ex:- If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, show that

$$(i) \nabla \cdot \vec{r} = 3, \quad (ii) \nabla \times \vec{r} = \vec{0}$$

$$\text{Sol: } (i) \nabla \cdot \vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$$

$$(ii) \nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{i} \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z} \right) + \vec{j} \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x} \right) + \vec{k} \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) = \vec{0}$$

$$= \frac{\left(\frac{\partial}{\partial x} \left(\frac{2(x^2+y^2+z^2)}{2\sqrt{x^2+y^2+z^2}} \right) + \frac{\partial}{\partial y} \left(\frac{2xy}{2\sqrt{x^2+y^2+z^2}} \right) + \frac{\partial}{\partial z} \left(\frac{2xz}{2\sqrt{x^2+y^2+z^2}} \right) \right)}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}} = \frac{2}{\sqrt{x^2+y^2+z^2}}$$

(11) Find the value of 'a' if the vector $(axy + yz)\bar{i} + (xy^2 - xz^2)\bar{j} + (2xyz - 2x^2y^2)\bar{k}$ has a zero divergence. Find the curl of the above vector which has zero divergence.

Q. If $\vec{f}$ is solenoidal $\Rightarrow \nabla \cdot \vec{f} = 0$

$$\nabla \cdot \vec{f} = 0 \Rightarrow \frac{\partial}{\partial x}(axy + yz) + \frac{\partial}{\partial y}(xy^2 - xz^2) + \frac{\partial}{\partial z}(2xyz - 2x^2y^2) = 0$$

$$\Rightarrow axy + axy + axy = 0$$

$$\Rightarrow axy(a+2) = 0 \Rightarrow a = -2$$

$$\nabla \times \vec{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy + yz & xy^2 - xz^2 & 2xyz - 2x^2y^2 \end{vmatrix}$$

$$= \bar{i}(2xz - 4x^2y + 2xz) + \bar{j}(y - 2yz + 4xy^2) + \bar{k}(y^2 - x^2 - ax^2 - z)$$

show that each of the following vectors are solenoidal.

(i) let $\vec{P} = (-x^2 + yz)\vec{i} + (4y - z^2x)\vec{j} + (2xz - 4z)\vec{k}$

for solenoidal vector $\Rightarrow \text{div } \vec{P} = 0$

$$\begin{aligned}\text{div } \vec{P} &= \frac{\partial}{\partial x}(-x^2 + yz) + \frac{\partial}{\partial y}(4y - z^2x) + \frac{\partial}{\partial z}(2xz - 4z) \\ &= -2x + 4 + 2x - 4 \\ &= 0\end{aligned}$$

Hence the vector is solenoidal.

(ii) $\vec{P} = 3y^4z^2\vec{i} + 4x^3z^2\vec{j} + 3x^2y^2\vec{k}$

$$\begin{aligned}\text{div } \vec{P} &= \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(3x^2y^2) \\ &= 0 + 0 + 0 \\ &= 0\end{aligned}$$

(iii) If ϕ and ψ are scalar functions, then prove that $\nabla\phi \times \nabla\psi$ is solenoidal.

Given that ϕ and ψ are scalar function

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

$$\nabla\psi = \frac{\partial\psi}{\partial x}\vec{i} + \frac{\partial\psi}{\partial y}\vec{j} + \frac{\partial\psi}{\partial z}\vec{k}$$

$$\nabla\phi \times \nabla\psi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \\ \frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial y} & \frac{\partial\psi}{\partial z} \end{vmatrix}$$

$$= \bar{i} \left(\frac{\partial \phi}{\partial y} \cdot \frac{\partial \psi}{\partial z} - \frac{\partial \phi}{\partial z} \cdot \frac{\partial \psi}{\partial y} \right) + \bar{j} \left(\frac{\partial \phi}{\partial z} \cdot \frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial x} \cdot \frac{\partial \psi}{\partial z} \right) + \bar{k} \left(\frac{\partial \phi}{\partial x} \cdot \frac{\partial \psi}{\partial y} - \frac{\partial \phi}{\partial y} \cdot \frac{\partial \psi}{\partial x} \right)$$

A. Vector field $\bar{F}$ is solenoidal if $\text{div } \bar{F} = 0$ if

$$\bar{F} = F_1 \bar{i} + F_2 \bar{j} + F_3 \bar{k} \text{ then}$$

$$\text{div } \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

To prove that $\nabla \cdot (\nabla \phi \times \nabla \psi) = 0$

$$\text{So } \nabla \cdot (\nabla \phi \times \nabla \psi) = \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial y} \cdot \frac{\partial \psi}{\partial z} - \frac{\partial \psi}{\partial y} \cdot \frac{\partial \phi}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial z} \cdot \frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial x} \cdot \frac{\partial \psi}{\partial z} \right) + \frac{\partial}{\partial z} \left(\frac{\partial \phi}{\partial x} \cdot \frac{\partial \psi}{\partial y} - \frac{\partial \phi}{\partial y} \cdot \frac{\partial \psi}{\partial x} \right)$$

$$= \frac{\partial \phi}{\partial y} \cdot \frac{\partial^2 \psi}{\partial x \partial z} + \frac{\partial \psi}{\partial z} \cdot \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial \psi}{\partial y} \cdot \frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial \phi}{\partial z} \cdot \frac{\partial^2 \psi}{\partial x \partial y} +$$

$$\frac{\partial \phi}{\partial z} \cdot \frac{\partial^2 \psi}{\partial y \partial x} + \frac{\partial \psi}{\partial x} \cdot \frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial \phi}{\partial x} \cdot \frac{\partial^2 \psi}{\partial y \partial z} - \frac{\partial \psi}{\partial z} \cdot \frac{\partial^2 \phi}{\partial y \partial x}$$

$$+ \frac{\partial \phi}{\partial x} \cdot \frac{\partial^2 \psi}{\partial z \partial y} + \frac{\partial \psi}{\partial y} \cdot \frac{\partial^2 \phi}{\partial z \partial x} - \frac{\partial \phi}{\partial y} \cdot \frac{\partial^2 \psi}{\partial z \partial x} - \frac{\partial \psi}{\partial x} \cdot \frac{\partial^2 \phi}{\partial z \partial y}$$

$$\nabla \cdot (\nabla \phi \times \nabla \psi) = 0$$

$$\left(\because \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x} \right)$$

$$\nabla \phi \times \nabla \psi \neq 0$$

are solenoidal.

$\vec{A}$ and $\vec{B}$ are irrotational, prove that $\vec{A} \times \vec{B}$ is solenoidal.

Sol:

Given $\vec{A}$ and $\vec{B}$ are irrotational

$$\nabla \times \vec{A} = \vec{0}, \quad \nabla \times \vec{B} = \vec{0}$$

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

$$= \vec{B} \cdot \vec{0} - \vec{A} \cdot \vec{0}$$

$$= 0$$

$$\nabla \cdot (\vec{A} \times \vec{B}) = 0 \Rightarrow \vec{A} \times \vec{B} \text{ is solenoidal.}$$

(4) If $U = x^2 + y^2 + z^2$ and $\vec{V} = x\vec{i} + y\vec{j} + z\vec{k}$, show that

$$\text{div}(\vec{U}\vec{V}) = 5U.$$

$$\vec{U}\vec{V} = (x^2 + y^2 + z^2)(x\vec{i} + y\vec{j} + z\vec{k})$$

$$= x(x^2 + y^2 + z^2)\vec{i} + y(x^2 + y^2 + z^2)\vec{j} + z(x^2 + y^2 + z^2)\vec{k}$$

$$\text{div}(\vec{U}\vec{V}) = \frac{\partial}{\partial x}(x^3 + y^2 + z^2) + \frac{\partial}{\partial y}(x^2 + y^3 + z^2) + \frac{\partial}{\partial z}(x^2 + y^2 + z^3)$$

$$= 3x^2 + y^2 + z^2 + x^2 + 3y^2 + z^2 + x^2 + y^2 + 3z^2$$

$$= 5(x^2 + y^2 + z^2)$$

$$= 5U.$$

(5) If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $n \neq 0$, show that

$$(b) \quad \nabla \left(\frac{1}{r^n} \right) = -\frac{\vec{r}}{r^{n+2}}$$

$$\nabla \left(\frac{1}{r^n} \right) = \vec{r} \cdot \frac{\partial}{\partial x} \left(\frac{1}{r^n} \right)$$

$$= \vec{r} \cdot \frac{1}{r^n} \cdot \frac{\partial}{\partial x} (1) = \frac{\vec{r} \cdot \frac{\partial \vec{r}}{\partial x}}{(r^n)^2}$$

$$= - \frac{\sum \bar{r} \cdot \frac{\partial}{\partial x} \frac{\bar{r}}{r^2}}{r^4} = - \frac{2\bar{r}}{r^4}$$

$$(ii) \quad \nabla \cdot \left(\frac{\bar{r}}{r^2} \right) = \frac{1}{r^2}$$

$$\sum \bar{r} \cdot \frac{\partial}{\partial x} \left(\frac{\bar{r}}{r^2} \right) = \sum \bar{r} \cdot \frac{\left[r^2 \frac{\partial \bar{r}}{\partial x} - \bar{r} \cdot 2r \frac{\partial r}{\partial x} \right]}{(r^2)^2}$$

$$= \sum \bar{r} \cdot \frac{\left[r^2 \cdot \bar{r} - \bar{r} \cdot 2r \cdot \frac{r}{r} \right]}{r^4}$$

$$= \frac{3r^2 - 2\bar{r} \cdot \bar{r}}{r^4} = \frac{3r^2 - 2r^2}{r^4}$$

$$\Rightarrow \frac{r^2}{r^4} = \frac{1}{r^2}$$

$$(iii) \quad \text{div} (r^n \bar{r}) = (n+3)r^n ; \text{ curl } r^n \bar{r} = \vec{0}$$

$$r^n = (x^2 + y^2 + z^2)^{n/2} ; \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$\text{div} (r^n \bar{r}) = \frac{\partial}{\partial x} (x^2 + y^2 + z^2)^{n/2} x + \frac{\partial}{\partial y} (x^2 + y^2 + z^2)^{n/2} y + \frac{\partial}{\partial z} (x^2 + y^2 + z^2)^{n/2} z$$

$$= (x^2 + y^2 + z^2)^{n/2} + \frac{n}{2} \cdot x(x^2 + y^2 + z^2)^{\frac{n}{2}-1} \cdot 2x$$

$$+ (x^2 + y^2 + z^2)^{n/2} + \frac{n}{2} \cdot y(x^2 + y^2 + z^2)^{\frac{n}{2}-1} \cdot 2y$$

$$+ (x^2 + y^2 + z^2)^{n/2} + \frac{n}{2} \cdot z(x^2 + y^2 + z^2)^{\frac{n}{2}-1} \cdot 2z$$

$$= (x^2 + y^2 + z^2)^{n/2} \left[3(x^2 + y^2 + z^2) + n(x^2 + y^2 + z^2) \right]$$

$$= (x^2 + y^2 + z^2)^{n/2} [3 + n]$$

$$= (n+3)r^n$$

$$\vec{r}^n = (x^2 + y^2 + z^2)^{n/2} (x\vec{i} + y\vec{j} + z\vec{k})$$

$$= x(x^2 + y^2 + z^2)^{n/2} \vec{i}$$

$$\text{curl}(\vec{r}^n) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x(x^2 + y^2 + z^2)^{n/2} & y(x^2 + y^2 + z^2)^{n/2} & z(x^2 + y^2 + z^2)^{n/2} \end{vmatrix}$$

$$= \vec{i} \left[\frac{\partial}{\partial y} (z(x^2 + y^2 + z^2)^{n/2}) - \frac{\partial}{\partial z} (y(x^2 + y^2 + z^2)^{n/2}) \right]$$

$$= \vec{0}$$

$$(v) \quad \nabla \left(\frac{\vec{r}}{r} \right) = -\frac{\vec{r}}{r^3}$$

$$\vec{r} = (x\vec{i} + y\vec{j} + z\vec{k}) \Rightarrow r = \sqrt{x^2 + y^2 + z^2}$$

$$\text{div} \left(\frac{\vec{r}}{r} \right) = \nabla \cdot \frac{\vec{r}}{r} = \nabla \cdot \frac{(x\vec{i} + y\vec{j} + z\vec{k})}{\sqrt{x^2 + y^2 + z^2}}$$

$$= \frac{\partial}{\partial x} \left(\frac{x}{\sqrt{x^2 + y^2 + z^2}} \right) + \frac{\partial}{\partial y} \left(\frac{y}{\sqrt{x^2 + y^2 + z^2}} \right) + \frac{\partial}{\partial z} \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)$$

$$= \frac{\sqrt{x^2 + y^2 + z^2} - \frac{x^2}{\sqrt{x^2 + y^2 + z^2}}}{(x^2 + y^2 + z^2)} + \frac{\sqrt{x^2 + y^2 + z^2} - \frac{y^2}{\sqrt{x^2 + y^2 + z^2}}}{(x^2 + y^2 + z^2)}$$

$$+ \frac{\sqrt{x^2 + y^2 + z^2} - \frac{z^2}{\sqrt{x^2 + y^2 + z^2}}}{(x^2 + y^2 + z^2)}$$

$$\Rightarrow \frac{x^2 + y^2 + z^2 - x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{3/2}} = 0$$

$$\frac{2(x^2+y^2+z^2)^{-1/2}}{(x^2+y^2+z^2)^{3/2}}$$

$$2(x^2+y^2+z^2)^{-1/2}$$

$$\begin{aligned}\nabla \left(\nabla \cdot \frac{\vec{R}}{r} \right) &= 2 \frac{\partial}{\partial x} (x^2+y^2+z^2)^{-1/2} \cdot \vec{i} + 2 \frac{\partial}{\partial y} (x^2+y^2+z^2)^{-1/2} \cdot \vec{j} \\ &\quad + 2 \frac{\partial}{\partial z} (x^2+y^2+z^2)^{-1/2} \cdot \vec{k} \\ &= -2 \cdot \frac{1}{2} (x^2+y^2+z^2)^{-3/2} \cdot 2x \vec{i} + 2 \left(-\frac{1}{2} \right) (x^2+y^2+z^2)^{-3/2} \cdot 2y \vec{j} \\ &\quad + 2 \left(-\frac{1}{2} \right) (x^2+y^2+z^2)^{-3/2} \cdot 2z \vec{k} \\ &= \frac{-2(x\vec{i} + y\vec{j} + z\vec{k})}{(x^2+y^2+z^2)^{3/2}} = \frac{-2\vec{R}}{r^3}\end{aligned}$$

(9) If $\vec{V}_1$ and $\vec{V}_2$ be the vectors joining the fixed point (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively to a variable point (x, y, z) . prove that (i) $\text{div}(\vec{V}_1 + \vec{V}_2) = 0$

(ii) $\text{grad}(\vec{V}_1 \cdot \vec{V}_2) = \vec{V}_1 + \vec{V}_2$ (iii) $\text{curl}(\vec{V}_1 \times \vec{V}_2) = 2(\vec{V}_1 - \vec{V}_2)$

$$\text{Here } \vec{V}_1 = (x-x_1)\vec{i} + (y-y_1)\vec{j} + (z-z_1)\vec{k} = \sum \vec{i}(x-x_1)$$

$$\vec{V}_2 = (x-x_2)\vec{i} + (y-y_2)\vec{j} + (z-z_2)\vec{k} = \sum \vec{i}(x-x_2)$$

$$\vec{V}_1 \cdot \vec{V}_2 = (x-x_1)(x-x_2) + (y-y_1)(y-y_2) + (z-z_1)(z-z_2)$$

$$\text{grad}(\vec{V}_1 \cdot \vec{V}_2) = \nabla(\vec{V}_1 \cdot \vec{V}_2) = \sum \vec{i} \frac{\partial}{\partial x} \{ (x-x_1)(x-x_2) + (y-y_1)(y-y_2) + (z-z_1)(z-z_2) \}$$

$$= \sum \vec{i} \{ (x-x_1) + (x-x_2) \}$$

$$= \sum \vec{i} (x-x_1) + \sum \vec{i} (x-x_2)$$

$$= \vec{V}_1 + \vec{V}_2$$

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x-x_1 & y-y_1 & z-z_1 \\ x-x_2 & y-y_2 & z-z_2 \end{vmatrix}$$

$$= \bar{i}[(y-y_1)(z-z_2) - (z-z_1)(y-y_2)] + \bar{j}[(z-z_1)(x-x_2) - (x-x_1)(z-z_2)] + \bar{k}[(x-x_1)(y-y_2) - (y-y_1)(x-x_2)]$$

$$\text{div}(\bar{v}_1 \times \bar{v}_2) = 0$$

$$\text{curl}(\bar{v}_1 \times \bar{v}_2) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y_1 z_2 - y_2 z_1 - y z_1 & -x_1 z_2 + x z_1 + x z_2 & x_1 y_2 - x_2 y_1 - x y_1 \\ -y_2 z_1 + y z_2 + y_1 z_1 & +x_2 z_1 - x z_2 - x z_1 & -x_2 y_1 + x y_2 + x y_1 \end{vmatrix}$$

$$= \bar{i}[-x_1 + x_2 - x_1 + x_2] + \bar{j}[-y_1 + y_2 + y_2 - y_1] + \bar{k}[z_2 - z_1 + z_2 - z_1]$$

$$= 2(x_2 - x_1)\bar{i} + 2(y_2 - y_1)\bar{j} + 2(z_2 - z_1)\bar{k}$$

$$= 2[x_2 \bar{i} + y_2 \bar{j} + z_2 \bar{k}] - 2[x_1 \bar{i} + y_1 \bar{j} + z_1 \bar{k}]$$

$$= 2[\bar{v}_2 - \bar{v}_1]$$

2) Show that (i) $\nabla^2 f(x) = f''(x) + \frac{2}{x} f'(x)$

$$\nabla^2 = \nabla \cdot (\nabla f(x)) = \nabla \cdot [f'(x) \nabla x]$$

$$= \nabla \cdot \left[f'(x) \frac{\bar{x}}{x} \right] = \nabla \cdot \left[\frac{f'(x)}{x} \cdot \bar{x} \right]$$

$$\nabla \cdot (\phi \bar{r}) = \nabla \phi \cdot \bar{r} + \phi (\nabla \cdot \bar{r}) \quad ; \quad \phi = \frac{f'(x)}{x}, \quad \bar{r} = \bar{x}$$

$$\nabla^2 f(x) = \nabla \cdot \left[\frac{f'(x)}{x} \cdot \bar{x} \right] = \nabla \frac{f'(x)}{x} \cdot \bar{x} + \frac{f'(x)}{x} (\nabla \cdot \bar{x})$$

$$= \bar{x} \cdot \left[-\frac{1}{x^2} f'(x) + \frac{1}{x} f''(x) \right] \frac{\bar{x}}{x} + \frac{3}{x} f'(x)$$

$$= \left[-\frac{1}{n^3} f'(x) + \frac{1}{n^2} f''(x) \right] (0, n) + \frac{2}{n} f'(n)$$

$$= \left[-\frac{1}{n^3} f'(n) + \frac{1}{n^2} f''(n) \right] n^2 + \frac{2}{n} f'(n)$$

$$= -\frac{1}{n} f'(n) + f''(n) + \frac{2}{n} f'(n)$$

$$= f''(n) + \frac{1}{n} f'(n)$$

$$\nabla^2 f(x) = f''(n) + \frac{1}{n} f'(n)$$

(OR)

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \text{ and } |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

$$\text{or } \frac{\partial r}{\partial x} = \frac{x}{r} \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}$$

Similarly,

$$\frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\frac{\partial}{\partial x} (f(r)) = f'(r) \frac{\partial r}{\partial x} = f'(r) \frac{x}{r}$$

$$\begin{aligned} \frac{\partial^2}{\partial x^2} (f(r)) &= f''(r) \frac{\partial}{\partial x} \left(\frac{x}{r} \right) + f'(r) \left[\frac{\partial}{\partial x} \left(\frac{x}{r} \right) \right] \\ &= f''(r) \left(\frac{r - x \frac{\partial r}{\partial x}}{r^2} \right) + f'(r) \left(\frac{r - x \frac{\partial r}{\partial x}}{r^2} \right) \\ &= f''(r) \left(\frac{r - x \frac{x}{r}}{r^2} \right) + f'(r) \left(\frac{r - x \frac{x}{r}}{r^2} \right) \\ &= f''(r) \left(\frac{r^2 - x^2}{r^3} \right) + f'(r) \left(\frac{r^2 - x^2}{r^3} \right) \\ &= f''(r) \left(\frac{y^2 + z^2}{r^3} \right) + f'(r) \left(\frac{y^2 + z^2}{r^3} \right) \end{aligned}$$

Similarly,

$$\frac{\partial^2}{\partial y^2} [f(x)] = f''(x) \frac{y^2}{x^2} - \frac{y^2}{x^3} f'(x) + \frac{1}{x} f'(x)$$

$$\frac{\partial^2}{\partial z^2} [f(x)] = f''(x) \frac{z^2}{x^2} - \frac{z^2}{x^3} f'(x) + \frac{1}{x} f'(x)$$

adding the above values, we get

$$\begin{aligned} & \frac{\partial^2}{\partial x^2} f(x) + \frac{\partial^2}{\partial y^2} f(x) + \frac{\partial^2}{\partial z^2} f(x) \\ &= \frac{1}{x^2} f''(x) (x^2 + y^2 + z^2) - \frac{1}{x^3} f'(x) (x^2 + y^2 + z^2) + \frac{3}{x} f'(x) \\ &= \frac{1}{x^2} f''(x) x^2 - \frac{1}{x^3} f'(x) x^2 + \frac{3}{x} f'(x) \\ &= f''(x) - \frac{1}{x} f'(x) + \frac{3}{x} f'(x) \\ &= f''(x) + \frac{2}{x} f'(x) \\ &= \nabla^2 f(x) \end{aligned}$$

$$\nabla^2 f(x) = \left[\frac{\partial^2}{\partial x^2} f(x) + \frac{\partial^2}{\partial y^2} f(x) + \frac{\partial^2}{\partial z^2} f(x) \right]$$

(ii) prove that $\nabla \cdot [\phi \nabla \psi - \psi \nabla \phi] = \phi \nabla^2 \psi - \psi \nabla^2 \phi$

$$\nabla \cdot (\phi \bar{F}) = \nabla \phi \cdot \bar{F} + \phi \nabla \cdot \bar{F}$$

$$\nabla \cdot (\phi \nabla \psi - \psi \nabla \phi) = \nabla \cdot (\phi \nabla \psi) - \nabla \cdot (\psi \nabla \phi) \rightarrow (1)$$

$$\nabla \cdot (\phi \nabla \psi) = \nabla \phi \cdot \nabla \psi + \phi \nabla \cdot \nabla \psi \rightarrow (2)$$

Here ϕ is scalar function, $\nabla \psi$ is a vector

$$\nabla \cdot (\psi \nabla \phi) = \nabla \psi \cdot \nabla \phi + \psi \nabla \cdot \nabla \phi \rightarrow (3)$$

Here ψ is a scalar and $\nabla\phi$ is a vector.

Substituting (2) & (3) in (1)

$$\nabla \cdot (\phi \nabla \psi - \psi \nabla \phi) = \nabla \phi \cdot \nabla \psi + \phi \nabla^2 \psi - \nabla \psi \cdot \nabla \phi - \psi \nabla^2 \phi$$

$$= \phi \nabla^2 \psi - \psi \nabla^2 \phi$$

(II) If $\bar{A}$ is a constant vector and $\bar{R} = x\bar{i} + y\bar{j} + z\bar{k}$, prove that

(i) $\text{grad}(\bar{A} \cdot \bar{R}) = \bar{A}$ (ii) $\text{div}(\bar{A} \times \bar{R}) = 0$

(iii) $\text{curl}(\bar{A} \times \bar{R}) = 2\bar{A}$ (iv) $\text{curl}[(\bar{A} \cdot \bar{R})\bar{R}] = \bar{A} \times \bar{R}$

Let $\bar{A} = A_1\bar{i} + A_2\bar{j} + A_3\bar{k}$

(i) $\text{grad}(\bar{A} \cdot \bar{R}) = \nabla [A_1\bar{i} + A_2\bar{j} + A_3\bar{k}] \cdot [x\bar{i} + y\bar{j} + z\bar{k}]$

$$= \nabla [A_1x + A_2y + A_3z]$$

$$= \bar{i} \frac{\partial}{\partial x} [A_1x + A_2y + A_3z]$$

$$= \bar{i} \frac{\partial}{\partial x} (A_1x) + \bar{j} \frac{\partial}{\partial y} (A_2y) + \bar{k} \frac{\partial}{\partial z} (A_3z)$$

$$= A_1\bar{i} + A_2\bar{j} + A_3\bar{k}$$

$$= \bar{A}$$

(ii) $\bar{A} \times \bar{R} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ A_1 & A_2 & A_3 \\ x & y & z \end{vmatrix}$

$$= \bar{i} [A_2z - A_3y] + \bar{j} [A_3x - A_1z] + \bar{k} [A_1y - A_2x]$$

$$\frac{\partial}{\partial z} (A_1 y - A_2 x) = 0$$

$$\begin{aligned} \nabla \times (\vec{A} \times \vec{R}) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_2 z - A_3 y & A_3 x - A_1 z & A_1 y - A_2 x \end{vmatrix} \\ &= \vec{i} (A_1 + A_1) + \vec{j} (A_2 + A_2) + \vec{k} (A_3 + A_3) \\ &= 2(A_1 \vec{i} + A_2 \vec{j} + A_3 \vec{k}) \\ &= 2\vec{A} \end{aligned}$$

$$(\vec{A} \cdot \vec{R}) = (A_1 \vec{i} + A_2 \vec{j} + A_3 \vec{k}) \cdot (x \vec{i} + y \vec{j} + z \vec{k})$$

$$= A_1 x + A_2 y + A_3 z$$

$$(\vec{A} \cdot \vec{R}) \vec{R} = (A_1 x + A_2 y + A_3 z) (x \vec{i} + y \vec{j} + z \vec{k})$$

$$= (A_1 x^2 + A_2 xy + A_3 zx) \vec{i} + (A_1 xy + A_2 y^2 + A_3 yz) \vec{j} + (A_1 xz + A_2 yz + A_3 z^2) \vec{k}$$

$$\nabla \times (\vec{A} \cdot \vec{R}) \vec{R} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 x^2 + A_2 xy + A_3 zx & A_1 xy + A_2 y^2 + A_3 yz & A_1 xz + A_2 yz + A_3 z^2 \end{vmatrix}$$

$$= \vec{i} (A_2 z - A_3 y) + \vec{j} (A_3 x - A_1 z) + \vec{k} (A_1 y - A_2 x)$$

$$= \vec{A} \times \vec{R}$$

Calculate (i) curl(grad f) given $f(x, y, z) = x^2 + y^2 - z$

$$\text{grad } f = 2x \vec{i} + 2y \vec{j} - \vec{k}$$

$$\nabla \times \vec{A} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & y^2z & z^2y \end{vmatrix} = 0$$

$$(iii) \vec{A} = x^2y\bar{i} + y^2z\bar{j} + z^2y\bar{k}$$

$$\nabla \times \vec{A} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & y^2z & z^2y \end{vmatrix}$$

$$= \bar{i}(z^2 - y^2) + \bar{j}(0 - 0) + \bar{k}(0 - x^2)$$

$$= (z^2 - y^2)\bar{i} - x^2\bar{k}$$

$$\nabla \times (\nabla \times \vec{A}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 - y^2 & 0 & -x^2 \end{vmatrix}$$

$$= \bar{i}(0 - 0) + \bar{j}(2z + 2x) + \bar{k}(2y)$$

$$= 2(x + z)\bar{j} + 2y\bar{k}$$

(iv) If $\phi = (x^2 + y^2 + z^2)^{-n}$, find $\text{div}(\text{grad } \phi)$ and determine

if $\text{div grad } \phi = 0$

Given that $\phi = (x^2 + y^2 + z^2)^{-n}$

$$(i) \nabla \phi = \text{grad } \phi = \frac{\partial \phi}{\partial x} \bar{i} + \frac{\partial \phi}{\partial y} \bar{j} + \frac{\partial \phi}{\partial z} \bar{k}$$

$$\nabla \cdot (\nabla \phi) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$\frac{\partial \phi}{\partial x} = -n(x^2 + y^2 + z^2)^{-n-1} \cdot 2x$$

$$\frac{\partial^2 \phi}{\partial x^2} = -2n \left[(x^2 + y^2 + z^2)^{-(n+1)} - x(n+1)(x^2 + y^2 + z^2)^{-n-2} \cdot 2x \right]$$

$$= -2n(x^2+y^2+z^2)^{-(n+2)} [y^2+z^2-x^2(n+1)]$$

$$= -2n(x^2+y^2+z^2)^{-(n+2)} [y^2+z^2-x^2(n+1)]$$

Similarly by symmetry,

$$\frac{\partial^2 \phi}{\partial y^2} = -2n(x^2+y^2+z^2)^{-(n+2)} [x^2+z^2-y^2(n+1)]$$

$$\frac{\partial^2 \phi}{\partial z^2} = -2n(x^2+y^2+z^2)^{-(n+2)} [x^2+y^2-z^2(n+1)]$$

$$\nabla \cdot \nabla \phi = -2n(x^2+y^2+z^2)^{-(n+2)} [y^2+z^2-2nx^2-x^2+x^2+z^2-2ny^2-y^2+y^2-x^2-2nz^2-z^2]$$

$$= -2n(x^2+y^2+z^2)^{-(n+2)} [x^2+y^2+z^2-2n(x^2+y^2+z^2)]$$

$$= -2n(x^2+y^2+z^2)^{-(n+1)} (1-2n)$$

$$= \frac{2n(2n-1)}{(x^2+y^2+z^2)^{n+1}}$$

If $\text{div grad } \phi = 0$

$$\frac{2n(2n-1)}{(x^2+y^2+z^2)^{n+1}} = 0 \quad ; \quad n = \frac{1}{2}$$

Since 'n' can't be zero

S.T $\text{div}(\text{grad } r^n) = n(n+1)r^{n-2}$ where $r^2 = x^2+y^2+z^2$

By definition $\text{grad}(r^n) = \bar{i} \frac{\partial}{\partial x}(r^n) + \bar{j} \frac{\partial}{\partial y}(r^n) + \bar{k} \frac{\partial}{\partial z}(r^n)$

$$= \bar{i} n r^{n-1} \frac{\partial r}{\partial x} + \bar{j} n r^{n-1} \frac{\partial r}{\partial y} + \bar{k} n r^{n-1} \frac{\partial r}{\partial z}$$

$$= n r^{n-1} \left[\bar{i} \frac{\partial r}{\partial x} + \bar{j} \frac{\partial r}{\partial y} + \bar{k} \frac{\partial r}{\partial z} \right]$$

$$\Rightarrow 2x \frac{\partial x}{\partial x} = 2x \Rightarrow \frac{\partial x}{\partial x} = \frac{x}{x}$$

$$\text{grad } r^n = nr^{n-1} \left[\bar{i} \frac{x}{r} + \bar{j} \frac{y}{r} + \bar{k} \frac{z}{r} \right]$$

$$= nr^{n-2} [x\bar{i} + y\bar{j} + z\bar{k}]$$

$$\text{div}(\text{grad } r^n) = \frac{\partial}{\partial x}(nr^{n-2}x) + \frac{\partial}{\partial y}(nr^{n-2}y) + \frac{\partial}{\partial z}(nr^{n-2}z)$$

$$= \left(nr^{n-2} + nx(n-2)r^{n-3} \frac{\partial x}{\partial x} \right) + \left(nr^{n-2} + ny(n-2)r^{n-3} \frac{\partial y}{\partial y} \right) + \left(nr^{n-2} + nz(n-2)r^{n-3} \frac{\partial z}{\partial z} \right)$$

$$= 3nr^{n-2} + n(n-2)r^{n-3} \left[x \frac{\partial x}{\partial x} + y \frac{\partial y}{\partial y} + z \frac{\partial z}{\partial z} \right]$$

$$= 3nr^{n-2} + n(n-2)r^{n-3} \left[x \left(\frac{x}{r} \right) + y \left(\frac{y}{r} \right) + z \left(\frac{z}{r} \right) \right]$$

$$= 3nr^{n-2} + n(n-2)r^{n-4} [x^2 + y^2 + z^2]$$

$$= 3nr^{n-2} + n(n-2)r^{n-4} \cdot r^2$$

$$= 3nr^{n-2} + n(n-2)r^{n-2}$$

$$= r^{n-2} [n^2 + 3n - 2n]$$

$$= r^{n-2} [n^2 + n]$$

$$= r^{n-2} n(n+1)$$

If we put $n = -1$ then

$$\text{div}(\text{grad } r^n) = -1(-1+1)r^{-1-2}$$

$$\nabla^2 \left(\frac{1}{r} \right) = 0$$

Let $\vec{f}(t)$ and $\vec{F}(t)$ be two vector functions of a scalar variable

't' such that $\frac{d}{dt} [\vec{F}(t)] = \vec{f}(t)$, then $\vec{F}(t)$ is called an integral of $\vec{f}(t)$

w.r.t to 't' and we write $\int \vec{f}(t) dt = \vec{F}(t)$

If $\vec{c}$ is any arbitrary constant vector independent of t,

$$\text{then } \frac{d}{dt} [\vec{F}(t) + \vec{c}] = \vec{f}(t)$$

This is equivalent to $\int \vec{f}(t) dt = \vec{F}(t) + \vec{c}$

$\vec{F}(t)$ is called the indefinite integral of $\vec{f}(t)$. The constant vector

$\vec{c}$ is called constant of integration and can be determined if

some initial conditions are given.

The definite integral of $\vec{f}(t)$ between limits $t=a$ and $t=b$

$$\text{written as } \int_a^b \vec{f}(t) dt = [\vec{F}(t)]_a^b = \vec{F}(b) - \vec{F}(a)$$

Note 1- If $\vec{f}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$, then $\int \vec{f}(t) dt = \vec{i} \int f_1(t) dt + \vec{j} \int f_2(t) dt + \vec{k} \int f_3(t) dt$

Note 2- we can obtain some standard results for integration of vector functions by considering the derivatives of suitable

vector functions. For example,

$$(i) \frac{d}{dt} (\vec{r} \cdot \vec{a}) = \frac{d\vec{r}}{dt} \cdot \vec{a} + \vec{r} \cdot \frac{d\vec{a}}{dt} \Rightarrow \int \left(\frac{d\vec{r}}{dt} \cdot \vec{a} + \vec{r} \cdot \frac{d\vec{a}}{dt} \right) dt = \vec{r} \cdot \vec{a} + c$$

Here 'c' is scalar quantity since the integrand is a scalar

$$(ii) \frac{d}{dt} (\vec{r}^2) = 2\vec{r} \cdot \frac{d\vec{r}}{dt} \Rightarrow \int (2\vec{r} \cdot \frac{d\vec{r}}{dt}) dt = \vec{r}^2 + c \text{ where 'c' is scalar quantity}$$

$$(iii) \frac{d}{dt} (\vec{r} \times \frac{d\vec{r}}{dt}) = \frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} + \vec{r} \times \frac{d^3\vec{r}}{dt^3} \Rightarrow \int \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} + \vec{r} \times \frac{d^3\vec{r}}{dt^3} \right) dt = \vec{r} \times \frac{d^2\vec{r}}{dt^2} + c$$

Here $\vec{c}$ is vector quantity since the integrand is a vector

$$(iv) \text{ If } \vec{a} \text{ is constant vector, then } \frac{d}{dt} (\vec{a} \times \vec{r}) = \vec{a} \times \frac{d\vec{r}}{dt} + \vec{r} \times \frac{d\vec{a}}{dt} = \vec{a} \times \frac{d\vec{r}}{dt} \Rightarrow \int (\vec{a} \times \frac{d\vec{r}}{dt}) dt = \vec{a} \times \vec{r} + c \text{ where 'c' is scalar quantity}$$

① Q8 $F(t) = (3t^2 - 2t)\hat{i} + (6t - 4)\hat{j} + 4t\hat{k}$ evaluate $\int_2^3 F(t) dt$ www.FirstRanker.com

$$\begin{aligned}\int_2^3 \vec{F}(t) dt &= \int_2^3 [(3t^2 - 2t)\hat{i} + (6t - 4)\hat{j} + 4t\hat{k}] dt \\&= \hat{i} \int_2^3 (3t^2 - 2t) dt + \hat{j} \int_2^3 (6t - 4) dt + \hat{k} \int_2^3 4t dt \\&= \hat{i} \left[t^3 - t^2 \right]_2^3 + \hat{j} \left[3t^2 - 4t \right]_2^3 + \hat{k} \left[2t^2 \right]_2^3 \\&= 14\hat{i} + 11\hat{j} + 10\hat{k}\end{aligned}$$

3
1
27 - 8 - 9
27 - 12
- 12 +
35
24
11

④ Q9 $\vec{r}(t) = \begin{cases} 2\hat{i} - \hat{j} + 2\hat{k} & \text{when } t=2 \\ 4\hat{i} - 2\hat{j} + 3\hat{k} & \text{when } t=3 \end{cases}$ Show that $\int_2^3 \left(\vec{r} \cdot \frac{d\vec{r}}{dt} \right) dt = 10$.

Sol Since $\frac{d}{dt} [\vec{r}^2] = 2\vec{r} \cdot \frac{d\vec{r}}{dt}$

$$\int_2^3 \left(\vec{r} \cdot \frac{d\vec{r}}{dt} \right) dt = \frac{1}{2} [\vec{r}^2]_2^3 \quad \text{--- (1)}$$

when $t=3$, $\vec{r}^2 = (4\hat{i} - 2\hat{j} + 3\hat{k}) \cdot (4\hat{i} - 2\hat{j} + 3\hat{k}) = 16 + 4 + 9 = 29$

when $t=2$, $\vec{r}^2 = (2\hat{i} - \hat{j} + 2\hat{k}) \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = 4 + 1 + 4 = 9$

From (1), we have $\int_2^3 \left(\vec{r} \cdot \frac{d\vec{r}}{dt} \right) dt = \frac{1}{2} [29 - 9] = 10$

3) Given $\vec{r}(t) = 5t^2 \hat{i} + t\hat{j} - t^3 \hat{k}$, Prove that $\int_1^2 \left(\vec{r} \times \frac{d^2 \vec{r}}{dt^2} \right) dt = -14\hat{i} + 75\hat{j} - 15\hat{k}$

Sol:- $\frac{d}{dt} \left[\vec{r} \times \frac{d\vec{r}}{dt} \right] = \frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} + \vec{r} \times \frac{d^2 \vec{r}}{dt^2} = \vec{r} \times \frac{d^2 \vec{r}}{dt^2}$

$\int_1^2 \left(\vec{r} \times \frac{d^2 \vec{r}}{dt^2} \right) dt = \left[\vec{r} \times \frac{d\vec{r}}{dt} \right]_1^2 \quad \text{--- (1)}$

Let us now find $\vec{r} \times \frac{d\vec{r}}{dt} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t & -t^3 \\ 10t & 1 & -3t^2 \end{vmatrix}$

$= \hat{i} [-3t^3 + t^3] + \hat{j} [-10t^4 + 15t^4] + \hat{k} [5t^2 - 10t^2]$

$= -2t^3 \hat{i} + 5t^4 \hat{j} - 5t^2 \hat{k}$

From (1) we have $\int_1^2 \left(\vec{r} \times \frac{d^2 \vec{r}}{dt^2} \right) dt = \left[-2t^3 \hat{i} + 5t^4 \hat{j} - 5t^2 \hat{k} \right]_1^2$

$= \left[-2t^3 \right]_1^2 \hat{i} + \left[5t^4 \right]_1^2 \hat{j} - \left[5t^2 \right]_1^2 \hat{k}$

$= -14\hat{i} + 75\hat{j} - 15\hat{k}$

The acceleration of a particle at any time $t > 0$ is given by

$\vec{a} = \frac{d\vec{v}}{dt} = 12 \cos 2t \hat{i} - 8 \sin 2t \hat{j} + 16t \hat{k}$. If the velocity $\vec{v}$ and the

displacement $\vec{r}$ are zero at $t=0$, find $\vec{v}$ and $\vec{r}$ at any time.

Sol:- $\vec{a} = \frac{d\vec{v}}{dt} = 12 \cos 2t \hat{i} - 8 \sin 2t \hat{j} + 16t \hat{k}$

$\vec{v} = \int \frac{d\vec{v}}{dt} dt = \hat{i} \int 12 \cos 2t dt - \hat{j} \int 8 \sin 2t dt + \hat{k} \int 16t dt$

$\vec{v} = 6 \sin 2t \hat{i} + 4 \cos 2t \hat{j} + 8t^2 \hat{k} + \vec{c}_1$

when $t=0$, $\vec{v}=0$ $0 = 4\hat{j} + \vec{c}_1 \Rightarrow \vec{c}_1 = -4\hat{j}$

Then $\vec{v} = 6\sin 2t \hat{i} + (4\cos 2t - 4)\hat{j} + 8t^2 \hat{k} - 4\hat{j}$

$$\frac{d\vec{r}}{dt} = \vec{v} = 6\sin 2t \hat{i} + (4\cos 2t - 4)\hat{j} + 8t^2 \hat{k}$$

So $d\vec{r}$

Integration $\vec{r} = \int \frac{d\vec{r}}{dt} dt = \hat{i} \int 6\sin 2t dt + \hat{j} \int (4\cos 2t - 4) dt + \hat{k} \int 8t^2 dt$

$$\vec{r} = -3\cos 2t \hat{i} + (2\sin 2t - 4t)\hat{j} + \frac{8}{3}t^3 + \vec{C}_2$$

when $t=0$, $\vec{r}=0$.

$$0 = -3\hat{i} + \vec{C}_2 \Rightarrow \vec{C}_2 = 3\hat{i}$$

$$\vec{r} = -3\cos 2t \hat{i} + (2\sin 2t - 4t)\hat{j} + \frac{8}{3}t^3 + 3\hat{i}$$

$$\vec{r} = (3 - 3\cos 2t) \hat{i} + (2\sin 2t - 4t)\hat{j} + \frac{8}{3}t^3 \hat{k}$$

2

$$\vec{r} = 18\cos 3t \hat{i} - 8\sin 2t \hat{j} + 6t \hat{k}$$

$\nabla \times \mathbf{H} = \frac{1}{c} \left(\mathbf{E} + \frac{\partial \mathbf{D}}{\partial t} \right)$ Prove that.

(i) $\nabla^2 \mathbf{D} - \frac{1}{c^2} \frac{\partial^2 \mathbf{D}}{\partial t^2} = \nabla \rho + \frac{1}{c^2} \frac{\partial}{\partial t} (\nabla \times \mathbf{E})$ (ii) $\nabla^2 \mathbf{H} - \frac{1}{c^2} \frac{\partial^2 \mathbf{H}}{\partial t^2} = -\frac{1}{c} \nabla \times \mathbf{E}$

Sol:- (i) we have $\frac{1}{c^2} \left[\frac{\partial^2 \mathbf{D}}{\partial t^2} + \frac{\partial}{\partial t} (\nabla \times \mathbf{E}) \right] = \frac{1}{c^2} \frac{\partial}{\partial t} \left(\frac{\partial \mathbf{D}}{\partial t} + \mathbf{E} \right)$
 $= \frac{1}{c^2} \frac{\partial}{\partial t} (\nabla \times \mathbf{H}) = \frac{1}{c^2} (\nabla \times \frac{\partial \mathbf{H}}{\partial t})$
 $= -\nabla \times (\nabla \times \mathbf{D})$

$= -[\nabla(\nabla \cdot \mathbf{D}) - (\nabla \cdot \nabla) \mathbf{D}]$
 $= -\nabla(\nabla \cdot \mathbf{D}) + \nabla^2 \mathbf{D}$
 $= -\nabla \rho + \nabla^2 \mathbf{D}$

Hence $\nabla^2 \mathbf{D} - \frac{1}{c^2} \frac{\partial^2 \mathbf{D}}{\partial t^2} = \nabla \rho + \frac{1}{c^2} \frac{\partial}{\partial t} (\nabla \times \mathbf{E})$

(ii) L.H.S. $= \nabla^2 \mathbf{H} - \frac{1}{c^2} \frac{\partial^2 \mathbf{H}}{\partial t^2} = \nabla^2 \mathbf{H} + \frac{1}{c} \frac{\partial}{\partial t} \left(-\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \right)$

$= \nabla^2 \mathbf{H} + \frac{1}{c} \frac{\partial}{\partial t} (\nabla \times \mathbf{D})$

$= \nabla^2 \mathbf{H} + \frac{1}{c} (\nabla \times \frac{\partial \mathbf{D}}{\partial t})$

$= \nabla^2 \mathbf{H} + \nabla \times (\nabla \times \mathbf{H} - \frac{1}{c} \mathbf{E})$

$= \nabla^2 \mathbf{H} + \nabla \times (\nabla \times \mathbf{H}) - \frac{1}{c} \nabla \times \mathbf{E}$

$= \nabla^2 \mathbf{H} + \nabla(\nabla \cdot \mathbf{H}) - (\nabla \cdot \nabla) \mathbf{H} - \frac{1}{c} \nabla \times \mathbf{E}$

$= \nabla^2 \mathbf{H} - 0 - \nabla^2 \mathbf{H} - \frac{1}{c} \nabla \times \mathbf{E}$

$= -\frac{1}{c} \nabla \times \mathbf{E} = \text{R.H.S.}$

Hence $\nabla^2 \mathbf{H} - \frac{1}{c^2} \frac{\partial^2 \mathbf{H}}{\partial t^2} = -\frac{1}{c} \nabla \times \mathbf{E}$

Prove that $\vec{A} \cdot \nabla \left(\vec{B} \cdot \frac{1}{r} \right) = 3(\vec{A} \cdot \vec{B})(\vec{B} \cdot \vec{A}) - \vec{A} \cdot \vec{B} \frac{1}{r^3} + \frac{3}{r^5} (\vec{A} \cdot \vec{B})(\vec{B} \cdot \vec{A})$

Sol:- $\nabla \frac{1}{r} = \nabla(r^{-1}) = -r^{-2} \frac{dr}{dr} \vec{r} + (-r^{-2} \frac{dr}{dr}) \left(\frac{dr}{dr} \frac{\partial}{\partial r} \right) \vec{r}$

$$= -r^{-2} \frac{1}{r} - r^{-4} \frac{1}{r} \vec{r} = -r^{-3} \frac{1}{r} \vec{r}$$

$$= -r^{-4} (\vec{r} \cdot \vec{r}) = -\frac{1}{r^3} \vec{r}$$

$$r^2 = x^2 + y^2 + z^2$$

$$2r \frac{dr}{dr} = 2r \Rightarrow \frac{dr}{dr} = \frac{r}{r} = 1$$

$$\nabla (\vec{B} \cdot \nabla \frac{1}{r}) = \nabla (\vec{B} \cdot (-\frac{1}{r^3} \vec{r}))$$

$$= -\nabla \left[\frac{\vec{B} \cdot \vec{r}}{r^3} \right]$$

$$= -\nabla \left(\frac{1}{r^3} \right) (\vec{B} \cdot \vec{r}) - \frac{1}{r^3} \nabla (\vec{B} \cdot \vec{r})$$

$$= \frac{3}{r^5} \vec{r} (\vec{B} \cdot \vec{r}) - \frac{1}{r^3} \vec{B}$$

$$\nabla (\vec{B} \cdot \vec{r}) = \vec{B}$$

$$\vec{A} \cdot \nabla (\vec{B} \cdot \nabla \frac{1}{r}) = \frac{3}{r^5} (\vec{B} \cdot \vec{r}) (\vec{A} \cdot \vec{r}) - \frac{1}{r^3} (\vec{A} \cdot \vec{B})$$

Line Integrals:- Any integral which is to be evaluate along a curve is called a line integral.

The line integral of $\vec{F}$ along the curve 'c' is defined as $\int_C \vec{F} \cdot d\vec{r}$

Note:- $\int_C \vec{F} \cdot d\vec{r} = \int_C (F_1 i + F_2 j + F_3 k) \cdot (dx i + dy j + dz k)$

$$= \int_C F_1 dx + F_2 dy + F_3 dz$$

② $\oint_C \vec{F} \cdot d\vec{r}$ is called line integral of $\vec{F}$ along the curve 'c'.

circulation:- Let 'c' be a simple, closed curve, the line integral of $\vec{F}$ along 'c' is called circulation $\vec{F}$ i.e. $\oint_C \vec{F} \cdot d\vec{r}$

1) If $\vec{F} = 3xy\vec{i} - y^2\vec{j}$, evaluate $\int_C \vec{F} \cdot d\vec{r}$ where 'C' is the parabola $y = 2x^2$ from $(0,0)$ to $(1,2)$ in the xy -plane.

Sol:- Since the particle moves in the xy -plane ($z=0$) we take $\vec{r} = x\vec{i} + y\vec{j}$. Then $\int_C \vec{F} \cdot d\vec{r}$, where 'C' is the parabola $y = 2x^2$.

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (3xy\vec{i} - y^2\vec{j}) \cdot (dx\vec{i} + dy\vec{j}) = \int_C (3xy dx - y^2 dy) \quad \text{--- (1)}$$

Substituting $y = 2x^2$ where x goes from 0 to 1.

$$\text{(1) becomes } \int_{x=0}^1 3x(2x^2) dx - (2x^2)^2 d(2x^2) = \int_0^1 (6x^3 - 16x^5) dx = -7/6$$

2) A vector field is given by $\vec{F} = \sin y\vec{i} + x(1 + \cos y)\vec{j}$. Evaluate the line integral over a circular path given by $x^2 + y^2 = a^2$, $z=0$.

Sol:- As the particles moves in xy -plane ($z=0$), let $\vec{r} = x\vec{i} + y\vec{j}$ so that $d\vec{r} = dx\vec{i} + dy\vec{j}$. Also the circular path is $x = a \cos t$, $y = a \sin t$, $z=0$, where 't' varies from 0 to 2π .

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{r} &= \oint_C (\sin y\vec{i} + x(1 + \cos y)\vec{j}) \cdot (dx\vec{i} + dy\vec{j}) \\ &= \oint_C \sin y dx + x(1 + \cos y) dy = \oint_C (\sin y dx + x \cos y dy + x dy) \\ &= \oint_C [d(x \sin y) + x dy] = \int_0^{2\pi} d(a \cos t \sin(a \sin t) + a^2 \cos^2 t dt) \\ &= [a \cos t \sin(a \sin t)]_0^{2\pi} + \frac{a^2}{2} \int_0^{2\pi} 1 + \cos 2t dt \\ &= \frac{a^2}{2} \left[t + \frac{\sin 2t}{2} \right]_0^{2\pi} = \pi a^2 \end{aligned}$$

work done in moving a particle in the force field $\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ along (a) the straight

line from $(0,0,0)$ to $(2,1,3)$.

(b) the curve defined by $x^2 = 4y$, $3x^3 = 8z$ from $x=0$ to $x=2$.

Sol:- $\int_C \vec{F} \cdot d\vec{r} = \int_C (3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$
 $= \int_C 3x^2 dx + (2xz - y) dy + z dz$

(a) The equations of the straight line from $(0,0,0)$ to $(2,1,3)$ are $\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$ say.

$\therefore x=2t, y=t, z=3t$ are its parametric equations. The points $(0,0,0)$ and $(2,1,3)$ corresponding to $t=0$ and $t=1$ respectively.

$\therefore \text{work done} = \int_C \vec{F} \cdot d\vec{r} = \int_0^1 3(2t)^2 dt + (4t)(3t) - t) dt + (3t) 3 dt$
 $= \int_0^1 (36t^2 + 8t) dt = 16$

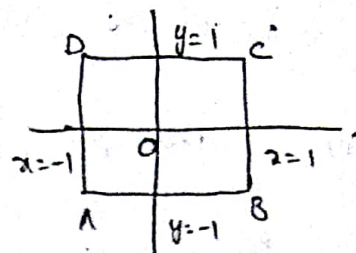
(b) Let $x=t$ in $x^2 = 4y$, $3x^3 = 8z$. Then the parametric equations of 'c' are $x=t, y = \frac{t^2}{4}, z = \frac{3t^3}{8}$ and 't' varies from 0 to 2.

$\therefore \text{work done} = \int_C \vec{F} \cdot d\vec{r} = \int_0^2 3t^2 dt + \left\{ 2t \left(\frac{3t^3}{8} \right) - \frac{t^2}{4} \right\} d\left(\frac{t^2}{4}\right) + \frac{3t^3}{8} d\left(\frac{3t^3}{8}\right)$
 $= \int_0^2 \left(3t^2 - \frac{t^3}{8} + \frac{51}{64} t^5 \right) dt$
 $= \left[t^3 - \frac{t^4}{32} + \frac{17}{128} t^6 \right]_0^2 = 16$

Q. Evaluate the line integral $\int_C (x^2 + y^2) dx + (x^2 + y^2) dy$ where C is the square formed by the lines $y = \pm 1$ and $x = \pm 1$.

Sol:- The given vector

$$\vec{F} = (x^2 + xy) \vec{i} + (x^2 + y^2) \vec{j}$$



To evaluate $\int_C (x^2 + xy) dx + (x^2 + y^2) dy$

C is the square formed by the lines $y = \pm 1$ and $x = \pm 1$.

The square is ABCD.

Along AB, x varies from -1 to 1 .

$$y = -1 \text{ and } dy = 0 \Rightarrow \int_C (x^2 - x) dx = \int_{-1}^1 x^2 dx - \int_{-1}^1 x dx$$

Along BC, $x = 1$, $dx = 0$ y varies -1 to 1 . $= \left[\frac{x^3}{3} \right]_{-1}^1 - \left[\frac{x^2}{2} \right]_{-1}^1$

$$\begin{aligned} \int_{BC} \vec{F} \cdot d\vec{r} &= \int_{-1}^1 (1 + y^2) dy = \left[y + \frac{y^3}{3} \right]_{-1}^1 = \left(\frac{1}{3} + \frac{1}{3} \right) - \left(-\frac{1}{3} - \frac{1}{3} \right) \\ &= [1 + 1] + \left[\frac{1}{3} + \frac{1}{3} \right] = 2 + \frac{2}{3} = \frac{8}{3} \end{aligned}$$

Along CD, $y = 1$, $dy = 0$, x varies from 1 to -1 .

$$\int_{CD} (x^2 + x) dx = 0 \Rightarrow \int_1^{-1} x^2 dx = \left[\frac{x^3}{3} \right]_1^{-1} = -\frac{1}{3} - \frac{1}{3} = -\frac{2}{3}$$

Along DA, $x = -1$, $dx = 0$, y varies from 1 to -1 .

$$\int_{DA} (1 + y^2) dy = \left[y + \frac{y^3}{3} \right]_1^{-1} = (-1 - \frac{1}{3}) - \left(\frac{1}{3} - \frac{1}{3} \right) = -\frac{8}{3}$$

$$\begin{aligned} \oint_C (x^2 + xy) dx + (x^2 + y^2) dy &= \int_{AB} (x^2 + xy) dx + (x^2 + y^2) dy + \int_{BC} (x^2 + xy) dx + (x^2 + y^2) dy + \int_{CD} (x^2 + xy) dx + (x^2 + y^2) dy \\ &\quad + \int_{DA} (x^2 + xy) dx + (x^2 + y^2) dy \\ &= \frac{8}{3} + \frac{8}{3} - \frac{2}{3} - \frac{8}{3} = 0 \end{aligned}$$

2) If $\vec{F} = (5xy - 6x^2)\vec{i} + (2y - 4x)\vec{j}$; evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the curve 'C' in the xy plane $y = x^3$ from the point (1, 1) to (2, 8).

Sol:- The given function $\vec{F} = (5xy - 6x^2)\vec{i} + (2y - 4x)\vec{j}$

$$y = x^3 \Rightarrow dy = 3x^2 dx$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (5xy - 6x^2) dx + (2y - 4x) dy$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_1^2 (5xy^3 - 6x^2) dx + (2x^3 - 4x) 3x^2 dx$$

$$= \left[\frac{5x^5}{5} - 6\frac{x^3}{3} + \frac{6x^6}{6} - 12\frac{x^4}{4} \right]_1^2$$

$$= \left[x^5 - 2x^3 + x^6 - 3x^4 \right]_1^2 = 32 - 16 + 64 - 48 - 1 + 2 - 1 + 3 = 35$$

3) compute the line integral $\int_C y^2 dx - x^2 dy$ about the triangle whose vertices are (1,0), (0,1) and (-1,0).



Sol:- Let the points (1,0), (0,1) and (-1,0) be A, B and C respectively.

$$\int_C y^2 dx - x^2 dy = \int_{AB} y^2 dx - x^2 dy + \int_{BC} y^2 dx - x^2 dy + \int_{CA} y^2 dx - x^2 dy = I_1 + I_2 + I_3$$

$I_1 = \int_{AB} y^2 dx - x^2 dy$. The equation of the line AB is $y = 1 - x$ $dy = -dx$

$$y = 1 - x \quad dy = -dx$$

x varies from 1 to 0

$$\int_1^0 (x^2 + x^2) dx = \int_1^0 (2x^2) dx = \int_1^0 (2x^2) dx = -\frac{2}{3}$$

$$I_2 = \int_{BC} y^2 dx - x^2 dy, \text{ the equation of the line BC is } y-x=1$$

$$y = 1+x, dy = dx, x \text{ varies from } 0 \text{ to } -1.$$

$$I_2 = \int_0^{-1} [(1+x)^2 - x^2] dx = \int_0^{-1} (1+2x) dx = \left[x + \frac{2x^2}{2} \right]_0^{-1} = -1 + 1 = 0.$$

$$I_3 = \int_{CA} y^2 dx - x^2 dy, y=0, dy=0 \quad I_3 = 0.$$

$$\therefore \int_C y^2 dx - x^2 dy = -\frac{2}{3} \text{ along } ABCA.$$

4) If $\vec{F} = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$ evaluate $\int_C \vec{F} \cdot d\vec{r}$ from (0,0,0) to (1,1,1) along the path $x=t, y=t^2, z=t^3$.

Sol: The given function $\vec{F} = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$.

The limits are (0,0,0) to (1,1,1), $x=0$ to 1, $y=0$ to 1, $z=0$ to 1.

Given was $x=t, y=t^2, z=t^3$ therefore the limits t are 0 and 1.

$$dx = dt, dy = 2t dt, dz = 3t^2 dt$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (3x^2 + 6y) dt - (14yz) 2t dt + (20xz^2) 3t^2 dt$$

$$= \int_0^1 (3t^2 + 6t^2) dt - (14t^5) 2t dt + (20t^4) 3t^2 dt$$

$$= \int_0^1 9t^2 - 28t^6 + 60t^6 dt = \int_0^1 9t^2 + 32t^6 dt$$

$$= \left[9 \frac{t^3}{3} + 32 \frac{t^7}{7} \right]_0^1$$

$$= 3 + \frac{32}{7} = \frac{53}{7}$$

6) Using the line integral compute the work done by the force $\vec{F} = (2y+3)\vec{i} + xz\vec{j} + (yz-x)\vec{k}$ when it moves a particle from the point $(0,0,0)$ to the point $(2,1,1)$ along the curve $x=2t^2, y=t, z=t^3$.

Sol: $\vec{F} = (2y+3)\vec{i} + xz\vec{j} + (yz-x)\vec{k}$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (2y+3)dx + \int_C xzdy + \int_C (yz-x)dz$$

$$= \int_0^1 (2t+3)4t dt + \int_0^1 (2t^5) dt + \int_0^1 (t^4 - 2t^2)3t^2 dt$$

$$= \int_0^1 (8t^2 + 12t) dt + \int_0^1 2t^5 dt + \int_0^1 (3t^6 - 6t^4) dt$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (8t^2 + 12t) dt + \int_0^1 2t^5 dt + \int_0^1 (3t^6 - 6t^4) dt$$

$$= \left[\frac{8t^3}{3} + \frac{12t^2}{2} + \frac{2t^6}{6} + \frac{3t^7}{7} - \frac{6t^5}{5} \right]_0^1$$

$$= \frac{8}{3} + 6 + \frac{1}{3} + \frac{3}{7} - \frac{6}{5} = 9 + \frac{3}{7} - \frac{6}{5} = \frac{315 + 15 - 42}{105}$$

$$= \frac{288}{105}$$

6) If $\vec{F} = 2y\vec{i} - z\vec{j} + x\vec{k}$, evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the curve $x = \cos t, y = \sin t, z = 2\cos t$ from $t=0$ to $t=\pi/2$.

$z = 2\cos t$ from $t=0$ to $t=\pi/2$

Sol: $\vec{F} = 2y\vec{i} - z\vec{j} + x\vec{k}$, $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\vec{F} \cdot d\vec{r} = \int_C (x dy + z dx) + \int_C (x dy - zy dz)\vec{j} + \int_C (zy dy + 2x dz)\vec{k}$

$x = \cos t, y = \sin t, z = 2\cos t$

$dx = -\sin t dt, dy = \cos t dt, dz = -2\sin t dt$

$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} [-\cos t(\cos t dt + 2\cos t(-\sin t dt)]\vec{i} + \int_0^{\pi/2} [\cos t(-\sin t dt) - 2\sin t(-2\sin t) dt]\vec{j}$
 $+ \int_0^{\pi/2} [2\sin t \cos t dt + 2\cos t(-\sin t dt)]\vec{k}$

$= \int_0^{\pi/2} [-\cos^2 t dt + 2\sin^2 t dt]\vec{i} + \int_0^{\pi/2} \left(-\frac{\sin 2t}{2} + 4\sin^2 t\right) dt \vec{j}$
 $+ \int_0^{\pi/2} (\sin 2t - \sin 2t) dt \vec{k}$

$= \vec{i} \int_0^{\pi/2} \left[-\frac{1+\cos 2t}{2} dt + 2\sin^2 t dt\right] + \vec{j} \int_0^{\pi/2} \left[-\frac{\sin 2t}{2} + 4\left(\frac{1-\cos 2t}{2}\right)\right] dt$

$= \vec{i} \left[-\frac{t}{2} - \frac{\sin 2t}{4} + \cos 2t \right]_0^{\pi/2} + \left[\frac{\cos 2t}{4} + 2t - \frac{2\sin 2t}{2} \right]_0^{\pi/2} \vec{j}$

$= \left[-\frac{\pi}{4} + 0 + 2(1-1) \right] \vec{i} + \left[\frac{1}{4} + \pi + 0 - \frac{1}{4} \right] \vec{j}$

$= \left[-\frac{\pi}{4} \right] \vec{i} + \left[\pi \right] \vec{j}$

Surface Integrals:- Let $\vec{F}$ be a continuous vector point function defined over the smooth surface

Surface Integrals:- Any integral which is to be evaluated over a surface is called a surface integral.

Then the surface integral of $\vec{F}$ over S is defined as $\int_S \vec{F} \cdot \vec{N}$ where $\vec{N}$ is the unit outward normal to S .

Cartesian form of surface integrals:- Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$\vec{N} = \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}$ where $\cos\alpha, \cos\beta, \cos\gamma$ are the d.c's of $\vec{N}$

$$\vec{F} \cdot \vec{N} = F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma$$

$$\text{Now } \int_S \vec{F} \cdot \vec{N} dS = \int_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma) dS$$

$$= \int_S F_1 \cos\alpha dS + \int_S F_2 \cos\beta dS + \int_S F_3 \cos\gamma dS$$

$$\int_S \vec{F} \cdot \vec{N} dS = \int_S F_1 dydz + \int_S F_2 dzdx + \int_S F_3 dxdy$$

Flux:- If 'S' be a closed surface then $\oint_S \vec{F} \cdot \vec{N} dS$ is called the flux over 'S'.

Note 1:- ① Let R_1 be the projection of 'S' on xy-plane then $\int_{R_1} \vec{F} \cdot \vec{N} dS = \int_{R_1} \vec{F} \cdot \vec{N} \frac{dxdy}{|\cos\gamma|} = \int_{R_1} \vec{F} \cdot \vec{N} \frac{dxdy}{|\vec{N} \cdot \hat{k}|}$

② Let R_2 be the projection of 'S' on yz-plane then $\int_{R_2} \vec{F} \cdot \vec{N} dS = \int_{R_2} \vec{F} \cdot \vec{N} \frac{dydz}{|\cos\alpha|} = \int_{R_2} \vec{F} \cdot \vec{N} \frac{dydz}{|\vec{N} \cdot \hat{i}|}$

③ Let R_3 be the projection of 'S' on xz-plane then $\int_{R_3} \vec{F} \cdot \vec{N} dS = \int_{R_3} \vec{F} \cdot \vec{N} \frac{dxdz}{|\cos\beta|} = \int_{R_3} \vec{F} \cdot \vec{N} \frac{dxdz}{|\vec{N} \cdot \hat{j}|}$

an integral which is to be evaluated over a surface is called a surface integral.

Suppose 'S' is a two sided integral, consider one side as positive. If 'S' is a closed surface we choose the outside as the positive. Let $\vec{F}$ be a vector point function. Let ds be an element of area in the surface and $\vec{n}$ be the unit vector normal to the surface, drawn on the positive side. Let θ be the angle between $\vec{F}$ and $\vec{n}$. The normal component of $\vec{F}$ is given by the dot product $\vec{F} \cdot \vec{n} = F \cos \theta$. The integral of this normal component i.e. of $\vec{F} \cdot \vec{n}$ through an element of area ds over the surface 'S' is called the surface integral of $\vec{F}$ over S.

So the surface integral is $\int_S \vec{F} \cdot \vec{n} ds$. If $d\vec{s}$ is a vector whose magnitude is ds and whose direction is that of $\vec{n}$, then $d\vec{s} = \vec{n} ds$.

Hence surface integral is $\int_S \vec{F} \cdot d\vec{s}$.

Other surface integrals are $\int_S \phi ds$ or $\int_S F \cos \theta ds$.

To evaluate surface integrals it is convenient to express them as double integrals taken over the projected area of the surfaces on one of the coordinate planes.

A.B : 11/11/2020
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Let $\vec{n} = \cos\alpha \vec{i} + \cos\beta \vec{j} + \cos\gamma \vec{k}$

where $\cos\alpha, \cos\beta, \cos\gamma$ are the d.c.s of $\vec{n}$.

$ds \cos\alpha, ds \cos\beta, ds \cos\gamma$ are the orthogonal projections of ds on the coordinate planes $x=0, y=0, z=0$ respectively.

$$ds \cos\alpha = dy dz, \quad ds \cos\beta = dz dx, \quad ds \cos\gamma = dx dy.$$

considering subdivisions formed by planes parallel to the coordinate planes

Forming dot products $\vec{n}$ and unit vectors $\vec{i}, \vec{j}, \vec{k}$

$$|\vec{i} \cdot \vec{n}| = \cos\alpha, \quad |\vec{j} \cdot \vec{n}| = \cos\beta, \quad |\vec{k} \cdot \vec{n}| = \cos\gamma.$$

$$ds = \frac{dy dz}{\cos\alpha} = \frac{dy dz}{|\vec{i} \cdot \vec{n}|}, \quad ds = \frac{dz dx}{\cos\beta} = \frac{dz dx}{|\vec{j} \cdot \vec{n}|}, \quad ds = \frac{dx dy}{\cos\gamma} = \frac{dx dy}{|\vec{k} \cdot \vec{n}|}$$

Surface Integral $\iint_S \vec{F} \cdot \vec{n} ds$

$$\iint_S \vec{F} \cdot \vec{n} ds = \iint_{S_1} \vec{F} \cdot \vec{n} \frac{dy dz}{|\vec{i} \cdot \vec{n}|} = \iint_{S_2} \vec{F} \cdot \vec{n} \frac{dz dx}{|\vec{j} \cdot \vec{n}|} = \iint_{S_3} \vec{F} \cdot \vec{n} \frac{dx dy}{|\vec{k} \cdot \vec{n}|}$$

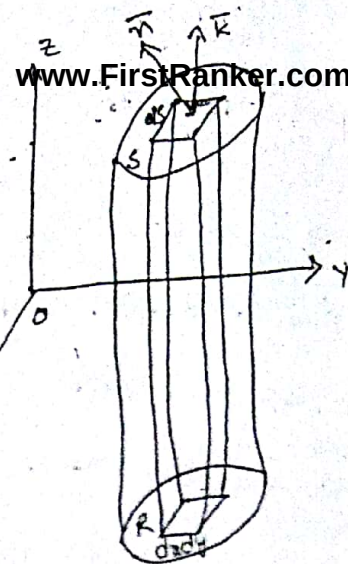
where S_1, S_2, S_3 are the projections of S on the planes

$x=0, y=0, z=0$ respectively

2

cosine

cos



1) Evaluate $\int_S \vec{F} \cdot \vec{n} \, ds$ where $\vec{F} = z\vec{i} + x\vec{j} - 3y^2z\vec{k}$ and S is the surface $x^2 + y^2 = 16$ included in the first octant between $z=0$ and $z=5$.

Sol:- $\vec{F} = z\vec{i} + x\vec{j} - 3y^2z\vec{k}$

Here 'S' is the surface $x^2 + y^2 = 16$ included in the first octant between $z=0$ and $z=5$.

$\vec{n}$ = unit outward normal to S $= \frac{\nabla \phi}{|\nabla \phi|}$ where $\phi = x^2 + y^2 = 16$

$$\vec{n} = \frac{2x\vec{i} + 2y\vec{j}}{\sqrt{4x^2 + 4y^2}} = \frac{x\vec{i} + y\vec{j}}{2} \quad x^2 + y^2 = 16$$

Let 'R' be the projection of S on the yz-plane.

$$\int_S \vec{F} \cdot \vec{n} \, ds = \int_R \vec{F} \cdot \vec{n} \frac{dy \, dz}{|\vec{n} \cdot \vec{k}|}$$

$$= \int_R (z\vec{i} + x\vec{j} - 3y^2z\vec{k}) \cdot \left(\frac{x\vec{i} + y\vec{j}}{2} \right) \frac{dy \, dz}{\left| \frac{x\vec{i} + y\vec{j}}{2} \cdot \vec{k} \right|}$$

$$= \int_R (zx + xy) \frac{dy \, dz}{2}$$

$$= \int_{z=0}^5 \int_{y=0}^4 (zx + xy) \, dy \, dz = \int_{z=0}^5 \left[\frac{zx^2}{2} + \frac{xy^2}{2} \right]_{y=0}^4 \, dz$$

$$= \int_{z=0}^5 (4xz + 8z) \, dz = \left[2xz^2 + 4z^2 \right]_0^5$$

$$= 2 \times 5 \times 25 + 4 \times 25$$

$$= 90$$

Q. Evaluate $\int_S \vec{F} \cdot \vec{n} \, ds$ where $\vec{F} = 18z\vec{i} - 12\vec{j} + 3y\vec{k}$ and 'S' is the part of the plane $2x+3y+6z=12$ located in the first octant.

Sol:- $\vec{F} = 18z\vec{i} - 12\vec{j} + 3y\vec{k}$

Here 'S' is the part of the plane $2x+3y+6z=12$ located in the first octant.

$\vec{n}$ = unit outward normal to S = $\frac{\nabla\phi}{|\nabla\phi|}$ where $\phi = 2x+3y+6z-12$

$$\vec{n} = \frac{2\vec{i}+3\vec{j}+6\vec{k}}{\sqrt{4+9+36}} = \frac{1}{7}(2\vec{i}+3\vec{j}+6\vec{k})$$

Let 'R' be the projection of S on xy-plane then

$$\int_S \vec{F} \cdot \vec{n} \, ds = \int_R \vec{F} \cdot \vec{n} \frac{dx \, dy}{|\vec{n} \cdot \vec{E}|}$$

$$\int_S \vec{F} \cdot \vec{n} \, ds = \int_R (18z\vec{i} - 12\vec{j} + 3y\vec{k}) \cdot \frac{1}{7}(2\vec{i}+3\vec{j}+6\vec{k}) \frac{dx \, dy}{|\frac{1}{7}(2\vec{i}+3\vec{j}+6\vec{k}) \cdot \vec{E}|}$$

$$= \int_R (36z - 36 + 18y) \frac{dx \, dy}{6} = \int_R (6z - 6 + 3y) dx \, dy$$

$$= \int_{x=0}^6 \int_{y=0}^{\frac{12-2x}{3}} \left[6\left(\frac{12-2x-3y}{6}\right) - 6 + 3y \right] dy \, dx$$

$$= \int_{x=0}^6 \int_{y=0}^{\frac{12-2x}{3}} [(6-2x) dy] dx = \int_{x=0}^6 [6y - 2xy]_0^{\frac{12-2x}{3}} dx$$

$$= \int_{x=0}^6 \left(6\left(\frac{12-2x}{3}\right) - 2x\left(\frac{12-2x}{3}\right) \right) dx$$

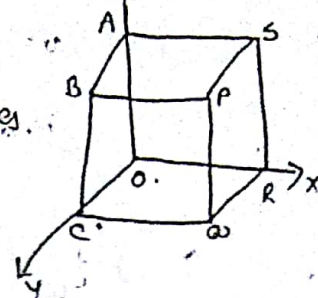
$$= \int_{x=0}^6 \left(24 - 12x + \frac{4x^2}{3} \right) dx = \left[24x - 6x^2 + \frac{4x^3}{9} \right]_0^6$$

$$= 24 \times 6 - 6 \times 36 + \frac{4 \times 6^3}{9} = 144 - 216 + 96 = 24$$

$$\begin{aligned} 2x+3y+6z &= 12 \\ 6z &= 12-2x-3y \\ z &= \frac{12-2x-3y}{6} \\ 3y &= 12-2x \\ y &= \frac{12-2x}{3} \\ x &= 0 \text{ to } 6 \end{aligned}$$

$$\begin{array}{r} 144 \\ -216 \\ \hline 24 \end{array}$$

of the bounded by $x=0, x=a, y=0, y=a, z=0, z=a$. Evaluate $\int_S \vec{F} \cdot \vec{n} \, ds$, where 'S' is the surface of the cube.



Sol:- Consider the cube surrounded by the following faces.

(i) For the face PQRS, $\vec{i}$ is the outward normal

$$\vec{N} = \vec{i}; x=a, ds = dy \, dz$$

$$\int_S \vec{F} \cdot \vec{N} \, ds = \int_{y=0}^a \int_{z=0}^a (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot \vec{i} \, dy \, dz = \int_{y=0}^a \int_{z=0}^a 4xz \, dy \, dz$$

$$= \int_{y=0}^a \left[4xz \cdot y \right]_{y=0}^y \, dz = \int_{y=0}^a \left[4a^2 \cdot \frac{z^2}{2} \right]_{z=0}^z \, dy = \int_{y=0}^a 2a^2 z^2 \, dy = 2a^2 \int_{y=0}^a dy = 2a^2 \cdot a = 2a^3$$

$\therefore x=a$ on this face

(ii) For the face OABC, $-\vec{i}$ is the outward normal.

$$\vec{N} = -\vec{i}, x=0, \text{ and } ds = dy \, dz.$$

$$\int_{R_2} \vec{F} \cdot \vec{N} \, ds = \iint_{R_2} (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot (-\vec{i}) \, dy \, dz = - \iint_{R_2} 4xz \, dy \, dz = 0.$$

(iii) For the face BPQR, $\vec{j}$ is the outward normal.

$$\vec{N} = \vec{j}, y=a, \text{ and } ds = dx \, dz.$$

$$\int_{R_3} \vec{F} \cdot \vec{N} \, ds = \iint_{R_3} (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot \vec{j} \, dx \, dz = - \iint_{R_3} y^2 \, dx \, dz = -a^2 \int_{z=0}^a \int_{x=0}^a dx \, dz = -a^4.$$

(iv) For the face ORSA, $-\vec{j}$ is the outward normal.

$$\vec{N} = -\vec{j}, y=0, \text{ and } ds = dx \, dz.$$

$$\int_{R_4} \vec{F} \cdot \vec{N} \, ds = \iint_{R_4} (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot (-\vec{j}) \, dx \, dz = \iint_{R_4} y^2 \, dx \, dz = 0.$$

(v) For the face ABPS, $\vec{k}$ is the outward normal.

$$\vec{N} = \vec{k}, z=a, \text{ and } ds = dx \, dy. \quad \vec{F} \cdot \vec{N} = (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot \vec{k} = yz$$

$$\int_{R_5} \vec{F} \cdot \vec{N} \, ds = \iint_{R_5} yz \, dx \, dy = a \int_{y=0}^a \int_{x=0}^a y \, dx \, dy = a \int_{y=0}^a y \, dy = a \cdot \frac{y^2}{2} \Big|_{y=0}^y = \frac{a^4}{2}$$

(vi) For the face $z=0$,

$$\vec{n} = -\vec{k}, \quad z=0, \quad ds = dx dy.$$

$$\vec{F} \cdot \vec{n} = (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot (-\vec{k}) = -yz.$$

$$\iint_{R_6} \vec{F} \cdot \vec{n} ds = - \iint_{R_6} yz dx dy = 0 \quad [z=0 \text{ on this face}]$$

Hence adding all we get $\int \vec{F} \cdot \vec{n} ds = 2a^4 + 0 - a^4 + \frac{1}{2}a^4 + 0 = \frac{3}{2}a^4.$

4) $\vec{F} = 2y\vec{i} - 3\vec{j} + x^2\vec{k}$ and 'S' is surface $y^2 = 8x$ in first octant bounded by the planes $y=4$ and $z=6$, evaluate $\int_S \vec{F} \cdot \vec{n} ds$

Soln: $\vec{F} = 2y\vec{i} - 3\vec{j} + x^2\vec{k}$

Here 'S' is the surface $y^2 = 8x$ in first octant bounded by the planes $y=4$ and $z=6$.

$\vec{n}$ = unit outward normal to S = $\frac{\nabla\phi}{|\nabla\phi|}$

$$\vec{n} = \frac{8\vec{i} - 2y\vec{j}}{\sqrt{64 + 4y^2}} = \frac{4\vec{i} - y\vec{j}}{\sqrt{16 + y^2}}$$

Let 'R' be the projection of 'S' on the yz -plane then

$$\int_S \vec{F} \cdot \vec{n} ds = \int_R \vec{F} \cdot \vec{n} \frac{dy dz}{|\vec{n} \cdot \vec{k}|}$$

For R: $z = y^2/8$, $y = 0$ to 4 , $z = 0$ to 6 .

$$\int_S \vec{F} \cdot \vec{n} ds = \int_R \vec{F} \cdot \vec{n} \frac{dy dz}{|\vec{n} \cdot \vec{k}|} = \int_{z=0}^6 \int_{y=0}^4 (2y\vec{i} - 3\vec{j} + x^2\vec{k}) \cdot \left(\frac{4\vec{i} - y\vec{j}}{\sqrt{16 + y^2}} \right) \frac{dy dz}{1}$$

$$= \int_{z=0}^6 \int_{y=0}^4 (8y - 3y) \frac{dy dz}{4} = \frac{11}{4} \int_{z=0}^6 \left[\int_{y=0}^4 y \right] dz$$

$$= \frac{11}{4} \int_{z=0}^6 \left[\frac{y^2}{2} \right]_0^4 dz = \frac{11}{4} \int_{z=0}^6 8y dz = \frac{88}{4} \left[z \right]_0^6$$

$$= 22 \times 6 = 132$$

At velocity vector is $\vec{F} = y\vec{i} + 2\vec{j} + xz\vec{k}$ m/sec. u
 show that the flux of water through the parabolic cylinder

$y = x^2$, $0 \leq x \leq 3$, $0 \leq z \leq 2$ is $69 \text{ m}^3/\text{sec}$.

Sol:- $\vec{F} = y\vec{i} + 2\vec{j} + xz\vec{k}$

Here 's' is the surface $y = x^2$ first octant bounded by the planes $x = 3$ and $z = 2$.

$\vec{N}$ = unit outward normal to $S = \frac{\nabla \phi}{|\nabla \phi|}$

$\vec{N} = \frac{2x\vec{i} - \vec{j}}{\sqrt{4x^2 + 1}}$ Let 'R' be the projection of 's' on the xz -plane.

$\int_S \vec{F} \cdot \vec{N} \, ds = \int_R \vec{F} \cdot \vec{N} \frac{dx \, dz}{|\vec{N} \cdot \vec{j}|}$

$\vec{F} \cdot \vec{N} = (y\vec{i} + 2\vec{j} + xz\vec{k}) \cdot \left(\frac{2x\vec{i} - \vec{j}}{\sqrt{4x^2 + 1}} \right)$

$\vec{N} \cdot \vec{j} = \frac{2x\vec{i} - \vec{j}}{\sqrt{4x^2 + 1}} \cdot \vec{j} = \frac{-1}{\sqrt{4x^2 + 1}}$

For R: $y = x^2$, $x = 0$ to 3 , $z = 0$ to 2

$\int_S \vec{F} \cdot \vec{N} \, ds = \int_R \vec{F} \cdot \vec{N} \frac{dx \, dz}{|\vec{N} \cdot \vec{j}|} = \int_{x=0}^3 \int_{z=0}^2 \frac{2xy - 2}{\sqrt{4x^2 + 1}} \cdot \frac{dx \, dz}{\left| \frac{-1}{\sqrt{4x^2 + 1}} \right|}$

$= +2 \int_{x=0}^3 \int_{z=0}^2 (xy + 1) \, dx \, dz = +2 \int_{x=0}^3 \int_{z=0}^2 (x^3 + 1) \, dx \, dz \quad [\because y = x^2]$

$= +2 \int_{z=0}^2 \left[\frac{x^4}{4} + x \right]_0^3 \, dz = +2 \int_{z=0}^2 \left(\frac{81}{4} + 3 \right) \, dz$

$= +2 \left[\frac{81}{4} z + 3z \right]_0^2$

$= +2 \left[\frac{81}{4} \cdot 2 + 3(2) \right]$

$= +2 \cdot \frac{81}{2} + 2(3 \cdot 2)$

$= +81 + 12 = +69 \text{ m}^3/\text{sec}$

Surface of $x^2 + y^2 + z^2 = 1$ in the first octant.

Sol:-

$$\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$$

Here 'S' is the surface of $x^2 + y^2 + z^2 = 1$ in the first octant.

$$\vec{N} = \text{unit outward normal} = \frac{\nabla \phi}{|\nabla \phi|}$$

$$\text{where } \phi = x^2 + y^2 + z^2 - 1$$

$$\vec{N} = \frac{2x\vec{i} + 2y\vec{j} + 2z\vec{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{2(x\vec{i} + y\vec{j} + z\vec{k})}{2\sqrt{x^2 + y^2 + z^2}} = x\vec{i} + y\vec{j} + z\vec{k}$$

Let 'R' be the projection of any plane $\int_S \vec{F} \cdot \vec{N} ds = \int_R \vec{F} \cdot \vec{N} \frac{ds}{|\vec{N} \cdot \vec{E}|}$

$$\int_S \vec{F} \cdot \vec{N} ds = \int_R \vec{F} \cdot \vec{N} \frac{dxdy}{|\vec{N} \cdot \vec{E}|}$$

$$= \int_R (yz\vec{i} + zx\vec{j} + xy\vec{k}) \cdot (x\vec{i} + y\vec{j} + z\vec{k}) \cdot \frac{dxdy}{|x\vec{i} + y\vec{j} + z\vec{k} \cdot \vec{E}|}$$

$$\begin{cases} x=0 \text{ to } 1 \\ y=0 \text{ to } \sqrt{1-x^2} \\ z=0 \text{ to } \sqrt{1-x^2-y^2} \end{cases}$$

$$= \int_R (xyz + xyz + xyz) \frac{dxdy}{z} = \int_R 3xyz \frac{dxdy}{z}$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} 3xy \cdot dxdy = \int_0^1 \int_0^{\sqrt{1-x^2}} (3xy) dy dx = \int_0^1 \left[\frac{3xy^2}{2} \right]_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 \frac{3x}{2} (1-x^2) dx = \frac{3}{2} \int_0^1 (x - x^3) dx$$

$$= \frac{3}{2} \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{3}{2} \left[\frac{1}{2} - \frac{1}{4} \right] = \frac{3}{2} \left[\frac{1}{4} \right] = \frac{3}{8}$$

over the region 'V'.

Another form of volume Integrals are $\int \phi dv$ and

$$\iiint_V (F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}) dx dy dz$$

Q. 9d. $\vec{F} = 2xz \hat{i} - x \hat{j} + y^2 \hat{k}$. Evaluate $\int \vec{F} dv$ where 'V' is the region bounded by the surfaces $x=0, x=2, y=0, y=6, z=x^2, z=4$.

Sol:- $\vec{F} = 2xz \hat{i} - x \hat{j} + y^2 \hat{k}$

Here 'V' is the region bounded by the surfaces $x=0, x=2, y=0, y=6, z=x^2, z=4$

For V: $x=0$ to $2, y=0$ to $6, z=x^2$ to 4 .

$$\iiint_V \vec{F} dv = \iiint_V (2xz \hat{i} - x \hat{j} + y^2 \hat{k}) dx dy dz$$

$$= \int_{x=0}^2 \int_{y=0}^6 \left[\int_{z=x^2}^4 (2xz \hat{i} - x \hat{j} + y^2 \hat{k}) dz \right] dx dy$$

$$= \int_{x=0}^2 \int_{y=0}^6 \left[x \left(\frac{z^2}{2} \right)_{x^2}^4 \hat{i} - x \left(z \right)_{x^2}^4 \hat{j} + y^2 \left(z \right)_{x^2}^4 \hat{k} \right] dy dx$$

$$= \int_{x=0}^2 \int_{y=0}^6 \left[(16x - x^5) \hat{i} - (4x - x^3) \hat{j} + (4y^2 - x^2 y^2) \hat{k} \right] dy dx$$

$$= \int_{x=0}^2 \left[(16xy - x^5 y) \hat{i} - (4xy - x^3 y) \hat{j} + \left(\frac{4}{3} y^3 - x^2 \frac{y^3}{3} \right) \hat{k} \right]_0^6 dx$$

$$= \int_{x=0}^2 \left[(96x - 6x^5) \hat{i} - (24x - 6x^3) \hat{j} + (288 - 72x^2) \hat{k} \right] dx$$

$$= \left[96 \frac{x^2}{2} - 6 \frac{x^6}{6} \right]_0^2 + \left[12x^2 - \frac{3}{2}x^4 \right]_0^2 + \left[288x - 24x^3 \right]_0^2$$

$$= (48x^2 - x^6)_0^2 - (12x^2 - \frac{3}{2}x^4)_0^2 + (288x - 24x^3)_0^2$$

$$= (192 - 64)\hat{i} - (48 - 24)\hat{j} + (576 - 192)\hat{k}$$

$$= 128\hat{i} - 24\hat{j} + 384\hat{k}$$

② Evaluate $\iiint_V (2x+y) dv$ where V is closed region bounded by the cylinder $z = 4-x^2$ and the plane $x=0, y=0, y=2$ and $z=0$.

Sol - V is the closed region bounded by the cylinder $z = 4-x^2$ and the plane $x=0, y=0, y=2$ and $z=0$.

$$\text{For } V: z = 0 \text{ to } 4-x^2$$

$$y = 0 \text{ to } 2$$

$$x = 0 \text{ to } 2$$

$$\iiint_V (2x+y) dv = \int_0^2 \int_0^2 \left[\int_0^{4-x^2} (2x+y) dz \right] dy dx$$

$$= \int_0^2 \int_0^2 \left[(2x+y)z \right]_0^{4-x^2} dy dx = \int_0^2 \int_0^2 (2x+y)(4-x^2) dy dx$$

$$= \int_0^2 \left[(8x - 2x^3 + 4y - x^3y) \right]_0^2 dx$$

$$= \int_0^2 \left[8x(2)^2 - 2x^3(2)^2 + 4(2)^2 - x^3(2)^2 \right] dx = \int_0^2 (16x - 4x^3 + 8 - 2x^3) dx$$

$$= \left[8x^2 - x^4 + 8x - \frac{2}{4}x^4 \right]_0^2$$

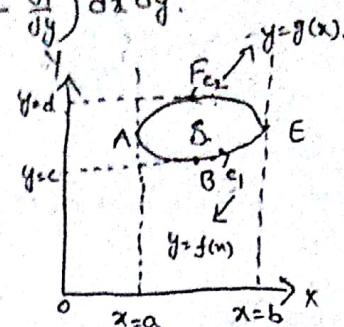
$$= 32 - 16 + 16 - \frac{16}{3}$$

$$= \frac{96 - 16}{3} = \frac{80}{3}$$

curve "c". Let p and q be continuous and differentiable scalar point functions of x and y in S . Then $\oint_C p dx + q dy = \iint_S \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right) dx dy$.

Proof:- 'S' be a closed region in xy -plane enclosed by a closed curve "c".

p and q are continuous and differentiable scalar point functions of x and y in S .



Let any line parallel to either co-ordinate axis cut 'c' in at most two points.

Let 'S' lie between the lines $x=a, x=b, y=c, y=d$.

Let $y=f(x)$ be the curve "c1" [ie ABE]. and

$y=g(x)$ " " " "c2" [ie EFA].

Generally $f(x) \leq g(x)$.

$$\iint_S \frac{\partial p}{\partial y} dx dy = \int_{x=a}^b \int_{y=f(x)}^{y=g(x)} \left[\frac{\partial p}{\partial y} dy \right] dx = \int_{x=a}^b [p(x, y)]_{f(x)}^{g(x)} dx.$$

$$= \int_{x=a}^b [p(x, g) - p(x, f)] dx = \int_{x=a}^b p(x, g) dx - \int_{x=a}^b p(x, f) dx.$$

$$= - \int_{x=b}^a p(x, g) dx - \int_{x=a}^b p(x, f) dx.$$

$$= - \left[\int_{x=b}^a p(x, g) dx + \int_{x=a}^b p(x, f) dx \right]$$

$$= - \left[\int_{c_2} p(x, y) dx + \int_{c_1} p(x, y) dx \right]$$

$$= - \left[\int_{c_1} p dx + \int_{c_2} p dx \right] = - \oint_C p dx.$$

$$\iint_S \frac{\partial p}{\partial y} dx dy = - \oint_C p dx \Rightarrow \oint_C p dx = - \iint_S \frac{\partial p}{\partial y} dx dy. \quad \text{--- (I)}$$

$$\oint_C \omega dy = \iint_S \frac{\partial \omega}{\partial x} dz dy. \quad \text{--- (II)}$$

$$\oint_C p dx + \oint_C \omega dy = - \iint_S \frac{\partial p}{\partial y} dx dy + \iint_S \frac{\partial \omega}{\partial x} dz dy.$$

$$\oint_C p dx + \omega dy = \iint_S \left(\frac{\partial \omega}{\partial x} - \frac{\partial p}{\partial y} \right) dz dy.$$

Note ① :- Let "S" be closed region in yz-plane enclosed by a curve "C". Let p and ω continuous and differentiable scalar functions are y and z in S.

$$\text{Then } \oint_C p dy + \omega dz = \iint_S \left(\frac{\partial \omega}{\partial y} - \frac{\partial p}{\partial z} \right) dy dz.$$

Note ② :- Let "S" be closed region xz-plane enclosed by a curve "C". Let p and ω continuous and differentiable scalar functions are z and x in S.

$$\text{Then } \oint_C p dz + \omega dx = \iint_S \left(\frac{\partial \omega}{\partial z} - \frac{\partial p}{\partial x} \right) dz dx.$$

2

1) Verify Green's Theorem for $\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$

boundary of the region bounded by $x \geq 0$, $y \geq 0$, and $x + y = 1$.

Sol:- $\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$

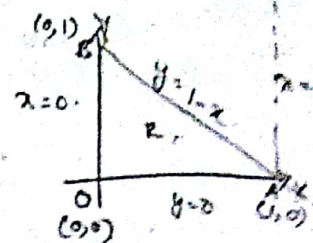
By Green's Theorem $\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$

$P = 3x^2 - 8y^2$, $Q = 4y - 6xy \Rightarrow \frac{\partial P}{\partial y} = -16y$; $\frac{\partial Q}{\partial x} = -6y$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -6y + 16y = 10y$

and 'S' is the boundary of region bounded by $x=0$, $y=0$, $x+y=1$.

For S: $y = 0$ to $1-x$; $x = 0$ to 1 .



$\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy = \iint_S 10y dx dy = \int_{x=0}^1 \int_{y=0}^{1-x} [10y dy] dx$

$= \int_{x=0}^1 [5y^2]_0^{1-x} dx = 5 \int_{x=0}^1 (1+x^2-2x) dx = 5 \left[x + \frac{x^3}{3} - x^2 \right]_0^1 = 5 \left[1 + \frac{1}{3} - 1 \right] = \frac{5}{3}$

The region $R = OAB$

Along OA, $y=0$, $dy=0$, x varies from 0 to 1.

$\int_{OA} \vec{F} \cdot d\vec{r} = \int_0^1 3x^2 dx = \left[\frac{3x^3}{3} \right]_0^1 = \frac{3}{3}$

Along AB, $x+y=1$, $y=1-x$, $dy=-dx$, x varies from 1 to 0.

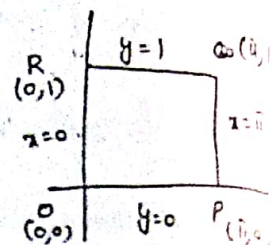
$\int_{AB} \vec{F} \cdot d\vec{r} = \int_1^0 [3x^2 - 8(1-x)^2] dx - [4(1-x) - 6x(1-x)](-dx) = \int_1^0 (-4x^2 + 29x - 12) dx$
 $= \int_0^1 (4x^2 - 29x + 12) dx = \left[\frac{4x^3}{3} - \frac{29x^2}{2} + 12x \right]_0^1 = \frac{4}{3} - \frac{29}{2} + 12 = \frac{13}{6}$

Along BO $x=0$, $y=1$ to 0.

$\int_{BO} \vec{F} \cdot d\vec{r} = \int_1^0 4y dy = \left[\frac{4y^2}{2} \right]_1^0 = -\frac{4}{2} = -2$

$\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy = \frac{3}{3} + \frac{13}{6} - 2 = \frac{5}{3}$

the rectangle with vertices $(0,0)$ $(\pi,0)$ $(\pi,1)$ $(0,1)$



sol:- By Green's theorem $\oint_C p dx + q dy = \iint_S \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right) dx dy$

Here $p = x^2 - \cosh y$, $q = y + \sin x$

$$\frac{\partial p}{\partial y} = -\sinh y \quad \frac{\partial q}{\partial x} = \cos x \quad \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} = \cos x + \sinh y$$

and 'S' is the surface of rectangle with vertices $(0,0)$ $(\pi,0)$ $(\pi,1)$ and $(0,1)$. For S; $x = 0$ to π , $y = 0$ to 1 .

$$\oint_C (x^2 - \cosh y) dx + (y + \sin x) dy = \iint_S (\cos x + \sinh y) dx dy$$

$$= \int_{y=0}^1 \left[\int_{x=0}^{\pi} (\cos x + \sinh y) dx \right] dy = \int_{y=0}^1 \left[(\sin x)_0^{\pi} + \sinh y (x)_0^{\pi} \right] dy$$

$$= \int_{y=0}^1 (0-0) + (\pi-0) \sinh y dy = \int_{y=0}^1 \pi \sinh y dy = \pi [\cosh y]_0^1$$

$$= \pi [\cosh 1 - \cosh 0]$$

$$= \pi [\cosh 1 - 1]$$

2) Find the area of a circle of radius a using Green's Theorem

sol:- Area = $\frac{1}{2} \oint_C (x dy - y dx)$

choosing $x = a \cos \theta$, $y = a \sin \theta$ we get

$$A = \frac{1}{2} \int_0^{2\pi} (a^2 \cos^2 \theta + a^2 \sin^2 \theta) d\theta$$

$$= \frac{1}{2} a^2 \int_0^{2\pi} d\theta = \pi a^2$$

Soln:- By Green's theorem. $\oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$.

Let $P = -y$, $Q = x$, $\frac{\partial P}{\partial y} = -1$, $\frac{\partial Q}{\partial x} = 1$.

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - (-1) = 2$$

$$\oint_C -y dx + x dy = \iint_R 2 dx dy$$

$$\int ds = \int dx dy$$

$$\int dx dy = S$$

$$\frac{1}{2} \oint_C x dy - y dx = \iint_R dx dy$$

$$S = \frac{1}{2} \oint_C x dy - y dx$$

∴ The area bounded by a simple closed curve C is

$$\frac{1}{2} \oint_C x dy - y dx$$

∴ The area of the given ellipse

$$= \frac{1}{2} \oint_C x dy - y dx$$

Given that of ellipse

$$x = a \cos \theta, y = b \sin \theta, 0 \leq \theta \leq 2\pi$$

$$dx = -a \sin \theta d\theta, dy = b \cos \theta d\theta$$

$$\theta = 0 \text{ to } 2\pi$$

$$= \frac{1}{2} \int_0^{2\pi} a \cos \theta (b \cos \theta) d\theta - b \sin \theta (-a \sin \theta) d\theta$$

$$= \frac{1}{2} ab \int_0^{2\pi} (\cos^2 \theta + \sin^2 \theta) d\theta = \frac{1}{2} ab \int_0^{2\pi} 1 d\theta$$

$$= \frac{1}{2} ab [\theta]_0^{2\pi} = \pi ab \text{ sq. units}$$

Answer Area of the ellipse is πab sq. units

of the region defined by $y^2 = 8x$ and $x = 2$.

Sol:-

By Green's theorem $\oint_C P dx + Q dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$

$$P = x^2 - 2xy, \quad Q = x^2y + 2$$

$$\frac{\partial P}{\partial y} = -2x, \quad \frac{\partial Q}{\partial x} = 2xy$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2xy + 2x$$



and $\oint_C$ is the surface of the region bounded $y^2 = 8x$ and $x = 2$.

$$\text{For } x: y = -\sqrt{8x} \text{ to } \sqrt{8x} \quad x = 0 \text{ to } 2$$

$$\oint_C (x^2 - 2xy) dx + (x^2y + 2) dy = \iint_A (2xy + 2x) dx dy$$

$$= \int_{x=0}^2 \left[\int_{y=-\sqrt{8x}}^{\sqrt{8x}} (2xy + 2x) dy \right] dx = \int_{x=0}^2 \left[xy^2 + 2xy \right]_{y=-\sqrt{8x}}^{\sqrt{8x}} dx$$

$$= \int_{x=0}^2 2(8x - 8x) + 2x [\sqrt{8x} + \sqrt{8x}] dx$$

$$= \int_{x=0}^2 4\sqrt{8} x^{\frac{3}{2}} dx = 4\sqrt{8} \left[\frac{x^{\frac{5}{2}}}{\frac{5}{2}} \right]_0^2$$

$$= 4\sqrt{8} \cdot \frac{2}{5} \cdot 2^{\frac{5}{2}} = \frac{4 \cdot 2\sqrt{2} \cdot 2}{5} \cdot 2^{\frac{5}{2}}$$

$$= \frac{16\sqrt{2} \cdot 2^{\frac{5}{2}}}{5}$$

$$= \frac{16\sqrt{2} \cdot 2^2 \sqrt{2}}{5}$$

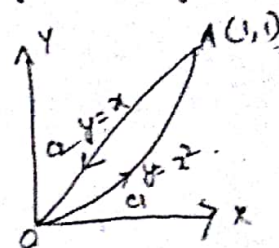
$$= \frac{128}{5}$$

1) Verify Green's theorem in the plane for $\oint_C (xy + y^2) dx + x^2 dy$.

where 'c' is the closed curve of the region bounded by $y=x$ and $y=x^2$

Sol:- By Green's theorem, we have

$$\oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy.$$



The two curves $y=x$ and $y=x^2$ intersect at $O(0,0)$ and

$A(1,1)$

$$\oint_C (xy + y^2) dx + x^2 dy \text{ where } P = xy + y^2, Q = x^2.$$

$$\frac{\partial Q}{\partial x} = 2x, \quad \frac{\partial P}{\partial y} = x + 2y.$$

$$\begin{aligned} \oint_C P dx + Q dy &= \iint_R (2x - x - 2y) dx dy = \iint_R (x - 2y) dx dy = \int_0^1 \int_{y=x^2}^x (x - 2y) dy dx \\ &= \int_0^1 \left(xy - 2 \frac{y^2}{2} \right)_{y=x^2}^x dx = \int_0^1 (x^4 - x^3) dx = \left(\frac{x^5}{5} - \frac{x^4}{4} \right)_0^1 = -\frac{1}{20} \quad \text{--- (i)} \end{aligned}$$

consider the integral.

$$\text{Along } c, y = x^2 \Rightarrow dy = 2x dx.$$

$$\oint_C (xy + y^2) dx + x^2 dy$$

$$\int_0^1 (x(x^2) + x^4) dx + x^2(2x dx) = \int_0^1 (3x^3 + 2x^4) dx = \frac{19}{20}$$

$$\left[\frac{3x^4}{4} + \frac{2x^5}{5} \right]_0^1 = \frac{3}{4} + \frac{2}{5} = \frac{15+8}{20} = \frac{23}{20}$$

Along $y=x$, $dy=dx$.

$$\int_1^0 (x(x) + x^2) dx + x^2 dx = \int_1^0 3x^2 dx = \left[x^3 \right]_1^0 = 0 - 1 = -1$$

$$\text{so } \oint_C (xy + y^2) dx + x^2 dy = \frac{19}{20} - 1 = -\frac{1}{20} \quad \text{--- (ii)}$$

Hence the Green's theorem is verified from the equality of (i) and (ii).

2) If C is a simple closed curve in the xy -plane enclosing the origin, show that $\oint_C \vec{F} \cdot d\vec{r} = 0$, where $\vec{F} = \frac{y\vec{i} - x\vec{j}}{x^2 + y^2}$.

Sol:- $\oint_C \vec{F} \cdot d\vec{r} = \oint_C \frac{y\vec{i} - x\vec{j}}{x^2 + y^2} \cdot (dx\vec{i} + dy\vec{j})$ $\frac{dx}{dy}$

$$\frac{dQ}{dx} = \frac{-(x^2 + y^2) + x(2x)}{(x^2 + y^2)^2}$$

$$P = \frac{y}{x^2 + y^2}, \quad Q = \frac{-x}{x^2 + y^2}$$

$$\frac{dP}{dy} = \frac{(x^2 + y^2) - y(2y)}{(x^2 + y^2)^2}$$

$$\frac{dQ}{dx} - \frac{dP}{dy} = -\frac{x^2 - y^2 + 2x^2}{(x^2 + y^2)^2}$$

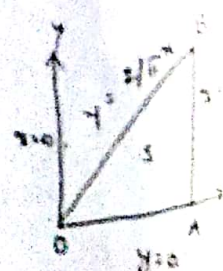
$$\frac{dQ}{dx} - \frac{dP}{dy} = 0$$

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{r} &= \iint_R \left(\frac{dQ}{dx} - \frac{dP}{dy} \right) dx dy \\ &= \iint_R \left[\frac{-(x^2 + y^2) + x(2x)}{(x^2 + y^2)^2} - \frac{(x^2 + y^2) - y(2y)}{(x^2 + y^2)^2} \right] dx dy = 0 \end{aligned}$$

3) Evaluate using Green's theorem, evaluate $\oint_C (y - \sin x) dx + \cos x dy$ where C is the plane triangle enclosed by the lines $y=0$, $x=\pi/2$ and $y=x$.

Sol:- By Green's theorem.

$$\oint_C P dx + Q dy = \iint_R \left(\frac{dQ}{dx} - \frac{dP}{dy} \right) dx dy$$



Here $P = y - \sin x$, $Q = \cos x$. $\frac{dP}{dy} = 1$, $\frac{dQ}{dx} = -\sin x$.

$$\frac{dQ}{dx} - \frac{dP}{dy} = -\sin x - 1$$

and R is the surface of a triangle bounded by the lines $y=0$, $x=\pi/2$, $y=x$. For S : $y=0$ to $\frac{\pi}{2}$, $x=0$ to $\pi/2$.

$$\oint_C (y \sin x) dx + (\cos x) dy = \int_0^{\pi/2} \int_0^{2x/\pi} (-\sin x - y) dy dx$$

$$= \int_{x=0}^{\pi/2} \left[\int_{y=0}^{2x/\pi} (-\sin x - y) dy \right] dx = \int_{x=0}^{\pi/2} \left[(-\sin x - y) y \right]_0^{2x/\pi} dx$$

$$= \int_{x=0}^{\pi/2} (-\sin x - 1) \frac{2x}{\pi} dx = \int_{x=0}^{\pi/2} -\frac{2x}{\pi} \sin x dx - \int_{x=0}^{\pi/2} \frac{2x}{\pi} dx$$

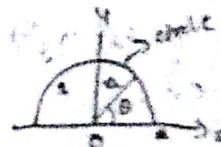
$$= \frac{2}{\pi} \int_{x=0}^{\pi/2} x d(\cos x) - \frac{1}{\pi} [x^2]_0^{\pi/2}$$

$$= \frac{2}{\pi} \left[(x \cos x) - \int_0^{\pi/2} \cos x dx \right] - \frac{1}{\pi} \frac{\pi^2}{4}$$

$$= \frac{2}{\pi} \left[\frac{\pi}{2} \cos \frac{\pi}{2} - [\sin x]_0^{\pi/2} - \frac{\pi}{4} \right]$$

$$= \frac{2}{\pi} \left[0 - (1-0) - \frac{\pi}{4} \right] = -\frac{2}{\pi} - \frac{\pi}{4}$$

4) Apply Green's theorem to evaluate $\int_C (2x^2 - y^2) dx + (x^2 + y^2) dy$, where 'C' is the boundary of the area enclosed by the x-axis and the upper-half of the circle $x^2 + y^2 = a^2$



$$\int_C (2x^2 - y^2) dx + (x^2 + y^2) dy = \iint_R \left[\frac{\partial}{\partial x} (x^2 + y^2) - \frac{\partial}{\partial y} (2x^2 - y^2) \right] dx dy$$

$$= 2 \iint_R (x + y) dx dy = 2 \int_0^a \int_0^{\pi} r(\cos \theta + \sin \theta) r d\theta dr$$

changing the polar coordinates (r, θ) , (r, θ) from 0 to a and θ varies from 0 to π .

$$= 2 \int_0^a r^2 dr \int_0^{\pi} (\cos \theta + \sin \theta) d\theta = 2 \frac{a^3}{3} (1+1) = \frac{4a^3}{3}$$

$$\int_0^{\pi} (\cos \theta + \sin \theta) d\theta = [\sin \theta - \cos \theta]_0^{\pi}$$

$$= [\sin \pi - \cos \pi] - [\sin 0 - \cos 0]$$

$$= [0 - (-1)] - [0 - 1]$$

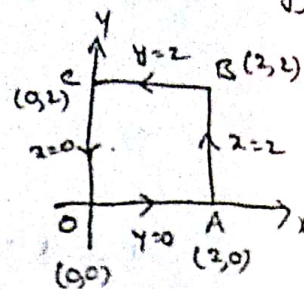
$$= 1 + 1$$

1) is a square with vertices $(0,0)$ $(2,0)$ $(2,2)$ $(0,2)$

$$P = x^2 - xy^3, \quad Q = y^2 - 2xy.$$

$$\frac{\partial Q}{\partial x} = -2y, \quad \frac{\partial P}{\partial y} = -3xy^2.$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -2y + 3xy^2.$$



$$\oint_C (x^2 - xy^3) dx + (y^2 - 2xy) dy = \iint_R (-2y + 3xy^2) dx dy.$$

1) Along OA: $y=0$ hence $dy=0$, x varies from 0 to 2.

$$\int_{OA} (x^2 - xy^3) dx + (y^2 - 2xy) dy = \int_{x=0}^2 x^2 dx = \left[\frac{x^3}{3} \right]_0^2 = \frac{8}{3}$$

2) Along AB: $x=2$, $dx=0$, y varies from 0 to 2.

$$\int_{AB} (x^2 - xy^3) dx + (y^2 - 2xy) dy = \int_{y=0}^2 (y^2 - 4y) dy = \left[\frac{y^3}{3} - 4 \frac{y^2}{2} \right]_0^2 = \frac{8}{3} - 8 = -\frac{16}{3}$$

3) Along BC: $y=2$, $dy=0$, x varies from 2 to 0.

$$\int_{BC} (x^2 - xy^3) dx + (y^2 - 2xy) dy = \int_{x=2}^0 (x^2 - 2xy^3) dx = \left[\frac{x^3}{3} - 8 \frac{x^2}{2} \right]_2^0 = -\frac{8}{3} + 16 = \frac{40}{3}$$

4) Along CO: $x=0$, $dx=0$, y varies from 2 to 0.

$$\int_{CO} (x^2 - xy^3) dx + (y^2 - 2xy) dy = \int_{y=2}^0 y^2 dy = \left[\frac{y^3}{3} \right]_2^0 = -\frac{8}{3}$$

$$\therefore \oint_C (x^2 - xy^3) dx + (y^2 - 2xy) dy = \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3} = 8$$

Now RHS $\iint_R (-2y + 3xy^2) dx dy = \int_{x=0}^2 \int_{y=0}^2 (-2y + 3xy^2) dy dx$

$$= \int_{x=0}^2 \left[-2 \frac{y^2}{2} + 3x \frac{y^3}{3} \right]_0^2 dx = \int_{x=0}^2 (-4 + 8x) dx$$

$$= \left[-4x + 8 \frac{x^2}{2} \right]_0^2 = -8 + 16 = 8$$

Let $\vec{F}$ be any vector point function having continuous partial derivatives.

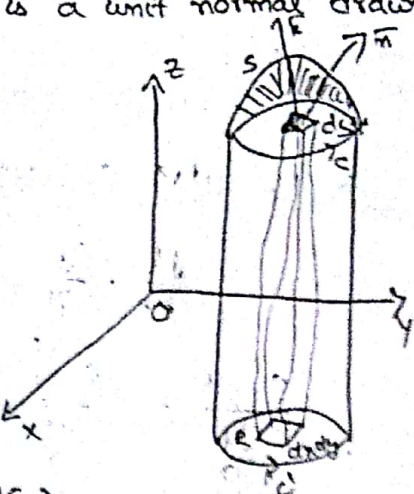
Let 'S' be an open surface bounded by closed curve C, then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n} \, ds \quad \text{where } \vec{n} \text{ is a unit normal drawn outward at any point to S.}$$

Proof:- Let $\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{k}$$



Let the unit normal vector makes angles α, β, γ with positive direction of x, y, z axes, then $\vec{n} = \cos \alpha \hat{i} + \cos \beta \hat{j} + \cos \gamma \hat{k}$.

$$\text{Also } \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$(\nabla \times \vec{F}) \cdot \vec{n} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \cos \alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \cos \beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \cos \gamma$$

$$\text{and } \vec{F} \cdot d\vec{r} = (F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k}) = F_1 dx + F_2 dy + F_3 dz$$

now Stokes theorem can also be written as,

$$\oint_C (F_1 dx + F_2 dy + F_3 dz) = \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \cos \alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \cos \beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \cos \gamma \right] \cdot ds \quad \text{--- (1)}$$

Let $z = f(x, y)$ be the equation of the surface S whose projection on the xy -plane is R. Then the projection of curve C on the xy -plane is the curve C' which is the boundary of the region R.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_C F_1 dx + F_2 dy + F_3 dz = \int_C F_1(x, y, z) dx + F_2(x, y, z) dy + F_3(x, y, z) dz$$

$$= - \iint_R \frac{\partial}{\partial y} F_1(x, y, z) dx dy \quad [\text{by Green's th.}]$$

$$= - \iint_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z} \frac{dz}{dy} \right) dx dy \quad (2)$$

Now the dir's of the normal $\hat{n}$ to the surface $z = f(x, y)$ are

$$\frac{dz}{dx}, \frac{dz}{dy}, -1.$$

$$\frac{\cos \alpha}{\frac{dz}{dx}} = \frac{\cos \beta}{\frac{dz}{dy}} = \frac{\cos \gamma}{-1} \Rightarrow \frac{dz}{dy} = - \frac{\cos \beta}{\cos \gamma}$$

Since $dx dy$ is the projection of ds on the xy plane.
 $dx dy = ds \cos \gamma$

$$\text{By (2)} \quad \oint_C F_1(x, y, z) dx = - \iint_S \left(\frac{\partial F_1}{\partial y} - \frac{\partial F_1}{\partial z} \frac{\cos \beta}{\cos \gamma} \right) ds \cos \gamma$$

$$= - \iint_S \left(\frac{\partial F_1}{\partial y} \cos \gamma - \frac{\partial F_1}{\partial z} \cos \beta \right) ds$$

$$= \iint_S \left(\frac{\partial F_1}{\partial z} \cos \beta - \frac{\partial F_1}{\partial y} \cos \gamma \right) ds \quad (3)$$

By we can prove that.

$$\oint_C F_2(x, y, z) dy = \iint_S \left(\frac{\partial F_2}{\partial x} \cos \gamma - \frac{\partial F_2}{\partial z} \cos \alpha \right) ds \quad (4)$$

$$\oint_C F_3(x, y, z) dz = \iint_S \left(\frac{\partial F_3}{\partial x} \cos \gamma - \frac{\partial F_3}{\partial y} \cos \alpha \right) ds \quad (5)$$

Add (3) (4) and (5) and using (1) we get

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot \hat{n} ds$$

1. verify Stoke's theorem for $\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$ taken around the rectangle bounded by the lines $x = \pm a, y = 0, y = b$.

Sol: By Stoke's theorem $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n} \, ds$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = \vec{k}(-2y-2y) - 4yx\vec{j} = -4yx\vec{j} - 4y\vec{k}$$

vector $\perp$ to xy -plane $\vec{n} = \vec{k}$

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, ds = \int_{y=0}^b \int_{x=-a}^a (-4yx\vec{k}) \cdot \vec{k} \, dx \, dy = -4 \int_{y=0}^b \int_{x=-a}^a y \, dx \, dy$$

Along AB; $y=0, dy=0$.

x varies from $-a$ to a .

$$\oint_C \vec{F} \cdot d\vec{r} = \int (x^2+y^2)dx - 2xydy = \int_{-a}^a x^2 dx = \left[\frac{x^3}{3} \right]_{-a}^a = \frac{2a^3}{3}$$

Along BC $x=a, dx=0$, y varies from 0 to b .

$$\oint_C \vec{F} \cdot d\vec{r} = \int (x^2+y^2)dx - 2xydy = -\int_0^b 2ay \, dy = -2a \left[\frac{y^2}{2} \right]_0^b = -ab^2$$

Along CD $y=b, dy=0$ x varies from a to $-a$.

$$\oint_C \vec{F} \cdot d\vec{r} = \int_a^{-a} (x^2+b^2)dx = \left[\frac{x^3}{3} + bx^2 \right]_a^{-a} = -\frac{a^3}{3} + ab^2 - \left(\frac{a^3}{3} + ab^2 \right) = -\frac{2a^3}{3} - 2ab^2$$

Along DA $x=-a, dx=0$, y varies from b to 0 .

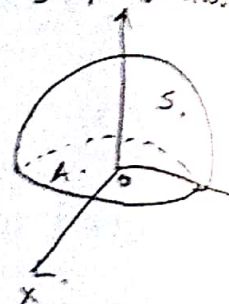
$$\oint_C \vec{F} \cdot d\vec{r} = \int_b^0 2ay \, dy = a \left[\frac{y^2}{2} \right]_b^0 = -\frac{ab^2}{2}$$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CD} \vec{F} \cdot d\vec{r} + \int_{DA} \vec{F} \cdot d\vec{r} = \frac{2a^3}{3} - ab^2 - \frac{2a^3}{3} - 2ab^2 - \frac{ab^2}{2} = -\frac{5ab^2}{2}$$

Stoke's theorem is verified

over the upper half surface $x^2+y^2+z^2=1$, bounded by its projection on xy -plane.

Sol: The projection of the upper half of given sphere on the xy -plane ($z=0$) is the circle c ($x^2+y^2=1$)



$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C (2x-y)dx - yz^2dy - y^2zdz$$

put $x = \cos\theta, y = \sin\theta$
 $dx = -\sin\theta d\theta$

$$= \oint_C (2x-y)dx$$

$$= \int_0^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta d\theta) = \int_0^{2\pi} (-\sin 2\theta + \sin^2\theta) d\theta$$

$$= 4 \int_0^{\pi/2} \frac{1-\cos 2\theta}{2} d\theta = 4 \left[\frac{\theta}{2} - \frac{\sin 2\theta}{2} \right]_0^{\pi/2} = 4 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \pi$$

now $\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$

$$= (-2yz + 2yz)\vec{i} + 0\vec{j} + \vec{k}(0+1) = \vec{k}$$

$$\int \text{curl } \vec{F} \cdot \vec{n} \, dS = \int \vec{F} \cdot \vec{n} \, dS = \iint_A \vec{k} \cdot \vec{n} \, \frac{dx dy}{|\vec{n} \cdot \vec{k}|}$$

where A is projection of S on xy -plane and

$$= \int dx dy \text{ area of circle } c = \pi r^2 \quad [r=1] \\ = \pi \quad \text{--- (2)}$$

$$dS = \frac{dx dy}{|\vec{n} \cdot \vec{k}|}$$

hence the Stokes's theorem is verified from the equality of (i) and (ii)

5) Apply Stokes theorem for calculation $\int_C \vec{F} \cdot d\vec{s}$ where

$\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$ and 'C' is the curve of intersection

$$x^2 + y^2 + z^2 = a^2, \quad x + z = a.$$

Given $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$ and

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = \vec{i}(0-1) + \vec{j}(0-1) + \vec{k}(0-1) = -\vec{i} - \vec{j} - \vec{k}$$

$\vec{N}$ = unit normal to the plane $(x+z=a)$. $= \frac{\nabla \phi}{|\nabla \phi|} = \frac{\vec{i} + \vec{k}}{\sqrt{2}}$

By Stokes theorem. $\int_C \vec{F} \cdot d\vec{s} = \int_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS$

$$= \int_S (-\vec{i} - \vec{j} - \vec{k}) \cdot \frac{\vec{i} + \vec{k}}{\sqrt{2}} \, dS$$

$$= -\sqrt{2} \int_S dS = -\sqrt{2} (\text{area of } S)$$

$$= -\sqrt{2} (\text{area of given circle})$$

Now $r^2 = a^2 - p^2$ where 'p' is the

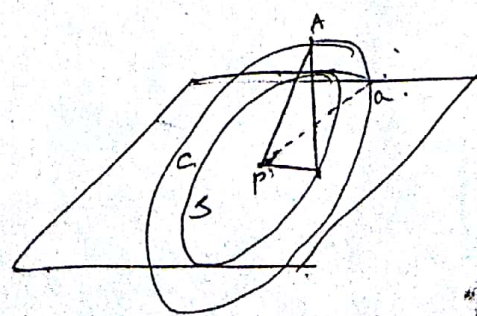
distance of centre (0,0,0) of the sphere

$= -\sqrt{2} \pi r^2$ where r is the radius of given circle

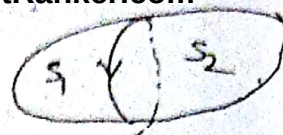
from the plane $(x+z-a=0)$. $= \left| \frac{-a}{\sqrt{1+1}} \right| = \left(\frac{a}{\sqrt{2}} \right)$

$$\therefore r^2 = a^2 - \frac{a^2}{2} = \frac{a^2}{2}$$

The required value of given integral is $-\sqrt{2} \left(\pi \frac{a^2}{2} \right) = -\frac{\pi a^2}{\sqrt{2}}$



Sol:- cut open the surface 'S' by any



plane and let S_1, S_2 denote it's upper and lower portions,

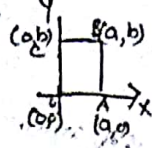
let C be the common line boundary, both these portions.

$$\int_S \nabla \times \vec{F} \cdot \vec{n} \, dS = \int_{S_1} \nabla \times \vec{F} \cdot \vec{n} \, dS + \int_{S_2} (\nabla \times \vec{F}) \cdot \vec{n} \, dS$$

$$= \int_C \vec{F} \cdot d\vec{r} - \int_C \vec{F} \cdot d\vec{r} = 0$$

on applying Stokes's theorem. The second integral is negative because it is traversed in a direction opposite to that of C .

vector field (i) $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$
the box bounded by the planes
 $x=0, x=a, y=0, y=b, z=0, z=c$, if
the face $z=0$ is cut.



Sol: $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$

Along OA, $y=0, dy=0, x$ varies from 0 to a

$$\int_0^a \vec{F} \cdot d\vec{r} = \int_0^a x^2 dx = \frac{a^3}{3}$$

Along AB, $x=a, dx=0, y$ varies from 0 to b

$$\int_0^b \vec{F} \cdot d\vec{r} = \int_0^b (a^2 - y^2)(0) + 2ay dy = \int_0^b 2ay dy = ab^2$$

Along BC, $y=b, dy=0, x$ varies from a to 0

$$\int_a^0 \vec{F} \cdot d\vec{r} = \int_a^0 (x^2 - b^2) dx = -\frac{a^3}{3} + ab^2$$

Along CO, $x=0, dx=0, y$ varies from b to 0

$$\int_b^0 \vec{F} \cdot d\vec{r} = \int_b^0 0 dy = 0$$

$$\oint \vec{F} \cdot d\vec{r} = \int_0^a \vec{F} \cdot d\vec{r} + \int_0^b \vec{F} \cdot d\vec{r} + \int_a^0 \vec{F} \cdot d\vec{r} + \int_b^0 \vec{F} \cdot d\vec{r}$$

$$= \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 + 0 = 2ab^2 \quad \text{L.H.S.}$$

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} ds$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix}$$

$$= \vec{i}(0) + \vec{j}(0-0) + \vec{k}(2y+2y)$$

$$= 4y\vec{k}$$

$$\int_0^a \int_0^b 4y\vec{k} \cdot \vec{k} dx dy = 4 \int_0^a \int_0^b y dx dy$$

$$= 4 \int_0^a \frac{b^2}{2} dy = 4 \cdot \frac{b^2}{2} \int_0^a dy$$

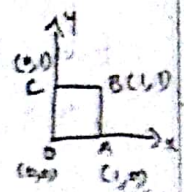
$$= 2ab^2 = \text{R.H.S.}$$

$$\therefore \oint \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} ds$$

Stokes's theorem is verified.

(ii) $\vec{F} = (z^2, 5x, 0)$ and $S: 0 \leq x \leq 1, 0 \leq y \leq 1, z=1$

$$\vec{F} = z^2\vec{i} + 5x\vec{j} + 0\vec{k} \quad z=1$$



$$\vec{F} = \vec{i} + 5x\vec{j}$$

Along OA, $y=0, dy=0, x$ varies from 0 to 1

$$\int_0^1 \vec{F} \cdot d\vec{r} = \int_0^1 1 dx = 1$$

Along AB, $x=1, dx=0, y$ varies from 0 to 1

$$\int_0^1 \vec{F} \cdot d\vec{r} = \int_0^1 5 dy = 5y \Big|_0^1 = 5$$

Along BC, $y=1, dy=0, x$ varies from 1 to 0

$$\int_1^0 \vec{F} \cdot d\vec{r} = \int_1^0 1 dx = -1$$

Along CO, $x=0, dx=0, y$ varies from 1 to 0

$$\int_1^0 \vec{F} \cdot d\vec{r} = \int_1^0 5(0) dy = 0$$

$$\oint \vec{F} \cdot d\vec{r} = \int_{OA} \vec{F} \cdot d\vec{r} + \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CO} \vec{F} \cdot d\vec{r}$$

$$= 1 + 5 - 1 + 0 = 5$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & 5x & 0 \end{vmatrix}$$

$$= \vec{i}(0) + \vec{j}(2z) + \vec{k}(5) = 2z\vec{j} + 5\vec{k}$$

along z-axis

$$\int_0^1 \int_0^1 s \, dx \, dy = 5$$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS$$

② Verify Stoke's theorem for a vector field defined by $\vec{F} = -y^3 \hat{i} + x^3 \hat{j}$ in the region $x^2 + y^2 \leq 1, z=0$.

Sol:- By Stoke's theorem $\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS$

Here $\vec{F} = -y^3 \hat{i} + x^3 \hat{j}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) + \hat{j}(0-0) + \hat{k}(3x^2+3y^2)$$

and 'S' is the circular disc

$$x^2 + y^2 \leq 1, z=0$$

for S:- $\vec{n} = \hat{k}, x = r \cos \theta, y = r \sin \theta$

$$dS = dx \, dy, z=0$$

$$dx \, dy = r \, dr \, d\theta$$

$$r = 0 \text{ to } 1, \theta = 0 \text{ to } 2\pi$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS$$

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS = \iint_S (3x^2 + 3y^2) \hat{k} \cdot \hat{k} \, dx \, dy$$

$$= \iint_S (3x^2 + 3y^2) \, dx \, dy$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^1 (3r^2 \cos^2 \theta + 3r^2 \sin^2 \theta) r \, dr \, d\theta$$

verification:- $\vec{F} = -y^3 \hat{i} + x^3 \hat{j}$

and 'C' is boundary of a circular disc $x^2 + y^2 \leq 1, z=0$

For C: $x = 1 \cdot \cos \theta, y = 1 \cdot \sin \theta$
 $dx = -\sin \theta \, d\theta, dy = \cos \theta \, d\theta$
 $z=0, dz=0, \theta = 0 \text{ to } 2\pi$

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C (-y^3 \hat{i} + x^3 \hat{j}) \cdot (dx \hat{i} + dy \hat{j})$$

$$= \oint_C -y^3 dx + x^3 dy$$

$$= \int_0^{2\pi} -\sin^3 \theta (-\sin \theta \, d\theta) + \cos^3 \theta \cos \theta \, d\theta$$

$$= \int_0^{2\pi} (\sin^4 \theta + \cos^4 \theta) d\theta = \int_0^{2\pi} (\sin^2 \theta + \cos^2 \theta - 2 \times \frac{1}{2} \sin^2 \theta) d\theta$$

$$= \int_0^{2\pi} (1 - \frac{1}{2} \sin^2 2\theta) d\theta = \int_0^{2\pi} (1 - \frac{1}{2} \frac{1-\cos 4\theta}{2}) d\theta$$

$$= \theta - \frac{1}{4} \theta + \frac{1}{4} \frac{\sin 4\theta}{4} \Big|_0^{2\pi}$$

$$= 2\pi - \frac{1}{4} 2\pi + \frac{1}{16} \sin 4(2\pi)$$

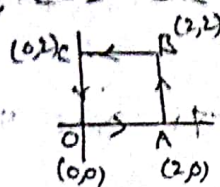
$$= \frac{3\pi}{2} + 0 = \frac{3\pi}{2}$$

Stoke's theorem for verified

$$\vec{F} = (y-z+2)\vec{i} + (yz+4)\vec{j} - xz\vec{k} \quad \text{where}$$

S is the surface of the cube

$x=0, y=0, z=0, x=2, y=2, z=2$ above the xy -plane.



Sol: By Stoke's theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, ds$$

Along $\vec{OA}$, $y=0, dy=0, z=2$, x varies from 0 to 2

$$\int_{OA} \vec{F} \cdot d\vec{r} = \int_0^2 (0-2+2) dx = 0$$

Along $\vec{AB}$, $x=2, dx=0, z=2, y$ varies from 0 to 2

$$\int_{AB} \vec{F} \cdot d\vec{r} = \int_0^2 (2y+4) dy = \left[y^2 + 4y \right]_0^2 = 4 + 8 = 12$$

Along $\vec{BC}$, $y=2, z=2, dy=0, x$ varies from 2 to 0

$$\int_{BC} \vec{F} \cdot d\vec{r} = \int_2^0 2 dx = 2x \Big|_2^0 = -4$$

Along $\vec{CO}$, $x=0, dx=0, z=2, y$ varies from 2 to 0

$$\int_{CO} \vec{F} \cdot d\vec{r} = \int_2^0 (2y+4) dy = \left[y^2 + 4y \right]_2^0 = -12$$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{OA} \vec{F} \cdot d\vec{r} + \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CO} \vec{F} \cdot d\vec{r}$$

$$= 0 + 12 - 4 - 12 = -4$$

$\therefore$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z+2 & yz+4 & -xz \end{vmatrix}$$

$$= \vec{i}(-y) + \vec{j}(-1+z) + \vec{k}(0-1)$$

$$= -y\vec{i} + (z-1)\vec{j} - \vec{k}$$

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, ds = \int_0^2 \int_0^2 (-y\vec{i} + (z-1)\vec{j} - \vec{k}) \cdot \vec{k} \, dx \, dy$$

$$= \int_0^2 \int_0^2 (-1) \, dx \, dy = - \int_0^2 dx \int_0^2 dy = -2 \int_0^2 dy = -2y \Big|_0^2 = -4$$

Stoke's theorem verified.

4) Evaluate $\oint_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = y\vec{i} + xz\vec{j} - yz\vec{k}$ and C is the circle $x^2 + y^2 = 4, z=1.5$.

By Stoke's theorem $\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, ds$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz & -yz \end{vmatrix} = \vec{i}(-3yz - 3xz) + \vec{j}(z^3 - 1) + \vec{k}(z^3 - 1)$$

$$(\nabla \times \vec{F}) \cdot \vec{n} \, ds = (\nabla \times \vec{F}) \cdot \vec{k} \, ds = (z^3 - 1) \, ds$$

Let $x = r \cos \theta, y = r \sin \theta, dz = 0, dxdy = r dr d\theta$

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, ds = \int_{r=0}^2 \int_{\theta=0}^{2\pi} \left(\left(\frac{3}{2} \right)^3 - 1 \right) r \, dr \, d\theta$$

$$= \frac{19}{8} \int_0^2 r \, dr \int_0^{2\pi} d\theta = \frac{19}{8} \left(\frac{r^2}{2} \right)_0^2 \cdot 2\pi$$

$$= \frac{19}{4} \times \frac{4}{2} \times \pi$$

$$= \frac{19}{2} \pi$$

Applying Stokes's theorem

$$\oint_C (yz dx + zx dy + xy dz) \text{ where}$$

$$\text{is the circle } x^2 + y^2 = 1, z = y^2.$$

Sol:- By Stokes's theorem.

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS.$$

$$\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}.$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= \vec{i}(x-z) + \vec{j}(y-y) + \vec{k}(z-z)$$

$$= \vec{0}.$$

$$\iint_S \vec{0} \cdot \vec{n} \, dS = 0.$$

6) If 'S' be the surface of the sphere $x^2 + y^2 + z^2 = 1$, prove that

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = 0$$

Sol:- Let S_1 and S_2 be the upper and lower portions of the sphere

$$x^2 + y^2 + z^2 = 1 \text{ by any plane.}$$

Let 'C' be the boundary for S_1 and S_2 .

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS = \iint_{S_1} (\nabla \times \vec{F}) \cdot \vec{n} \, dS + \iint_{S_2} (\nabla \times \vec{F}) \cdot \vec{n} \, dS$$

Reversing the direction of S_2 in the direction of S_1

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS = \int_C \vec{F} \cdot d\vec{r} - \int_C \vec{F} \cdot d\vec{r} = 0.$$

7) Prove that $\int_C (\vec{A} \times \vec{R}) \cdot d\vec{R} = 2\vec{A} \cdot \vec{R}$

$\vec{A}$ being any constant vector, and deduce that $\oint_C \vec{R} \times d\vec{R}$ is twice the vector area of the surface enclosed.

Sol:- By Stokes's theorem

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS$$

Here $\vec{F} = \vec{A} \times \vec{R}$, $\therefore \vec{A}$ is constant

$$\therefore \int_C (\vec{A} \times \vec{R}) \cdot d\vec{R} = \iint_S \nabla \times (\vec{A} \times \vec{R}) \cdot \vec{n} \, dS$$

$$\boxed{\nabla \times (\vec{A} \times \vec{R}) = (\nabla \cdot \vec{R})\vec{A} - (\nabla \cdot \vec{A})\vec{R} = 3\vec{A} - \vec{A} = 2\vec{A}}$$

Here $\vec{F} = \vec{A} \times \vec{R}$ where $\vec{A}$ is constant

$$\oint_C (\vec{A} \times \vec{R}) \cdot d\vec{R} = \int_S \nabla \times (\vec{A} \times \vec{R}) \cdot \vec{n} \, dS$$

$$\oint_C \vec{A} \cdot (\vec{R} \times d\vec{R}) = \int_S (\nabla \cdot \vec{R})\vec{A} \cdot \vec{n} \, dS$$

$$\vec{A} \cdot \oint_C \vec{R} \times d\vec{R} = \int_S (3\vec{A} \cdot \vec{n}) \, dS$$

$$= \int_S 2\vec{A} \cdot \vec{n} \, dS$$

$$= \int_S \vec{A} \cdot 2\vec{n} \, dS = \vec{A} \cdot \oint_C \vec{R} \times d\vec{R}$$

$$\oint_C \vec{R} \times d\vec{R} = \int_S 2\vec{n} \, dS$$

$$\oint_C \vec{R} \times d\vec{R} = 2 \int_S d\vec{S}$$

Use Stokes's theorem to prove that
(i) $\text{curl}(\text{grad } \phi) = 0$ (ii) $\text{div curl } \vec{F} = 0$

Sol: Let $\vec{F} = \nabla \phi$, so by Stokes's

$$\int_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS = \int_S (\nabla \times \nabla \phi) \cdot \vec{n} \, dS$$

$$= \oint_C \nabla \phi \cdot d\vec{r}$$

$$= \oint_C d\phi = 0$$

$$\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = d\phi$$

also circulation $\oint_C d\phi = 0$ along

closed curve being initial and

points are one and the same.

$$\int_S (\nabla \times \nabla \phi) \cdot \vec{n} \, dS = 0 \text{ over the}$$

surface S , which implies that

$$\text{curl}(\text{grad } \phi) = 0$$

Let V be any volume enclosed

by a closed curve S . Then by

Gauss's divergence theorem

$$\int_V \nabla \cdot (\text{curl } \vec{F}) \, dv = \int_S (\text{curl } \vec{F}) \cdot \vec{n} \, dS \quad \text{--- (1)}$$

divide the surface S into two
portions S_1 and S_2 by a closed
curve C

$$\int_S (\text{curl } \vec{F}) \cdot \vec{n} \, dS$$

$$\int_{S_1} (\text{curl } \vec{F}) \cdot \vec{n} \, dS + \int_{S_2} (\text{curl } \vec{F}) \cdot \vec{n} \, dS$$

$$= \int_C \vec{F} \cdot d\vec{r} - \int_C \vec{F} \cdot d\vec{r} = 0$$



By Stokes's theorem, the directions
along the boundaries of S_1 and S_2
are opposite

$$\textcircled{1} \Rightarrow \int_V \nabla \cdot (\text{curl } \vec{F}) \, dv = 0$$

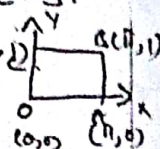
since this is true for all volume
elements V , we have

$$\nabla \cdot \text{curl } \vec{F} = 0 \Rightarrow \text{div}(\text{curl } \vec{F}) = 0$$

9) Evaluate $\oint_C (\sin z \, dx - \cos x \, dy + \sin y \, dz)$
where C is the boundary of the
rectangle $0 \leq x \leq \pi$, $0 \leq y \leq 1$, $z = 3$.

Sol: Along OA, $y=0, dy=0$

x varies from 0 to π .



$$\int_{OA} \vec{F} \cdot d\vec{r} = \int_0^\pi \sin 3 \, dx = \sin 3 \cdot \pi$$

Along AB, $z=3, dz=0$, y varies from 0 to 1

$$\int_{AB} \vec{F} \cdot d\vec{r} = \int_0^1 -\cos \pi \, dy = -\cos \pi = 1$$

Along BC, $y=1, dy=0$, x varies from π to 0

$$\int_{BC} \vec{F} \cdot d\vec{r} = \int_\pi^0 -\sin 3 \, dx = -\sin 3 \cdot \pi$$

Along CO, $x=0, dx=0$, y varies from 1 to 0

$$\int_{CO} \vec{F} \cdot d\vec{r} = \int_1^0 -\cos 0 \, dy = -y \Big|_1^0 = 1$$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{OA} \vec{F} \cdot d\vec{r} + \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CO} \vec{F} \cdot d\vec{r}$$

$$= \sin 3/\pi + 1 - \sin 3/\pi + 1$$

$(\nabla \times \vec{F}) \cdot \vec{n} \, dS$ where $\vec{F} = y\vec{i} + (x - z^2)\vec{j}$
 $= \nabla \times \vec{F}$

and 'S' is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy-plane.

Sol: By Stokes's theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x - z^2 & -xy \end{vmatrix}$$

$$= \vec{i}(-z + 2z) + \vec{j}(0 + y) + \vec{k}(1 - z^2 - x)$$

$$= 2z\vec{i} + y\vec{j} - z^2\vec{k}$$

above xy-plane of the sphere
 $\vec{n}$ is unit outward normal along z-axis i.e. $\vec{k}$. $x^2 + y^2 = a^2$ ($\because z=0$)

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS = \int_0^{2\pi} \int_0^{2\pi} (2a\vec{i} + y\vec{j} - z^2\vec{k}) \cdot \vec{k} \, dndy$$

$$= \int_0^{2\pi} \int_0^{2\pi} -z^2 \, dndy \quad [z=0]$$

$$= 0$$

Evaluate $\int (\nabla \times \vec{v}) \cdot d\vec{s}$ over the surface of the paraboloid $z = 1 - x^2 - y^2$, $z \geq 0$ where $\vec{v} = y\vec{i} + z\vec{j} + x\vec{k}$

Sol: By Stokes's theorem

$$\oint_C \vec{v} \cdot d\vec{r} = \iint_S (\nabla \times \vec{v}) \cdot d\vec{s}$$

$$\nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$= -\vec{i} - \vec{j} - \vec{k} = -(\vec{i} + \vec{j} + \vec{k})$$

$\vec{n} = \vec{k}$, for $z \geq 0$

$x^2 + y^2 = 1$, $x = r \cos \theta$, $y = r \sin \theta$
 $dx = -r \sin \theta \, d\theta$, $dy = r \cos \theta \, d\theta$

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS = \int_0^{2\pi} \int_0^1 -(\vec{i} + \vec{j} + \vec{k}) \cdot \vec{k} \, r \, dr \, d\theta$$

$$= - \int_0^{2\pi} \int_0^1 r \, dr \, d\theta = - \int_0^{2\pi} \left[\frac{r^2}{2} \right]_0^1 d\theta$$

$$= - \int_0^{2\pi} \frac{1}{2} d\theta = - \frac{1}{2} \int_0^{2\pi} d\theta = - \frac{1}{2} [2\pi]_0^{2\pi} = -\pi$$

'c' is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

sol:- $\vec{F} = -y^3 \hat{i} + x^3 \hat{j}$ Let, $x = a \cos \theta$, $y = b \sin \theta$
 $dx = -a \sin \theta d\theta$, $dy = b \cos \theta d\theta$.

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_C (-y^3 \hat{i} + x^3 \hat{j}) (dx \hat{i} + dy \hat{j}) = \int_C -y^3 dx + x^3 dy \\ &= \int_0^{2\pi} (-b^3 \sin^3 \theta) (-a \sin \theta d\theta) + (a^3 \cos^3 \theta) (b \cos \theta d\theta) \\ &= ab^3 \int_0^{2\pi} \sin^4 \theta d\theta + a^3 b \int_0^{2\pi} \cos^4 \theta d\theta \\ &= ab^3 \cdot 4 \int_0^{\pi/2} \sin^4 \theta d\theta + a^3 b \cdot 4 \int_0^{\pi/2} \cos^4 \theta d\theta \\ &= ab^3 \cdot 4 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} + a^3 b \cdot 4 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi ab}{4} (a^2 + b^2) \end{aligned}$$

curl $\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = \hat{k} (3x^2 + 3y^2)$

$$\int_S (\nabla \times \vec{F}) \cdot \vec{n} ds = \int_S \hat{k} (3x^2 + 3y^2) \cdot \hat{k} ds = 3 \int_S (x^2 + y^2) ds$$

polar coordinates be $x = a \cos \theta$, $y = b \sin \theta$

$dx dy = ab r dr d\theta$

'r' varies from 0 to $\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}$
' θ ' varies from 0 to 2π

$$\int_S (\nabla \times \vec{F}) \cdot \vec{n} ds = 3 \int_0^{2\pi} \int_0^{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}} (a^2 r^2 \cos^2 \theta + b^2 r^2 \sin^2 \theta) r ab dr d\theta$$

$$= 3a^3b \left[\frac{r^4}{4} \right]_0^1 \int_0^{2\pi} \cos^2 \theta d\theta + 3ab^3 \left[\frac{r^4}{4} \right]_0^1 \int_0^{2\pi} \sin^2 \theta d\theta$$

$$= 3 \frac{a^3b}{4} \int_0^{\pi/2} \cos^2 \theta d\theta + 3 \frac{ab^3}{4} \int_0^{\pi/2} \sin^2 \theta d\theta$$

$$= 3a^3b \cdot \frac{1}{2} \cdot \frac{\pi}{2} + 3ab^3 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3ab\pi}{4} (a^2 + b^2)$$

L.H.S = R.H.S Stoke's theorem verified.

2.
Q) For a solenoidal vector $\vec{F}$, show that

$$\text{curl curl curl curl } \vec{F} = \nabla^4 \vec{F}$$

Sol: A vector $\vec{F}$ is said to be solenoidal if $\nabla \cdot \vec{F} = 0$

$$\text{we know that } \nabla \times (\nabla \times \vec{F}) = (\nabla \cdot \vec{F}) \nabla - (\nabla \cdot \nabla) \vec{F}$$

$$= (\nabla \cdot \vec{F}) \nabla - \nabla^2 \vec{F}$$

$$\nabla \times (\nabla \times \vec{F}) = -\nabla^2 \vec{F} \quad [\because \nabla \cdot \vec{F} = 0]$$

$$\nabla \times \nabla \times [\nabla \times (\nabla \times \vec{F})] = \nabla \times [\nabla \times (-\nabla^2 \vec{F})]$$

$$= \nabla \cdot (-\nabla^2 \vec{F}) \nabla - (\nabla \cdot \nabla) (-\nabla^2 \vec{F})$$

$$= \nabla [\nabla \cdot (\nabla \times (\nabla \times \vec{F}))] + \nabla^4 \vec{F}$$

$$= \nabla [\text{div} (\text{curl} (\nabla \times \vec{F}))] + \nabla^4 \vec{F}$$

$$= \nabla^4 \vec{F} \quad [\because \text{div} (\text{curl} (\nabla \times \vec{F})) = 0]$$

Gauss's Divergence Theorem: Let S be a closed surface enclosed a volume V . If $\vec{F}$ is a continuously differentiable vector point function, then $\int_V \text{div } \vec{F} \, dv = \int_S \vec{F} \cdot \vec{N} \, ds$ where $\vec{N}$ is the outward drawn unit normal vector at any point of S .

Cartesian form: Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ and $\vec{N} = \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}$ where $\cos\alpha, \cos\beta, \cos\gamma$ are the d.c.s of $\vec{N}$.

$$\vec{F} \cdot \vec{N} = F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma$$

$$\text{Also } \text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Hence the divergence theorem can be written as.

$$\begin{aligned} \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx \, dy \, dz &= \int_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma) \, ds \\ &= \iiint_V (F_1 \, dy \, dz + F_2 \, dz \, dx + F_3 \, dx \, dy) \end{aligned}$$

Proof: Let S be a closed surface. Let us choose the coordinate axes so that any line parallel to the coordinate axes cuts S in at most two points.

Let R be the projection of S on xy -plane.

S_1 and S_2 be the lower and upper parts of S .

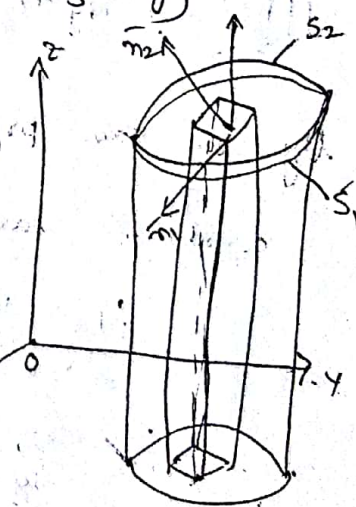
$z = f(x, y)$ and $z = g(x, y)$ be the equations of S_1 and S_2 .

The region can be put in the form $f(x, y) \leq z \leq g(x, y)$.

$$\int_V \frac{\partial F_3}{\partial z} \, dv = \iiint_V \frac{\partial F_3}{\partial z} \, dz \, dy \, dx = \iint_R \left[\int_{z=f(x,y)}^{z=g(x,y)} \frac{\partial F_3}{\partial z} \, dz \right] dx \, dy$$

$$= \iint_R [F_3(x, y, g) - F_3(x, y, f)] \, dx \, dy$$

$$= \iint_R [F_3(x, y, g) - F_3(x, y, f)] \, dx \, dy$$



$$\iiint_V \frac{\partial F_3}{\partial z} dx dy dz = \iint_R F_3(x, y, z) dx dy \quad (1)$$

For the upper part S_2 , $z = g(x, y)$, $dx dy = ds \cos \gamma$
 $= (\vec{n} \cdot \vec{e}_3) ds$

Since the normal to S_2 makes an acute angle γ with $\vec{e}_3$.

$$\iint_R F_3(x, y, z) dx dy = \iint_{S_2} F_3(x, y, z) (\vec{n} \cdot \vec{e}_3) ds \quad (2)$$

For the lower portion S_1 , $z = f(x, y)$, $dx dy = -ds \cos \gamma$
 $= -(\vec{n} \cdot \vec{e}_3) ds$

Since the normal to S_1 makes an obtuse angle γ with $\vec{e}_3$.

$$\iint_R F_3(x, y, z) dx dy = - \iint_{S_1} F_3 (\vec{n} \cdot \vec{e}_3) ds \quad (3)$$

from (1) (2) and (3).

$$\iiint_V \frac{\partial F_3}{\partial z} dx dy dz = \iint_{S_2} F_3 (\vec{n} \cdot \vec{e}_3) ds + \iint_{S_1} F_3 (\vec{n} \cdot \vec{e}_3) ds = \iint_S F_3 (\vec{n} \cdot \vec{e}_3) ds$$

My use can prove that, by projecting S on the other coordinate planes

$$\iiint_V \frac{\partial F_1}{\partial x} dx dy dz = \iint_S F_1 (\vec{n} \cdot \vec{e}_1) ds \quad (I)$$

$$\iiint_V \frac{\partial F_2}{\partial y} dx dy dz = \iint_S F_2 (\vec{n} \cdot \vec{e}_2) ds \quad (II)$$

Adding (I) (II) and (III)

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx dy dz = \iint_S (F_1 \vec{e}_1 + F_2 \vec{e}_2 + F_3 \vec{e}_3) \cdot \vec{n} ds$$

$$\iiint_V \text{div } \vec{F} dv = \iint_S \vec{F} \cdot \vec{n} ds$$

$\square$

Verify divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ taken over a rectangular parallelepiped $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c$.

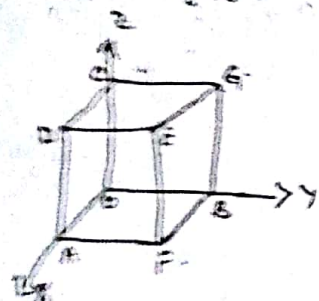
$\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ taken over a rectangular parallelepiped $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c$.

Sol:- $\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy) = 2x + 2y + 2z$

$$\iiint_V (\nabla \cdot \vec{F}) dV = \iiint_V 2(x + y + z) dxdydz = 2 \int_{z=0}^c \int_{y=0}^b \int_{x=0}^a (x + y + z) dxdydz$$

$$= 2 \int_{z=0}^c \int_{y=0}^b \left(\frac{x^2}{2} + xy + xz \right) \Big|_0^a dy dz = 2 \int_{z=0}^c \int_{y=0}^b \left(\frac{a^2}{2} + ay + az \right) dy dz$$

$$= 2 \int_{z=0}^c \left[\frac{a^2}{2}y + \frac{ay^2}{2} + azy \right]_0^b dz = 2 \int_{z=0}^c \left[\frac{a^2b}{2} + \frac{ab^2}{2} + abz \right] dz$$



$$= 2 \left[\frac{a^2b}{2}z + \frac{ab^2}{2}z + ab \frac{z^2}{2} \right]_0^c = abc(a + b + c)$$

To calculate $\iiint_V \vec{F} \cdot \vec{n} dS$ over the six faces of the rectangular parallelepiped over the face

over the face DEFA, $\vec{n} = \vec{i}$, $x = a$, $ds = dy dz$.

$$\iiint_{DEFA} \vec{F} \cdot \vec{n} dS = \int_{z=0}^c \int_{y=0}^b [(a^2 - yz)\vec{i} + (y^2 - za)\vec{j} + (z^2 - ay)\vec{k}] \cdot \vec{i} dy dz$$

$$= \int_{z=0}^c \int_{y=0}^b (a^2 - yz) dy dz = \int_{z=0}^c \left[a^2y - \frac{yz^2}{2} \right]_0^b dz = \int_{z=0}^c (a^2b - \frac{bz^2}{2}) dz$$

$$= \left[a^2bz - \frac{b^2z^3}{6} \right]_0^c = a^2bc - \frac{b^2c^3}{6}$$

over the face BGCD, $\vec{n} = -\vec{i}$, $ds = dy dz$, $x = 0$.

$$\iiint_{BGCD} \vec{F} \cdot \vec{n} dS = \int_{z=0}^c \int_{y=0}^b yz dy dz = \int_{z=0}^c \left[\frac{y^2}{2} z \right]_0^b dz = \int_{z=0}^c \frac{b^2}{2} z dz$$

$$= \left[\frac{b^2}{2} \frac{z^2}{2} \right]_0^c = \frac{b^2c^2}{4}$$

$$\begin{aligned} \iiint_{BGEF} \vec{F} \cdot \vec{n} ds &= \int_{z=0}^c \int_{x=0}^a (b^2 - x^2) dx dz = \int_{z=0}^c \left[b^2 x - \frac{x^3}{3} \right]_0^a dz = \int_{z=0}^c \left(b^2 a - \frac{a^3}{3} \right) dz \\ &= \left[b^2 a z - \frac{a^3}{3} z \right]_0^c = ab^2 c - \frac{a^3 c}{3} \end{aligned}$$

4) over the face OADC, $\vec{n} = -\vec{j}$, $y = 0$

$$\iiint_{OADC} \vec{F} \cdot \vec{n} ds = \int_{z=0}^c \int_{x=0}^a -x^2 dx dz = - \int_{z=0}^c \left[\frac{x^3}{3} \right]_0^a dz = - \int_{z=0}^c \frac{a^3}{3} dz = - \frac{a^3}{3} z \Big|_0^c = - \frac{a^3 c}{3}$$

5) over the face GCDE, $\vec{n} = \vec{k}$, $z = c$, $ds = dx dy$

$$\iiint_{GCDE} \vec{F} \cdot \vec{n} ds = \int_{y=0}^b \int_{x=0}^a (c^2 - xy) dx dy = \int_{y=0}^b \left[c^2 x - \frac{xy^2}{2} \right]_0^a dy = \int_{y=0}^b \left(c^2 a - \frac{a^2 y}{2} \right) dy$$

6) over the face AFBO, $\vec{n} = -\vec{k}$, $z = 0$, $ds = dx dy$

$$\begin{aligned} \iiint_{AFBO} \vec{F} \cdot \vec{n} ds &= \int_{y=0}^b \int_{x=0}^a -xy dx dy = - \int_{y=0}^b \left[\frac{x^2 y}{2} \right]_0^a dy = - \int_{y=0}^b \frac{a^2 y}{2} dy = - \frac{a^2}{2} \int_{y=0}^b y dy \\ &= - \frac{a^2}{2} \left[\frac{y^2}{2} \right]_0^b = - \frac{a^2 b^2}{4} \end{aligned}$$

Add the Integrals over the side faces

$$\begin{aligned} \oint \vec{F} \cdot \vec{n} ds &= ab^2 c - \frac{c^3 a}{3} + \frac{c^3 b}{3} + b^2 c a - \frac{a^3 c}{3} + c^2 ab - \frac{a^2 b^2}{4} + \frac{a^3}{4} \\ &= ab^2 c + b^2 c a + c^2 ab - \frac{c^3 a}{3} + \frac{c^3 b}{3} - \frac{a^3 c}{3} + \frac{a^3}{4} + \frac{a^2 b^2}{4} \\ &= abc(a+b+c) \end{aligned}$$

$$\text{Hence } \oint \vec{F} \cdot \vec{n} ds = \iiint_V (\nabla \cdot \vec{F}) dv$$

by the surface 'S'.

Sol:- By Gauss divergence theorem $\int_S \vec{r} \cdot \vec{n} \, ds = \iiint_V \text{div } \vec{r} \, dv$

$\nabla \cdot \vec{r} = \text{div } \vec{r} = 3$ where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$\int_S \vec{r} \cdot \vec{n} \, ds = \iiint_V 3 \, dv = 3V$

3) For any closed surface 'S', $\int_S (\nabla \times \vec{F}) \cdot \vec{n} \, ds = 0$

Sol:- By divergence theorem $\int_S (\nabla \times \vec{F}) \cdot \vec{n} \, ds = \iiint_V \text{div}(\text{curl } \vec{F}) \, dv$

where 'V' is the volume enclosed by 'S'.

But we know that $\text{div}(\text{curl } \vec{F}) = 0$.

$\therefore \int_S (\nabla \times \vec{F}) \cdot \vec{n} \, ds = 0$

4) use divergence theorem to show that $\int_S \nabla r^2 \cdot \vec{n} \, ds = 6V$

Sol:- $\nabla r^2 = \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) r^2 = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$

$= 2\vec{r}$

$= 2(x\vec{i} + y\vec{j} + z\vec{k}) = 2\vec{r}$

$\nabla \cdot \nabla r^2 = \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot (2x\vec{i} + 2y\vec{j} + 2z\vec{k})$

$= 2 + 2 + 2 = 6$

$\int_S \nabla r^2 \cdot \vec{n} \, ds = \iiint_V \nabla \cdot \nabla r^2 \, dv = \iiint_V 6 \, dv = 6V$

5) If $\vec{F} = ax\vec{i} + by\vec{j} + cz\vec{k}$ a, b, c are constants, show that

$\int_S \vec{F} \cdot \vec{n} \, ds = \frac{4}{3} \pi (a+b+c)$ where 'S' is the surface of a unit sphere.

Sol:- By Gauss divergence theorem $\int_S \vec{F} \cdot \vec{n} \, ds = \iiint_V \text{div } \vec{F} \, dv$

$\text{div } \vec{F} = \frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz) = a+b+c$

where 'v' is the volume of the unit sphere $V = \frac{4}{3}\pi r^3$

$$\therefore \int_S \vec{F} \cdot \vec{n} \, ds = \frac{4}{3}\pi (a+b+c)$$

6) $\oint_S (a^2x^2 + b^2y^2 + c^2z^2)^{1/2} \, ds$ over the ellipsoid $ax^2 + by^2 + cz^2 = 1$

Sol:- By G.D.T $\int_S \vec{F} \cdot \vec{n} \, ds = \iiint_V \text{div } \vec{F} \, dv$

Here $\vec{F} \cdot \vec{n} = (a^2x^2 + b^2y^2 + c^2z^2)^{1/2}$

and 's' is the surface of an ellipsoid $ax^2 + by^2 + cz^2 = 1$

$\vec{n}$ = unit outward normal $= \frac{\nabla \phi}{|\nabla \phi|}$ where $\phi = ax^2 + by^2 + cz^2 - 1$

$$= \frac{2ax\vec{i} + 2by\vec{j} + 2cz\vec{k}}{\sqrt{4a^2x^2 + 4b^2y^2 + 4c^2z^2}}$$

Given $\vec{F} = ax\vec{i} + by\vec{j} + cz\vec{k}$

$\text{div } \vec{F} = a+b+c$

$$= \frac{ax\vec{i} + by\vec{j} + cz\vec{k}}{(a^2x^2 + b^2y^2 + c^2z^2)^{1/2}}$$

and 'v' is the volume of an ellipsoid $ax^2 + by^2 + cz^2 = 1$

$$\int_S (a^2x^2 + b^2y^2 + c^2z^2)^{-1/2} \, ds = \iiint_V (a+b+c) \, dv = (a+b+c) V$$

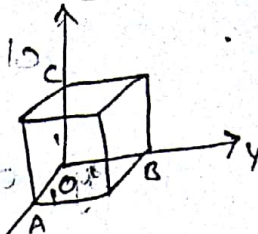
$$= (a+b+c) \frac{4\pi}{3} \frac{1}{\sqrt{abc}}$$

$$\therefore \vec{F} \cdot \vec{n} = (ax\vec{i} + by\vec{j} + cz\vec{k}) \cdot \frac{(ax\vec{i} + by\vec{j} + cz\vec{k})}{(a^2x^2 + b^2y^2 + c^2z^2)^{1/2}}$$

$$= \frac{(a^2x^2 + b^2y^2 + c^2z^2)}{(a^2x^2 + b^2y^2 + c^2z^2)^{1/2}} = (a^2x^2 + b^2y^2 + c^2z^2)^{1/2}$$

7) Evaluate $\int_S \vec{F} \cdot \vec{n} \, ds$ where 'S' is the surface of the cube bounded by $x=0, x=1, y=0, y=1, z=0, z=1$

and $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$



Sol: By Gauss divergence theorem $\int_S \vec{F} \cdot \vec{n} \, ds = \iiint_V \text{div} \vec{F} \, dv$

$$\text{div} \vec{F} = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)$$

$$= 4z - 2y + y = 4z - y$$

$$\int_S \vec{F} \cdot \vec{n} \, ds = \int_0^1 \int_0^1 \int_0^1 (4z - y) \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^1 (2z - \frac{y}{2}) \, dy \, dx$$

$$= \int_0^1 (2y - \frac{y^2}{2}) \, dx = \int_0^1 (2 - \frac{1}{2}) \, dx$$

$$= \frac{3}{2} \int_0^1 dx = \frac{3}{2}$$

8) Evaluate $\int_S (a^2x^2 + b^2y^2 + c^2z^2)^{-1/2} \, ds$, where 'S' is the surface of the ellipsoid $ax^2 + by^2 + cz^2 = 1$.

Sol: Taking $\phi = ax^2 + by^2 + cz^2 - 1 = 0$

$$\nabla \phi = 2ax\vec{i} + 2by\vec{j} + 2cz\vec{k}$$

unit vector normal to ellipsoid $= \vec{N} = \frac{\nabla \phi}{|\nabla \phi|}$

$$= \frac{ax\vec{i} + by\vec{j} + cz\vec{k}}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}}$$

since $\vec{F} \cdot \vec{N} = (a^2x^2 + b^2y^2 + c^2z^2)^{-1/2}$

$$\vec{F} \cdot (ax\vec{i} + by\vec{j} + cz\vec{k}) = (a^2x^2 + b^2y^2 + c^2z^2)^{-1/2}$$

$$\frac{1}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}}$$

$$\vec{F} = (ax\vec{i} + by\vec{j} + cz\vec{k})$$

$$ax^2 + by^2 + cz^2 = 1$$

clearly $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$

By divergence theorem

$$\begin{aligned} \int_S \vec{F} \cdot \vec{n} \, ds &= \int_V \text{div } \vec{F} \, dV = \int_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) \right] dV = 3 \int_V dV = 3V \\ &= 3 \cdot \frac{4\pi}{3} \frac{1}{\sqrt{abc}} = \frac{4\pi}{\sqrt{abc}} \quad [\because \text{vol of ellipsoid} = \frac{4\pi}{3} \frac{1}{\sqrt{abc}}] \end{aligned}$$

9) Apply Divergence theorem to evaluate $\int (lx^2 + my^2 + nz^2) \, ds$ taken over the sphere $(x-a)^2 + (y-b)^2 + (z-c)^2 = \rho^2$; l, m, n being the direction cosines of the external normal to the sphere.

Sol:- The parametric equations of the sphere are

$$x = a + \rho \sin \theta \cos \phi, \quad y = b + \rho \sin \theta \sin \phi, \quad z = c + \rho \cos \theta$$

and to cover the whole sphere, θ varies from 0 to π and ϕ varies from 0 to 2π .

$$\int_S (lx^2 + my^2 + nz^2) \, ds = \int_S (x^2\vec{i} + y^2\vec{j} + z^2\vec{k}) \cdot \vec{n} \, ds$$

$$= \int_V \text{div} (x^2\vec{i} + y^2\vec{j} + z^2\vec{k}) \, dV = 2 \int_V (x + y + z) \, dV$$

$$= 2 \int_0^{2\pi} \int_0^\pi \int_0^\rho [(a+b+c) + \rho(\sin \theta \cos \phi + \sin \theta \sin \phi + \cos \theta)] \times \rho^2 \sin \theta \, d\rho \, d\theta \, d\phi$$

$$= 2(a+b+c) \frac{\rho^3}{3} \Big|_0^\rho \int_0^\pi (1 - \cos \theta) \, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{8\pi}{3} (a+b+c) \rho^3$$

surface bounding the region $x^2 + y^2 = 4$, $z = 0$, and $z = 3$

sol: By divergence theorem $\int_S \vec{F} \cdot \vec{n} \, dS = \int_V \text{div } \vec{F} \, dV$

$$\nabla \cdot \vec{F} = \frac{d}{dx}(4x) + \frac{d}{dy}(-2y^2) + \frac{d}{dz}(2z)$$

$$\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$$

$$= 4 - 4y + 2z$$

$$\int_S \vec{F} \cdot \vec{n} \, dS = \iiint_V (4 - 4y + 2z) \, dx \, dy \, dz$$

$$= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - 4y + 2z) \, dx \, dy \, dz$$

$$y^2 = 4 - x^2$$

$$y = \pm \sqrt{4 - x^2}$$

$$x^2 = 2$$

$$x = \pm 2$$

$$= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4z - 4yz + 2 \frac{z^2}{2} \right]_0^3 \, dx \, dy$$

$$= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4z - 4yz + \frac{z^2}{2} \right]_0^3 \, dx \, dy = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [12 - 12y + 9] \, dx \, dy$$

$$= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21 - 12y) \, dx \, dy = \int_{-2}^2 \left[21y - \frac{12}{2} y^2 \right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \, dx$$

$$= 42 \int_{-2}^2 \sqrt{4-x^2} \, dx = 84 \int_0^2 \sqrt{4-x^2} \, dx$$

$$= 84 \left[\frac{x}{2} \sqrt{4-x^2} + \frac{4}{2} \sin^{-1} \frac{x}{2} \right]_0^2$$

$$= 84\pi$$

where $\vec{F} = 2x^2y \hat{i} - y^2 \hat{j} + 4xz \hat{k}$ and
 (ii) Evaluate $\oint_S \vec{F} \cdot \vec{n} \, ds$ where $\vec{F}$ is the closed surface of the region in the first octant
 bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x=0, x=2,$
 $y=0$ and $z=0$.

Sol: By Gauss divergence theorem $\oint_S \vec{F} \cdot \vec{n} \, ds = \iiint_V \text{div} \vec{F} \, dV$

$$= \int_{x=0}^2 \int_{y=0}^3 \int_{z=0}^{\sqrt{9-y^2}} (4xy - 2y + 8xz) \, dz \, dy \, dx$$

$x = 0 \text{ to } 2$
 $z = \sqrt{9-y^2}$
 $z = 0 \text{ to } \sqrt{9-y^2}$
 $y = 0 \text{ to } 3$

$$= \int_{y=0}^3 \int_{z=0}^{\sqrt{9-y^2}} \left[4 \frac{x^2}{2} y - 2xy + 8 \frac{x^2}{2} z \right]_0^2 dy \, dz = \int_{y=0}^3 \int_{z=0}^{\sqrt{9-y^2}} (2x^2y - 2xy + 4x^2z) \, dz \, dy$$

$$= \int_{y=0}^3 \int_{z=0}^{\sqrt{9-y^2}} (8y - 4y + 16z) \, dz \, dy = \int_{y=0}^3 \int_{z=0}^{\sqrt{9-y^2}} (4y + 16z) \, dz \, dy$$

$$= \int_{y=0}^3 \left[4yz + 16 \frac{z^2}{2} \right]_0^{\sqrt{9-y^2}} dy = \int_{y=0}^3 [4y\sqrt{9-y^2} + 8(9-y^2)] dy$$

$$\begin{aligned} 9-y^2 &= t \\ -2y \, dy &= dt \\ y \, dy &= -\frac{dt}{2} \end{aligned}$$

$$= 4 \int_{y=0}^3 [y\sqrt{9-y^2} + 2(9-y^2)] dy$$

$$= 4 \left[-\frac{1}{2} (9-y^2)^{3/2} \cdot \frac{2}{3} + 18y - \frac{2}{3} y^3 \right]_0^3$$

$$= 4 [54 - 18] - 4 \left[-\frac{1}{2} (9-9)^{3/2} \cdot \frac{2}{3} \right]$$

$$= 4 \times 36 + 36$$

$$= 144 + 36$$

$$= 180$$

Use the divergence theorem to evaluate $\iint_S \vec{F} \cdot \vec{n} ds$ where

$\vec{F} = x^3 \vec{i} + y^3 \vec{j} + z^3 \vec{k}$ and 'S' is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

According to Gauss divergence theorem,

$$\iiint_V \text{div } \vec{F} dv = \iint_S \vec{F} \cdot \vec{n} ds$$

$$\text{div } \vec{F} = 3(x^2 + y^2 + z^2)$$

$$\iint_S \vec{F} \cdot \vec{n} ds = \iiint_V 3(x^2 + y^2 + z^2) dx dy dz$$

Using the Spherical coordinates

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta$$

$$dv = r^2 \sin \theta dr d\theta d\phi$$

$$\iint_S \vec{F} \cdot \vec{n} ds = 3 \int_0^a \int_0^\pi \int_0^{2\pi} r^4 \sin \theta dr d\theta d\phi$$

$$= 3 \left[\int_0^a r^4 dr \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \right] = 3 \left[\frac{r^5}{5} \Big|_0^a \cdot (-\cos \theta) \Big|_0^\pi \cdot 2\pi \right]$$

$$= 3 \left[\frac{a^5}{5} \cdot (1+1) \cdot 2\pi \right]$$

$$= \frac{12\pi a^5}{5}$$

$$x^2 + y^2 + z^2 = r^2 \sin^2 \theta \cos^2 \phi + r^2 \sin^2 \theta \sin^2 \phi + r^2 \cos^2 \theta$$

$$= r^2 \sin^2 \theta (\cos^2 \phi + \sin^2 \phi) + r^2 \cos^2 \theta$$

$$= r^2 (\sin^2 \theta + \cos^2 \theta) = r^2$$

- 2) Find $\iint_S \vec{F} \cdot \vec{n} ds$ where $\vec{F} = (2x+3z)\vec{i} - (xz+y)\vec{j} + (y^2+xz)\vec{k}$ and 'S' is the surface of the sphere having centre at $(3, -1, 2)$ and radius 3.

Sol:- Let 'V' be the volume enclosed by the surface 'S'

Then Gauss divergence theorem, we have

$$\begin{aligned}\iint_S \vec{F} \cdot \vec{n} ds &= \iiint_V \text{div } \vec{F} dv \\ &= \iiint_V \left[\frac{\partial}{\partial x}(2x+3z) + \frac{\partial}{\partial y}(-xz+y) + \frac{\partial}{\partial z}(y^2+xz) \right] dv \\ &= \iiint_V (2-1+2) dv = 3 \iiint_V dv = 3V.\end{aligned}$$

But 'V' is the volume of a sphere of radius is 3.

$$V = \frac{4}{3} \pi r^3 \text{ [m}^3\text{]}$$

$$V = \frac{4}{3} \pi (3)^3$$

$$= 36\pi$$

Hence $\iint_S \vec{F} \cdot \vec{n} ds = 3(36\pi) = 108\pi$

- 3) Evaluate $\iint_S (x dy dz + y dz dx + z dx dy)$ over the surface of a sphere of radius 'a'.

Sol:- By Gauss divergence theorem $\iint_S \vec{F} \cdot \vec{n} ds = \iiint_V \text{div } \vec{F} dv$

(or) $\int_S F_1 dy dz + F_2 dz dx + F_3 dx dy = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dv$

Here $F_1 = x, F_2 = y, F_3 = z \Rightarrow \frac{\partial F_1}{\partial x} = 1, \frac{\partial F_2}{\partial y} = 1, \frac{\partial F_3}{\partial z} = 1$

$$\begin{aligned}\iint_S x dy dz + y dz dx + z dx dy &= \iiint_V (1+1+1) dv = 3V \\ &= 3 \times \frac{4}{3} \pi a^3 \\ &= 4\pi a^3\end{aligned}$$

[$\because$ 'V' is the volume of the sphere of radius is 'a']

If ϕ and ψ are scalar point functions which are uniform and continuous and possessing derivative in any direction within the volume V enclosed by the surface 'S' then show that
$$\iiint_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) dv = \int_S (\phi \nabla \psi - \psi \nabla \phi) \cdot \vec{n} ds$$

By Gauss divergence theorem we have
$$\int_S \vec{F} \cdot \vec{n} ds = \iiint_V \text{div } \vec{F} dv \quad \text{--- (1)}$$

taking $\vec{F} = \phi \nabla \psi$ then (1) becomes

$$\int_S (\phi \nabla \psi) \cdot \vec{n} ds = \iiint_V \nabla \cdot (\phi \nabla \psi) dv = \iiint_V (\nabla \phi \cdot \nabla \psi + \phi \nabla^2 \psi) dv \quad \text{--- (2)}$$

Interchanging ϕ and ψ in (2) we have

$$[\because \nabla \phi \cdot \nabla \psi = \nabla \psi \cdot \nabla \phi]$$

$$\int_S (\psi \nabla \phi) \cdot \vec{n} ds = \iiint_V [\nabla \psi \cdot \nabla \phi + \psi \nabla^2 \phi] dv \quad \text{--- (3)}$$

on subtracting
$$\int_S (\phi \nabla \psi - \psi \nabla \phi) \cdot \vec{n} ds = \iiint_V [\phi \nabla^2 \psi - \psi \nabla^2 \phi] dv \quad \text{--- (4)}$$

which is known as Green's Second Identity.

Irrrotational fields :- An irrotational field $\vec{F}$ is characterised by one of the following conditions (i) $\nabla \times \vec{F} = \vec{0}$ (ii) circulation $\int \vec{F} \cdot d\vec{r}$ along every closed surface is zero.

(i) $\vec{F} = \nabla \phi$ (for some scalar point function ϕ) if the domain is simply connected. (ii) $\nabla \times \vec{F} = \vec{0}$ by Stokes theorem $\int_C \vec{F} \cdot d\vec{r} = \int_S (\nabla \times \vec{F}) \cdot \vec{n} ds = 0$ (ie circulation along every closed surface is zero).

Again since $\text{curl}(\text{grad } \phi) = 0$ is an irrotational field for which $\nabla \times \vec{F} = \vec{0}$;

the vector $\vec{F}$ can always be expressed as the gradient of a scalar point function ϕ provided the domain is simply connected.

Thus $\vec{F} = \nabla \phi$ has a scalar function ϕ is called the potential. In a irrotational field $\vec{F}$ cannot be expressed as the gradient of a scalar potential.

Result :- On a irrotational field, the line integral of $\vec{F}$ between two points is independent of the path of integration and is equal to the potential difference between these points.



Let A and B be any two points in the region R . Let A and B in R then we have

region R and C be any path joining A and B

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla \phi \cdot d\vec{r} = \int_C \left(\hat{i} \frac{d\phi}{dx} + \hat{j} \frac{d\phi}{dy} + \hat{k} \frac{d\phi}{dz} \right) \cdot (dx \hat{i} + dy \hat{j} + dz \hat{k})$$

$$= \int_C \left(\frac{d\phi}{dx} dx + \frac{d\phi}{dy} dy + \frac{d\phi}{dz} dz \right) = \int_C d\phi = \int_A^B d\phi = [\phi]_A^B = \phi_B - \phi_A$$

which shows that the line integral depends only on point A and B and not on path C

Note 1:- If $\vec{F}$ is a vector force acting on a particle then $\oint \vec{F} \cdot d\vec{r}$ represents the work done in moving the particle around the closed path.

when $\oint \vec{F} \cdot d\vec{r} = 0$ the field is said to be conservative. i.e. no work is done in displacement from a point to another point in the field, and back to that point and the mechanical energy is conserved. Thus every irrotational field is conservative.

Solenoidal Fields:- A solenoidal field $\vec{F}$ is characterised by any one of the following conditions:-

- (i) $\nabla \cdot \vec{F} = 0$ (ii) Flux $\int \vec{F} \cdot \vec{n} \cdot d\vec{s}$ across every closed surface is zero since $\nabla \cdot (\nabla \times \vec{v}) = 0$.
- i.e. $\text{div}(\text{curl } \vec{v}) = 0$.

Therefore, in a solenoidal field for which $\nabla \cdot \vec{F} = 0$, then vector $\vec{F}$ can always be expressed as the curl of a vector function $\vec{v}$ thus

$$\vec{F} = \nabla \times \vec{v}$$

Any point P, we have $x = \rho \cos \phi$, $y = \rho \sin \phi$, $z = z$ so that

$$\vec{R} = \rho \cos \phi \vec{i} + \rho \sin \phi \vec{j} + z \vec{k}$$

If T_ρ, T_ϕ, T_z are unit vectors at P in the directions of the tangents to the ρ, ϕ, z curves then

$$T_\rho = \frac{\frac{\partial \vec{R}}{\partial \rho}}{|\frac{\partial \vec{R}}{\partial \rho}|} = \frac{\cos \phi \vec{i} + \sin \phi \vec{j}}{\sqrt{\cos^2 \phi + \sin^2 \phi}} = \cos \phi \vec{i} + \sin \phi \vec{j}$$

$$T_\phi = \frac{\frac{\partial \vec{R}}{\partial \phi}}{|\frac{\partial \vec{R}}{\partial \phi}|} = \frac{-\rho \sin \phi \vec{i} + \rho \cos \phi \vec{j}}{\sqrt{\rho^2 \sin^2 \phi + \rho^2 \cos^2 \phi}} = -\sin \phi \vec{i} + \cos \phi \vec{j}$$

$$T_z = \frac{\frac{\partial \vec{R}}{\partial z}}{|\frac{\partial \vec{R}}{\partial z}|} = \frac{\vec{k}}{1} = \vec{k}$$

$$\text{Now } T_\rho \cdot T_\phi = (\cos \phi \vec{i} + \sin \phi \vec{j}) \cdot (-\sin \phi \vec{i} + \cos \phi \vec{j}) = 0$$

$$T_\phi \cdot T_z = (-\sin \phi \vec{i} + \cos \phi \vec{j}) \cdot \vec{k} = 0$$

$$T_\rho \cdot T_z = (\cos \phi \vec{i} + \sin \phi \vec{j}) \cdot \vec{k} = 0$$

$\therefore$ The cylindrical coordinate system is orthogonal.

$$\text{Also } T_\rho \times T_\phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \end{vmatrix} = \vec{k} (\cos^2 \phi + \sin^2 \phi) = \vec{k} = T_z$$

$$T_\phi \times T_z = T_\rho, \quad T_z \times T_\rho = T_\phi$$

$\therefore$ The cylindrical coordinate system is a right handed

Del applied to functions in cylindrical

① Expression for grad f

$$\nabla f = \frac{\partial f}{\partial \rho} \hat{T}_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \hat{T}_\phi + \frac{\partial f}{\partial z} \hat{T}_z$$

② Expression for div $\vec{F}$ where $\vec{F} = f_1 \hat{T}_\rho + f_2 \hat{T}_\phi + f_3 \hat{T}_z$

$$\nabla \cdot \vec{F} = \frac{1}{\rho} \left[\frac{d}{d\rho} (\rho f_1) + \frac{\partial f_2}{\partial \phi} + \frac{d}{dz} (\rho f_3) \right]$$

③ Expression for curl $\vec{F}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{T}_\rho/\rho & \hat{T}_\phi & \hat{T}_z/\rho \\ \frac{d}{d\rho} & \frac{d}{d\phi} & \frac{d}{dz} \\ f_1 & \rho f_2 & f_3 \end{vmatrix}$$

④ expression for $\nabla^2 f$

$$\nabla^2 f = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial f}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Problem ① Express the vector $z\hat{i} - 2x\hat{j} + y\hat{k}$ in cylindrical coordinates.

Sol:- we have $x = \rho \cos \phi$, $y = \rho \sin \phi$, $z = z$

$$\text{So that } \vec{R} = \rho \cos \phi \hat{i} + \rho \sin \phi \hat{j} + z \hat{k}$$

If $\hat{T}_\rho$, $\hat{T}_\phi$, $\hat{T}_z$ are unit vectors along the tangents to ρ , ϕ and z respectively

$$\hat{T}_\rho = \frac{\partial \vec{R}}{\partial \rho} / \left| \frac{\partial \vec{R}}{\partial \rho} \right| = \cos \phi \hat{i} + \sin \phi \hat{j}$$

$$\hat{T}_\phi = \frac{\partial \vec{R}}{\partial \phi} / \left| \frac{\partial \vec{R}}{\partial \phi} \right| = -\sin \phi \hat{i} + \cos \phi \hat{j}$$

$$\hat{T}_z = \frac{\partial \vec{R}}{\partial z} / \left| \frac{\partial \vec{R}}{\partial z} \right| = \hat{k}$$

Given $\vec{F} = z\hat{i} - 2x\hat{j} + y\hat{k}$

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Let the expression for $\vec{F}$ in cylindrical coordinates be

$$\vec{F} = f_1 \hat{T}_r + f_2 \hat{T}_\phi + f_3 \hat{T}_z$$

where $f_1 = \vec{F} \cdot \hat{T}_r$, $f_2 = \vec{F} \cdot \hat{T}_\phi$, $f_3 = \vec{F} \cdot \hat{T}_z$

$$f_1 = (z\hat{i} - 2x\hat{j} + y\hat{k}) \cdot (\cos\phi \hat{i} + \sin\phi \hat{j})$$

$$= z\cos\phi - 2x\sin\phi$$

$$f_2 = (z\hat{i} - 2x\hat{j} + y\hat{k}) \cdot (-\sin\phi \hat{i} + \cos\phi \hat{j})$$

$$= -z\sin\phi - 2x\cos\phi$$

$$f_3 = (z\hat{i} - 2x\hat{j} + y\hat{k}) \cdot \hat{k} = y = r\sin\phi$$

required expression is $\vec{F} = f_1 \hat{T}_r + f_2 \hat{T}_\phi + f_3 \hat{T}_z$

$$\vec{F} = (z\cos\phi - 2x\sin\phi) \hat{T}_r - (z\sin\phi + 2x\cos\phi) \hat{T}_\phi + y\hat{T}_z$$

$$\vec{F} = (z\cos\phi - r\sin 2\phi) \hat{T}_r - (z\sin\phi + 2r\cos^2\phi) \hat{T}_\phi + r\sin\phi \hat{T}_z$$

②

Express the vector $2y\hat{i} - z\hat{j} + 3x\hat{k}$ in cylindrical coordinates

Sol: we have $x = r\cos\phi$, $y = r\sin\phi$, $z = z$

So that $\vec{r} = r\cos\phi \hat{i} + r\sin\phi \hat{j} + z\hat{k}$

If $\hat{T}_r, \hat{T}_\phi, \hat{T}_z$ are unit vectors along the tangents to r, ϕ and z respectively

$$\hat{T}_r = \frac{\partial \vec{r}}{\partial r} / \left| \frac{\partial \vec{r}}{\partial r} \right| = \cos\phi \hat{i} + \sin\phi \hat{j}$$

$$\hat{T}_\phi = \frac{\partial \vec{r}}{\partial \phi} / \left| \frac{\partial \vec{r}}{\partial \phi} \right| = -r\sin\phi \hat{i} + r\cos\phi \hat{j}$$

$$\hat{T}_z = \frac{\partial \vec{r}}{\partial z} / \left| \frac{\partial \vec{r}}{\partial z} \right| = \hat{k}$$

Given $\vec{F} = 2y\hat{i} - z\hat{j} + 3x\hat{k}$

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Let the expression for $\vec{F}$ in cylindrical coordinates be

$$\vec{F} = f_1\hat{T}_\rho + f_2\hat{T}_\phi + f_3\hat{T}_z$$

where $f_1 = \vec{F} \cdot \hat{T}_\rho$, $f_2 = \vec{F} \cdot \hat{T}_\phi$, $f_3 = \vec{F} \cdot \hat{T}_z$

$$f_1 = (2y\hat{i} - z\hat{j} + 3x\hat{k}) \cdot (\cos\phi\hat{i} + \sin\phi\hat{j})$$

$$= 2y\cos\phi - z\sin\phi$$

$$= 2(\rho\sin\phi)\cos\phi - z\sin\phi$$

$$= \rho\sin 2\phi - z\sin\phi$$

$$f_2 = (2y\hat{i} - z\hat{j} + 3x\hat{k}) \cdot (-\sin\phi\hat{i} + \cos\phi\hat{j})$$

$$= -2y\sin\phi - z\cos\phi$$

$$= -2(\rho\sin\phi)\sin\phi - z\cos\phi = -2\rho\sin^2\phi - z\cos\phi$$

$$f_3 = (2y\hat{i} - z\hat{j} + 3x\hat{k}) \cdot \hat{k} = 3x = 3\rho\cos\phi$$

Required expression is $\vec{F} = f_1\hat{T}_\rho + f_2\hat{T}_\phi + f_3\hat{T}_z$

$$\vec{F} = (\rho\sin 2\phi - z\sin\phi)\hat{T}_\rho - (2\rho\sin^2\phi + z\cos\phi)\hat{T}_\phi + 3\rho\cos\phi\hat{T}_z$$

③ Express the following vectors in cylindrical coordinates

$$2x\hat{i} - 3y^2\hat{j} + 2x\hat{k}$$

Sol:- we have $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$

$$\text{So that } \vec{r} = \rho\cos\phi\hat{i} + \rho\sin\phi\hat{j} + z\hat{k}$$

If $\hat{T}_\rho$, $\hat{T}_\phi$, $\hat{T}_z$ are unit vectors along the tangents to

ρ , ϕ and z respectively $\hat{T}_\rho = \frac{\partial \vec{r}}{\partial \rho} / \left| \frac{\partial \vec{r}}{\partial \rho} \right| = \cos\phi\hat{i} + \sin\phi\hat{j}$

$$\hat{T}_\phi = \frac{\partial \vec{r}}{\partial \phi} / \left| \frac{\partial \vec{r}}{\partial \phi} \right| = -\sin\phi\hat{i} + \cos\phi\hat{j}$$

$$\hat{T}_z = \frac{\partial \vec{r}}{\partial z} / \left| \frac{\partial \vec{r}}{\partial z} \right| = \hat{k}$$

Find the expression for $\vec{F}$ in cylindrical coordinates

$$\vec{F} = f_1 \hat{T}_r + f_2 \hat{T}_\phi + f_3 \hat{T}_z$$

where $f_1 = \vec{F} \cdot \hat{T}_r$, $f_2 = \vec{F} \cdot \hat{T}_\phi$, $f_3 = \vec{F} \cdot \hat{T}_z$

$$f_1 = (2x\hat{i} - 3y^2\hat{j} + 2z\hat{k}) \cdot (\cos\phi\hat{i} + \sin\phi\hat{j})$$

$$f_1 = 2x \cos\phi - 3y^2 \sin\phi$$

$$= 2(\rho \cos\phi) \cos\phi - 3(\rho \sin\phi)^2 \sin\phi$$

$$= 2\rho \cos^2\phi - 3\rho^2 \sin^3\phi$$

$$f_2 = (2x\hat{i} - 3y^2\hat{j} + 2z\hat{k}) \cdot (-\sin\phi\hat{i} + \cos\phi\hat{j})$$

$$= -2x \sin\phi - 3y^2 \cos\phi$$

$$= -2(\rho \cos\phi) \sin\phi - 3(\rho \sin\phi)^2 \cos\phi$$

$$= -\rho \sin 2\phi - 3\rho^2 \sin^2\phi \cos\phi$$

$$f_3 = (2x\hat{i} - 3y^2\hat{j} + 2z\hat{k}) \cdot \hat{k} = 2z = 2\rho \cos\phi$$

Required expression is

$$\vec{F} = f_1 \hat{T}_r + f_2 \hat{T}_\phi + f_3 \hat{T}_z$$

$$\vec{F} = (2\rho \cos^2\phi - 3\rho^2 \sin^3\phi) \hat{T}_r + (-\rho \sin 2\phi - 3\rho^2 \sin^2\phi \cos\phi) \hat{T}_\phi + 2\rho \cos\phi \hat{T}_z$$

✓

Problem (i) :-

s.t. $\nabla(\log \rho)$ and $\nabla\phi$, $\rho \neq 0$, $\phi \neq 0$ are

Solenoidal vectors.

Sol: $f = \log \rho$ is a function of ρ only

To p.t. $\nabla \cdot (\nabla f) = 0 \Rightarrow \nabla^2 f$

$$\nabla^2 f = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial f}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \frac{\partial^2}{\partial \rho^2} (\log \rho) + \frac{1}{\rho} \frac{\partial}{\partial \rho} (\log \rho) + 0 + 0$$

$$= -\frac{1}{\rho^2} + \frac{1}{\rho^2} = 0$$

$\therefore \nabla \log \rho$ is Solenoidal.

(ii) $f = \nabla\phi$ is a function of ϕ only. we have to show that $\nabla \cdot (\nabla f)$ i.e. $\nabla^2 f = 0$

$$\nabla^2 f = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial f}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= 0 + 0 + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + 0$$

$$= 0$$

At any point P, we have $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$.

$$\Rightarrow \vec{R} = r \sin \theta \cos \phi \hat{i} + r \sin \theta \sin \phi \hat{j} + r \cos \theta \hat{k}$$

If T_r, T_θ, T_ϕ be the unit vectors at 'P', in the directions of tangents to the r, θ, ϕ -curves respectively then:

$$T_r = \frac{\frac{\partial \vec{R}}{\partial r}}{\left| \frac{\partial \vec{R}}{\partial r} \right|} = \frac{\sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k}}{\sqrt{\sin^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \phi + \cos^2 \theta}}$$

$$\cos^2 \phi + \sin^2 \phi = 1$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$= \frac{\sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k}}{\sqrt{\sin^2 \theta (\cos^2 \phi + \sin^2 \phi) + \cos^2 \theta}}$$

$$= \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k}$$

$$T_\theta = \frac{\frac{\partial \vec{R}}{\partial \theta}}{\left| \frac{\partial \vec{R}}{\partial \theta} \right|} = \frac{r \cos \theta \cos \phi \hat{i} + r \cos \theta \sin \phi \hat{j} - r \sin \theta \hat{k}}{\sqrt{r^2 \cos^2 \theta \cos^2 \phi + r^2 \cos^2 \theta \sin^2 \phi + r^2 \sin^2 \theta}}$$

$$= \frac{r (\cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k})}{r \sqrt{\cos^2 \theta (\cos^2 \phi + \sin^2 \phi) + \sin^2 \theta}}$$

$$= \cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k}$$

$$T_\phi = \frac{\frac{\partial \vec{R}}{\partial \phi}}{\left| \frac{\partial \vec{R}}{\partial \phi} \right|} = \frac{-r \sin \theta \sin \phi \hat{i} + r \sin \theta \cos \phi \hat{j}}{\sqrt{r^2 \sin^2 \theta \sin^2 \phi + r^2 \sin^2 \theta \cos^2 \phi}}$$

$$= \frac{r \sin \theta (-\sin \phi \hat{i} + \cos \phi \hat{j})}{r \sin \theta \sqrt{\sin^2 \phi + \cos^2 \phi}}$$

$$= -\sin \phi \hat{i} + \cos \phi \hat{j}$$

$$= \sin \theta \cos \theta (\cos^2 \phi + \sin^2 \phi) - \sin \theta \cos \theta$$

$$= 0$$

lly $T_\theta \cdot T_\phi = 0$ and $T_\phi \cdot T_r = 0$.

$$T_r \times T_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \end{vmatrix}$$

$$= \hat{i} [-\sin^2 \theta \sin \phi - \cos^2 \theta \sin \phi] + \hat{j} [\cos^2 \theta \cos \phi + \sin^2 \theta \cos \phi]$$

$$+ \hat{k} [\sin \theta \cos \phi \cos \theta \sin \phi - \sin \theta \cos \theta \sin \phi \cos \phi]$$

$$= -\sin \phi \hat{i} + \cos \phi \hat{j} = T_\phi$$

lly $T_\theta \times T_\phi = T_r$ and $T_\phi \times T_r = T_\theta$.

$\therefore$ The spherical polar coordinate system is a right handed orthogonal coordinate system.

Del applied to functions in spherical polar coordinates

Let T_r, T_θ, T_ϕ be the unit vectors in the directions of the tangents to the r, θ, ϕ -curves.

(i) Expression for grad f:-

$$\nabla f = \frac{\partial f}{\partial r} T_r + \frac{1}{r} \frac{\partial f}{\partial \theta} T_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} T_\phi$$

(ii) Expression for $\text{div } \vec{F}$, where $\vec{F} = f_1 T_r + f_2 T_\theta + f_3 T_\phi$

$$\nabla \cdot \vec{F} = \frac{1}{r^2} \frac{\partial}{\partial r} (f_1 r^2) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (f_2 \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial f_3}{\partial \phi}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ \frac{1}{r^2 \sin \theta} & \frac{1}{r \sin \theta} & \frac{1}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ r_1 & r_2 & r \sin \theta \end{vmatrix}$$

(iv) Expression for $\nabla^2 f$:-

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} + \frac{2}{r} \frac{\partial f}{\partial r} + \frac{\cot \theta}{r} \frac{\partial f}{\partial \theta}$$

(i) Express the vector field $xy\hat{i} - 2z\hat{j} + 3xz\hat{k}$ in spherical polar coordinate system.

Soln we have $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$.

So that $\vec{r} = r \sin \theta \cos \phi \hat{i} + r \sin \theta \sin \phi \hat{j} + r \cos \theta \hat{k}$.

Let $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$ be the unit vectors along the tangents to r, θ, ϕ curves respectively, then

$$\begin{aligned} \hat{r} &= \frac{\frac{\partial \vec{r}}{\partial r}}{\left| \frac{\partial \vec{r}}{\partial r} \right|} = \frac{\sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k}}{\sqrt{\sin^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \phi + \cos^2 \theta}} \\ &= \frac{\sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k}}{\sqrt{\sin^2 \theta (\cos^2 \phi + \sin^2 \phi) + \cos^2 \theta}} \\ &= \boxed{\sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k}} \end{aligned}$$

$$\begin{aligned} \hat{\theta} &= \frac{\frac{\partial \vec{r}}{\partial \theta}}{\left| \frac{\partial \vec{r}}{\partial \theta} \right|} = \frac{r \cos \theta \cos \phi \hat{i} + r \cos \theta \sin \phi \hat{j} - r \sin \theta \hat{k}}{\sqrt{r^2 \cos^2 \theta \cos^2 \phi + r^2 \cos^2 \theta \sin^2 \phi + r^2 \sin^2 \theta}} \\ &= \frac{r [\cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k}]}{r \sqrt{\cos^2 \theta (\cos^2 \phi + \sin^2 \phi) + \sin^2 \theta}} \end{aligned}$$

$$T_\phi = \frac{\frac{\partial \vec{r}}{\partial \phi}}{\left| \frac{\partial \vec{r}}{\partial \phi} \right|} = \frac{-r \sin \theta \sin \phi \hat{i} + r \sin \theta \cos \phi \hat{j}}{\sqrt{r^2 \sin^2 \theta \sin^2 \phi + r^2 \sin^2 \theta \cos^2 \phi}}$$

$$T_\phi = \frac{r \sin \theta [-\sin \phi \hat{i} + \cos \phi \hat{j}]}{r \sin \theta [\sqrt{\sin^2 \phi + \cos^2 \phi}]} = \boxed{-\sin \phi \hat{i} + \cos \phi \hat{j}}$$

Let the expression for $\vec{F} = 2y\hat{i} - 2\hat{j} + 3z\hat{k}$ in spherical polar coordinates be $\vec{F} = f_1 T_r + f_2 T_\theta + f_3 T_\phi$ — (1)

$$f_1 = \vec{F} \cdot T_r, \quad f_2 = \vec{F} \cdot T_\theta \quad \text{and} \quad f_3 = \vec{F} \cdot T_\phi$$

$$f_1 = (2y\hat{i} - 2\hat{j} + 3z\hat{k}) \cdot (\sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k})$$

$$f_1 = 2y \sin \theta \cos \phi - 2 \sin \theta \sin \phi + 3z \cos \theta$$

$$f_1 = 2(r \sin \theta \sin \phi) \sin \theta \cos \phi - 2 \cos \theta \sin \theta \sin \phi + 3r \sin \theta \cos \theta \cos \phi$$

$$f_1 = r \sin^2 \theta \sin 2\phi - r \cos^2 \theta \sin \phi + 3r \sin \theta \cos \theta \cos \phi$$

$$f_2 = \vec{F} \cdot T_\theta = [2y\hat{i} - 2\hat{j} + 3z\hat{k}] \cdot [\cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k}]$$

$$= 2y \cos \theta \cos \phi - 2 \cos \theta \sin \phi - 3z \sin \theta$$

$$= 2(r \sin \theta \sin \phi) \cos \theta \cos \phi - r \cos^2 \theta \cos \phi - 3r \sin \theta \cos \theta \sin \theta$$

$$= r \sin \theta \cos \theta \sin 2\phi - r \cos^2 \theta \cos \phi - 3r \sin^2 \theta \cos \phi$$

$$f_3 = \vec{F} \cdot T_\phi = (2y\hat{i} - 2\hat{j} + 3z\hat{k}) \cdot [-\sin \phi \hat{i} + \cos \phi \hat{j}]$$

$$= -2y \sin \phi + 2 \cos \phi$$

$$= -2(r \sin \theta \sin \phi) \sin \phi + r \cos^2 \theta \cos \phi$$

$$= -2r \sin \theta \sin^2 \phi + r \cos^2 \theta \cos \phi$$

Substituting the values f_1, f_2, f_3 in (1) we get

$$\begin{aligned}\vec{F} &= [r \sin^2 \theta \sin 2\phi + r \sin \theta \cos \theta \sin \phi + 3r \sin \theta \cos \theta \cos \phi] T_r \\ &+ [r \sin \theta \cos \theta \sin 2\phi - r \cos^2 \theta \sin \phi - 3r \sin^2 \theta \cos \phi] T_\theta \\ &+ [-2r \sin \theta \sin^2 \phi - r \cos \theta \cos \phi] T_\phi \\ \vec{F} &= [r \sin^2 \theta \sin 2\phi - r \sin \theta \cos \theta \sin \phi + 3r \sin \theta \cos \theta \cos \phi] T_r \\ &+ [r \sin \theta \cos \theta \sin 2\phi - r \cos^2 \theta \sin \phi - 3r \sin^2 \theta \cos \phi] T_\theta \\ &- [2r \sin \theta \sin^2 \phi + r \cos \theta \cos \phi] T_\phi\end{aligned}$$

2) P.T $\nabla(\cos \theta) \times \nabla \phi = \nabla\left(\frac{1}{r}\right)$, $r \neq 0$

Sol:- In spherical polar coordinates

$$\nabla f = \frac{\partial f}{\partial r} T_r + \frac{1}{r} \frac{\partial f}{\partial \theta} T_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} T_\phi$$

$$\nabla(\cos \theta) = 0 + \frac{1}{r} \frac{\partial (\cos \theta)}{\partial \theta} T_\theta + 0 = -\frac{1}{r} \sin \theta T_\theta$$

$$\nabla \phi = 0 + 0 + \frac{1}{r \sin \theta} \frac{\partial (\phi)}{\partial \phi} T_\phi = \frac{1}{r \sin \theta} T_\phi$$

$$\nabla\left(\frac{1}{r}\right) = \frac{\partial}{\partial r}\left(\frac{1}{r}\right) T_r + 0 + 0 = -\frac{1}{r^2} T_r$$

$$\nabla(\cos \theta) \times \nabla \phi = \left(-\frac{1}{r} \sin \theta T_\theta\right) \times \left(\frac{1}{r \sin \theta} T_\phi\right)$$

$$= -\frac{1}{r^2} (T_\theta \times T_\phi) = -\frac{1}{r^2} T_r = \nabla\left(\frac{1}{r}\right)$$

3) If $\vec{F} = r^2 \cos \theta T_r - \frac{1}{r} T_\theta + \frac{1}{r \sin \theta} T_\phi$ find the value of $\text{curl } \vec{F}$

Sol:- In spherical polar coordinates

$$\text{curl } \vec{F} = \begin{vmatrix} T_r/r \sin \theta & T_\theta/r \sin \theta & T_\phi/r \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\text{curl } \vec{F} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{T}_r & r\hat{T}_\theta & r\sin \theta \hat{T}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ r^2 \cos \theta & -1 & 1 \end{vmatrix}$$

$$= \frac{1}{r^2 \sin \theta} \left[r\sin \theta \hat{T}_\phi (0 - r^2(-\sin \theta)) \right] = \frac{r^3 \sin^2 \theta \hat{T}_\phi}{r^2 \sin \theta} = r\sin \theta \hat{T}_\phi$$

$$\vec{F} \times \text{curl } \vec{F} = \left[r^2 \cos \theta \hat{T}_r - \frac{1}{r} \hat{T}_\theta + \frac{1}{r \sin \theta} \hat{T}_\phi \right] \times \left[r\sin \theta \hat{T}_\phi \right]$$

$$= -r^3 \sin \theta \cos \theta \hat{T}_\theta - \sin \theta \hat{T}_r$$

$$= -(r^3 \sin \theta \cos \theta \hat{T}_\theta + \sin \theta \hat{T}_r)$$

4) Express the following vectors in spherical polar coordinates

$$x\hat{i} + 2y\hat{j} + yz\hat{k}$$

Sol: we have $x = r\sin \theta \cos \phi$, $y = r\sin \theta \sin \phi$, $z = r\cos \theta$.

$$\text{So that } \vec{R} = r\sin \theta \cos \phi \hat{i} + r\sin \theta \sin \phi \hat{j} + r\cos \theta \hat{k}$$

If $\hat{T}_r, \hat{T}_\theta, \hat{T}_\phi$ be the unit vectors along the tangents to r, θ, ϕ curves respectively, then

$$\hat{T}_r = \frac{\partial \vec{R}}{\partial r} \Big/ \left| \frac{\partial \vec{R}}{\partial r} \right| = \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k}$$

$$\hat{T}_\theta = \frac{\partial \vec{R}}{\partial \theta} \Big/ \left| \frac{\partial \vec{R}}{\partial \theta} \right| = \cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k}$$

$$\hat{T}_\phi = \frac{\partial \vec{R}}{\partial \phi} \Big/ \left| \frac{\partial \vec{R}}{\partial \phi} \right| = -\sin \phi \hat{i} + \cos \phi \hat{j}$$

Let the expression for $\vec{F} = x\hat{i} + 2y\hat{j} + yz\hat{k}$ in spherical

polar coordinates be $\vec{F} = f_1 \hat{T}_r + f_2 \hat{T}_\theta + f_3 \hat{T}_\phi$

$$\text{Then } f_1 = \vec{F} \cdot \hat{T}_r = (x\hat{i} + 2y\hat{j} + yz\hat{k}) \cdot (\sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k})$$

$$= r \sin \theta \cos \phi \sin \theta \cos \phi + 2(r \sin \theta \sin \phi) \sin \theta \sin \phi + (r \cos \theta) \cos \theta$$

$$= r \sin^2 \theta \cos^2 \phi + 2r \sin^2 \theta \sin^2 \phi + r^2 \sin^2 \theta \cos^2 \theta$$

$$f_2 = \vec{F} \cdot \vec{T}_\theta = (x\vec{i} + 2y\vec{j} + yz\vec{k}) \cdot (\cos \theta \cos \phi \vec{i} + \cos \theta \sin \phi \vec{j} - \sin \theta \vec{k})$$

$$= x \cos \theta \cos \phi + 2y \cos \theta \sin \phi - yz \sin \theta$$

$$= (r \sin \theta \cos \phi) \cos \theta \cos \phi + 2(r \sin \theta \sin \phi) \cos \theta \sin \phi - (r \cos \theta) \sin \theta$$

$$= r \sin \theta \cos \theta \cos^2 \phi + r \sin 2\theta \sin^2 \phi - r^2 \sin^2 \theta \cos \theta \sin \phi$$

$$f_3 = \vec{F} \cdot \vec{T}_\phi = (x\vec{i} + 2y\vec{j} + yz\vec{k}) \cdot (-\sin \phi \vec{i} + \cos \phi \vec{j})$$

$$= -x \sin \phi + 2y \cos \phi$$

$$= -r \sin \theta \cos \phi \sin \phi + 2(r \sin \theta \sin \phi) \cos \phi$$

$$= -r \sin \theta \sin \phi \cos \phi + r \sin \theta \sin 2\phi$$

$$\vec{F} = f_1 \vec{T}_r + f_2 \vec{T}_\theta + f_3 \vec{T}_\phi$$

$$\vec{F} = [r \sin^2 \theta \cos^2 \phi + 2r \sin^2 \theta \sin^2 \phi + r^2 \sin^2 \theta \cos^2 \theta] \vec{T}_r$$

$$+ [r \sin \theta \cos \theta \cos^2 \phi + r \sin 2\theta \sin^2 \phi - r^2 \sin^2 \theta \cos \theta \sin \phi] \vec{T}_\theta$$

$$+ [-r \sin \theta \sin \phi \cos \phi + r \sin \theta \sin 2\phi] \vec{T}_\phi$$

5) $2y\vec{i} + yz\vec{j} + 2z\vec{k}$

$$\vec{R} = r \sin \theta \cos \phi \vec{i} + r \sin \theta \sin \phi \vec{j} + r \cos \theta \vec{k}$$

$$\vec{T}_r = \sin \theta \cos \phi \vec{i} + \sin \theta \sin \phi \vec{j} + \cos \theta \vec{k}$$

$$\vec{T}_\theta = \cos \theta \cos \phi \vec{i} + \cos \theta \sin \phi \vec{j} - \sin \theta \vec{k}$$

$$\vec{T}_\phi = -\sin \phi \vec{i} + \cos \phi \vec{j}$$

$$\text{Then } f_1 = \vec{F} \cdot \vec{T}_r = xy \sin \theta \cos \phi + yz \sin \theta \sin \phi + zx \cos \theta$$

$$= (r \sin \theta \cos \phi) (r \sin \theta \sin \phi) \sin \theta \cos \phi + (r \sin \theta \cos \phi) (r \cos \theta) \sin \theta \sin \phi + (r \cos \theta) (r \sin \theta \cos \phi) \cos \theta$$

$$= r^2 \sin^3 \theta \sin \phi \cos^2 \phi + r^2 \sin^2 \theta \cos \theta \sin^2 \phi + r^2 \sin \theta \cos^2 \theta \cos \phi$$

$$f_2 = \vec{F} \cdot \vec{T}_\theta = (xy \vec{i} + yz \vec{j} + zx \vec{k}) \cdot (\cos \theta \cos \phi \vec{i} + \cos \theta \sin \phi \vec{j} - \sin \theta \vec{k})$$

$$= xy \cos \theta \cos \phi + yz \cos \theta \sin \phi - zx \sin \theta$$

$$= (r \sin \theta \cos \phi) (r \sin \theta \sin \phi) \cos \theta \cos \phi + (r \sin \theta \cos \phi) (r \cos \theta) \cos \theta \sin \phi - [r \cos \theta] [r \sin \theta \cos \phi] \sin \theta$$

$$= r^2 \sin^2 \theta \cos \theta \sin \phi \cos^2 \phi + r^2 \sin \theta \cos^2 \theta \sin^2 \phi - r^2 \sin^2 \theta \cos \theta \cos \phi$$

$$f_3 = \vec{F} \cdot \vec{T}_\phi = (xy \vec{i} + yz \vec{j} + zx \vec{k}) \cdot (-\sin \phi \vec{i} + \cos \phi \vec{j})$$

$$= -xy \sin \phi + \cos \phi yz$$

$$= -(r \sin \theta \cos \phi) (r \sin \theta \sin \phi) \sin \phi + (r \sin \theta \cos \phi) (r \cos \theta) \cos \phi$$

$$= -r^2 \sin^2 \theta \sin^2 \phi \cos \phi + r^2 \sin \theta \cos \theta \sin \phi \cos \phi$$

$$\vec{F} = d_1 \vec{T}_r + d_2 \vec{T}_\theta + d_3 \vec{T}_\phi$$

$$= [r^2 \sin^3 \theta \sin \phi \cos^2 \phi + r^2 \sin^2 \theta \cos \theta \sin^2 \phi + r^2 \sin \theta \cos^2 \theta \cos \phi] \vec{T}_r$$

$$+ [r^2 \sin^2 \theta \cos \theta \sin \phi \cos^2 \phi + r^2 \sin \theta \cos^2 \theta \sin^2 \phi - r^2 \sin^2 \theta \cos \theta \cos \phi] \vec{T}_\theta$$

$$- [r^2 \sin^2 \theta \sin^2 \phi \cos \phi - r^2 \sin \theta \cos \theta \sin \phi \cos \phi] \vec{T}_\phi$$

BETA FUNCTION: The definite integral $\int_0^1 x^m (1-x)^{n-1} dx$, for $m > 0, n > 0$ is called the Beta function (or) The Eulerian integral of the first kind and is denoted by $\beta(m, n)$ and read as "Beta m, n"

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx; m > 0, n > 0$$

PROPERTIES OF BETA FUNCTION

① $\beta(m, n) = \beta(n, m)$

Proof:- By the definition

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

$$= \int_0^1 (1-x)^{m-1} (1-(1-x))^{n-1} dx$$

$$= \int_0^1 (1-x)^{m-1} x^{n-1} dx$$

$$= \beta(n, m)$$

$$\therefore \boxed{\beta(m, n) = \beta(n, m)}$$

2. $\beta(m, n) = 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta$

Proof:- By the definition

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

$$\text{put } x = \sin^2 \theta$$

$$dx = 2 \sin \theta \cos \theta d\theta$$

$$\text{as } x \rightarrow 0, \theta \rightarrow 0 \text{ and } x \rightarrow 1, \theta \rightarrow \pi/2$$

$$\beta(m, n) = \int_0^{\pi/2} (\sin^2 \theta)^{m-1} (1 - \sin^2 \theta)^{n-1} 2 \sin \theta \cos \theta d\theta$$

$$= 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta$$

$$\therefore \beta(m, n) = 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta$$

③ $\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$

proof:

We know that

$$\beta(m, n) = 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta$$

put $m = n = \frac{1}{2}$

$$\beta\left(\frac{1}{2}, \frac{1}{2}\right) = 2 \int_0^{\pi/2} \sin^{2\left(\frac{1}{2}\right)-1} \theta \cos^{2\left(\frac{1}{2}\right)-1} \theta d\theta$$

$$= 2 \int_0^{\pi/2} d\theta$$

$$= 2 [\theta]_0^{\pi/2} = 2 \left[\frac{\pi}{2} \right] - 2(0)$$

$$\therefore \beta\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$$

DIFFERENT FORMS OF BETA FUNCTION

Form I: Show that $\beta(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx$

proof:

By the definition $\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$

put $x = \frac{y}{1+y} \Rightarrow 1-x = \frac{1}{1+y}$

$$\Rightarrow dx = \frac{1}{(1+y)^2} dy$$

As $x \rightarrow 0$; $y \rightarrow 0$ and $x \rightarrow 1$; $y \rightarrow \infty$

$$\begin{aligned} \beta(m, n) &= \int_0^{\infty} \left[\frac{y}{1+y} \right]^{m-1} \left[1 - \frac{y}{1+y} \right]^{n-1} \frac{1}{(1+y)^2} dy \\ &= \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m-1}} \cdot \frac{1}{(1+y)^{n-1}} \cdot \frac{1}{(1+y)^2} dy \end{aligned}$$

Show that $\beta(m, n) = a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}} dx$

proof :- consider $a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}} dx$

put $ax = bt$

$$\Rightarrow x = \frac{b}{a} t \Rightarrow dx = \frac{b}{a} dt$$

As $x \rightarrow 0; t \rightarrow 0$ and as $x \rightarrow \infty; t \rightarrow \infty$

$$= a^m b^n \int_0^\infty \frac{\frac{b^{m-1}}{a^{m-1}} t^{m-1}}{(bt+b)^{m+n}} \cdot \frac{b}{a} dt$$

$$= \frac{a^m b^n}{a^m b^{m+n}} b^m \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}} dt$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$$= \beta(m, n)$$

Hence $\beta(m, n) = a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}} dx$

FORM: IV

Show that $\int_a^b (x-a)^{m-1} (b-x)^{n-1} dx = (b-a)^{m+n-1} \beta(m, n)$

proof :- now $\int_a^b (x-a)^{m-1} (b-x)^{n-1} dx$

put $x = a + (b-a)t \Rightarrow dx = (b-a)dt$

as $x \rightarrow a, t \rightarrow 0$ and as $x \rightarrow b, t \rightarrow 1$

$$= \int_0^1 [(b-a)t]^{m-1} [b-a-(b-a)t]^{n-1} (b-a) dt$$

$$= (b-a)^{m+n-1} \int_0^1 t^{m-1} (1-t)^{n-1} dt$$

$$= (b-a)^{m+n-1} \beta(m, n)$$

Hence $\int_a^b (x-a)^{m-1} (b-x)^{n-1} dx = (b-a)^{m+n-1} \beta(m, n)$

$$= \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$$= \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$\therefore \int_a^b f(x) dx = \int_a^b f(t) dt$$

By property
 $\beta(m, n) = \beta(n, m)$

$$\therefore \beta(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

Form: II

Show that $\beta(m, n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$

proof:-

We know that

$$\beta(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_1^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx \quad \text{--- ①}$$

Consider $\int_1^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$

put $x = \frac{1}{y} \Rightarrow dx = -\frac{1}{y^2} dy$

as $x \rightarrow 1; y \rightarrow 1$ and $x \rightarrow \infty; y \rightarrow 0$

$$= \int_1^0 \frac{\left(\frac{1}{y}\right)^{m-1}}{\left(1+\frac{1}{y}\right)^{m+n}} \left(-\frac{1}{y^2}\right) dy$$

$$= \int_0^1 \frac{y^{m+n}}{y^{m-1}(1+y)^{m+n}} \cdot \frac{1}{y^2} dy$$

$$= \int_0^1 \frac{y^{m+n-m+1-2}}{(1+y)^{m+n}} dy$$

$$= \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}} dy = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

From ①, we have

$$\beta(m, n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

Hence $\beta(m, n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$

Soln: By the definition

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

$$\beta(1, 1) = \int_0^1 x^{1-1} (1-x)^{1-1} dx = \int_0^1 dx = [x]_0^1 = 1 - 0 = 1$$

$$\therefore \beta(1, 1) = 1$$

② Evaluate $\int_0^1 \frac{x^2}{\sqrt{1-x^5}} dx$ in terms of Beta-function

Soln: Consider $\int_0^1 \frac{x^2}{\sqrt{1-x^5}} dx$

$$\text{put } x^5 = t \Rightarrow x = t^{1/5}$$

$$dx = \frac{1}{5} t^{-4/5} dt$$

as $x \rightarrow 0$; $t \rightarrow 0$ and $x \rightarrow 1$; $t \rightarrow 1$

$$= \int_0^1 \frac{t^{2/5}}{\sqrt{1-t}} \cdot \frac{1}{5} t^{-4/5} dt$$

$$= \frac{1}{5} \int_0^1 t^{-2/5} (1-t)^{-1/2} dt$$

$$= \frac{1}{5} \int_0^1 t^{3/5-1} (1-t)^{1/2-1} dt$$

$$= \frac{1}{5} \beta\left(\frac{3}{5}, \frac{1}{2}\right)$$

③ Evaluate $\int_0^1 x^5 (1-x^3)^{10} dx$

Soln: consider $\int_0^1 x^5 (1-x^3)^{10} dx$

$$\text{put } x^3 = t \Rightarrow x = t^{1/3}$$

$$dx = \frac{1}{3} t^{-2/3} dt$$

as $x \rightarrow 0$; $t \rightarrow 0$ and $x \rightarrow 1$; $t \rightarrow 1$

$$\begin{aligned}
 &= \int_0^1 t^{2/3} (1-t)^0 \frac{1}{3} t^{-2/3} dt \\
 &= \frac{1}{3} \int_0^1 t (1-t)^0 dt \\
 &= \frac{1}{3} \int_0^1 t^{2-1} (1-t)^{1-1} dt \\
 &= \frac{1}{3} \beta(2, 1)
 \end{aligned}$$

④ Evaluate $\int_0^1 \frac{1}{(1-x^3)^{1/3}} dx$ using Beta function.

Soln:

Consider $\int_0^1 \frac{1}{(1-x^3)^{1/3}} dx$

$$\begin{aligned}
 \text{put } x^3 &= t \Rightarrow x = t^{1/3} \\
 &\Rightarrow dx = \frac{1}{3} t^{-2/3} dt
 \end{aligned}$$

as $x \rightarrow 0$; $t \rightarrow 0$ and as $x \rightarrow 1$; $t \rightarrow 1$

$$\begin{aligned}
 &= \int_0^1 \frac{1}{(1-t)^{1/3}} \frac{1}{3} t^{-2/3} dt \\
 &= \frac{1}{3} \int_0^1 t^{-2/3} (1-t)^{-1/3} dt \\
 &= \frac{1}{3} \int_0^1 t^{\frac{1}{3}-1} (1-t)^{\frac{2}{3}-1} dt \\
 &= \frac{1}{3} \beta\left(\frac{4}{3}, \frac{2}{3}\right)
 \end{aligned}$$

Def:- The definite integral $\int_0^{\infty} e^{-x} x^{n-1} dx$; $n > 0$ is called the Gamma function (& Eulerian integral of the second kind) and is denoted by $\Gamma(n)$ and read as 'Gamma n'.

$$\text{Thus } \Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx; n > 0$$

PROPERTIES OF GAMMA FUNCTION

1. Show that $\Gamma(1) = 1$

Proof: By the definition of Gamma function

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$$

$$\Gamma(1) = \int_0^{\infty} e^{-x} dx = \left[\frac{e^{-x}}{-1} \right]_0^{\infty} = -[e^{-\infty} - e^{-0}] = -[0 - 1] = 1$$

$$\therefore \boxed{\Gamma(1) = 1}$$

② Show that $\Gamma(n+1) = n \Gamma(n)$ for $n > 0$

proof:-

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$$

$$\Gamma(n+1) = \int_0^{\infty} e^{-x} x^n dx$$

$$= \left[-x^n e^{-x} \right]_0^{\infty} + n \int_0^{\infty} e^{-x} x^{n-1} dx$$

$$= 0 + n \Gamma(n)$$

$$\boxed{\Gamma(n+1) = n \Gamma(n)}$$

③ Show that $\Gamma(n) = 2 \int_0^{\infty} e^{-x^2} x^{2n-1} dx$

proof:

By the definition,

$$\text{we have } \Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$$

$$\text{put } x = t^2$$

$$dx = 2t dt$$

as $x \rightarrow 0$, $t \rightarrow 0$ and as $x \rightarrow \infty$, $t \rightarrow \infty$

$$\Gamma(n) = \int_0^{\infty} e^{-t} t^{n-1} dt$$

$$= 2 \int_0^{\infty} e^{-t^2} t^{2n-1} dt$$

$$\text{Hence } \Gamma(n) = 2 \int_0^{\infty} e^{-x^2} x^{2n-1} dx$$

④ Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Proof:-

By the def; we have

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$$

$$\text{We have } \Gamma(n) = 2 \int_0^{\infty} e^{-x^2} x^{2n-1} dx$$

$$\Gamma(\frac{1}{2}) = 2 \int_0^{\infty} e^{-x^2} dx \quad \text{--- (1)}$$

$$\text{Also, we have } \Gamma(\frac{1}{2}) = 2 \int_0^{\infty} e^{-y^2} dy \quad \text{--- (2)}$$

$$\text{①} \times \text{②} \Rightarrow [\Gamma(\frac{1}{2})]^2 = 4 \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy$$

Changing into polar coordinates we have

$$x = r \cos \theta ; y = r \sin \theta ; dx dy = r dr d\theta$$

θ varies from 0 to $\frac{\pi}{2}$ and r varies from 0 to ∞

$$[\Gamma(\frac{1}{2})]^2 = 4 \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-r^2} r dr d\theta$$

$$\text{put } r^2 = t \Rightarrow 2r dr = dt$$

$$\Rightarrow r dr = \frac{dt}{2}$$

$$= 2 \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-t} dt d\theta$$

$$= 2 \int_0^{\frac{\pi}{2}} \left[\frac{-e^{-t}}{-1} \right]_0^{\infty} d\theta$$

$$= 2 \int_0^{\frac{\pi}{2}} -[e^{-\infty} - e^{-0}] d\theta$$

$$= 2 \int_0^{\frac{\pi}{2}} d\theta$$

$$= 2[0]_0^{\frac{\pi}{2}} = 2(\frac{\pi}{2}) - 0 = \pi$$

$$[\Gamma(\frac{1}{2})]^2 = \pi$$

$$\therefore \Gamma(\frac{1}{2}) = \sqrt{\pi}$$

Soln:

consider $\int_0^{\infty} x^v e^{-x} dx$

put $x^v = t \Rightarrow x = t^{1/2}$
 $dx = \frac{1}{2} t^{-1/2} dt$

as $x \rightarrow 0$; $t \rightarrow 0$ and as $x \rightarrow \infty$, $t \rightarrow \infty$

$$\begin{aligned} &= \int_0^{\infty} t e^{-t} \frac{1}{2} t^{-1/2} dt \\ &= \frac{1}{2} \int_0^{\infty} e^{-t} t^{1/2} dt \\ &= \frac{1}{2} \int_0^{\infty} e^{-t} t^{3/2-1} dt \\ &= \frac{1}{2} \Gamma\left(\frac{3}{2}\right) \\ &= \frac{1}{2} \left(\frac{3}{2}-1\right) \Gamma\left(\frac{3}{2}-1\right) \\ &= \frac{1}{2} \left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right) \\ &= \frac{\sqrt{\pi}}{4} \end{aligned}$$

②

Show that $\Gamma(n) = \int_0^1 (\log \frac{1}{x})^{n-1} dx$, $n > 0$

Soln:

R.H.S $\int_0^1 (\log \frac{1}{x})^{n-1} dx$

put $t = \log \frac{1}{x} = -\log x$

$$\begin{aligned} &= \int_0^{\infty} t^{n-1} (-e^{-t}) dt \\ &= \int_0^{\infty} e^{-t} t^{n-1} dt \\ &= \Gamma(n) \\ &= \text{L.H.S} \end{aligned}$$

$$\begin{aligned} \Rightarrow -t &= \log x \\ \Rightarrow e^{-t} &= x \\ \therefore x &= e^{-t} \\ dx &= -e^{-t} dt \end{aligned}$$

as $x \rightarrow 0$, $t \rightarrow \infty$, as $x \rightarrow 1$, $t \rightarrow 0$

Soln:-

$$\text{L.H.S } \int_0^1 x^3 (\log \frac{1}{x})^4 dx$$

$$= \int_0^{\infty} e^{-3t} t^4 (-e^{-t}) dt$$

$$= \int_0^{\infty} e^{-4t} t^4 dt$$

$$\text{put } \log \frac{1}{x} = t$$

$$\Rightarrow \frac{1}{x} = e^t$$

$$\Rightarrow x = e^{-t}$$

$$dx = -e^{-t} dt$$

$$\text{as } x \rightarrow 0; t \rightarrow \infty \text{ and}$$

$$\text{as } x \rightarrow 1; t \rightarrow 0$$

$$\text{again put } 4t = u$$

$$4 dt = du$$

$$dt = \frac{1}{4} du$$

$$= \int_0^{\infty} e^{-u} \left(\frac{u}{4}\right)^4 d\frac{1}{4} du$$

$$= \int_0^{\infty} e^{-u} \frac{u^4}{4^5} du$$

$$= \frac{1}{4^5} \int_0^{\infty} e^{-u} u^{5-1} du$$

$$= \frac{1}{4^5} \Gamma(5)$$

$$= \frac{4!}{4^5} = \frac{(4 \cdot 3 \cdot 2 \cdot 1)}{4 \times 4 \times 4 \times 64} = \frac{3}{128}$$

$$= \text{R.H.S}$$

$$\int_0^1 (x^3 \log \frac{1}{x})^4 dx = \frac{3}{128}$$

Soln:-

consider

$$\begin{aligned} & \int_0^{\infty} \frac{4x^v}{3} dx \\ &= \int_0^{\infty} e^{\log 3} \cdot 4x^v dx \\ &= \int_0^{\infty} e^{4x^v \log 3} dx \end{aligned}$$

$$\boxed{e^{\log f(x)} = f(x)}$$

$$4x^v \log 3 = t$$

$$\Rightarrow x^v = \frac{1}{4 \log 3} t$$

$$\Rightarrow x = \frac{1}{2} \frac{1}{\sqrt{\log 3}} \cdot t^{\frac{1}{2}}$$

$$\Rightarrow dx = \frac{1}{2} \frac{1}{\sqrt{\log 3}} \cdot \frac{1}{2} t^{-\frac{1}{2}} dt$$

$$= dx = \frac{1}{4} \frac{1}{\sqrt{\log 3}} t^{-\frac{1}{2}} dt$$

$$= \int_0^{\infty} e^{-t} \frac{1}{4} \frac{1}{\sqrt{\log 3}} t^{-\frac{1}{2}} dt$$

$$= \frac{1}{4 \sqrt{\log 3}} \int_0^{\infty} e^{-t} t^{\frac{1}{2}-1} dt$$

$$= \frac{1}{4 \sqrt{\log 3}} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{4 \sqrt{\log 3}}$$

$$\therefore \int_0^{\infty} \frac{4x^v}{3} dx = \frac{\sqrt{\pi}}{4 \sqrt{\log 3}}$$

(5)

Show that

$$\int_0^{\infty} \frac{x^a}{a^x} dx = \frac{\Gamma(a+1)}{(\log a)^{a+1}} \quad (a > 1)$$

Soln

L.H.S $\int_0^{\infty} \frac{x^a}{a^x} dx$

put $a^x = e^t$

$\Rightarrow x \log a = t$

$$x = \frac{1}{\log a} t$$

$$dx = \frac{1}{\log a} dt$$

as $x \rightarrow 0$; $t \rightarrow 0$ and $x \rightarrow \infty$, $t \rightarrow \infty$

$$= \int_0^{\infty} \left(\frac{t}{\log a} \right)^a e^{-t} \frac{1}{\log a} dt$$

$$= \frac{1}{(\log a)^{a+1}} \int_0^{\infty} e^{-t} dt \cdot t^a$$

$$= \frac{1}{(\log a)^{a+1}} \int_0^{\infty} e^{-t} t^{(a+1)-1} dt$$

$$= \frac{1}{(\log a)^{a+1}} \Gamma(a+1)$$

= R.H.S

$$\therefore \int_0^{\infty} \frac{x^a}{a^x} dx = \frac{\Gamma(a+1)}{(\log a)^{a+1}}$$

⑤

Show that $\int_0^{\infty} e^{-kx} x^{n-1} dx = \frac{\Gamma(n)}{k^n}$

Soln:

L.H.S $\int_0^{\infty} e^{-kx} x^{n-1} dx$

put $kx = t \Rightarrow x = \frac{1}{k} t$

$dx = \frac{1}{k} dt$

$$= \int_0^{\infty} e^{-t} \left(\frac{t}{k} \right)^{n-1} \frac{1}{k} dt$$

as $x \rightarrow 0$; $t \rightarrow 0$ and $x \rightarrow \infty$, $t \rightarrow \infty$

$$= \int_0^{\infty} e^{-t} \frac{t^{n-1}}{k^{n-1}} \cdot \frac{1}{k} dt$$

$$= \frac{1}{k^n} \int_0^{\infty} e^{-t} t^{n-1} dt$$

$$= \frac{\Gamma(n)}{k^n}$$

= R.H.S

$$\therefore \int_0^{\infty} e^{-kx} x^{n-1} dx = \frac{\Gamma(n)}{k^n}$$

prove that $\beta(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$ where $m > 0, n > 0$

Proof:

We know that

$$\Gamma(m) = 2 \int_0^{\infty} e^{-x^2} x^{2m-1} dx \quad \text{--- (1)}$$

$$\text{Ily } \Gamma(n) = 2 \int_0^{\infty} e^{-y^2} y^{2n-1} dy \quad \text{--- (2)}$$

$$\begin{aligned} \text{(1) x (2)} \Rightarrow \Gamma(m) \Gamma(n) &= 4 \int_0^{\infty} e^{-x^2} x^{2m-1} dx \int_0^{\infty} e^{-y^2} y^{2n-1} dy \\ &= 4 \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} x^{2m-1} y^{2n-1} dy dx \quad \text{--- (3)} \end{aligned}$$

changing to polar co-ordinates by writing

$$x = r \cos \theta ; y = r \sin \theta ; dx dy = r dr d\theta$$

r varies from 0 to ∞ and θ varies from 0 to $\pi/2$

$$\begin{aligned} \Gamma(m) \Gamma(n) &= 4 \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2} (r \cos \theta)^{2m-1} (r \sin \theta)^{2n-1} r dr d\theta \\ &= 2 \int_{\theta=0}^{\pi/2} \cos^{2m-1} \theta \sin^{2n-1} \theta d\theta \cdot 2 \int_{r=0}^{\infty} e^{-r^2} r^{2(m+n)-1} dr \\ &= \beta(m, n) \Gamma(m+n) \end{aligned}$$

$$\frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} = \beta(m, n)$$

$$\therefore \beta(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$$

prove that $\int_0^1 x^{n-1} (1-x)^{n-1} dx = \frac{\pi}{\sin n\pi}$ if n is not an integer

Soln:

we know that

$$\beta(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} \quad \text{--- (1)}$$

$$\text{and also } \beta(m, n) = \int_0^1 \frac{x^{m-1} (1-x)^{n-1}}{(1-x)^{m+n}} dx \quad \text{--- (2)}$$

$$\therefore \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} = \int_0^1 \frac{x^{m-1} (1-x)^{n-1}}{(1-x)^{m+n}} dx$$

$$\text{put } m = 1-n$$

$$\Rightarrow m+n = 1$$

$$\frac{\Gamma(1-n) \Gamma(n)}{\Gamma(1)} = \int_0^1 \frac{x^{n-1} (1-x)^{-n}}{(1-x)^1} dx$$

$$= \Gamma(n) \Gamma(1-n) = \int_0^1 \frac{x^{n-1}}{(1-x)} dx$$

$$\Rightarrow \text{But we have } \int_0^1 \frac{x^{n-1}}{(1-x)} dx = \frac{\pi}{\sin n\pi} \text{ for non integer } n.$$

$$\therefore \Gamma(n) \Gamma(1-n) = \frac{\pi}{\sin n\pi} \text{ for non integer } n.$$

(2) Prove that $\int_0^1 \frac{x^8 (1-x^6)}{(1+x)^{24}} dx = 0$ using β and Γ functions

Soln:

$$\text{LHS} = \int_0^1 \frac{x^8}{(1+x)^{24}} dx - \int_0^1 \frac{x^{14}}{(1+x)^{24}} dx$$

$$= \int_0^1 \frac{x^{9-1}}{(1+x)^{9+15}} dx - \int_0^1 \frac{x^{15-1}}{(1+x)^{15+9}} dx$$

$$= \beta(9, 15) - \beta(15, 9)$$

$$= \beta(9, 15) - \beta(9, 15)$$

$$= 0$$

$$= \text{R.H.S}$$

$$(\because \beta(m, n) = \beta(n, m))$$

Soln:

Consider

$$\int_0^2 (8-x^3)^{-1/3} dx$$

Put $x^3 = 8t$

$$\Rightarrow x = 2t^{1/3}$$

$$dx = \frac{2}{3} t^{-2/3} dt$$

as $x \rightarrow 0$, $t \rightarrow 0$ and as $x \rightarrow 2$, $t \rightarrow 1$

$$= \int_0^1 (8-8t)^{-1/3} \cdot \frac{2}{3} t^{-2/3} dt$$

$$= \frac{2}{3} (8)^{-1/3} \int_0^1 t^{-2/3} (1-t)^{-1/3} dt$$

$$= \frac{1}{3} \int_0^1 t^{\frac{1}{3}-1} (1-t)^{\frac{2}{3}-1} dt$$

$$= \frac{1}{3} B\left(\frac{1}{3}, \frac{2}{3}\right)$$

$$= \frac{1}{3} \frac{\Gamma_{1/3} \Gamma_{2/3}}{\Gamma_{1/3+2/3}}$$

$$= \frac{1}{3} \Gamma_{1/3} \Gamma_{1-1/3}$$

$$= \frac{1}{3} \frac{\pi}{\sin \pi/3}$$

$$= \frac{\pi}{3} \cdot \frac{2}{\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$

④

Express $\int_0^1 x^m (1-x^n)^p dx$ in terms of Γ function.

Soln:

consider $\int_0^1 x^m (1-x^n)^p dx$

put $x^n = t$

$$\Rightarrow x = t^{1/n}$$

$$dx = \frac{1}{n} t^{\frac{1}{n}-1} dt$$

$$= \int_0^1 t^{m/n} (1-t)^p \cdot \frac{1}{n} t^{\frac{1}{n}-1} dt$$

$$= \frac{1}{n} \int_0^1 t^{\left(\frac{m}{n} + \frac{1}{n} - 1\right)} (1-t)^p dt$$

$$= \frac{1}{n} \int_0^1 t^{\left(\frac{m+1}{n} - 1\right)} (1-t)^{p+1} dt$$

$$= \frac{1}{n} B\left(\frac{m+1}{n}, p+1\right)$$

⑤ Evaluate $\int_0^1 x^5 (1-x^3)^{10} dx$

Soln: We know that

$$\int_0^1 x^m (1-x^n)^p dx = \frac{1}{n} \frac{\Gamma\left(\frac{m+1}{n}\right) \Gamma(p+1)}{\Gamma\left(\frac{m+1}{n} + p+1\right)}$$

Here $m=5$; $n=3$; $p=10$

$$\begin{aligned} \int_0^1 x^5 (1-x^3)^{10} dx &= \frac{1}{3} \frac{\Gamma\left(\frac{6}{3}\right) \Gamma(11)}{\Gamma\left(\frac{6}{3} + 11\right)} = \frac{1}{3} \frac{\Gamma(2) \Gamma(11)}{\Gamma(13)} \\ &= \frac{1}{396} \end{aligned}$$

⑥ Show that $\Gamma(x) \Gamma(-x) = \frac{-\pi}{x \sin \pi x}$

Soln: We know that

$$\Gamma(n) \Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

$$\begin{aligned} \text{Now } \Gamma(x) \Gamma(-x) &= \Gamma(x) \left[-\frac{1}{x} \Gamma(-x+1) \right] \quad \boxed{\Gamma(n) = \frac{1}{n} \Gamma(n+1)} \\ &= -\frac{1}{x} \Gamma(x) \Gamma(1-x) \\ &= -\frac{1}{x} \frac{\pi}{\sin \pi x} \\ &= \frac{-\pi}{x \sin \pi x} \end{aligned}$$

⑦ Show that $\Gamma\left(\frac{1}{2}+x\right) \Gamma\left(\frac{1}{2}-x\right) = \pi \sec \pi x$

Soln: We know that $\Gamma(n) \Gamma(1-n) = \frac{\pi}{\sin n\pi}$

$$\begin{aligned} \Gamma\left(\frac{1}{2}+x\right) \Gamma\left(\frac{1}{2}-x\right) &= \Gamma\left(\frac{1}{2}+x\right) \Gamma\left(1-\left(\frac{1}{2}+x\right)\right) \\ &= \frac{\pi}{\sin\left[\left(\frac{1}{2}+x\right)\pi\right]} \\ &= \frac{\pi}{\sin\left(\frac{\pi}{2} + \pi x\right)} = \frac{\pi}{\cos \pi x} = \pi \sec \pi x \end{aligned}$$

Soln:-

consider $4 \int_0^{\infty} \frac{x^2}{1+x^4} dx$ put $x^4 = t$

$$\Rightarrow x = t^{1/4}$$

$$dx = \frac{1}{4} t^{-3/4} dt$$

as $x \rightarrow 0, t \rightarrow 0$ and as $x \rightarrow \infty, t \rightarrow \infty$

$$= 4 \int_0^{\infty} \frac{t^{2/4} t^{-3/4}}{4(1+t)} dt$$

$$= \int_0^{\infty} \frac{t^{-1/4}}{(1+t)} dt$$

$$= \int_0^{\infty} \frac{t^{3/4-1}}{(1+t)^{3/4+1/4}} dt$$

$$= \beta\left(\frac{3}{4}, \frac{1}{4}\right)$$

$$= \frac{\Gamma_{3/4} \Gamma_{1/4}}{\Gamma_{3/4+1/4}}$$

$$= \Gamma_{3/4} \Gamma_{1/4}$$

$$= \Gamma_{1/4} \Gamma_{1-1/4} = \frac{\pi}{\sin \frac{\pi}{4}} = \sqrt{2}\pi$$

⑨

Evaluate $\int_0^{\pi/2} \sin^4 \theta \cos^2 \theta d\theta$

Soln:-

We know that

$$\int_0^{\pi/2} \sin^m \theta \cos^n \theta d\theta = \frac{1}{2} \beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

put $m=4$; $n=2$

$$\int_0^{\pi/2} \sin^4 \theta \cos^2 \theta d\theta = \frac{1}{2} \beta\left(\frac{4+1}{2}, \frac{2+1}{2}\right)$$

$$= \frac{1}{2} \beta\left(\frac{5}{2}, \frac{3}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma_{5/2} \Gamma_{3/2}}{\Gamma_{5/2+3/2}}$$

$$= \frac{1}{2} \left(\frac{5}{2} - 1 \right) \left(\frac{3}{2} - 1 \right) \sqrt{\frac{3}{2} - 1} \left(\frac{3}{2} - 1 \right) \sqrt{\frac{3}{2} - 1}$$

$$= \frac{1}{6} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi} \cdot \frac{1}{2} \sqrt{\pi} = \frac{\pi}{16}$$

(10) Show that $\int_0^{\infty} \sin x^2 dx = \int_0^{\infty} \cos x^2 dx = \frac{1}{2} \sqrt{\frac{\pi}{2}}$

Soln: Let $A = \int_0^{\infty} \cos x^2 dx$ and $B = \int_0^{\infty} \sin x^2 dx$

Consider $A - iB = \int_0^{\infty} [\cos x^2 - i \sin x^2] dx = \int_0^{\infty} e^{-ix^2} dx$

put $ix^2 = t$

$\Rightarrow e^{i\pi/2} x^2 = t$

$\Rightarrow e^{i\pi/2} x = \sqrt{t}$

$\Rightarrow x = e^{-i\pi/2} \sqrt{t}$

$\Rightarrow dx = e^{-i\pi/2} \frac{1}{2\sqrt{t}} dt$

as $x \rightarrow 0; t \rightarrow 0$ and as $x \rightarrow \infty; t \rightarrow \infty$

$$A - iB = \int_0^{\infty} e^{-t} e^{-i\pi/2} \frac{1}{2\sqrt{t}} dt$$

$$= \frac{e^{-i\pi/2}}{2} \int_0^{\infty} e^{-t} t^{\frac{1}{2}-1} dt = \frac{e^{-i\pi/2}}{2} \Gamma_{\frac{1}{2}}$$

$$= \frac{\sqrt{\pi}}{2} [\cos \frac{\pi}{4} - i \sin \frac{\pi}{4}]$$

$$= \frac{\sqrt{\pi}}{2} \left[\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}} \right]$$

$$A - iB = \frac{1}{2} \sqrt{\frac{\pi}{2}} - i \frac{1}{2} \sqrt{\frac{\pi}{2}}$$

Comparing real and imaginary parts we have

$$A = \frac{1}{2} \sqrt{\frac{\pi}{2}} \text{ and } B = \frac{1}{2} \sqrt{\frac{\pi}{2}}$$

Hence $\int_0^{\infty} \cos x^2 dx = \int_0^{\infty} \sin x^2 dx = \frac{1}{2} \sqrt{\frac{\pi}{2}}$