

I Year - II Semester

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MATHEMATICS-II (Mathematical Methods)**Course Objectives:**

1. The course is designed to equip the students with the necessary mathematical skills and techniques that are essential for an engineering course.
2. The skills derived from the course will help the student from a necessary base to develop analytic and design concepts.
3. Understand the most basic numerical methods to solve simultaneous linear equations.

Course Outcomes: At the end of the Course, Student will be able to:

1. Calculate a root of algebraic and transcendental equations. Explain relation between the finite difference operators.
2. Compute interpolating polynomial for the given data.
3. Solve ordinary differential equations numerically using Euler's and RK method.
4. Find Fourier series and Fourier transforms for certain functions.
5. Identify/classify and solve the different types of partial differential equations.

UNIT I: Solution of Algebraic and Transcendental Equations:

Introduction- Bisection method – Method of false position – Iteration method – Newton-Raphson method (One variable and simultaneous Equations).

UNIT II: Interpolation:

Introduction- Errors in polynomial interpolation – Finite differences- Forward differences- Backward differences – Central differences – Symbolic relations and separation of symbols - Differences of a polynomial-Newton's formulae for interpolation – Interpolation with unequal intervals - Lagrange's interpolation formula.

UNIT III: Numerical Integration and solution of Ordinary Differential equations:

Trapezoidal rule- Simpson's $1/3^{\text{rd}}$ and $3/8^{\text{th}}$ rule-Solution of ordinary differential equations by Taylor's series-Picard's method of successive approximations-Euler's method - Runge-Kutta method (second and fourth order).

UNIT IV: Fourier Series:

Introduction- Periodic functions – Fourier series of -periodic function - Dirichlet's conditions – Even and odd functions – Change of interval– Half-range sine and cosine series.

UNIT V: Applications of PDE:

Method of separation of Variables- Solution of One dimensional Wave, Heat and two-dimensional Laplace equation.

UNIT VI: Fourier Transforms:

Fourier integral theorem (without proof) – Fourier sine and cosine integrals - sine and cosine transforms – properties – inverse transforms – Finite Fourier transforms.

Text Books:

1. **B.S.Grewal**, Higher Engineering Mathematics, 43rd Edition, Khanna Publishers.
2. **N.P.Bali**, Engineering Mathematics, Lakshmi Publications.

Reference Books:

1. **Dean G. Duffy**, Advanced engineering mathematics with MATLAB, CRC Press
2. **V.Ravindranath and P.Vijayalakshmi**, Mathematical Methods, Himalaya Publishing House.
3. **Erwin Kreyszig**, Advanced Engineering Mathematics, 10th Edition, Wiley-India
4. **David Kincaid, Ward Cheney**, Numerical Analysis-Mathematics of Scientific Computing, 3rd Edition, Universities Press.
5. **Srimanta Pal, Subodh C.Bhunia**, Engineering Mathematics, Oxford University Press.
6. **Dass H.K., Rajnish Verma. Er.**, Higher Engineering Mathematics, S. Chand Co. Pvt. Ltd, Delhi.

Unit - I

Method 1:

SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS.

Bisection Method,

Consider an equation $f(x) = 0$, find a, b , so that

$f(a), f(b)$ have opposite sign

$$x_1 = \frac{a+b}{2}$$

i) If $f(x_1) \leq 0$ is negative then root lies b/w x_1, b

$$f(x_2) = \frac{x_1+b}{2}$$

ii) If $f(x_1) > 0$ is positive, then root lies b/w a and x_1

$$f(x_2) = \frac{a+x_1}{2}$$

Problem:

1) Find the positive root of $x^3 - x - 1 = 0$ correct two decimal places by bisection method.

Sol
Let $f(x) = x^3 - x - 1 = 0$

Consider $f(x) = x^3 - x - 1$

$$f(0) = 0 - 0 - 1 = -1 \text{ (-ve)}$$

$$f(1) = 1 - 1 - 1 = -1 \text{ (-ve)}$$

$$f(2) = 8 - 2 - 1 = 5 \text{ (+ve)}$$

Since $f(1) < 0$ and $f(2) > 0$

$\therefore$ The root lies b/w 1 and 2.

Let $a=1$, $b=2$

by bisection method,

$$x_1 = \frac{a+b}{2} = \frac{1+2}{2} = \frac{3}{2} = 1.5$$

$$f(x_1) = f(1.5) = (1.5)^3 - (1.5) - 1 = 0.875 \text{ (+ve)}$$

Since $f(1) < 0$ and $f(1.5) > 0$

$\therefore$ The root lies b/w 1 & 1.5

$$x_2 = \frac{x_1+b}{2} = \frac{1.5+1}{2} = \frac{2.5}{2} = 1.25 \text{ (+ve)}$$

$$\begin{aligned} f(x_2) &= f(1.25) = (1.25)^3 - (1.25) - 1 \\ &= -0.296 \text{ (-ve)} \end{aligned}$$

Since $f(1.25) < 0$ and $f(1.5) > 0$

$\therefore$ The root lies b/w 1.25 and 1.5

$$\therefore x_3 = \frac{1.25+1.5}{2} = \frac{2.75}{2} = 1.375 \text{ (+ve)}$$

$$\begin{aligned} f(x_3) &= f(1.375) = (1.375)^3 - (1.375) - 1 \\ &= 0.2246 \text{ (+ve)} \end{aligned}$$

Since $f(1.25) < 0$ and $f(1.375) > 0$

$\therefore$ The root lies b/w 1.25 & 1.375

$$\therefore x_4 = \frac{1.25+1.375}{2} = \frac{2.625}{2} = 1.3125 \text{ (+ve)}$$

$$\begin{aligned} f(x_4) &= f(1.3125) = (1.3125)^3 - (1.3125) - 1 \\ &= -0.0515 \text{ (-ve)} \end{aligned}$$

$\therefore$ Since $f(1.3125) < 0$ and $f(1.375) > 0$,

Let $a=1$, $b=2$

by bisection method,

$$x_1 = \frac{a+b}{2} = \frac{1+2}{2} = \frac{3}{2} = 1.5$$

$$f(x) = f(1.5) = (1.5)^3 - (1.5) - 1 = 0.875 \text{ (+ve)}$$

Since $f(1) < 0$ and $f(2) > 0$

$\therefore$ The root lies b/w 1 & 1.5

$$x_2 = \frac{x_1+b}{2} = \frac{1.5+1}{2} = \frac{2.5}{2} = 1.25 \text{ (+ve)}$$

$$f(x_2) = f(1.25) = (1.25)^3 - (1.25) - 1 = -0.296 \text{ (-ve)}$$

Since $f(1.25) < 0$ and $f(1.5) > 0$

$\therefore$ The root lies b/w 1.25 and 1.5

$$\therefore x_3 = \frac{1.25+1.5}{2} = \frac{2.75}{2} = 1.375 \text{ (+ve)}$$

$$f(x_3) = f(1.375) = (1.375)^3 - (1.375) - 1 = 0.2246 \text{ (+ve)}$$

Since $f(1.25) < 0$ and $f(1.375) > 0$

$\therefore$ The root lies b/w 1.25 & 1.375

$$\therefore x_4 = \frac{1.25+1.375}{2} = \frac{2.625}{2} = 1.3125 \text{ (+ve)}$$

$$f(x_4) = f(1.3125) = (1.3125)^3 - (1.3125) - 1 = -0.0515 \text{ (-ve)}$$

$\therefore$ Since $f(1.3125) < 0$ and $f(1.375) > 0$,

∴ The root lies b/w 1.3125 & 1.375

$$\therefore x_5 = \frac{1.3125 + 1.375}{2} = \frac{2.6875}{2} = 1.343 \text{ (ve)}$$

$$f(x_5) = f(1.343) = (1.343)^3 - (1.343) - 1 = 0.062 \text{ (ve)}$$

Since $f(1.343) > 0$ and $f(1.3125) < 0$

∴ The root lies b/w 1.3125 & 1.343

$$\therefore x_6 = \frac{1.3125 + 1.343}{2} = \frac{2.6563}{2} = 1.3281$$

$$f(x_6) = f(1.3281) = (1.3281)^3 - (1.3281) - 1 = 0.0145$$

Since $f(1.3125) < 0$ and $f(1.3281) > 0$

∴ The root lies b/w 1.3125 & 1.3281

$$\therefore x_7 = \frac{1.3125 + 1.3281}{2} = \frac{2.6406}{2} = 1.3203$$

upto 3 decimal places x_7, x_6 are same.

∴ The root of the given eq. is 1.32.

Method 2: Method of false position:-

Consider an Equation $f(x) = 0$, find a, b , so that

$f(a), f(b)$ have opposite signs

$$x = \frac{a \cdot f(b) - b \cdot f(a)}{f(b) - f(a)}$$

Problem:

- Find the root of the Eqn. $x \log_{10} x = 1.2$. Using false position method.

84 let $f(x) = x \log_{10} x - 1.2$

Consider

$$f(2) = 2 \times \log_{10}(2) - 1.2 = 2 \times 0.3010 - 1.2 = -0.5979 \text{ (ve)}$$

$$f(3) = 3 \times \log_{10}(3) - 1.2 = 3 \times 0.47712 - 1.2 = 0.23136 \text{ (ve)}$$

Since $f(2) < 0$ and $f(3) > 0$ here opposite sign

$\therefore$ The root lies b/w 2 and 3.

By false position method.

$$a = 2, b = 3.$$

$$x = \frac{a \cdot f(b) - b \cdot f(a)}{f(b) - f(a)} = \frac{2 \times (0.23136) - 3 \times (-0.59794)}{0.23136 - (-0.59794)}$$

$$= 2.7210$$

$$f(x) = f(2.7210) = 2.7210 \log_{10}(2.7210) - 1.2 = -0.0171$$

$\therefore$ The root lies b/w 2.721 & 3

$$x_1 = \frac{2.721 \times 0.23136 - 3 \times (-0.0171)}{0.23136 - (-0.0171)} = 2.740$$

$$f(x_1) = f(2.740) = 2.740 \log_{10}(2.740) - 1.2 = -0.00056$$

The root lies b/w 2.740 and 3.

$$\therefore x_2 = \frac{2.740 \times (0.23166) - 3 \times (-0.00056)}{0.23136 - (-0.00056)} = 2.7406$$

upto two decimal places x_1, x_2 are same
 $\therefore$ The root is 2.74

Method 3:- Iteration method:-

Given equation $f(x) = 0$

We write $x = \phi(x)$, so that $|\phi'(x)| < 1$

Find a, b so that $f(a), f(b)$ have opposite sign

Initial approximation by bisection method,

$$x_0 = \frac{a+b}{2}$$

1st approximation $x_1 = \phi(x_0)$

2nd approximation $x_2 = \phi(x_1)$

$\vdots$

n^{th} approximation $x_n = \phi(x_{n-1})$

Problem:

Find the positive root of $x^4 - x - 10 = 0$ by iteration method.

Given that $x^4 - x - 10 = 0$

$$x = x^4 - 10$$

$$x = \frac{10}{x^3 - 1}$$

It can be written as $x = \phi(x)$

$x = (x+10)^{1/4}$, satisfies the converge criteria

$$|\phi'(x)| < 1$$

we take iteration formula as

$$x_{i+1} = (x_i + 10)^{1/4}; i = 0, 1, 2, \dots$$

we observe $f(1) < 0$, $f(2) > 0$ for the given equation

hence the root lies b/w 1 and 2.

$$x_0 = \frac{1+2}{2} = 1.5$$

$$x_1 = (1.5 + 10)^{1/4} = 1.8415$$

$$x_2 = (1.8415 + 10)^{1/4} = 1.8550$$

$$x_3 = (1.8550 + 10)^{1/4} = 1.8556$$

$$x_4 = (1.8556 + 10)^{1/4} = 1.8556.$$

hence the root is 1.8556 correct to 4 decimal places.

Method 4: Newton-Raphson method:

Given $f(x) = 0$

Find a, b , so that $f(a), f(b)$ have opposite sign.

Find initial approximation, $x_0 = \frac{a+b}{2}$

1st approximation, $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

2nd approximation $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$\vdots$

n^{th} approximation $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$

Problem

Find the real root of the equation $x \cdot e^x - \cos x = 0$ using Newton-Raphson method.

Given $x e^x - \cos x = 0$

Let $f(x) = x e^x - \cos x$

Find x_1 and x_2 such that $f(x_1)$ and $f(x_2)$ are of opposite signs

$\therefore$ Root of the equation lies b/w x_1 and x_2 .

$f(x) = x e^x - \cos x$

$f'(x) = (x+1)e^x + \sin x$

Now $f(0) = 0 - \cos 0 = -1$ (-ve)

$f'(0) = 1 + \sin 0 = 1$

$f(1) = e - \cos 1 = 2.17797$ (+ve)

$f'(1) = 6.27807$

$\therefore$ The root lies b/w 0 and 1.

Let $x_0 = \frac{x_1 + x_2}{2} = \frac{0+1}{2} = 0.5$

by Newton Raphson method.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

put $i=0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{f(0.5)}{f'(0.5)} = 0.5310$$

$$f(x_1) = 0.0407, \quad f'(x_1) = 3.1106$$

put $i=1$,

$$\begin{aligned} \therefore x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 0.5310 - \frac{0.04073}{3.1106} \\ &= 0.5179 \end{aligned}$$

$$f(x_2) = 0.0004, \quad f'(x_2) = 3.0428.$$

put $i=2$,

$$\begin{aligned} \therefore x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} \\ &= 0.5179 - \frac{0.0004}{3.0428} \\ &= 0.5177 \end{aligned}$$

$$f(x_3) = 0.000001106873, \quad f'(x_3) = 3.042121852.$$

put $i=3$,

$$\begin{aligned} \therefore x_4 &= x_3 - \frac{f(x_3)}{f'(x_3)} \\ &= 0.517757 - \frac{0.000001106873}{3.042121852} \\ &= 0.517757 \end{aligned}$$

$\therefore$ Since $x_3 = x_4 = 0.517757$.

$\therefore$ The Root of the equation is 0.517757 .

⇒ obtain Newton-Raphson formula to find the $\sqrt{N}$ and hence deduce the value of $\sqrt{8}$

A. Let $x = \sqrt{N}$

$$\Rightarrow x^2 = N \text{ (squaring on both sides)}$$

$$\Rightarrow x^2 - N = 0$$

$$\Rightarrow f(x) = x^2 - N$$

$$f'(x) = 2x$$

$$\text{Newton-Raphson formula } x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\begin{aligned} x_{n+1} &= x_n - \frac{x_n^2 - N}{2x_n} \\ &= \frac{2x_n^2 - (x_n^2 - N)}{2x_n} \end{aligned}$$

$$x_{n+1} = \frac{x_n^2 + N}{2x_n} \quad \text{--- (1)}$$

To find $\sqrt{8}$, take $N = 8$ in (1).

$$\begin{aligned} f(x) = x^2 - N &\Rightarrow f(2.8) = (2.8)^2 - 8 = -0.16 < 0 \\ f(2.9) &= (2.9)^2 - 8 = 0.14 > 0 \end{aligned}$$

root lies b/w 2.8 and 2.9

$$\therefore \text{Take } x_0 = \frac{2.8 + 2.9}{2} = 2.85$$

From (1) Put $n=0$; $x_0 = 2.85$, $N = 8$

$$x_1 = \frac{x_0^2 + 8}{2x_0} = \frac{(2.85)^2 + 8}{2(2.85)} = \frac{16.1225}{5.7} = 2.8285$$

$$x_2 = \frac{(2.8285)^2 + 8}{2(2.8285)} = \frac{16.0004}{5.657} = 2.8284$$

x_1 & x_2 are equal upto 2 decimal places.

⇒ Obtain Newton-Raphson formula to find $\sqrt[3]{N}$ and hence deduce the value of $\sqrt[3]{12}$.

A). Let $x = (N)^{1/3} \Rightarrow x^3 = N \Rightarrow x^3 - N = 0$

Let $f(x) = x^3 - N$

$f'(x) = 3x^2$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$= x_n - \frac{x_n^3 - N}{3x_n^2} = \frac{3x_n^3 - (x_n^3 - N)}{3x_n^2}$

$x_{n+1} = \frac{2x_n^3 + N}{3x_n^2} \quad \text{--- (1)}$

To find $\sqrt[3]{12}$, put $N = 12$ in (1).

$x_0 = \frac{2 \cdot 2 + 12}{3} = 2.25$

Put $n=0$, $N=12$, $x_0 = 2.25$ in (1).

$x_1 = \frac{2(2.25)^3 + 12}{3(2.25)^2} = \frac{34.7812}{15.1875} = 2.2901$

$x_2 = \frac{2(2.2901)^3 + 12}{3(2.2901)^2} = \frac{36.0211}{15.7336} = 2.2894$

$x_3 = \frac{2(2.2894)^3 + 12}{3(2.2894)^2} = \frac{35.9991}{15.724} = 2.2894$

x_2 and x_3 are equal upto 4 decimals.

cube root of 12 = 2.2894.

Obtain Newton-Raphson formula for reciprocal of N and hence deduce the value of $1/31$

Sol:-

$$\text{Let } x = \frac{1}{N} \Rightarrow N = \frac{1}{x}$$

$$f(x) = N - \frac{1}{x} = 0 \Rightarrow f'(x) = \frac{1}{x^2}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(N - \frac{1}{x_n})}{\frac{1}{x_n^2}} = x_n - \left(\frac{Nx_n - 1}{\frac{1}{x_n}} \right) x_n^2$$

$$x_{n+1} = x_n - (Nx_n - 1)x_n$$

$$= x_n - Nx_n^2 + x_n$$

$$x_{n+1} = 2x_n - Nx_n^2 \quad \text{--- (1)}$$

To find the value of $\frac{1}{31}$ put $N = 31$

$$x_{n+1} = 2x_n - 31x_n^2$$

$$f(x) = 31 - \frac{1}{x}$$

root lies b/w 0.03 & 0.04

$$x_0 = \frac{0.03 + 0.04}{2} = 0.035$$

Put $n=0$, $x_0 = 0.035$, $N=31$ in (1).

$$x_1 = 2x_0 - 31x_0^2 = 2(0.035) - 31(0.035)^2$$

$$x_1 = 0.0320$$

$$x_2 = 2(0.0320) - 31(0.0320)^2$$

$$= 0.0323$$

$$x_3 = 2(0.0323) - 31(0.0323)^2$$

$$= 0.0323$$

→ Find a real root for $x \tan x + 1 = 0$ using Newton-Raphson method.

Sol. Given $f(x) = x \tan x + 1 = 0$.

$$f'(x) = x \sec^2 x + \tan x$$

$$\text{Now } f(2) = 2 \tan 2 + 1 = -3.370079 < 0.$$

$$f(3) = 3 \tan 3 + 1 = 0.572370 > 0.$$

root lies b/w '2' and '3'.

$$\text{Let } x_0 = \frac{2+3}{2} = 2.5$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.5 - \frac{-0.86755}{3.14808} = 2.77558$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.77558 - \frac{(-0.06383)}{2.80004} = 2.798$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.798 - \frac{-0.0010803052}{2.7983} = 2.798$$

Since $x_2 = x_3$; the real root is 2.798.

→ Errors in polynomial Interpolation

Let the function $y(x)$ defined by $(n+1)$ points (x_i, y_i) $i = 0, 1, 2, \dots, n$ be continuous and differentiable $(n+1)$ times and let $y(x)$ be approximated by a polynomial $\phi_n(x)$ of degree not exceeding 'n' such that

$$\phi_n(x_i) = y_i; \quad i = 0, 1, 2, \dots, n$$

required expression for error is

$$y(x) - \phi_n(x) = \frac{\prod_{n+1}(x)}{(n+1)!} y^{(n+1)}(\xi), \quad x_0 < \xi < x_n$$

Newton-Raphson Method for Solution of Simultaneous Equations

Let (x_0, y_0) be an initial approximation to the root of the equation. If (x_0+h, y_0+k) is the root of the system, then we must have $f(x_0+h, y_0+k) = 0$, $g(x_0+h, y_0+k) = 0$ — (1)

Assuming that f and g are sufficiently differentiable, we expand both the functions in (1), by Taylor's series

$$\left. \begin{aligned} f_0 + h \frac{df}{dx_0} + k \frac{df}{dy_0} + \dots &= 0 \\ g_0 + h \frac{dg}{dx_0} + k \frac{dg}{dy_0} + \dots &= 0 \end{aligned} \right\} \text{--- (2)}$$

where $\frac{df}{dx_0} = \left[\frac{df}{dx} \right]_{x=x_0}$, $f_0 = f(x_0, y_0)$ etc

Neglecting the second and higher-order derivative terms, we obtain the following system of linear equations.

$$\left. \begin{aligned} h \frac{df}{dx_0} + k \frac{df}{dy_0} &= -f_0 \\ \text{and } h \frac{dg}{dx_0} + k \frac{dg}{dy_0} &= -g_0 \end{aligned} \right\} \text{--- (3)}$$

Equation (3) possesses a unique solution if

$$D = \begin{vmatrix} \frac{df}{dx_0} & \frac{df}{dy_0} \\ \frac{dg}{dx_0} & \frac{dg}{dy_0} \end{vmatrix} \neq 0 \text{ --- (4)}$$

By Cramer's rule, the solution of Eq. (3) is given by.

$$h = \frac{1}{D} \begin{vmatrix} -f_0 & \frac{df}{dy_0} \\ -g_0 & \frac{dg}{dy_0} \end{vmatrix} \text{ and } k = \frac{1}{D} \begin{vmatrix} \frac{df}{dx_0} & -f_0 \\ \frac{dg}{dx_0} & -g_0 \end{vmatrix}$$

The new approximations are $x_1 = x_0 + h$, and $y_1 = y_0 + k$

The process is to be repeated and we obtain the roots

⇒ Solve the system of Equations $3y^2 - 10x + 7 = 0$, $y^2 - 5y + 4 = 0$.
 by using Newton-Raphson method.

A) we have $f(x) = 3y^2 - 10x + 7 = 0$, $g(x) = y^2 - 5y + 4 = 0$.

Then $\frac{df}{dx} = 6yx - 10$, $\frac{df}{dy} = 3x^2$, $\frac{dg}{dx} = 0$, $\frac{dg}{dy} = 2y - 5$.

Taking $x_0 = y_0 = 0.5$, we obtain

$$\frac{df}{dx_0} = -8.5, \quad \frac{df}{dy_0} = 0.75, \quad f_0 = 2.375$$

$$\frac{dg}{dx_0} = 0, \quad \frac{dg}{dy_0} = -4, \quad g_0 = 1.75$$

Hence $D = \begin{vmatrix} -8.5 & 0.75 \\ 0 & -4 \end{vmatrix} = 34 \neq 0$.

$$h = \frac{1}{34} \begin{vmatrix} -2.375 & 0.75 \\ -1.75 & -4 \end{vmatrix} = 0.3180$$

$$k = \frac{1}{34} \begin{vmatrix} -8.5 & -2.375 \\ 0 & -1.75 \end{vmatrix} = 0.4375$$

It follows that $x_1 = 0.5 + 0.3180 = 0.8180$
 $y_1 = 0.5 + 0.4375 = 0.9375$

For the second approximation, we have

$$f_1 = 0.7019, \quad g_1 = 0.1914$$

$$\frac{df}{dx_1} = -5.3988, \quad \frac{df}{dy_1} = 2.0074$$

$$\frac{dg}{dx_1} = 0, \quad \frac{dg}{dy_1} = -3.125$$

$$D = \begin{vmatrix} -5.3988 & 2.0074 \\ 0 & -3.125 \end{vmatrix} = 16.8712 \neq 0$$

$$h = \frac{1}{16.8712} \begin{vmatrix} -0.7019 & 2.0074 \\ -0.1914 & -3.125 \end{vmatrix} = 0.1528$$

$$K = \frac{1}{16.8712} \begin{vmatrix} -5.3918 & -0.7019 \\ 0 & -0.1914 \end{vmatrix} = 0.0612$$

It follows that

$$x_2 = 0.8180 + 0.1528 = 0.9708$$

$$y_2 = 0.9375 + 0.0612 = 0.9987$$

⇒ solve the system $x^2 + y^2 = 1$ and $y = x^2$ by Newton-Raphson method.

Sol Let $f = x^2 + y^2 - 1$ and $g = y - x^2$

From the graphs of the curves, we find that there are two points of intersection, one each in the first and second quadrants. we shall approximate to the solution in the first quadrant. we have.

$$\frac{df}{dx} = 2x, \quad \frac{df}{dy} = 2y, \quad \frac{dg}{dx} = -2x, \quad \frac{dg}{dy} = 1.$$

we start with $x_0 = y_0 = 0.7071$ obtained the approximation $y = x$, Then we compute.

$$\frac{df}{dx_0} = 1.4142, \quad \frac{df}{dy_0} = 1.4142.$$

$$-\frac{dg}{dx_0} = -1.4142, \quad \frac{dg}{dy_0} = 1.$$

$$D = \begin{vmatrix} 1.4142 & 1.4142 \\ -1.4142 & 1 \end{vmatrix} = 3.4142 \neq 0, \quad f_0 = 0$$

$$h = \frac{1}{3.4142} \begin{vmatrix} 0 & 1.4142 \\ -0.2071 & 1 \end{vmatrix} = 0.0858$$

$$K = \frac{1}{3.4142} \begin{vmatrix} 1.4142 & 0 \\ -1.4142 & -0.2071 \end{vmatrix} = -0.0858$$

It follows, therefore

$$x_1 = 0.7071 + 0.0858 = 0.7858$$

$$y_1 = 0.7071 - 0.0858 = 0.6213$$

The process can be repeated to obtain a better approximation.

⇒ solve the system $\sin x - y = -0.9793$
 $\cos y - x = -0.6703$

with $x_0 = 0.5$ and $y_0 = 1.5$ as the initial approximation.

Sol.

$$f(x, y) = \sin x - y + 0.9793$$

$$g(x, y) = \cos y - x + 0.6703$$

$$f_0 = -0.0413, \quad g_0 = 0.2410$$

$$D = f_x g_y - g_x f_y = \cos(0.5)(-1) - (-1) = -1.8754$$

$$h = \frac{f g_y - g f_y}{D} = 0.1505, \quad k = \frac{g f_x - f g_x}{D} = -0.0908$$

$$x = 0.5 + 0.1505 = 0.6505$$

$$y = 1.5 + 0.0908 = 1.5908$$

For the second iteration, we have $x_0 = 0.6505$, and

$$y_0 = 1.5908; \text{ we obtain } D = -1.7956$$

$$h = -0.003181 \quad \text{and} \quad k = 0.003384$$

Hence the new approximation is

$$x = 0.6505 + 0.0032 = 0.6537$$

UNIT-II

INTERPOLATION

* Forward Differences

Consider a function $y = f(x)$ of an independent variable x .
Let $y_0, y_1, y_2, \dots, y_r$ be the value of y corresponding to the values $x_0, x_1, x_2, \dots, x_r$ of x respectively. Then the differences $y_1 - y_0, y_2 - y_1, \dots$ are called the first forward Differences of y , and we denote them by $\Delta y_0, \Delta y_1, \dots$; that is $\Delta y_0 = y_1 - y_0, \Delta y_1 = y_2 - y_1, \dots$

In general, $\Delta y_r = y_{r+1} - y_r, r = 0, 1, 2, \dots$

Here, the symbol ' Δ ' is called the Forward Difference Operator.

This is Rewrite

$$\Delta f(x) = f(x+h) - f(x)$$

* Backward Differences

Consider a function $y = f(x)$ of an independent variable x .
Let $y_0, y_1, y_2, \dots, y_r$ be the value of y corresponding to the values $x_0, x_1, x_2, \dots, x_r$ of x respectively. Then

$\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \nabla y_3 = y_3 - y_2, \dots$ are called the first backward Differences.

In general, $\nabla y_r = y_r - y_{r-1}, r = 1, 2, 3, \dots$

The symbol ' ∇ ' is called Backward Difference operator.

$$\nabla f(x) = f(x) - f(x-h)$$

* Newton's Forward Formula:-

We write x_0, x, h find $P = \frac{x - x_0}{h}$

$$y(x) = y(x_0 + ph) = y_0 + \frac{P}{1!} \Delta y_0 + \frac{P(P-1)}{2!} \Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!} \Delta^3 y_0 + \dots$$

Problems:-

1. If $y_0 = 1, y_1 = 0, y_2 = 5, y_3 = 22, y_4 = 57$. Find $y_{0.5}$ by using Newton Forward Interpolation Formula.

sol Forward Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1	-1			
1	0	5	6		
2	5	17	12	6	
3	22	35	18	6	0
4	57				

Given that $x = 0.5, x_0 = 0, h = 1$

$$P = \frac{x - x_0}{h} = \frac{0.5 - 0}{1} = 0.5$$

$$P = 0.5$$

By Newton's Forward Interpolation Formula

$$y_n = f(x) = f(x_0 + ph)$$

$$= y_0 + \frac{P}{1!} \Delta y_0 + \frac{P(P-1)}{2!} \Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!} \Delta^3 y_0 + \dots$$

$$y_{0.5} = 1 + \frac{0.5}{1}(-1) + \frac{0.5(-0.5)}{2}(6) + \frac{0.5(-0.5)(-1.5)}{6}(6)$$

$$y_{0.5} = 0.125$$

* Newton's Backward Formula:-

We write x_n, x, h find $\boxed{p = \frac{x - x_n}{h}}$

$$y(x) = y(x_n + ph)$$

$$= y_n + \frac{p}{1!} \nabla y_0 + \frac{p(p+1)}{2!} \nabla^2 y_0 + \frac{p(p+1)(p+2)}{3!} \nabla^3 y_0 + \dots$$

Problem:-

Apply Newton Backward Interpolation Formula to compute the $y_{5.5}$ given that $y_5 = 2.236, y_6 = 2.449, y_7 = 2.648, y_8 = 2.828$

Sol Backward Difference Table

x	$y = \sqrt{x}$	∇y	$\nabla^2 y$	$\nabla^3 y$
5	2.236			
6	2.449	0.213		
7	2.648	0.199	-0.04	
8	2.828	0.18	-0.019	-0.005

Given that

$$x_n = x_3 = 8, x = 5.5, h = 1$$

$$p = \frac{x - x_n}{h} = \frac{5.5 - 8}{1} = -2.5$$

By Newton Backward Interpolation Formula,

$$\begin{aligned}
 y_{5.5} = f(5.5) &= y_3 + \frac{p}{1!} \nabla y_3 + \frac{p(p+1)}{2!} \nabla^2 y_3 + \frac{p(p+1)(p+2)}{3!} \nabla^3 y_3 \\
 &= 2.828 - \frac{2.5}{1} (0.180) + \frac{(-2.5)(-2.5+1)}{2!} (0.014) \\
 &\quad + \frac{(-2.5)(-2.5+1)(-2.5+2)}{3!} (0.005) \\
 &= 2.828 - 0.4500 + 0.0263 + 0.0016 \\
 &= 2.4059.
 \end{aligned}$$

* Gauss Forward Interpolation:-

Let x_0 near to x and $p = \frac{x-x_0}{h}$

$$\begin{aligned}
 y(x) = y(x_0 + ph) &= y_0 + \frac{p}{1!} \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!} \Delta^3 y_{-2} \\
 &\quad + \frac{(p+1)p(p-1)(p-2)}{4!} \Delta^4 y_{-2} + \dots
 \end{aligned}$$

Problem:-

Q Find the value of y at $x=1936$ using the following data by Gauss Forward Interpolation.

x	1901	1911	1921	1931	1941	1951
y	12	15	20	27	39	52

Sol: Given that $x = 1936$, $x_0 = 1931$, $h = 10$

$$p = \frac{x - x_0}{h} = \frac{1936 - 1931}{10} = 0.5$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1901	12	3				
1911	15	5	2			
1921	20	7	2	0	3	
1931	27	12	5	3	-7	-10
1941	39	13	1	-4		
1951	52					

Now Gauss Forward Interpolation Formula.

$$y_x = f(x) = f(x_0 + ph)$$

$$= y_0 + \frac{p}{1!} \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!} \Delta^3 y_{-2}$$

$$= 8 + 0.5 \times 19 + \frac{0.5(-0.5)}{2} (12) + \frac{(1.5)(0.5)(-0.5)6}{6}$$

$$= 8 + 9.5 - 1.5 - 0.3750$$

$$= 15.6250$$

Gauss Backward Interpolation:-

$$y(x) = y(x_0 + ph) = y_0 + \frac{p}{1!} \Delta y_{-1} + \frac{(p+1)p}{2!} \Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!} \Delta^3 y_{-2} + \frac{(p+1)p(p-1)(p-2)}{4!} \Delta^4 y_{-2} + \dots$$

where $p = \frac{x - x_0}{h}$

Problem:-

Q) Using Gauss Backward Interpolation Formula. Given that

$$\sqrt{6500} = 80.6223, \sqrt{6510} = 80.6846, \sqrt{6520} = 80.7456$$

Sol	x	y = $\sqrt{x}$	Δy	$\Delta^2 y$	$\Delta^3 y$
	6500	80.6223			
			0.0623		
	6510	80.6846		-0.0013	
			0.0610		0.003
	6520	80.7456		-0.0018	
			0.0628		
	6530	80.8084			

Given that

$$x = 6526, \quad x_0 = 6520, \quad h = 10$$

$$p = \frac{x - x_0}{h} = \frac{6526 - 6520}{10} = 0.6$$

Now Gauss Backward Interpolation Formula,

$$y_x = f(x) = f(x_0 + ph)$$

$$= y_0 + \frac{p}{1!} \Delta y_1 + \frac{p(p+1)}{2!} \Delta^2 y_1 + \frac{p(p+1)(p+2)}{3!} \Delta^3 y_1$$

$$= 80.7456 + 0.6 (0.0610) + \frac{(1.6)(0.6)(0.0018)}{2}$$

$$+ \frac{(1.6)(0.6)(-0.4)}{6} \times 0.003$$

$$= 80.7456 + 0.0366 + 0.0009 - 0.0002$$

$$= 80.7829$$

Lagrange's Interpolation Formula:-

Used for Un Equally Spread data

$y_0, y_1, y_2, y_3, \dots \rightarrow x_0, x_1, x_2, x_3, \dots$ Then

$$y_x = y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) \\ + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3) \\ + \dots$$

Problem:-

Q. Using Lagrange's Formula, find $f(3)$

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

Sol. Given $x_0 = 0$ $f(x_0) = 1$

$x_1 = 1$ $f(x_1) = 14$

$x_2 = 2$ $f(x_2) = 15$

$x_3 = 4$ $f(x_3) = 5$

$x_4 = 5$ $f(x_4) = 6$

$x_5 = 6$ $f(x_5) = 19$

From Lagrange's Interpolation Formula

$$\begin{aligned}
 f(x) = & \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)} f(x_0) \\
 & + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)} f(x_1) \\
 & + \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)} f(x_2) \\
 & + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)} f(x_3) \\
 & + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_5)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)} f(x_4) \\
 & + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)} f(x_5)
 \end{aligned}$$

Here $x = 3$

$$\begin{aligned}
 f(3) = & \frac{(3-1)(3-2)(3-4)(3-5)(3-6)}{(0-1)(0-2)(0-4)(0-5)(0-6)} \times 1 + \frac{(3-0)(3-2)(3-4)(3-5)(3-6)}{(1-0)(1-2)(1-4)(1-5)(1-6)} \times 14 \\
 & + \frac{(3-0)(3-1)(3-4)(3-5)(3-6)}{(2-0)(2-1)(2-4)(2-5)(2-6)} \times 15 + \frac{(3-0)(3-1)(3-2)(3-5)(3-6)}{(4-0)(4-1)(4-2)(4-5)(4-6)} \times 5 \\
 & + \frac{(3-0)(3-1)(3-2)(3-4)(3-6)}{(5-0)(5-1)(5-2)(5-4)(5-6)} \times 9 + \frac{(3-0)(3-1)(3-2)(3-4)(3-5)}{(6-0)(6-1)(6-2)(6-4)(6-5)} \times 9
 \end{aligned}$$

$$= \frac{12}{240} - \frac{18}{16} \times 14 + \frac{36}{48} \times 15 + \frac{36}{48} \times 5 - \frac{18}{60} \times 6 + \frac{12}{240} \times 19$$

$$= 0.05 - 4.2 + 11.25 + 3.75 - 1.8 + 0.95$$

$$= 10$$

$$f(3) = \underline{\underline{10}}$$

Problem

Find the parabola passing through the points $(0, 1)$, $(1, 3)$ and $(3, 55)$ using Lagrange's interpolation formula.

Sol:- Given points are $(0, 1)$, $(1, 3)$, $(3, 55)$

Lagrange's interpolation formula is

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2).$$

$$f(x) = \frac{(x-1)(x-3)}{(0-1)(0-3)} (1) + \frac{(x-0)(x-3)}{(1-0)(1-3)} (3)$$

$$+ \frac{(x-0)(x-1)}{(3-0)(3-1)} (55).$$

$$= \frac{x^2 - 4x + 3}{3} + \frac{x^2 - 3x}{(-2)} (3) + \frac{x^2 - x}{6} (55)$$

$$= \frac{2x^2 - 8x + 6 - 9x^2 + 27x + 55x^2 - 55x}{6}$$

$$= \frac{1}{6} [48x^2 - 36x + 6]$$

$$f(x) = 8x^2 - 6x + 1$$

2.

Interpolation and Extrapolation

Suppose, we are given the values of $y = f(x)$ for a set of values of x :

$$x : x_0 \quad x_1 \quad x_2 \quad \dots \quad x_n$$

$$y : y_0 \quad y_1 \quad y_2 \quad \dots \quad y_n$$

The process of finding the value of 'y' corresponding to any value of $x = x_i$ between x_0 and x_n is called interpolation.

Hence, interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable while the process of computing the value of the function outside the given range is called extrapolation.

Central Differences :-

with $y_0, y_1, y_2, \dots, y_r$ as the values of a function $y = f(x)$ corresponding to the values $x_1, x_2, x_3, \dots, x_r$ of x , we define the first central differences $\delta y_{1/2}, \delta y_{3/2}, \delta y_{5/2}, \dots$ as follows $\delta y_{1/2} = y_1 - y_0, \delta y_{3/2} = y_2 - y_1, \delta y_{5/2} = y_3 - y_2, \dots$

$$\delta y_{r-1/2} = y_r - y_{r-1}$$

The symbol δ is called central difference operation. This operator is a linear operator.

Shift operator :- The shift operator ' E ' is defined by

the equation $E y_r = y_{r+1} \quad [E f(x) = f(x+h)]$.

$$E^2 y_r = E[E y_r] = E[y_{r+1}] = y_{r+2}; \text{ Generally } E^n y_r = y_{r+n}$$

Inverse operator E^{-1} is defined as $E^{-1} y_r = y_{r-1}$.

$$\text{In general } E^{-n} y_r = y_{r-n}$$

Relation b/w Δ and E

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Delta f(x) = E f(x) - f(x)$$

$$\Delta f(x) = [E - I] f(x)$$

$$\boxed{\Delta = E - I} \text{ or } \boxed{E = I + \Delta}$$

Relation b/w ' ∇ ' and ' E '

$$\nabla f(x) = f(x) - f(x-h)$$

$$= f(x) - E^{-1} f(x)$$

$$\nabla f(x) = [I - E^{-1}] f(x)$$

$$\boxed{\nabla = I - E^{-1}}$$

Some more Relations

$$\Delta^3 y_0 = [E - I]^3 y_0 = (E^3 - 3E^2 + 3E - I) y_0$$

$$= E^3 y_0 - 3E^2 y_0 + 3E y_0 - y_0 = y_3 - 3y_2 + 3y_1 - y_0$$

$$\Delta^4 y_0 = [E - I]^4 y_0 = (E^4 - 4E^3 + 6E^2 - 4E + I) y_0$$

Central Difference operator :- The central difference operator 'δ' is defined by $\delta f(x) = f(x + \frac{h}{2}) - f(x - \frac{h}{2})$

Relation b/w 'E' and 'δ'

$$\begin{aligned}\delta f(x) &= f(x + \frac{h}{2}) - f(x - \frac{h}{2}) \\ &= E^{1/2} f(x) - E^{-1/2} f(x)\end{aligned}$$

$$E^{1/2} f(x) = f(x + \frac{h}{2})$$

$$E^{-1/2} f(x) = f(x - \frac{h}{2})$$

$$\delta f(x) = [E^{1/2} - E^{-1/2}] f(x)$$

$$\boxed{\delta = E^{1/2} - E^{-1/2}}$$

Average operator (μ) :-

The Average operator 'μ' is defined as

$$\mu f(x) = \frac{f(x + \frac{h}{2}) + f(x - \frac{h}{2})}{2}$$

Relation b/w 'μ' and 'E' :-

$$\mu f(x) = \frac{f(x + \frac{h}{2}) + f(x - \frac{h}{2})}{2}$$

$$= \frac{E^{1/2} f(x) + E^{-1/2} f(x)}{2}$$

$$\mu = \frac{E^{1/2} + E^{-1/2}}{2}$$

Differential operator :-

The differential operator 'D' is defined by

$$Dy_x = \frac{d}{dx} [y_x] = y'(x).$$

$$D f(x) = \frac{d}{dx} [f(x)]$$

Relation b/w 'D' and 'E'

by using Taylor's formula.

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \dots$$

$$y(a+h) = y(a) + \frac{h}{1!} y'(a) + \frac{h^2}{2!} y''(a) + \frac{h^3}{3!} y'''(a) + \dots$$

This can be written as.

$$Ey_x = y_x + \frac{h}{1!} Dy_x + \frac{h^2}{2!} D^2 y_x + \frac{h^3}{3!} D^3 y_x + \dots$$

$$= \left[1 + \frac{hD}{1!} + \frac{h^2 D^2}{2!} + \frac{h^3 D^3}{3!} + \dots \right] y_x.$$

$$E = 1 + \frac{hD}{1!} + \frac{h^2 D^2}{2!} + \frac{h^3 D^3}{3!} + \dots$$

$$\boxed{E = e^{hD}} \quad \left[e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right]$$

Pb Show that $\mu^2 = 1 + \frac{\delta^2}{4}$ (or) $\mu = \sqrt{1 + \frac{\delta^2}{4}}$.

Sol:- $\mu = \frac{E^{1/2} + E^{-1/2}}{2}$ and $\delta = E^{1/2} - E^{-1/2}$.

$$\mu^2 = \left[\frac{E^{1/2} + E^{-1/2}}{2} \right]^2 = \frac{[E^{1/2} + E^{-1/2}]^2}{4}$$

$$(a+b)^2 = (a-b)^2 + 4ab.$$

$$\mu^2 = \frac{(E^{1/2} - E^{-1/2})^2 + 4E^{1/2}E^{-1/2}}{4}$$

$$= \frac{[E^{1/2} - E^{-1/2}]^2 + 4}{4}$$

$$= \frac{\delta^2}{4} + 1 \quad \text{www.FirstRanker.com}$$

Prob- Evaluate (i) $\Delta \cos x$ (ii) $\Delta \log f(x)$ (iii) $\Delta^2 \sin(Px+Q)$
(iv) $\Delta \tan^{-1} x$ (v) $\Delta^n e^{ax+b}$.

$$\begin{aligned} \text{(i)} \quad \Delta \cos x &= \cos(x+h) - \cos x & \left[\cos A - \cos B = \cancel{-2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}} \right] \\ &= -2 \sin\left(x+\frac{h}{2}\right) \sin \frac{h}{2} & = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \Delta \log f(x) &= \log f(x+h) - \log f(x) \\ &= \log \left[\frac{f(x+h)}{f(x)} \right] \end{aligned}$$

$$= \log \left[\frac{f(x) + \Delta f(x)}{f(x)} \right] = \log \left[1 + \frac{\Delta f(x)}{f(x)} \right]$$

$$\text{(iii)} \quad \Delta \sin Px+Q = \sin(P(x+h)+Q) - \sin(Px+Q)$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$= 2 \cos \frac{Px+Ph+Q+Px+Q}{2} \sin \frac{Px+Ph+Q-Px-Q}{2}$$

$$= 2 \cos \left(Px+Q+\frac{Ph}{2} \right) \sin \frac{Ph}{2}$$

$$= 2 \sin \frac{Ph}{2} \sin \left(\frac{\pi}{2} + Px+Q+\frac{Ph}{2} \right)$$

$$\Delta^2 \sin(Px+Q) = 2 \sin \frac{Ph}{2} \Delta \left[\sin \left(Px+Q+\frac{1}{2}(\pi+Ph) \right) \right]$$

$$= \left[2 \sin \frac{Ph}{2} \right]^2 \sin \left[Px+Q+2 \cdot \frac{1}{2}(\pi+Ph) \right]$$

Proceeding in the same manner, we get.

$$\Delta^3 \sin(Px+Q) = \left(2 \sin \frac{Ph}{2} \right)^3 \sin \left[Px+Q+3 \frac{(\pi+Ph)}{2} \right]$$

$$\Delta^n \sin(Px+Q) = \left(2 \sin \frac{Ph}{2} \right)^n \sin \left[Px+Q+\frac{n(\pi+Ph)}{2} \right]$$

$$\begin{aligned}
 \text{(iv)} \quad \Delta \tan^{-1} x &= \tan^{-1}(x+h) - \tan^{-1} x \\
 &= \tan^{-1} \left[\frac{x+h-x}{1+x(x+h)} \right] \\
 &= \tan^{-1} \left[\frac{h}{1+x(x+h)} \right]
 \end{aligned}
 \quad \left[\begin{aligned} &\tan^{-1} A - \tan^{-1} B \\ &= \tan^{-1} \frac{A-B}{1+AB} \end{aligned} \right]$$

$$\begin{aligned}
 \text{(v)} \quad \Delta e^{ax+b} &= e^{a(x+h)+b} - e^{ax+b} \\
 &= e^{ax+ah+b} - e^{ax+b} \\
 &= e^{(ax+b)+ah} - e^{ax+b} \\
 &= e^{ax+b} [e^{ah} - 1]
 \end{aligned}$$

$$\begin{aligned}
 \Delta^2 e^{ax+b} &= \Delta [\Delta e^{ax+b}] \\
 &= \Delta [e^{ax+b} (e^{ah} - 1)] \\
 &= (e^{ah} - 1) \Delta e^{ax+b} \\
 &= (e^{ah} - 1) [(e^{ah} - 1) e^{ax+b}] \\
 &= (e^{ah} - 1)^2 e^{ax+b} \quad [e^{ah} - 1 \text{ is constant}]
 \end{aligned}$$

Proceeding in this manner, we get.

$$\Delta^3 e^{ax+b} = (e^{ah} - 1)^3 e^{ax+b}$$

$$\Delta^4 e^{ax+b} = (e^{ah} - 1)^4 e^{ax+b}$$

$$\Delta^n e^{ax+b} = (e^{ah} - 1)^n e^{ax+b}$$

Problem:- If the interval of differencing is

$$\text{unity, Prove that } \Delta \tan^{-1} \left(\frac{n-1}{n} \right) = \tan^{-1} \left(\frac{1}{2n^2} \right)$$

Sol:- we know that $\Delta \tan^{-1} x = \tan^{-1}(x+h) - \tan^{-1} x$.

$$\begin{aligned}
 \Delta \tan^{-1} \left(\frac{n-1}{n} \right) &= \Delta \tan^{-1} \left(1 - \frac{1}{n} \right) \\
 &= \tan^{-1} \left(1 - \frac{1}{n+1} \right) - \tan^{-1} \left(1 - \frac{1}{n} \right) \\
 &= \tan^{-1} \left[\frac{\left(1 - \frac{1}{n+1} \right) - \left(1 - \frac{1}{n} \right)}{1 + \left(1 - \frac{1}{n+1} \right) \left(1 - \frac{1}{n} \right)} \right] = \tan^{-1} \left[\frac{\left(\frac{1}{n} - \frac{1}{n+1} \right)}{1 + \left(\frac{n}{n+1} \right) \left(\frac{n-1}{n} \right)} \right] \\
 &= \tan^{-1} \left(\frac{1}{2n^2} \right) \text{ on simplification}
 \end{aligned}$$

Problem:- using the method of separation of symbols, show that $\Delta^n u_{x-n} = u_x - nu_{x-1} + \frac{n(n-1)}{2} u_{x-2} - \dots + (-1)^n u_{x-n}$

Solution:- To prove the result, we start with the right hand side.

$$\begin{aligned}
 &u_x - nu_{x-1} + \frac{n(n-1)}{2} u_{x-2} + \dots + (-1)^n u_{x-n} \\
 &= u_x - n\bar{E}u_x + \frac{n(n-1)}{2} \bar{E}^2 u_x + \dots + (-1)^n \bar{E}^n u_x \\
 &= \left[1 - n\bar{E} + \frac{n(n-1)}{2} \bar{E}^2 + \dots + (-1)^n \bar{E}^n \right] u_x \\
 &= \left[1 - \bar{E} \right]^n u_x = \left(1 - \frac{1}{E} \right)^n u_x = \frac{(E-1)^n}{E^n} u_x \\
 &= \frac{\Delta^n}{E^n} u_x = \Delta^n \bar{E}^n u_x \\
 &= \Delta^n u_{x-n} \text{ which is left hand side.}
 \end{aligned}$$

Problem:- show that $\Delta f_i^2 = (f_i + f_{i+1}) \Delta f_i$

Solution:- $\Delta f_k = f_{k+1} - f_k$

$$\Delta f_i^2 = f_{i+1}^2 - f_i^2 = (f_{i+1} + f_i)(f_{i+1} - f_i)$$

$$\begin{aligned}
 &\text{www.FirstRanker.com} \\
 &= (f_{i+1} + f_i) \Delta f_i
 \end{aligned}$$

Problem:- Show that $e^x (u_0 + x \Delta u_0 + \frac{x^2}{2!} \Delta^2 u_0 + \dots)$
 $= u_0 + u_1 x + u_2 \frac{x^2}{2!} + \dots$

Solution:- $e^x (u_0 + x \Delta u_0 + \frac{x^2}{2!} \Delta^2 u_0 + \dots) = e^x (1 + x \Delta + \frac{x^2}{2} \Delta^2 + \dots) u_0$
 $= e^x \cdot e^{x \Delta} u_0 = e^{x(1+\Delta)} u_0 = e^{xE} u_0$
 $= [1 + xE + \frac{x^2 E^2}{2!} + \dots] u_0$
 $= u_0 + x E u_0 + \frac{x^2 E^2 u_0}{2!} + \dots$
 $= u_0 + x u_1 + \frac{x^2}{2!} u_2 + \dots$

Problem:- Evaluate (i) $\Delta [f(x)g(x)]$ (ii) $\Delta \left[\frac{f(x)}{g(x)} \right]$.

(i) $\Delta f(x)g(x) = f(x+h)g(x+h) - f(x)g(x)$
 $= f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)$
 $= f(x+h) [g(x+h) - g(x)] + g(x) [f(x+h) - f(x)]$
 $= f(x+h) \Delta g(x) + g(x) \Delta f(x)$

(ii) $\Delta \frac{f(x)}{g(x)} = \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}$
 $= \frac{f(x+h)g(x) - g(x+h)f(x)}{g(x+h)g(x)}$
 $= \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - g(x+h)f(x)}{g(x+h)g(x)}$
 $= \frac{g(x) [f(x+h) - f(x)] + f(x) [g(x) - g(x+h)]}{g(x+h)g(x)}$
 $= \frac{g(x) [f(x+h) - f(x)] - f(x) [g(x+h) - g(x)]}{g(x+h)g(x)}$

$$= \frac{g(x) \Delta f(x) - f(x) \Delta g(x)}{g(x+h) g(x)}.$$

Problem:- show that $\sum_{k=0}^{n-1} \Delta^2 f_k = \Delta f_n - \Delta f_0$.

Solution:- $\sum_{k=0}^{n-1} \Delta^2 f_k = \Delta^2 f_0 + \Delta^2 f_1 + \Delta^2 f_2 + \dots + \Delta^2 f_{n-1}$ ①

$$\left[\begin{array}{l} \Delta f_0 = f_1 - f_0 \Rightarrow \Delta^2 f_0 = \Delta f_1 - \Delta f_0 \\ \Delta f_1 = f_2 - f_1 \Rightarrow \Delta^2 f_1 = \Delta f_2 - \Delta f_1 \end{array} \right]$$

From ①

$$\begin{aligned} \sum_{k=0}^{n-1} \Delta^2 f_k &= \Delta f_1 - \Delta f_0 + \Delta f_2 - \Delta f_1 + \Delta f_3 - \Delta f_2 + \dots + \Delta f_n - \Delta f_{n-1} \\ &= \Delta f_n - \Delta f_0. \end{aligned}$$

Problem:- If the interval of differencing is unity,

Prove that $\Delta \frac{1}{f(x)} = \frac{-\Delta f(x)}{f(x)f(x+1)}.$

Sol:- we know that $\Delta \left(\frac{1}{f(x)} \right) = \frac{1}{f(x+h)} - \frac{1}{f(x)}.$

$$= \frac{f(x) - f(x+h)}{f(x+h)f(x)}$$

$$= - \left[\frac{f(x+h) - f(x)}{f(x+h)f(x)} \right] = \frac{-\Delta f(x)}{f(x)f(x+h)}$$

Taking $h=1$, we get.

$$\Delta \frac{1}{f(x)} = \frac{-\Delta f(x)}{f(x)f(x+1)}$$

Prove the results

(i). $E\nabla = \Delta = \nabla E$.

$$\begin{aligned}(E\nabla)f(x) &= E[\nabla f(x)] \\ &= E[f(x) - f(x-h)] \\ &= Ef(x) - Ef(x-h) \\ &= f(x+h) - f(x) \\ &= \Delta f(x)\end{aligned}$$

$\therefore E\nabla = \Delta$.

$$\begin{aligned}\text{Also } (\nabla E)f(x) &= \nabla[Ef(x)] \\ &= \nabla f(x+h) \\ &= f(x+h) - f(x) \\ &= \Delta f(x)\end{aligned}$$

$\nabla E = \Delta$.

$\therefore E\nabla = \Delta = \nabla E$.

(ii) $\delta E^{1/2} = \Delta$.

$$\begin{aligned}\delta f(x + \frac{h}{2}) &= (E^{1/2} - E^{-1/2})f(x + \frac{h}{2}) \\ &= E^{1/2}f(x + \frac{h}{2}) - E^{-1/2}f(x + \frac{h}{2}) \\ &= E^{1/2}y_{x+h/2} - E^{-1/2}y_{x+h/2} \\ &= y_{x+h} - y_x = \Delta y_x.\end{aligned}$$

$\delta f(x + \frac{h}{2}) = \Delta f(x)$

$\delta E^{1/2}f(x) = \Delta f(x)$

$\delta E^{1/2} = \Delta$

(iii). $h\Delta = \log(1+\Delta) = -\log(1-\nabla)$
 $= \sinh(\mu\delta)$

Sol:- we know $e^{hd} = E = 1 + \Delta$.

Taking logarithm

$hd \log_e = \log(1+\Delta)$

Also $\nabla = 1 - E^{-1} \Rightarrow E^{-1} = 1 - \nabla$

$E^{-hd} = (1 - \nabla)$

Taking logarithms.

$-hd = \log(1 - \nabla)$

$hd = -\log(1 - \nabla)$

$$\begin{aligned}\sinh(hd) &= \frac{e^{hd} - e^{-hd}}{2} \\ &= \frac{E - E^{-1}}{2}\end{aligned}$$

$= \left[\frac{E^{1/2} + E^{-1/2}}{2} \right] \left[\frac{E^{1/2} - E^{-1/2}}{2} \right] = \mu\delta$

$hd = \sinh^{-1}(\mu\delta)$

(iv). $\nabla\Delta = \Delta - \nabla = \delta^2$.

$$\begin{aligned}\nabla\Delta &= (1 - E^{-1})(E - 1) \\ &= E - 1 - EE^{-1} + E^{-1} \\ &= E + E^{-1} - 2 \\ &= [E^{1/2} - E^{-1/2}]^2 = \delta^2\end{aligned}$$

Also $\Delta - \nabla$

$$\begin{aligned}&= (E - 1) - (1 - E^{-1}) \\ &= E + E^{-1} - 2 = \delta^2.\end{aligned}$$

www.FirstRanker.com $\nabla\Delta = \Delta - \nabla = \delta^2$.

$\Delta - \nabla = \delta^2$

$$(v) (1+\Delta)(1-\nabla) = 1.$$

$$\begin{aligned} (1+\Delta)(1-\nabla) &= E[1-(1-E^{-1})] \\ &= E[1-1+E^{-1}] \\ &= EE^{-1} = 1. \end{aligned}$$

$$(vi) \frac{1}{2}(\Delta + \nabla) = \mu\delta.$$

$$\begin{aligned} \frac{1}{2}(\Delta + \nabla) &= \frac{1}{2}[E-1+1-E^{-1}] \\ &= \frac{1}{2}[E-E^{-1}] \\ &= \mu\delta. \end{aligned}$$

$$(vii) 1 + \mu^2\delta^2 = \left(1 + \frac{\delta^2}{2}\right)^2.$$

$$\begin{aligned} 1 + \mu^2\delta^2 &= 1 + \left(\frac{E^{1/2} + E^{-1/2}}{2}\right)^2 (E^{1/2} - E^{-1/2})^2 \\ &= 1 + \left(\frac{E - E^{-1}}{4}\right)^2 \\ &= \frac{4 + (E - E^{-1})^2}{4} = \left[\frac{E + E^{-1}}{2}\right]^2. \end{aligned}$$

$$\begin{aligned} \text{Now } \left(1 + \frac{\delta^2}{2}\right)^2 &= \left[1 + \frac{1}{2}[E^{1/2} - E^{-1/2}]^2\right]^2 \\ &= \left[1 + \frac{1}{2}(E + E^{-1} - 2)\right]^2 \\ &= \left[\frac{E + E^{-1}}{2}\right]^2 \quad \text{--- (2)} \end{aligned}$$

From (1) and (2).

$$1 + \mu^2\delta^2 = \left(1 + \frac{\delta^2}{2}\right)^2$$

$$\begin{aligned} (viii) \mu + \frac{\delta}{2} &= \frac{(E^{1/2} + E^{-1/2})}{2} + \frac{(E^{1/2} - E^{-1/2})}{2} \\ &= \frac{2E^{1/2}}{2} = E^{1/2}. \end{aligned}$$

$$(ix) \mu - \frac{\delta}{2} = E^{-1/2}$$

$$\begin{aligned} \mu - \frac{\delta}{2} &= \frac{E^{1/2} + E^{-1/2}}{2} - \frac{(E^{1/2} - E^{-1/2})}{2} \\ &= \frac{2E^{-1/2}}{2} = E^{-1/2} \end{aligned}$$

$$(x) \mu\delta = \frac{1}{2}\Delta E^{-1} + \frac{1}{2}\Delta.$$

$$\begin{aligned} \frac{1}{2}\Delta E^{-1} + \frac{1}{2}\Delta &= \frac{1}{2}\Delta[E^{-1} + 1] \\ &= \frac{1}{2}(E-1)(E^{-1} + 1) \\ &= \frac{1}{2}(E - E^{-1}) \\ &= \mu\delta. \end{aligned}$$

$$(xi) \Delta = \frac{\delta^2}{2} + \delta\sqrt{1 + \frac{\delta^2}{4}}.$$

$$\begin{aligned} \frac{\delta^2}{2} + \delta\sqrt{1 + \frac{\delta^2}{4}} &= \frac{1}{2}\delta\left[\delta + 2\sqrt{1 + \frac{\delta^2}{4}}\right] \\ &= \frac{1}{2}\delta\left[\delta + \sqrt{4 + \delta^2}\right] \\ &= \frac{1}{2}\delta\left[(E^{1/2} - E^{-1/2}) + \sqrt{4 + (E^{1/2} - E^{-1/2})^2}\right] \\ &= \frac{1}{2}\delta\left[(E^{1/2} - E^{-1/2}) + \sqrt{(E^{1/2} + E^{-1/2})^2}\right] \\ &= \frac{1}{2}\left[(E^{1/2} - E^{-1/2})[E^{1/2} - E^{-1/2} + E^{1/2} + E^{-1/2}]\right] \\ &= \frac{1}{2}\left[(E^{1/2} - E^{-1/2})2E^{1/2}\right] \\ &= E - E^{-1} = \Delta \end{aligned}$$

$$\frac{\delta^2}{2} + \delta\sqrt{1 + \frac{\delta^2}{4}} = \Delta.$$

$$(xii) \Delta^n \cos(ax + b).$$

$$= \left(2\sin\frac{ah}{2}\right)^n \cos\left[ax + b + n\left(\frac{ah + \pi}{2}\right)\right]$$

By we can prove that.

Problem:- Estimate the missing term in the following table.

x :	0	1	2	3	4
$f(x)$:	1	3	9	?	81.

Solution:- we are given four values.

$$\Delta^4 f(x) = 0 \Rightarrow \Delta^4 y_0 = 0.$$

$$\Rightarrow (E-1)^4 y_0 = 0$$

$$\Rightarrow (E^4 - 4E^3 + 6E^2 - 4E + 1) y_0 = 0.$$

$$E^4 y_0 - 4E^3 y_0 + 6E^2 y_0 - 4E y_0 + y_0 = 0.$$

$$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0.$$

$$81 - 4y_3 + 6(9) - 4(3) + 1 = 0.$$

$$81 - 4y_3 + 54 - 12 + 1 = 0.$$

$$4y_3 = 123 \Rightarrow y_3 = 31.$$

Problem:- Estimate the production for 1964 and 1966 from the following data.

Year :	1961	1962	1963	1964	1965	1966	1967
population :	200	220	260	-	350	-	430.

Solution:- Since five figures are known, assume fifth differences as zero. i.e. $\Delta^5 y_0 = 0$ and $\Delta^5 y_1 = 0$.

$$(E-1)^5 y_0 = 0 \text{ and } (E-1)^5 y_1 = 0.$$

$$(E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1) y_0 = 0.$$

$$E^5 y_0 - 5E^4 y_0 + 10E^3 y_0 - 10E^2 y_0 + 5E y_0 - y_0 = 0.$$

$$(E-1)^5 y_1 = 0 \Rightarrow (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)y_1 = 0.$$

$$E^5 y_1 - 5E^4 y_1 + 10E^3 y_1 - 10E^2 y_1 + 5E y_1 - y_1 = 0.$$

$$y_6 - 5y_5 + 10y_4 - 10y_3 + 5y_2 - y_1 = 0.$$

Substituting the known values, we get.

$$y_5 - 1750 + 10y_3 - 2600 + 1100 - 200 = 0.$$

$$430 - 5y_5 + 3500 - 10y_3 + 1300 - 220 = 0.$$

$$y_5 + 10y_3 = 3450. \quad \text{--- (1)}$$

$$-5y_5 - 10y_3 = -5010. \quad \text{--- (2)}$$

Adding (1) and (2) we get.

$$-4y_5 = -1560 \Rightarrow y_5 = 390.$$

From (1) $390 + 10y_3 = 3450.$

$$10y_3 = 3060 \Rightarrow y_3 = 306.$$

Hence production for year 1964 = 306.

and production for year 1966 = 390.

Problem:- $\Delta^3 y_2 = \nabla^3 y_5.$

$$\Delta^3 y_2 = (E-1)^3 y_2 = (E^3 - 3E^2 + 3E - 1)y_2$$

$$E^3 y_2 - 3E^2 y_2 + 3E y_2 - y_2 = 0.$$

$$y_5 - 3y_4 + 3y_3 - y_2 = 0.$$

$$\nabla^3 y_5 = (1-E)^3 y_5 = (1 - 3E^{-1} + 3E^{-2} - E^{-3})y_5$$

$$y_5 - 3E^{-1} y_5 + 3E^{-2} y_5 - E^{-3} y_5 = y_5 - 3y_4 + 3y_3 - y_2$$

$$\Delta^3 y_2 = \nabla^3 y_5$$

Problem:- Given $u_0 + u_8 = 1.9243$, $u_1 + u_7 = 1.9590$.

$u_2 + u_6 = 1.9823$, $u_3 + u_5 = 1.9956$ find u_4 .

Sol:- Taking $(E-1)^8 u_0 = 0$.

$$(E^8 - 8E^7 + 28E^6 - 56E^5 + 70E^4 - 56E^3 + 28E^2 - 8E + 1)u_0 = 0$$

$$E^8 u_0 - 8E^7 u_0 + 28E^6 u_0 - 56E^5 u_0 + 70E^4 u_0 - 56E^3 u_0 + 28E^2 u_0 - 8E u_0 + u_0 = 0$$

$$u_8 - 8u_7 + 28u_6 - 56u_5 + 70u_4 - 56u_3 + 28u_2 - 8u_1 + u_0 = 0$$

$$(u_0 + u_8) - 8(u_7 + u_1) + 28(u_6 + u_2) - 56(u_5 + u_3) + 70u_4 = 0$$

$$1.9243 - 8(1.9590) + 28(1.9823) - 56(1.9956) + 70u_4 = 0$$

$$u_4 = 0.99996$$

Problem:- For the second difference of the polynomial $x^4 - 12x^3 + 42x^2 - 30x + 9$ with interval of differencing

$$h=2$$

Solution:- Let $f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$.

First difference is given by $= \Delta f(x)$

$$f(x+h) - f(x) = f(x+2) - f(x)$$

$$= (x+2)^4 - 12(x+2)^3 + 42(x+2)^2 - 30(x+2) + 9 - x^4 + 12x^3 - 42x^2 + 30x - 9$$

$$= 8x^3 - 48x^2 + 56x + 28$$

Second difference $\Delta^2 f(x) = \Delta[\Delta f(x)]$

$$= 8(x+2)^3 - 48(x+2)^2 + 56(x+2) + 28$$

$$= -8x^3 + 48x^2 - 56x + 28$$

40

Prove the results. (i) $\Delta \nabla = \Delta - \nabla$ (ii) $\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} = \Delta + \nabla$

$$\begin{aligned}
 \text{(i)} \quad \Delta \nabla &= (E-1)(1-E^{-1}) \\
 &= E - EE^{-1} - 1 + E^{-1} \\
 &= E - 1 - 1 + E^{-1} \\
 &= (E-1) - (1-E^{-1}) = \Delta - \nabla \\
 \therefore \Delta \nabla &= \Delta - \nabla
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} &= \frac{\Delta^2 - \nabla^2}{\Delta \nabla} \\
 &= \frac{(\Delta + \nabla)(\Delta - \nabla)}{\Delta \nabla} \\
 &= \frac{(\Delta + \nabla)(\cancel{\Delta - \nabla})}{(\cancel{\Delta - \nabla})} \quad \text{From (i)} \\
 &= \Delta + \nabla \\
 \therefore \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} &= \Delta + \nabla
 \end{aligned}$$

Problem:- $\Delta^2 \cos 2x$

$$\begin{aligned}
 \Delta^2 \cos 2x &= \Delta [\cos 2(x+h) - \cos 2x] \\
 &= [\cos 2(x+2h) - \cos 2(x+h)] - [\cos 2(x+h) - \cos 2x] \\
 &= -2 \sin(2x+3h) \sin h + 2 \sin(2x+h) \sin h \\
 &= -2 \sin h [\sin(2x+3h) - \sin(2x+h)]
 \end{aligned}$$

$$\begin{aligned}
 &= -2 \sin h \left[\frac{2 \cos(2x+2h) \sin h}{\text{www.FirstRanker.com}} \right] \left[\frac{\sin A - \sin B}{2} = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2} \right] \\
 &= -4 \sin^2 h \cos 2(x+h)
 \end{aligned}$$

40

Prove the results. (i) $\Delta \nabla = \Delta - \nabla$ (ii) $\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} = \Delta + \nabla$

$$\begin{aligned}
 \text{(i)} \quad \Delta \nabla &= (E-1)(1-E^{-1}) \\
 &= E - EE^{-1} - 1 + E^{-1} \\
 &= E - 1 - 1 + E^{-1} \\
 &= (E-1) - (1-E^{-1}) = \Delta - \nabla \\
 \therefore \Delta \nabla &= \Delta - \nabla
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} &= \frac{\Delta^2 - \nabla^2}{\Delta \nabla} \\
 &= \frac{(\Delta + \nabla)(\Delta - \nabla)}{\Delta \nabla} \\
 &= \frac{(\Delta + \nabla)(\cancel{\Delta - \nabla})}{(\cancel{\Delta - \nabla})} \quad \text{From (i)} \\
 &= \Delta + \nabla \\
 \therefore \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} &= \Delta + \nabla
 \end{aligned}$$

Problem:- $\Delta^2 \cos 2x$

$$\begin{aligned}
 \Delta^2 \cos 2x &= \Delta [\cos 2(x+h) - \cos 2x] \\
 &= [\cos 2(x+2h) - \cos 2(x+h)] - [\cos 2(x+h) - \cos 2x] \\
 &= -2 \sin(2x+3h) \sin h + 2 \sin(2x+h) \sin h \\
 &= -2 \sin h [\sin(2x+3h) - \sin(2x+h)]
 \end{aligned}$$

$$\begin{aligned}
 &= -2 \sin h \left[\frac{2 \cos(2x+2h) \sin h}{\text{www.FirstRanker.com}} \right] \left[\frac{\sin A - \sin B}{2} = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2} \right] \\
 &= -4 \sin^2 h \cos 2(x+h)
 \end{aligned}$$

UNIT :- 3 NUMERICAL SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

* Taylor's series:

Given differential Equation is $\frac{dy}{dx} = f(x, y)$ with initial condition is that $y = y_0$ at $x = x_0$

By Taylor's series

$$y(x) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \dots$$

* problem:

1. Using Taylor's series method to Find $y(0.1)$ and $y(0.2)$ if $y' = x - y^2$, $y = 1$ and $x = 0$.

sol:

Given $y' = f(x, y)$

$$x_1 = x_0 + h$$

$$y' = x - y^2, \quad x_0 = 0, \quad y_0 = 1.$$

$$x_1 = 0 + 0.1 = 0.1$$

$$y' = x - y^2, \quad y_0' = x_0 - y_0^2 = 0 - (1)^2 = -1.$$

$$y'' = 1 - 2yy' \Rightarrow y_0'' = 1 - 2y_0 y_0' = 3.$$

$$y''' = 0 - 2[y y'' + y' y'] \Rightarrow y_0''' = -2[y_0 y_0'' + y_0' y_0'] = -8$$

$$y^{(4)} = -2[y y''' + y'' y' + y' y'' + y' y'']$$

$$\Rightarrow y_0^{(4)} = -2[y_0 y_0''' + 3 y_0' y_0''] = 34.$$

$$y^{(5)} = -2[y y^{(4)} + y''' y' + y'' y'' + y' y'''] + 2(y' y'' + y'' y')]$$

$$= -2[y y^{(4)} + 4 y' y''' + 3(y'')^2]$$

$$\Rightarrow y_0^{(5)} = -2[y_0 y_0^{(4)} + 4 y_0' y_0''' + 3(y_0'')^2]$$

$$= -186$$

By using Taylor's series.

$$y(x) = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{(4)} + \frac{h^5}{5!} y_0^{(5)} + \dots$$

$$y_1 = y(0.1) = 1 + (0.1)(-1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(-8) + \frac{(0.1)^4}{24}(34) + \frac{(0.1)^5}{120}(-186)$$

$$y_1 = 1 - 0.1 + 0.015 - 0.00133 + 0.0001416 - 0.0000155$$

$$y(0.1) = 0.9138$$

$$x_1 = 0.1, h = 0.1, y_1 = 0.9138$$

$$y_1' = x_1 - y_1^2 = 0.1 - (0.9138)^2 = 0.1 - 0.8350487 = -0.735$$

$$y_1'' = 1 - 2y_1 y_1' = 1 - 2(0.9138)(-0.735) = 2.3433$$

$$y_1''' = -2[(y_1')^2 + y_1 y_1''] = -2[(-0.735)^2 + (0.9138)(2.3433)] = -5.363112$$

$$y_1^{IV} = -2[3y_1' y_1'' + y_1 y_1'''] = -2[3(-0.735)(2.3433) + (0.9138)(-5.363112)]$$

$$= 20.133953$$

$$y_2 = y(0.2)$$

$$= y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{IV} + \frac{h^5}{5!} y_1^{V}$$

$$y_2 = y(0.2)$$

$$= 0.9138 + (0.1)(-0.735) + \frac{(0.1)^2}{2}(2.3433)$$

$$+ \frac{(0.1)^3}{6}(-5.363112) + \frac{(0.1)^4}{24}(20.133953) + \dots$$

$$= 0.9138 - 0.0735 + 0.0117 - 0.00089 + 0.00008$$

$$y_2 = 0.8512$$

* picard's method of successive Approximations:

Given Differential Equation is $\frac{dy}{dx} = f(x, y)$

$y = y_0$ at $x = x_0$.

1st Approximation $y^{(1)} = y_0 + \int_{x_0}^x f(x, y_0) dx$

2nd Approximation $y^{(2)} = y_0 + \int_{x_0}^x f(x, y^{(1)}) dx$

3rd Approximation $y^{(3)} = y_0 + \int_{x_0}^x f(x, y^{(2)}) dx$

proceeding in this way after finite no. of steps, we get the value of y when x is given.

* problem:

Find an approximate value of y for $x = 0.2$ if $\frac{dy}{dx} = x + y$, $y(0) = 1$ using picard's method compare the numerical solution obtained with Exact solution.

Given Equation $\frac{dy}{dx} = x + y$, $x_0 = 0$, $y_0 = 1$.

1st approximation, $y^{(1)} = y_0 + \int_{x_0}^x f(x + y_0) dx$
 $= y_0 + \int_{x_0}^x (x + 1) dx$

$y^{(1)} = 1 + \frac{x^2}{2} + x = 1 + x + \frac{x^2}{2}$

2nd approximation, $y^{(2)} = y_0 + \int_{x_0}^x (x - y^{(1)}) dx$
 $= y_0 + \int_{x_0}^x (x - (1 + x + \frac{x^2}{2})) dx$

$= 1 + \frac{x^2}{2} + x + \frac{x^2}{2} + \frac{x^3}{6}$

$$y^{(2)} = 1 + x + x^2 + x^3/6$$

3rd approximation $y^{(3)} = y_0 + \int_{x_0}^x (x + y^{(2)}) dx$

$$= y_0 + \int_{x_0}^x (x + 1 + x + x^2 + x^3/6) dx$$

$$= 1 + x^2/2 + x + x^2/2 + x^3/3 + \frac{x^4}{24}$$

$$= 1 + x + x^2 + x^3/3 + x^4/24$$

when $x = 0.2$

$$y^{(1)} = 1 + (0.2) + \frac{(0.2)^2}{2} = 1.22$$

$$y^{(2)} = 1 + (0.2) + (0.2)^2 + \frac{(0.2)^3}{6} = 1.2413$$

$$y^{(3)} = 1 + (0.2) + (0.2)^2 + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{24} = 1.2427$$

$\therefore$ The value of y when $x = 0.2$ is 1.2427 .

* Exact value:

Given Equation is $\frac{dy}{dx} = x + y$.

$$\frac{dy}{dx} - y = x$$

comparing this Equation with $\frac{dy}{dx} + Py = Q$.

$$P = -1, Q = x$$

$$I.F = e^{\int P dx} = e^{-\int 1 dx} = e^{-x}$$

* General solution:

$$y(I.F) = \int Q(I.F) dx + C$$

$$y e^{-x} = \int x e^{-x} dx + C$$

$$= -x e^{-x} - e^{-x} + C$$

$$u = x \\ du = dx$$

$$v = -e^{-x}$$

$$y e^{-x} = -e^{-x} [x+1] + C$$

$$y = -(x+1) + C e^x$$

put $x=0, y=1$.

$$1 = -(0+1) + ce^0$$

$$c = 2.$$

$$\therefore y e^{-x} = (-x+1)e^{-x} + 2$$

$$y = -(x+1) + 2e^x$$

put $x=0.2; y=1.2427$

* Euler's method:

Given differential Equation is $y' = f(x, y)$.

at $y = y_0$ at $x = x_0$.

1st approximation $y_1 = y_0 + h f(x_0, y_0)$.

2nd approximation $y_2 = y_1 + h f(x_1, y_1)$, $x_1 = x_0 + h$.

proceeding in this way after a finite no. of steps we get the value of y when x is given.

* problem:

using Euler's method Find an approximate value of y when $x=1$ given that $\frac{dy}{dx} = x+y$ and $y=1$ when $x=0$ and $h=0.25$

Sol:

Given that $\frac{dy}{dx} = x+y$.

$x_0=0, y_0=1, f(x, y) = x+y, h=0.25$.

1st approximation, $y_1 = y_0 + h f(x_0, y_0)$

$$= 1 + (0.25)(x_0 + y_0)$$

$$= 1 + (0.25)(0+1)$$

$$= 1 + 0.25$$

$$= 1.25$$

2nd approximation $y_2 = y_1 + h f(x_1, y_1)$

$$1.25 + (0.25)(x_1 + y_1)$$

$$x_1 = x_0 + h$$

$$= 0 + 0.25$$

$$= 0.25$$

$$3^{\text{rd}} \text{ approximation, } y_3 = y_2 + h f(x_2, y_2) \\ = 1.625 + (0.25)(0.5 + 1.625) \left[\begin{array}{l} x_2 = x_1 + h \\ = 0.25 + 0.25 \\ = 0.5 \end{array} \right] \\ = 2.156.$$

$$4^{\text{th}} \text{ approximation, } y_4 = y_3 + h f(x_3, y_3) \\ = 2.156 + (0.25)(0.175 + 2.156) \left[\begin{array}{l} x_3 = x_2 + h \\ = 0.5 + 0.25 \\ = 0.75 \end{array} \right] \\ = 2.8825$$

$$5^{\text{th}} \text{ approximation, } y_5 = y_4 + h f(x_4, y_4) \\ = 2.8825 + (0.25)(1 + 2.8825) \\ = 3.853.$$

∴ The value of y when $x=0$ is 3.853.

* Euler's Modified method:

Given Differential Equation is $y' = f(x, y)$.

$y = y_0$ at $x = x_0$.

$$1^{\text{st}} \text{ approximation, } y_1^{(1)} = y_0 + h f(x_0, y_0)$$

$$2^{\text{nd}} \text{ approximation, } y_1^{(2)} = y_0 + h/2 [f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$3^{\text{rd}} \text{ approximation, } y_1^{(3)} = y_0 + h/2 [f(x_0, y_0) + f(x_1, y_1^{(2)})] \\ (x_1 = x_0 + h, x_2 = x_1 + h, x_3 = x_2 + h)$$

* Runge - Kutta Method:

* RK Method of 1st order:

Euler's Method is nothing but RK method of 1st order

$$y_1 = y_0 + h f(x_0, y_0).$$

* RK method of 2nd order:

Euler's modified method is nothing but RK method of 2nd order.

$$y_1^{(1)} = y_0 + h f(x_0, y_0)$$

$$y_1 = y_0 + h/2 (K_1 + K_2)$$

where $K_1 = h f(x_0, y_0)$

$$K_2 = h f(x_0 + h/2, y_0 + K_1/2)$$

* RK Method of 3rd order:

Given that $y' = f(x, y)$, $y = y_0$ at $x = x_0$.

$$K_1 = h f(x_0, y_0), \quad K_2 = h f(x_0 + h/2, y_0 + K_1/2)$$

$$K_3 = h f(x_0 + h, y_0 + K_1) \text{ (or) } h f(x_0 + h, y_0 + 2K_2 - K_1)$$

where $K_1' = f(x_0 + h, y_0 + K_1)$

$$\text{Finally } y_1 = y_0 + 1/6 (K_1 + 4K_2 + K_3)$$

* RK Method of 4th order:

Given that $\frac{dy}{dx} = f(x, y)$, $y = y_0$ at $x = x_0$.

$$K_1 = h f(x_0, y_0), \quad K_2 = h f(x_0 + h/2, y_0 + K_1/2)$$

$$K_3 = h f(x_0 + h/2, y_0 + K_2/2)$$

$$K_4 = h f(x_0 + h, y_0 + K_3)$$

$$K = 1/6 (K_1 + 2K_2 + 2K_3 + K_4)$$

$$\therefore y_1 = y_0 + K$$

$$y_1 = y_0 + 1/6 (K_1 + 2K_2 + 2K_3 + K_4)$$

* Problem:

Apply RK method of 4th order. Find an approximate value of y when $x = 0.2$ given that $\frac{dy}{dx} = x + y$ and $y = 1$ when $x = 0$.

Sol:
Given Equation

$$f(x, y) = x + y$$

$$x_0 = 0, \quad y_0 = 1, \quad h = 0.2$$

4th order:

$$K_1 = h f(x_0, y_0)$$

$$= h(x + y)$$

$$= (0.2)(0 + 1) = 0.2$$

$$\begin{aligned}
 K_2 &= h \left[f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) \right] \\
 &= 0.2 \left[f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right) \right] \\
 &= 0.2 [0.1 + 1.1] \\
 &= 0.24.
 \end{aligned}$$

$$\begin{aligned}
 K_3 &= h \left[f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) \right] \\
 &= 0.2 \left[f\left(0 + \frac{0.2}{2}, 1 + \frac{0.24}{2}\right) \right] \\
 &= 0.2 [0.1 + 1 + 0.12] \\
 &= 0.244.
 \end{aligned}$$

$$\begin{aligned}
 K_4 &= h \left[f(x_0 + h, y_0 + K_3) \right] \\
 &= 0.2 \left[f(0 + 0.2, 1 + 0.244) \right] \\
 &= 0.2 [0.2 + 1.244] \\
 &= 0.2888.
 \end{aligned}$$

$$\begin{aligned}
 \therefore K &= \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4) \\
 &= \frac{1}{6} (0.2 + 2(0.24) + 2(0.244) + 0.2888) \\
 &= 0.2428
 \end{aligned}$$

$$\begin{aligned}
 y_1 &= y_0 + K \\
 &= 1 + 0.2428 \\
 &= 1.2428.
 \end{aligned}$$

2. solve the differential equation $\frac{dy}{dx} = x^2 + y$, $y(0) = 1$ by modified Euler's Method and compute $y(0.02)$ and $y(0.04)$

Sol. Here $f(x, y) = x^2 + y$, $x_0 = 0$, $y_0 = 1$, $h = 0.02$.

To find y_1 i.e. $y(0.02)$.

$$f(x_0, y_0) = f(0, 1) = 0 + 1 = 1.$$

$$\text{using Euler's formula } y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + 0.02(1) = 1.02.$$

$$\text{Now } x_1 = 0.02 \text{ and } f(x_1, y_1^{(0)}) = f(0.02, 1.02) = (0.02)^2 + 1.02 = 1.0204.$$

$$y_1^{(1)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(0)})] = 1 + 0.01 [1 + 1.0204] = 1.0202.$$

$$\begin{aligned} y_1^{(2)} &= y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(1)})] = 1 + 0.01 [1 + f(0.02, 1.0202)] \\ &= 1 + 0.01 [1 + (0.02)^2 + 1.0202] \\ &= 1.0202. \end{aligned}$$

$$y_1^{(1)} = y_1^{(2)} = 1.0202$$

$$\text{we take } y_1 = y(0.02) = 1.0202.$$

To find y_2 i.e. $y(0.04)$.

$$\text{Now } x_1 = 0.02, y_1 = 1.0202, x_2 = 0.04, \text{ and } h = 0.02.$$

$$f(x_1, y_1) = f(0.02, 1.0202) = (0.02)^2 + 1.0202 = 1.0206.$$

$$\text{using Euler's formula } y_2^{(0)} = y_1 + hf(x_1, y_1).$$

$$= 1.0202 + 0.02(1.0206) = 1.0406.$$

$$y_2^{(1)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(0)})]$$

$$= 1.0202 + (0.01) [1.0206 + f(0.04, 1.0406)]$$

$$= 1.0202 + 0.01 [1.0206 + (0.04)^2 + 1.0406]$$

$$= 1.0408.$$

$$y_2^{(2)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(1)})]$$

$$= 1.0202 + (0.01) [1.0206 + f(0.04, 1.0408)]$$

problem:- using Runge-Kutta method of second order, compute $y(2.5)$ from $\frac{dy}{dx} = \frac{x+y}{x}$, $y(2) = 2$ taking $h=0.25$

solution:- $f(x, y) = \frac{x+y}{x}$

$$x_0 = 2, y_0 = 2 \text{ and } h = 0.25$$

$$K_1 = h f(x_0, y_0) = 0.25 f(2, 2) = 0.25(2) = 0.5$$

$$\begin{aligned} K_2 &= h f(x_0 + h, y_0 + K_1) = 0.25 [f(2.25, 2.5)] \\ &= 0.25 \left[\frac{2.25 + 2.5}{2.25} \right] = 0.528. \end{aligned}$$

$$\begin{aligned} y_1 &= y(2.25) = y_0 + \frac{1}{2} (K_1 + K_2) \\ &= 2 + \frac{1}{2} (0.5 + 0.528) \end{aligned}$$

$$y_1 = 2.514.$$

$$\text{Here } x_1 = x_0 + h = 2 + 0.25 = 2.25$$

$$y_1 = 2.514, \quad h = 0.25.$$

$$\begin{aligned} K_1 &= h f(x_1, y_1) = 0.25 f(2.25, 2.514) \\ &= 0.25 \left(\frac{2.25 + 2.514}{2.25} \right) \end{aligned}$$

$$K_1 = 0.5293$$

$$\begin{aligned} K_2 &= h f(x_1 + h, y_1 + K_1) \\ &= 0.25 [f(2.25) + 0.25, 2.514 + 0.5293] \\ &= 0.25 [f(2.5, 3.0433)] \\ &= 0.25 \left[\frac{2.5 + 3.0433}{2.5} \right] = 0.55433 \end{aligned}$$

$$y_2 = y(2.5) = y_1 + \frac{1}{2} (K_1 + K_2) = 2.514 + \frac{1}{2} (0.5293 + 0.55433)$$

TRAPEZOIDAL RULE

The formula is $\int_{x_0}^{x_n} f(x) dx = \frac{h}{2} [(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})]$

Thus $\int_{x_0}^{x_n} f(x) dx = \frac{h}{2} [(\text{sum of the first and last ordinates}) + 2(\text{sum of the remaining ordinates})]$

Problem:- Evaluate $\int_0^{\pi/2} e^{\sin x} dx$ taking $h = \pi/6$

Solution:- Let $y = e^{\sin x}$.

Length of interval is $(\frac{\pi}{2} - 0) = \frac{\pi}{2}$

The values of 'y' are calculated as follows

taking $h = \pi/6$.

x	:	0	$\pi/6$	$\frac{2\pi}{6} = \frac{\pi}{3}$	$\frac{3\pi}{6} = \frac{\pi}{2}$
$y = e^{\sin x}$	:	1	1.6487	2.3774	2.71828
		y_0	y_1	y_2	y_3

Here $n=3$, we will use Trapezoidal Rule.

By Trapezoidal Rule,

$$\begin{aligned} \int_0^{\pi/2} e^{\sin x} dx &= \frac{h}{2} [(y_0 + y_3) + 2(y_1 + y_2)] \\ &= \frac{\pi}{12} [(1 + 2.71828) + 2(1.6487 + 2.3774)] \\ &= \frac{\pi}{12} (11.77048) = 3.0815 \end{aligned}$$

SIMPSON'S $\frac{1}{3}$ -RULE

The formula is.

$$\int_{x_0}^{x_n} f(x) dx = \frac{h}{3} \left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \dots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \dots + y_{n-2}) \right]$$

$$\int_{x_0}^{x_n} f(x) dx = \frac{h}{3} \left[\begin{aligned} &(\text{sum of the first and last ordinates}) \\ &+ 4(\text{sum of the odd ordinates}) \\ &+ 2(\text{sum of the remaining even ordinates}) \end{aligned} \right]$$

Problem:- Evaluate $\int_0^2 e^{-x^2} dx$ using Simpson's $\frac{1}{3}$ -Rule taking $h=0.25$.

Solution:- The values of $y = f(x) = e^{-x^2}$ are given below.

x :	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
x^2 :	0.0625	0.25	0.5625	1.00	1.5625	2.25	3.0625	4.00
y :	0.93941	0.7788	0.56978	0.36788	0.20961	0.1054	0.04677	0.0183
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7

By Simpson's $\frac{1}{3}$ -Rule, we have.

$$\begin{aligned} \int_0^2 e^{-x^2} dx &= \frac{h}{3} \left[(y_0 + y_7) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4 + y_6) \right] \\ &= \frac{0.25}{3} \left[(0.93941 + 0.0183) + 4(0.7788 + 0.36788 + 0.1054) \right. \\ &\quad \left. + 2(0.56978 + 0.20961 + 0.04677) \right] \end{aligned}$$

$$= \frac{0.25}{3} [0.95771 + 7.61835] = \frac{0.25}{3} [8.57606] = 0.71467$$

$$= 0.71467$$

Problem :-

A train is moving at the speed of 30 m/sec. Suddenly brakes are applied. The speed of the train per second after 't' seconds is given by.

Time (t) :	0	5	10	15	20	25	30	35	40	45
speed (v) :	30	24	19	16	13	11	10	8	7	5

Apply Simpson's $\frac{3}{8}$ - rule to determine the distance moved by the train in 45 seconds.

Sol:- If 's' metres is the distance covered in 't' seconds

$$\text{then } \frac{ds}{dt} = v \Rightarrow \left[s \right]_{t=0}^{t=45} = \int_0^{45} v dt$$

Since the no. of sub intervals is 9 (a multiple of 3) hence by using Simpson's $\frac{3}{8}$ rule.

$$\int_0^{45} v dt = \frac{3h}{8} \left[(v_0 + v_9) + 3(v_1 + v_2 + v_4 + v_5 + v_7 + v_8) + 2(v_3 + v_6) \right]$$

$$= \frac{15}{8} \left[(30 + 5) + 3(24 + 19 + 13 + 11 + 8 + 7) + 2(16 + 10) \right]$$

$$= 624.375 \text{ metres.}$$

Hence the distance moved by the train in 45 seconds is 624.375 metres.

2

Def:- The function $f(x)$ defined in $[-\pi, \pi]$ or $[0, 2\pi]$ or in any other interval can be expressed as a trigonometric series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where a 's and b 's are constants. Such series is known as the Fourier series for $f(x)$ and the constants a_0, a_n, b_n ($n=1, 2, 3, \dots$) are called Fourier coefficients of $f(x)$.

Periodic function:- A function $f(x)$ is said to be of period ' T ' or to be periodic with period $T > 0$ if for all real x , $f(x+T) = f(x)$, and ' T ' is the least of such values.

On other words, if the value of each ordinate $f(x)$ repeats itself at equal intervals in the abscissa $f(x)$ is said to be a periodic function.

If $f(x) = f(x+T) = f(x+2T) = f(x+3T) = \dots$ then ' T ' is called the period of the function $f(x)$.

Ex:- since $\sin x = \sin(x+2\pi) = \sin(x+4\pi) = \sin(x+6\pi) = \dots$

the function $\sin x$ is periodic with period 2π

There is no positive value T , $0 < T < 2\pi$ such

that $\sin(x+T) = \sin x$ for all x .

Note:- 1) $\sin n\pi = 0$, $\sin 2n\pi = 0$ for $n \in \mathbb{Z}$, where ' $\mathbb{Z}$ ' is the integers.

$$2) \cos n\pi = (-1)^n, \cos 2n\pi = 1 \quad \forall n \in \mathbb{Z}.$$

EULER'S FORMULAE :- The Fourier series for a

function $f(x)$ in the interval $c \leq x \leq c+2\pi$ is

$$\text{given by } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$\text{where } a_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x) dx \quad \text{--- (A)}, \quad a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx dx \quad \text{--- (A)}$$

$$b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx dx \quad \text{--- (A)}$$

These values of a_0, a_n, b_n are known as Euler's formula.

Corollary 1 :- If $f(x)$ to be expanded as a

Fourier series in the interval $0 \leq x \leq 2\pi$,

Put $c=0$, then the formulae (A) reduces to.

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx \quad \text{--- (B)}, \quad a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx \quad \text{--- (B)}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx \quad \text{--- (B)}$$

Corollary 2 :- If $f(x)$ to be expanded as a

Fourier series in the interval $[-\pi, \pi]$, put $c = -\pi$

the interval becomes $-\pi \leq x \leq \pi$ and the formula

(A) reduces to

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

Even and Odd functions :- A function $f(x)$ is

said to be even function if $f(-x) = f(x)$ and

odd function $f(-x) = -f(x)$

Ex:- $x^2, x^4 + x^2 + 1, e^x + e^{-x}, \cos x, \sec x$ are all even function of 'x'

$x, x^3, x^5 + 2x^3 + 3, \sin x, \csc x$ are all odd

1. Generalized rule of integration by parts.

If u, v are function of x , then

$$\int uv dx = uv_1 - u^1 v_2 + u^2 v_3 - u^3 v_4 + \dots$$

Here $v_1, v_2, v_3, v_4, \dots$ Integration

$u^1, u^2, u^3, \dots$ Differentiation.

$$\int \sin x dx = -\cos x + c$$

$$\int \cos x dx = \sin x$$

$$\begin{aligned} \int x^2 \sin 2x dx &= x^2 \left[-\frac{\cos 2x}{2} \right] - 2x \left[-\frac{\sin 2x}{2^2} \right] + 2 \left[\frac{\cos 2x}{2^3} \right] \\ &= -\frac{x^2}{2} \cos 2x + \frac{x}{2} \sin 2x + \frac{1}{4} \cos 2x \end{aligned}$$

$$2. \int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2} [a \sin bx - b \cos bx]$$

$$\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} [a \cos bx + b \sin bx]$$

$$3. \sin(A+B) + \sin(A-B) = 2 \sin A \cos B$$

$$\sin(A+B) - \sin(A-B) = 2 \cos A \sin B$$

$$\cos(A+B) + \cos(A-B) = 2 \cos A \cos B$$

$$\cos(A-B) - \cos(A+B) = 2 \sin A \sin B$$

$$(or) \cos(A+B) - \cos(A-B) = -2 \sin A \sin B$$

$$4. \sin(-\theta) = -\sin \theta \quad \text{and} \quad \cos(-\theta) = \cos \theta$$

$$5. \int_{-\pi}^{\pi} \cos mx \cos nx dx = \begin{cases} 0, & \text{for } m \neq n \\ \pi, & \text{for } m = n > 0 \\ 2\pi & \text{for } m = n = 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = \begin{cases} 0, & \text{for } m \neq n \text{ and } m = n = 0 \\ \pi, & \text{for } m = n > 0 \end{cases}$$

$$6. \int_{-a}^a f(x) dx = 0, \text{ if } f(x) \text{ is odd function}$$

$$= 2 \int_0^a f(x) dx, \text{ if } f(x) \text{ is even function.}$$

Pr:- Express $f(x) = x - \pi$ as Fourier series in the interval $-\pi < x < \pi$

Sol:- Let the function $f(x) = x - \pi$ be represented by the Fourier series $x - \pi = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

$$x - \pi = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

$$\text{Then } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x - \pi) dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{\pi} x dx - \pi \int_{-\pi}^{\pi} dx \right]$$

$$= \frac{1}{\pi} \left[0 - \pi \cdot 2 \int_0^{\pi} dx \right] \quad \left[\because x \text{ is odd function} \right]$$

$$= \frac{1}{\pi} \left[-2\pi (x)_0^{\pi} \right] = -2(\pi - 0) = -2\pi$$

$$\text{and } a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x - \pi) \cos nx dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{\pi} x \cos nx dx - \pi \int_{-\pi}^{\pi} \cos nx dx \right]$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{\pi} 0 - 2\pi \int_0^{\pi} \cos nx dx \right] \quad \left[x \cos nx \text{ is odd function and } \cos nx \text{ is even function} \right]$$

$$= \frac{1}{\pi} (-2\pi) \int_0^{\pi} \cos nx dx$$

$$= -2 \int_0^{\pi} \cos nx dx = -2 \left[\frac{\sin nx}{n} \right]_0^{\pi}$$

$$= -\frac{2}{n} [\sin n\pi - \sin 0]$$

$$= -\frac{2}{n} [0 - 0]$$

$$= 0 \text{ for } n = 1, 2, 3, \dots$$

Finally $b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x - \pi) \sin nx \, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{\pi} x \sin nx \, dx - \pi \int_{-\pi}^{\pi} \sin nx \, dx \right]$$

$$= \frac{1}{\pi} \left[2 \int_0^{\pi} x \sin nx \, dx - \pi (0) \right] \quad \left[\begin{array}{l} x \sin nx \text{ is even} \\ \text{function and} \\ \sin nx \text{ is odd function} \end{array} \right]$$

$$b_m = \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi} \left[x \left(\frac{-\cos nx}{n} \right) - 1 \left(\frac{-\sin nx}{n^2} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\left(\frac{-\pi \cos n\pi}{n} + 0 \right) - (0 + 0) \right] \quad [\sin n\pi = 0]$$

$$= -\frac{2}{\pi} \cos n\pi = -\frac{2}{\pi} (-1)^n = \frac{2}{\pi} (-1)^{n+1} \text{ for } n=1, 2, 3, \dots$$

Substituting the values of a_0 , a_n and b_n in (1), we get the required Fourier series for $f(x)$ in the interval $(-\pi, \pi)$ as

$$x - \pi = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= -\frac{2\pi}{2} + \sum_{n=1}^{\infty} (0) \cos nx + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{\pi} \sin nx$$

$$= -\pi + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{\pi} \sin nx$$

$$x - \pi = -\pi + 2 \left[\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \dots \right]$$

Pr:

Find a Fourier series to represent $f(x) = x^2$ in the interval $(0, 2\pi)$. (or) Expand $f(x) = x^2$, $0 < x < 2\pi$ as a Fourier series.

Sol:

$$\text{Let } x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) \, dx = \frac{1}{\pi} \int_0^{2\pi} x^2 \, dx = \frac{1}{\pi} \left[\frac{x^3}{3} \right]_0^{2\pi}$$

$$= \frac{1}{3\pi} [8\pi^3 - 0] = \frac{8}{3} \pi^2$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x^2 \cos nx \, dx$$

$$= \frac{1}{\pi} \left[x^2 \left(\frac{\sin nx}{n} \right) - 2x \left(\frac{-\cos nx}{n^2} \right) + 2 \left(\frac{-\sin nx}{n^3} \right) \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[x^2 \left(\frac{\sin nx}{n} \right) + \frac{2x \cos nx}{n^2} - \frac{2 \sin nx}{n^3} \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\left(0 + \frac{4\pi \cos 2n\pi}{n^2} - 0 \right) - \left(0 + 0 - 0 \right) \right]$$

$$= \frac{1}{\pi} \left[\frac{4\pi \cos 2n\pi}{n^2} \right] = \frac{4}{n^2} \left[\because \cos 2n\pi = 1 \right]$$

Finally $b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx$

$$= \frac{1}{\pi} \int_0^{2\pi} x^2 \sin nx \, dx$$

$$= \frac{1}{\pi} \left[x^2 \left(\frac{-\cos nx}{n} \right) - 2x \left(\frac{-\sin nx}{n^2} \right) + 2 \left(\frac{\cos nx}{n^3} \right) \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[-\frac{x^2 \cos nx}{n} + \frac{2x \sin nx}{n^2} + \frac{2 \cos nx}{n^3} \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\left(-\frac{4\pi^2 \cos 2n\pi}{n} + 0 + \frac{2 \cos 2n\pi}{n^3} \right) - \left(0 + 0 + \frac{2}{n^3} \right) \right]$$

$$= \frac{1}{\pi} \left[-\frac{4\pi^2}{n} + \frac{2}{n^3} - \frac{2}{n^3} \right] = -\frac{4\pi}{n}$$

Substituting the values a_0 , a_n and b_n in ① we get the required Fourier Series as.

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{8}{6} \pi^2 + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx - \sum_{n=1}^{\infty} \frac{4\pi}{n} \sin nx$$

$$= \frac{4}{3} \pi^2 + 4 \left[\cos x + \frac{1}{2^2} \cos 2x + \frac{1}{3^2} \cos 3x + \dots \right]$$

$$- 4\pi \left[\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots \right]$$

Pr: Obtain the Fourier series for the function $f(x)$ from $x = 0$ to $x = 2\pi$

Sol

$$\text{Let } e^x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$\text{Then } a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{\pi} \int_0^{2\pi} e^x dx.$$

$$= \frac{1}{\pi} \left[e^x \right]_0^{2\pi} = \frac{1}{\pi} \left[e^{2\pi} - e^0 \right] = \frac{1}{\pi} \left[e^{2\pi} - 1 \right].$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_0^{2\pi} e^x \cos nx dx.$$

$$= \frac{1}{\pi} \left[\frac{e^x}{1+n^2} (\cos nx + n \sin nx) \right]_0^{2\pi} \quad \left[\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx) \right]$$

$$= \frac{1}{\pi} \left[\frac{e^{2\pi}}{1+n^2} (\cos 2n\pi + n \sin 2n\pi) - \frac{e^0}{1+n^2} (\cos 0 + n \sin 0) \right]$$

$$= \frac{1}{\pi} \left[\frac{e^{2\pi}}{1+n^2} [(1+0) - \frac{1}{1+n^2} (1+0)] \right]$$

$$= \frac{1}{\pi} \left[\frac{e^{2\pi}}{1+n^2} [1] - \frac{1}{1+n^2} (1) \right]$$

$$= \frac{e^{2\pi} - 1}{\pi (1+n^2)} \quad [\sin 2n\pi = 0, \cos 2n\pi = 1]$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_0^{2\pi} e^x \sin nx dx.$$

$$= \frac{1}{\pi} \left[\frac{e^x}{1+n^2} [\sin nx - n \cos nx] \right]_0^{2\pi} \quad \left[\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx) \right]$$

$$= \frac{1}{\pi} \left[\frac{e^{2\pi}}{1+n^2} (\sin 2n\pi - n \cos 2n\pi) - \frac{e^0}{1+n^2} (\sin 0 - n \cos 0) \right]$$

$$= \frac{1}{\pi} \left[\frac{e^{2\pi}}{1+n^2} (0 - n) - \frac{1}{1+n^2} (-n) \right] \quad \left[\begin{array}{l} \sin 2n\pi = 0 \\ \cos 2n\pi = 1 \end{array} \right]$$

$$b_n = \frac{(-n)(e^{2\pi} - 1)}{\pi (1+n^2)}.$$

$$\text{Hence } e^x = \frac{e^{2\pi} - 1}{2\pi} + \sum_{n=1}^{\infty} \frac{e^{2\pi} - 1}{\pi(1+n^2)} \cos nx + \sum_{n=1}^{\infty} \frac{(-n)(e^{2\pi} - 1)}{\pi(1+n^2)} \sin nx$$

$$= \frac{e^{2\pi} - 1}{\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{\cos nx}{1+n^2} - \sum_{n=1}^{\infty} \frac{n \sin nx}{1+n^2} \right]$$

2.

Pb:- obtain the Fourier series for $f(x) = x - x^2$ in the interval $[-\pi, \pi]$ Show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$

$$(\text{or}) \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$$

Sol:- The Fourier series of $f(x) = x - x^2$ in $[-\pi, \pi]$ is given by.

$$x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{--- (1)}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) dx = \frac{1}{\pi} \left[\int_{-\pi}^{\pi} x dx - \int_{-\pi}^{\pi} x^2 dx \right]$$

$$= \frac{1}{\pi} \left[0 - \int_{-\pi}^{\pi} x^2 dx \right] \quad \left[x \text{ is odd function and } x^2 \text{ even function} \right]$$

$$= \frac{1}{\pi} \left[-2 \int_0^{\pi} x^2 dx \right]$$

$$= -\frac{2}{\pi} \int_0^{\pi} x^2 dx = -\frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi}$$

$$= -\frac{2}{\pi} \left[\frac{\pi^3}{3} \right] = -\frac{2\pi^2}{3}$$

$$\boxed{a_0 = -\frac{2\pi^2}{3}}$$

$$\text{and } a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) \cos nx dx$$

$$= \frac{1}{\pi} \left[(x - x^2) \frac{\sin nx}{n} - (1 - 2x) \left(-\frac{\cos nx}{n^2} \right) + (-2) \left(-\frac{\sin nx}{n^3} \right) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[(x - x^2) \frac{\sin nx}{n} + (1 - 2x) \frac{\cos nx}{n^2} + \frac{2 \sin nx}{n^3} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\left((\pi - \pi^2) \frac{\sin n\pi}{n} + (1 - 2\pi) \frac{\cos n\pi}{n^2} + \frac{2 \sin n\pi}{n^3} \right) - \left((-\pi - \pi^2) \frac{\sin n\pi}{n} + (1 + 2\pi) \frac{\cos n\pi}{n^2} + \frac{2 \sin n(-\pi)}{n^3} \right) \right]$$

$$\frac{\sin n\pi}{n^2}$$

$$= \frac{(-1)^n}{n^2} \quad (n \neq 0)$$

$$\therefore \cos n\pi = (-1)^n$$

Finally $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x-x^2) \sin nx \, dx$

$$= \frac{1}{\pi} \left[(x-x^2) \left[-\frac{\cos nx}{n} \right] - (1-2x) \left(-\frac{\sin nx}{n^2} \right) + (-2) \frac{\cos nx}{n^3} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\left[(\pi-\pi^2) \left[-\frac{\cos n\pi}{n} \right] + (1-2\pi) \frac{\sin n\pi}{n^2} - 2 \frac{\cos n\pi}{n^3} \right] \right.$$

$$\left. - \left[(-\pi-\pi^2) \left[-\frac{\cos n(-\pi)}{n} \right] + (1-2(-\pi)) \frac{\sin n(-\pi)}{n^2} + (-2) \frac{\cos n(-\pi)}{n^3} \right] \right]$$

$$= \frac{1}{\pi} \left[(\pi-\pi^2) \frac{-\cos n\pi}{n} + (-\pi-\pi^2) \frac{\cos n\pi}{n} \right]$$

$$= \frac{1}{\pi} \left[\frac{\cos n\pi}{n} [-\pi + \pi^2 - \pi - \pi^2] \right]$$

$$= \frac{1}{\pi} \left[\frac{\cos n\pi}{n} (-2\pi) \right] = -\frac{2 \cos n\pi}{n} = -\frac{2(-1)^n}{n}$$

Substituting the values of a_0 , a_n and b_n in (1) we get

$$x-x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= -\frac{2\pi^2}{2 \times 3} + \sum_{n=1}^{\infty} \left[\frac{-4(-1)^n}{n^2} \cos nx + \frac{(-2)(-1)^n}{n} \sin nx \right]$$

$$= -\frac{\pi^2}{3} + \sum_{n=1}^{\infty} \left[\frac{+4(-1)^{n+1}}{n^2} \cos nx + \frac{2(-1)^{n+1}}{n} \sin nx \right]$$

$$= -\frac{\pi^2}{3} + 4 \left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \dots \right]$$

$$+ 2 \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots \right] \quad \text{--- (2)}$$

Deduction:- Taking $x=0$ in the above Fourier series, we have

$$0 = -\frac{\pi^2}{3} + 4 \left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \right]$$

$$\frac{\pi^2}{3} = 4 \left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \right]$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \quad \text{--- (3)} \quad \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

Sol.

Given $f(x) = \left(\frac{\pi-x}{2}\right)^2 = \frac{(\pi-x)^2}{4}$ in $(0, 2\pi)$

The Fourier series of $f(x)$ in $[0, 2\pi]$ is given by

$$\frac{(\pi-x)^2}{4} = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{--- (1)}$$

where a_n, b_n are obtained through Euler's formulae.

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{\pi} \int_0^{2\pi} \frac{(\pi-x)^2}{4} dx = \frac{1}{4\pi} \int_0^{2\pi} (\pi-x)^2 dx$$

$$= \frac{1}{4\pi} \left[\frac{(\pi-x)^3}{3} (-1) \right]_0^{2\pi} = \frac{1}{4\pi} \left[-\frac{1}{3} (\pi-x)^3 \right]_0^{2\pi}$$

$$= -\frac{1}{12\pi} \left[(\pi-2\pi)^3 - \pi^3 \right]$$

$$= -\frac{1}{12\pi} \left[-\pi^3 - \pi^3 \right] = \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} \frac{(\pi-x)^2}{4} \cos nx dx = \frac{1}{4\pi} \int_0^{2\pi} (\pi-x)^2 \cos nx dx$$

$$= \frac{1}{4\pi} \left[(\pi-x)^2 \left(\frac{\sin nx}{n} \right) - 2(\pi-x)(-1) \left(-\frac{\cos nx}{n^2} \right) + 2 \left(-\frac{\sin nx}{n^3} \right) \right]_0^{2\pi}$$

$$= \frac{1}{4\pi} \left[\left(0 + \frac{2\pi \cos 2n\pi}{n^2} + 0 \right) - \left(0 - \frac{2\pi \cos 0}{n^2} + 0 \right) \right]$$

$$= \frac{1}{4\pi} \left[\frac{2\pi \cos 2n\pi}{n^2} + \frac{2\pi}{n^2} \right]$$

$$= \frac{1}{4\pi} \left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2} \right] \quad \left[\because \cos 2n\pi = \cos 0 = 1 \right]$$

$$= \frac{1}{4\pi} \left[\frac{4\pi}{n^2} \right] = \frac{1}{n^2} \quad \left[\text{if } n \neq 0 \right]$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} \frac{(\pi-x)^2}{4} \sin nx dx$$

$$= \frac{1}{4\pi} \int_0^{2\pi} (\pi-x)^2 \sin nx dx$$

$$\begin{aligned}
 b_n &= \frac{1}{4\pi} \left[(\pi-x)^2 \left[-\frac{\cos nx}{n} \right] - 2(\pi-x)(-1) \left[-\frac{\sin nx}{n^2} \right] + 2 \left[\frac{\cos nx}{n^3} \right] \right]_0^{\pi} \\
 &= \frac{1}{4\pi} \left[\left(-\frac{\pi^2 \cos 2n\pi}{n} - 0 + \frac{2 \cos 2n\pi}{n^3} \right) - \left(-\frac{\pi^2}{n} - 0 + \frac{2}{n^3} \right) \right] \\
 &= \frac{1}{4\pi} \left[-\frac{\pi^2}{n} + \frac{2}{n^3} + \frac{\pi^2}{n} - \frac{2}{n^3} \right] \quad [\because \cos 2n\pi = 1] \\
 &= 0
 \end{aligned}$$

Substituting the values of a_0, a_n, b_n in (1) we get.

$$\left(\frac{\pi-x}{2} \right)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx = \frac{\pi^2}{12} + \frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \dots \quad (2)$$

which is required Fourier series.

Deduction:- Putting $x=\pi$ in (2), we obtain.

$$\begin{aligned}
 0 &= \frac{\pi^2}{12} - \frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \dots \\
 &= \frac{\pi^2}{12} - \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \right) \\
 \frac{\pi^2}{12} &= \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots
 \end{aligned}$$

Pb:- obtain the Fourier series for the function

$$f(x) = x \sin x, \quad 0 < x < 2\pi$$

Sol:- Let $x \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ — (1)

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{\pi} \int_0^{2\pi} x \sin x dx$$

$$= \frac{1}{\pi} \left[x(-\cos x) - 1(-\sin x) \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[-x \cos x + \sin x \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[(-2\pi \cos 2\pi + \sin 2\pi) - (0+0) \right]$$

$$= \frac{1}{\pi} \left[-2\pi(1) \right]$$

$$= -2$$

$$\begin{aligned}
 a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \int_0^{2\pi} x \sin x \cos nx \, dx \\
 &= \frac{1}{2\pi} \int_0^{2\pi} x (2 \sin x \cos nx) \, dx = \frac{1}{2\pi} \int_0^{2\pi} x [\sin(n+1)x - \sin(n-1)x] \, dx \\
 &= \frac{1}{2\pi} \left[x \left[\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1} \right] - \left[\frac{-\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2} \right] \right]_0^{2\pi} \\
 &= \frac{1}{2\pi} \left[2\pi \left[\frac{\cos(n+1)2\pi}{n+1} + \frac{\cos(n-1)2\pi}{n-1} \right] \right] \quad n \neq 1 \\
 &= \frac{1}{n+1} + \frac{1}{n-1} = \frac{2}{n^2-1} \quad (n \neq 1) \quad [\cos 2(n+1)\pi = 1]
 \end{aligned}$$

$n=1$, we have

$$\begin{aligned}
 a_1 &= \frac{1}{2\pi} \int_0^{2\pi} x \sin 2x \, dx \\
 &= \frac{1}{2\pi} \left[x \left(\frac{-\cos 2x}{2} \right) - \left(\frac{-\sin 2x}{4} \right) \right]_0^{2\pi} \\
 &= \frac{1}{2\pi} \left[2\pi \left(\frac{-\cos 4\pi}{2} \right) - 1(0) \right] \\
 &= -\frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_0^{2\pi} x \sin x \sin nx \, dx \\
 &= \frac{1}{2\pi} \int_0^{2\pi} x (2 \sin x \sin nx) \, dx \\
 &= \frac{1}{2\pi} \int_0^{2\pi} x [\cos(n-1)x - \cos(n+1)x] \, dx \\
 \text{(or)} &= \frac{1}{2\pi} \int_0^{2\pi} x [\cos(1-n)x - \cos(1+n)x] \, dx \\
 &= \frac{1}{2\pi} \left[x \left(\frac{\sin(1-n)x}{1-n} - \frac{\sin(1+n)x}{1+n} \right) \right. \\
 &\quad \left. - \left[\frac{-\cos(1-n)x}{(1-n)^2} + \frac{\cos(1+n)x}{(1+n)^2} \right] \right]_0^{2\pi} \quad (n \neq 1)
 \end{aligned}$$

$$= \frac{1}{2\pi} \left[\frac{\cos \delta(1-n)\pi}{1-n} - \frac{\cos \delta(1+n)\pi}{(1+n)^2} \right]_{\delta=0}^{\delta=2\pi} \quad n \neq 1$$

$$= \frac{1}{2\pi} \left[\left(\frac{\cos 2\pi(1-n)}{(1-n)} - \frac{\cos 2\pi(1+n)}{(1+n)^2} \right) - \left(\frac{1}{1-n} - \frac{1}{(1+n)^2} \right) \right]$$

$$= \frac{1}{2\pi} \left[\frac{1}{1-n} - \frac{1}{(1+n)^2} - \frac{1}{1-n} + \frac{1}{(1+n)^2} \right]$$

$$= \frac{1}{2\pi} (0)$$

$$b_n = 0 \text{ for } n \neq 1$$

$$\text{If } n=1, \text{ then } b_1 = \frac{1}{2\pi} \int_0^{2\pi} x \cdot 2 \sin^2 x \, dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x(1 - \cos 2x) \, dx$$

$$= \frac{1}{2\pi} \left[x \left(x - \frac{\sin 2x}{2} \right) - 1 \cdot \left(\frac{x^2}{2} + \frac{\cos 2x}{4} \right) \right]_0^{2\pi}$$

$$= \frac{1}{2\pi} \left[2\pi(2\pi - 0) - \left(\frac{4\pi^2}{2} + \frac{\cos 4\pi}{4} - 0 + \frac{1}{4} \right) \right]$$

$$= \frac{1}{2\pi} \left[4\pi^2 - \frac{4\pi^2}{2} - \frac{1}{4} + \frac{1}{4} \right]$$

$$= \frac{4\pi^2}{2\pi} \left[1 - \frac{1}{2} \right] = 2\pi \left[\frac{1}{2} \right] = \pi$$

Substituting the values of a_0, a_n, b_n in (1) we get

$$x \sin x = -1 - \frac{1}{2} \cos x + \sum_{n=2}^{\infty} \left(\frac{2 \cos nx}{n^2-1} + \pi \sin nx \right)$$

$$x \sin x = -1 - \frac{1}{2} \cos x + \sum_{n=2}^{\infty} \left(\frac{2 \cos nx}{n^2-1} + \pi \sin nx \right)$$

Deductions:- Taking $x=0$ in the above Fourier series

$$0 = -1 - \frac{1}{2} + \sum_{n=2}^{\infty} \frac{2}{n^2-1} (1) + \pi(0)$$

$$\frac{3}{2} = \sum_{n=2}^{\infty} \frac{2}{n^2-1}$$

$$\frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} + \dots = \frac{3}{4}$$

2) Taking $x = \pi/2$ in the above Fourier series

$$\frac{\pi}{2} (1) = -1 - \frac{1}{2} \cos\left(\frac{\pi}{2}\right) + \sum_{n=2}^{\infty} \frac{2}{n^2-1} \cos\frac{n\pi}{2} + \pi \sin\frac{n\pi}{2}$$

$$\frac{\pi}{2} - \pi = -1 + 2 \left[\frac{1}{1 \cdot 3} \cos\pi + \frac{1}{2 \cdot 4} \cos\frac{3\pi}{2} + \frac{1}{3 \cdot 5} \cos 2\pi + \dots \right]$$

$$\frac{-\pi}{2} + 1 = 2 \left[\dots \right]$$

pb:- Find the Fourier expansion of $f(x) = x \cos x$, $0 < x < 2\pi$

sol:- The Fourier series of $f(x) = x \cos x$ is given by.

$$f(x) = x \cos x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

$$\text{Here } a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{\pi} \int_0^{2\pi} x \cos x dx$$

$$= \frac{1}{\pi} \left[x(\sin x) - 1 \cdot (-\cos x) \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[x \sin x + \cos x \right]_0^{2\pi} = \frac{1}{\pi} \left[(2\pi \sin 2\pi + \cos 2\pi) - (0 + \cos 0) \right]$$

$$= \frac{1}{\pi} [0 + 1 - 1] = 0$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \cos x \cos nx dx \quad \text{--- (2)}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x (2 \cos x \cos nx) dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x [\cos(1+n)x + \cos(1-n)x] dx$$

$$= \frac{1}{2\pi} \left[\frac{\cos(1+n)\alpha}{(1+n)^2} + \frac{\cos(1-n)\alpha}{(1-n)^2} \right]_{0}^{2\pi}$$

$$= \frac{1}{2\pi} \left[\frac{1}{(1+n)^2} + \frac{1}{(1-n)^2} - \frac{1}{(1+n)^2} - \frac{1}{(1-n)^2} \right]$$

$$a_n = 0 \quad (n \neq 1)$$

If $n=1$ in above eq. (2),

$$a_1 = \frac{1}{\pi} \int_0^{2\pi} x \cos^2 x \, dx = \frac{1}{\pi} \int_0^{2\pi} x \left(\frac{1 + \cos 2x}{2} \right) dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x(1 + \cos 2x) \, dx$$

$$= \frac{1}{2\pi} \left[x \left(x + \frac{\sin 2x}{2} \right) - \frac{1}{2} \left(\frac{x^2}{2} - \frac{\cos 2x}{4} \right) \right]_{0}^{2\pi}$$

$$= \frac{1}{2\pi} \left[2\pi \left(2\pi + \frac{\sin 4\pi}{2} \right) - \frac{1}{2} \left(\frac{4\pi^2}{2} - \frac{\cos 4\pi}{4} \right) \right. \\ \left. - \left[0 \left(0 + \frac{\sin 0}{2} \right) - \frac{1}{2} \left(\frac{0^2}{2} - \frac{\cos 0}{4} \right) \right] \right]$$

$$= \frac{1}{2\pi} \left[4\pi^2 - \frac{4\pi^2}{2} + \frac{1}{4} - \frac{1}{4} \right]$$

$$= \frac{1}{2\pi} \left[\frac{4\pi^2}{2} \right] = \pi$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_0^{2\pi} x \cos x \sin nx \, dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x (2 \cos x \sin nx) \, dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x [\sin(1+n)x - \sin(1-n)x] \, dx$$

$$\sin(1+n)2\pi = 0$$

$$\sin(1-n)2\pi = 0$$

$$\cos 2(n+1)\pi = 1$$

$$\cos 2(n-1)\pi = 1$$

$$\begin{aligned}
 &= \frac{1}{2\pi} \left[x \left(\frac{-\cos(1+n)x}{(1+n)} + \frac{\cos(1-n)x}{(1-n)} \right) \right. \\
 &\quad \left. - 1 \left(\frac{-\sin(1+n)x}{(1+n)^2} + \frac{\sin(1-n)x}{(1-n)^2} \right) \right]_{0}^{2\pi} \quad (n \neq 1) \\
 &= \frac{1}{2\pi} \left[2\pi \left(\frac{-\cos(1+n)2\pi}{1+n} + \frac{\cos 2(1-n)\pi}{(1-n)^2} \right) \right] \\
 &= \frac{1}{2\pi} \left[2\pi \left(-\frac{1}{1+n} \right) + \frac{1}{1-n} \right] \\
 &= -\frac{1}{1+n} + \frac{1}{1-n} \\
 &= \frac{2n}{1-n^2} \quad (n \neq 1)
 \end{aligned}$$

If $n=1$, then

$$\begin{aligned}
 b_1 &= \frac{1}{\pi} \int_0^{2\pi} x \cos x \sin x \, dx \\
 &= \frac{1}{2\pi} \int_0^{2\pi} x \sin 2x \, dx \\
 &= \frac{1}{2\pi} \left[x \left(-\frac{\cos 2x}{2} \right) - 1 \left(-\frac{\sin 2x}{2} \right) \right]_{0}^{2\pi} \\
 &= \frac{1}{2\pi} \left[-\frac{x \cos 2x}{2} + \frac{\sin 2x}{2} \right]_{0}^{2\pi} \\
 &= \frac{1}{2\pi} \left[-\frac{2\pi}{2} + 0 \right] = -\frac{1}{2}
 \end{aligned}$$

Substituting a_0, a_1, a_n, b_1 and b_n , we get required

Fourier series of interval $(0, 2\pi)$ as

$$x \cos x = \pi \cos x - \frac{1}{2} \sin x + 2 \sum_{n=2}^{\infty} \frac{n}{1-n^2} \sin nx$$

$$x \cos x = \pi \cos x - \frac{1}{2} \sin x - 2 \sum_{n=2}^{\infty} \frac{n}{n^2-1} \sin nx$$

Find the Fourier series represent the function e^{-ax} from $x = -\pi$ to $x = \pi$. Deduce from this that.

$$\frac{\pi}{\sinh \pi} = 2 \left[\frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} - \dots \right]$$

Let the function e^{-ax} be represented by the Fourier series

$$e^{-ax} = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-ax} dx = \frac{1}{\pi} \left[\frac{e^{-ax}}{-a} \right]_{-\pi}^{\pi} = -\frac{1}{\pi a} \left[e^{-a\pi} - e^{a\pi} \right]$$

$$= \frac{e^{a\pi} - e^{-a\pi}}{\pi a}$$

$$\frac{a_0}{2} = \frac{1}{\pi a} \left[\frac{e^{a\pi} - e^{-a\pi}}{2} \right] = \frac{\sinh a\pi}{\pi a}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-ax} \cos nx dx = \frac{1}{\pi} \left[\frac{e^{-ax}}{a^2+n^2} (-a \cos nx + n \sin nx) \right]_{-\pi}^{\pi}$$

$$a_n = \frac{1}{\pi} \left[\frac{e^{-a\pi}}{a^2+n^2} (-a \cos n\pi + n \sin n\pi) - \frac{e^{a\pi}}{a^2+n^2} (-a \cos n\pi + 0) \right]$$

$$= \frac{1}{\pi} \left[\frac{e^{-a\pi}}{a^2+n^2} [-a(-1)^n + 0] - \frac{e^{a\pi}}{a^2+n^2} (-a(-1)^n) \right]$$

$$= \frac{(-1)^n}{\pi} \left[\frac{a e^{a\pi}}{a^2+n^2} - \frac{a e^{-a\pi}}{a^2+n^2} \right]$$

$$= \frac{(-1)^n}{\pi} \frac{a}{a^2+n^2} \left[e^{a\pi} - e^{-a\pi} \right]$$

$$= \frac{(-1)^n}{\pi} \frac{2a}{a^2+n^2} \left[\frac{e^{a\pi} - e^{-a\pi}}{2} \right] = \frac{(-1)^n 2a \sinh a\pi}{\pi(a^2+n^2)}$$

$$\frac{e^{a\pi} - e^{-a\pi}}{2} = \sinh a\pi$$

$$\cos n\pi = (-1)^n$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-ax} \sin nx dx$$

$$b_n = \frac{1}{\pi} \left[\frac{e^{-ax}}{a^2+n^2} (-a \sin nx - n \cos nx) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{e^{-a\pi}}{a^2+n^2} (-a \sin n\pi - n \cos n\pi) - \frac{e^{a\pi}}{a^2+n^2} (-a \sin(-\pi) - n \cos n\pi) \right]$$

$$= \frac{1}{\pi} \left[\frac{e^{-a\pi}}{a^2+n^2} (-n(-1)^n) - \frac{e^{a\pi}}{a^2+n^2} (0 - n(-1)^n) \right]$$

$$b_n = \frac{n(-1)^n [e^{a\pi} - e^{-a\pi}]}{\pi(\alpha^2 + n^2)} = \frac{(-1)^n 2n \sinh a\pi}{\pi(\alpha^2 + n^2)}$$

Substituting the values $\frac{a_0}{2}$, a_n , and b_n in (1) we get

$$e^{-ax} = \frac{\sinh a\pi}{a\pi} + \sum_{n=1}^{\infty} \frac{(-1)^n 2a \sinh a\pi}{\pi(\alpha^2 + n^2)} \cos nx + \sum_{n=1}^{\infty} \frac{(-1)^n 2n \sinh a\pi}{\pi(\alpha^2 + n^2)} \sin nx$$

$$= \frac{2 \sinh a\pi}{a\pi} \left[\left(\frac{1}{2a} - \frac{a \cos x}{\alpha^2 + 1^2} + \frac{a \cos 2x}{\alpha^2 + 2^2} - \frac{a \cos 3x}{\alpha^2 + 3^2} + \dots \right) - \left(\frac{3 \sin x}{1^2 + \alpha^2} - \frac{2 \sin 2x}{2^2 + \alpha^2} + \frac{3 \sin 3x}{3^2 + \alpha^2} - \dots \right) \right] \quad \text{--- (2)}$$

which is required Fourier series.

Deduction:-

Putting $x=0$ and $a=1$ in (2), we get.

$$1 = \frac{2 \sinh \pi}{\pi} \left[\frac{1}{2} - \frac{1}{2} + \frac{1}{2^2 + 1} - \frac{1}{3^2 + 1} + \frac{1}{4^2 + 1} - \dots \right]$$

$$\frac{\pi}{\sinh \pi} = 2 \left[\frac{1}{2^2 + 1} - \frac{1}{3^2 + 1} + \frac{1}{4^2 + 1} - \dots \right]$$

Pb. Expand $f(x) = \begin{cases} 1, & 0 < x < \pi \\ 0, & \pi < x < 2\pi \end{cases}$ as a Fourier series.

Sol:- The Fourier series of $f(x)$ in the interval $(0, 2\pi)$ is given by.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx. \quad \text{--- (1)}$$

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{\pi} \left[\int_0^{\pi} f(x) dx + \int_{\pi}^{2\pi} f(x) dx \right] \\ &= \frac{1}{\pi} \left[\int_0^{\pi} 1 dx + \int_{\pi}^{2\pi} 0 dx \right] = \frac{1}{\pi} \int_0^{\pi} 1 dx = \frac{1}{\pi} [x]_0^{\pi} \\ &= \frac{1}{\pi} [\pi] = 1 \end{aligned}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx.$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} 1 \cdot \cos nx dx + \int_{\pi}^{2\pi} 0 \cdot \cos nx dx \right]$$

$$= \frac{1}{\pi} \left[\frac{\sin nx}{n} \right]_0^{\pi} = \frac{1}{n\pi} (0 - 0) = 0.$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx = \frac{1}{\pi} \left[\int_0^{\pi} 1 \cdot \sin nx dx + \int_{\pi}^{2\pi} 0 \cdot \sin nx dx \right]$$

$$= \frac{1}{\pi} \int_0^{\pi} \sin nx dx = \frac{1}{\pi} \left[-\frac{\cos nx}{n} \right]_0^{\pi}$$

$$= \frac{1}{\pi n} [-\cos n\pi + \cos 0]$$

$$= \frac{1}{n\pi} [-(-1)^n + 1] = -\frac{1}{n\pi} [(-1)^n - 1]$$

$$b_n = 0, \text{ when 'n' is even.}$$

$$= \frac{2}{n\pi} \text{ when 'n' is odd.}$$

Substituting the values of a_0 , a_n , and b_n in (1) we get.

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n} \sin nx.$$

$$= \frac{1}{2} + \frac{2}{\pi} \left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right)$$

Pb

Find the Fourier series to represent the function $f(x)$

$$\text{given by } f(x) = -K, -\pi < x < 0 \\ K, 0 < x < \pi.$$

Hence show that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$.

Sol:-

The Fourier series is of the form.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \left[\int_{-\pi}^0 f(x) dx + \int_0^{\pi} f(x) dx \right]$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 (-K) dx + \int_0^{\pi} K dx \right]$$

$$= \frac{K}{\pi} \left[(-x)_{-\pi}^0 + (x)_0^{\pi} \right] = \frac{K}{\pi} [-\pi + \pi] = 0.$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx.$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 (-K) \cos nx dx + \int_0^{\pi} K \cos nx dx \right]$$

$$= \frac{K}{\pi} \left[-\left(\frac{\sin nx}{n}\right)_{-\pi}^0 + \left(\frac{\sin nx}{n}\right)_0^{\pi} \right]$$

$$= \frac{K}{\pi} [0 + 0] = 0.$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx.$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 -K \sin nx dx + \int_0^{\pi} K \sin nx dx \right]$$

$$= \frac{K}{\pi} \left[\left(\frac{\cos nx}{n}\right)_{-\pi}^0 + \left(-\frac{\cos nx}{n}\right)_0^{\pi} \right]$$

$$= \frac{K}{\pi} \left[\left(\frac{1 - (-1)^n}{n}\right) - \left[\frac{(-1)^n - 1}{n}\right] \right] \quad \cos n\pi = (-1)^n.$$

$$= \frac{K}{\pi} \left[\frac{2 - 2(-1)^n}{n} \right] = 0 \text{ when 'n' is even.}$$

$$\frac{4K}{\pi n} \text{ when 'n' is odd.}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{4K}{n\pi} \sin nx, \text{ where 'n' is odd.}$$

$$= \frac{4K}{\pi} \left[\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right] \quad \text{--- (2)}$$

Deduction:- At $x = \frac{\pi}{2}$, $f(x)$ is continuous. Hence the Fourier series converges to $f(\frac{\pi}{2})$. Thus putting $x = \frac{\pi}{2}$ in (2) we get

$$f(\frac{\pi}{2}) = K = \frac{4K}{\pi} \left(\sin \frac{\pi}{2} + \frac{1}{3} \sin \frac{3\pi}{2} + \frac{1}{5} \sin \frac{5\pi}{2} + \dots \right)$$

$$1 = \frac{4}{\pi} \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right]$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Pb:-

Find the Fourier series of the following function.

$$f(x) = 0, -\pi < x < 0. \quad \text{Also deduce that.}$$

$$\frac{\pi x}{4}, 0 < x < \pi. \quad \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Sol:-

The Fourier series of $f(x)$ is given by.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \left[\int_{-\pi}^0 0 dx + \int_0^{\pi} \frac{\pi x}{4} dx \right]$$

$$= \frac{1}{\pi} \left[\frac{\pi}{4} \left[\frac{x^2}{2} \right]_0^{\pi} \right] = \frac{\pi^2}{8}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \left[\int_{-\pi}^0 0 \cdot \cos nx dx + \int_0^{\pi} \frac{\pi x}{4} \cos nx dx \right]$$

$$= \frac{1}{\pi} \cdot \frac{\pi}{4} \int_0^{\pi} x \cos nx dx = \frac{1}{4} \int_0^{\pi} x \cos nx dx$$

$$= \frac{1}{4} \left[x \left(\frac{\sin nx}{n} \right) - \left(-\frac{\cos nx}{n^2} \right) \right]_0^{\pi}$$

$$= \frac{1}{4} \left[\left(\pi \frac{\sin n\pi}{n} + \frac{\cos n\pi}{n^2} \right) - \left[0 + \frac{1}{n^2} \right] \right]$$

$$a_n = \frac{1}{4} \left[+ \frac{(-1)^n}{n^2} - \frac{1}{n^2} \right] = \frac{(-1)^n - 1}{4n^2} = \begin{cases} 0 & \text{when 'n' is even} \\ -\frac{2}{4n^2} & \text{when 'n' is odd} \end{cases}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 0 \cdot \sin nx \, dx + \int_0^{\pi} \frac{\pi x}{4} \sin nx \, dx \right]$$

$$= \frac{1}{\pi} \cdot \frac{\pi}{4} \int_0^{\pi} x \sin nx \, dx$$

$$= \frac{1}{4} \left[x \left(\frac{-\cos nx}{n} \right) - \left(\frac{-\sin nx}{n^2} \right) \right]_0^{\pi}$$

$$= \frac{1}{4} \left[-\frac{x \cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^{\pi}$$

$$= \frac{1}{4} \left[\left(-\frac{\pi \cos n\pi}{n} + \frac{\sin n\pi}{n^2} \right) - (0+0) \right]$$

$$= \frac{-\pi (-1)^n}{4n}$$

Substituting the values of a_0 , a_n and b_n in (1) we get.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{\pi^2}{16} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{4n^2} \cos nx + \sum_{n=1}^{\infty} \frac{(-\pi)(-1)^n}{4n} \sin nx$$

$$f(x) = \frac{\pi^2}{16} - \frac{1}{2} \left(\cos x + \frac{1}{3^2} \cos 3x + \frac{1}{5^2} \cos 5x + \dots \right) + \frac{\pi}{4} \left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \right) \quad \text{--- (2)}$$

Deduction:- At $x=0$, $f(x)$ is continuous. Hence the Fourier series converges to $f(0)$ at $x=0$,

Putting $x=0$ in (2), we obtain

$$f(0) = 0 = \frac{\pi^2}{16} - \frac{1}{2} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right)$$

$$\frac{\pi^2}{16} = \frac{1}{2} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right)$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Pb Given $f(x) = 1 + \frac{2x}{\pi}$ when $-\pi \leq x \leq 0$.

$1 - \frac{2x}{\pi}$ when $0 \leq x \leq \pi$.

Show that $f(x) = \frac{8}{\pi^2} \left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right)$

Deduce from this that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$.

Sol:-

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$. — (1)

Then $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \left[\int_{-\pi}^0 f(x) dx + \int_0^{\pi} f(x) dx \right]$

$= \frac{1}{\pi} \left[\int_{-\pi}^0 \left(1 + \frac{2x}{\pi}\right) dx + \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) dx \right]$

$= \frac{1}{\pi} \left[\left(x + \frac{x^2}{\pi}\right)_{-\pi}^0 + \left(x - \frac{x^2}{\pi}\right)_0^{\pi} \right]$

$= \frac{1}{\pi} \left[\left[(0+0) - \left(-\pi + \frac{\pi^2}{\pi}\right) \right] + \left[\pi - \frac{\pi^2}{\pi} \right] - 0 \right] = 0$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$

$= \frac{1}{\pi} \left[\int_{-\pi}^0 f(x) \cos nx dx + \int_0^{\pi} f(x) \cos nx dx \right]$

$= \frac{1}{\pi} \left[\int_{-\pi}^0 \left(1 + \frac{2x}{\pi}\right) \cos nx dx + \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) \cos nx dx \right]$

$= \frac{1}{\pi} \left[\left(\frac{\sin nx}{n} \right)_{-\pi}^0 + \frac{2}{\pi} \left[x \left(\frac{\sin nx}{n} \right) - 1 \left(\frac{-\cos nx}{n^2} \right) \right]_{-\pi}^0 \right]$
 $+ \left[\left(\frac{\sin nx}{n} \right)_0^{\pi} - \frac{2}{\pi} \left(x \left(\frac{\sin nx}{n} \right) - 1 \left(\frac{-\cos nx}{n^2} \right) \right)_0^{\pi} \right]$

$= \frac{1}{\pi} \left[\left[0 + \frac{2}{\pi} \left(\frac{1}{n^2} - \frac{\cos n\pi}{n^2} \right) \right] + \left[0 - \frac{2}{\pi} \left(\frac{\cos n\pi}{n^2} - \frac{1}{n^2} \right) \right] \right]$

$= \frac{2}{\pi^2} \left[\frac{1}{n^2} - \frac{\cos n\pi}{n^2} - \frac{\cos n\pi}{n^2} + \frac{1}{n^2} \right]$

$$= \frac{2}{\pi^2} \left[\frac{2}{n^2} - \frac{2 \cos n\pi}{n^2} \right]$$

$$a_n = \frac{4}{\pi^2 n^2} [1 - \cos n\pi] = \frac{4 [1 - (-1)^n]}{\pi^2 n^2}$$

$$a_n = \begin{cases} 0, & \text{when 'n' is even} \\ \frac{8}{n^2 \pi^2} & \text{when 'n' is odd.} \end{cases}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 \left(1 + \frac{2x}{\pi}\right) \sin nx \, dx + \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) \sin nx \, dx \right]$$

$$= \frac{1}{\pi} \left[\left(-\frac{\cos nx}{n} \right)_{-\pi}^0 + \frac{2}{\pi} \left[x \left(-\frac{\cos nx}{n} \right) + 1 \left(-\frac{\sin nx}{n^2} \right) \right]_{-\pi}^0 \right]$$

$$+ \left[\left(-\frac{\cos nx}{n} \right)_{\pi}^0 - \frac{2}{\pi} \left[x \left(-\frac{\cos nx}{n} \right) - 1 \left(-\frac{\sin nx}{n^2} \right) \right]_{\pi}^0 \right]$$

$$= \frac{1}{\pi} \left[\left(-\frac{1}{n} + \frac{(-1)^n}{n} \right) + \frac{2}{\pi} \left[\pi \frac{\cos n\pi}{n} \right] \right]$$

$$+ \left(-\frac{(-1)^n}{n} + \frac{1}{n} \right) - \frac{2}{\pi} \left[\pi \frac{(-1)^n}{n} \right]$$

$$= \frac{1}{\pi} \left[-\frac{1}{n} + \frac{(-1)^n}{n} + \frac{2(-1)^n}{n} - \frac{(-1)^n}{n} + \frac{1}{n} - \frac{2(-1)^n}{n} \right]$$

$$= \frac{1}{\pi} (0) = 0$$

Substituting the values of a_0 , a_n , b_n in (1) we get

$$f(x) = \frac{8}{\pi^2} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right] \quad \text{--- (2)}$$

Deduction:- Putting $x=0$ in (2) we get

$$f(0) = \frac{8}{\pi^2} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

$$1 = \frac{8}{\pi^2} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

ProbFind the Fourier series in $[-\pi, \pi]$ for the function

$$f(x) = -\frac{1}{2}(\pi+x) \text{ for } -\pi \leq x < 0.$$

$$= \frac{1}{2}(\pi-x) \text{ for } 0 < x \leq \pi.$$

Sol: - Let the function $f(x)$ be represented by the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{--- (1)}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \left[\int_{-\pi}^0 f(x) dx + \int_0^{\pi} f(x) dx \right]$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 -\frac{1}{2}(\pi+x) dx + \int_0^{\pi} \frac{1}{2}(\pi-x) dx \right]$$

$$= \frac{1}{2\pi} \left[-\left(\pi x + \frac{x^2}{2}\right)_{-\pi}^0 + \left(\pi x - \frac{x^2}{2}\right)_0^{\pi} \right]$$

$$= \frac{1}{2\pi} \left[-\pi^2 + \frac{\pi^2}{2} + \frac{\pi^2}{2} - \frac{\pi^2}{2} \right]$$

$$= 0.$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 -\frac{1}{2}(\pi+x) \cos nx dx + \int_0^{\pi} \frac{1}{2}(\pi-x) \cos nx dx \right]$$

$$= -\frac{1}{2\pi} \left[\pi \left(\frac{\sin nx}{n} \right)_{-\pi}^0 + \left[x \left(\frac{\sin nx}{n} \right) - 1 \left(\frac{-\cos nx}{n^2} \right) \right]_{-\pi}^0 \right]$$

$$+ \frac{1}{2\pi} \left[\pi \left(\frac{\sin nx}{n} \right)_0^{\pi} - \left[x \left(\frac{\sin nx}{n} \right) - 1 \left(\frac{-\cos nx}{n^2} \right) \right]_0^{\pi} \right]$$

$$= -\frac{1}{2\pi} \left[\frac{1}{n^2} \right] + \frac{1}{2\pi} \left[\right]$$

$$= -\frac{1}{2\pi} \left[\frac{1}{n^2} - \frac{\cos n\pi}{n^2} \right] + \frac{1}{2\pi} \left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2} \right]$$

$$= -\frac{1}{2\pi n^2} + \frac{\cos n\pi}{2\pi n^2} - \frac{\cos n\pi}{2\pi n^2} + \frac{1}{2\pi n^2}$$

$$= 0.$$

Finally $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$.

$$b_n = \frac{1}{\pi} \left[\int_{-\pi}^0 -\frac{1}{2}(\pi+x) \sin nx \, dx + \int_0^{\pi} \frac{1}{2}(\pi-x) \sin nx \, dx \right]$$

$$= -\frac{1}{2\pi} \left[\int_{-\pi}^0 \pi \sin nx \, dx + \int_{-\pi}^0 x \sin nx \, dx \right]$$

$$+ \frac{1}{2\pi} \left[\int_0^{\pi} \pi \sin nx \, dx - \int_0^{\pi} x \sin nx \, dx \right]$$

$$= -\frac{1}{2\pi} \left[\pi \left(-\frac{\cos nx}{n} \right)_{-\pi}^0 + \left[x \left(-\frac{\cos nx}{n} \right) - 1 \left(-\frac{\sin nx}{n^2} \right) \right]_{-\pi}^0 \right]$$

$$+ \frac{1}{2\pi} \left[\pi \left(-\frac{\cos nx}{n} \right)_0^{\pi} - \left[x \left(-\frac{\cos nx}{n} \right) - 1 \left(-\frac{\sin nx}{n^2} \right) \right]_0^{\pi} \right]$$

$$= -\frac{1}{2\pi} \left[\left(-\frac{\pi}{n} + \frac{(-1)^n}{n} \right) + \left[-\frac{\pi x}{n} - \frac{(-1)^n}{n} \right] \right] \times$$

$$+ \frac{1}{2\pi} \left[-\frac{\pi(-1)^n}{n} + \frac{\pi}{n} - \left[-\frac{\pi(-1)^n}{n} \right] \right] \times$$

$$= -\frac{1}{2\pi} \left[-\frac{\pi}{n} \right] + \frac{1}{2\pi} \left[\frac{(-1)^n}{n} + \frac{\pi(-1)^n}{n} \right] \times$$

$$- \frac{\pi}{2\pi} \left[\frac{(-1)^n}{n} + \frac{1}{2n} + \frac{\pi}{2\pi} \frac{(-1)^n}{n} \right] \times$$

$$b_n = -\frac{1}{n} [(-1)^n - 1]$$

$$b_n = 0 \text{ if 'n' is even}$$

$$= \frac{2}{n} \text{ if 'n' is odd.}$$

Substituting the values of a_0 , a_n and b_n in (1) we get

$$f(x) = \sum_{n=1,3,5}^{\infty} \frac{2}{n} \sin nx$$

$$= 2 \left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right)$$

Fourier Series for Even function :- Let $f(x)$ be a function defined in the interval $(-\pi, \pi)$ then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = 0$$

$f(x) \sin nx$ is odd.

Fourier series for odd function :- Let $f(x)$ be a function defined in the interval $(-\pi, \pi)$ then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = 0$$

$f(x) \cos nx$ is odd.

Pb :- Express $f(x) = x$ as Fourier series in $(-\pi, \pi)$

Sol :- Since $f(-x) = -x = -f(x)$

$\therefore f(x)$ is an odd function in $(-\pi, \pi)$.

Hence in its Fourier series expansion, the cosine terms are absent and only sine terms are present.

$$x = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

where $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$

$$b_n = \frac{2}{\pi} \left[\int_0^{\pi} x \sin nx dx \right]$$

$$= \frac{2}{\pi} \left[x \left(\frac{-\cos nx}{n} \right) - 1 \left(\frac{-\sin nx}{n^2} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\pi \left(\frac{-\cos n\pi}{n} \right) - 0 \right] = -\frac{2}{n} (-1)^n = \frac{2}{n} (-1)^{n+1}$$

Substituting the values of b_n in (1) we get.

$$x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

$$= 2 \left(\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \dots \right)$$

Q. Expand the function $f(x) = x^2$ as a Fourier series in $(-\pi, \pi)$

Proof $f(-x) = (-x)^2 = x^2 = f(x)$

$f(x)$ is even function. in $(-\pi, \pi)$.

Hence in its Fourier series expansion, the sine terms are absent

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{--- (1)}$$

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} x^2 dx \\ &= \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi} = \frac{2}{\pi} \left[\frac{\pi^3}{3} \right] = \frac{2\pi^2}{3} \quad \text{--- (2)} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx \\ &= \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx \end{aligned}$$

$$= \frac{2}{\pi} \left[x^2 \left(\frac{\sin nx}{n} \right) - 2x \left(\frac{-\cos nx}{n^2} \right) + 2 \left(\frac{-\sin nx}{n^3} \right) \right]_0^{\pi}$$

$$\begin{aligned} &= \frac{2}{\pi} \left[\pi^2 \left(\frac{\sin n\pi}{n} \right) - 2\pi \left(\frac{-\cos n\pi}{n^2} \right) + 2 \left(\frac{-\sin n\pi}{n^3} \right) \right] \\ &\quad - [0 - 0 + 0] \end{aligned}$$

$$\begin{aligned} &= \frac{2}{\pi} \left[\frac{-2\pi(-1)^n}{n^2} \right] = \frac{2}{\pi} \left[\frac{2\pi(-1)^n}{n^2} \right] \\ &= \frac{4}{n^2} (-1)^n \quad \text{--- (3)} \end{aligned}$$

Substituting the of a_0 , and a_n from (2) and (3) in (1), we get

$$x^2 = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos nx$$

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}$$

$$x^2 = \frac{\pi^2}{3} + 4 \left[-\cos x + \frac{\cos 2x}{4} - \frac{\cos 3x}{9} + \dots \right] \quad \text{--- (4)}$$

Deductions :

(i) Putting $x=0$ in above equation.

From (4).

$$0 = \frac{\pi^2}{3} + 4 \left(-1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \dots \right)$$

$$-\frac{\pi^2}{3} = -4 \left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \right)$$

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \quad \text{--- (A)}$$

(ii) Putting $x=\pi$ in (4), we have.

$$\pi^2 = \frac{\pi^2}{3} + 4 \left(-\cos\pi + \frac{\cos 2\pi}{2^2} - \frac{\cos 3\pi}{3^2} + \dots \right)$$

$$\pi^2 = \frac{\pi^2}{3} + 4 \left(-(-1) + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right)$$

$$\pi^2 - \frac{\pi^2}{3} = 4 \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right)$$

$$\frac{2\pi^2}{3} = 4 \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right)$$

$$\frac{2\pi^2}{12 \times 6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots \quad \text{--- (B)}$$

Adding (A) and (B) and dividing by (2), we get

$$\frac{\pi^2}{12} + \frac{\pi^2}{6} = \left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \right) + \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right)$$

$$\frac{3\pi^2}{12 \times 4} = \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \frac{2}{7^2} + \dots$$

$$\text{i.e.} \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

pb: obtain the Fourier series for the function

$$f(x) = |x| \text{ in } -\pi < x < \pi.$$

$$\text{and deduce that } \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}.$$

Sol. Since $f(-x) = |-x| = x = |x| = f(x)$.

$f(x) = |x|$ is an even function.

Hence the Fourier series will consist of cosine terms only.

$$f(x) = |x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx. \quad \text{--- (1)}$$

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} |x| dx = \frac{2}{\pi} \int_0^{\pi} x dx \\ &= \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi} = \frac{\pi^2}{\pi} = \pi. \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx \\ &= \frac{2}{\pi} \int_0^{\pi} |x| \cos nx dx = \frac{2}{\pi} \int_0^{\pi} x \cos nx dx. \end{aligned}$$

$$\begin{aligned} &= \frac{2}{\pi} \left[x \left(\frac{\sin nx}{n} \right) - 1 \left(-\frac{\cos nx}{n^2} \right) \right]_0^{\pi} \\ &= \frac{2}{\pi} \left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2} \right] \\ &= \frac{2}{\pi} \left[\frac{(-1)^n - 1}{n^2} \right] = \frac{2}{\pi n^2} [(-1)^n - 1]. \end{aligned}$$

$$\begin{aligned} a_n &= 0 \text{ if } n \text{ is even} \\ &= -\frac{4}{\pi n^2} \text{ if } n \text{ is odd.} \end{aligned}$$

From (1).

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{\cos nx}{n^2}$$

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right]$$

Put $x=0$ in above eqn.

$$0 = \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Prob.

Find the Fourier series to represent the function

$$f(x) = x \sin x, \quad -\pi < x < \pi$$

Hence deduce that $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \frac{1}{7 \cdot 9} + \dots = \frac{1}{4}(\pi - 2)$

Proof

Since $f(x) = x \sin x$ is an even function $b_n = 0$.

$$x \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^{\pi} x \sin x \, dx = \frac{2}{\pi} \left[x(-\cos x) + 1(-\sin x) \right]_0^{\pi} \\ &= \frac{2}{\pi} \left[-x \cos x + \sin x \right]_0^{\pi} \\ &= \frac{2}{\pi} \left[-\pi \cos \pi \right] = 2 \quad (\cos \pi = -1) \end{aligned}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \sin x \cos nx \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x (2 \sin x \cos nx) \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x [\sin(1+n)x + \sin(1-n)x] \, dx$$

$$= \frac{1}{\pi} \left[x \left[\frac{-\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n} \right] - \left[\frac{-\sin(1+n)x}{(1+n)^2} - \frac{\sin(1-n)x}{1-n} \right] \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[-x \left[\frac{\cos(1+n)x}{1+n} + \frac{\cos(1-n)x}{1-n} \right] \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[-\pi \left[\frac{\cos(1+n)\pi}{1+n} + \frac{\cos(1-n)\pi}{1-n} \right] \right] \quad (n \neq 1)$$

$$= - \left[\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} \right]$$

$$= - \left[\frac{(-1)^{n+1}}{1+n} + \frac{(-1)^{1-n}}{1-n} \right] = - \left[\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}(-1)^2}{n-1} \right]$$

$$= - \left[\frac{(-1)^{1+n} + (-1)^{1-n}(-1)^2}{1-n^2} \right] = - \left[\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n+1}}{n-1} \right]$$

$$= - \left[\frac{(-1)^{1+n} + (-1)^{1+n}}{n^2-1} \right] = (-1)(-1)^{n+1} \left[\frac{n-1-n-1}{n^2-1} \right]$$

$$= \frac{2(-1)^{n+1}}{n^2-1}, \quad n \neq 1$$

Put $n=1$ in $a_n = \frac{2}{\pi} \int_0^{\pi} x \sin x \cos nx \, dx$.

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \sin x \cos nx \, dx$$

$$2 \sin A \cos A = \sin 2A$$

$$= \frac{2}{\pi} \int_0^{\pi} x \sin 2x \cos nx \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x \sin 2x \, dx$$

$$= \frac{1}{\pi} \left[x \left(-\frac{\cos 2x}{2} \right) - \left(-\frac{\sin 2x}{4} \right) \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[-\frac{\pi \cos 2\pi}{2} \right] = -\frac{1}{2} \quad [\cos 2\pi = 1]$$

$$x \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{2}{2} - \frac{1}{2} \cos x + \sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{n^2-1} \cos nx$$

$$= 1 - \frac{1}{2} \cos x + \sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{(n^2-1)(n+1)} \cos nx$$

$$= 1 - \frac{1}{2} \cos x + \left[-\frac{2}{1 \cdot 3} \cos 2x + \frac{2}{2 \cdot 4} \cos 3x - \frac{2}{3 \cdot 5} \cos 4x + \dots \right]$$

Putting $x = \pi/2$ in above equation.

$$\frac{\pi}{2} \sin \frac{\pi}{2} = 1 - \frac{1}{2} \cos \frac{\pi}{2} + \left[-\frac{2}{1 \cdot 3} \cos \left(2 \frac{\pi}{2} \right) + \frac{2}{2 \cdot 4} \cos 3 \frac{\pi}{2} - \frac{2}{3 \cdot 5} \cos 4 \left(\frac{\pi}{2} \right) + \dots \right]$$

$$\frac{\pi}{2} = 1 + \left[-\frac{2}{1 \cdot 3} (-1) + \frac{2}{2 \cdot 4} (0) - \frac{2}{3 \cdot 5} (1) + \dots \right]$$

$$\frac{\pi}{2} = 1 + \frac{2}{1 \cdot 3} - \frac{2}{3 \cdot 5} + \frac{2}{5 \cdot 7} - \frac{2}{7 \cdot 9} + \dots$$

$$\frac{\pi}{2} - 1 = \frac{2}{1 \cdot 3} - \frac{2}{3 \cdot 5} + \frac{2}{5 \cdot 7} - \frac{2}{7 \cdot 9} + \dots$$

$$\frac{\pi-2}{2} = \frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \frac{1}{7 \cdot 9} + \dots$$

Pb: - obtain Fourier series for the function $f(x)$ given by.

$$f(x) = -\frac{1}{2}(\pi+x) \text{ for } -\pi < x < 0 \\ \frac{1}{2}(\pi-x) \text{ for } 0 < x < \pi$$

Sol: - since $f(-x) = -\frac{1}{2}(\pi-x)$ in $(-\pi, 0)$
 $= -f(x)$ in $(0, \pi)$

$$f(-x) = -f(x)$$

$f(x)$ is an odd function in $(-\pi, \pi)$.

Hence $a_0 = a_n = 0$.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \frac{1}{2}(\pi-x) \sin nx \, dx = \frac{1}{\pi} \int_0^{\pi} (\pi-x) \sin nx \, dx$$

$$= \frac{1}{\pi} \left[(\pi-x) \left(-\frac{\cos nx}{n} \right) - (-1) \left(-\frac{\sin nx}{n^2} \right) \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[(\pi-x) \left(-\frac{\cos nx}{n} \right) - \frac{\sin nx}{n^2} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[(\pi-0) \left(\frac{1}{n} \right) \right] = \frac{1}{n}$$

Substituting the values of b_n in (1), we get.

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx$$

$$f(x) = \sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots$$

Pb

$$\text{Is } f(x) = \frac{1+2x}{\pi}, \quad -\pi < x < 0 \\ = 1 - \frac{2x}{\pi}, \quad 0 < x < \pi \quad \text{even?}$$

(b) If so, find the Fourier series for the function.

$$(c) \text{ Deduce that } \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}$$

(a)

$$f(-x) = 1 + \frac{2x}{\pi} \sin(0, \pi)$$

$$= f(x) \sin(-\pi, 0)$$

$\therefore f(x)$ is even function in $(-\pi, \pi)$.

(b) Since $f(x)$ is even function in $(-\pi, \pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{--- (1)}$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) dx = \frac{2}{\pi} \left(x - \frac{x^2}{\pi}\right)_0^{\pi}$$

$$= 0$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) \cos nx dx$$

$$= \frac{2}{\pi} \left[\left(1 - \frac{2x}{\pi}\right) \frac{\sin nx}{n} - \left(-\frac{2}{\pi}\right) \left(-\frac{\cos nx}{n^2}\right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[-\frac{2}{\pi} \frac{\cos nx}{n^2} \right]_0^{\pi}$$

$$= \frac{-4}{\pi^2} \left[\frac{\cos nx}{n^2} \right]_0^{\pi} = \frac{-4}{\pi^2} \left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2} \right]$$

$$= \frac{-4}{n^2 \pi^2} \left[(-1)^n - 1 \right]$$

$$= \frac{4}{n^2 \pi^2} \left[1 - (-1)^n \right]$$

$$a_n = \frac{8}{n^2 \pi^2} \text{ if } n \text{ is odd.}$$

$$= 0 \text{ if } n \text{ is even.}$$

$$a_1 = \frac{8}{\pi^2}, a_3 = \frac{8}{9\pi^2}, a_5 = \frac{8}{25\pi^2}, \dots$$

$$a_2 = a_4 = a_6 = \dots = 0$$

Thus substituting the values of a 's in (1), we get.

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin nx \\ &= \left[\frac{8}{\pi^2} \cos x + \frac{8 \cos 3x}{3^2 \pi^2} + \frac{8 \cos 5x}{5^2 \pi^2} + \dots \right] \\ &= \frac{8}{\pi^2} \left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right] \quad \text{--- (2)} \end{aligned}$$

as the required Fourier series.

(c) Putting $x=0$ in (2), we get.

$$f(0) = 1 = \frac{8}{\pi^2} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

$$2$$

Half-Range Fourier Series

The Half-Range sine series in $(0, \pi)$ is given by.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\text{where } b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

The Half-Range cosine series in $(0, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\text{where } a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) \, dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$

Pb. Find ^{Half-Range} cosine and sine series for $f(x) = (\pi - x)$ in $(0, \pi)$

Sol. The Half-Range cosine series of $f(x) = \pi - x$ in $(0, \pi)$ is given by.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx. \quad \text{--- (1)}$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} (\pi - x) dx.$$

$$= \frac{2}{\pi} \left[\pi x - \frac{x^2}{2} \right]_0^{\pi} = \frac{2}{\pi} \left[\pi^2 - \frac{\pi^2}{2} \right]$$

$$= \frac{2}{\pi} \left[\frac{\pi^2}{2} \right] = \pi.$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi - x) \cos nx dx.$$

$$= \frac{2}{\pi} \left[(\pi - x) \frac{\sin nx}{n} - (-1) \frac{\cos nx}{n^2} \right]_0^{\pi}.$$

$$= \frac{2}{\pi} \left[(\pi - x) \frac{\sin nx}{n} - \frac{\cos nx}{n^2} \right]_0^{\pi}.$$

$$= \frac{2}{\pi} \left[-\frac{\cos n\pi}{n^2} + \frac{1}{n^2} \right].$$

$$= \frac{2}{\pi} \left[-\frac{(-1)^n}{n^2} + \frac{1}{n^2} \right].$$

$$= \frac{2}{\pi n^2} \left[1 - (-1)^n \right]$$

$$a_n = 0 \text{ if } n \text{ is even.}$$

$$= \frac{4}{\pi n^2} \text{ if } n \text{ is odd.}$$

Substituting the values of a_0 and a_n in (1) we get

$$\begin{aligned}\pi - x &= \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1,3,5}^{\infty} \frac{1}{n^2} \cos nx \\ &= \frac{\pi}{2} + \frac{4}{\pi} \left[\frac{1}{1^2} \cos x + \frac{1}{3^2} \cos 3x + \frac{1}{5^2} \cos 5x + \dots \right]\end{aligned}$$

Sine Series.

The Half-Range sine series of $f(x) = \pi - x$ in $[0, \pi]$ is given by.

$$f(x) = \pi - x = \sum_{n=1}^{\infty} b_n \sin nx.$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx.$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi - x) \sin nx \, dx.$$

$$= \frac{2}{\pi} \left[(\pi - x) \left(-\frac{\cos nx}{n} \right) - (-1) \left(-\frac{\sin nx}{n^2} \right) \right]_0^{\pi}.$$

$$= \frac{2}{\pi} \left[(\pi - x) \left(-\frac{\cos nx}{n} \right) - \frac{\sin nx}{n^2} \right]_0^{\pi}.$$

$$= \frac{2}{\pi} \left[-(\pi - 0) \left(-\frac{\cos 0}{n} \right) \right]$$

$$= \frac{2}{\pi} \frac{\pi}{n} = \frac{2}{n}.$$

$$\pi - x = 2 \sum_{n=1}^{\infty} \frac{1}{n} \sin nx.$$

$$\pi - x = 2 \left[\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots \right]$$

Pb

Express $f(x) = \sin x$ as Fourier cosine series in $(0, \pi)$. Hence show that $\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$

Sol:

The half-range Fourier cosine series is given by.

$$f(x) = \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} \sin x dx \\ &= \frac{2}{\pi} (-\cos x)_0^{\pi} = \frac{2}{\pi} (-\cos \pi + \cos 0) \\ &= \frac{2}{\pi} (-(-1) + 1) = \frac{4}{\pi}. \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx \\ &= \frac{2}{\pi} \int_0^{\pi} \sin x \cos nx dx \end{aligned} \quad \left| \begin{array}{l} 2 \sin A \cos B = \\ \sin(A+B) + \sin(A-B) \end{array} \right.$$

$$= \frac{1}{\pi} \int_0^{\pi} 2 \sin x \cos nx dx \quad \text{--- (1)}$$

$$= \frac{1}{\pi} \int_0^{\pi} [\sin(1+n)x + \sin(1-n)x] dx.$$

$$= \frac{1}{\pi} \left[\frac{-\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n} \right]_0^{\pi}.$$

$$= -\frac{1}{\pi} \left[\frac{\cos(1+n)x}{1+n} + \frac{\cos(1-n)x}{1-n} \right]_0^{\pi}.$$

$$= -\frac{1}{\pi} \left[\left(\frac{\cos(1+n)\pi}{1+n} + \frac{\cos(1-n)\pi}{1-n} \right) - \left(\frac{1}{1+n} + \frac{1}{1-n} \right) \right] \quad n \neq 1.$$

$$= -\frac{1}{\pi} \left[\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} \right] - \left(\frac{1}{1+n} + \frac{1}{1-n} \right)$$

$$= -\frac{1}{\pi} \left[\left(\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}}{n-1} \right) - \left(\frac{1}{1+n} + \frac{1}{1-n} \right) \right]$$

$$= -\frac{1}{\pi} \left[\left(\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}(-1)^2}{n-1} \right) - \left(\frac{1}{n+1} - \frac{1}{n-1} \right) \right]$$

$$= -\frac{1}{\pi} \left[\left(\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n+1}}{n-1} \right) - \left(\frac{n-1 - n-1}{n^2-1} \right) \right]$$

$$= -\frac{1}{\pi} \left[(-1)^{n+1} \left[\frac{1}{n+1} - \frac{1}{n-1} \right] + \frac{2}{n^2-1} \right]$$

$$= -\frac{1}{\pi} \left[(-1)^{n+1} \left[\frac{n-1 - n-1}{n^2-1} \right] + \frac{2}{n^2-1} \right]$$

$$= -\frac{1}{\pi} \left[(-1)^{n+1} \frac{(-2)}{n^2-1} + \frac{2}{n^2-1} \right]$$

$$= \frac{2}{\pi(n^2-1)} \left[\frac{1 - (-1)^{n+1}}{1} \right]$$

$$a_n = \frac{2}{\pi(n^2-1)} \left[(-1)^{n+1} - 1 \right]$$

$$a_n = 0 \text{ if } n \text{ is odd.}$$

$$= -\frac{4}{\pi(n^2-1)} \text{ if } n \text{ is even.}$$

Put $n=1$ in ① we get.

$$a_1 = \frac{1}{\pi} \int_0^{\pi} 2 \sin x \cos x \, dx.$$

$$a_1 = \frac{1}{\pi} \int_0^{\pi} \sin 2x \, dx = \frac{1}{\pi} \left(-\frac{\cos 2x}{2} \right)_0^{\pi}$$

$$= \frac{1}{\pi} \left(\frac{-\cos 2\pi}{2} + \frac{\cos 0}{2} \right) = \frac{1}{2\pi} [\cos 2\pi - \cos 0]$$

$$= \frac{1}{2\pi} [1 - 1] = 0.$$

$$a_1 = a_3 = a_5 = \dots = 0.$$

$$a_2 = \frac{-4}{\pi(4-1)} = -\frac{4}{3\pi}, \quad a_4 = \frac{-4}{\pi(16-1)} = -\frac{4}{15\pi}$$

Thus the half range cosine series is

$$\sin x = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=2,4,6}^{\infty} \frac{\cos nx}{(n^2-1)(n+1)}$$

$$\sin x = \frac{2}{\pi} - \frac{4}{\pi} \left(\frac{\cos 2x}{1 \cdot 3} + \frac{\cos 4x}{3 \cdot 5} + \frac{\cos 6x}{5 \cdot 7} \right)$$

$$= \frac{4}{\pi} \left[\frac{1}{2} - \frac{\cos 2x}{1 \cdot 3} - \frac{\cos 4x}{3 \cdot 5} - \frac{\cos 6x}{5 \cdot 7} - \dots \right]$$

$$\sin x = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2-1} \quad \text{--- (2)}$$

Deduction $x=0$ in (2) we obtain

$$0 = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2-1}$$

$$\frac{2}{\pi} = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2-1}$$

$$\frac{2\pi}{4\pi} = \sum_{n=1}^{\infty} \frac{1}{4n^2-1}$$

$$\frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{4n^2-1}$$

Prob

Obtain the half-range series for

$$e^x \text{ in } (0, \pi)$$

Sol

The half range sine series expansion of

e^x in $(0, \pi)$ is given by.

$$e^x = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} e^x \sin nx \, dx$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} [a \sin bx - b \cos bx]$$

$$\begin{aligned}
 b_n &= \frac{2}{\pi} \left[\frac{e^x}{1+n^2} (\sin n\pi - n \cos n\pi) \right]_0^\pi \\
 &= \frac{2}{\pi} \left[\left[\frac{e^\pi}{1+n^2} (0 - n \cos n\pi) \right] - \frac{1}{1+n^2} (0 - n) \right] \\
 &= \frac{2}{\pi} \left[\frac{e^\pi}{1+n^2} - n(-1)^n + \frac{n}{1+n^2} \right] \\
 &= \frac{2n}{\pi(1+n^2)} [1 + (-1)^{n+1} e^\pi].
 \end{aligned}$$

Thus the half-range sine series for e^x is

$$\begin{aligned}
 e^x &= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{n [1 + (-1)^{n+1} e^\pi]}{1+n^2} \sin nx \\
 &= \frac{2}{\pi} \left[\frac{1+e^\pi}{1+1^2} \sin x + \frac{2(1-e^\pi)}{1+2^2} \sin 2x \right. \\
 &\quad \left. + \frac{3(1+e^\pi)}{1+3^2} \sin 3x + \dots \right].
 \end{aligned}$$

2.

Pb If $f(x) = 1$ in $0 < x < \frac{\pi}{2}$ expand $f(x)$ in series
 -1 in $\frac{\pi}{2} < x < \pi$ of cosines

Sol.

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

$$a_0 = \frac{2}{\pi} \int_0^\pi f(x) dx = \frac{2}{\pi} \left[\int_0^{\pi/2} f(x) dx + \int_{\pi/2}^\pi f(x) dx \right]$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} 1 dx + \int_{\pi/2}^\pi -1 dx \right]$$

$$= \frac{2}{\pi} \left[(x)_0^{\pi/2} - (x)_{\pi/2}^\pi \right]$$

$$= \frac{2}{\pi} \left[\frac{\pi}{2} - 0 - \pi + \frac{\pi}{2} \right] = 0.$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} 1 \cdot \cos nx \, dx + \int_{\pi/2}^{\pi} (-1) \cos nx \, dx \right]$$

$$= \frac{2}{\pi} \left[\left(\frac{\sin nx}{n} \right)_0^{\pi/2} - \left(\frac{\sin nx}{n} \right)_{\pi/2}^{\pi} \right]$$

$$= \frac{2}{\pi n} \left[(\sin nx)_0^{\pi/2} - (\sin nx)_{\pi/2}^{\pi} \right]$$

$$= \frac{2}{\pi n} \left[\sin \frac{n\pi}{2} - 0 - \sin n\pi + \sin \frac{n\pi}{2} \right]$$

$$= \frac{2}{\pi n} 2 \sin \frac{n\pi}{2}$$

$$= \frac{4}{\pi n} \sin \frac{n\pi}{2}$$

$$a_n = 0 \text{ when 'n' is even}$$

$$a_2 = a_4 = a_6 = \dots = 0$$

$$f(x) = \sum_{n=1,3,5,\dots}^{\infty} \frac{4}{\pi n} \sin \frac{n\pi}{2} \cos nx$$

$$= \frac{4}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{\sin \frac{n\pi}{2}}{n} \cos nx$$

$$= \frac{4}{\pi} \left[\sin \frac{\pi}{2} \cos x + \frac{1}{3} \sin \frac{3\pi}{2} \cos 3x + \frac{1}{5} \sin \frac{5\pi}{2} \cos 5x + \dots \right]$$

$$= \frac{4}{\pi} \left[\cos x - \frac{1}{3} \cos 3x + \frac{1}{5} \cos 5x - \dots \right]$$

Pb.

Represent the following function by Fourier sine series

$$f(x) = \frac{\pi}{2} \text{ for } 0 < x < \frac{\pi}{2}$$

$$= \pi - x \text{ for } \frac{\pi}{2} < x < \pi$$

Sol.

$$\text{Let } f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} f(x) \sin nx \, dx + \int_{\pi/2}^{\pi} f(x) \sin nx \, dx \right]$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} \left(\frac{\pi}{2}\right) \sin nx \, dx + \int_{\pi/2}^{\pi} (\pi-x) \sin nx \, dx \right]$$

$$= \frac{2}{\pi} \left[\left(\frac{\pi}{2}\right) \left(-\frac{\cos nx}{n}\right) \Big|_0^{\pi/2} + \left[(\pi-x) \left(-\frac{\cos nx}{n}\right) - (-1) \left(-\frac{\sin nx}{n^2}\right) \right] \Big|_{\pi/2}^{\pi} \right]$$

$$= \frac{2}{\pi} \left[-\frac{\pi}{2n} (\cos \frac{n\pi}{2} - 1) + \left[\frac{\pi}{2} \left(-\frac{\cos n\pi}{n}\right) - \frac{1}{n^2} \sin \frac{n\pi}{2} \right] \right]$$

$$= \frac{2}{\pi} \left[-\frac{\pi}{2n} (0-1) - \left[\frac{\pi}{2} (-0) - \frac{1}{n^2} \sin \frac{n\pi}{2} \right] \right] \frac{1}{\pi} =$$

$$= \frac{2}{\pi} \left[\frac{\pi}{2n} + \frac{1}{n^2} \sin \frac{n\pi}{2} \right]$$

$$\text{i.e. } b_n = \frac{2}{\pi} \left[\frac{\pi}{2} + \frac{\sin \frac{n\pi}{2}}{n} \right]$$

$$= \frac{2}{\pi} \left(\frac{\pi}{2} + 1 \right) = 1 + \frac{2}{\pi}$$

$$b_2 = \frac{2}{\pi} \left[\frac{\pi}{4} + \frac{1}{4} \sin \pi \right] = \frac{2}{\pi} \left[\frac{\pi}{4} + 0 \right] = \frac{1}{2}$$

$$b_3 = \frac{2}{\pi} \left[\frac{\pi}{6} + \frac{1}{9} \sin \frac{3\pi}{2} \right] = \frac{2}{\pi} \left[\frac{\pi}{6} - \frac{1}{9} \right] = \frac{1}{3} - \frac{2}{9\pi}$$

$$b_4 = \frac{2}{\pi} \left[\frac{\pi}{8} + \frac{1}{16} \sin 2\pi \right] = \frac{2}{\pi} \left[\frac{\pi}{8} + 0 \right] = \frac{1}{4}$$

Substituting values of 'b's in (1), we get.

$$f(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + b_4 \sin 4x + \dots$$

$$f(x) = \left(1 + \frac{2}{\pi}\right) \sin x + \frac{1}{2} \sin 2x + \left(\frac{1}{3} - \frac{2}{9\pi}\right) \sin 3x + \frac{1}{4} \sin 4x + \dots$$

Pb Find the half-range cosine series for $f(x) = \cos \alpha x$,
 α is an integer in the range $0 < x < \pi$.

Sol:

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{--- (1)}$$

$$\begin{aligned} \text{Then } a_0 &= \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} \cos \alpha x dx \\ &= \frac{2}{\pi} \left[\frac{\sin \alpha x}{\alpha} \right]_0^{\pi} = \frac{2}{\pi \alpha} \sin \alpha \pi. \end{aligned}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx.$$

$$= \frac{2}{\pi} \int_0^{\pi} \cos \alpha x \cos nx dx$$

$$\begin{aligned} 2 \cos A \cos B \\ = \cos(A+B) + \cos(A-B) \end{aligned}$$

$$= \frac{1}{\pi} \int_0^{\pi} [\cos(\alpha x + nx) + \cos(\alpha x - nx)] dx.$$

$$= \frac{1}{\pi} \int_0^{\pi} [\cos(\alpha + n)x + \cos(\alpha - n)x] dx.$$

$$= \frac{1}{\pi} \left[\frac{\sin(\alpha + n)x}{\alpha + n} + \frac{\sin(\alpha - n)x}{\alpha - n} \right]_0^{\pi}.$$

$$= \frac{1}{\pi} \left[\frac{\sin(\alpha + n)\pi}{\alpha + n} + \frac{\sin(\alpha - n)\pi}{\alpha - n} \right]$$

$$= \frac{1}{\pi} \left[\frac{\sin(\alpha\pi + n\pi)}{\alpha + n} + \frac{\sin(\alpha\pi - n\pi)}{\alpha - n} \right] \quad \sin n\pi = 0.$$

$$= \frac{1}{\pi} \left[\frac{1}{\alpha + n} \sin \alpha\pi \cos n\pi + \frac{1}{\alpha - n} \sin \alpha\pi \cos n\pi \right]$$

$$= \frac{\sin \alpha\pi \cos n\pi}{\pi} \left(\frac{1}{\alpha + n} + \frac{1}{\alpha - n} \right).$$

$$= \frac{2\alpha \sin \alpha\pi (-1)^n}{\pi(\alpha^2 - n^2)}.$$

Substituting the values of a_0 and a_n in (1), we get

$$f(x) = \frac{1}{\pi x} \sin \pi x + \frac{2x \sin \pi x}{\pi(n^2 - 1)} \sum_{n=1}^{\infty} \frac{(-1)^n}{x^2 - n^2} \cos n x.$$

Pb

Find the Fourier cosine series of

Expand $f(x) = \cos x$, $0 < x < \pi$ in half-range sine series

Sol

$$\text{Let } f(x) = \sum_{n=1}^{\infty} b_n \sin n x. \quad (1)$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin n x dx = \frac{2}{\pi} \int_0^{\pi} \cos x \sin n x dx.$$

$$= \frac{1}{\pi} \int_0^{\pi} 2 \sin n x \cos x dx.$$

$$= \frac{1}{\pi} \int_0^{\pi} [\sin(n+1)x + \sin(n-1)x] dx.$$

$$= \frac{1}{\pi} \left[-\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1} \right]_0^{\pi} \quad (n \neq 1)$$

$$= \frac{1}{\pi} \left[-\frac{\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} + \frac{1}{n-1} \right]$$

$$= \frac{1}{\pi} \left[-\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}}{n-1} + \frac{n+1+n-1}{n^2-1} \right]$$

$$= \frac{1}{\pi} (-1)^{n+1} \left[-\frac{1}{n+1} - \frac{1}{n-1} + \frac{n}{n^2-1} \right]$$

$$= \frac{1}{\pi} (-1)^{n+1} \left[\frac{-n+1-n-1}{n^2-1} + \frac{2n}{n^2-1} \right]$$

$$= \frac{1}{\pi} \left[\frac{(-1)^n}{n+1} + \frac{(-1)^n}{n-1} + \frac{2n}{n^2-1} \right]$$

$$= \frac{1}{\pi} \left[(-1)^n \left[\frac{1}{n+1} + \frac{1}{n-1} \right] + \frac{2n}{n^2-1} \right]$$

$$= \frac{1}{\pi} \left[\frac{(-1)^n 2n}{n^2-1} + \frac{2n}{n^2-1} \right] = \frac{2n}{\pi} \left[\frac{1+(-1)^n}{n^2-1} \right]$$

$(n \neq 1)$

$\therefore b_n = 0$ when 'n' is odd.

$$= \frac{4n}{\pi(n^2-1)} \text{ when 'n' is even.}$$

If $n=1$, then.

$$\begin{aligned} b_1 &= \frac{2}{\pi} \int_0^{\pi} \cos x \sin x \, dx = \frac{1}{\pi} \int_0^{\pi} \sin 2x \, dx \\ &= \frac{1}{\pi} \left[-\frac{\cos 2x}{2} \right]_0^{\pi} = -\frac{1}{2\pi} [\cos 2\pi - \cos 0] \\ &= -\frac{1}{2\pi} [1 - 1] = 0. \end{aligned}$$

$b_n = 0$ when 'n' is odd.

Substituting the values of b 's in (1) we get.

$$f(x) = \sum_{n=2,4,6,\dots}^{\infty} \frac{4n}{\pi(n^2-1)} \sin nx.$$

$$= \frac{4}{\pi} \sum_{n=2,4,6,\dots}^{\infty} \frac{n}{n^2-1} \sin nx.$$

$$\begin{aligned} \cos x &= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{2n}{4n^2-1} \sin 2nx \quad \left[\because 'n' \text{ is even} \right. \\ &\quad \left. \text{replace 'n' by '2n'} \right] \\ &= \frac{8}{\pi} \left[\frac{1}{3} \sin 2x + \frac{2}{15} \sin 4x + \frac{3}{35} \sin 6x + \dots \right] \end{aligned}$$

Change of IntervalIntervals other than $(-\pi, \pi)$ and $(0, 2\pi)$ Fourier Series for functions having period $2l$ Fourier expansion for $f(x)$ in the interval $c < x < c+2l$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$ Fourier series of $f(x)$ is defined on $[0, 2l]$

$$a_0 = \frac{1}{l} \int_0^{2l} f(x) dx$$

$$a_n = \frac{1}{l} \int_0^{2l} f(x) \cos \frac{n\pi x}{l} dx, \quad b_n = \frac{1}{l} \int_0^{2l} f(x) \sin \frac{n\pi x}{l} dx.$$

Fourier series of $f(x)$ is defined in $[-l, l]$

$$a_0 = \frac{1}{l} \int_{-l}^l f(x) dx.$$

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx, \quad b_n = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx.$$

Fourier series for even and odd functions $(-l, l)$ in the interval $(-l, l)$, then the Fourier cosineseries is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$

$$\text{where } a_0 = \frac{2}{l} \int_0^l f(x) dx \quad \text{and} \quad a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx.$$

 $f(x)$ is odd in $(-l, l)$, then the Fourier sine series

is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$\text{where } b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx.$$

Pb. Express $f(x) = x^2$ as a Fourier series in $(-l, l)$

Sol. Since $f(-x) = (-x)^2 = x^2 = f(x)$.

$f(x)$ is an even function.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \quad \text{--- (1)}$$

$$\text{Here } a_0 = \frac{2}{l} \int_0^l x^2 dx = \frac{2}{l} \left(\frac{x^3}{3} \right)_0^l = \frac{2}{l} \left(\frac{l^3}{3} \right) = \frac{2l^2}{3}.$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx = \frac{2}{l} \int_0^l x^2 \cos \frac{n\pi x}{l} dx$$

$$= \frac{2}{l} \left[x^2 \left(\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - 2x \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right) + 2 \left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}} \right) \right]_0^l$$

$$= \frac{2}{l} \left[2x \cdot \frac{\cos \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right]_0^l$$

$$a_n = \frac{2}{l} \left[2l \cdot \frac{\cos n\pi}{\frac{n^2\pi^2}{l^2}} \right] = \frac{4l^2 \cos n\pi}{n^2\pi^2} = \frac{(-1)^n 4l^2}{n^2\pi^2}.$$

(3) Substituting these values in (1) we get.

$$x^2 = \frac{l^2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n 4l^2}{n^2\pi^2} \cos \frac{n\pi x}{l}.$$

$$= \frac{l^2}{3} + \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{l}.$$

$$= \frac{l^2}{3} + \frac{4l^2}{3} \left[-\frac{1}{12} \cos \left(\frac{\pi x}{l} \right) + \frac{1}{2^2} \cos \frac{2\pi x}{l} - \frac{1}{3^2} \cos \frac{3\pi x}{l} + \dots \right]$$

$$= \frac{l^2}{3} - \frac{4l^2}{3} \left[\cos \left(\frac{\pi x}{l} \right) - \frac{1}{2^2} \cos \frac{2\pi x}{l} + \frac{1}{3^2} \cos \frac{3\pi x}{l} - \dots \right]$$

$\approx$

Pb

Find the Fourier series to represent $1-x^2$ in the interval $-1 \leq x \leq 1$.

Sol

$$1-x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \quad \text{Here } l=1$$

$$1-x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x \quad \text{--- (1)}$$

$$a_0 = \frac{2}{l} \int_0^l f(x) dx = \frac{2}{1} \int_0^1 (1-x^2) dx$$

$$= 2 \left[x - \frac{x^3}{3} \right]_0^1 = 2 \left[1 - \frac{1}{3} \right] = \frac{4}{3}$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx$$

$$= \frac{2}{1} \int_0^1 (1-x^2) \cos n\pi x dx \quad [l=1]$$

$$= 2 \left[(1-x^2) \frac{\sin n\pi x}{n\pi} - (-2x) \left(-\frac{\cos n\pi x}{n^2\pi^2} \right) + (-2) \left(\frac{-\sin n\pi x}{n^3\pi^3} \right) \right]_0^1$$

$$= 2 \left[-\frac{2 \cos n\pi}{n^2\pi^2} \right] = \frac{-4 \cos n\pi}{n^2\pi^2} = \frac{-4(-1)^n}{n^2\pi^2} = \frac{(-1)^{n+1} 4}{n^2\pi^2}$$

Substituting these values in (1) we get.

$$1-x^2 = \frac{2}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos n\pi x$$

$$= \frac{2}{3} + \frac{4}{\pi^2} \left[\cos \pi x - \frac{1}{2^2} \cos 2\pi x + \frac{1}{3^2} \cos 3\pi x - \dots \right]$$

Pb

Find the Fourier series with period '3' to represent

$$f(x) = x + x^2 \text{ in } (0, 3).$$

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right) \quad \text{--- (1)}$$

$$\text{Here } 2l = 3 \Rightarrow l = \frac{3}{2}$$

$$f(x) = x + x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2n\pi x}{3} + b_n \sin \frac{2n\pi x}{3} \right) \quad \text{--- (2)}$$

$$a_0 = \frac{1}{l} \int_0^{2l} f(x) dx = \frac{2}{3} \int_0^3 (x + x^2) dx = \frac{2}{3} \left(\frac{x^2}{2} + \frac{x^3}{3} \right)_0^3 = 9$$

$$a_n = \frac{1}{l} \int_0^{2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{3} \int_0^3 (x+x^2) \cos\left(\frac{2n\pi x}{3}\right) dx$$

$$= \frac{2}{3} \left[(x+x^2) \left[\frac{\sin \frac{2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)} \right] + (1+2x) \left[\frac{-\cos \frac{2n\pi x}{3}}{\frac{4n^2\pi^2}{9}} \right] - 2 \left[\frac{-\sin \frac{2n\pi x}{3}}{\frac{8n^3\pi^3}{27}} \right] \right]_0^3$$

$$= \frac{2}{3} \left[(1+2x) \frac{-\cos \frac{2n\pi x}{3}}{\frac{4n^2\pi^2}{9}} \right]_0^3$$

$$= \frac{2}{3} \left[-7 \frac{\cos 6n\pi}{\frac{4n^2\pi^2}{9}} + \frac{1}{\frac{4n^2\pi^2}{9}} \right]$$

$$\cos n\pi = (-1)^n$$

$$\cos 6n\pi = (-1)^{6n}$$

$$= \frac{6}{4n^2\pi^2} [-7+1] = \frac{6(6)}{4n^2\pi^2} = \frac{36}{4n^2\pi^2} = \frac{9}{n^2\pi^2}$$

$$b_n = \frac{1}{l} \int_0^{2l} f(x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{3} \int_0^3 (x+x^2) \sin\left(\frac{2n\pi x}{3}\right) dx$$

$$= \frac{2}{3} \left[(x+x^2) \left[\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{\frac{2n\pi}{3}} \right] - (1+2x) \left[\frac{-\sin\left(\frac{2n\pi x}{3}\right)}{\frac{4n^2\pi^2}{9}} \right] \right. \\ \left. + 2 \left[\frac{-\cos \frac{2n\pi x}{3}}{\frac{8n^3\pi^3}{27}} \right] \right]_0^3$$

$$= \frac{2}{3} \left[\frac{3}{2n\pi} (3+9) \left(-\cos \frac{6n\pi}{3} \right) \right] = \frac{-12}{n\pi}$$

$$+ \frac{2}{3} \left[-2 \left(\frac{\cos}{16n^3\pi^3} \right) + \frac{2 \times 81}{16n^3\pi^3} \right]$$

Substituting the values of a 's and b 's in (2), we get

$$x + x^2 = \frac{9}{2} + \sum_{n=1}^{\infty} \left(\frac{9}{n^2 \pi^2} \cos \frac{2n\pi x}{3} - \frac{12}{n\pi} \sin \frac{2n\pi x}{3} \right)$$

$$= \frac{9}{2} + \frac{9}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \frac{2n\pi x}{3} - \frac{12}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{2n\pi x}{3}$$

pb

Qf $f(x) = |x|$, expand $f(x)$ as a Fourier series in the interval $(-2, 2)$

sol: Here $l = 2$ since $|x|$ is even function.

$$|x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2} \quad \text{--- (1)}$$

$$a_0 = \frac{2}{l} \int_0^l f(x) dx = \frac{2}{2} \int_0^2 |x| dx$$

$$= \int_0^2 x dx = \left(\frac{x^2}{2} \right)_0^2 = \frac{4}{2} = 2 \quad \text{--- (2)}$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx = \int_0^2 |x| \cos \frac{n\pi x}{2} dx$$

$$= \int_0^2 x \cos \frac{n\pi x}{2} dx$$

$$= \left[x \frac{\sin \frac{n\pi x}{2}}{\frac{n\pi}{2}} - 1 \left[\frac{-\cos \frac{n\pi x}{2}}{\frac{n^2 \pi^2}{4}} \right] \right]_0^2$$

$$= \left(0 + \frac{\cos n\pi}{\frac{n^2 \pi^2}{4}} \right) - \left(0 + \frac{1}{\frac{n^2 \pi^2}{4}} \right)$$

$$= \frac{4}{n^2 \pi^2} [(-1)^n - 1] \Rightarrow \left. \begin{aligned} a_n &= 0 \text{ when 'n' is even} \\ &= -\frac{8}{n^2 \pi^2} \text{ when 'n' is odd} \end{aligned} \right\} \text{--- (3)}$$

Substituting (2) and (3) in (1) we get

$$|x| = 1 - \frac{8}{\pi^2} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} \cos \frac{n\pi x}{2}$$

$$= 1 - \frac{8}{\pi^2} \left[\cos \frac{\pi x}{2} + \frac{1}{3^2} \cos \frac{3\pi x}{2} + \frac{1}{5^2} \cos \frac{5\pi x}{2} + \dots \right]$$

Pb

Find the Fourier series of the function.

$$f(x) = \frac{1}{2} + x \quad \text{when } -1 \leq x \leq 0.$$

$$\frac{1}{2} - x \quad \text{when } 0 \leq x \leq 1.$$

sol

$$f(-x) = \frac{1}{2} - x \quad \text{in } (-1, 0) = f(x) \quad \text{in } (0, 1).$$

$$f(-x) = \frac{1}{2} + x \quad \text{in } (0, 1) = f(x) \quad \text{in } (-1, 0).$$

$f(x)$ is even function.

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x.$$

$$a_0 = \frac{2}{l} \int_0^l f(x) dx = \frac{2}{1} \int_0^1 \left(\frac{1}{2} - x\right) dx.$$

$$= 2 \left[\frac{x}{2} - \frac{x^2}{2} \right]_0^1 = 2 \left[\frac{1}{2} - \frac{1}{2} \right] = 0.$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx.$$

$$= \frac{2}{1} \int_0^1 \left(\frac{1}{2} - x\right) \cos n\pi x dx.$$

$$= 2 \left[\left(\frac{1}{2} - x\right) \left(\frac{\sin n\pi x}{n\pi}\right) - (-1) \left[\frac{-\cos n\pi x}{n^2 \pi^2}\right] \right]_0^1$$

$$= 2 \left[-\frac{\cos n\pi}{n^2 \pi^2} + \frac{1}{n^2 \pi^2} \right]$$

$$= \frac{2}{n^2 \pi^2} [1 - \cos n\pi] = \frac{2}{n^2 \pi^2} [1 - (-1)^n]$$

$$a_n = 0 \quad \text{if 'n' is even}$$

$$= \frac{4}{n^2 \pi^2} \quad \text{if 'n' is odd.}$$

Substituting the values a_0 and a_n in (1) we get

$$f(x) = \frac{4}{\pi^2} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} \cos n\pi x = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)\pi x$$

Pb

A sinusoidal voltage $E \sin \omega t$ is passed through a half-wave rectifier which clips the negative portion of the wave. Develop the resulting periodic function

$$v(t) = 0 \quad \text{when} \quad -\frac{T}{2} < t < 0.$$

$$= E \sin \omega t \quad \text{when} \quad 0 < t < \frac{T}{2} \quad \text{and} \quad T = \frac{2\pi}{\omega} \quad \text{is a Fourier series}$$

Soln

$$\text{Let } v(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{2n\pi t}{T} + \sum_{n=1}^{\infty} b_n \frac{2n\pi t}{T}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

$$\therefore T = \frac{2\pi}{\omega} \\ \omega = \frac{2\pi}{T}$$

$$a_0 = \frac{2}{T} \int_{-T/2}^{T/2} v(t) dt = \frac{2}{T} \left[\int_{-T/2}^0 v(t) dt + \int_0^{T/2} E \sin \omega t dt \right]$$

$$= \frac{2}{T} \left[\int_0^{T/2} E \sin \omega t dt \right]$$

$$= \frac{2}{T} \left[-\frac{E \cos \omega t}{\omega} \right]_0^{T/2} = -\frac{2E}{\omega T} \left[\cos \omega \frac{T}{2} - 1 \right]$$

$$\omega T = 2\pi$$

$$= -\frac{2E}{\omega T} [\cos 2\pi - 1] = -\frac{2E}{\omega T} (-2) = \frac{2E(2)}{2\pi} = \frac{2E}{\pi}$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} v(t) \cos n\omega t dt$$

$$= \frac{2}{T} \int_0^{T/2} E \sin \omega t \cos n\omega t dt$$

$$= \frac{E}{T} \int_0^{T/2} [\sin(1+n)\omega t + \sin(1-n)\omega t] dt$$

$$= \frac{E}{T} \left[-\frac{\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega} \right]_0^{T/2} \quad [n \neq 1]$$

$$= -\frac{E}{T} \left[\frac{\cos(1+n)\pi}{(1+n)\omega} + \frac{\cos(1-n)\pi}{(1-n)\omega} - \frac{1}{(1+n)\omega} - \frac{1}{(1-n)\omega} \right]$$

$$= -\frac{E}{\omega T} \left[-\frac{1}{(1+n)} - \frac{1}{(1-n)} - \frac{1}{(1+n)} - \frac{1}{(1-n)} \right]$$

$$= \frac{E}{2\pi} \left[\frac{2}{1+n} + \frac{2}{1-n} \right] = \frac{E}{2\pi} \left[\frac{2}{n+1} - \frac{2}{n-1} \right]$$

$$= \frac{E}{2\pi} \left[\frac{2(n-1) - 2(n+1)}{n^2-1} \right]$$

$$= \frac{E}{2\pi} \left[\frac{2n-2-2n-2}{n^2-1} \right] = \frac{-4E}{2\pi(n^2-1)}$$

$$= \frac{-2E}{\pi(n^2-1)} \quad \text{when 'n' is even}$$

If 'n' is odd ($n \neq 1$) $a_n = 0$.

In case $n=1$ we have.

$$a_1 = \frac{2}{T} \int_0^{T/2} E \sin \omega t \cos \omega t dt = \frac{E}{T} \int_0^{T/2} \sin 2\omega t dt$$

$$= \frac{E}{T} \left[-\frac{\cos 2\omega t}{2\omega} \right]_0^{T/2} = -\frac{E}{2\omega T} \left[\cos 2\omega \frac{T}{2} - 1 \right]$$

$$= -\frac{E}{2\omega T} \left[\cos \omega T - 1 \right] \quad (\omega T = 2\pi)$$

$$= -\frac{E}{2\omega T} [\cos 2\pi - 1]$$

$$= -\frac{E}{2\omega T} [1 - 1] = 0$$

$$b_1 = \frac{2}{T} \int_0^{T/2} E \sin \omega t \sin \omega t dt = \frac{E}{T} \int_0^{T/2} 2 \sin^2 \omega t dt$$

$$= \frac{E}{T} \int_0^{T/2} (1 - \cos 2\omega t) dt$$

$$= \frac{E}{T} \left[t - \frac{\sin 2\omega t}{2\omega} \right]_0^{T/2} = \frac{E}{T} \left[\frac{T}{2} - \frac{\sin 2\omega \frac{T}{2}}{2\omega} \right]$$

$$= \frac{E}{2} [\sin \omega T - \sin 2\pi = 0]$$

If $n \neq 1$, then we have

$$b_n = \frac{2}{T} \int_0^{T/2} E \sin \omega t \sin n\omega t dt \quad [2\pi = T\omega]$$

$$= \frac{E}{T} \int_0^{T/2} (\cos(n-1)\omega t - \cos(n+1)\omega t) dt = 0$$

Substituting the values of a's and b's in (1) we get.

$$v(t) = \frac{E}{\pi} + \frac{E}{2} \sin \omega t - \frac{2E}{\pi} \sum_{n=2,4,\dots} \frac{1}{n^2-1} \cos n\omega t$$

The Half-range sine series expansion of $f(x)$ in $(0, l)$ is given by. $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

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$$\text{where } b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx.$$

The Half-range cosine series expansion of $f(x)$ in $(0, l)$

$$\text{is given by } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}.$$

$$a_0 = \frac{2}{l} \int_0^l f(x) dx, \quad a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx.$$

Pb:-

Express $f(x) = x$ as half-range cosine series in the interval $0 < x < 2$

Sol:-

The Half-range Fourier cosine series is given by.

$$f(x) = x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}.$$

Here $l = 2$

$$a_0 = \frac{2}{l} \int_0^l f(x) dx = \frac{2}{2} \int_0^2 x dx = \left[\frac{x^2}{2} \right]_0^2 = \frac{4}{2} = 2.$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx = \frac{2}{2} \int_0^2 x \cos \frac{n\pi x}{2} dx.$$

$$= \left[x \frac{\sin \frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)} - \left(- \cos \frac{n\pi x}{2} \right) \frac{2}{\left(\frac{n\pi}{2}\right)^2} \right]_0^2$$

$$= \left[x \frac{\sin \frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)} + \frac{\cos \frac{n\pi x}{2}}{\frac{n^2 \pi^2}{4}} \right]_0^2$$

$$= \frac{\cos n\pi - 1}{\frac{n^2 \pi^2}{4}} = \frac{4[(-1)^n - 1]}{n^2 \pi^2}.$$

$a_n = 0$ when 'n' is even

$$= \frac{-8}{n^2 \pi^2} \text{ when 'n' is odd.}$$

Substituting the values of a_0 and a_n in (1) we get

$$x = 1 - \frac{8}{\pi^2} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} \cos \frac{n\pi x}{2}.$$

$$= 1 - \frac{8}{\pi^2} \left[\frac{\cos \pi x}{2} + \frac{1}{3^2} \cos \frac{3\pi x}{2} + \frac{1}{5^2} \cos \frac{5\pi x}{2} + \dots \right]$$

obtain the half-range sine series for e^x in $0 < x < 1$

Sol:-

The half-range sine series of e^x is given by.

$$e^x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$e^x = \sum_{n=1}^{\infty} b_n \sin n\pi x \quad [\because l=1]$$

$$\begin{aligned} \text{Here } b_n &= \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx \\ &= 2 \int_0^1 e^x \sin n\pi x dx \end{aligned}$$

$$\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2} [a \sin bx - b \cos bx]$$

$$= 2 \left[\frac{e^x}{1+n^2\pi^2} [\sin n\pi x - n\pi \cos n\pi x] \right]_0^1$$

$$= 2 \left[\frac{e}{1+n^2\pi^2} (0 - n\pi \cos n\pi) - \frac{1}{1+n^2\pi^2} (0 - n\pi) \right]$$

$$= 2 \left[\frac{1}{1+n^2\pi^2} (n\pi - n\pi e \cos n\pi) \right] = \frac{2n\pi}{1+n^2\pi^2} [1 - e(-1)^n]$$

$$\text{Hence } e^x = \sum_{n=1}^{\infty} \frac{2n\pi}{1+n^2\pi^2} (1 - e(-1)^n)$$

$$e^x = 2\pi \sum_{n=1}^{\infty} \frac{n}{1+n^2\pi^2} [1 - e(-1)^n]$$

2

Prob:- Expand $\cos \pi x$ in $(0,1)$ as Fourier sine series.

Show that the interval $(0,1)$, $\cos \pi x = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n}{4n^2-1} \sin 2n\pi x$

Sol:-

$$\text{Let } f(x) = \cos \pi x = \sum_{n=1}^{\infty} b_n \sin n\pi x \quad [\because l=1] \quad \text{--- (1)}$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx \quad [\because l=1]$$

$$b_n = 2 \int_0^1 \cos \pi x \sin n\pi x dx = \int_0^1 2 \cos \pi x \sin n\pi x dx$$

$$= \int_0^1 [\sin (n+1)\pi x + \sin (n-1)\pi x] dx$$

$$\begin{aligned} 2 \sin A \cos B &= \sin (A+B) + \sin (A-B) \end{aligned}$$

$$= \left[-\frac{\cos(n+1)\pi x}{(n+1)\pi} - \frac{\cos(n-1)\pi x}{(n-1)\pi} \right]_0^1 \quad (n \neq 1).$$

$$= -\frac{1}{\pi} \left[\frac{\cos(n+1)\pi}{(n+1)} + \frac{\cos(n-1)\pi}{(n-1)} \right] - \left[\frac{1}{n+1} + \frac{1}{n-1} \right]$$

$$= -\frac{1}{\pi} \left[\left(\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} \right) - \left(\frac{1}{n+1} + \frac{1}{n-1} \right) \right] \quad n \neq 1$$

$$= -\frac{1}{\pi} \left[\left(\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n+1}}{n-1} \right) - \left(\frac{1}{n+1} + \frac{1}{n-1} \right) \right]$$

$$= -\frac{1}{\pi} \left[(-1)^{n+1} \left[\frac{1}{n+1} + \frac{1}{n-1} \right] - \left(\frac{1}{n+1} + \frac{1}{n-1} \right) \right]$$

$$= -\frac{1}{\pi} \left[(-1)^{n+1} \left(\frac{n-x+n+x}{n^2-1} \right) - \left(\frac{n-x+n+x}{n^2-1} \right) \right]$$

$$= -\frac{1}{\pi} \frac{2n}{n^2-1} \left[(-1)^{n+1} - 1 \right] \quad n \neq 1.$$

~~if~~ $b_n = 0$ if 'n' is odd.

$$= \frac{4n}{\pi(n^2-1)} \quad \text{if 'n' is even.}$$

for $n=1$ then $b_1 = \int_0^1 (2\sin \pi x \cos \pi x) dx = \int_0^1 \sin 2\pi x dx.$

$$= -\frac{1}{2\pi} [\cos 2\pi x]_0^1 = -\frac{1}{2\pi} [1-1] = 0.$$

Substituting the values of b_1 and b_n in (1) we get

$$\cos \pi x = \sum_{n=2,4,6,\dots}^{\infty} \frac{4n}{\pi(n^2-1)} \sin n\pi x.$$

$$= \frac{4}{\pi} \sum_{n=2,4,6,\dots}^{\infty} \frac{2n}{n^2-1} \sin 2n\pi x \quad \left[\text{Replacing 'n' with } 2n \right].$$

$$= \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n}{4n^2-1} \sin 2n\pi x.$$

which is the required series.

UNIT - V

APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

* Method of Separation of Variables:

1. Solve $y^3 \frac{\partial z}{\partial x} + x^2 \frac{\partial z}{\partial y} = 0$.

sol. Given Equation is $y^3 \frac{\partial z}{\partial x} + x^2 \frac{\partial z}{\partial y} = 0 \rightarrow \textcircled{1}$.

Let $z = X(x) Y(y)$

$z = xy$.

$\frac{\partial z}{\partial x} = X'Y, \frac{\partial z}{\partial y} = XY'$

From $\textcircled{1} \Rightarrow y^3 X'Y + x^2 XY' = 0$.

$y^3 X'Y = -x^2 XY'$

$\frac{X'}{x^2 X} = -\frac{Y'}{y^3 Y} = \lambda \text{ (say)}$

consider, $\frac{X'}{x^2 X} = \lambda$.

$\frac{X'}{X} = \lambda x^2$

I.O.B.S

$\int \frac{X'}{X} dx = \int \lambda x^2 dx$

$\int \frac{X'}{X} dx = \lambda \int x^2 dx$

$\log X = \lambda \frac{x^3}{3} + \log A$

$\log X - \log A = \lambda \frac{x^3}{3}$

$\log \left(\frac{X}{A} \right) = \lambda \frac{x^3}{3}$

$\frac{X}{A} = e^{\frac{x^3 \lambda}{3}}$

$X = A e^{\frac{x^3 \lambda}{3}}$

consider

$$\frac{-y'}{y^3 y} = \lambda$$

$$\frac{y'}{y} = -y^3 \lambda$$

I.O.B.S

$$\int \frac{y'}{y} dy = \int -y^3 \lambda dy$$

$$\int \frac{y'}{y} dy = -\lambda \int y^3 dy$$

$$\log y = -\lambda \frac{y^4}{4} + \log B$$

$$\log y - \log B = -\lambda \frac{y^4}{4}$$

$$\log \left(\frac{y}{B} \right) = -\lambda \frac{y^4}{4}$$

$$\frac{y}{B} = e^{-\lambda y^4/4}$$

$$y = B e^{-\lambda y^4/4}$$

$$\therefore \text{solution is } z = xy^{x^3 \lambda/3} \cdot B e^{-\lambda y^4/4}$$

$$z = A e^{x^3 \lambda/3} \cdot B e^{-\lambda y^4/4}$$

$$z = C e^{\lambda(x^3/3 - y^4/4)}$$

2. solve $\frac{\partial u}{\partial x} = 2 \frac{\partial u}{\partial t} + u$ where $u(x,0) = 6e^{-3x}$ by the method of separation of variables.

Sol. Given Equation is $\frac{\partial u}{\partial x} = 2 \frac{\partial u}{\partial t} + u \rightarrow \textcircled{1}$

$$\text{and } u(x,0) = 6e^{-3x} \rightarrow \textcircled{2}$$

$$\text{Let } u = X(x) T(t)$$

$$u = XT$$

$$\frac{\partial u}{\partial x} = X' T \quad \Bigg| \quad \frac{\partial u}{\partial t} = X T'$$

$$\text{From } \textcircled{1} \Rightarrow X' T = 2 X T' + XT$$

$$x'T = (2T' + T) x$$

$$\frac{x'}{x} = \frac{2T' + T}{T} = \lambda \text{ (say).}$$

consider, $\frac{x'}{x} = \lambda.$

I.O.B.S.

$$\int \frac{x'}{x} dx = \int \lambda dx$$

$$\int \frac{x'}{x} dx = \lambda \int dx$$

$$\log x = \lambda x + \log A.$$

$$\log x - \log A = \lambda x$$

$$\log\left(\frac{x}{A}\right) = \lambda x.$$

$$\frac{x}{A} = e^{\lambda x}$$

$$x = A e^{\lambda x} \rightarrow \textcircled{3}.$$

consider, $\frac{2T' + T}{T} = \lambda.$

$$\frac{2T'}{T} + \frac{T}{T} = \lambda.$$

$$\frac{2T'}{T} + 1 = \lambda.$$

$$\frac{T'}{T} = \frac{\lambda - 1}{2}$$

I.O.B.S

$$\int \frac{T'}{T} dt = \int \left(\frac{\lambda - 1}{2}\right) dt$$

$$\int \frac{T'}{T} dt = \frac{\lambda - 1}{2} \int dt$$

$$\log T = \left(\frac{\lambda - 1}{2}\right) t + \log B.$$

$$\log T - \log B = \left(\frac{\lambda - 1}{2}\right) t.$$

$$\log\left(\frac{T}{B}\right) = \left(\frac{\lambda-1}{2}\right)t$$

$$\frac{T}{B} = e^{\left(\frac{\lambda-1}{2}\right)t}$$

$$T = B e^{\left(\frac{\lambda-1}{2}\right)t} \rightarrow (4)$$

$\therefore$ solution is $u = XT$

$$u = A e^{\lambda x} B e^{\left(\frac{\lambda-1}{2}\right)t}$$

$$u = C e^{\lambda x + \left(\frac{\lambda-1}{2}\right)t}$$

$$\therefore u(x,0) = C e^{\lambda x} \rightarrow (5)$$

From (2) and (5)

$$6e^{-3x} = C e^{\lambda x}$$

$$C = 6, \lambda = -3$$

$$\therefore \text{solution is } u(x,t) = 6 e^{-3x + \left(-\frac{3-1}{2}\right)t}$$

$$u(x,t) = 6 e^{-3x-2t}$$

$$u(x,t) = 6 e^{-(3x+2t)}$$

3. Solve $2xz_x - 3yz_y = 0$.

Sol:

Given Equation is $2xz_x - 3yz_y = 0 \rightarrow (1)$

$$2x \frac{\partial z}{\partial x} - 3y \frac{\partial z}{\partial y} = 0$$

$$\text{Let } z = x(x) y(y)$$

$$z = xy$$

$$\frac{\partial z}{\partial x} = x'y \quad \left| \quad \frac{\partial z}{\partial y} = xy' \right.$$

$$\text{From (1)} \Rightarrow 2x(x'y) - 3y(xy') = 0$$

$$2x x'y = 3y xy'$$

$$\frac{2xx'}{x} = \frac{3yy'}{y} = \lambda(\text{say})$$

consider $\frac{2x x'}{x} = \lambda$.

$$\frac{x'}{x} = \frac{\lambda}{2x}$$

I.O.B.S

$$\int \frac{x'}{x} dx = \int \frac{\lambda}{2x} dx$$

$$\int \frac{x'}{x} dx = \frac{\lambda}{2} \int \frac{1}{x} dx$$

$$\log x = \frac{\lambda}{2} \log x + \log A$$

$$\log x = \log x^{\lambda/2} + \log A$$

$$\log x = \log A x^{\lambda/2}$$

$$x = A x^{\lambda/2}$$

consider $\frac{3y y'}{y} = \lambda$.

$$\frac{y'}{y} = \frac{\lambda}{3y}$$

I.O.B.S

$$\int \frac{y'}{y} dy = \int \frac{\lambda}{3y} dy$$

$$\int \frac{y'}{y} dy = \frac{\lambda}{3} \int \frac{1}{y} dy$$

$$\log y = \frac{\lambda}{3} \log y + \log B$$

$$\log y = \log y^{\lambda/3} + \log B$$

$$\log y = \log B y^{\lambda/3}$$

$$y = B y^{\lambda/3}$$

∴ solution is $z = xy$

$$z = A x^{\lambda/2} B y^{\lambda/3}$$

$$z = c x^{\lambda/2} y^{\lambda/3}$$

* one dimensional Wave Equation:

An Equation of the form $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2}$ with the initial conditions $y(0, t) = 0$; $y(l, t) = 0$; $y(x, 0) = f(x)$; $0 \leq x \leq l$, $\left(\frac{\partial y}{\partial t}\right)_{(x, 0)} = g(x)$, $0 \leq x \leq l$ is called one dimensional wave Equation.

* one dimensional heat Equation:

An Equation of the form $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \left(\frac{\partial u}{\partial t}\right)$ where $c^2 = K/\rho s$ with the initial conditions $y(0, t) = 0$; $y(l, t) = 0$; $y(x, 0) = f(x)$; $0 \leq x \leq l$, $\left(\frac{\partial u}{\partial t}\right)_{(x, 0)} = g(x)$, $0 \leq x \leq l$ is called one dimensional heat Equation.

* Two dimensional steady state heat blow (Laplace Equation in two directions):

An Equation of the form $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ with the initial conditions $y(0, t) = 0$; $y(l, t) = 0$; $y(x, 0) = f(x)$; $0 \leq x \leq l$; $\left(\frac{\partial u}{\partial x}\right)_{(x, 0)} = g(x)$; $0 \leq x \leq l$ is called Laplace Equation in two dimensional.

* Solution of one dimensional Wave Equation:

consider the Equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} \rightarrow \textcircled{1}$

Let a solution of $\textcircled{1}$ is

$$y = X(x) T(t)$$

$$\text{i.e., } y = XT$$

$$\Rightarrow \frac{\partial y}{\partial x} = X' T \Rightarrow \frac{\partial^2 y}{\partial x^2} = X'' T$$

$$\Rightarrow \frac{\partial y}{\partial t} = X T' \Rightarrow \frac{\partial^2 y}{\partial t^2} = X T''$$

From ① $\Rightarrow X'' T = \frac{1}{c^2} X T''$

$$\frac{X''}{X} = \frac{1}{c^2} \frac{T''}{T} = \lambda \text{ (say)}$$

Case: i

If $\lambda > 0 \Rightarrow \lambda = p^2$

$$\therefore \frac{X''}{X} = \frac{1}{c^2} \frac{T''}{T} = p^2$$

consider $\frac{X''}{X} = p^2$

$$X'' = p^2 X$$

$$X'' - p^2 X = 0.$$

$\therefore$ A.E is $m^2 - p^2 = 0.$

$$m = \pm p.$$

$\therefore$ solution is $X = A_1 e^{px} + B_1 e^{-px}$

Now consider $\frac{1}{c^2} \frac{T''}{T} = p^2$

$$T'' = p^2 c^2 T$$

$$T'' - p^2 c^2 T = 0.$$

A.E is $m^2 - p^2 c^2 = 0$

$$m = \pm pc.$$

$\therefore$ solution is $T = c_1 e^{pct} + d_1 e^{-pct}$

$\therefore$ complete solution is

$$Y = XT$$

$$Y = (A_1 e^{px} + B_1 e^{-px}) (c_1 e^{pct} + d_1 e^{-pct}) \rightarrow \text{①}$$

Case: ii

If $\lambda = 0$ then

$$\frac{X''}{X} = \frac{1}{c^2} \frac{T''}{T} = 0.$$

consider $\frac{X''}{X} = 0.$

$$\Rightarrow X'' = 0$$

$$\text{A.E is } m^2 = 0 \\ m = 0, 0.$$

$$\therefore \text{ solution is } X = (A_2 + B_2 x) e^{0x}$$

$$\text{Now consider } \frac{1}{c^2} \frac{T''}{T} = 0.$$

$$T'' = 0.$$

$$\text{A.E is } m^2 = 0 \\ m = 0, 0.$$

$$\therefore \text{ Solution is } T = (C_2 + D_2 x) e^{0x}$$

$\therefore$ complete solution is

$$y = XT$$

$$y = (A_2 + B_2 x) (C_2 + D_2 t).$$

case: iii

$$\text{If } A < 0 \Rightarrow A = -p^2$$

$$\therefore \frac{X''}{X} = \frac{1}{c^2} \frac{T''}{T} = -p^2$$

$$\text{consider } \frac{X''}{X} = -p^2$$

$$X'' = -p^2 X.$$

$$X'' + p^2 X = 0.$$

$$\therefore \text{ A.E is } m^2 + p^2 = 0.$$

$$m = \pm ip.$$

$$\text{solution is } X = A_3 \cos px + B_3 \sin px.$$

Now consider

$$\frac{1}{c^2} \frac{T''}{T} = -p^2$$

$$T'' = -p^2 c^2 T$$

$$T'' + p^2 c^2 T = 0$$

$$\therefore \text{A.E. is } m^2 + p^2 c^2 = 0$$

$$m = \pm i p c$$

$$\text{solution is } T = C_3 \cos p c t + D_3 \sin p c t$$

$\therefore$ complete solution is

$$y = x T$$

$$= (A_3 \cos p x + B_3 \sin p x) (C_3 \cos p c t + D_3 \sin p c t)$$

NOTE:

The most general solution of (1) by above solution satisfying the initial conditions $y(0,t) = 0$, $y(l,t) = 0$ is $y(x,t) = \sum_{n=1}^{\infty} \left[C_n \cos \frac{n\pi x}{l} + D_n \sin \frac{n\pi x}{l} \right] \sin \frac{n\pi c t}{l}$

Solve the boundary value problem $u_{tt} = a^2 u_{xx}$ for $0 < x < l$, $t > 0$ with $u(0,t) = 0$, $u(l,t) = 0$, $u(x,0) = 0$, $u_t(x,0) = \sin^3\left(\frac{\pi x}{l}\right)$.

Given Equation is $u_{tt} = a^2 u_{xx}$

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \rightarrow (1)$$

Given $u(0,t) = 0 \rightarrow (2)$

$u(l,t) = 0 \rightarrow (3)$

$u(x,0) = 0 \rightarrow (4)$

$u_t(x,0) = \sin^3\left(\frac{\pi x}{l}\right) \rightarrow (5)$

Let the solution of (1) is

$$u(x,t) = (A \cos p x + B \sin p x) (C \cos p c t + D \sin p c t) \rightarrow (6)$$

$$u(0,t) = A [C \cos p c t + D \sin p c t] = 0$$

$$\Rightarrow A = 0$$

$$(6) \Rightarrow u(x,t) = B \sin p x [c \cos p c t + D \sin p c t] \rightarrow (7)$$

$$u(x,t) = B \sin p x [c \cos p c t + D \sin p c t]$$

$$\Rightarrow \sin p l = 0$$

$$\Rightarrow p = \frac{n\pi}{l}$$

From (7)

$$\Rightarrow u(x,t) = B \sin \frac{n\pi x}{l} \left[c \cos \frac{n\pi c t}{l} + D \sin \frac{n\pi c t}{l} \right] \rightarrow (8)$$

$$u(x,0) = B \sin \frac{n\pi x}{l} \cdot c$$

$$= 0.$$

$$\Rightarrow c = 0.$$

$$(8) \Rightarrow u(x,t) = B \sin \frac{n\pi x}{l} D \sin \frac{n\pi c t}{l}$$

$$= c_n \sin \frac{n\pi x}{l} \sin \frac{n\pi c t}{l}$$

The most general solution of (1) is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} \sin \frac{n\pi c t}{l} \rightarrow (9)$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} \cos \frac{n\pi c t}{l} \left(\frac{n\pi c}{l} \right)$$

$$\left(\frac{\partial u}{\partial t} \right)_{(x,0)} = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} \frac{n\pi c}{l} \rightarrow (10)$$

From (9) & (10).

$$\Rightarrow \sin^3 \left(\frac{\pi x}{l} \right) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} \frac{n\pi c}{l}$$

$$\sin 3A = 3 \sin A - 4 \sin^3 A.$$

$$\frac{3}{4} \sin \frac{\pi x}{l} - \frac{1}{4} \sin \frac{3\pi x}{l} = \frac{c_1 \pi c}{l} \sin \frac{\pi x}{l} +$$

$$c_2 \frac{2\pi c}{l} \sin \frac{2\pi x}{l} + c_3 \frac{3\pi c}{l} \sin \frac{3\pi x}{l} + \dots$$

$$\Rightarrow \frac{c_1 \pi c}{l} = \frac{3}{4} \quad \text{and} \quad c_3 \frac{3\pi c}{l} = -\frac{1}{4}, \quad c_2 = 0.$$

$$c_1 = \frac{3l}{4\pi c}, \quad c_3 = -\frac{l}{12\pi c}, \quad c_2 = 0.$$

$$\therefore u(x, t) = \left(\frac{3l}{4\pi c} \right) \sin \frac{\pi x}{l} \sin \frac{\pi ct}{l} = \frac{l}{12\pi c} \sin \frac{3\pi x}{l} \sin \frac{3\pi ct}{l}$$

A tightly stretched string with fixed end-points $x=0$ and $x=10$ cm is initially at rest in its equilibrium position and at every point x on it is given an initial velocity $g(x) = \lambda x(10-x)$. Find the displacement of the string at any distance x from one end at any time t .

The displacement $y(x, t)$ is given by

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} \rightarrow (1)$$

with initial conditions

$$y(0, t) = 0 \text{ for all } t \rightarrow (2)$$

$$y(10, t) = 0 \text{ for all } t \rightarrow (3)$$

$$y(x, 0) = 0 \text{ for } 0 \leq x \leq 10 \rightarrow (4)$$

$$\text{and } \left(\frac{\partial y}{\partial t} \right)_{t=0} = \lambda x(10-x) \text{ for } 0 \leq x \leq 10 \rightarrow (5)$$

The most general solution of (1), (2), (3) is

$$y(x, t) = \sum_{n=1}^{\infty} \left(c_n \cos \frac{n\pi ct}{l} + D_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}$$

using condition (4) we get

$$y(x, 0) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} = 0.$$

Hence $c_n = 0$ for all n .

$$\Rightarrow D_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx.$$

$$\begin{aligned}
 \therefore D_n &= \frac{2}{n\pi c} \int_0^{10} \lambda x(10-x) \sin \frac{n\pi x}{2} dx \\
 &= \frac{2\lambda}{n\pi c} \int_0^{10} x(10-x) \sin \frac{n\pi x}{2} dx \\
 &= \frac{2\lambda}{n\pi c} \left[x(10-x) \left(\frac{-\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}} \right) - (10-2x) \left(\frac{-\sin \frac{n\pi x}{2}}{\frac{n^2\pi^2}{2}} \right) \right. \\
 &\quad \left. + (-2) \left(\frac{\cos \frac{n\pi x}{2}}{\frac{n^3\pi^3}{2}} \right) \right] \\
 &= \frac{2\lambda}{n\pi c} \left[-\frac{2x^3}{n^3\pi^3} \cos n\pi + \frac{2x^3}{n^3\pi^3} \right] \\
 &= \frac{2\lambda}{n\pi c} \cdot \frac{2x^3}{n^3\pi^3} [1 - \cos n\pi] \\
 &= \frac{4\lambda x^3}{n^4\pi^4 c} (1 - \cos n\pi)
 \end{aligned}$$

If n is even, $D_n = 0$.

If n is odd, $n = 1, 3, 5, \dots, \infty$

$$D_n = \frac{4\lambda x^3}{n^4\pi^4 c} \cdot 2 = \frac{8\lambda x^3}{n^4\pi^4 c} \quad \text{where } n = 1, 3, 5, \dots$$

Hence

$$y(x,t) = \frac{8\lambda x^3}{\pi^4 c} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^4} \sin\left(\frac{n\pi x}{2}\right) \sin\left(\frac{n\pi t}{2}\right)$$

where $l = 10$.

The Fourier Transform is defined as $\bar{f}(p)$ is denoted by $\bar{f}(p) = F\{f(t)\} = \int_{-\infty}^{\infty} e^{ipt} f(t) dt$

Fourier Integral Theorem.

Fourier Integral Theorem states that

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} \cos p(t-x) f(t) dt dp.$$

Fourier sine Integrals.

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin px \int_0^{\infty} f(t) \sin pt dt dp.$$

Fourier cosine Integrals.

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos px \int_0^{\infty} f(t) \cos pt dt dp.$$

Pb:- using Fourier integral show that.

$$e^{-ax} - e^{-bx} = \frac{2(b^2 - a^2)}{\pi} \int_0^{\infty} \frac{\lambda \sin \lambda x}{(\lambda^2 + a^2)(\lambda^2 + b^2)} d\lambda, \quad a, b > 0.$$

Sol

Since the integrand on R.H.S contains sine term, we use Fourier sine integral formula.

we know that Fourier sine integral for $f(x)$ is given by

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin px \int_0^{\infty} f(t) \sin pt dt dp.$$

Replace 'p' with ' λ ', we get

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \int_0^{\infty} f(t) \sin \lambda t dt d\lambda \quad \text{--- (1)}$$

$$\text{Here } f(x) = e^{-ax} - e^{-bx} \Rightarrow f(t) = e^{-at} - e^{-bt} \quad \text{--- (2)}$$

Substituting (2) in (1) we get

$$\begin{aligned} f(x) &= \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \left[\int_0^{\infty} (e^{-at} - e^{-bt}) \sin \lambda t dt \right] d\lambda \\ &= \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \left[\int_0^{\infty} e^{-at} \sin \lambda t dt - \int_0^{\infty} e^{-bt} \sin \lambda t dt \right] d\lambda \end{aligned}$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \left[\left(\frac{e^{-at}}{\lambda^2 + a^2} (-a \sin \lambda t - \lambda \cos \lambda t) \right) \Big|_0^{\infty} - \left(\frac{e^{-bt}}{\lambda^2 + b^2} (-b \sin \lambda t - \lambda \cos \lambda t) \right) \Big|_0^{\infty} \right]$$

$$\left[\text{using } \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) \right]$$

$$= \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \left[\frac{\lambda}{\lambda^2 + a^2} - \frac{\lambda}{\lambda^2 + b^2} \right] d\lambda$$

$$= \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \cdot \lambda \left[\frac{1}{\lambda^2 + a^2} - \frac{1}{\lambda^2 + b^2} \right] d\lambda$$

$$= \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \cdot \lambda \left[\frac{\lambda^2 + b^2 - \lambda^2 - a^2}{(\lambda^2 + a^2)(\lambda^2 + b^2)} \right] d\lambda$$

$$= \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \cdot \frac{\lambda(b^2 - a^2)}{(\lambda^2 + a^2)(\lambda^2 + b^2)} d\lambda$$

$$= \frac{2(b^2 - a^2)}{\pi} \int_0^{\infty} \frac{\lambda \sin \lambda x}{(\lambda^2 + a^2)(\lambda^2 + b^2)} d\lambda$$

$$\frac{e^{-ax} - e^{-bx}}{a - b} = \frac{2(b^2 - a^2)}{\pi} \int_0^{\infty} \frac{\lambda \sin \lambda x}{(\lambda^2 + a^2)(\lambda^2 + b^2)} d\lambda$$

Pb

using Fourier integral, show that $\int_0^{\infty} \frac{1 - \cos \pi \lambda}{\lambda} \sin \lambda x \, d\lambda$

Sol

$$\text{Let } f(x) = \frac{\pi}{2} \text{ if } 0 < x < \pi$$

$$= 0 \text{ if } x > \pi$$

$$= \frac{\pi}{2} \text{ if } 0 < x < \pi$$

$$= 0 \text{ if } x > \pi$$

$$\text{Then } f(t) = \frac{\pi}{2} \text{ if } 0 < t < \pi$$

$$= 0 \text{ if } t > \pi$$

} - (1)

we know that Fourier sine integral of $f(x)$ is given by

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \int_0^{\infty} f(t) \sin \lambda t \, dt \, d\lambda \quad \text{--- (2)}$$

Substituting (1) in (2) we get

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \left[\int_0^{\pi} f(t) \sin \lambda t \, dt \right] d\lambda$$

$$= \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \left[\int_0^{\pi} \frac{\pi}{2} \sin \lambda t \, dt \right] d\lambda$$

$$= \frac{2}{\pi} \cdot \frac{\pi}{2} \int_0^{\infty} \sin \lambda x \cdot \left(\frac{1 - \cos \lambda \pi}{\lambda} \right) d\lambda$$

$$f(x) = \int_0^{\infty} \sin \lambda x \cdot \left(\frac{1 - \cos \lambda \pi}{\lambda} \right) d\lambda$$

$$\int_0^{\infty} \frac{1 - \cos \lambda \pi}{\lambda} \sin \lambda x d\lambda = \begin{cases} \frac{\pi}{2} & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$$

Qb) using Fourier integral, show that $\frac{e^{-ax}}{e} = \frac{2a}{\pi} \int_0^{\infty} \frac{\cos \lambda x}{\lambda^2 + a^2} d\lambda$ ($a > 0$)

sol)

Since the integrand contains cosine term, we use Fourier cosine integral formula.

By Fourier cosine Integral formula.

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos p x \int_0^{\infty} f(t) \cos p t dt dp$$

Replace 'p' with 'λ' we get.

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} f(t) \cos \lambda t \cos \lambda x dt d\lambda$$

$$\text{Let } f(x) = e^{-ax} \Rightarrow f(t) = e^{-at}$$

$$e^{-ax} = \frac{2}{\pi} \int_0^{\infty} \left[\int_0^{\infty} e^{-at} \cos \lambda t dt \right] \cos \lambda x d\lambda$$

$$= \frac{2}{\pi} \int_0^{\infty} \cos \lambda x \left[\frac{e^{-at}}{a^2 + \lambda^2} (-a \cos \lambda t + \lambda \sin \lambda t) \right]_0^{\infty} d\lambda$$

$$= \frac{2}{\pi} \int_0^{\infty} \cos \lambda x \cdot \frac{a}{a^2 + \lambda^2} d\lambda$$

$$e^{-ax} = \frac{2a}{\pi} \int_0^{\infty} \frac{\cos \lambda x}{a^2 + \lambda^2} d\lambda$$

Put $a=1$ in the above result.

$$e^{-x} = \frac{2}{\pi} \int_0^{\infty} \frac{\cos \lambda x}{1 + \lambda^2} d\lambda$$

$$\frac{\pi}{2} e^{-x} = \int_0^{\infty} \frac{\cos \lambda x}{1 + \lambda^2} d\lambda$$

Pr using Fourier integral, show that

$$e^{-x} \cos x = \frac{2}{\pi} \int_0^{\infty} \frac{\lambda^2 + 2}{\lambda^4 + 4} \cos \lambda x \, d\lambda.$$

Sol using Fourier cosine integral formula

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos p x \int_0^{\infty} f(t) \cos p t \, dt \, dp.$$

Replace 'p' with 'λ' we get

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} f(t) \cos \lambda t \cos \lambda x \, dt \, d\lambda.$$

$$\text{Let } f(x) = e^{-x} \cos x$$

$$e^{-x} \cos x = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} e^{-t} \cos t \cos \lambda t \cos \lambda x \, dt \, d\lambda.$$

$$= \frac{1}{\pi} \int_0^{\infty} \left[\int_0^{\infty} e^{-t} (2 \cos t \cos \lambda t) \, dt \right] \cos \lambda x \, d\lambda.$$

$$= \frac{1}{\pi} \int_0^{\infty} \left[\int_0^{\infty} e^{-t} (\cos(\lambda+1)t + \cos(\lambda-1)t) \, dt \right] \cos \lambda x \, d\lambda$$

$$= \frac{1}{\pi} \int_0^{\infty} \left[\frac{1}{(\lambda+1)^2 + 1} + \frac{1}{(\lambda-1)^2 + 1} \right] \cos \lambda x \, d\lambda \left[\int_0^{\infty} e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2} \right]$$

$$= \frac{1}{\pi} \int_0^{\infty} \left[\frac{1}{\lambda^2 + 2\lambda + 2} + \frac{1}{\lambda^2 - 2\lambda + 2} \right] \cos \lambda x \, d\lambda.$$

$$= \frac{1}{\pi} \int_0^{\infty} \left[\frac{\lambda^2 - 2\lambda + 2 + \lambda^2 + 2\lambda + 2}{[\lambda^2 + 2 + 2\lambda][\lambda^2 + 2 - 2\lambda]} \right] \cos \lambda x \, d\lambda$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{(\lambda^2 + 2) \cos \lambda x}{(\lambda^2 + 2)^2 - (2\lambda)^2} \, d\lambda.$$

$$e^{-x} \cos x = \frac{2}{\pi} \int_0^{\infty} \frac{\lambda^2 + 2}{\lambda^4 + 4} \cos \lambda x \, d\lambda.$$

∴

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ip(t-x)} f(t) dt dp.$$

Infinite Fourier Transform of $f(x)$

The Fourier Transform of a function $f(x)$ is given by.

$$F\{f(x)\} = F(p) = \int_{-\infty}^{\infty} f(x) e^{ipx} dx.$$

The inverse Fourier Transform of $F(p)$ is given by.

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(p) e^{-ipx} dp.$$

Fourier Sine Transform

Fourier sine Transform as.

$$F_s(p) = F_s\{f(x)\} = \int_0^{\infty} f(x) \sin px dx.$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_s(p) \sin px dp.$$

Fourier cosine Transform

Fourier cosine Transform as.

$$F_c(p) = F_c\{f(x)\} = \int_0^{\infty} f(x) \cos px dx.$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_c(p) \cos px dp.$$

Properties of Fourier Transforms.

1) Linear Property of Fourier Transform.

If $F(p)$ and $G(p)$ are Fourier transforms of $f(x)$ and $g(x)$ respectively, then

$$F\{af(x) + bg(x)\} = aF(p) + bG(p).$$

where 'a' and 'b' are constant.

Proof It is given that $F\{f(x)\} = \bar{F}(P)$ & $F\{g(x)\} = \bar{G}(P)$.

$$F\{f(x)\} = \int_{-\infty}^{\infty} e^{ipx} f(x) dx = \bar{F}(P)$$

$$F\{g(x)\} = \int_{-\infty}^{\infty} e^{ipx} g(x) dx = \bar{G}(P)$$

$$F\{af(x) + bg(x)\} = \int_{-\infty}^{\infty} e^{ipx} [af(x) + bg(x)] dx$$

$$= \int_{-\infty}^{\infty} a e^{ipx} f(x) dx + \int_{-\infty}^{\infty} b e^{ipx} g(x) dx$$

$$= a \int_{-\infty}^{\infty} e^{ipx} f(x) dx + b \int_{-\infty}^{\infty} e^{ipx} g(x) dx$$

$$= a \bar{F}(P) + b \bar{G}(P)$$

Th 2)

change of scale property.

If $\bar{F}(P)$ is the Fourier Transform of $f(x)$ then
fourier transformation of $F\{f(ax)\} = \frac{1}{a} \bar{F}\left(\frac{P}{a}\right)$.

Proof

$$\text{we know that } F\{f(x)\} = \bar{F}(P) = \int_{-\infty}^{\infty} e^{ipx} f(x) dx$$

$$F\{f(ax)\} = \int_{-\infty}^{\infty} e^{ipx} f(ax) dx$$

$$= \int_{-\infty}^{\infty} e^{ip\left(\frac{t}{a}\right)} f(t) \frac{dt}{a}$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} e^{i\left(\frac{P}{a}\right)t} f(t) dt = \frac{1}{a} \bar{F}\left(\frac{P}{a}\right)$$

$$F\{f(ax)\} = \frac{1}{a} \bar{F}\left(\frac{P}{a}\right)$$

3) Shifting property :- If $\bar{F}(P)$ is the Fourier transformation of $f(x)$ then $F\{f(x-a)\} = e^{ipa} \bar{F}(P)$.

Proof

$$\text{w.k.t } F\{f(x)\} = \int_{-\infty}^{\infty} e^{ipx} f(x) dx$$

$$F\{f(x-a)\} = \int_{-\infty}^{\infty} e^{ipx} f(x-a) dx = \int_{-\infty}^{\infty} e^{ip(t+a)} f(t) dt$$

$$= \int_{-\infty}^{\infty} e^{ipt} e^{ipa} f(t) dt = e^{ipa} \int_{-\infty}^{\infty} e^{ipt} f(t) dt$$

$$= e^{ipa} \bar{F}(P)$$

$$F\{f(x-a)\} = e^{ipa} \bar{F}(P)$$

4)

Modulation Theorem

of $f(x)$ then $F\{f(x) \cos ax\} = \frac{1}{2} [\bar{F}(P+a) + \bar{F}(P-a)]$.

Proof

$$w.k.T \quad F\{f(x)\} = \int_{-\infty}^{\infty} e^{ipx} f(x) dx = \bar{F}(P).$$

$$F\{f(x) \cos ax\} = \int_{-\infty}^{\infty} e^{ipx} f(x) \cos ax dx.$$

$$= \int_{-\infty}^{\infty} e^{ipx} f(x) \left[\frac{e^{iax} + e^{-iax}}{2} \right] dx.$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} e^{i\alpha(P+a)} f(x) dx + \frac{1}{2} \int_{-\infty}^{\infty} e^{i\alpha(P-a)} f(x) dx.$$

$$= \frac{1}{2} [\bar{F}(P+a)] + \frac{1}{2} [\bar{F}(P-a)]$$

Hence $\bar{F}\{f(x) \cos ax\} = \frac{1}{2} [\bar{F}(P+a) + \bar{F}(P-a)]$.

Note:- If $F_s(P)$, $F_c(P)$ are Fourier sine, cosine transformations of $f(x)$ respectively then

(i) $F_s\{f(x) \cos ax\} = \frac{1}{2} [F_s(P+a) + F_s(P-a)]$

(ii) $F_c\{f(x) \sin ax\} = \frac{1}{2} [F_s(P+a) - F_s(P-a)]$

(iii) $F_s\{f(x) \sin ax\} = \frac{1}{2} [F_c(P-a) - F_c(P+a)]$

(iv) $F_c\{f(x) \cos ax\} = \frac{1}{2} [F_c(P+a) + F_c(P-a)]$.

5)

$F\{x^n f(x)\} = (-i)^n \frac{d^n}{dP^n} [\bar{F}(P)]$.

Proof

$$\bar{F}(P) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{ipx} dx.$$

$$\frac{d}{dP} [\bar{F}(P)] = \frac{d}{dP} \left[\int_{-\infty}^{\infty} f(x) e^{ipx} dx \right]$$

$$= \int_{-\infty}^{\infty} f(x) \frac{d}{dP} (e^{ipx}) dx.$$

$$= \int_{-\infty}^{\infty} f(x) \cdot (ix) e^{ipx} dx.$$

$$\frac{d^2}{dP^2} [\bar{F}(P)] = \frac{d}{dP} \int_{-\infty}^{\infty} f(x) (ix) e^{ipx} dx$$

$$\begin{aligned}\frac{d^2}{dp^2} [\bar{F}(p)] &= \int_{-\infty}^{\infty} f(x) \cdot \frac{d}{dp} [e^{ipx}] (ix) dx \\ &= \int_{-\infty}^{\infty} f(x) (ix)^2 e^{ipx} dx = (i)^2 \int_{-\infty}^{\infty} x^2 f(x) e^{ipx} dx.\end{aligned}$$

On general

$$\begin{aligned}\frac{d^n}{dp^n} [\bar{F}(p)] &= (i)^n \int_{-\infty}^{\infty} x^n f(x) e^{ipx} dx \\ &= (i)^n F\{x^n f(x)\}.\end{aligned}$$

$$\begin{aligned}F\{x^n f(x)\} &= \frac{1}{i^n} \frac{d^n}{dp^n} [\bar{F}(p)] & \frac{i^n}{(i^n)^2} &= \left(\frac{i^n}{i^2}\right) \\ &= \frac{i^n}{(i^n)^2} \frac{d^n}{dp^n} [\bar{F}(p)] & &= \frac{i^n}{-1}.\end{aligned}$$

$$F\{x^n f(x)\} = (-i)^n \frac{d^n}{dp^n} [\bar{F}(p)].$$

6.

$$F\left\{\frac{d^n}{dx^n} f(x)\right\} = (-ip)^n \bar{F}(p).$$

Sol:-

$$\begin{aligned}F\{f'(x)\} &= F\left\{\frac{d}{dx} f(x)\right\} = \int_{-\infty}^{\infty} f'(x) e^{ipx} dx \\ &= \int_{-\infty}^{\infty} e^{ipx} d[f(x)]\end{aligned}$$

$$\boxed{\int u dv = uv - \int v du.}$$

$$= \left[e^{ipx} f(x) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) (ip) e^{ipx} dx.$$

$$= -ip \int_{-\infty}^{\infty} f(x) e^{ipx} dx \quad \left[\text{Assuming } f(x) \rightarrow 0 \text{ as } x \rightarrow \pm\infty \right]$$

$$= -ip F\{f(x)\}$$

$$= -ip \bar{F}(p).$$

$$\text{Now } F\{f''(x)\} = F\left\{\frac{d^2}{dx^2} f(x)\right\} = \int_{-\infty}^{\infty} f''(x) e^{ipx} dx.$$

$$= \int_{-\infty}^{\infty} e^{ipx} d[f'(x)].$$

$$F\{f''(x)\} = \left[e^{ipx} f'(x) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x) e^{ipx} (ip) dx$$

Assuming that

$f'(x) \rightarrow 0$ as $x \rightarrow \pm\infty$.

$$= -ip \int_{-\infty}^{\infty} e^{ipx} d[f(x)]$$

$$= -ip \left[(e^{ipx} f(x))_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) (ip) e^{ipx} dx \right]$$

$$= (-ip)^2 \int_{-\infty}^{\infty} f(x) e^{ipx} dx \quad \left[\text{Assuming } f(x) \rightarrow 0 \text{ as } x \rightarrow \pm\infty \right]$$

$$= (-ip)^2 F\{f(x)\}$$

$$= (-ip)^2 \bar{F}(p)$$

Similarly $F\{f^{(n)}(x)\} = F\left\{\frac{d^n}{dx^n} f(x)\right\} = (-ip)^n \bar{F}(p)$.

7

$$F_S\{f'(x)\} = -p F_C(p)$$

Sol

By definition $F_S\{f'(x)\} = \int_0^{\infty} f'(x) \sin px \, dx$

$$= \int_0^{\infty} \sin px \, d[f(x)]$$

$$= [\sin px \cdot f(x)]_0^{\infty} - \int_0^{\infty} f(x) p \cos px \, dx$$

$$= -p \int_0^{\infty} f(x) \cos px \, dx \quad \left[\text{Assuming } f(x) \rightarrow 0 \text{ as } x \rightarrow \infty \right]$$

$$= -p F_C(p)$$

8)

$$F_C\{f'(x)\} = -f(0) + p F_S(p)$$

By definition $F_C\{f'(x)\} = \int_0^{\infty} f'(x) \cos px \, dx$

$$= \int_0^{\infty} \cos px \, d[f(x)]$$

$$= [\cos px \cdot f(x)]_0^{\infty} - \int_0^{\infty} f(x) p (-\sin px) \, dx$$

$$= -f(0) + p \int_0^{\infty} f(x) \sin px \, dx$$

$$= -f(0) + p F_S(p)$$

$$\therefore F_C\{f'(x)\} = -f(0) + p F_S(p)$$

$$= 0 \text{ for } |x| > a.$$

and hence evaluate $\int_0^x \frac{\sin x}{x} dx$ and $\int_{-a}^a \frac{\sin ap \cos xp}{p} dp$

Sol.

we have

$$F\{f(x)\} = \int_{-\infty}^{\infty} e^{ipx} f(x) dx.$$

$$= \int_{-\infty}^{-a} e^{ipx} f(x) dx + \int_{-a}^a e^{ipx} f(x) dx + \int_a^{\infty} e^{ipx} f(x) dx.$$

$$= \int_{-\infty}^{-a} e^{ipx} (1) dx = \left[\frac{e^{ipx}}{ip} \right]_{-\infty}^{-a} = \frac{e^{-ipa} - e^{-i\infty p}}{ip}$$

$$= \frac{2i \sin pa}{ip} = \frac{2 \sin pa}{p} \quad \left[\sin pa = \frac{e^{ipa} - e^{-ipa}}{2i} \right]$$

$$= \bar{F}(p).$$

Apply the inversion formula. $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{F}(p) e^{-ipx} dp.$ — (1)

$$\bar{F}(p) = \frac{2 \sin pa}{p} \text{ in equation (1).}$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2 \sin pa}{p} e^{-ipx} dp.$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin pa}{p} (\cos xp - i \sin xp) dp.$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin pa \cos xp}{p} dp - \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\sin pa \sin xp}{p} dp.$$

The integrand in the second integral is an odd function and integral vanishes.

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin pa \cos xp}{p} dp.$$

$$\int_{-\infty}^{\infty} \frac{\sin pa \cos xp}{p} dp = \pi f(x) \quad \text{--- (2)}$$

$$= \pi \text{ for } |x| \leq a$$

$$= 0 \text{ for } |x| > a.$$

If $\alpha = 0$ then equation (2) becomes

$$\int_{-\infty}^{\infty} \frac{\sin pa}{p} dp = \pi \Rightarrow 2 \int_0^{\infty} \frac{\sin pa}{p} dp = \pi$$

$$\Rightarrow \int_0^{\infty} \frac{\sin pa}{p} dp = \frac{\pi}{2}$$

Put $a=1$ in above equation

$$\int_0^{\infty} \frac{\sin p}{p} dp = \frac{\pi}{2} \quad (\text{or}) \quad \int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

2) Find the Fourier Transform of $f(x)$ defined by

$$f(x) = 1-x^2, \text{ if } |x| \leq 1$$

$$= 0 \text{ if } |x| > 1.$$

Hence evaluate $\int_0^{\infty} \frac{x \cos x - \sin x}{x^3} \cos \frac{x}{2} dx$ (ii) $\int_0^{\infty} \frac{x \cos x - \sin x}{x^3} dx$

Sol

we have $F\{f(x)\} = \int_{-\infty}^{\infty} e^{ipx} f(x) dx$

$$= \int_{-\infty}^{-1} e^{ipx} \cdot 0 dx + \int_{-1}^1 e^{ipx} (1-x^2) dx + \int_1^{\infty} e^{ipx} (0) dx$$

$$= \int_{-1}^1 (1-x^2) \frac{e^{ipx}}{ip} - (-2x) \frac{e^{ipx}}{i^2 p^2} + (-2) \frac{e^{ipx}}{i^3 p^3} \Big|_{-1}^1$$

$$= \frac{2e^{ip}}{-p^2} - 2 \left[\frac{e^{ip}}{-ip^3} + \frac{e^{-ip}}{-ip^3} \right] - 2 \frac{e^{-ip}}{-p^2}$$

$$e^{ip} + e^{-ip} = 2 \cos p$$

$$= \frac{2e^{ip}}{-p^2} + \frac{2}{ip^3} [e^{ip} + e^{-ip}] + 2 \frac{e^{-ip}}{-p^2}$$

$$e^{ip} - e^{-ip} = 2i \sin p$$

$$= \frac{2}{p^2} [e^{-ip} - e^{ip}] + \frac{2}{ip^3} [e^{ip} + e^{-ip}]$$

$$= \frac{2}{p^2} (2i \sin p) + \frac{2}{ip^3} (2 \cos p)$$

$$= \left[2x \frac{e^{ipx}}{i^2 p^2} - 2 \frac{e^{ipx}}{i^3 p^3} \right]_{-1}^1$$

$$= \left(2 \frac{e^{ip}}{-p^2} - 2 \frac{e^{ip}}{-ip^3} \right) - \left(2(-1) \frac{e^{-ip}}{-p^2} - 2 \frac{e^{-ip}}{-ip^3} \right)$$

$$= \left(2 \frac{e^{ip}}{-p^2} + 2 \frac{e^{-ip}}{-p^2} \right) - 2 \left(\frac{e^{ip}}{-ip^3} - \frac{e^{-ip}}{-ip^3} \right)$$

$$= 2 \left(\frac{2 \cos p}{-p^2} \right) - 2 \left(\frac{2i \sin p}{-p^3} \right)$$

$$= -\frac{4 \cos p}{p^2} + \frac{4 \sin p}{p^3}$$

$$= -\frac{4}{p^3} (p \cos p - \sin p) = \bar{F}(p)$$

Now by inversion formula, we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{F}(p) e^{-ipx} dp$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} -\frac{4}{p^3} (p \cos p - \sin p) e^{-ipx} dp = \begin{cases} 1-x^2; |x| \leq 1 \\ 0; |x| > 1 \end{cases} \quad \text{--- (1)}$$

Putting $x = \frac{1}{2}$ in (1), we get

$$= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4}{p^3} (p \cos p - \sin p) e^{-ip/2} dp = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\frac{3}{4} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4}{p^3} (p \cos p - \sin p) \left(\cos \frac{p}{2} - i \sin \frac{p}{2} \right) dp$$

$$= \frac{3\pi}{8} = \int_{-\infty}^{\infty} \frac{1}{p^3} (p \cos p - \sin p) \left(\cos \frac{p}{2} - i \sin \frac{p}{2} \right) dp$$

$$\int_{-\infty}^{\infty} \frac{p \cos p - \sin p}{p^3} \cos \frac{p}{2} dp = -\frac{3\pi}{8} \quad \left[\text{Equating real parts} \right]$$

$$2 \int_0^{\infty} \frac{p \cos p - \sin p}{p^3} \cos \frac{p}{2} dp = -\frac{3\pi}{8}$$

$$\int_0^{\infty} \frac{x \cos x - \sin x}{x^3} \cos \frac{x}{2} dx = -\frac{3\pi}{16}$$

Putting $x=0$ in (1) we get.

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} -\frac{4}{p^3} (p \cos p - \sin p) dp = 1.$$

$$-\frac{2}{\pi} \int_{-\infty}^{\infty} \frac{1}{p^3} (p \cos p - \sin p) dp = 1.$$

$$2 \int_0^{\infty} \frac{p \cos p - \sin p}{p^3} dp = -\pi/2 \quad [\because \text{Integrand is even}]$$

$$\text{or} \int_0^{\infty} \frac{x \cos x - \sin x}{x^3} dx = -\frac{\pi}{4}$$

$$(\text{os}) \int_0^{\infty} \frac{\sin x - x \cos x}{x^3} dx = \frac{\pi}{4}$$

Prob:- Find the Fourier Transform of $f(x)$ defined by

$$f(x) = e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty. \text{ Show that the Fourier}$$

Transform of $e^{-\frac{x^2}{2}}$ is reciprocal.

Sol:- we have $F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{ipx} dx.$

$$= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{ipx} dx = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-ip)^2 - \frac{p^2}{2}} dx.$$

Put $\frac{1}{\sqrt{2}}(x-ip) = t$ so that $\frac{dx}{\sqrt{2}} = dt \Rightarrow dx = \sqrt{2} dt$

$$F\{f(x)\} = e^{-\frac{p^2}{2}} \int_{-\infty}^{\infty} e^{-t^2} \sqrt{2} dt.$$

$$= \sqrt{2} e^{-\frac{p^2}{2}} \int_{-\infty}^{\infty} e^{-t^2} dt.$$

$$= \sqrt{2} e^{-\frac{p^2}{2}} \sqrt{\pi}$$

$$= \sqrt{2\pi} e^{-\frac{p^2}{2}}.$$

Thus the Fourier transform of $e^{-\frac{x^2}{2}}$ is $e^{-\frac{p^2}{2}}$

Hence $f(x)$ is "self reciprocal".

If the function $f(x)$ is equal to $\bar{F}(p)$ then the function $f(x)$ is called self-reciprocal.

Sol: we have

$$|P| = \begin{cases} -P & \text{if } P < 0 \\ P & \text{if } P \geq 0. \end{cases}$$

From the inverse Fourier Transform, we have.

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ipx} \bar{F}(p) dp \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ipx} e^{-|p|y} dp \\ &= \frac{1}{2\pi} \left[\int_{-\infty}^0 e^{-ipx} e^{py} dp + \int_0^{\infty} e^{-ipx} e^{-py} dp \right] \\ &= \frac{1}{2\pi} \left[\int_{-\infty}^0 e^{(y-ix)p} dp + \int_0^{\infty} e^{-(y+ix)p} dp \right] \\ &= \frac{1}{2\pi} \left[\frac{e^{(y-ix)p}}{y-ix} \right]_{-\infty}^0 + \frac{1}{2\pi} \left[\frac{e^{-(y+ix)p}}{-(y+ix)} \right]_0^{\infty} \\ &= \frac{1}{2\pi} \left[\frac{1}{y-ix} \right] + \frac{1}{2\pi} \left[\frac{1}{y+ix} \right] \\ &= \frac{1}{2\pi} \left[\frac{y+ix+y-ix}{y^2+x^2} \right] = \frac{y}{\pi(y^2+x^2)} \end{aligned}$$

Pb:-

show that the Fourier Transform of $f(x) = a - |x|$ for $|x| < a$ is $\frac{1 - \cos ax}{p^2}$. Hence deduce that $\int_0^{\infty} \left(\frac{\sin t}{t} \right)^2 dt = \frac{\pi}{2}$.

Sol:-

$$\begin{aligned} \text{Given } f(x) &= a - |x| \text{ for } |x| < a \\ &= 0 \text{ for } |x| > a > 0. \end{aligned}$$

The Fourier transform of $f(x)$ is

$$\bar{F}(p) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{ipx} dx.$$

$$F\{f(x)\} = \int_{-a}^a f(x) e^{ipx} dx + \int_{-a}^a f(x) e^{ipx} dx + \int_a^\infty f(x) e^{ipx} dx.$$

$$= \int_{-a}^a f(x) e^{ipx} dx$$

$$= \int_{-a}^a (a - |x|) e^{ipx} dx.$$

$$\left[\int_{-a}^a f(x) dx = 0 \right. \\ \left. \int_{-\infty}^{\infty} f(x) dx = 0 \right]$$

$$= \int_{-a}^a a e^{ipx} dx - \int_{-a}^a |x| e^{ipx} dx.$$

$$= a \int_{-a}^a e^{ipx} dx - \left[\int_{-a}^0 |x| e^{ipx} dx + \int_0^a |x| e^{ipx} dx \right]$$

$$= a \left[\frac{e^{ipx}}{ip} \right]_{-a}^a - \left[\int_{-a}^0 -x e^{ipx} dx + \int_0^a x e^{ipx} dx \right] \quad \begin{matrix} |x| = x \text{ for } x > 0 \\ |x| = -x \text{ for } x < 0 \end{matrix}$$

$$= a \left[\frac{e^{ipa} - e^{-ipa}}{ip} \right] + \left[x \frac{e^{ipx}}{ip} - \frac{e^{ipx}}{(ip)^2} \right]_{-a}^0 - \left[x \frac{e^{ipx}}{ip} - \frac{e^{ipx}}{(ip)^2} \right]_0^a$$

$$= a \left(\frac{2i \sin ap}{ip} \right) + \left[\left(0 + \frac{1}{p^2} \right) + \left(a \frac{e^{-ipa}}{ip} - \frac{e^{-ipa}}{p^2} \right) \right] \\ - \left[a \frac{e^{ipa}}{ip} + \frac{e^{ipa}}{p^2} \right] - \left(0 + \frac{1}{p^2} \right)$$

$$= \frac{2a \sin ap}{p} - a \left(\frac{e^{ipa} - e^{-ipa}}{ip} \right) - \left(\frac{e^{ipa} + e^{-ipa}}{p^2} \right) + \frac{2}{p^2}$$

$$= \frac{2a \sin ap}{p} - a \left(\frac{2i \sin ap}{ip} \right) - \frac{2 \cos ap}{p^2} + \frac{2}{p^2}$$

$$\bar{F}(p) = \frac{2}{p^2} (1 - \cos ap). \quad (p \neq 0).$$

By the inversion formula, we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ipx} \bar{F}(p) dp.$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ipx} \frac{2}{p^2} (1 - \cos ap) dp.$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{p^2} (1 - \cos ap) e^{-ipx} dp.$$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{p^2} (1 - \cos ap) e^{-ipx} dp = \begin{cases} a - |x| & \text{for } |x| < a \\ 0 & \text{for } |x| > a > 0. \end{cases}$$

Deduction :-

Taking $a=2$ and $x=0$ we get

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(1 - \cos 2p)}{p^2} \cdot 1 \cdot dp = 2$$

$$\frac{2}{\pi} \int_0^{\infty} \left(\frac{1 - \cos 2p}{p^2} \right) dp = 2.$$

$$= \int_0^{\infty} \frac{2 \sin^2 p}{p^2} dp = \pi.$$

$$\Rightarrow \int_0^{\infty} \left(\frac{\sin p}{p} \right)^2 dp = \frac{\pi}{2}.$$

Replacing 'p' by 't' we have.

$$\int_0^{\infty} \left(\frac{\sin t}{t} \right)^2 dt = \frac{\pi}{2}.$$

Pb.

Find the Fourier transform of $\frac{1}{\sqrt{|x|}}$

Sol:

$$\text{Let } f(x) = \frac{1}{\sqrt{|x|}}$$

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{ipx} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{|x|}} e^{ipx} dx.$$

$$= \int_{-\infty}^0 \frac{1}{\sqrt{-x}} e^{ipx} dx + \int_0^{\infty} \frac{1}{\sqrt{x}} e^{ipx} dx.$$

Put $-x=t$ in the first integral $dx = -dt$.

Also $x=0, t=0$ and $x=-\infty, t=\infty$.

$$F\{f(x)\} = \int_{-\infty}^0 \frac{1}{\sqrt{t}} e^{-ipt} (-dt) + \int_0^{\infty} \frac{1}{\sqrt{x}} e^{ipx} dx.$$

$$F\{f(x)\} = \int_0^{\infty} \frac{1}{\sqrt{x}} e^{-ipx} dx + \int_0^{\infty} \frac{1}{\sqrt{x}} e^{ipx} dx$$

$$= \int_0^{\infty} \frac{1}{\sqrt{x}} e^{-ipx} dx + \int_0^{\infty} \frac{1}{\sqrt{x}} e^{ipx} dx$$

$$= \int_0^{\infty} \frac{e^{ipx} + e^{-ipx}}{\sqrt{x}} dx = \int_0^{\infty} \frac{2 \cos px}{\sqrt{x}} dx$$

$$= \text{R.P. of } \int_0^{\infty} \frac{e^{-ipx}}{\sqrt{x}} \cdot \frac{d(ipx)}{ip}$$

$$= \text{R.P. of } \int_0^{\infty} \frac{e^{-ipx}}{\sqrt{ipx}} \cdot \frac{d(ipx)}{\sqrt{ip}}$$

$$F\{f(x)\} = \text{R.P. of } \frac{2}{\sqrt{ip}} \int_0^{\infty} \frac{e^{-t}}{\sqrt{t}} dt \quad (\text{putting } ipx = t)$$

$$= \text{R.P. of } \frac{2}{\sqrt{ip}} \int_0^{\infty} e^{-t} t^{\frac{1}{2}-1} dt$$

$$= \text{R.P. of } \frac{2}{\sqrt{p}} i^{-1/2} \Gamma\left(\frac{1}{2}\right) \quad [\text{using definition of Gamma function}]$$

$$= \text{R.P. of } \frac{2}{\sqrt{p}} \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)^{-1/2} \sqrt{\pi}$$

$$= \text{R.P. of } \frac{2\sqrt{\pi}}{\sqrt{p}} \left(\cos \frac{\pi}{4} - i \sin \frac{\pi}{4} \right)$$

$$= \frac{2\sqrt{\pi}}{\sqrt{p}} \cos \frac{\pi}{4}$$

$$F\{f(x)\} = \frac{2\sqrt{\pi}}{\sqrt{p}} \cdot \frac{1}{\sqrt{2}}$$

$$F\{f(x)\} = \sqrt{\frac{2\pi}{p}}$$

Pb

Find the Fourier transform of $f(x) = x$ if $|x| \leq a$
 $= 0$ if $|x| > a$.

sol:

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{ipx} dx.$$

$f(x)$ in $-a \leq x \leq a$ and $f(x) = 0$ in $-\infty < x < -a$
 and $a < x < \infty$.

$$F\{f(x)\} = \int_{-\infty}^{-a} f(x) e^{ipx} dx + \int_{-a}^a f(x) e^{ipx} dx + \int_a^{\infty} f(x) e^{ipx} dx.$$

$$= \int_{-\infty}^{-a} 0 \cdot e^{ipx} dx + \int_{-a}^a x e^{ipx} dx + \int_a^{\infty} 0 \cdot e^{ipx} dx.$$

$$= \int_{-a}^a x e^{ipx} dx.$$

$$= \left[x \frac{e^{ipx}}{ip} - 1 \cdot \frac{e^{ipx}}{i^2 p^2} \right]_{-a}^a = \left[x \frac{e^{ipx}}{ip} + \frac{e^{ipx}}{p^2} \right]_{-a}^a.$$

$$= \left(a \frac{e^{ipa}}{ip} + \frac{e^{ipa}}{p^2} \right) - \left(-a \frac{e^{-ipa}}{ip} + \frac{e^{-ipa}}{p^2} \right).$$

$$= \frac{a}{ip} (e^{ipa} + e^{-ipa}) + \frac{1}{p^2} (e^{ipa} - e^{-ipa}).$$

$$= \frac{a}{ip} (2 \cos pa) + \frac{1}{p^2} (2i \sin pa).$$

$$= 2 \left(\frac{a \cos pa}{ip} + \frac{i \sin pa}{p^2} \right).$$

$$\text{Hence } F\{f(x)\} = 2 \left[\frac{iap \cos pa}{(ip)^2} + \frac{i \sin pa}{p^2} \right]$$

$$= \frac{2i}{p^2} (\sin pa - a p \cos pa).$$

Pb

Find the Fourier transform of $f(x) = \cos x$ for $|x| \leq a$
 $= 0$ for $|x| > a > 0$.

$$\text{sol: } F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{ipx} dx = \int_{-\infty}^{-a} f(x) e^{ipx} dx + \int_{-a}^a f(x) e^{ipx} dx + \int_a^{\infty} f(x) e^{ipx} dx$$

$$= \int_{-a}^{-a} 0 \cdot e^{ipx} dx + \int_{-a}^a \cos x \cdot e^{ipx} dx + \int_a^{\infty} 0 \cdot e^{ipx} dx$$

$$= \int_{-a}^a e^{ipx} \cos x dx = \left[\frac{e^{ipx}}{ip^2+1} (ip \cos x + \sin x) \right]_{-a}^a$$

$$= \frac{e^{ipa}}{1-p^2} (ip \cos a + \sin a) - \frac{e^{-ipa}}{1-p^2} (ip \cos a + \sin a)$$

$$= \frac{1}{1-p^2} [(e^{ipa} - e^{-ipa}) ip \cos a + (e^{ipa} + e^{-ipa}) \sin a]$$

$$= \frac{1}{1-p^2} [(2i \sin pa)(ip \cos a) + 2 \cos pa \sin a]$$

$$= \frac{2}{1-p^2} [\sin a \cos pa - p \cos a \sin pa]$$

Pb

Find the Fourier Transform of $f(x) = \frac{1}{2a}$ for $|x| \leq a$
 $= 0$ for $|x| > a$.

Sol

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{ipx} dx$$

$$= \int_{-\infty}^{-a} f(x) e^{ipx} dx + \int_{-a}^a f(x) e^{ipx} dx + \int_a^{\infty} f(x) e^{ipx} dx$$

$$= \int_{-\infty}^{-a} 0 \cdot e^{ipx} dx + \int_{-a}^a \frac{1}{2a} e^{ipx} dx + \int_a^{\infty} 0 \cdot e^{ipx} dx$$

$$F\{f(x)\} = \frac{1}{2a} \int_{-a}^a e^{ipx} dx = \frac{1}{2a} \left[\frac{e^{ipx}}{ip} \right]_{-a}^a$$

$$= \frac{1}{2aip} [e^{ipa} - e^{-ipa}]$$

$$= \frac{1}{ap} \left[\frac{e^{ipa} - e^{-ipa}}{2i} \right] = \frac{\sin pa}{pa}$$

Pb

find the Fourier Transform of $e^{-a|x|}$ ($a > 0$)

hence show that $\int_0^{\infty} \frac{\cos px}{a^2 + p^2} dp = \frac{\pi}{2a} e^{-a|x|}$

$$F\{e^{-a|x|}\} = \int_{-\infty}^{\infty} e^{-a|x|} e^{ipx} dx$$

$$= \int_{-\infty}^{\infty} e^{-a|x|} (\cos px + i \sin px) dx$$

$$= \int_{-\infty}^{\infty} e^{-a|x|} \cos px dx + i \int_{-\infty}^{\infty} e^{-a|x|} \sin px dx$$

$$= 2 \int_0^{\infty} e^{-ax} \cos px dx + i(0)$$

[Integrand in first integral is even and odd in second integral]

$$= 2 \int_0^{\infty} e^{-ax} \cos px dx$$

$$= 2 \left[\frac{e^{-ax}}{a^2 + p^2} (-a \cos px + p \sin px) \right]_0^{\infty}$$

$$= 2 \left[0 - \frac{(-a)}{a^2 + p^2} \right] = 2 \cdot \frac{a}{a^2 + p^2} = \frac{2a}{a^2 + p^2}$$

Deduction

(i) By inversion formula, we have

$$e^{-a|x|} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2a}{a^2 + p^2} e^{-ipx} dp$$

$$= \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{1}{a^2 + p^2} (\cos px - i \sin px) dp$$

$$e^{-a|x|} = \frac{2a}{\pi} \int_0^{\infty} \frac{\cos px}{a^2 + p^2} dp$$

$$\int_0^{\infty} \frac{\cos px}{a^2 + p^2} dx = \frac{\pi}{2a} e^{-a|x|}$$

Put $a=1$ in the above problem

$$\int_0^{\infty} \frac{\cos px}{1 + p^2} dx = \frac{\pi}{2} e^{-|x|}$$

Prob

Find the Fourier transform of $f(x) = a^2 - x^2$ if $|x| < a$
 0 if $|x| > a > 0$.

Hence show that $\int_0^{\infty} \frac{\sin x - x \cos x}{x^3} dx = \frac{\pi}{4}$

sol

Fourier Transform of a function $f(x)$ is given by.

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{ipx} dx = \int_{-\infty}^{-a} f(x) e^{ipx} dx + \int_{-a}^a f(x) e^{ipx} dx + \int_a^{\infty} f(x) e^{ipx} dx$$

$$= \int_{-\infty}^{-a} 0 \cdot e^{ipx} dx + \int_{-a}^a (a^2 - x^2) e^{ipx} dx + \int_a^{\infty} 0 \cdot e^{ipx} dx$$

$$= \int_{-a}^a (a^2 - x^2) e^{ipx} dx = \left[(a^2 - x^2) \frac{e^{ipx}}{ip} - (-2x) \frac{e^{ipx}}{ip^2} + (-2) \frac{e^{ipx}}{(ip)^3} \right]_{-a}^a$$

$$= -\frac{2a}{p^2} e^{ipa} - \frac{2ie^{ipa}}{p^3} - \frac{2ae^{-ipa}}{p^2} + \frac{2ie^{-ipa}}{p^3}$$

$$= -\frac{2a}{p^2} [e^{ipa} + e^{-ipa}] + \frac{2i}{p^3} [e^{-ipa} - e^{ipa}]$$

$$= -\frac{2a}{p^2} [e^{ipa} + e^{-ipa}] - \frac{2i}{p^3} [e^{ipa} - e^{-ipa}]$$

$$= -\frac{2a}{p^2} [2 \cos ap] - \frac{2i}{p^3} [2i \sin ap]$$

$$F\{f(x)\} = -\frac{4a \cos ap}{p^2} + \frac{4}{p^3} \sin ap$$

$$= \frac{4}{p^3} [\sin ap - ap \cos ap] = \bar{F}(p) \text{ say.}$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{F}(p) e^{-ipx} dp$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4}{p^3} (\sin ap - ap \cos ap) e^{-ipx} dp$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin ap - ap \cos ap}{p^3} \right) (\cos p x - i \sin p x) dx$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin ap - ap \cos ap}{p^3} \cos p x dx$$

The second integral is zero because the integrand is odd.

$$f(n) = \frac{4}{\pi} \int_0^{\infty} \frac{\sin ap - a \cos ap}{p^3} \cos pn \, dp.$$

Put $a=1$ in above equation, we get

$$f(n) = \frac{4}{\pi} \int_0^{\infty} \frac{\sin p - p \cos p}{p^3} \cos pn \, dp.$$

Putting $x=0$ in above eqn.

$$f(0) = \frac{4}{\pi} \int_0^{\infty} \frac{\sin p - p \cos p}{p^3} \, dp$$

$$1 = \frac{4}{\pi} \int_0^{\infty} \frac{\sin p - p \cos p}{p^3} \, dp.$$

$$\frac{\pi}{4} = \int_0^{\infty} \frac{\sin p - p \cos p}{p^3} \, dp.$$

$$(or) \int_0^{\infty} \frac{\sin x - x \cos x}{x^3} \, dx = \frac{\pi}{4}.$$

Problems Related to Infinite Fourier Sine and Cosine Transforms.

Pb Find the Fourier cosine transform of the function $f(x)$ defined by $f(x) = \cos x$; $0 < x < a$
 $= 0$; $x \geq a$.

Sol

$$\begin{aligned} \text{we have } F_c\{f(x)\} &= \int_0^{\infty} f(x) \cos px \, dx \\ &= \int_0^a \cos x \cos px \, dx + \int_a^{\infty} 0 \cdot \cos px \, dx \\ &= \frac{1}{2} \int_0^a 2 \cos x \cos px \, dx = \frac{1}{2} \int_0^a [\cos(1+p)x + \cos(1-p)x] \, dx \\ &= \frac{1}{2} \left[\frac{\sin(1+p)x}{1+p} + \frac{\sin(1-p)x}{1-p} \right]_0^a \\ &= \frac{1}{2} \left[\frac{\sin(1+p)a}{1+p} + \frac{\sin(1-p)a}{1-p} \right] \end{aligned}$$

Pb Find Fourier transform of $f(x) = \sin x$; $0 < x < a$
 $= 0$; $x \geq a$.

(ii) $f(x) = e^{-x}$.

Sol

$$\begin{aligned} F_s\{f(x)\} &= \int_0^{\infty} f(x) \sin px \, dx = \int_0^a f(x) \sin px \, dx + \int_a^{\infty} f(x) \sin px \, dx \\ &= \int_0^a \sin x \sin px \, dx + \int_a^{\infty} 0 \cdot \sin px \, dx \\ &= \frac{1}{2} \int_0^a 2 \sin x \sin px \, dx = \frac{1}{2} \int_0^a [\cos(1-p)x - \cos(1+p)x] \, dx \\ &= \frac{1}{2} \left[\frac{\sin(1-p)x}{1-p} - \frac{\sin(1+p)x}{1+p} \right]_0^a = \frac{1}{2} \left[\frac{\sin(1-p)a}{1-p} - \frac{\sin(1+p)a}{1+p} \right] \end{aligned}$$

(ii) $F_s\{e^{-x}\} = \int_0^{\infty} e^{-x} \sin px \, dx$.

$$= \frac{e^{-x}}{1+p^2} [-\sin px - p \cos px]_0^{\infty}$$

$$= \frac{p}{1+p^2}$$

Pb Find the Fourier sine and cosine transforms

of $f(x) = \frac{e^{-ax}}{x}$ and deduce that $\int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} \sin sx dx$
 $= \tan^{-1}\left(\frac{s}{a}\right) - \tan^{-1}\left(\frac{s}{b}\right)$

Sol

Fourier sine Transform:

we have $F_s\{f(x)\} = \int_0^{\infty} f(x) \sin px dx$

$$= \int_0^{\infty} \frac{e^{-ax}}{x} \sin px dx.$$

Differentiating both sides w.r.t to 'p', we get.

$$\frac{d}{dp} [F_s\{f(x)\}] = \int_0^{\infty} \frac{x e^{-ax} \cos px}{x} dx.$$

$$= \int_0^{\infty} e^{-ax} \cos px dx = \frac{a}{a^2 + p^2}$$

$$\frac{d}{dp} [F_s\{f(x)\}] = \frac{a}{a^2 + p^2}$$

Integrating w.r.t to 'p'

$$F_s\{f(x)\} = \int \frac{a}{a^2 + p^2} dp = \tan^{-1}\left(\frac{p}{a}\right) + c.$$

$$F_s\{f(x)\} = \tan^{-1}\left(\frac{p}{a}\right) + c.$$

If $p=0$ then $F_s\{f(x)\} = 0$ and therefore $c=0$.

$$F_s\{f(x)\} = \tan^{-1}\left(\frac{p}{a}\right). \quad \text{--- (A)}$$

we know that Fourier sine transform of $f(x)$ is

$$\text{given by } F_s\{f(x)\} = \int_0^{\infty} f(x) \sin px dx.$$

$$F_s\{f(x)\} = \int_0^{\infty} f(x) \sin px dx.$$

$$= \int_0^{\infty} f(x) \sin sx dx. \quad \text{--- (1) } [\because p=s]$$

$$\text{Let } f(x) = \frac{e^{-ax} - e^{-bx}}{x} \quad \text{--- (2)}$$

Substituting ② in ①, we get ..

$$\begin{aligned} \int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} \sin sx \, dx &= F_s \left[\frac{e^{-ax} - e^{-bx}}{x} \right] \\ &= F_s \left[\frac{e^{-ax}}{x} \right] - F_s \left[\frac{e^{-bx}}{x} \right] \\ &= \tan^{-1} \left(\frac{s}{a} \right) - \tan^{-1} \left(\frac{s}{b} \right). \end{aligned}$$

Fourier cosine Transform

$$\begin{aligned} F_c \{ f(x) \} &= \int_0^{\infty} f(x) \cos px \, dx \\ &= \int_0^{\infty} \frac{e^{-ax}}{x} \cos px \, dx \end{aligned}$$

$$\left[\int_0^{\infty} e^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2} \right]$$

$$\begin{aligned} \frac{d}{dp} [F_c \{ f(x) \}] &= \int_0^{\infty} \frac{e^{-ax}}{x} (-\sin px) x \, dx \\ &= - \int_0^{\infty} e^{-ax} \sin px \, dx = - \frac{p}{a^2 + p^2} \end{aligned}$$

Integrating w.r.t to 'p' we get.

$$\begin{aligned} F_c \{ f(x) \} &= - \int \frac{p}{a^2 + p^2} dp = - \frac{1}{2} \int \frac{2p}{a^2 + p^2} dp \\ &= - \frac{1}{2} \log (p^2 + a^2) + c. \end{aligned}$$

Pb

Find the Fourier cosine transform of $f(x) = \frac{1}{1+x^2}$ and hence find Fourier sine transform of

$$f(x) = \frac{x}{1+x^2}$$

Sol

$$\begin{aligned} \text{Let } I &= F_c \{ f(x) \} = \int_0^{\infty} f(x) \cos px \, dx \\ &= \int_0^{\infty} \frac{\cos px}{1+x^2} dx. \end{aligned}$$

Differentiating both sides w.r.t to 'p' we get.

$$\frac{dI}{dp} = \frac{d}{dp} \{ F_c \{ f(x) \} \} = - \int_0^{\infty} \frac{x \sin px}{1+x^2} dx. \quad \text{--- (A)}$$

$$= - \int_0^{\infty} \frac{x^2 \sin px}{x(1+x^2)} dx = - \int_0^{\infty} \frac{(x^2 + 1 - 1) \sin px}{x(1+x^2)} dx.$$

$$= \int_0^{\infty} \left[\frac{1}{x} - \frac{1}{x(1+x^2)} \right] \sin px \, dx.$$

$$= - \int_0^{\infty} \frac{\sin px}{x} \, dx + \int_0^{\infty} \frac{\sin px}{x(1+x^2)} \, dx.$$

$$= -\frac{\pi}{2} + \int_0^{\infty} \frac{\sin px}{x(1+x^2)} \, dx. \quad \text{--- (1)}$$

Differentiating again w.r.t 'p' we get

$$\frac{d^2}{dp^2} [F_c\{f(x)\}] = \int_0^{\infty} \frac{\cos px}{1+x^2} \, dx = F_c\{f(x)\}$$

$$\text{which gives } (D^2-1) F_c\{f(x)\} = 0.$$

general solution is $F_c\{f(x)\} = c_1 e^p + c_2 e^{-p}$ --- (2)
now when $p=0$

$$F_c\{f(x)\} = \int_0^{\infty} \frac{1}{1+x^2} \, dx = \left[\tan^{-1} x \right]_0^{\infty} = \tan^{-1} \infty - \tan^{-1} 0 = \frac{\pi}{2}$$

$$\text{Also } \frac{d}{dp} [F_c\{f(x)\}] = -\frac{\pi}{2} \quad [\text{from (1)}]$$

$$\text{From (2) we have } c_1 + c_2 = \frac{\pi}{2}$$

$$c_1 - c_2 = -\frac{\pi}{2}$$

$$\text{By solving } c_1 = 0, c_2 = \frac{\pi}{2}$$

$$\text{then from (2) we have } F_c\{f(x)\} = I = \frac{\pi}{2} e^{-p}$$

second part

$$\frac{dI}{dp} = -\frac{\pi}{2} e^{-p}$$

so from (A) we get

$$F_s\left\{\frac{-x}{1+x^2}\right\} = -\frac{\pi}{2} e^{-p} \quad (\text{or}) \quad F_s\left\{\frac{x}{1+x^2}\right\} = \frac{\pi}{2} e^{-p}$$

Prob Find Fourier cosine and sine transform of e^{-ax} ($a > 0$) and hence deduce the inversion formula

(or) Deduce the integrals (i) $\int_0^{\infty} \frac{\cos px}{a^2 + p^2} dp$ (ii) $\int_0^{\infty} \frac{p \sin px}{a^2 + p^2} dp$

Sol Let $f(x) = e^{-ax}$

$$F_c \{ f(x) \} = \int_0^{\infty} f(x) \cos px \, dx = \int_0^{\infty} e^{-ax} \cos px \, dx$$

$$= \left[\frac{e^{-ax}}{a^2 + p^2} (-a \cos px + p \sin px) \right]_0^{\infty}$$

$$F_c \{ e^{-ax} \} = \frac{a}{a^2 + p^2}$$

$$F_s \{ f(x) \} = \int_0^{\infty} f(x) \sin px \, dx = \int_0^{\infty} e^{-ax} \sin px \, dx$$

$$= \left[\frac{e^{-ax}}{a^2 + p^2} (-a \sin px - p \cos px) \right]_0^{\infty}$$

$$F_s \{ e^{-ax} \} = \frac{p}{a^2 + p^2}$$

(ii) Now by the inverse Fourier cosine transform

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_c \{ f(x) \} \cos px \, dp$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{a}{a^2 + p^2} \cos px \, dp$$

Since $f(x) = e^{-ax}$

$$e^{-ax} = \frac{2a}{\pi} \int_0^{\infty} \frac{\cos px}{a^2 + p^2} dp$$

$$\frac{\pi}{2a} e^{-ax} = \int_0^{\infty} \frac{\cos px}{a^2 + p^2} dp$$

Now by the inverse Fourier sine transform

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_s \{ f(x) \} \sin px \, dp = \frac{2}{\pi} \int_0^{\infty} \frac{p}{a^2 + p^2} \sin px \, dp$$

Since $f(x) = e^{-ax}$, the above integral takes the form

$$\int_0^{\infty} \frac{p \sin px}{a^2 + p^2} dp = \frac{\pi}{2} e^{-ax}$$

Pbshow that (i) $F_s \{ x f(x) \} = -\frac{d}{dp} [F_c(p)]$

(ii) $F_c \{ x f(x) \} = \frac{d}{dp} [F_s(p)]$

Sol

(i) we have $F_c(p) = \int_0^{\infty} f(x) \cos px \, dx$.

Diff. w.r.t to 'p' we get

$$\frac{d}{dp} [F_c(p)] = \int_0^{\infty} f(x) (-x \sin px) \, dx$$

$$= - \int_0^{\infty} [x f(x)] \sin px \, dx$$

$$= -F_s \{ x f(x) \}$$

$$F_s \{ x f(x) \} = -\frac{d}{dp} [F_c(p)]$$

(ii) we have $F_s(p) = \int_0^{\infty} f(x) \sin px \, dx$.

Diff. w.r.t to 'p', we get

$$\frac{d}{dp} [F_s(p)] = \int_0^{\infty} f(x) (x \cos px) \, dx$$

$$= \int_0^{\infty} [x f(x)] \cos px \, dx = F_c \{ x f(x) \}$$

$$F_c \{ x f(x) \} = \frac{d}{dp} [F_s(p)]$$

PbFind the Fourier sine transform of $e^{-|x|}$ and hence evaluate $\int_0^{\infty} \frac{x \sin mx}{1+x^2} \, dx$ Sol

Let $f(x) = e^{-|x|}$

$$F_s \{ f(x) \} = \int_0^{\infty} f(x) \sin px \, dx$$

$$= \int_0^{\infty} e^{-|x|} \sin px \, dx = \int_0^{\infty} e^{-x} \sin px \, dx \quad [\because |x| = x \text{ for } x > 0]$$

$$= e^{-x} \left(-\sin px - p \cos px \right) \Big|_0^{\infty} = \frac{p}{1+p^2}$$

Taking the inverse sine transform

$$\begin{aligned} \bar{e}^{-x} = f(x) &= \frac{2}{\pi} \int_0^{\infty} F_s \{f(x)\} \sin px \, dp \\ &= \frac{2}{\pi} \int_0^{\infty} \frac{p}{1+p^2} \sin px \, dp. \end{aligned}$$

changing x to ' m ' $\bar{e}^{-x} = \frac{2}{\pi} \int_0^{\infty} \frac{p \sin pm}{1+p^2} \, dp.$

(or) $\bar{e}^{-x} = \frac{2}{\pi} \int_0^{\infty} \frac{x \sin xm}{1+x^2} \, dx$, where ' p ' is replaced by ' x '.

$$\int_0^{\infty} \frac{x \sin xm}{1+x^2} \, dx = \frac{\pi}{2} \bar{e}^{-x}.$$

$\therefore$

Pb:- Find Fourier sine and cosine transforms of $x e^{-ax}$.

Sol:- Let $f(x) = e^{-ax}$.

Fourier sine transform:-

we know that $F_s \{x f(x)\} = -\frac{d}{dp} [F_c \{f(x)\}]$.

$$\begin{aligned} F_s \{x e^{-ax}\} &= -\frac{d}{dp} [F_c \{e^{-ax}\}] = -\frac{d}{dp} \left[\frac{a}{a^2+p^2} \right] \\ &= -a \left[-\frac{1}{(a^2+p^2)^2} \right] \cdot 2p = \frac{2ap}{(a^2+p^2)^2}. \end{aligned}$$

Fourier cosine Transform

Also we know that.

$$F_c \{x f(x)\} = \frac{d}{dp} [F_s \{f(x)\}]$$

$$F_c \{x e^{-ax}\} = \frac{d}{dp} [F_s \{e^{-ax}\}] = \frac{d}{dp} \left[\frac{p}{a^2+p^2} \right]$$

$$= \frac{(a^2+p^2) \cdot 1 - p \cdot 2p}{(a^2+p^2)^2} = \frac{a^2+p^2-2p^2}{(a^2+p^2)^2} = \frac{a^2-p^2}{(a^2+p^2)^2}.$$

$\therefore$