UNIT-1

Introduction

Definitions

Signal: A function of one (or) more independent variable which contain some information is called signal.

Ex:- Electric Voltage (or) Current, such as radio signals, TV signal.

System: A system is set of elements (or) functional block that are connected together and produce an o/p in response to an i/p signal.

Ex:- A audio amplifier, attenuator, TV set, transmitter, receiver etc.

Classification of signals:-

The signals can be classified into two parts depending upon independent variable (time)

a) Continous Time (CT) signal.

b) Discrete Time (DT) signal.

Both the CT and DT signals can be classified into following parts.

a) periodic & non-periodic signals

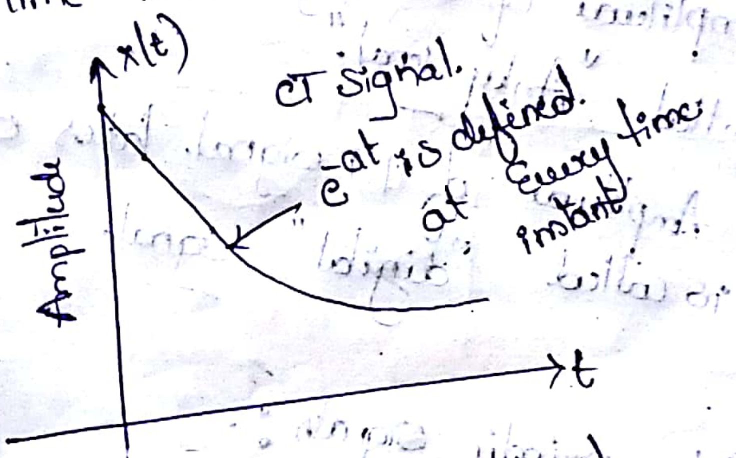
b) Even & Odd signals

c) Energy & power signals.

d) Deterministic & random signals.

<u>CT & DT signals</u> :- 1) A "CT" signal is defined continuously w.r.t time.

2) A "DT" signal is defined only at specific & regular time instant.



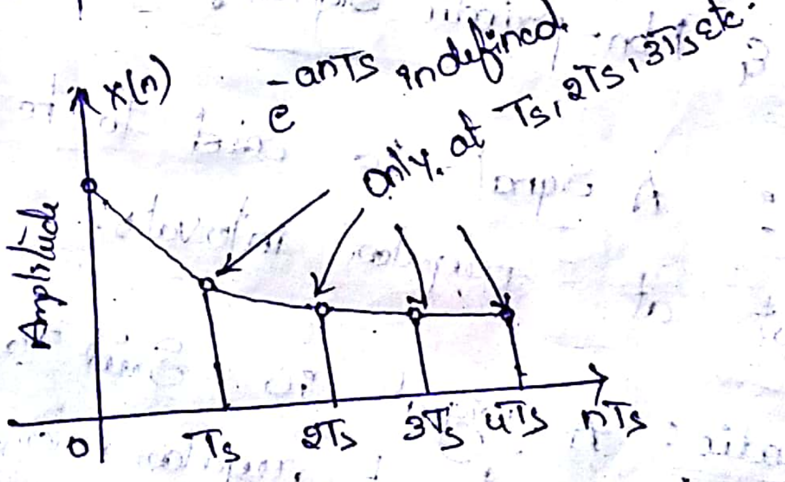


Fig: CT & DT signals.

Continous fun of time : $x(t) = e^{-at}$

~~Continous~~ Descrete fun of time : $x(n) = e^{-anTs}$

Significance:

1) Anlog Circuit process CT signal. Such ckt are op-amp, filters, amplifier etc.

2) Digital ckt process DT signal. Such circuit are micro processors, Counters, flip-flops etc.

Anlog & Digital system:-

⇒ When amplitude of CT signal varies Continously, it is called "Anlog signal."

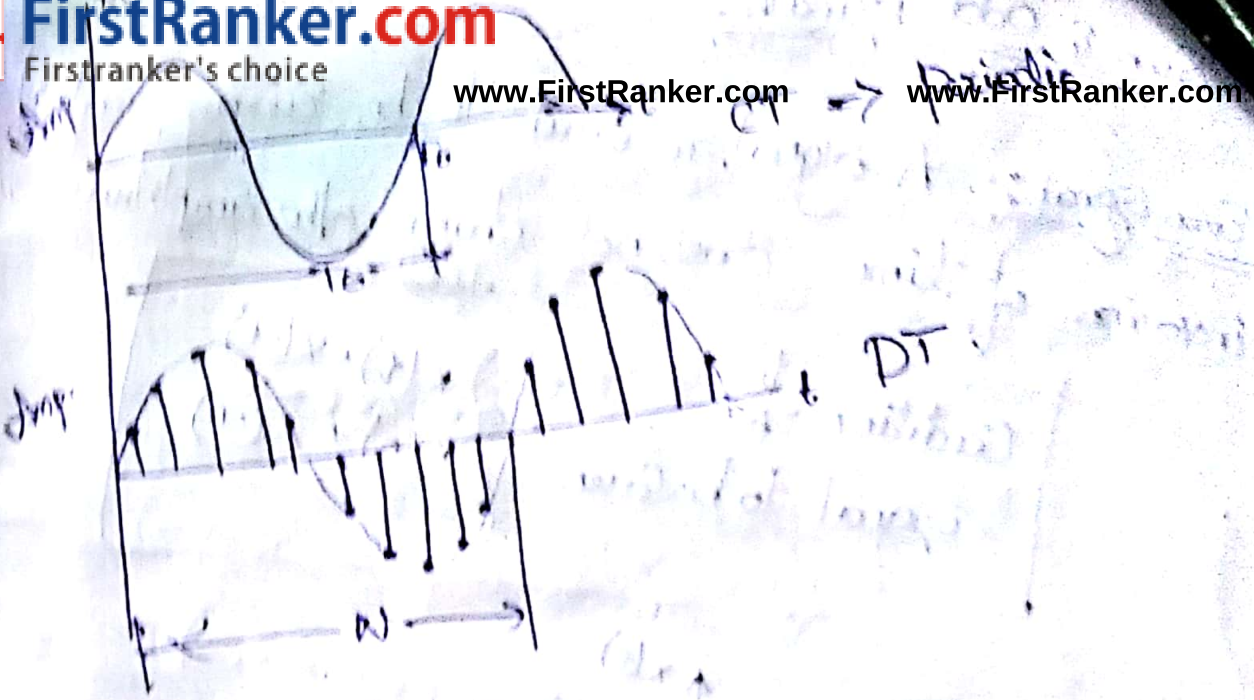
⇒ When Amplitude of DT signal takes only finite values it is called "digital" signal.

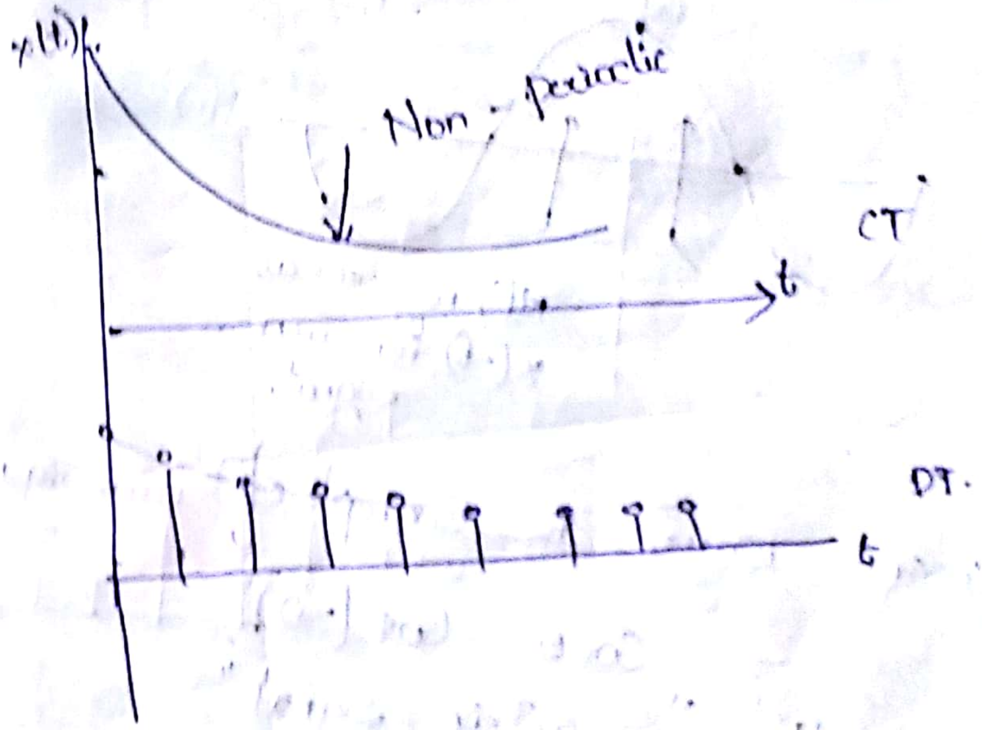
d) Periodic & Non-periodic signals:-

Periodic: A signal is said to be periodic if it repeats at regular intervals.

Non-periodic: A signal is said to be non-periodic if it not repeat at regular intervals.

Ex: CT for Periodic
DT for Non-periodic





Condition for periodicity of CT signal.

The CT signal repeat after certain period T_0 i.e

$x(t) = x(t+T_0)$ & $x(n) = x_1(t) + x_2(t)$

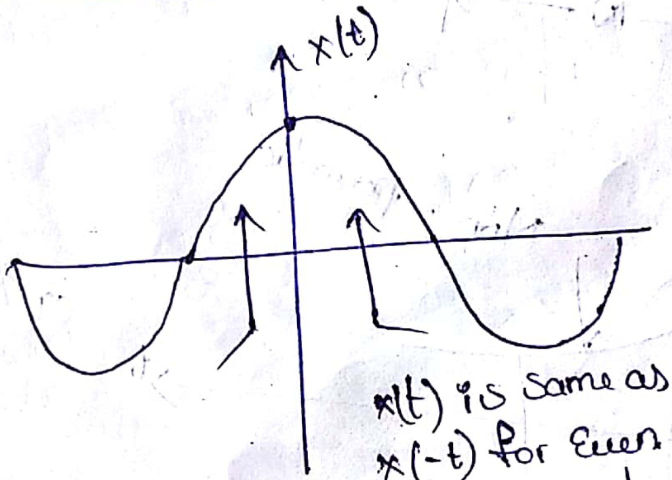
Condition for periodicity of DT signal.

Consider DT cosine wave. $x(n) = \cos(2\pi f_0 n)$

Even & Odd Signals :-

Even Signal :- A signal is said to be Even Signal if inversion of time does not change the amplitude.

Condition for signal to be Even. : $\begin{cases} x(t) = x(-t) \\ x(n) = x(-n) \end{cases}$



x(t) is same as x(-t) for Even signal.

Cosine wave is Example of Even signal

$$\cos\theta = \cos(-\theta)$$

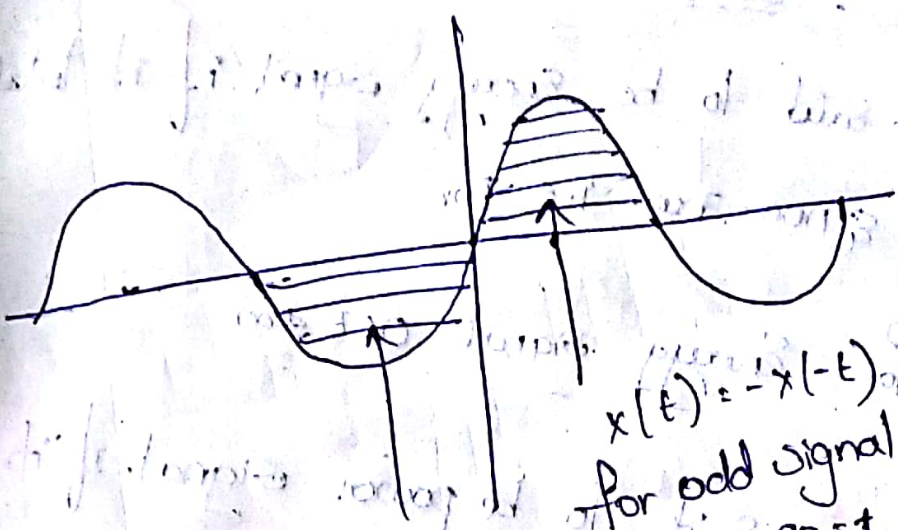
* also called "Symmetric signal"

Odd Signal :-

A signal is said to be odd signal if inversion of time axis also inverts Amplitude of the signal.

Condition for signal to be odd. $\begin{cases} x(t) = -x(-t) \\ x(n) = -x(-n) \end{cases}$

* It is also called anti-symmetric signal.



$x(t) = -x(-t)$

for odd signal

$\sin 90 = 1$

$\sin(-90) = -1$

sine wave is odd.

⇒ Sine wave is Example of odd signal

~~Sign~~ ⇒ Even & odd symmetry of the signal have specific harmonic (or) freq content.

⇒ Even & odd symmetry property is used in filter design.

$$x(t) = x_e(t) + x_o(t)$$

Even ↑ ↑ odd

Continuous time signal

Even part $= x_e(t) = \frac{1}{2}[x(t) + x(-t)]$

odd part $x_o(t) = \frac{1}{2}[x(t) - x(-t)]$

and $E = \sum_{n=-\infty}^{\infty} |x(n)|^2$ for DT signal.

Power of CT & DT Signal :-

$$\boxed{\begin{aligned} &\text{Power } P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\, dt \quad \text{for CT signal.} \\ &\& \quad P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 \quad \text{for DT signal.} \end{aligned}}$$

---o---

Deterministic and Random Signal.

→ A Deterministic signal can be completely represented by Mathematical Equation any time

Ex $x(t) = \cos \omega t$

$x(n) = \cos 2\pi f n$

→ A signal which can't be represented by any Mathematical Equ is called random signal.

Here we are taking
→ Variance
→ Co-Variance.

Problems

Determine whether, the following DT signal are Periodic (or) not ? if Periodic determine fundamental Period.

i) $\cos(0.01\pi n) = x(n)$

ii) $\cos(3\pi n)$

iii) $\sin(3n)$

iv) $\cos\frac{2\pi n}{5} + \cos\frac{2\pi n}{7}$

v) $\cos(n/8)\cos n\pi/8$

vi) $\sin(\pi + 0.2n)$

vii) $e^{(j\pi/4)n}$

i) $x(n) = \cos(0.01\pi n)$

Compare with $x(n) = \cos 2\pi f n$

$2\pi f n = 0.01\pi n$

$f = \frac{0.01}{2} = \frac{1}{200} = k/N$

Here f is expressed as ratio of two integers with $k=1$ & $N=200$

Periodic, $N=200$

ii) $x(n) = \cos(3\pi n)$

Compare with $x(n) = \cos 2\pi f n$.

$\cancel{\cos} 2\pi f \cancel{n} = \cancel{\cos}(3\pi \cancel{n})$

$f = \frac{3\cancel{\pi}}{2\cancel{\pi}} = 3/2$

$f = k/N \quad N=2$

Hence

iii) $x(n)$: ...

Compare with $x(n) = \cos 2\pi fn$.

$\cos 2\pi fn = \sin 3n$

$f = \frac{\sin 3}{\cos 2\pi} = k/N.$

Which is not ratio of two integers.

The signal is non-periodic

iv) $x(n) = \cos\frac{2\pi n}{5} + \cos\frac{2\pi n}{7}$

$x(n) = \cos 2\pi f_1 n + \cos 2\pi f_2 n.$

$2\pi f_1 n = \frac{2\pi n}{5}$

$\boxed{f = 1/5}$; $N_1 = 5$

$2\pi f_2 n = \frac{2\pi n}{7}$

$\boxed{f_2 = 1/7}$ $N_2 = 7$

$\frac{N_1}{N_2} = 5/7$ is the ratio of two integers.

The sequence is periodic. The periodic of $x(n)$ is least common multiple of N_1 & N_2. Here least common multiple of $N_1 = 5$ and $N_2 = 7$

Therefore this sequence is $\boxed{\text{Periodic with } N = 35}$

v) $x(n) = \cos(n/8)\cos\frac{n\pi}{8}$

Here $2\pi f_1 n = n/8 \Rightarrow f_1 = \frac{1}{16\pi}$ which is not rational

$2\pi f_2 n = n\pi/8 \Rightarrow f_1 = 1/16$ which is rational.

Thus $\cos(n/8)$ is non-periodic and $\cos(n\pi/8)$ is Periodic. $x(n)$ is non-periodic since it is the product of periodic & non-periodic signal.

vi) $x(n) = \sin(\pi + 0.2n)$

Compare with $x(n) = \sin(2\pi f n + \theta)$

$\theta = \pi$ i.e phase shift &

$2\pi f n = 0.2n$

$f = \frac{0.2}{2\pi} = \frac{1}{10\pi}$ which is not rational.

Hence this signal is non-periodic.

vii) $x(n) = e^{j\pi/4 n}$

$\cos \pi/4 n + j\sin\pi/4 n$

Compare with $x(n) = \cos 2\pi f n + j\sin 2\pi f n$

Here $2\pi f n = \pi/4 n \Rightarrow f = 1/8 = k/N$

which is rational

Hence this signal is

Periodic with $\boxed{N=8}$

1) Determine whether the following signals are Energy signals (or) power signals and calculate Energy (or) power.

a) $x(n) = \left(\frac{1}{2}\right)^n u(n)$ c) $x(t) = \text{rect}\left(t/T_0\right)$

b) $x(t) = \cos^2 \omega_0 t$ d) $x(t) = \text{rect}\left(t/T_0\right)\cos\omega_0 t$.

We have follow the given steps:-

① Observe the signal carefully. if it is periodic & infinite duration then it can be power signal. Hence calculate its power directly.

② if the signal is periodic but of finite duration, then it can be Energy signal. Hence calculate its Energy directly

③ if the signal is not periodic, then it can be Energy signal. Hence calculate it Energy directly..

i) $x(n) = \left(\frac{1}{2}\right)^n u(n)$.

This signal is not periodic. Hence as per step 3.

Calculate its Energy directly

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} \left[\left(\frac{1}{2}\right)^n\right]^2 = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n$$

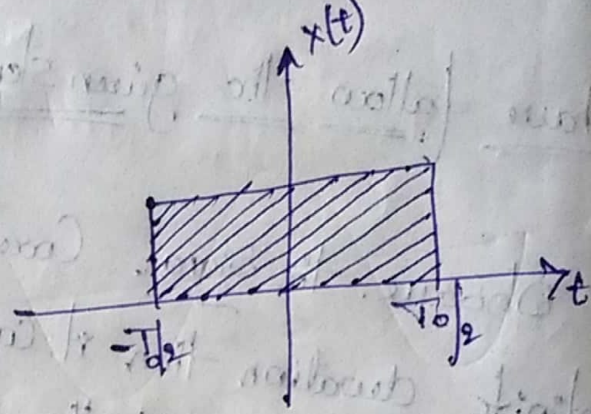
Here case $\sum_{n=0} a^n = \frac{1}{1-a}$ for $|a|<1$ The above Eqn will be.

$$E = \frac{1}{1-1/4} = 4/3$$

Since Energy is finite & non-zero, it is Energy signal with $E = 4/3$

ii) $x(t) = rect(t/T_0)$

The $rect(t/T_0)$



$$rect(t/T_0) = \begin{cases} 1 & \text{for } -T_0/2 \le t \le T_0/2 \\ 0 & \text{Else where} \end{cases}$$

it non-periodic. Hence it can be Energy signal as per signal as per step 3 Hence, Calculate Energy directly

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-T_0/2}^{T_0/2} (1)^2 dt$$

$$t\Big]_{-T_0/2}^{T_0/2} = T_0.$$

The Energy is finite and non-zero. it is Energy signal with $E = T_0$

ii) $x(t) = \cos^2 \omega_0 t$

This is Squared Cosine wave, hence it is Periodic. Therefore this can be periodic signal. As per step 1, Calculate power of this signal directly

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

The given signal $x(t) = \cos^2 \omega_0 t$. has Some period. T_0 & it is real signal.

$$P = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} [\cos^2 \omega_0 t]^2 dt$$

Hence $[\cos^2 \omega_0 t]^2 = \cos^4 \omega_0 t$. it can be Expanded by standard trigonometric relation.

$$P = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \frac{1}{8} [3 + 4\cos 2\omega_0 t + \cos 4\omega_0 t] dt$$

$$\lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} 3/8 \, dt + \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} 4\cos 2\omega_0 t \, dt +$$

$$\underbrace{\lim_{T_0 \to \infty} \int_{-T_0/2}^{T_0/2} \cos 4\omega_0 t \, dt}$$

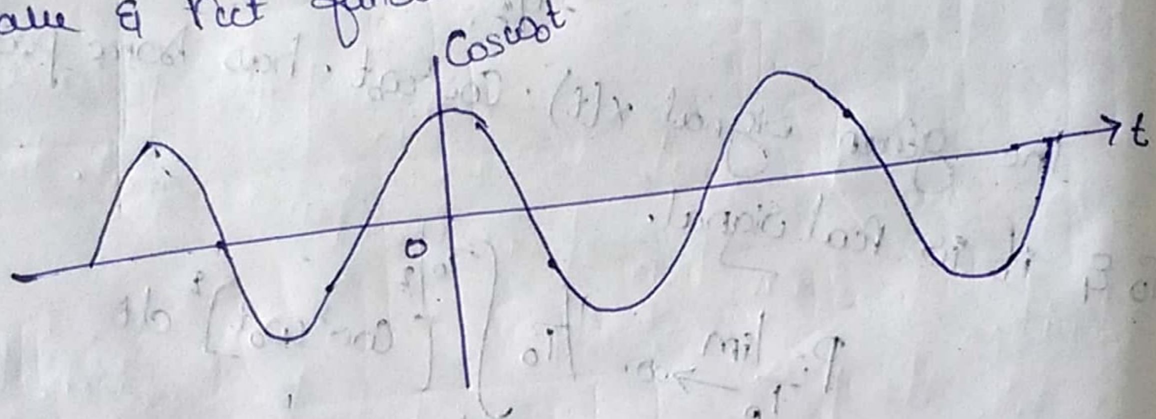
This term will be also zero Since it is integration of cosine wave over "full cycle".

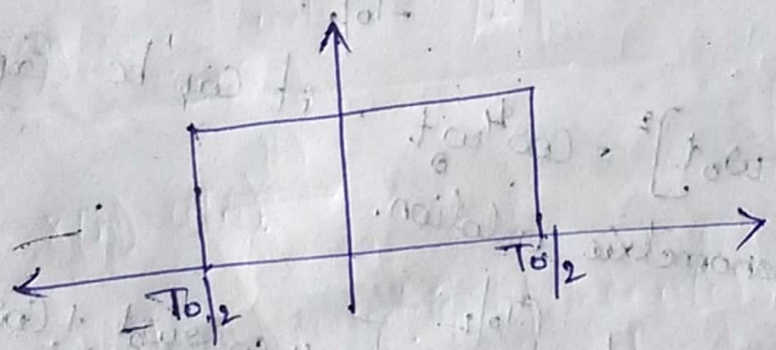
$$\lim_{T_0 \to \infty} \frac{1}{T_0} \cdot \frac{3}{8} (t) \Big|_{-T_0/2}^{T_0/2} + 0 + 0$$

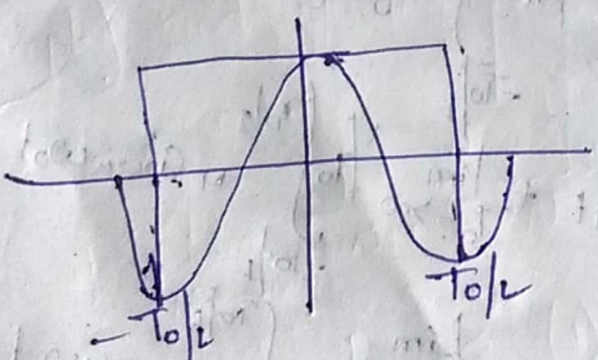
$$\therefore \lim_{T_0 \to \infty} \frac{1}{T_0} \cdot \frac{3}{8} T_0 = 3/8$$

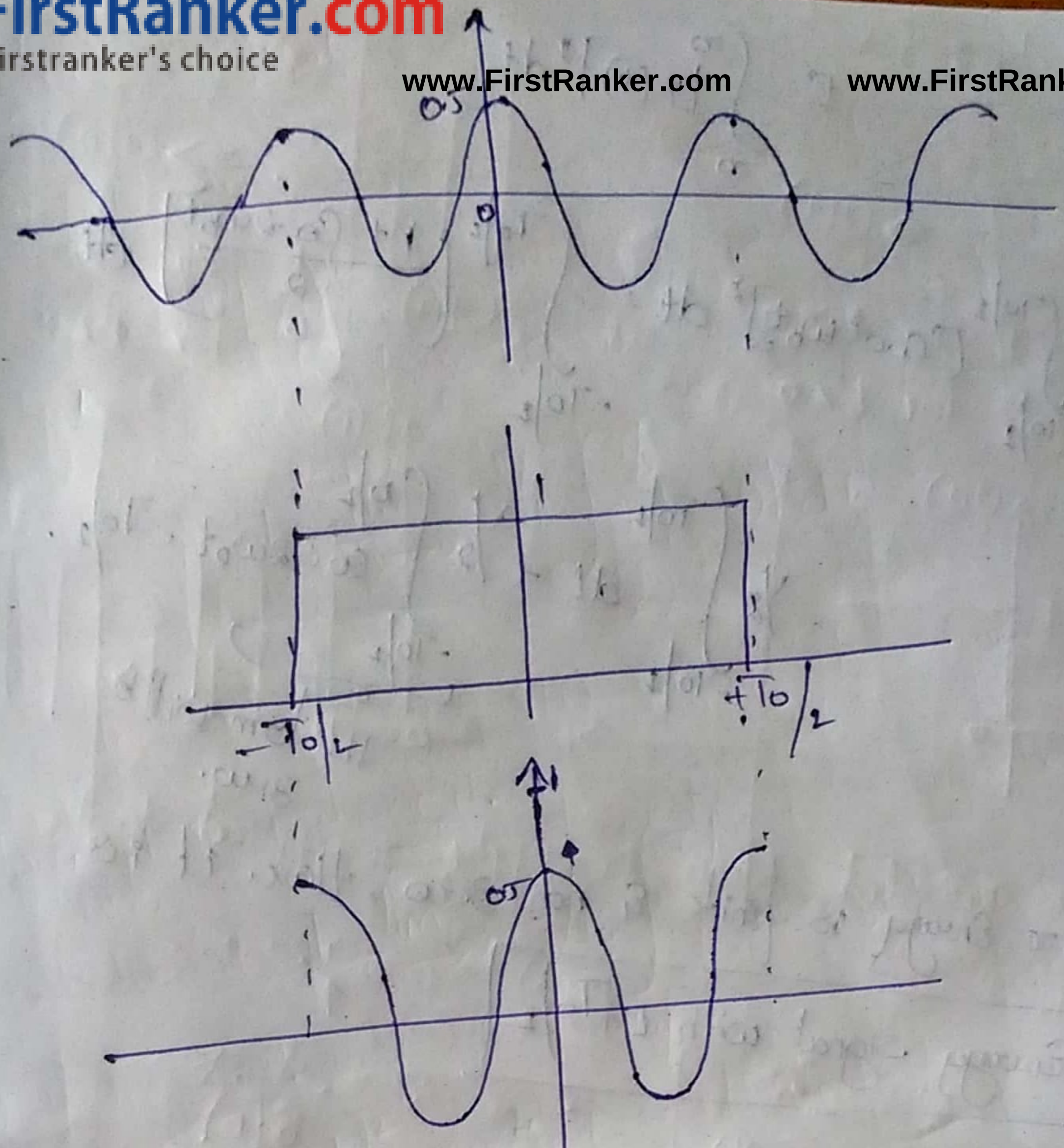
IV) $x(t) = \text{rect}(t/T_0) \cos \omega_0 t$

The given function is the product of cosine wave & rect function.









→ $\cos\omega_0 t$ is periodic & infinite duration signal.
Basically it is power signal.

→ $\cos\omega_0 t$ is Multiplied with the rectangular pulse.

→ Hence the resultant signal is Cosine wave of duration $-T_0/2 \le t \le T_0/2$

it is assumed that there are Multiple No. of cycle of cosine wave in $-T_0/2 \le t \le T_0/2$

The final signal is periodic but finite duration. Hence it is Can be Energy Signal.

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-T_0/2}^{T_0/2} [\cos^2 \omega_0 t]^2 dt = \int_{-T_0/2}^{T_0/2} \left[\frac{1 + \cos 2\omega_0 t}{2} \right] dt$$

$$= \frac{1}{2} \int_{-T_0/2}^{T_0/2} dt + \frac{1}{2} \underbrace{\int_{-T_0/2}^{T_0/2} \cos 2\omega_0 t}_{0} = T_0/2.$$

This term will be zero.

Here Energy is finite & non-zero, Hence it is

$$\boxed{\text{Energy signal with } E = T_0/2}$$

v) $x(n) = u(n)$.

This signal is periodic (since $u(n)$) repeat after Every sample) and of infinite duration. Hence it may be power signal.

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$\lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=0}^{N} (1)^2$$

Here $\sum_{n=0}^{N} (1)^2$ mean $1+1+1+1 \ldots$ for $n=0$ to N

In other words

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1+1+1+1+ .... (N+1) times: (N+1) Therefore above Equ will be

$$P = \lim_{N\to\infty} \frac{1}{2N+1}(N+1)$$

$$\lim_{N\to\infty} \frac{N+1}{2N+1} = \lim_{N\to\infty} \frac{1+1/N}{2+1/N} = 1/2$$

The power is finite & non-zero. hence unit step function is power signal with = P = 1/2

Elementry Signal:

→ standard signal are used for the analysis of System

→ These standard signal are.

a) unit step function.

b) unit impulse function.

c) unit ramp function.

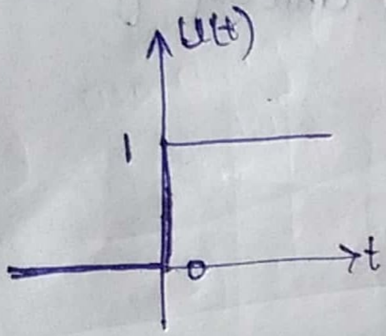
d) Complex Exponential function.

e) Sinusoidal function.

1) Unit Step:

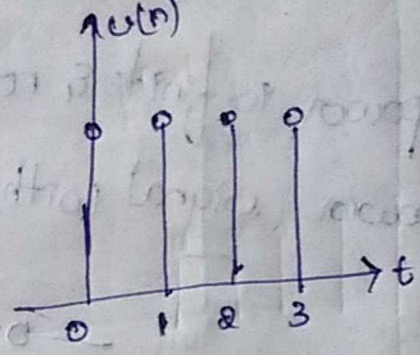
CT $\rightarrow u(t)$

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$



DT $\rightarrow u(n)$

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$



$\Rightarrow$ it is generated when DC supply is applied to the circuit

$$u(n) = \{0, 0, \underset{\uparrow}{1}, 1, 1, 1, \ldots\}$$

2) **Unit impulse:-**

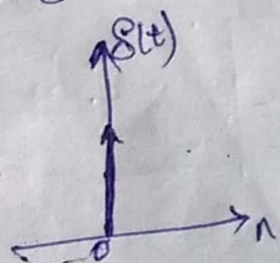
δ(t)

Area under unit impulse approaches 1 as its width approaches zero. Thus it has zero value every where except $t = 0$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \quad \& \quad t \to 0$$

$$\delta(t) = 0 \quad \text{for } t \neq 0$$

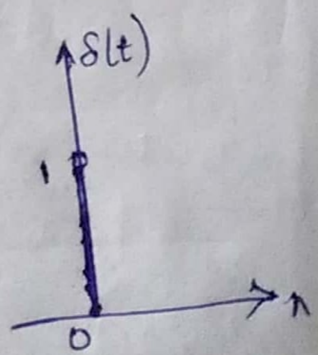
Unit Step



δ(n)

Amplitude of unit sample is '1' at $n = 0$ & it has zero value at all other value of n.

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$



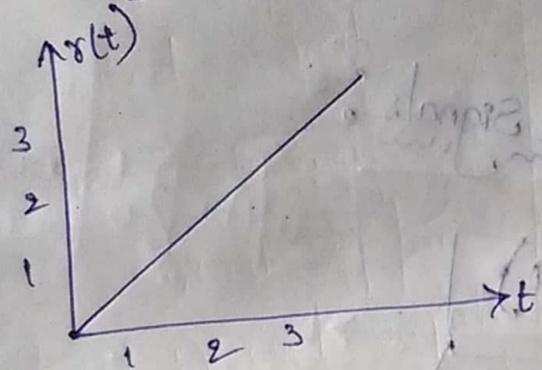
$\delta(n) = \{\ldots 0, 0, 1, 0, 0, 0 \ldots\}$

3) Unit Ramp :-

CT

it is linearly growing fun for positive value of independent variable

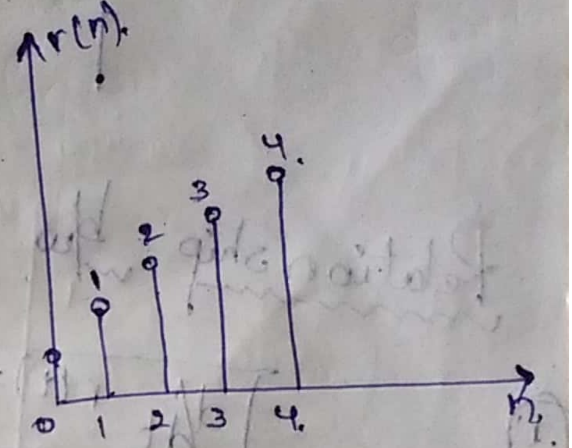
$$r(t) = \begin{cases} t & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$



DT

The amplitude of every sample increase linearly with its number for positive value of "n".

$$r(n) = \begin{cases} n & \text{for } n \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$



→ The ramp fun. indicate linear relationship.
→ It indicate constant current charging of the capacitor.

Complex Exponential & Sinusoidal signals:-

CT

1) it is exponentially growing (or) decaying signal

$x(t) = be^{at}$

b & a are real.

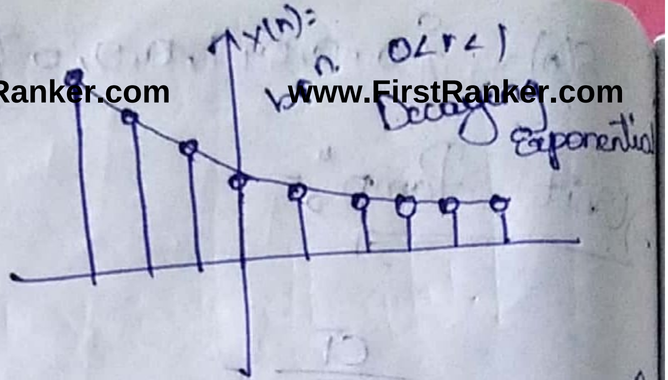
DT

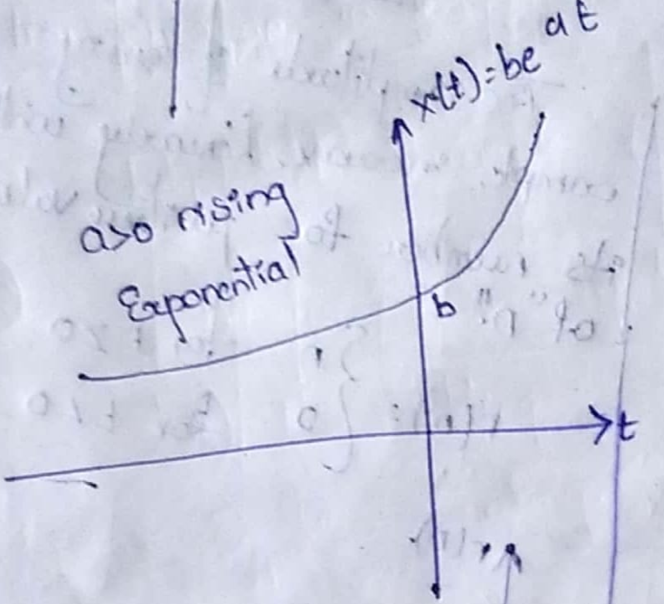
$x(n) = br^{n}$

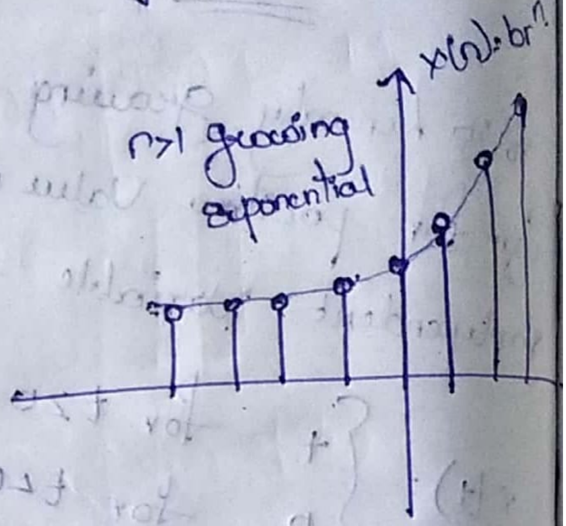
if $r = e^{\alpha}$ then

$x(n) = be^{\alpha n}$

Here b & α are real







Relationship b/w the signals:-

(1) $\boxed{\frac{d}{dt} r(t) = u(t)}$

$dr(t) = dt\, u(t)$

$\int dr(t) = \int u(t)\, dt$

$\boxed{r(t) = \int u(t)\, dt}$

Relation b/w unit step & unit ramp

Signal

(2) $\frac{d}{dt} u(t) = \delta(t)$

(or) $u(t) = \int \delta(t)\,dt$

Ex: Find the derivative of the following signal.

1) $x(t) = u(t) - u(t-a)$, $a>0$

2) $x(t) = t\,[u(t) - u(t-a)]$, $a>0$

3) $x(t) = \text{sgn}(t) = \begin{cases} 1 & t>0 \\ -1 & t<0 \end{cases}$

1) $\frac{d}{dt} x(t) = \frac{d}{dt}[u(t) - u(t-a)]$

$= \frac{d}{dt} u(t) - \frac{d}{dt} u(t-a)$

$= \delta(t) - \delta(t-a)$

2) $\frac{d}{dt} x(t) = \frac{d}{dt}\left(t[u(t) - u(t-a)]\right)$

$y(t) = u(t) - u(t-a)$

$\frac{d}{dt} x(t) = \frac{d}{dt}[t\,y(t)]$

$\frac{d}{dt} t\,y(t) = t\frac{d}{dt} y(t) + y(t)\frac{d}{dt} t$

$= t[\delta(t) - \delta(t-a)] + y(t)\cdot 1 =$

$t(\delta(t) - \delta(t-a)) + u(t) - u(t-a)$

Operations on signals :-

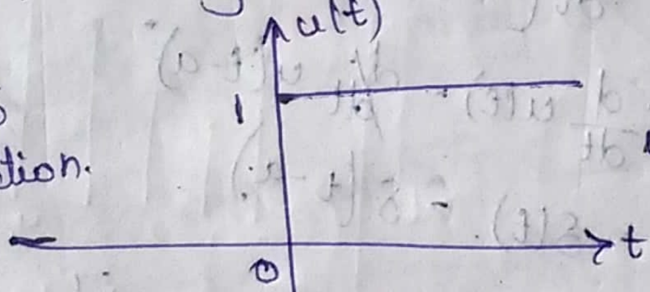
Transformation in independent variable of signal.

Independent variable 't (or) n' can be Manipulated by

1) Delay / Advancing
2) Time folding,
3) Time Scaling.

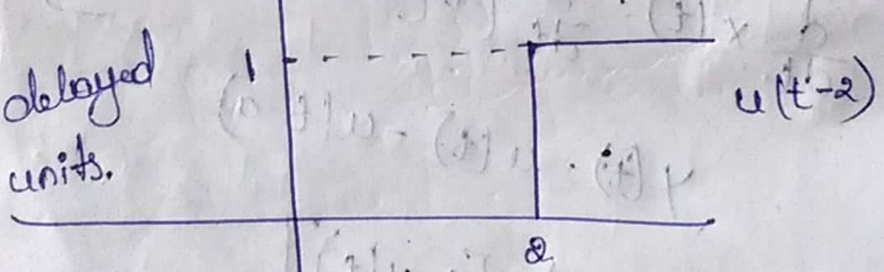
1) Delay / Advancing

unit step function.

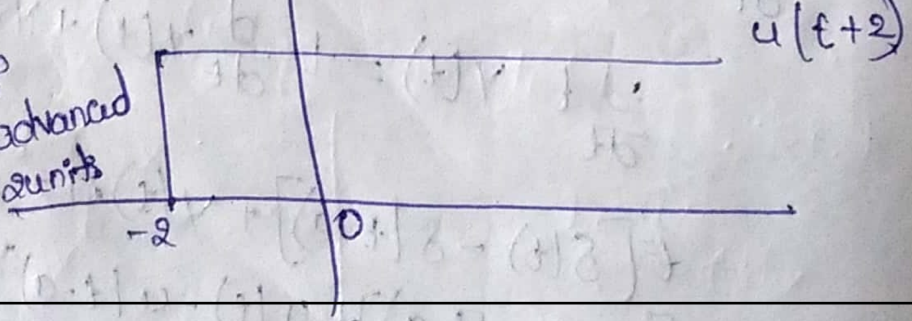


$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$

unit step function delayed by 2 units.



unit step function advanced by 2 units

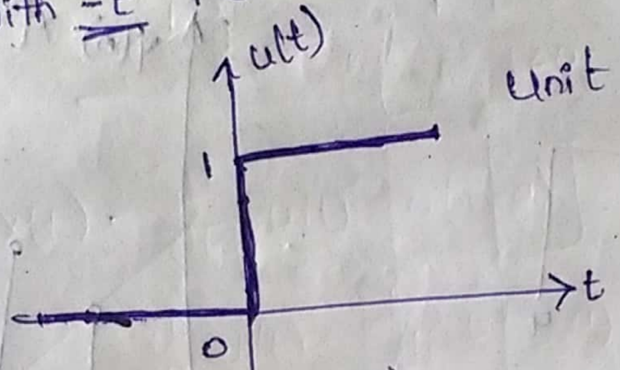


1) when function is advanced it is shifted left.

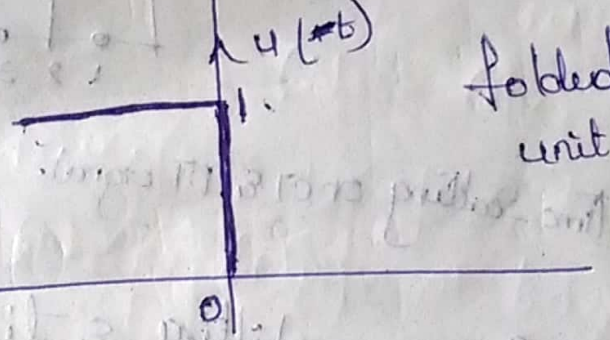
2) when function is delay it is shifted Right.

2) Time folding :-

The time folding operation is used in Convolution. Consider the continous time signal $x(t)$. Then it's time folded signal is obtained by replacing "t" with "-t" i.e



unit step signal.



folded unit step signal.

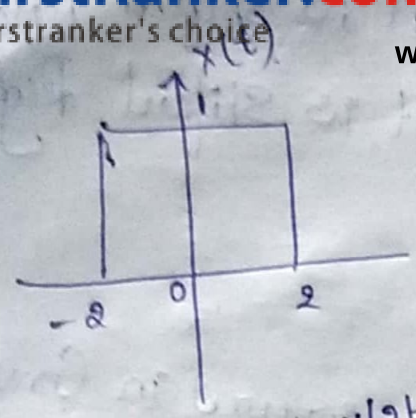
Time Scalling :

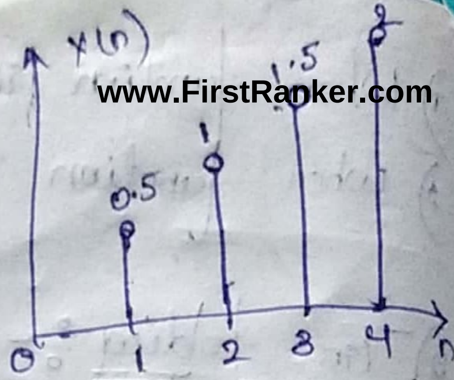
Two types of time scalling

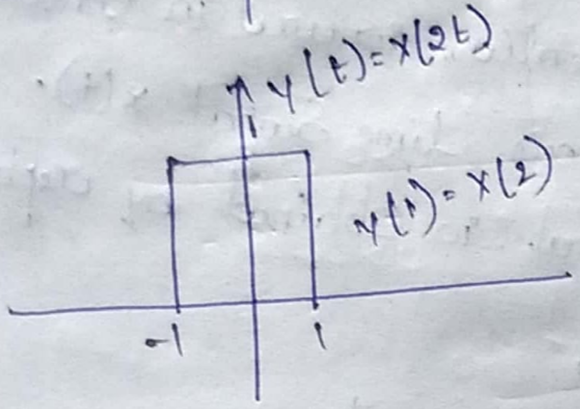
1) Time Compression : The time axis is compressed.

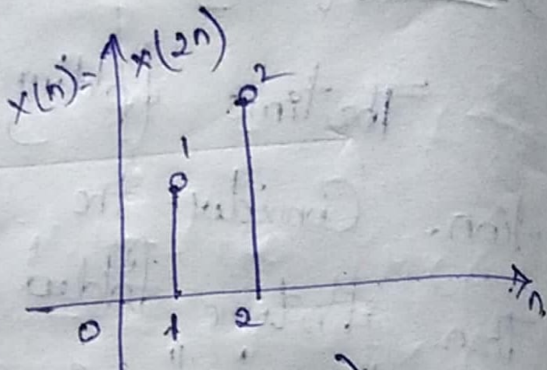
$$y(t) = x(2t)$$

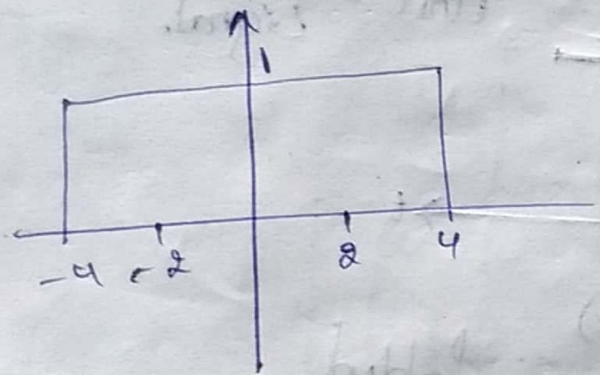
2) Time Expansion : The time axis is Expanded.

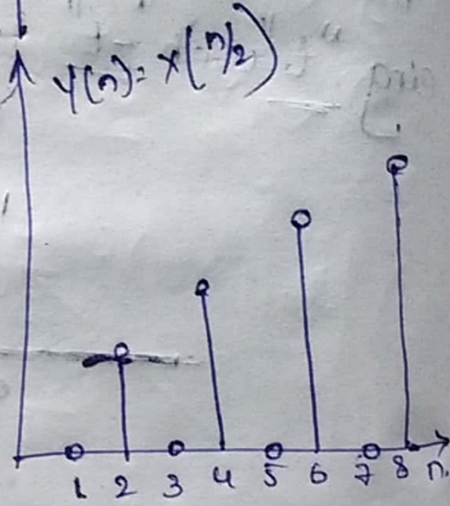












→ Time Scalling on CT & DT Signal.

Precedence Rule for Time shifting & time Scaling

Rules

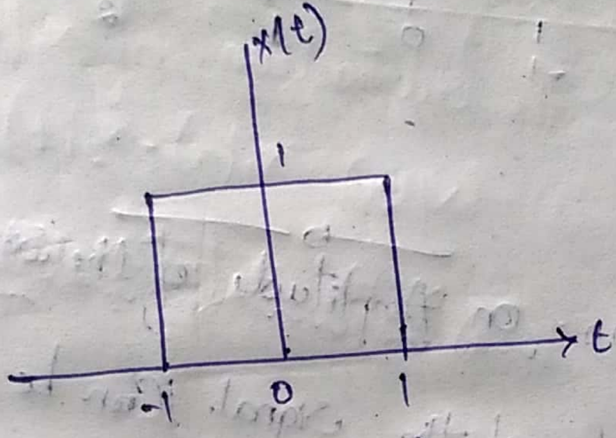
⇒ first do the shifting operation.

⇒ then do the time scalling operation.

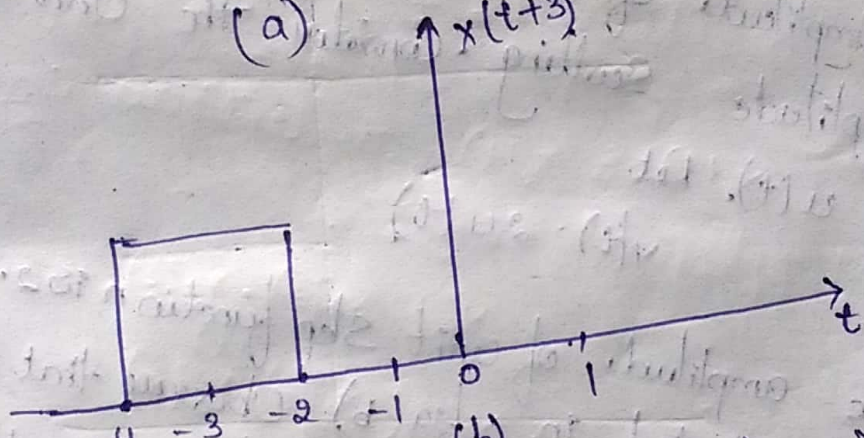
Example:-

⇒ Let us consider $y(t) = x(-2t+3)$. The $x(t)$ is rectangular pulse of amplitude duration $-1 \le t \le 1$.

Step 1:- Shift $x(t)$ to Left by 3, to get $x(t+3)$.

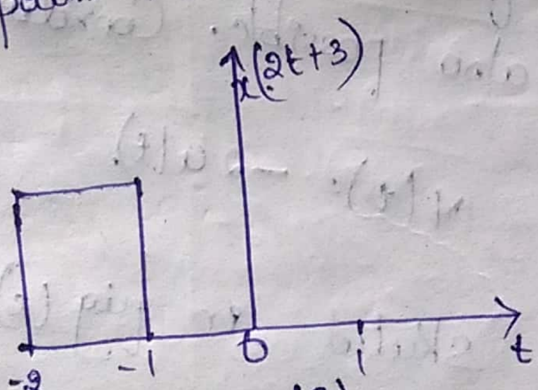


(a)



(b)

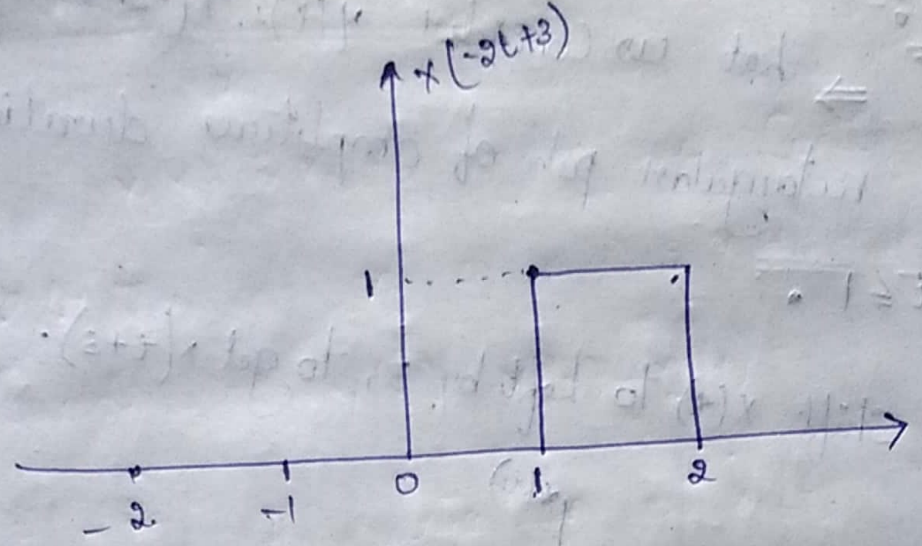
Step 2: Compress $x(t+3)$ by '2' to get $x(2t+3)$.



(c)

Step 3 The $x(2t+3)$ of fig (c) is folded in time to

Sol: $x(-2t+3)$ This is shown fig given below.



Transformation on Amplitude of the Signals.

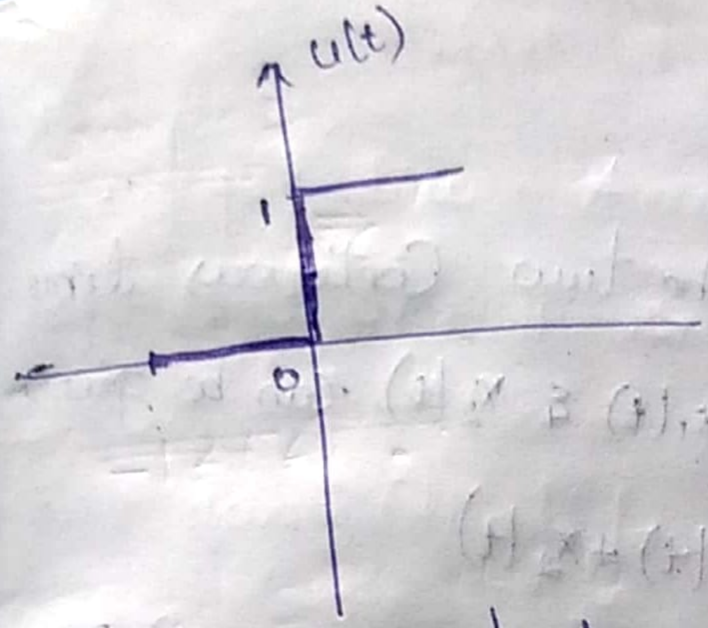
The Amplitude of the Signal. Can be changed with amplitude Scalling: Consider the unit step function $u(t)$. Let

$$y(t) = 2u(t)$$

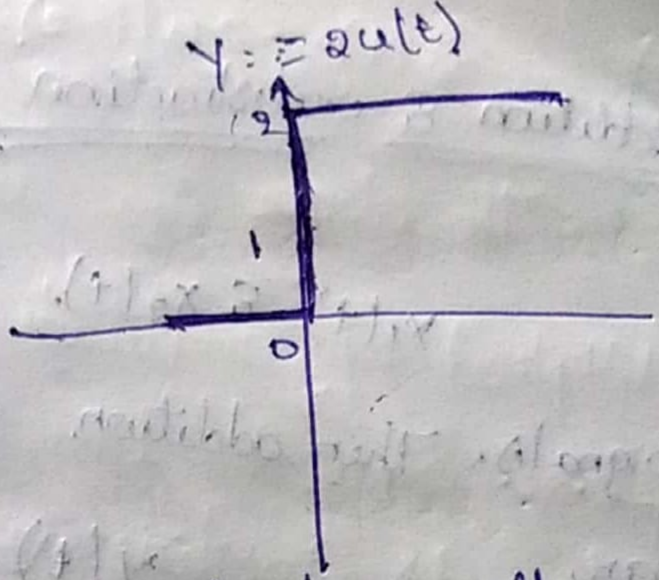
Here amplitude of unit step function is 2. This function is sketched in fig(b) Observe that the amplitude of step function '2'. Similarly negative amplitudes are also possible. Consider

$$y(t) = -2u(t)$$

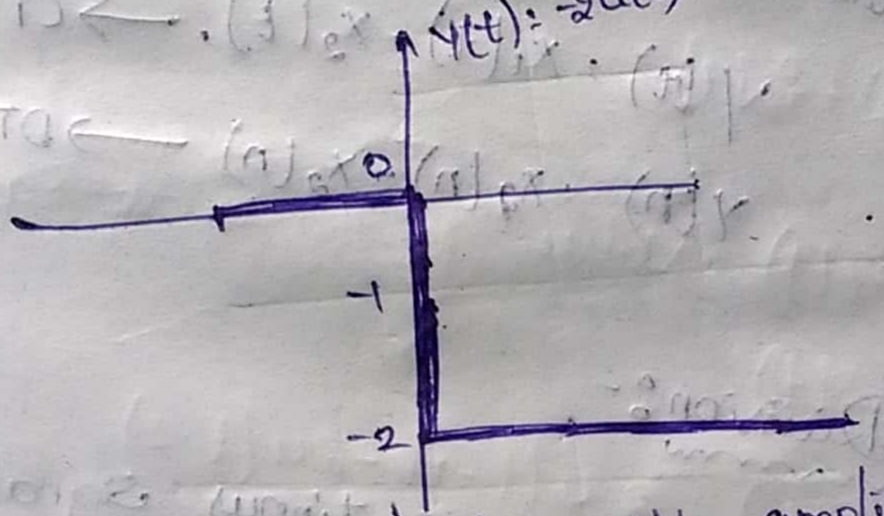
This fun is sketched in fig (c) Observe that the step function has -ve amplitude i.e "-2"



Unit step function



Step fun with amplitude '2' (Positive)



Step function with amplitude "2" (Negative)

Amplitude Scalling can also be performed on discrete time signal. Consider the unit step sequence u(n) Let.

$y(n) = 2u(n)$.

Addition & Substraction

$x_1(t)$ & $x_2(t)$ be the two Continous time signals. Then addition of $x_1(t)$ & $x_2(t)$ can be given as,

$$y(t) = x_1(t) + x_2(t)$$

Similarly, the substraction of $x_1(t)$ & $x_2(t)$ is given as.

$$y(t) = x_1(t) - x_2(t). \longrightarrow CT$$

$$y(n) = x_1(n) - x_2(n) \longrightarrow DT$$

Multiplication & Division:-

Let $x_1(t)$ & $x_2(t)$ are continous signal then their Multiplication given as:

$$y(t) = x_1(t) \cdot x_2(t)$$

$$y(n) = x_1(n) \cdot x_2(n)$$

$$y(t) = \frac{x_1(t)}{x_2(t)}$$

$$y(n) = \frac{x_1(n)}{x_2(n)}.$$

Differentiation & Integration:-

Let $x(t)$ be the Continous time signal. Then its differentiation. w.r.to given as.

$$y(t) = \frac{d}{dt} x(t).$$

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

Let the current i(t) is following through an inductor the voltage across it will be

$$V(t) = L \frac{d}{dt} i(t).$$

Here y(t) is integration of x(t). integration is used to represent voltage across the capacitor "C"

$$V(t) = \frac{1}{C}\int_{-\infty}^{t} f(\tau)\, d\tau.$$

Problem :-

Draw the waveform represented by following step function.

a) $f_1(t) = 2u(t-1)$

b) $f_2(t) = -2u(t-2)$

c) $f(t) = f_1(t) + f_2(t)$

d) $f(t) = f_1(t) - f_2(t)$

1) $f_1(t) = 2u(t-1)$

The above Eqn. Represents a unit step function multiplied by amplitude of 2. There is a time shift of 1sec. This time shift will be towards positive value of t.

2) $f_2(t) = -2u(t-2)$.

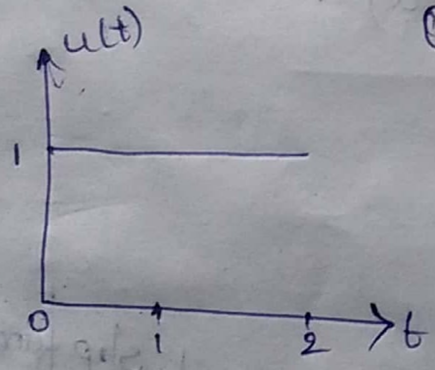
The above Equ represents a unit step fun Multiplied by amplitude of -2. There is time shift of 2sec. Since the time shift is "Subtracted" it will be towards positive value. of fig (b) shows the generation of $f_2(t)$ of above Equ.

3) $f(t) = f_1(t) + f_2(t)$

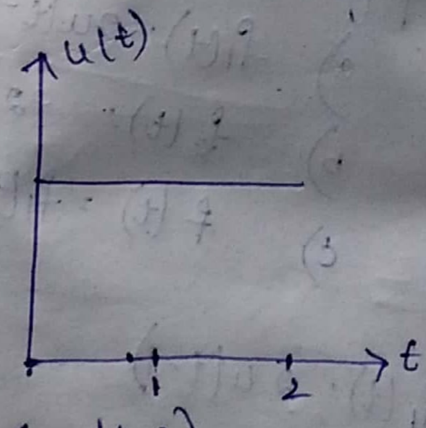
$f_1(t)$ & $f_2(t)$ values in the above Equation we are getting $f(t)$

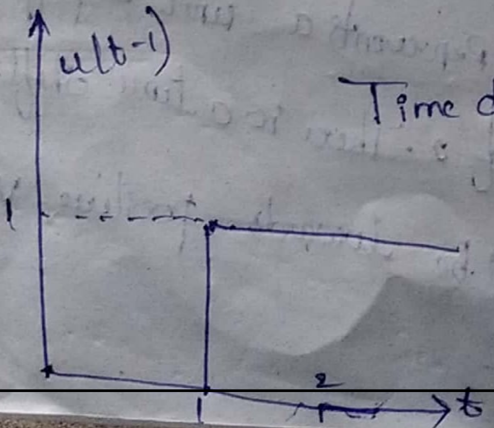
$$f(t) = 2u(t-1) - 2u(t-2)$$

IV) $f(t) = f_1(t) - f_2(t)$.

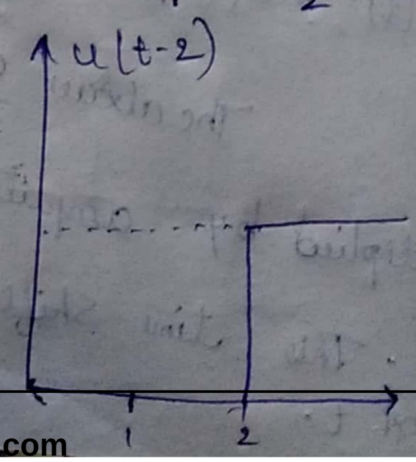


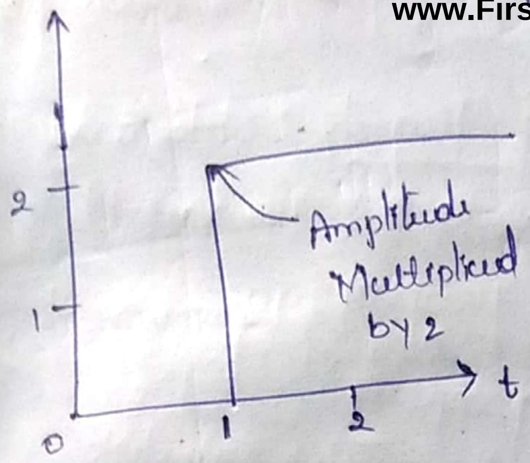
Original.

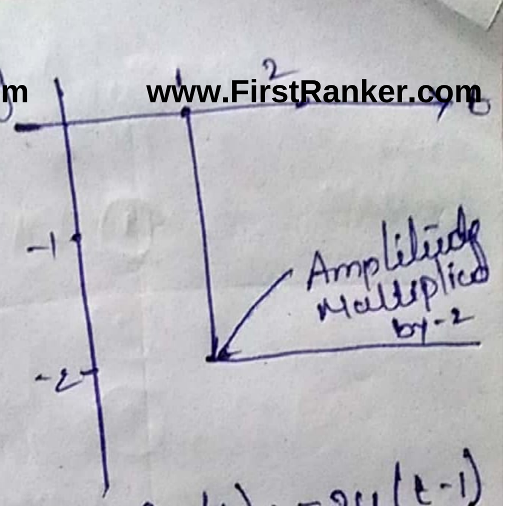




Time delay.

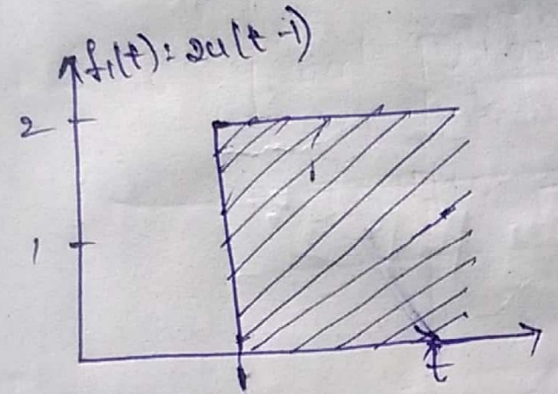


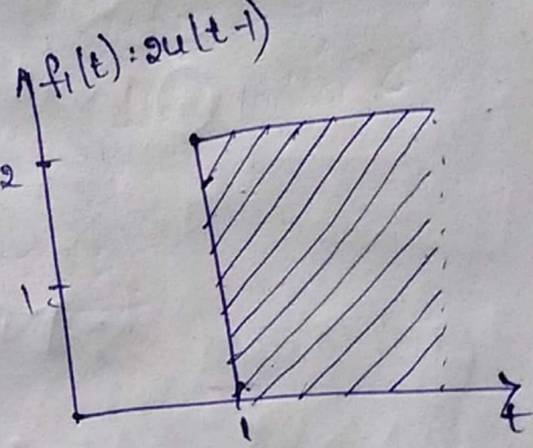


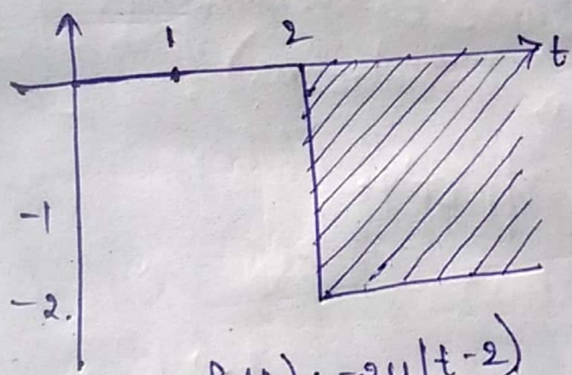


$f_2(t) : -2u(t-1)$

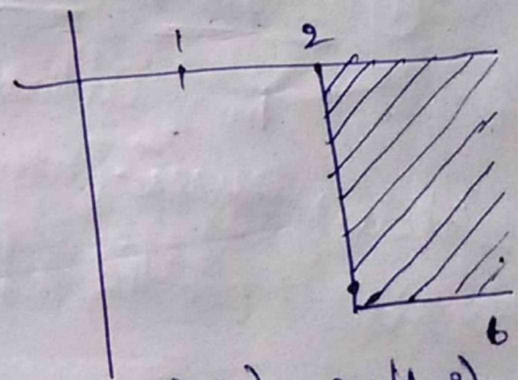
$f_1(t) : 2u(t-1)$



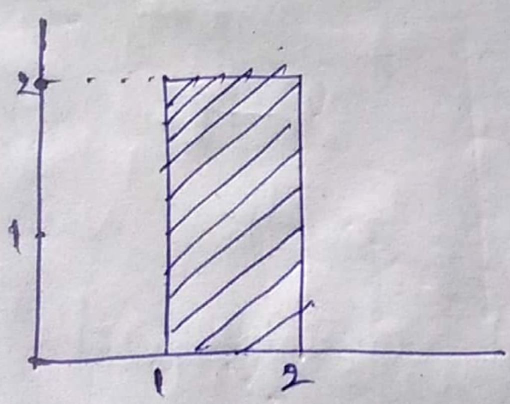


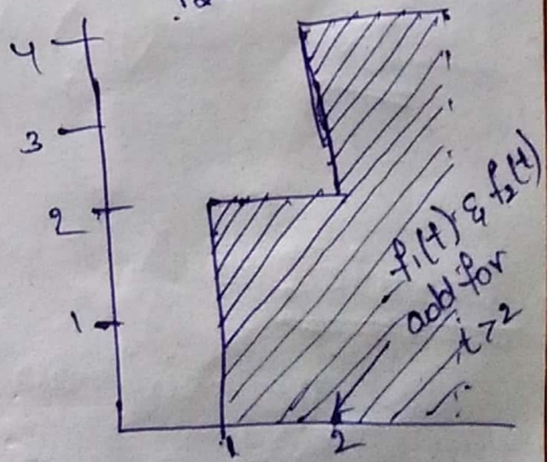


$f_2(t) : -2u(t-2)$



$f_2(t) : -2u(t-2)$





$f(t) : 2u(t-1) - (-2u(t+2))$

$f_1(t) + f_2(t) : f(t)$

$= 2u(t-1) + 2u(t-2)$

System & their properties

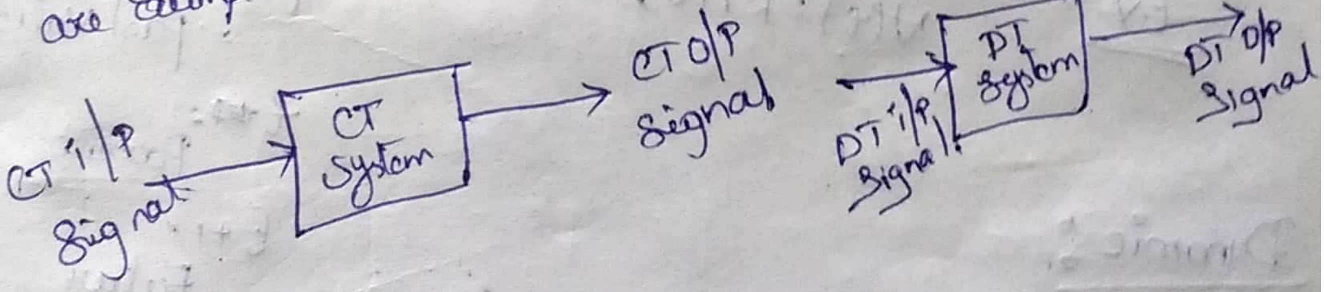
→ A system is a set of Elements (or) functional blocks that are connected together & produce an o/p in response to an i/p signal.

Classification

two types of systems
1. Continous time system
2. Discrete time system.

CT: It handle continous time signals. Anlog filters, amplifiers, attenuators, anlog transmitter & receiver etc.

DT: it handle discrete time signal. Computers, Printers, Microprocessor, Memories, Shift registers etc. are examples of discrete time system.



CT's Can be further classified based on properties

1. Dynamicity Property : static & dynamic
2. Shift invariance : Time Variant & Time Invariant
3. Linearity property : Linear & non linear
4. Causality property : Causal & non-Causal
5. stability property : stable & unstable System
6. Invertibility property : Inversible & non-inversible

Dynamicity property :-

① Static System :-

The Continous time system is said to be Static (or) ~~Dynamic~~ (memory less. instantaneous) if its o/p depends upon the present i/p only.

Ex $v(t) = Ri(t)$

$t-1$: past value

t : Present

$t+1$: Future Value

Dynamic :

The Continous time system is said to be Dynamic if its o/p values depend upon the present i/p & past values.

(Or)

In dynamic system the n^{th} o/p sample value depend upon n^{th} i/p sample & just previous i.e $(n-1)^{th}$ i/p samples. This system need to be store the previous sample value.

$$y(n) = x(n) + x(n-1)$$

② Time Invariant & Time Variant System:-

Time invariant :- A continous time system is time invariant if the time shift in the i/p signal result in corresponding time shift in the o/p. Ex Nigh & day with time

$$f(x(t-t_1)) = y(t-t_1)$$

Time Variant: A continous system is time variant if the time shift in the i/p signal result so there is no time shift in the o/p then it is said to be time variant system.

$$f[x(n-k)] = y(t).$$

Ex temprature in a day. ~~time~~ temperature is varie with the time.

Causal & Non-Causal System:

Causal:- The system is said to be Causal if its o/p at any time depends upon present & past i/p's only.

Ex $y(n) = x(n) \cdot x(n-1)$.

Non Causal:

The system is said to be Non Causal if o/p at any time depends up on present, past, future i/p values.

$$y(n) = x(n) + x(n-1) + x(n+1).$$

Linear & Non-Linear System:-

Linear: A system is said to be Linear if it satisfy the super position principle.

Super position principle - Sum of ~~produc~~ two i/p is Equal to the sum of the two individual i/p.

$$f[a_1 x_1(t) + a_2 x_2(t)] = a_1 y_1(t) + a_2 y_2(t)$$

Non-linear:- A system is said to be Non-linear if it don't satisfy the supper position principer.

$y_1(t) = f(x_1(t))$ $y_2(t) = f(x_2(t))$

$f[a_1 x_1(t) + a_2 x_2(t)] = a_1 y_1(t) + a_2 y_2(t)]$

Stable & unstable System:

When Every bounded i/p produces bounded o/p then the system is "stable."

If the system produce unbounded o/p for bounded i/p then it is "unstable."

Problem

Determine whether the following Continous time system are stable (or) not?

1) $y(t) = t\,x(t)$ 2) $y(t) = x(t) \sin 100\pi t$

i) $y(t) = t\,x(t)$
⇒ Here Let $x(t)$ be bounded And, $t \to \infty$, $y(t) \to \infty$

$x(t)$ is Multiplied by 't'.

2) $y(t) = x(t) \sin 100\pi t$

Let $x(t)$ is bounded. Here $x(t)$ is Multiplied by $\sin 100\pi t$. We know that value of the sine function

Hence o/p $y(t)$ is bounded as long as $x(t)$ is bounded. Hence this system is stable.

Ex

Determine whether the following discrete time system are stable (or) not?

i) $y(n) = x(n) + x(n-1) + x(n-1)$ ii) $y(n) = r^n x(n)$ $r>1$

Problems Determine whether the following Continuous time system are Causal (or) non-Causal.

1) $y(t) = x(t)\cos(t+1)$ 2) $y(t) = x(2t)$ 3) $y(t) = x(-t)$

4) $\frac{dy(t)}{dt} + 10y(t) + 5 = x(t)$ 5) $y(t) = \int_{-\infty}^{t} x(t)\,dt$

1) $y(t) = x(t)\cos(t+1)$

Here observe that $y(t)$ depends upon present i/p $x(t)$. A Cosine function can be Calculated at $t+1$. Hence this is Causal system.

2) $y(t) = x(2t)$

Here, if $t=2$ then,

$y(2) = x(2\cdot 2)$

$= x(4)$

Thus o/p at $t=2$ depends on feature i/p $x(4)$. Hence this is non-Causal system.

iii) $y(t) = x(-t)$

Here if $t=-2$ then $y(-2) = x(-(-2))$.

Thus o/p depends upon feature i/p. Hence this is non-Causal system.

iv) $\frac{dy(t)}{dt} + 10y(t) + 5 = x(t)$

Here Observe that o/p $y(t)$ depends present i/p.

Hence this is Causal system.

v) $y(t) = \int_{-\infty}^{t} x(t)\,dt$.

Here o/p depends upon present & past i/p. Hence this is a Causal system.

Check whether the following Continous time system as Linear (or) non-linear.

1) $y(t) = t\,x(t)$ 2) $y(t) = x^2(t)$

$y_1(t) = f[x_1(t)] = t x_1(t)$, $y_2(t) = f[x_2(t)] = t x_2(t)$

Hence Linear Combination of o/p become.

$y_3(t) = a_1 y_1(t) + a_2 y_2(t)$

$= a_1 t x_1(t) + a_2 t x_2(t)$

Now response to the Linear Combination of i/p becomes.

$$y_3'(t) = f[a_1 x_1(t) + a_2 x_2(t)]$$

$$= t[a_1 x_1(t) + a_2 x_2(t)]$$

$$= a_1 t x_1(t) + a_2 t x_2(t)$$

Equ $y_3(t) = y_3'(t)$

On Comparing above

Hence this is a Linear system.

ii) $y(t) = x^2(t)$

The o/p of the system to two i/p $x_1(t)$ & $x_2(t)$

$$f[x_1(t)] = x_1^2(t)$$

$$f(x_2(t)) = x_2^2(t)$$

Hence Linear Combination of these o/p becomes

$$y_3(t) = a_1 y_1(t) + a_2 y_2(t)$$

$$= a_1 x_1^2(t) + a_2 x_2^2(t)$$

Now let us find the response of the System to Combination of i/p

$$y_3'(t) = f[a_1 x_1(t) + a_2 x_2(t)]$$

$$= [a_1 x_1(t) + a_2 x_2(t)]^2$$

$a_2^2 x_2^2(t) + 2a_1a_2x_1(t)x_2(t)$.

— o —

check whether the following Continous time system are ~~time~~ time invariant (or) time Variant.

1) $y(t) = \sin x(t)$ 2) $y(t) = t\,x(t)$ 3) $y(t) = x(t) \cos 200\pi t$.

Anlogy b/w Vectors & Signals:-

Signals can be represented in terms of Orthogonal function. These Orthogonal fun. Satisfy specific properties.

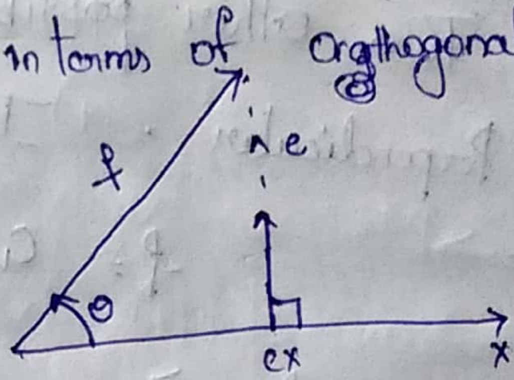


fig-1.

Orthogonality Concept in Vector:-

All the Signals are basically Vectors. A Vector can be represented in terms of its Co-ordinate. System. For Example Consider the Vector f. as shown fig 1. There is another Vector x. The dot product of 'f' and 'x'

$$f \cdot x = |f|\,|x|\cos\theta$$

Here 'θ' is angle b/w f & x.

In the above fig 'cx' is the Component of Vector 'f' along 'x'. In other words 'cx' is the Projection of 'f' on 'x'. Here 'f' can be Expressed as Vector addition as:

$$f = cx + e$$

Here 'e' is an Error Vector. Note 'e' is mini mum only it is perpendicular.

Two other possibilities where it's not Perpendicular. In this Case observe that

$f = c_1 x + e_1 = c_2 x + e_2$

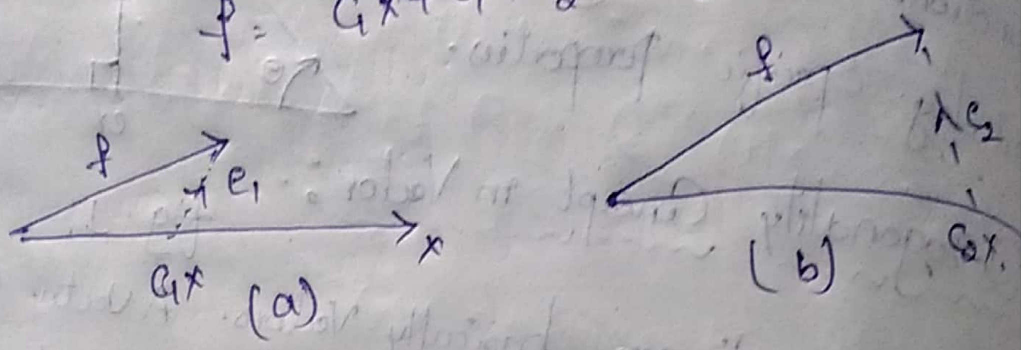


fig 2. c_1 & c_2 are greater than c.

But c_1 & c_2 are greater than c. Here 'c' is minimum only when it is ⊥r to 'x'. The Component of f along x is cx. It is also given as $|f|\cos\theta$.

$$c|x| = f\cos\theta$$

Multiplying both side by $|x|$

$$c|x|^2 = |f|\,|x|\cos\theta$$

R.H.S of above equ. represents the dot product of Vector f & x. Hence

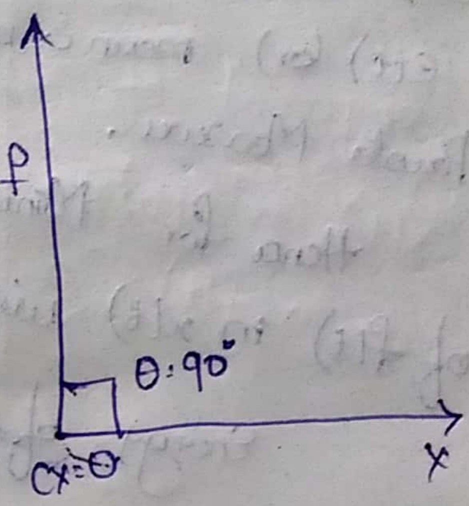
$$c|x|^2 = f\cdot x$$

$$c = \frac{1}{|x|^2} f\cdot x$$

$$x\cdot x = |x|^2, \quad c = \frac{f\cdot x}{x\cdot x}.$$

... will be Cancelled. Since $f.x$ & $x.x$ are Vector products.

fig 2(a) & 2(b) Observe that C_1x will be zero when 'f' is ⊥r to 'x'. In other words, will not have Component along 'x' then 'f' and 'x' are ⊥r to each other.



Hence the dot product $f.x$ will zero i.e

$$f.x = |f||x| \cos\theta$$

$$|f||x| \cos 90° = 0.$$

The vector 'f' and 'x' are said to be Orthogonal if their dot product is zero. In other word, Vectors are Orthogonal if they are Mutually perpendicular.

Orthogonality in signals:-

Now let us apply the Orthogonality Concept of Vectors to real signals. Let us Consider signal $f(t)$ to be represented in terms of $x(t)$ over an interval t_1 & t_2

$f(t) \approx c\,x(t) + e(t)$ → (a)

$e(t) = f(t) - c\,x(t)$ → (b)

→ Minimum value of e(t) will give best approximation of f(t) in x(t).

→ Minimum value of e(t), minimum Energy of e(t) (or) mean square value of e(t) serves appropriate measure.

Hence for Minimum Energy of c(t), representation of f(t) in x(t) will better.

Energy of e(t) will be.

$$E_e = \int_{t_1}^{t_2} e^2(t)\,dt$$

And Mean square value of e(t) will be given

$$\overline{e^2 t} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} e^2(t)\,dt$$

$$\overline{e^2(t)} = \frac{E_e}{t_2 - t_1}$$

Here E_e is Energy of e(t) over the interval of t_1 to t_2. And $\overline{e^2(t)}$ is Mean square value e(t).

from Eqn (5)

$$E_c = \int_{t_1}^{t_2} [f(t) - c\,x(t)]^2 dt$$

[Here the value of $e(t)^2$ over the interval t_1 to t_2

And $\overline{e^2(t)}$ is Mean Square value $e(t)$]

Here the value of 'c' should be selected such that E_c will be Minimum. This can obtained by differentiating E_c w.r.to c & Equating it to zero i.e

for Minimum E_c, $\frac{dE_c}{dc} = 0$

$$\text{i.e } \frac{d}{dc}\left[\int_{t_1}^{t_2} [f(t) - c\,x(t)]^2\right] dt = 0$$

$$= \frac{d}{dc}\int_{t_1}^{t_2} f^2(t)\,dt - \frac{d}{dc}\,2c\,f(t)\cdot x(t)\,dt + \frac{d}{dc}\int_{t_1}^{t_2} c^2 x^2(t)$$

$dt = 0$

= first term is independent of 'c' hence it will be zero.

$$-2\int_{t_1}^{t_2} f(t)\cdot x(t)\,dt + 2c\int_{t_1}^{t_2} x^2(t)\,dt = 0$$

$$= 2\left[-\cancel{2}\int_{t_1}^{t_2} f(t)\cdot x(t)\,dt + c\int_{t_1}^{t_2} x^2(t)\,dt = 0\right.$$

$$\int_{t_1}^{t_2} f(t)\,x(t)\,dt + c\int_{t_1}^{t_2} x^2(t)\,dt = 0$$

$$c\int_{t_1}^{t_2} x^2(t)\,dt = \int_{t_1}^{t_2} f(t)\,x(t)\,dt$$

Component of $x(t)$ Contained in $f(t)$:

$$\boxed{c = \frac{\int_{t_1}^{t_2} f(t)\,x(t)\,dt}{\int_{t_1}^{t_2} x^2(t)\,dt}}$$

Here are clearly observe that above Eqn is Similar to the system Equation.

The Denominator of the above Eqn represent Energy of $x(t)$. it Can't be zero. Hence numerator must be zero to make 'c' zero. If 'c' is zero there will be no Component of $f(t)$ along $x(t)$. then $f(t)$ and $x(t)$ are said to be Orthogonal Over an interval $[t_1\ t_2]$ i.e

Orthogonality $\int_{t_1}^{t_2} f(t)\,x(t)\,dt = 0$

Similarly if $f(t)$ and $x(t)$ are complex signals, then they are Orthogonal over an interval $[t_1, t_2]$

$$\text{for} \int_{t_1}^{t_2} f(t)\, x(t)\, dt = 0$$

if $x(t)$ & $f(t)$ are Orthogonal signal then they are Orthogonal over an terval $[t_1, t_2]$ if

$$f(t)\, x^*(t)\, dt = 0 \text{ (or) } \int_{t_1}^{t_2} f^*(t)\, x(t)\, dt = 0.$$

$x^*(t)$ is complex conjugate of $x(t)$.

Problem Show that the following signal are Orthogonal over an interval $[0,1]$

$$f(t) = 1, \quad x(t) = \sqrt{3}(1-2t)$$

Sol We know that the signals are Orthogonal

$$\text{if} \int_{t_1}^{t_2} f(t)\, x(t) = 0$$

$$\int_{t_1}^{t_2} f(t)\, x(t)\, dt = \int_0^1 1\left[\sqrt{3}\,(1-2t)\right] dt$$

$$= \int_0^1 \sqrt{3}\, dt - \int_0^1 2\sqrt{3}\, t\, dt$$

$$\sqrt{3}\int_0^1 dt - 2\sqrt{3}\int_0^1 t\,dt$$

$$\sqrt{3}\cdot[t]_0^1 - 2\sqrt{3}\left[\frac{t^2}{2}\right]_0^1 = 0$$

Thus the two given signal are Orthogonal over interval $[0,1]$.

2) A rectangular function is defined as.

$$f(t) = \begin{cases} A & \text{for } 0 \le t \le \pi/2 \\ -A & \text{for } \pi/2 \le t \le 3\pi/2 \\ A & \text{for } 3\pi/2 \le t \le 2\pi \end{cases}$$

Approximate above fun by $A\cos t$ b/w the interval $(0, 2\pi)$ Such that Mean Sauare Error is Minimum.

Sol

$f(t) = c\,x(t)$

$x(t) = A\cos t$

Here $c = \dfrac{\int_{t_1}^{t_2} f(t)x(t)\,dt}{\int_{t_1}^{t_2} x^2(t)\,dt} = \dfrac{\int_0^{\pi/2} A\cdot A\cos t\,dt + \int_{\pi/2}^{3\pi/2}(-A)A\cos t\,dt + \int_{3\pi/2}^{\pi/2} A\cdot A\cos t\,dt}{\int_0^{2\pi}(A\cos t)^2\,dt}$

$$A^2\left[\sin t\right]_0^{\pi/2} - A^2\left[\sin t\right]_{\pi/2}^{3\pi/2} + A^2\left[\sin t\right]_{3\pi/2}^{2\pi}$$

$$A^2\int_0^{2\pi}\frac{1+\cos 2t}{2}\,dt$$

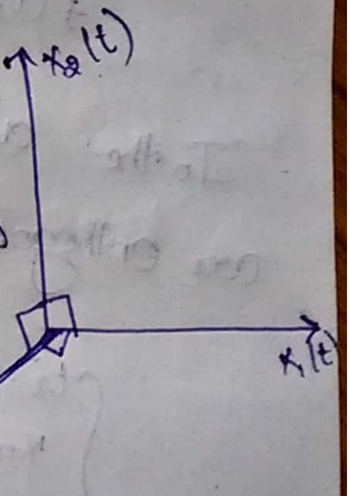
$$= A^2\left[\sin \pi/2 - \sin 0\right] - A^2\left[\sin\frac{3\pi}{2} - \sin \pi/2\right] + A^2\left[\sin 2\pi - \sin 3\pi/2\right]$$

$$\frac{A^2}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= 4/\pi$$

Thus $f(t) = 4/\pi \, A\cos t$ is the required Approximation.

Orthogonal Signal Space:-



Let $x_1(t)$, $x_2(t)$ & $x_3(t)$ be Orthogonal to each other. This mean these three signals will be mutually $\perp$r to each other.

it forms a three dimensional signal space. Such signal space. This signal space is used to represent any signal lying in that space.

If there are ... Mutually Orthogonal Signal Space. Any signal f(t) Can be represented in this dimensional signal space.

Signal Approximation using Orthogonal functions

Let us Consider the set of signal which are mutually Orthogonal Over an interval $[t_1\ t_2]$. These signals Can represents any signal f(t) as,

$$f(t) \cong c_1 x_1(t) + c_2 x_2(t) + \ldots\ldots + c_N x_N(t)$$

$$f(t) = \sum_{n=1}^{N} c_n x_n(t)$$

In the above Equ any two signals $x_m(t)$ & $x_n(t)$ are orthogonal Over an interval $[t_1\ t_2]$ i.e

$$\int_{t_1}^{t_2} x_m(t) x_n(t)\, dt = \begin{cases} 0 & \text{for } m \neq n \\ E_n & \text{for } m = n \end{cases}$$

In the above Equ Observe that any two different signals are Orthogonal, when m=n it is the Same Signal.

$$\int_{t_1}^{t_2} x_n(t) x_n(t)\, dt = \int_{t_1}^{t_2} x_n^2(t)\, dt = E_n$$

... Energy of the signals i.e E_n.

Error $e(t)$ in the approximation of Eqn is given as $e(t) = f(t) - \sum_{n=1}^{N} C_n x_n(t)$

Hence Error Energy will be

$$E_e = \int_{t_1}^{t_2} e^2(t)dt = \int_{t_1}^{t_2} \left[f(t) - \sum_{n=1}^{N} C_n x_n(t)\right]^2 dt.$$

Here E_e is the fun. of $C_1, C_2, C_3 \ldots C_N$

Hence E_e will be Minimized w.r.to C_i if

$$\frac{\partial E_e}{\partial C_j} = 0$$

$$\frac{\partial}{\partial C_j}\left\{\int_{t_1}^{t_2} \left[f(t) - \sum_{n=1}^{N} C_n x_n(t)\right]^2 dt\right\} = 0 \longrightarrow (1)$$

Above Eq. will be Executed for $i = 1, 2, 3 \ldots N$

$$\frac{\partial}{\partial C_j}\left[\int_{t_1}^{t_2} f^2(t)dt - \int_{t_1}^{t_2} \sum_{n=1}^{N} 2C_n f(t)\, x_n(t)dt + \int_{t_1}^{t_2} \sum_{n=1}^{N} C_n^2 x_n^2(t)\right\} = 0.$$

Above Eqn is Executed for $i = 1, 2, 3 \ldots N$

Here Observe that first integration term is independent of C_i. Hence its derivative will be zero

The derivative of second and third integration terms will be non zero only when $n=i$ these terms will be constant & their derivatives are zero.

$$\frac{\partial}{\partial c_i}\left\{-\int_{t_1}^{t_2} 2c_i f(t)\, x_i(t) + \int_{t_1}^{t_2} c_i^2 x_i^2(t)\,dt\right\} = 0$$

$$-2\int_{t_1}^{t_2} f(t)\, x_i(t)\,dt + 2c_i\int_{t_1}^{t_2} x_i^2(t)\,dt = 0$$

$i = 1, 2 \ldots N$

$$c_i = \frac{\int_{t_1}^{t_2} f(t)\, x_i(t)\,dt}{\int_{t_1}^{t_2} x_i^2(t)\,dt}$$

We know that $\int_{t_1}^{t_2} x_i^2(t)\,dt = E_i$ i.e Energy.

Hence above Eqn becomes.

$$\boxed{c_i = \frac{1}{E_i}\int_{t_1}^{t_2} f(t)\, x_i(t)\,dt.} \longrightarrow (a)$$

— o —

Mean Square Error :-

Now let us Consider the Mean Square Error in signal approximation using Orthogonal functions.

The Error Energy is given by Eqn.

$$E_e = \int_{t_1}^{t_2} \left[f(t) - \sum_{n=1}^{N} C_n x_n(t) \right]^2 dt.$$

$$= \int_{t_1}^{t_2} f^2(t)\, dt - 2 \int_{t_1}^{t_2} \sum_{n=1}^{N} C_n f(t)\, x_n(t) dt + \int_{t_1}^{t_2} \sum_{n=1}^{N} C_n^2 x_n^2(t) dt$$

last integration term is Energy of $x(n)$ i.e E_n.

And with the help of Eqn ⓐ we can write middle term of above Equation as.

$$\int_{t_1}^{t_2} f(t)\, x_n(t) dt = C_n E_n$$

$$E_e = \int_{t_1}^{t_2} f^2(t) dt - 2 \sum_{n=1}^{N} C_n C_n E_n + \sum_{n=1}^{N} C_n^2 E_n.$$

$$= \int_{t_1}^{t_2} f^2(t) dt - 2 \sum_{n=1}^{N} C_n^2 E_n + \sum_{n=1}^{N} C_n^2 E_n.$$

$$= \int_{t_1}^{t_2} f^2(t)\,dt - \sum_{n=1}^{N} C_n^2 E_n$$

The Mean Square Error & Error Energy are related as.

$$\overline{e^2(t)} = \frac{E_e}{t_2 - t_1} = \frac{1}{t_2 - t_1}\left[\int_{t_1}^{t_2} f^2(t)\,dt - \sum_{n=1}^{N} C_n^2 E_n\right]$$

In the above Eqn. $C_n^2 E_n$ is always positive

Hence Error Energy E_e Can be reduced if number of terms. N used for representation are increased

ideally. $E_e \to 0$. & $N \to \infty$ under this Condition, the Orthogonal signal set is said to be Complex.

Closed (or) Complete set of Orthogonal function

The Mean square Error approaches zero as number of terms $C_n^2 E_n$ are Made infinite

$$0 = \frac{1}{t_2 - t_1}\left[\int_{t_1}^{t_2} f^2(t)\,dt - \sum_{n=1}^{\infty} C_n^2 E_n\right] \text{ with}$$

$$\overline{e^2(t)} = 0 \text{ as } N = \infty$$

$$\int_{t_1} f^2(t)\,dt = \sum_{n=1}^{\infty} C_n^2 E_n$$

With N approaching infinity Equ can be written as.

$$f(t) = \sum_{n=1}^{\infty} C_n x_n(t)$$

Here $x_1(t), x_2(t), \ldots x_n(t)$ is a set of Mutually Orthogonal function. it is Said to be Complete (or) Closed set if there Exists no function $p(t)$ for which

$$\int_{t_1}^{t_2} p(t)\, x_n(t)\,dt = 0 \quad \text{for } n = 1, 2 \ldots$$

if $p(t)$ Exists & above integral is zero. then Obviously. $p(t)$ must be a member of set $\{x_n(t)\}$

For the set of mutually Orthogonal signals. $x_n(t)$ over an interval (t_1, t_2).

$$\int_{t_1}^{t_2} x_m(t)\, x_n(t)\,dt = \begin{cases} 0 & \text{if } m \neq n \\ E_n & \text{if } m = n \end{cases}$$

For this Complete set, the function $f(t)$ is Expressed as.

$f(t) = C_1 x_1(t) + C_2 x_2(t) + \ldots$

$$C_i = \frac{\int_{t_1}^{t_2} f(t)\, x_i(t)\, dt}{\int_{t_1}^{t_2} x_i^2(t)\, dt} = \frac{1}{E_i}\int_{t_1}^{t_2} f(t)\, x_i(t)\, dt$$

The set of $x_n(t)$ is called orthogonal basis functions.

—o—

Orthogonality in Complex functions:

Consider that the set of signals $x_1(t), x_2(t), x_3(t) \ldots$ are complex then they are mutually orthogonal if

$$\int_{t_1}^{t_2} x_m(t)\, x_n^*(t)\, dt = \int_{t_1}^{t_2} x_m^*(t)\, x_n(t)\, dt = \begin{cases} 0 & \text{for} \\ E_n & \text{for} \end{cases}$$

Then $f(t)$ can be expressed as,

$$\boxed{f(t) = \sum_{n=1}^{\infty} C_n x_n(t)}$$

where C_n is given in the similar fashion of above eqn.

$$C_n = \frac{1}{E_n}\int_{t_1}^{t_2} f(t)\, x_n^*(t)\, dt$$

Where E_n is given for Complex signals are.

$$E_n = \int_{t_1}^{t_2} x_n(t) \cdot x_n^*(t)\, dt$$

---o---

Trigonometric Fourier Series:-

We know that any fun. $f(t)$ can be Expressed as

$$f(t) = \sum_{n=1}^{\infty} C_n x_n(t)$$

Here $x_n(t)$ represent Orthogonal signal set. They are also Called basic function. This Equ is Called generalized Fourier Series.

$m \neq n$
$m = n$

We have seen that the set.

$\{1, \cos\omega_0 t, \cos 2\omega_0 t \ldots \cos n\omega_0 t, \ldots \sin\omega_0 t, \sin 2\omega_0 t \ldots \sin n\omega_0 t \ldots\}$

is Orthogonal Over the period T_0. Here ω_0 is called fundamental frequency. and $n\omega_0$ is Called n^{th} harmonic. There is DC Component of $\cos n\omega_0 t$ at $n=0$ i.e

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→ Trigonometric fourier Series:

As we know that $\sin n\omega_0 t$ & $\cos n\omega_0 t$ both are orthogonal over the given interval, Now we choose a composite set of function consisting of a set $\cos n\omega_0 t$ & $\sin n\omega_0 t$ for $(n=0,1,2,\ldots)$ & forms a complete orthogonal set.

∴ for $n=0$, $\sin n\omega_0 t = 0$ & for $n=1$ $\cos n\omega_0 t = 1$

& the set of orthogonal fn are given as $1, \cos\omega_0 t, \cos 2\omega_0 t \ldots \cos n\omega_0 t \ldots, \sin\omega_0 t, \sin 2\omega_0 t \ldots \sin n\omega_0 t \ldots$

Now any fn $f(t)$ can be represented in terms of these functions over any interval

$$(0,T) \text{ (or) } (t_0, t_0+T) \text{ (or) } \left(t_0, t_0+\frac{2\pi}{\omega_0}\right)$$

$$\Rightarrow f(t) = a_0 + a_1\cos\omega_0 t + \ldots a_n\cos n\omega_0 t + \ldots + b_1 \sin\omega_0 t + b_2 \sin 2\omega_0 t \ldots + b_n \sin n\omega_0 t$$

$$\left(t_0, t_0+\frac{2\pi}{\omega_0}\right)$$

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \quad \text{—— (1)}$$

$$(t_0 < t_0 < t_0 + T)$$

eq (1) is the say trignometric fourier series representation of $f(t)$ over the interval (t_0, t_0+T)

where $a_0, a_1 \ldots a_n, b_1, b_2 \ldots b_n$ are the components of $f(t)$ along the mutually orthogonal set (or) the constant values, & are given by

As we have,

$$C_{12} = \frac{\int_{t_1}^{t_2} f_1(t) f_2(t)\, dt}{\int_{t_1}^{t_2} f_2^2(t)\, dt}$$

Illy

$$a_n = \frac{\int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t\, dt}{\int_{t_0}^{t_0+T} \cos^2 n\omega_0 t\, dt}$$

$$\int_{t_0}^{t_0+T} \cos^2 n\omega_0 t \, dt = \int_{t_0}^{t_0+T} \left[\frac{1+\cos 2n\omega_0 t}{2}\right] dt$$

$$= \frac{1}{2}\int_{t_0}^{t_0+T} \left[1+\cos 2n\omega_0 t\right] dt$$

$$= \frac{1}{2}\left[t - \frac{\sin 2n\omega_0 t}{2n\omega_0}\right]_{t_0}^{t_0+T}$$

$$= \frac{1}{2}\left[t_0+T-t_0+\frac{\sin 2n\omega_0(t_0+T)}{2n\omega_0} - \frac{\sin 2n\omega_0 t_0}{2n\omega_0}\right]$$

$$= \frac{1}{2}\left[T + \frac{\sin(2n\omega_0 t_0 + 2n\omega_0 \frac{2\pi}{\omega_0})}{2n\omega_0} - \frac{\sin 2n\omega_0 t}{2n\omega_0}\right]$$

$$= \frac{1}{2}\left[T + \frac{1}{2n\omega_0}\left\{\sin(2n\omega_0 t + 4n\pi) - \sin(2n\omega_0 t_0)\right\}\right]$$

$$= \frac{1}{2}\left[T + \frac{1}{2n\omega_0}\left\{\sin(2n\omega_0 t_0) - \sin(2n\omega_0 t_0)\right\}\right]$$

$$= \frac{1}{2}\left[T+0\right] = T/2$$

$$a_n = \frac{\int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t\, dt}{\int_{t_0}^{t_0+T} \cos^2 n\omega_0 t\, dt} = \frac{\int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t\, dt}{T/2}$$

$$\boxed{a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t\, dt}$$

let $n=0$,

$$a_0 = \frac{\int_{t_0}^{t_0+T} f(t)\cos(0)\, dt}{\int_{t_0}^{t_0+T} \cos^2(0)\, dt}$$

$$a_0 = \frac{\int_{t_0}^{t_0+T} f(t)\, dt}{\int_{t_0}^{t_0+T} (1)\, dt}$$

$$a_0 = \frac{\int_{t_0}^{t_0+T} f(t)\, dt}{\left[t\right]_{t_0}^{t_0+T}} = \int_{t_0}^{t_0+T} f(t)\, dt$$

$$\boxed{a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,dt}$$

& $$b_n = \frac{\int_{t_0}^{t_0+T} f(t)\sin n\omega_0 t\,dt}{\int_{t_0}^{t_0+T} \sin^2\omega_0 t\,dt}$$

$$\boxed{b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin n\omega_0 t\,dt}$$

The constant term a_0 is the average value of $f(t)$ over the interval (t_0, t_0+T), & thus a_0 is the dc component of $f(t)$ over this interval.

→ Alternate form of the trigonometric series :-

we have

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n\cos n\omega_0 t + b_n\sin n\omega_0 t)$$

$$a_n\cos n\omega_0 t + b_n\sin n\omega_0 t = A_n\cos[n\omega_0 t + \phi_n]$$

where $A_n = \sqrt{a_n^2 + b_n^2}$ &

$$\phi_n = -\tan^{-1}\left(\frac{b_n}{a_n}\right)$$

$$\boxed{f(t) = a_0 + \sum_{n=1}^{\infty} A_n\cos(n\omega_0 t + \phi_n)}$$

the coefficients A_n are called spectral amplitudes & ϕ_n is the spectral phase.

||ly
$$F_n = \frac{\int_{t_0}^{t_0+T} f(t)\,(e^{jn\omega_0 t})\,dt}{\int_{t_0}^{t_0+T} (e^{jn\omega_0 t})(e^{-jn\omega_0 t})\,dt}$$

$$\boxed{F_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\, e^{-jn\omega_0 t}\, dt}$$

Thus any fn may be expressed as a discrete sum of exponential functions $\{e^{jn\omega_0 t}\}$, $(n = 0, \pm 1, \pm 2 \ldots)$ over an interval $t_0 < t < t_0 + T$.

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$\& \quad F_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\, e^{-jn\omega_0 t}\, dt$$

These two eq are referred as Fourier series pair

There exponential fourier series:

→ Relation b/w the trigonometric & the exponential fourier series:

Now consider an exponential fourier series:

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \quad (t_0 < t < t_0 + T)$$

$$= F_0 + F_1 e^{j\omega_0 t} + F_2 e^{j2\omega_0 t} \ldots F_n e^{jn\omega_0 t} \ldots$$
$$F_{-1} e^{j\omega_0 t} + F_{-2} e^{j2\omega_0 t} \ldots F_{-n} e^{-jn\omega_0 t} \ldots \quad \text{—— (A)}$$

where $F_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\, e^{-jn\omega_0 t}\, dt$ —— (1)

||ly $F_{-n} = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\, e^{jn\omega_0 t}\, dt$ —— (2)

from (1) & (2) F_n & F_{-n} are complex conjugates

i.e $F_n = F_{-n}^*$

Now let $F_n = \alpha_n + j\beta_n$ —— (3)

$F_{-n} = \alpha_n - j\beta_n$ —— (4)

adding these two eq we get & subtracting

$$\alpha_n = \frac{1}{2}(F_n + F_{-n}) \quad ③$$

$$\beta_n = \frac{1}{2j}(F_n - F_{-n})$$

(or) $2\alpha_n = F_n + F_{-n}$ ③

$-2\beta_n = +j(F_n - F_{-n})$

sub ③ & ④ in eq Ⓐ

$$f(t) = F_0 + (\alpha_1 + j\beta_1)e^{j\omega_0 t} + (\alpha_2 + j\beta_2)e^{j2\omega_0 t} \ldots + (\alpha_n + j\beta_n)e^{jn\omega_0 t} + \ldots (\alpha_1 - j\beta_1)e^{-j\omega_0 t} + (\alpha_2 - j\beta_2)e^{-j2\omega_0 t} \ldots (\alpha_n - j\beta_n)e^{-jn\omega_0 t} + \ldots$$

$$f(t) = F_0 + \Big[(\alpha_1 e^{j\omega_0 t} + \alpha_2 e^{j2\omega_0 t} \ldots \alpha_{+1} e^{-j\omega_0 t} + \alpha_{+2} e^{-j2\omega_0 t} \ldots \alpha_n e^{jn\omega_0 t} + \alpha_{+n} e^{-jn\omega_0 t})$$

$$+ j(\beta_1 e^{j\omega_0 t} + \beta_2 e^{j2\omega_0 t} + \ldots \beta_n e^{jn\omega_0 t} + \beta_{+1} e^{-j\omega_0 t} + \beta_{+2} e^{-j2\omega_0 t} \ldots - \beta_{+n} e^{-jn\omega_0 t} \ldots)\Big]$$

$$f(t) = F_0 + \sum_{n=-\infty}^{\infty} \left[\alpha_n(e^{jn\omega_0 t} + e^{-jn\omega_0 t}) + j\beta_n(e^{jn\omega_0 t} - e^{-jn\omega_0 t})\right]$$

$$f(t) = F_0 + \sum_{n=-\infty}^{\infty} \left[2\alpha_n\left(\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}\right) + j^2 2\beta_n\left(\frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}\right)\right]$$

$$f(t) = F_0 + \sum_{n=-\infty}^{\infty} \left[2\alpha_n \cos n\omega_0 t - 2\beta_n \sin n\omega_0 t\right]$$

Now compare this with the standard trignometric eq

$$f(t) = a_0 + \sum_{n=-\infty}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$\Rightarrow F_0 = a_0 \;\Big|\; 2\alpha_n = a_n \;\Big|\; -2\beta_n = b_n$$

$a_0 = F_0$

$a_n = 2\alpha_n = F_n + F_{-n}$

$b_n = -2\beta_n = j(F_n - F_{-n})$

This is the representation of trigonometric in terms of exponential

Ily we can " exponential in terms of trigonometric as is

$a_n = F_n + F_{-n} \qquad b_n = j(F_n - F_{-n})$

$\Rightarrow \frac{b_n}{j} = F_n - F_{-n}$

$a_n = F_n + F_{-n}$ — (5) $\Rightarrow -jb_n = F_n - F_{-n}$ — (6)

Adding & subtracting (5) & (6) we get

$F_n = \frac{1}{2}[a_n - jb_n]$ &

$F_{-n} = \frac{1}{2}[a_n + jb_n]$

→ Representation of a periodic fn by the fourier series over the entire interval $(-\infty < t < \infty)$:

Upto know we represent a given fn $f(t)$ by the F S over a finite interval (t_0, t_0+T) & outside this interval, the fn & its corresponding F S are need not be equal. This equality b/w $f(t)$ & its series holds over the interval (t_0, t_0+T), Now we want that this equality holds over the entire interval $(-\infty < t < \infty)$

Now we consider some function $f(t)$ & its exponential F.S representation over an interval (t_0, t_0+T)

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \quad (t_0 < t < t_0+T) \text{ — (1)}$$

where $\omega_0 = \frac{2\pi}{T}$

The two sides of the equation need not be equal outside this interval.

Let the right-hand side of (1) be $\phi(t)$

Thus $f(t) = \phi(t) \quad (t_0 < t < t_0+T)$.

adding these two eqs & subtracting

$$\alpha_n = \frac{1}{2}(F_n + F_{-n}) \quad ③$$

$$\beta_n = \frac{1}{2j}(F_n - F_{-n})$$

(or) $2\alpha_n = F_n + F_{-n} \quad ③$

$$-2\beta_n = +j(F_n - F_{-n})$$

sub ③ & ④ in eq Ⓐ

$$f(t) = F_0 + (\alpha_1 + j\beta_1)e^{j\omega_0 t} + (\alpha_2 + j\beta_2)e^{j2\omega_0 t} \ldots + (\alpha_n + j\beta_n)e^{jn\omega_0 t} + \ldots (\alpha_1 - j\beta_1)e^{-j\omega_0 t} + (\alpha_2 - j\beta_2)e^{-j2\omega_0 t} \ldots (\alpha_n - j\beta_n)e^{-jn\omega_0 t} + \ldots$$

$$f(t) = F_0 + \Big[(\alpha_1 e^{j\omega_0 t} + \alpha_2 e^{j2\omega_0 t} \ldots \alpha_{+1} e^{-j\omega_0 t} + \alpha_{+2} e^{-j2\omega_0 t} \ldots \alpha_n e^{jn\omega_0 t} + \alpha_{+n} e^{-jn\omega_0 t}) + j(\beta_1 e^{j\omega_0 t} + \beta_2 e^{j2\omega_0 t} + \ldots \beta_n e^{jn\omega_0 t} + \beta_{+1} e^{-j\omega_0 t} + \beta_{+2} e^{-j2\omega_0 t} \ldots - \beta_{+n} e^{-jn\omega_0 t} \ldots)\Big]$$

$$f(t) = F_0 + \sum_{n=-\infty}^{\infty}\Big[\alpha_n(e^{jn\omega_0 t} + e^{-jn\omega_0 t}) + j\beta_n(e^{jn\omega_0 t} - e^{-jn\omega_0 t})\Big]$$

$$f(t) = F_0 + \sum_{n=-\infty}^{\infty}\Big[2\alpha_n\Big(\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}\Big) + j^2 2\beta_n\Big(\frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}\Big)$$

$$f(t) = F_0 + \sum_{n=-\infty}^{\infty}\Big[2\alpha_n \cos n\omega_0 t - 2\beta_n \sin n\omega_0 t\Big]$$

Now compare this with the standard trignometric eq

$$f(t) = a_0 + \sum_{n=-\infty}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$\Rightarrow F_0 = a_0 \;\Big|\; 2\alpha_n = a_n \;\Big|\; -2\beta_n = b_n$$

where $\phi(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$ (for all values of t)

Now consider the fn '$\phi(t+T)$',

$$\phi(t+T) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0(t+T)}$$

$$= \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} e^{jn\omega_0 T}$$

$$\phi(t+T) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} e^{j2\pi n}$$

$$\phi(t+T) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} = \phi(t)$$

$$\phi(t+T) = \phi(t)$$

i.e, the fn $\phi(t)$ repeats itself after every T seconds, such a fn is called a periodic fn.

ie the exponential (or Trignometric) F S ~~depend~~ repeats the -lues every T seconds. Thus if $f(t)$ be a periodic fn of period T, then it can be represented by an exponential (or) trignometric) F.S over the entire interval $(-\infty < t < \infty)$.

$\therefore$ A periodic fn $f(t)$ with period T can be rep by a F.S over the entire interval $(-\infty < t < \infty)$,

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \quad (-\infty < t < \infty)$$

where $\omega_0 = \frac{2\pi}{T}$

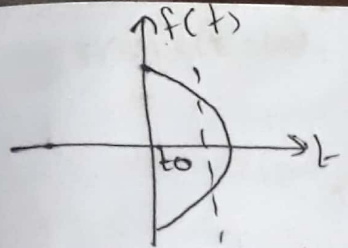
& $F_n = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) e^{-jn\omega_0 t} dt$

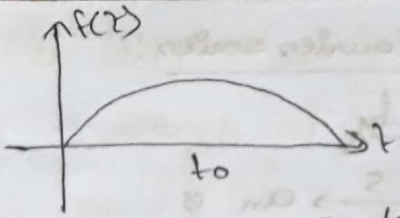
→ Fourier series - Dirichlet's conditions:

The sufficient conditions under which a s/g $f(t)$ can be represented in terms of its fourier series must satisfy are called dirichlet's conditions. They are

(i) The fn $f(t)$ is a single-valued fn of the variable 't' in the interval (t, t_n)

ie the fn $f(t)$ must have single value at any instant of time

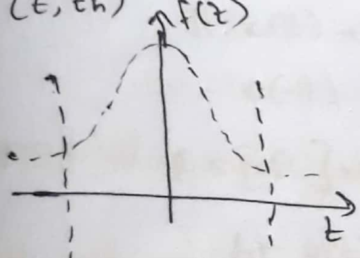


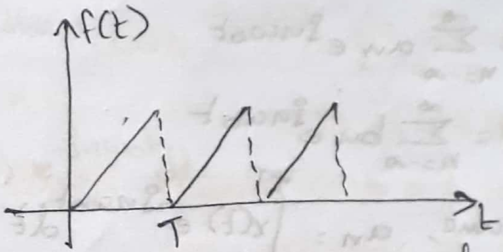


At t_0, it has 2 values so it, is not a single valued fn

it is a single valued fn

ii) The fn $f(t)$ has a finite number of discontinuities in the interval (t, t_n)



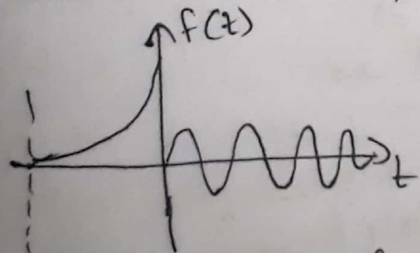


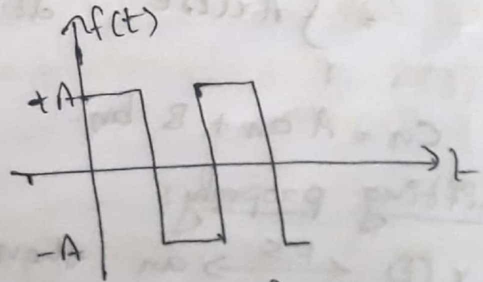
it has no finite number of discontinuities as it is not possible to find the value of $f(t)$ at such a number of discontinuities

It has finite number of discontinuities so the value of $f(t)$ at the discontinuities can be calculated

$$f(t=T) = \frac{f(T^-) + f(T^+)}{2}$$

iii) The fn $f(t)$ has a finite number of minima or maxima in the interval (t_1, t_2)



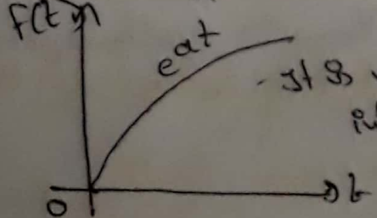


It has no fixed number of minima & maxima

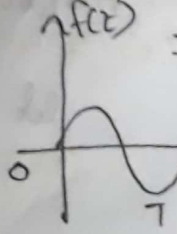
It has fixed number of minima & maxima (1)

iv) The fn $f(t)$ is absolutely integrable

$$\text{i.e. } \int_{t_1}^{t_2} |f(t)|\, dt < \infty$$



It is not absolutely integrable



It is absolutely integrable

where $\phi(t) = \sum_{n=-\infty} F_n e^{jn\omega_0 t}$ (for all values of t)

Now consider the fn '$\phi(t+T)$',

$$\phi(t+T) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0(t+T)}$$

$$= \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \cdot e^{jn\omega_0 T}$$

$$\phi(t+T) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} e^{j2\pi n}$$

$$\phi(t+T) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} = \phi(t)$$

$$\phi(t+T) = \phi(t)$$

i.e, the fn $\phi(t)$ repeats itself after every T seconds, such a fn is called a periodic fn.

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∴ A periodic fn f(t) with period T can be rep by a F.S over the entire interval $(-\infty < t < \infty)$,

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \quad (-\infty < t < \infty)$$

where $\omega_0 = \frac{2\pi}{T}$

$$\& \quad F_n = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) e^{-jn\omega_0 t} dt$$

→ Fourier series - Dirichlet's conditions:

The sufficient conditions under which a s/g f(t) can be represented in terms of its fourier series must satisfy are called dirichlet's conditions. They are

(i) The fn f(t) is a single-valued fn of the variable 't' in the interval (t, t_n)

i.e the fn f(t) must have single value at any instant of time

→ Properties of Fourier series:

1) linearity property:

If $x(t) \xleftrightarrow{F.S} a_n$ &

$y(t) \xleftrightarrow{F.S} b_n$ then

$$A x(t) + B y(t) \xleftrightarrow{FS} A a_n + B b_n$$

Proof: we have

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}$$

$$\& \ y(t) = \sum_{n=-\infty}^{\infty} b_n e^{jn\omega_0 t}$$

Also we have, $a_n = \int_{T} x(t) e^{-jn\omega_0 t} dt$

$$\& \ b_n = \int_T y(t) e^{-jn\omega_0 t} dt$$

let $c(t) = A x(t) + B y(t)$ then,

$$c_n = \int_T [A x(t) + B y(t)] e^{-jn\omega_0 t} dt$$

$$= \int_T A x(t) e^{-jn\omega_0 t} dt + \int_T B y(t) e^{-jn\omega_0 t} dt$$

$$c_n = A a_n + B b_n$$

2) time shifting property:

If $x(t) \xleftrightarrow{FS} a_n$ then

$$x(t - t_0) \xleftrightarrow{FS} e^{-jn\omega_0 t_0} a_n$$

proof: we have

$$FS[x(t)] = a_n = \int_T x(t) e^{-jn\omega_0 t} dt$$

$$FS[x(t-t_0)] = \int_T x(t-t_0) e^{-jn\omega_0 t} dt$$

$$F(S)[x(t-t_0)] = \int_T x(p)\, e^{-jn\omega_0(p+t)}\, dp$$

$$= \int_T x(p) e^{-jn\omega_0 p} \cdot e^{-jn\omega_0 t_0}\, dp$$

$$= e^{-jn\omega_0 t_0} \int_T x(p)\, e^{-jn\omega_0 p}\, dp$$

$$= e^{-jn\omega_0 t_0}\, a_n$$

3) <u>time</u> - <u>Reversal</u> <u>property</u>:

If $x(t) \xleftrightarrow{FS} a_n$ then

$x(-t) \xleftrightarrow{FS} a_{-n}$

proof: $FS[x(t)] = a_n = \int_T x(t)\, e^{-jn\omega_0 t}\, dt$

let $y(t) = x(-t)$

$$F.S[y(t)] = \int_T y(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \int_T x(-t)\, e^{-jn\omega_0 t}\, dt$$

let $p = -t \Rightarrow dp = -dt$

$$= \int_{t_0}^{t_0+T} x(-t)\, e^{-jn\omega_0 t}\, dt$$

we have

$p = -t \qquad dp = -dt$

$$= \int_{-t_0}^{-(t_0+T)} x(p)\, e^{jpn\omega_0} (-dp)$$

$$= \int_{t_0}^{t_0+T} x(p)\, e^{jn\omega_0 p}\, dp$$

$= a_{-n}$ is the FS coefficients of the time reversal of a s/g are time reversal of the F.S coefficients of the corresponding signal.

4) Time scaling:

$x(t) \xleftrightarrow{FS} a_n$

then $x(at) \xleftrightarrow{FS} a_n$ but the fundamental fq is $a\omega_0$

$FS[x(t)] =$

$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}$, then

$x(at) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 at}$

where 'a' is the scaling factor.

5) Frequency shifting:

If $x(t) \xleftrightarrow{FS} a_n$ then

$x(t) e^{jm\omega_0 t} \xleftrightarrow{FS} a_{n-m}$

proof: $FS[x(t)] = a_n = \int_{<T>} x(t) e^{-jn\omega_0 t} dt$

$FS[x(t) e^{jm\omega_0 t}] = \int_{<T>} x(t) e^{jm\omega_0 t} \cdot e^{-jn\omega_0 t} dt$

$= \int_T x(t) e^{-j(n-m)\omega_0 t} dt$

$= a_{n-m}$

6) Conjugation:

If $x(t) \xleftrightarrow{FS} a_n$ then

$x^*(t) \xleftrightarrow{FS} a^*_{-n}$

$FS[x(t)] = a_n = \int_{<T>} x(t) e^{-jn\omega_0 t} dt$

$FS[x^*(t)] = \int_{<T>} x^*(t) (e^{-jn\omega_0 t})^* dt$

$FS[x^*(t)] = \int_{<T>} x^*(t) e^{jn\omega_0 t} dt$

$= a^*_{-n}$

→ Symmetry conditions:

1) It is a periodic fn is symmetrical about the vertical axis, the corresponding fourier series contains only cosine terms.

2) It is a periodic fn is antisymmetrical about the vertical axis, the F.S contains sine terms only.

To prove this,

Consider a fn $f_e(t)$, it is said to be an even fn of it, if

$$f_e(t) = f_e(-t)$$

& $f_o(t)$ is said to be an odd fn of 'f' it

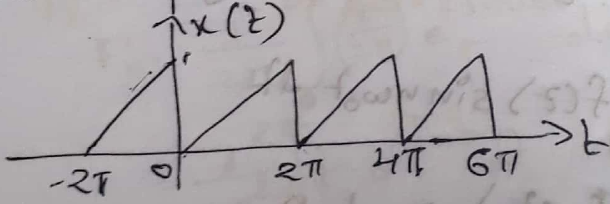
$$f_o(t) = -f_o(-t)$$

Properties of even & odd fn:

1) product of an even & an odd fn is an odd fn
2) " even & ~~odd~~ even fn "
3) " odd & even fn is odd fn
4) " odd & odd " even fn

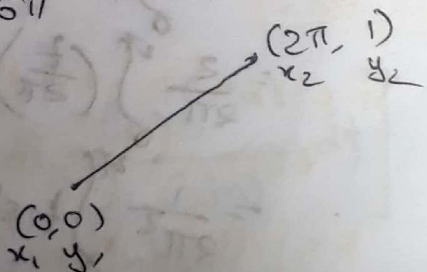
Problems

→ Find the cosine representation FS for the signal shown in fig



Time period, $T = 2\pi$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$



$$y - y_1 = m(x - x_1)$$

$$x(t) - 0 = \frac{1-0}{2\pi - 0}(t - 0)$$

$$x(t) = \frac{1}{2\pi}(t)$$

$$x(t) = \frac{t}{2\pi} \text{ for } 0 \le t < 2\pi \quad \text{for } 0 \le t < T$$

Multiply t with amplitude & divide with period

$$\frac{(1)t}{2\pi} = \frac{t}{2\pi} \text{ for a given interval}$$

we have the trigonometric F.S.

$$f(t) = a_0 + \sum_{n=1} \left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right)$$

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{t}{2\pi}\,dt = \frac{1}{2}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\,dt$$

$$= \frac{2}{2\pi}\int_0^{2\pi}\left(\frac{t}{2\pi}\right)\cos(nt)\,dt$$

$$= \frac{1}{2\pi^2}\left[t\int \cos nt\,dt - \int (1)\frac{\sin nt}{n}\,dt\right]_0^{2\pi}$$

$$= \frac{1}{2\pi^2}\left[\frac{t\sin nt}{n} + \frac{\cos nt}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi^2}\left[0 + \frac{1}{n^2} - 0 - \frac{1}{n^2}\right] = 0$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t\,dt$$

$$= \frac{2}{2\pi}\int_0^{2\pi}\left(\frac{t}{2\pi}\right)\sin nt\,dt$$

$$= \frac{1}{2\pi^2}\int_0^{2\pi} t\sin nt\,dt$$

$$b_n = \frac{1}{2\pi^2}\left[t\int \sin nt\,dt - \int (1)\frac{-\cos nt}{n}\,dt\right]_0$$

$$= \frac{1}{2\pi^2}\left[\frac{-t\cos nt}{n} + \frac{\sin nt}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi^2}\left[\frac{(-2\pi)\cos 2\pi n}{n} + 0 + 0 + 0\right]$$

$$\frac{1}{2\pi^2}\left[\frac{(-2\pi)\cos 2\pi n}{n} + 0 + 0 + 0\right]$$

$$= \frac{-1}{\pi n}\cos 2\pi n = \frac{-1}{n\pi}(1) = \frac{-1}{n\pi}$$

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right]$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty}\left[0 + \frac{-1}{n\pi}\sin nt\right]$$

$$= \frac{1}{2} - \sum_{n=1}^{\infty}\frac{1}{n\pi}\sin(nt)$$

$$\therefore f(t) = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin nt}{n}$$

→ find out above for exponential F.S.

$$f(t) = \frac{t}{2\pi} \text{ for } 0 \le t \le 2\pi$$

we have, $f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$

$$F_n = \frac{1}{T}\int_0^T f(t)\, e^{-jn\omega_0 t}\, dt$$

$$F_n = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{t}{2\pi}\right) e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{4\pi^2}\int_0^{2\pi} t \cdot e^{-jnt}\, dt$$

$$= \frac{1}{4\pi^2}\left[t\int e^{-jnt}\, dt - \int \frac{(1)e^{-jnt}}{(-jn)}\, dt\right]$$

$$= \frac{1}{4\pi^2}\left[\frac{+te^{-jnt}}{-jn} - \frac{e^{-jnt}}{j^2 n^2}\right]$$

$$= \frac{1}{4\pi^2}\left[\frac{te^{-jnt}}{-jn} + \frac{e^{-jnt}}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi^2}\left[\frac{2\pi e^{-j2\pi n}}{-jn} + \frac{e^{-j2\pi n}}{n^2} - 0 - \frac{1}{n^2}\right]$$

$$= \frac{1}{(4\pi)^2}\left[\frac{2\pi(1)}{-jn} + \frac{1}{n^2} - \frac{1}{n^2}\right]$$

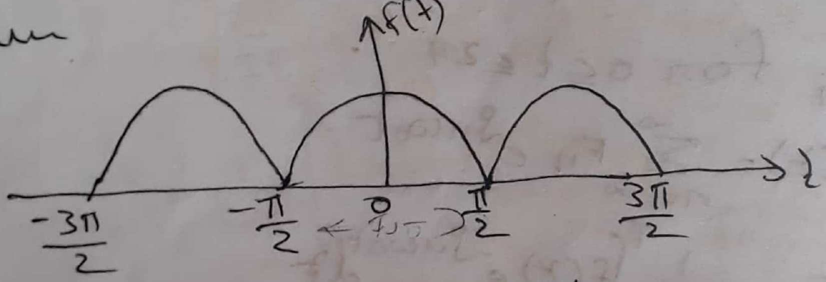
$$= \frac{1}{-j2\pi n} = \frac{j}{2\pi n}$$

$$\Rightarrow f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} \frac{j}{2\pi n} e^{jnt}$$

$$f(t) = \frac{j}{2\pi}\sum_{n=-\infty}^{\infty}\left(\frac{e^{jnt}}{n}\right)$$

→ Det the trignometric FS & exponential FS of a full rectified cosine fn shown in fig. & draw the complex spectrum



let $t_0 = -\frac{\pi}{2}$ | $f(t) = \cos\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

then $t_0 + T = \frac{\pi}{2}$

$$T = \frac{\pi}{2} - t_0 = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{\pi} = 2 \text{ rad/sec}$$

$$f(t) = a_0 + \sum_{n=-\infty}^{\infty}\left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right)$$

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,dt$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} (\cos t)\,dt$$

$$= \frac{1}{\pi}\left[+\sin t\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{\pi}\left[+\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)\right] = \frac{1}{\pi}(1+1)$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t\, dt$$

$$= \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \cos t \cos 2nt\, dt$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \left[\cos(2n+1)t + \cos(2n-1)t\right] dt$$

$$= \frac{1}{\pi}\left[\frac{\sin(2n+1)t}{(2n+1)} + \frac{\sin(2n-1)t}{(2n-1)}\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{\pi}\left[\frac{\sin(2n+1)\frac{\pi}{2}}{(2n+1)} - \frac{\sin(2n+1)(-\frac{\pi}{2})}{(2n+1)} + \frac{\sin(2n-1)\frac{\pi}{2}}{(2n-1)} - \frac{\sin(2n-1)(-\frac{\pi}{2})}{(2n-1)}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\sin(2n+1)\frac{\pi}{2}}{(2n+1)} + \frac{2\sin(2n-1)\frac{\pi}{2}}{(2n-1)}\right]$$

$$a_n = \frac{2}{\pi}\left[\frac{(-1)^n}{(2n+1)} + \frac{(-1)^{n+1}}{2n-1}\right]$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin n\omega_0 t\, dt$$

$$= \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \cos t \sin 2nt\, dt$$

$$b_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \cos t \sin 2t\, dt$$

$$= \frac{1}{\pi}\left[\int_{-\pi/2}^{\pi/2}\left[\sin(2n+1)t + \sin(2n-1)t\right\} dt\right]$$

$$= \frac{1}{\pi}\left[\frac{-\cos(2n+1)t}{2n+1} - \frac{\cos(2n-1)t}{2n-1}\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{\pi}\left[\frac{-\cos(2n+1)\frac{\pi}{2}}{2n+1} + \frac{\cos(2n+1)(\frac{-\pi}{2})}{(2n+1)} - \frac{\cos(2n-1)\frac{\pi}{2}}{(2n-1)} + \frac{\cos(2n-1)(-\frac{\pi}{2})}{2n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{-\cos(2n+1)\frac{\pi}{2}}{(2n+1)} + \frac{\cos(2n+1)\frac{\pi}{2}}{2n+1} - \frac{\cos(2n-1)\frac{\pi}{2}}{(2n-1)} + \frac{\cos(2n-1)\frac{\pi}{2}}{(2n-1)}\right]$$

$$= \frac{0}{2}$$

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right]$$

$$f(t) = \frac{2}{\pi} + \sum_{n=1}^{\infty}\left[\frac{2}{\pi}\left(\frac{(-1)^n}{2n+1} + \frac{(-1)^{n+1}}{2n-1}\right)\cos 2nt + 0\right]$$

Exponential F.S:

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$F_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\, e^{-jn\omega_0 t}\, dt$$

$$F_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos t \cdot e^{-j2nt}\, dt$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\left(\frac{e^{jt}+e^{-jt}}{2}\right)e^{-j2nt}\, dt$$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\left[e^{jt}\cdot e^{-j2nt} + e^{-jt}\cdot e^{-j2nt}\right] dt$$

$$= \frac{1}{2\pi}\left[\int_{-\pi/2}^{\pi/2} e^{-j(+2n-1)t}\, dt + \int_{-\pi/2}^{\pi/2} e^{-j(2n+1)t}\, dt\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j(2n-1)t}}{-j(2n-1)} + \frac{e^{-j(2n+1)t}}{-j(2n+1)}\right]_{-\pi/2}^{\pi/2}$$

$$\frac{1}{\ldots}\left[\frac{e^{-j(2n-1)t}}{\ldots} + e^{-j(2n+1)t}\ldots\right]^{\pi/2}$$

$$\frac{1}{2\pi}\left[\frac{e^{j(2n-1)\frac{\pi}{2}} - e^{-j(2n-1)\frac{\pi}{2}}}{+j(2n-1)} + \frac{e^{j(2n+1)\frac{\pi}{2}} - e^{-j(2n+1)\frac{\pi}{2}}}{j(2n+1)}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(2n-1)\frac{\pi}{2}}{(2n-1)} + \frac{\sin(2n+1)\frac{\pi}{2}}{(2n+1)}\right]$$

$$F_n = \frac{1}{\pi}\left[\frac{(-1)^{n+1}}{2n-1} + \frac{(-1)^n}{2n+1}\right]$$

$$F_0 = \frac{1}{\pi}\left[\frac{(-1)^1}{-1} + \frac{1}{1}\right] = \frac{1}{\pi}[1+1] = \frac{2}{\pi}$$

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{1}{\pi}\left[\frac{(-1)^{n+1}}{(2n-1)} + \frac{(-1)^n}{2n+1}\right] e^{j2nt}$$

$$F_n = \frac{1}{\pi}\left[\frac{(-1)^n}{2n+1} + \frac{(-1)^{n+1}}{2n-1}\right]$$

$$F_1 = \frac{1}{\pi}\left[\frac{(-1)^1}{3} + \frac{(-1)^2}{2-1}\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{3} + 1\right] = \frac{1}{\pi}\left[1 - \frac{1}{3}\right] = \frac{1}{\pi}\left[\frac{2}{3}\right] = \frac{2}{3\pi}$$

$$F_2 = \frac{1}{\pi}\left[\frac{(-1)^2}{4+1} + \frac{(-1)^{2+1}}{4-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{5} - \frac{1}{3}\right] = \frac{1}{\pi}\left[\frac{3-5}{15}\right] = \frac{-2}{15\pi}$$

Thus the F-S is

$$f(t) = \frac{2}{\pi} + \frac{2}{3\pi}e^{j2t} - \frac{2}{15\pi}e^{j4t} - \cdots$$

$$+ \frac{2}{3\pi}e^{-j2t} - \frac{2}{15\pi}e^{-j4t} + \cdots$$

$$= \frac{1}{\pi}\left[\frac{-\cos(2n+1)\frac{\pi}{2}}{(2n+1)} + \frac{\cos(2n+1)\frac{\pi}{2}}{2n+1} - \frac{\cos(2n-1)\frac{\pi}{2}}{(2n-1)} + \frac{\cos(2n-1)\frac{\pi}{2}}{(2n-1)}\right]$$

$$= \frac{0}{2}$$

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right)$$

$$f(t) = \frac{2}{\pi} + \sum_{n=1}^{\infty}\left[\frac{2}{\pi}\left(\frac{(-1)^n}{2n+1} + \frac{(-1)^{n+1}}{2n-1}\right)\cos 2nt + 0\right]$$

for exponential F.S :

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$F_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\, e^{-jn\omega_0 t}\, dt$$

$$F_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos t \cdot e^{-j2nt}\, dt$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\left(\frac{e^{jt}+e^{-jt}}{2}\right) e^{-j2nt}\, dt$$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\left[e^{jt} \cdot e^{-j2nt} + e^{-jt} \cdot e^{-j2nt}\right] dt$$

$$= \frac{1}{2\pi}\left[\int_{-\pi/2}^{\pi/2} e^{-j(+2n-1)t}\, dt + \int_{-\pi/2}^{\pi/2} e^{-j(2n+1)t}\, dt\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j(2n-1)t}}{-j(2n-1)} + \frac{e^{-j(2n+1)t}}{-j(2n+1)}\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j(2n-1)t}}{-j(2n-1)} + \frac{e^{-j(2n+1)t}}{-j(2n+1)}\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j(2n-1)\frac{\pi}{2}}}{-j(2n-1)} - \frac{e^{-j(2n-1)\left(-\frac{\pi}{2}\right)}}{-j(2n-1)} + \frac{e^{-j(2n+1)\pi/2}}{-j(2n+1)} - \ldots\right.$$

→ Fourier transforms:

Deriving fourier transform from fourier series (or) representation of an arbitary fn over the entire interval $(-\infty, \infty)$:

As we know that any non periodic signal can be represented in terms of its sum of exp fns over any finite interval $(t_0 < t < t_0+T)$ & any periodic signal can be represented in $(-\infty, \infty)$.

Now we want to represent an arbitary fn (non periodic) as a sum of exponential fn over the entire interval $(-\infty < t < \infty)$

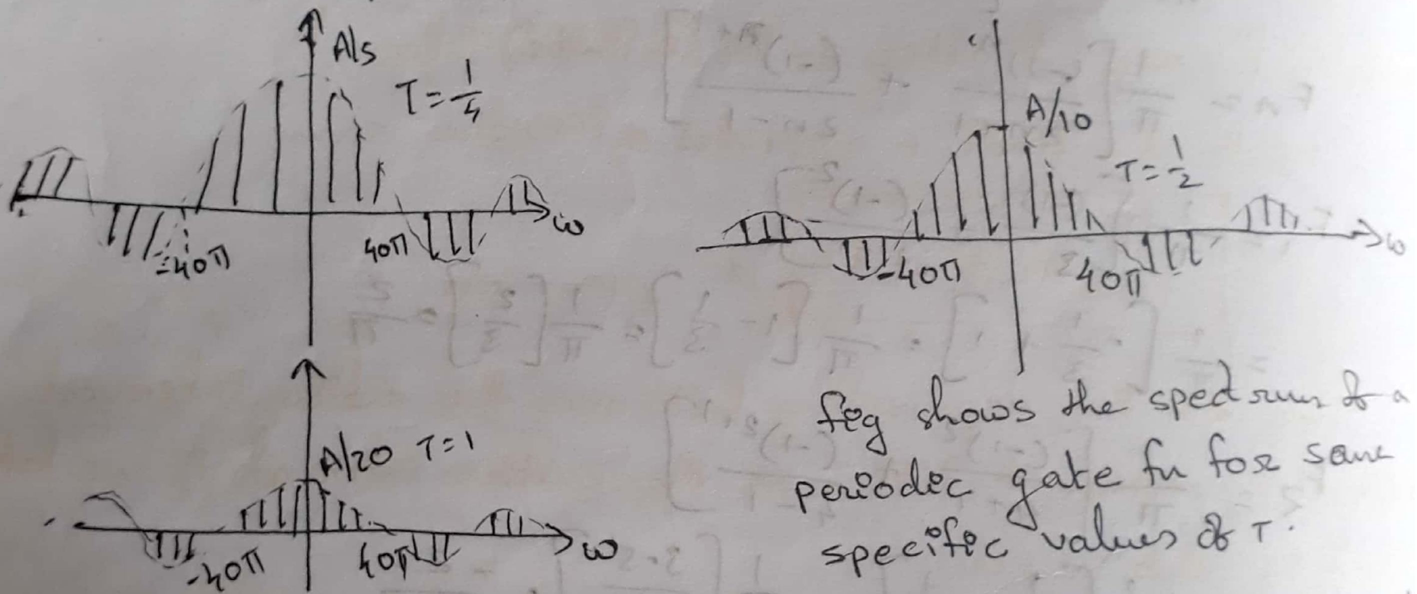


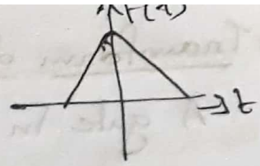
fig shows the spectrum of a periodic gate fn for some specific values of T.

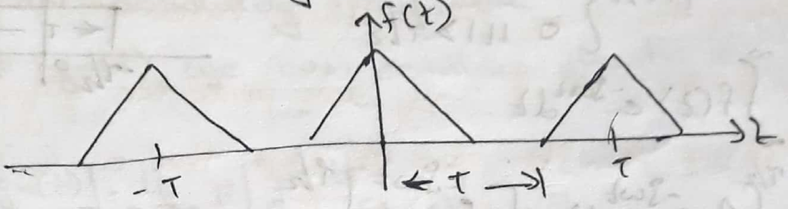
If we can observe the spectrum, then as the period 'T' is made larger, the fundamental fq becomes smaller. The fq spectrum becomes denser. But the amplitudes of the fq components becomes smaller.

The shape of the spectrum remains unaltered.

Now, consider an arbitary fn f(t), we want to represent this function as a sum of exponential fns over the entire interval $(-\infty < t < \infty)$

This can be achieved by constructing a new periodic fn $f_T(t)$ of period T, where the fn $f(t)$ repeats itself for every T seconds.





Now this fn $f_T(t)$ is a periodic fn & it can be represented with exponential FS over the entire interval $(-\infty,\infty)$

In the limit, if T becomes ∞, then the pulses in the periodic fn repeat after an ∞ (infinite) Interval

i.e in the limit $T\to\infty$ $f_T(t)$ & $f(t)$ are same

$$\boxed{\lim_{T\to\infty} f_T(t) = f(t)}$$

Thus the FS representing $f_T(t)$ over the entire interval will also represent $f(t)$ over the entire interval if we take $T\to\infty$ in this series

The expone FS for $f_T(t)$ can be represented by,

$$f_T(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

where $\omega_0 = \frac{2\pi}{T}$

$$\therefore F_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t)\, e^{-jn\omega_0 t}\, dt$$

→ Fourier Transform of signum fn:

$$\mathrm{sgn}(t) = 1,\ t>0$$
$$= -1,\ t<0$$

This is not absolutely integrable

so, instead of sgn(t), we can consider the fn $e^{-a|t|}\,\mathrm{sgn}(t)$ as the limit $a \to 0$

$$F[\mathrm{sgn}(t)] = \lim_{a\to 0} F\left[e^{-a|t|}\,\mathrm{sgn}(t)\right]$$

$$= \lim_{a\to 0} \int_{-\infty}^{\infty} e^{-a|t|}\,\mathrm{sgn}(t)\, dt\, e^{-j\omega t}$$

$$= \lim_{a\to 0}\left[\int_0^{\infty} e^{-at} e^{-j\omega t} dt - \int_{-\infty}^{0} e^{at}\cdot e^{-j\omega t} dt\right]$$

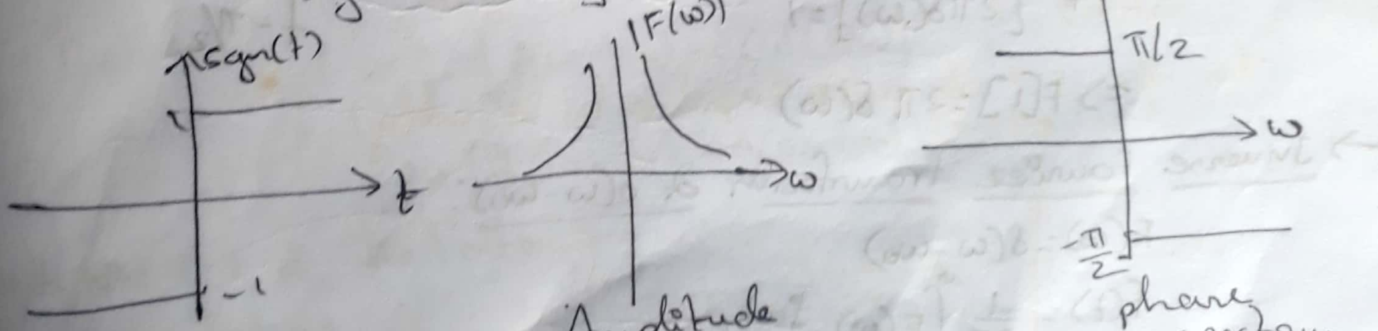
$$= \lim_{a\to 0}\left[\int_0^{\infty} e^{-(a+j\omega)t} dt - \int_{-\infty}^{0} e^{(a-j\omega)t} dt\right]$$

$$= \lim_{a\to 0}\left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\Bigg|_0^{\infty} - \frac{e^{(a-j\omega)t}}{a-j\omega}\Bigg|_{-\infty}^{0}\right]$$

$$= \lim_{a\to 0}\left[+\frac{1}{a+j\omega} - \frac{1}{a-j\omega}\right] = \lim_{a\to 0}\left[\frac{a-j\omega-a-j\omega}{a^2+\omega^2}\right]$$

$$= \lim_{a\to 0}\left[\frac{-2j\omega}{a^2+\omega^2}\right] = \frac{-2j\omega}{\omega^2} = \frac{-2j}{\omega} = \frac{2}{j\omega}$$

$$\mathrm{sgn}(t) \longleftrightarrow \frac{2}{j\omega}$$



Signum fn

Amplitude spectrum

phase spectrum

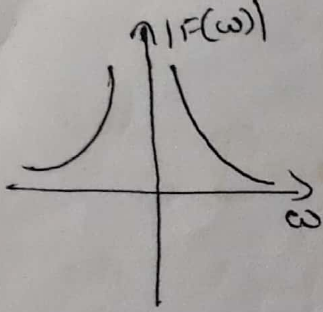
→ **Fourier transform of step fn:**

we have,

$$u(t) = \frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}(t)$$

$$\mathrm{sgn}(t) = 2u(t) - 1$$

$$F(\omega) = F[u(t)] = F\left[\frac{1}{2}\right] + F\left[\frac{1}{2}\,\mathrm{sgn}t\right]$$



$$= \frac{1}{2}F[1] + \frac{1}{2}F[\mathrm{sgn}t]$$

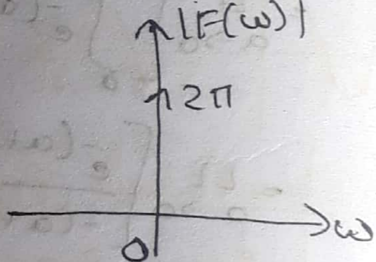
$$= \frac{1}{2}2\pi\,\delta(\omega) + \frac{1}{2}\frac{2}{j\omega}$$

$$= \pi\,\delta(\omega) + \frac{1}{j\omega}$$

→ **Inverse F.T of $\delta(\omega)$:**

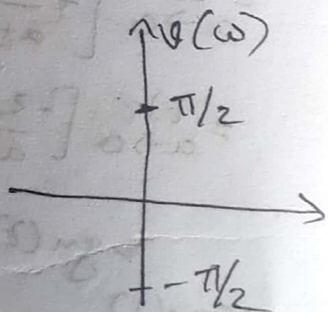
$$F(\omega) = \delta(\omega)$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,e^{j\omega t}\,d\omega$$



$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega)\,e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}(1) = \frac{1}{2\pi}$$



$$F^{-1}[\delta(\omega)] = f(t) = \frac{1}{2\pi}$$

$$F^{-1}[\delta(\omega)] = \frac{1}{2\pi}$$

$$F^{-1}[2\pi\,\delta(\omega)] = 1$$

$$\Rightarrow F[1] = 2\pi\,\delta(\omega)$$

→ **Inverse Fourier transform of $\delta(\omega-\omega_0)$:**

$$F(\omega) = \delta(\omega - \omega_0)$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega-\omega_0)\,e^{j\omega t}\,d\omega$$

$$F^{-1}[\delta(\omega-\omega_0)] = \frac{1}{2\pi}e^{j\omega_0 t}$$

$$F^{-1}[2\pi\delta(\omega-\omega_0)] = e^{j\omega_0 t}$$

$$F[e^{j\omega_0 t}] = 2\pi\delta(\omega-\omega_0)$$

→ **F.T of cosine signal:**

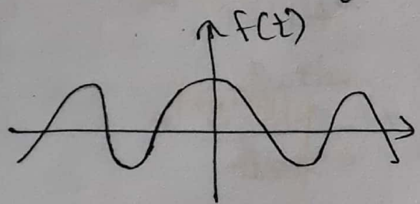
$$f(t) = \cos\omega_0 t = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$$

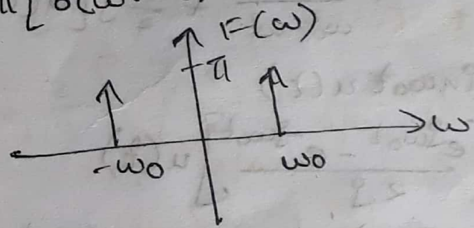
$$F[f(t)] = F(\omega) = F\left[\frac{1}{2}\left\{e^{j\omega_0 t} + e^{-j\omega_0 t}\right\}\right]$$

$$= \frac{1}{2}\left[F[e^{j\omega_0 t}] + F[e^{-j\omega_0 t}]\right]$$

$$= \frac{1}{2}\left[2\pi\delta(\omega+\omega_0) + 2\pi\,\delta(\omega-\omega_0)\right]$$

$$F[\cos\omega_0 t] = \pi\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]$$





→ **F.T of sinusoidal signal:**

$$f(t) = \sin\omega_0 t$$

$$= \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]$$

$$F\{f(t)\} = \frac{1}{2j}\left[F[e^{j\omega_0 t}] - F[e^{-j\omega_0 t}]\right]$$

$$= \frac{1}{2j}\left[2\pi\,\delta(\omega-\omega_0) - 2\pi\delta(\omega+\omega_0)\right]$$

$$= \frac{\pi}{j}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$

$$= j\pi\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]$$

→ Find the F.T of the following

$$f(t) = e^{j\omega_0 t}u(t)$$

$$F[u(t)] = \frac{1}{j\omega} + \pi\delta(\omega)$$

$$F[e^{j\omega_0 t}u(t)] = \frac{1}{j(\omega-\omega_0)} + \pi\,\delta(\omega-\omega_0)$$

→ $f(t) = \sin\omega_0 t\, u(t)$

$$F[\sin\omega_0 t\, u(t)] = F\left[\left\{\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right\} u(t)\right]$$

$$= \frac{1}{2j}\left[F[e^{j\omega_0 t} u(t)] - F[e^{-j\omega_0 t} u(t)]\right]$$

$$= \frac{1}{2j}\left[\left\{\frac{1}{j(\omega-\omega_0)} + \pi\delta(\omega-\omega_0)\right\} - \left\{\frac{1}{j(\omega+\omega_0)} + \pi\delta(\omega+\omega_0)\right\}\right]$$

$$= \frac{1}{2j}\left[\frac{1}{j}\left\{\frac{\omega+\omega_0-\omega+\omega_0}{(\omega-\omega_0)(\omega+\omega_0)}\right\} + \pi\left\{\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right\}\right]$$

$$= \frac{1}{2j}\left[\frac{1}{j}\left\{\frac{2\omega_0}{\omega^2-\omega_0^2}\right\} + \pi\,\delta(\omega-\omega_0) - \pi\,\delta(\omega+\omega_0)\right]$$

$$= \frac{\omega_0}{\omega_0^2-\omega^2} + \frac{\pi}{2} j\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]$$

→ $f(t) = A\sin\omega_0 t\, u(t)$

$$= A\left[\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right] u(t)$$

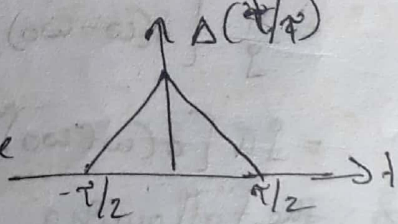
$$= \frac{A}{2j} e^{j\omega_0 t} u(t) - \frac{A}{2j} e^{-j\omega_0 t} u(t)$$

$$F[f(t)] = F\left[\frac{A}{2j} e^{j\omega_0 t} u(t) - \frac{A}{2j} e^{-j\omega_0 t} u(t)\right]$$

$$= \frac{A}{2j}\left[\pi\delta(\omega-\omega_0) + \frac{1}{j(\omega-\omega_0)}\right] - \frac{A}{2j}\left[\pi\delta(\omega+\omega_0) + \frac{1}{j(\omega-\omega_0)}\right]$$

$$= \frac{\pi A}{2j}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right] + \frac{A}{2}\left[\frac{2\omega_0}{\omega^2-\omega_0^2}\right]$$

→ F.T of triangular pulse:



$$\Delta_a(t) = \begin{cases} 1-|t| \;; & |t| < a \\ 0 \;; & \text{otherwise} \end{cases}$$

$$\Delta\left(\frac{t}{\tau}\right) = \begin{cases} 1 - \frac{2|t|}{\tau} \;; & |t| < \tau/2 \\ 0 \;; & \text{otherwise} \end{cases}$$

$$F[\omega] = F\left[\Delta\left(\frac{t}{\tau}\right)\right]$$

$$= \int_{-\tau/2}^{\tau/2}\left[1 - \frac{2|t|}{\tau}\right] e^{-j\omega t}\, dt$$

$$\int_{-\tau/2}^{0}\left(1+\frac{2t}{\tau}\right)e^{-j\omega t}dt+\int_{0}^{\tau/2}\left(1-\frac{2t}{\tau}\right)e^{-j\omega t}dt$$

$$=\int_{-\tau/2}^{0}\left(1+\frac{2t}{\tau}\right)e^{-j\omega t}dt+\int_{0}^{\tau/2}\left(1-\frac{2t}{\tau}\right)e^{-j\omega t}dt$$

$$=\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-\tau/2}^{0}+\frac{2}{\tau}\left[\frac{-te^{-j\omega t}}{j\omega}-\frac{e^{-j\omega t}}{(j\omega)^2}\right]_{-\tau/2}^{0}+\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{0}^{\tau/2}$$

$$-\frac{2}{\tau}\left[\frac{-te^{-j\omega t}}{j\omega}-\frac{e^{-j\omega t}}{(j\omega)^2}\right]_{0}^{\tau/2}$$

$$=-\cancel{\frac{1}{j\omega}}+\frac{e^{-j\omega(-\tau/2)}}{j\omega}+\frac{2}{\tau}\left[0-\frac{1}{(j\omega)^2}-\frac{\frac{\tau}{2}e^{+j\omega\tau/2}}{j\omega}+\frac{e^{j\omega\tau/2}}{(j\omega)^2}\right]$$

$$+\frac{e^{-j\omega\tau/2}}{-j\omega}+\cancel{\frac{1}{j\omega}}-\frac{2}{\tau}\left[\frac{-\frac{\tau}{2}e^{-j\omega\tau/2}}{j\omega}-\frac{e^{-j\omega\tau/2}}{(j\omega)^2}+0+\frac{1}{(j\omega)^2}\right]$$

$$=\cancel{\frac{e^{j\omega\tau/2}}{j\omega}}-\cancel{\frac{e^{-j\omega\tau/2}}{j\omega}}-\frac{2}{\tau(j\omega)^2}-\cancel{\frac{e^{j\omega\tau/2}}{j\omega}}+\frac{2}{\tau}\frac{e^{j\omega\tau/2}}{(j\omega)^2}+\cancel{\frac{e^{-j\omega\tau/2}}{j\omega}}$$

$$+\frac{2}{\tau}\frac{e^{-j\omega\tau/2}}{(j\omega)^2}-\frac{2}{\tau}\frac{1}{(j\omega)^2}$$

$$=\frac{-4}{\tau(j\omega)^2}+\frac{2}{\tau}\frac{e^{j\omega\tau/2}}{(j\omega)^2}+\frac{2}{\tau}\frac{e^{-j\omega\tau/2}}{(j\omega)^2}$$

$$=\frac{2}{\tau}\left[\frac{e^{j\omega\tau/2}}{(j\omega)^2}+\frac{e^{-j\omega\tau/2}}{(j\omega)^2}-\frac{2}{(j\omega)^2}\right]$$

$$=\frac{2}{\tau}\left[\left\{\frac{e^{j\omega\tau/4}}{j\omega}\right\}^2+\left\{\frac{e^{-j\omega\tau/4}}{j\omega}\right\}^2-2\left\{\frac{e^{j\omega\tau/4}}{j\omega}\right\}\left\{\frac{e^{-j\omega\tau}}{j\omega}\right\}\right]$$

$$=\frac{2}{\tau}\left[\left\{\frac{e^{j\omega\tau/4}}{j\omega}-\frac{e^{-j\omega\tau/4}}{j\omega}\right\}^2\right]$$

$$=4\frac{2}{\tau}\left[\frac{e^{j\omega\tau/4}-e^{-j\omega\tau/4}}{2j\omega}\right]^2$$

$$\frac{8}{\tau}\left[\frac{\sin(\omega\tau/4)}{\omega(\tau/4)}\right]^2 \times \left(\frac{\tau}{4}\right)^2$$

$$= \frac{8}{\tau}\cdot\frac{\tau^2}{16}\left[Sa(\omega\tau/4)\right]^2$$

$$= \frac{\cancel{8}}{\cancel{\tau}}\,\frac{\cancel{\tau}}{2}\,\frac{\tau}{2}\,Sa^2\left(\frac{\omega\tau}{4}\right)$$

$$= \frac{\tau}{2}\,\text{sinc}^2\left(\frac{\omega\tau}{4}\right)$$

$$F\left[\Delta\left(\frac{t}{\tau}\right)\right] = \frac{\tau}{2}\,\text{sinc}^2\left(\frac{\omega\tau}{4}\right)$$

→ F.T & Impulse train:

we have the exponential F.S of unit impulse train is

$$f(t) = \sum_{n=-\infty}^{\infty}\delta(t-nT_0) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$$

T_0 is the spacing b/w the

$$F\{f(t)\} = F\left[\sum_{n=-\infty}^{\infty}\delta(t-nT)\right]$$

$$= F\left[\frac{1}{T_0}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}\right]$$

$$= \frac{1}{T_0}\sum_{n=-\infty}^{\infty} F\left[e^{jn\omega_0 t}\right]$$

$$= \frac{1}{T_0}\sum_{n=-\infty}^{\infty}\left[2\pi\,\delta(\omega-n\omega_0)\right]$$

$$= \frac{2\pi}{T_0}\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$$

$$F(\omega) = \omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$$

→ F.T of aperiodic function:

Generally FT is applicable for aperiodic fn, & F.T of a periodic fn does not exist, ∴ it fails to satisfy the condition of absolutely integrability.

But the transform does exist in the limit, lly for $\cos\omega_0 t$ & $\sin\omega_0 t$.

i.e we can assume the periodic fn exists only in the finite interval $(-\tau/2, \tau/2)$ & in the limit let $\tau \to \infty$

with period ∞ as we can express a periodic fn $f(t)$

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

taking F.T on both sides

$$F[f(t)] = F\left[\sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}\right]$$

$$= \sum_{n=-\infty}^{\infty} F_n\, F\left[e^{jn\omega_0 t}\right]$$

$$= \sum_{n=-\infty}^{\infty} F_n\, 2\pi\,\delta(\omega - n\omega_0)$$

$$\therefore F[f(t)] = 2\pi \sum_{n=-\infty}^{\infty} F_n\, \delta(\omega - n\omega_0)$$

∴ The F.T of aperiodic s/g consists of impulses located at the harmonic fq of the signal & the strength of each impulse is same as 2π times the value of the corresponding coefficient in the exponential F.S

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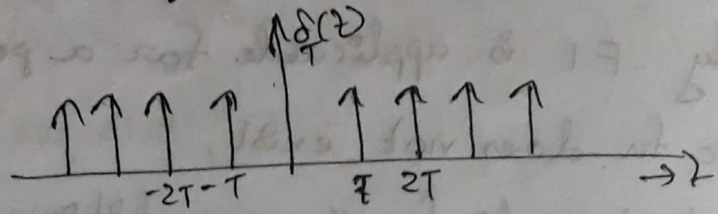
→ Find the F.T of sequence of equidistant impulses.



Now we consider a sequence of equidistant impulses of unit strength & seperated by T sec, & let it be $\delta_T(t)$

$$\delta_T(t) = \delta(t) + \delta(t-T) + \delta(t+2T) \cdots$$
$$+ \delta(t+T) + \delta(t+2T) \cdots$$

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

This is a periodic s/g with period T & then we can find its F.S

The F.S of $\delta_T(t)$ is

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

where $F_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta_T(t) e^{-jn\omega_0 t} dt$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jn\omega_0 t} dt$$

$$F_n = \frac{1}{T}(1) = \frac{1}{T}$$

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{jn\omega_0 t}$$

$$\boxed{\delta_T(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega$$

$$E = \int_{-\infty}^{\infty} f(t) f(t)\, dt$$

$$= \int_{-\infty}^{\infty} f(t) \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega\, dt$$

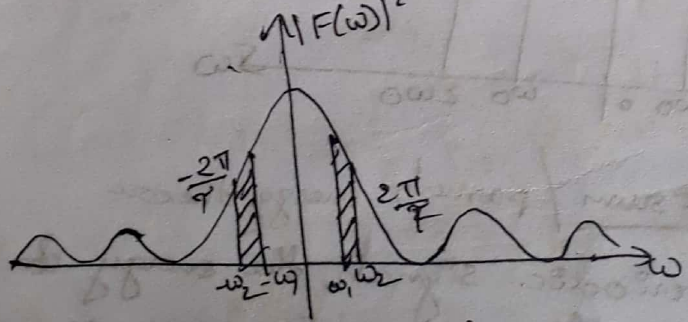
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, d\omega \int_{-\infty}^{\infty} f(t) e^{j\omega t}\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, F(-\omega)\, d\omega$$

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\, d\omega$$

$$\int_{-\infty}^{\infty} f^2(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\, d\omega$$

$\therefore$ The energy of a signal is given by $\frac{1}{2\pi}$ times the area under the curve $|F(\omega)|^2$



The energy contained in the freq components within a band of fq (ω_1, ω_2) is $\frac{1}{2\pi}$ times the area of $|F(\omega)|^2$ under the band (ω_1, ω_2)

There is also a band of (-ve $(-\omega_1, \omega_2)$ fq which also has ~~already~~ exactly the same amount of energy as that in (ω_1, ω_2)

thus The energy contained in the fz band (ω_1, ω_2) is given by

$$\Delta E = 2 \cdot \frac{1}{2\pi} \int_{\omega_1}^{\omega_2} |F(\omega)|^2 d\omega$$

$$= \frac{1}{\pi} \int_{\omega_1}^{\omega_2} |F(\omega)|^2 d\omega$$

$\frac{1}{\pi} |F(\omega)|^2 \rightarrow$ represents the energy per unit band width which represents the energy density denoted as $S(\omega)$

$$S(\omega) = \frac{1}{\pi} |F(\omega)|^2$$

$\therefore$ The energy ΔE associated with components of fz lieing in the interval (ω_1, ω_2) is

$$\Delta E = \int_{\omega_1}^{\omega_2} S(\omega)\, d\omega \text{ is}$$

$$E = \int_0^{\infty} S(\omega)\, d\omega$$

$\rightarrow$ Find FT of $e^{-2t} u(t-1)$

$$F[e^{-at} u(t)] = \frac{1}{a + j\omega}$$

$$F[e^{-2t} u(t)] = \frac{1}{2 + j\omega}$$

$$F[e^{-2(t-1)} u(t-1)] = \frac{1}{2 + j\omega} e^{-j\omega(1)}$$

$$e^2 F[e^{-2t} u(t-1)] = \frac{e^{-j\omega}}{2 + j\omega}$$

$$F[e^{-2t} u(t-1)] = \frac{e^{-(2+j\omega)}}{2 + j\omega}$$

→ Find $te^{-3t}u(t)$

$$F[e^{-3t}u(t)] = \frac{1}{3+j\omega}$$

$$f(t) \longleftrightarrow F(\omega)$$

$$-jtf(t) \longleftrightarrow \frac{dF}{d\omega}$$

$$tf(t) \longleftrightarrow j\frac{dF}{d\omega}$$

$$F[te^{-3t}u(t)] = j\frac{d}{d\omega}\left(\frac{1}{3+j\omega}\right)$$

$$= \frac{-j}{(3+j\omega)^2}(j) = \frac{1}{(3+j\omega)^2}$$

→ $f(t) = e^{-0.5t}$

$$f(at) = \frac{1}{|a|}F\left(\frac{\omega}{a}\right)$$

$$F[e^{-0.5t}] \longleftrightarrow \frac{1}{|0.5|}F\left(\frac{\omega}{0.5}\right)$$

→ F.T $\cos\omega_0 t$

$$F[\cos\omega_0 t] = \lim_{T\to\infty}\int_{-T/2}^{T/2}\cos\omega_0 t\, e^{-j\omega t}\,dt$$

$$= \lim_{T\to\infty}\int_{-T/2}^{T/2}\left\{\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}\right\}e^{-j\omega t}\,dt$$

$$= \frac{1}{2}\lim_{T\to\infty}\int_{-T/2}^{T/2}\left\{e^{-j(\omega-\omega_0)t}+e^{-j(\omega+\omega_0)t}\right\}dt$$

$$= \frac{1}{2}\lim_{T\to\infty}\left[\frac{e^{-j(\omega-\omega_0)t}}{-j(\omega-\omega_0)}+\frac{e^{-j(\omega+\omega_0)t}}{-j(\omega+\omega_0)}\right]_{-T/2}^{T/2}$$

$$= \frac{1}{2}\lim_{T\to\infty}\left[\frac{e^{-j(\omega-\omega_0)T/2}}{-j(\omega-\omega_0)}+\frac{e^{-j(\omega+\omega_0)T/2}}{-j(\omega+\omega_0)}+\frac{e^{+j(\omega-\omega_0)T/2}}{j(\omega-\omega_0)}\right.$$ $e^{j(\omega+\omega_0)T/2}$

$$\frac{2}{\tau} \underset{T\to\infty}{Lt} \left[\frac{\sin(\omega-\omega_0)\tau/2}{\omega-\omega_0} + \frac{\sin(\omega+\omega_0)\tau/2}{\omega+\omega_0} \right]$$

$$= \pi \underset{T\to\infty}{Lt} \left[\frac{2}{\cancel{\tau}\pi} Sa(\omega-\omega_0)\tau/2 + \frac{2}{2\pi} \{ Sa(\omega+\omega_0)\tau/2 \right]$$

$$= \pi \underset{T\to\infty}{Lt} \left[\frac{k}{\pi} Sa(\omega-\omega_0)\tau/2 + \frac{k}{\pi} Sa(\omega+\omega_0)\tau/2 \right]$$

$$= \pi [\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$$

$\rightarrow f(t) = t e^{-at} u(t)$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} t e^{-at} u(t) e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} t e^{-(a+j\omega)t} dt$$

$$= t \int e^{-(a+j\omega)t} dt - \int (1) \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} dt$$

$$= \left[t \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} + \frac{e^{-(a-j\omega)t}}{-(a+j\omega)^2} \right]_0^{\infty}$$

$$F(\omega) = \frac{1}{(a+j\omega)^2}$$

F-T properties

1) linearity property:

If $ax(t) ~~+b\,y(t)~~ \xleftrightarrow{F.T} a\,X(\omega) + b\,Y(\omega)$

$f(t) = a\,x(t) + b\,y(t)$

$F(\omega) = \int_{-\infty}^{\infty} [a\,x(t) + b\,y(t)]\, e^{-j\omega t}\, dt$

$= \left[a \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\right] + b \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$

$= a\,X(\omega) + b\,Y(\omega)$

2) time shift property:

$x(t \pm t_0) = e^{\pm jn\omega_0 t}\, ~~dt~~\, X(j\omega)$

$f(t - t_0) = \int_{-\infty}^{\infty} f(t - t_0)\, e^{-jn\omega_0 t}\, dt.$

put $t - t_0 = p$

$t = p + t_0$

$f(p) = \int_{-\infty}^{\infty} f(p)\, e^{-jn\omega_0 (p + t_0)}\, dt$

$= \int_{-\infty}^{\infty} f(p)\, e^{-jn\omega_0 p}\, ~~+~~ e^{-jn\omega_0 t}\, dt$

$= F(\omega)\, e^{-jn\omega_0 t}$

fq shifting: $F^{-1}[X(\omega - \omega_0)] = x(t)\, e^{\pm j\omega_0 t}$

$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$

$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega - \omega_0)\, e^{j\omega t}\, d\omega$

$\omega - \omega_0 = p$

$\frac{1}{2\pi} \int_{-\infty}^{\infty} F(p)\, e^{j(p + \omega_0)}\, d\omega$ $\quad \frac{1}{2\pi} \int_{-\infty}^{\infty} F(p)\, e^{jpt}\, e^{j\omega_0 t}\, dt$

$f(t)\, e^{\pm j\omega t}\, d\omega$

Time reversal:

$$x(-t) \xleftrightarrow{F.T} x(-\omega)$$

$$F(f(-t)) = \int_{-\infty}^{\infty} f(-t)\, e^{+j\omega t}\, dt$$

$$= F(-\omega)$$

Time scaling:

$$F[x(at)] = \frac{1}{a} x\left(\frac{\omega}{a}\right)$$

$$x(at) = \int_{-\infty}^{\infty} f(at) \cdot e^{-j\omega t}\, dt$$

$at = p$

$t = p/a$

$$\int_{-\infty}^{\infty} f(p)\, e^{-j\omega(p/a)}\, dt$$

$$= \int_{-\infty}^{\infty} f(p) \cdot e^{-\frac{1}{a}[j\omega p]}$$

$$= \frac{1}{a} x\left(\frac{\omega}{a}\right)$$

Convolution:

$$f(t) = h(t) * x(t) \xleftrightarrow{F.T} F(\omega) = X(\omega) \cdot Y(\omega)$$

$$h(t) * x(t) = \int_{-\infty}^{\infty} h(t) * x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} h(t) e^{j\omega t} * x(t) e^{j\omega t}\, dt$$

$$= H(\omega) X(\omega)$$

cos $\leftrightarrow$

→ Convolution:

$$f(t) = x(t) * y(t) \xleftrightarrow{FT} F(\omega) = X(\omega) \cdot Y(\omega)$$

A convolution operation is transformed to modulation in frequency domain.

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} [x(t) * y(t)] e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau) y(t-\tau) d\tau \right] e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} y(t-\tau) e^{-j\omega t} dt \right] d\tau.$$

put $t - \tau = \alpha$, then $t = \tau + \alpha$

$dt = d\alpha$

$$F(\omega) = \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} y(\alpha) e^{-j\omega(\tau + \alpha)} d\alpha \right] d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} y(\alpha) e^{-j\omega\tau} \cdot e^{-j\omega\alpha} d\alpha \right] d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau \int_{-\infty}^{\infty} y(\alpha) e^{-j\omega\alpha} d\alpha$$

$$F(\omega) = X(\omega) \cdot Y(\omega)$$

→ Frequency differentiation:

If $x(t) \xleftrightarrow{F.T} X(\omega)$, then

$$-jt\, x(t) \xleftrightarrow{F.T} \frac{d}{d\omega} X(\omega)$$

Differentiating the fq spectrum is equivalent to multiplying the time domain signal by complex number $-jt$:

proof: $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

$$\frac{d}{d\omega} X(\omega) = \int_{-\infty}^{\infty} x(t) \frac{d}{d\omega}\left[e^{-j\omega t}\right] dt$$

$$= \int_{-\infty}^{\infty} x(t)(-jt) e^{-j\omega t} dt$$

$$= -jt \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\frac{d}{d\omega} X(\omega) = -jt\, X(\omega)$$

→ Time Differentiation:

If $x(t) \xleftrightarrow{F.T} X(\omega)$, then

$$\frac{d}{dt} x(t) \xleftrightarrow{F.T} j\omega X(\omega)$$

Differentiation in time domain corresponds to multiplying by $j\omega$ in fq domain. It accentuates high frequency components of the signal.

proof: $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \left[\frac{d}{dt} e^{j\omega t}\right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) j\omega e^{j\omega t} d\omega$$

$$\int_{-\infty}^{\infty} [j\omega X(\omega)] e^{j\omega t} d\omega$$

$$\frac{d\,x(t)}{dt} \longleftrightarrow j\omega X(\omega)$$

→ Parseval's Theorem or Rayleigh's theorem:

If $x(t) \xleftrightarrow{F.T} X(\omega)$, then

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Energy of the signal can be obtained by interchanging its energy spectrum.

Proof:
$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t) \cdot x^*(t)\, dt \quad \text{—— (1)}$$

Inverse F.T states that
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t} d\omega$$

Taking conjugate of both sides
$$x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)\, e^{-j\omega t} d\omega$$

substitute $x^*(t)$ in eq (1)
$$E = \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)\, e^{-j\omega t}\right] d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega) \cdot X(\omega)\, d\omega$$

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

$\because \omega = 2\pi f,\ d\omega = 2\pi\, df$

$$E = \frac{1}{\cancel{2\pi}}\int_{-\infty}^{\infty} |X(\omega)|^2\, \cancel{2\pi}\, df = \int_{-\infty}^{\infty} |X(f)|^2 df$$

→ Introduction to Hilbert transform:

Hilbert transform of a signal $x(t)$ is defined as the transform in which phase angle of all components of the signal shifted by $\pm 90°$.

Hilbert transform of $x(t)$ is represented with $\bar{x}(t)$, as it is given by

$$\bar{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(k)}{t-k}\, dk$$

The inverse Hilbert transform is given by

$$\bar{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\bar{x}(k)}{(t-k)}\, dk$$

$x(t), \bar{x}(t)$ is called Hilbert transform pair

Properties of Hilbert Transform

A signal $x(t)$ and its Hilbert transform $\bar{x}(t)$ have

1) The same amplitude spectrum
2) The same autocorrelation function
3) The energy spectral density is same as $\bar{x}(t)$
4) $x(t)$ & $\bar{x}(t)$ are orthogonal
5) The Hilbert transform of $\bar{x}(t)$ is $-x(t)$
6) If fourier transform exist then Hilbert transform also exists for energy & power signals.

UNIT-3

Sampling Theorem

Sampling Theorem:

A continuous time signal can be completely represented in its samples and recovered back if the sampling frequency is twice of the highest frequency content of the signal. i.e

$$f_s \geq 2W$$

where f_s is the sampling frequency
W is the higher fq content.

→ proof of sampling theorem are two parts: 1) Representation of x(t) in terms of its samples
2) Reconstruction of x(t) from its samples.

→ Reconstruction of signal from its samples

step1: Take inverse Fourier transform of X(f) which is in terms of $X_\delta(f)$.

2: show that x(t) is obtained back with the help of interpolation function.

step1: Relation between X(f) & $X_\delta(f)$

let us assume $f_s = 2W$, then as per below diagram

$X_\delta(f) = f_s X(f)$ for $-W \leq f \leq W$

$X(f) = \frac{1}{f} X_\delta(f)$ ——— (1) $f_s = 2W$

DTFT $X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

$$X(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n} \quad \text{—— ②}$$

In above equation 'f' is the fq of DT signal. If we replace $X(f)$ by $X_\delta(f)$, then f becomes frequency of CT signal. i.e,

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi \frac{f}{f_s} n}$$

In above equation 'f' is frequency of CT signal. And $\frac{f}{f_s} = f_q$ of DT signal;

$\therefore$ $\cancel{x}(n) = x(nT_s)$

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f n T_s} \quad \text{—— ③}$$

$\because \frac{1}{f_s} = T_s$

substitute above equation in eq ①

$$X(f) = \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f n T_s}$$

Inverse Fourier transform of above equation gives $x(t)$ i.e,

$$x(t) = IFT\left\{\frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f n T_s}\right\}$$

$$x(t) = \int_{-\infty}^{\infty} \left[\frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f n T_s}\right] e^{j2\pi f t} df$$

Here the integration can be taken from $-\omega \le f \le \omega$. since $X(f) = \frac{1}{f_s} X_\delta(f)$ for $-\omega \le f \le$

$$\therefore x(t) = \int_{-\omega}^{\omega} \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f n T_s} . e^{j2\pi f t} df$$

Interchanging the order of summation & integration.

$$x(t) = \sum_{n=\infty}^{\infty} x(nT_s)\frac{1}{f_s}\int_{-\omega}^{\omega} e^{j2\pi f(t-nT_s)}\,df$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\cdot\frac{1}{f_s}\cdot\left[\frac{e^{j2\pi f(t-nT_s)}}{j2\pi(t-nT_s)}\right]_{-\omega}^{\omega}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\cdot\frac{1}{f_s}\left[\frac{e^{j2\pi\omega(t-nT_s)} - e^{-j2\pi\omega(t-nT_s)}}{j2\pi(t-nT_s)}\right]$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\cdot\frac{1}{f_s}\,\frac{\sin 2\pi W(t-nT_s)}{\pi(t-nT_s)}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\,\frac{\sin\pi(2Wt - 2W_nT_s)}{\pi(f_st - f_snT_s)}$$

Here $f_s = 2W$, hence $T_s = \frac{1}{f_s} = \frac{1}{2W}$

Simplifying above equation,

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\frac{\sin\pi(2Wt-n)}{\pi(2Wt-n)}$$

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\mathrm{sinc}(2Wt-n)$$

$$\because \frac{\sin\pi\theta}{\pi\theta} = \mathrm{sinc}\,\theta$$

step 2: Let us interpret the above equation. Expanding we get,

$$x(t) = \ldots + x(-2T_s)\,\mathrm{sinc}(2Wt+2) + x(-T_s)\,\mathrm{sinc}(2Wt+1) + x(0)\,\mathrm{sin}(2Wt) + x(T_s)\,\mathrm{sin}(2Wt-1) + \ldots$$

→ Graphical and analytical proof for Band limited signal:

1) A band limited signal of finite energy, which has no fq components higher than W Hertz, is completely described by specifying the values of the signal at instants of time seperated by $\frac{1}{2W}$ seconds and
2) A band limited signal of finite energy, which has no frequency components higher than W Hertz, may be completely recovered from the knowledge of its samples taken at the rate of 2W samples per second

The first part of above statement tells about sampling of the signal and 2nd part tells about reconstruction of the signal. Above statement can be combined & stated alternately as follows:

see the first page.

part I: Representation of $x(t)$ in its samples $x(nT_s)$

step 1: Define $x_\delta(t)$
2: Fourier transform of $x_\delta(t)$ i.e $X_\delta(f)$
3: Relation between $X(f)$ & $X_\delta(f)$
4: Relation between $x(t)$ & $x(nT_s)$

step 1: Define $x_\delta(t)$

step 2: Fourier transform of $x_\delta(t)$ i.e $X_\delta(f)$.

step 3 Relation between $x(f)$ & $x_\delta(f)$

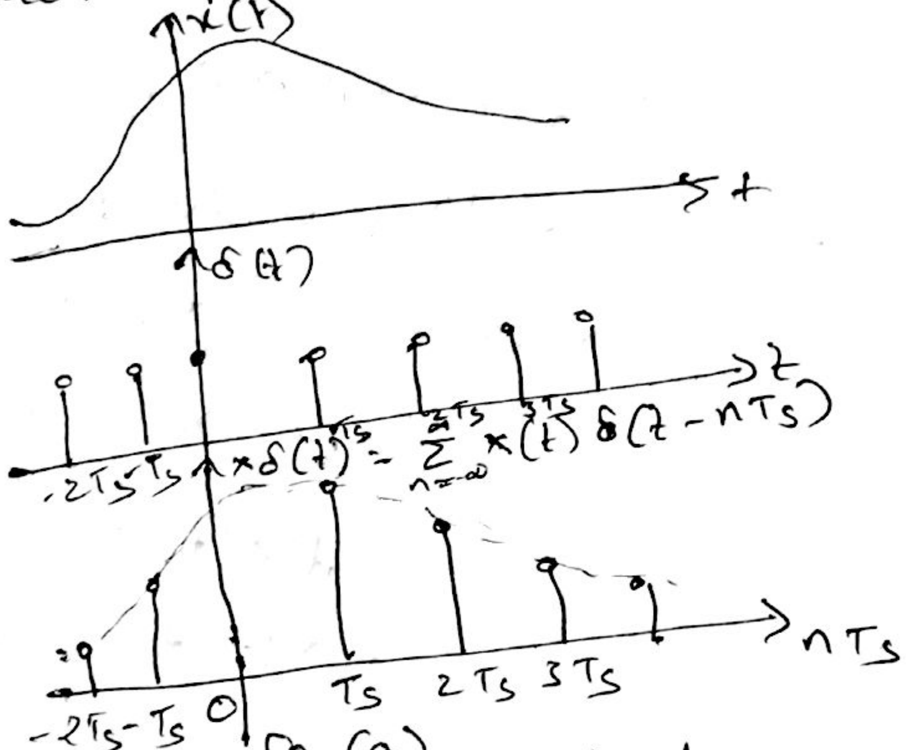
4 Relation between ... $x(nT_s)$

step 1 Define $x_\delta(t)$

The sampled signal $x_\delta(t)$ is given as

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(t)\,\delta(t - nT_s) \quad \text{—} \; ①$$

Here observe that $x\delta(t)$ is the product of $x\delta$ and impulse train $\delta(t)$ as shown in fig(a)



fig(a)

In equation ①, $\delta(t - nT_s)$ indicates the samples placed at $\pm T_s, \pm 2T_s, \pm 3T_s$ and so on.

step 2: F T of $x_\delta(t)$ i.e $x_\delta(f)$

Taking F.T of eq ①

$$x_\delta(f) = FT\left\{ \sum_{n=-\infty}^{\infty} x(t)\,\delta(t - nT_s) \right\}$$

$= F.T$ {product of $x(t)$ & impulse train}

We know that F.T of product of time domain becomes convolution in frequency domain i.e,

$$X_\delta(f) = F.T\{x(t)\} * F.T\{\delta(t-nT_s)\} \quad —(2)$$

By definitions, $x(t) \xleftrightarrow{F.T} X(f)$ &

$$\delta(t-nT_s) \xleftrightarrow{F.T} f_s \sum_{n=-\infty}^{\infty} \delta(f-nf_s)$$

∴ eq (2) becomes

$$X_\delta(f) = X(f) * f_s \sum_{n=-\infty}^{\infty} \delta(f-nf_s)$$

∵ convolution is linear,

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f) * \delta(f-nf_s)$$

$$= f_s \cdot \sum_{n=-\infty}^{\infty} X(f-nf_s)$$

$$= \dots, f_s X(f-2f_s) + f_s X(f-f_s) + f_s X(f) + f_s X(f-f_s) + \cancel{f_s X(f-2f_s)}$$

→ Sampling Techniques:

Here, We have different types of sampling the signals.

Ideal sampling (or) Instantaneous sampling (or) Impulse sampling:

Ideal sampling is same as instantaneous sampling. In fig(1) shows the

Switching sampler :

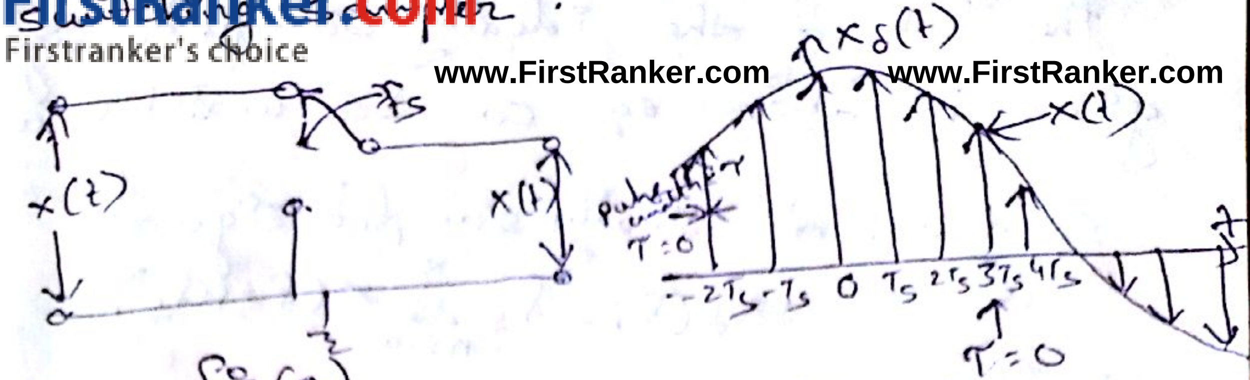


fig (a)

fig (b)

If closing time 't' of the switch approaches zero the output $x_\delta(t)$ gives only instantaneous value. The waveform shown in fig (b). Since the width of the pulse approaches zero, the instantaneous sampling gives train of impulses in $x_\delta(t)$. The area of each impulse in the sampled version is equal to instantaneous value of input signal $x(t)$.

We know that the train of impulses can be represented mathematically as,

$$s_\delta(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \quad \text{——— (1)}$$

This is called sampling function and its waveform is shown in fig (a). The sampled signal $x_\delta(t)$ is given by multiplication of $x(t)$ and $s_\delta(t)$.

$$\therefore x_\delta(t) = x(t)\, s_\delta(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\, \delta(t - nT_s) \quad \text{— (2)}$$

The FT of the ideally sampled signal given by above eq. can be written as

spectrum of ideally sampled signal

$$: X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

→ Natural sampling (or) chopper sampling

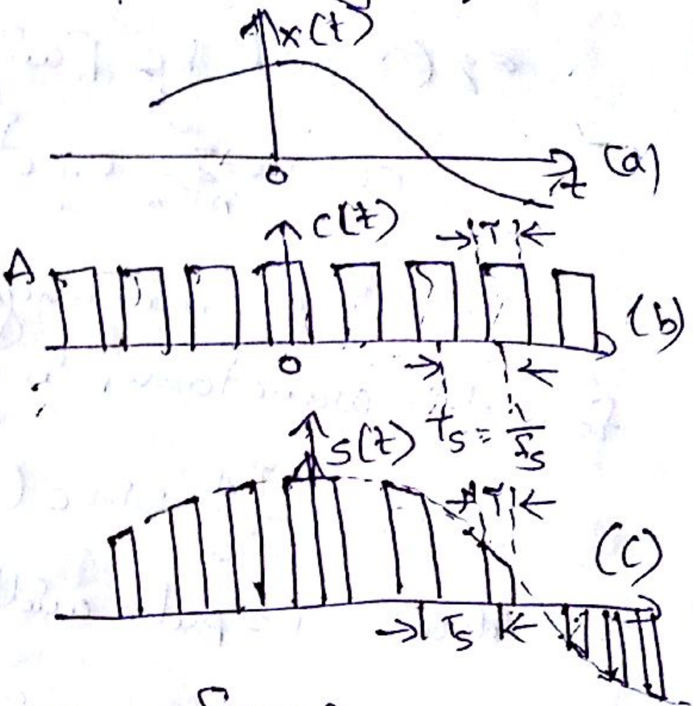
In instantaneous sampling, we have seen that the samples whose width τ approaches zero. Because of this impracticable method the power in the instantaneously sampled pulse is negligible hence it is not suitable for transmission. Therefore the possible methods like natural sampling & flat top sampling are used.

In natural sampling, the pulse has a finite width τ. The waveform of the sampled signal appears to be chopped off from the original signal waveform.

Let us consider an analog continuous time signal $x(t)$ to be sampled at the rate of f_s Hz and f_s is the higher than Nyquist rate such that sampling theorem is satisfied. A sampled signal $s(t)$ is obtained by multiplication of the sampling function & signal $x(t)$.

Sampling function $c(t)$ is a train of periodic pulses of width τ and frequency equal to f_s Hz.

fig 1



fig(2)

Fig (1) shows a functional diagram of natural sampler. When $c(t)$ goes high, a switch 's' is closed. Therefore,

$s(t) = x(t)$ when $c(t) = A$ (amplitude of $c(t)$)

$s(t) = 0$ when $c(t) = 0$

signal $s(t)$ can also be defined mathematically as $s(t) = c(t)\,x(t)$ —①

Here $c(t)$ is the periodic train of pulses of width τ & freq f_s.

Exponential F.S for periodic waveform is given as

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi nt/T_0} \quad \text{—②}$$

For the periodic pulse train of $c(t)$ we have $T_0 = T_s = \frac{1}{f_s}$ = period of $c(t)$

$f_0 = f_s = \frac{1}{T_0} = \frac{1}{T_s}$: Frequency of $c(t)$

eq ② will be [with $x(t) = c(t)$]

$$c(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi f_s n t}$$ putting $\frac{1}{T_0} = f_s$

$c(t)$ is a rectangular pulse train. c_n for this waveform is given as

$$\therefore c_n = \frac{TA}{T_0} \operatorname{sinc}(f_n T)$$

Here T = pulse width = τ

f_n = Harmonic fz

$f_n = nf_s$ or $f_n = \frac{n}{T_0} = nf_0$

$$\therefore c_n = \frac{\tau A}{T_s} \operatorname{sinc}(f_n \tau) \quad \text{④}$$

substitute c_n value in eq ③

$$c(t) = \sum_{n=-\infty}^{\infty} \frac{\tau A}{T_s} \operatorname{sinc}(f_n \tau) e^{j2\pi f_s n t}$$

on putting the value of $c(t)$ in

$$s(t) = c(t)\, x(t)$$

$$s(t) = \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} \operatorname{sinc}(f_n \tau) e^{j2\pi f_s n t} \cdot x(t)$$

This equation represents naturally sampled signals.

F.T of $s(t)$.

$S(f) = F.T\{s(t)\}$

$$= \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} \text{sinc}(f_n \tau)\, F.T\{e^{j2\pi f_n t} \cdot x(t)\}$$

By shifting property of F.T, that

$$e^{j2\pi f_s n t} x(t) \longleftrightarrow x(f - f_s n)$$

$$S(f) = \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} \text{sinc}(f_n \tau)\, x(f - f_s n)$$

we know that $f_n = n f_s$

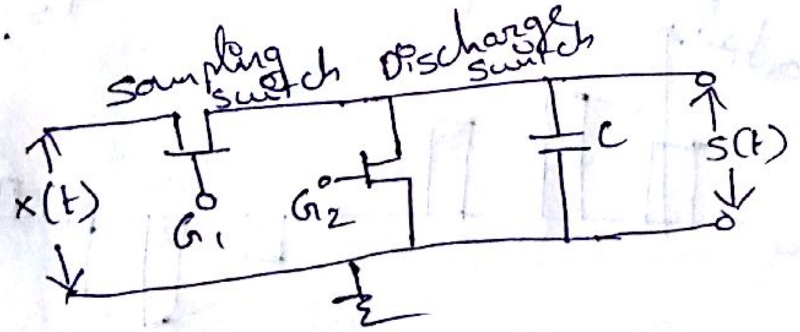
spectrum of naturally sampled signal

$$S(f) = \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} \text{sinc}(n f_s \tau)\, x(f - n f_s)$$

→ **Flat top sampling (or) Rectangular pulse sampling:**

Natural sampling is little complex, but it is very easy to get flat top samples. The top of the samples remains constant and equal to instantaneous value of base band signal $x(t)$ at the start of the sampling. The duration of each sample is τ and sampling rate is equal to

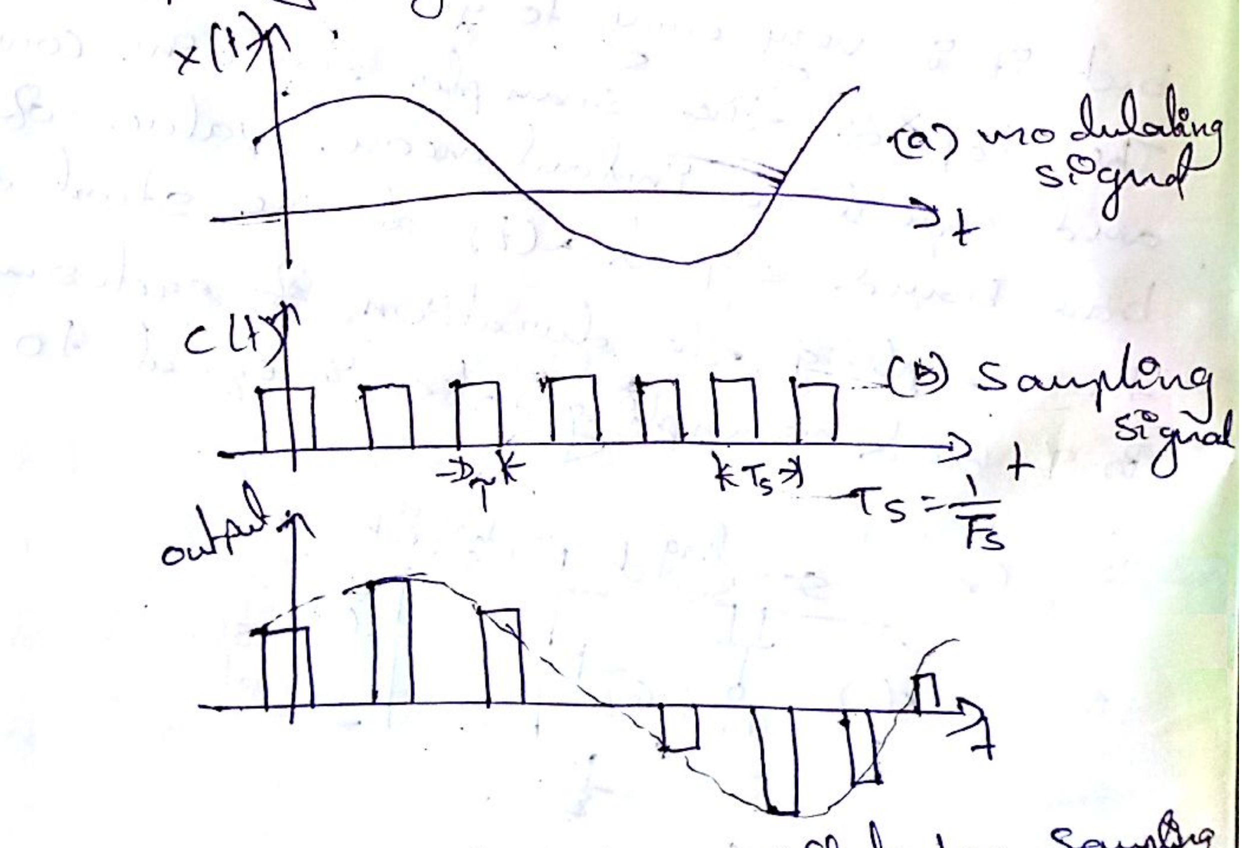
$f_s = \frac{1}{T_s}$.



In fig(1), the sample and hold circuit is used for generating flat top samples and waveforms shown in fig (2)

The switch S_1 closes at each sampling instant to sample the modulating signal. The capacitor 'C' holds the sampled voltage for period τ at the end of which switch S_2 is closed in order to discharge the capacitor.

Thus the signal generated as a result of sample & hold process is the flat top sampled signal. The spectrum of the generated flat top sampling signal along with the modulating signal and the sampling signal is shown below fig (2)



(c) flat top sampling signal

[Flat top sampling is mostly used in digital trans]

Flat top sampling $s(t)$ can be mathematically considered as convolution of the sampled signal and pulse signal $h(t)$.

$$\therefore s(t) = x_\delta(t) * h(t) \quad \text{——} (1)$$

$$x(t) * \delta(t) = x(t) \quad \text{——} (2)$$

Convolution of $x\delta(t)$ & $h(t)$, we get a pulse whose duration is equal to $h(t)$ only but amplitude is defined by $x\delta(t)$.

$x\delta(t)$ is given as

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\, \delta(t - nT_s) \quad \text{——} (3)$$

From eq (1)

$$s(t) = x_\delta(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x_\delta(u)\, h(t-u)\, du$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(nT_s)\, \delta(u - nT_s)\, h(t-u)\, du$$

From eq (3)

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \int_{-\infty}^{\infty} \delta(u - nT_s)\, h(t-u)\, du \quad \text{——} (4)$$

From the shifting property of delta function we know that,

$$\int_{-\infty}^{\infty} f(t)\, \delta(t - t_0) = f(t_0) \quad \text{——} (5)$$

Using this equation we can write eq ⑤ as

$$s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\, h(t - nT_s) \quad \text{——— ⑥}$$

$$s(t) = x_\delta(t) * h(t)$$

By taking F.T of both sides

$$S(f) = X_\delta(f)\, H(f) \quad \text{——— ⑦}$$

Convolution in time domain is converted to multiplication in fq domain.

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \quad \text{——— ⑧}$$

eq ⑦ becomes

Spectrum of flat top sampled signal

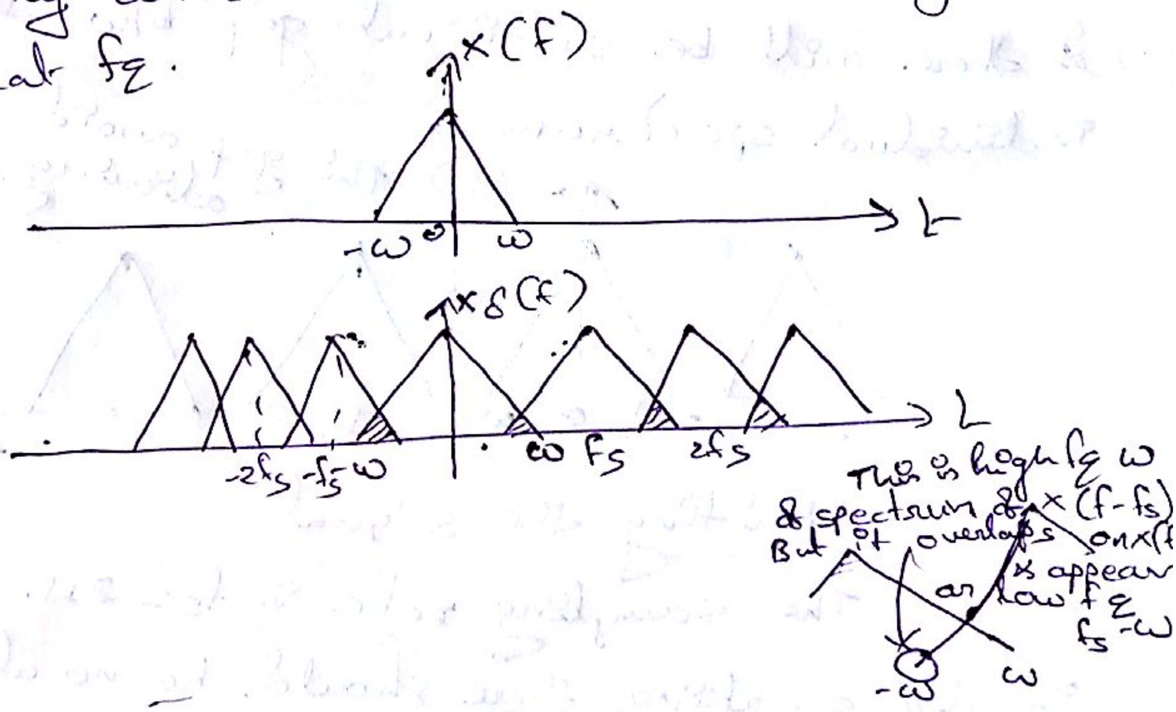
$$S(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)\, H(f)$$

→ Effects of undersampling (Aliasing):

When considering the reconstruction of a signal, you should already be familiar with the idea of nyquist rate. This concept allows us to find the sampling rate that will provide for perfect reconstruction of our signal. If we sample at too low of a rate (below the Nyquist rate), then problem will arise that will make perfect reconstruction impossible. This problem is known as aliasing.

Aliasing occurs when there is an overlap in the shifted, periodic copies of our original signal F.T. i.e, spectrum.

In fq domain, that part of the signal will overlap with the periodic signals next to it. In this overlap the values of the fq will be added together and the shape of the signals spectrum will be unwantingly altered. This overlapping, or aliasing, makes it impossible to correctly determine the correct strength of that fq.



Aliasing: When the high fq interferes with low fq & appears as low fq, then the phenomenon is called aliasing.

Effects of aliasing:

1) since high & low fz interfere with each other, distortion is generated.

2) The data is lost and it cannot be recovered

Different ways to avoid aliasing

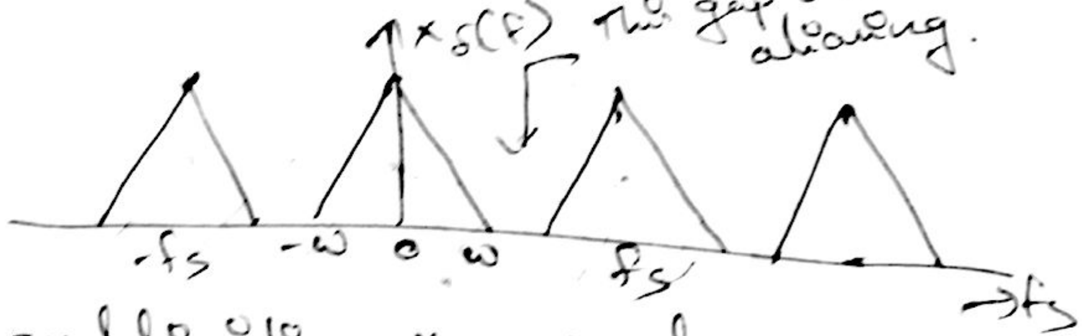
Aliasing can be avoided by two method

1) sampling rate $f_s \geq 2W$

2) strictly band limit the signal to 'W'

1) Sampling rate $f_s \geq 2W$

When the sampling rate is made higher than 2W, then the spectrum will not overlap & there will be sufficient gap b/w the individual spectrum.



2) Band limiting the signal:

The sampling rate is $f_s = 2W$. The ideally speaking there should be no aliasing. But there can be few components higher than 2W. These components create aliasing. Hence a LPF is used before sampling the signal. Thus the o/p of ~~Band limited~~ LPF ~~is strictly band limited & there are no fz~~

... higher than W. Then there is no aliasing.

→ Nyquist Rate & Nyquist Interval

Nyquist Rate:

When the sampling rate becomes exactly equal to 2W samples/sec, for a given bandwidth of W Hz, then it is called Nyquist rate.

Nyquist rate = 2W HZ.

Nyquist interval: It is the time interval between any two adjacent samples when sampling rate is Nyquist rate.

Nyquist ~~rate~~ Interval = $\frac{1}{2W}$ seconds.

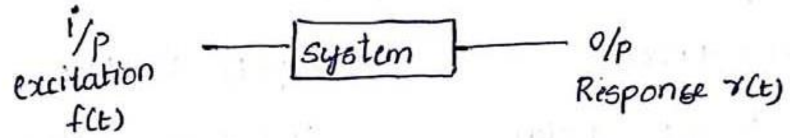
UNIT-4

SIGNAL TRANSMISSION THROUGH LINEAR SYSTEMS

System: A system is defined as set of rules that associates an o/p time function to every i/p time function.

(or)

A system is an interconnection of elements which produces expected o/p for available i/p.



$f(t) \longrightarrow r(t)$

→ System is an mathematical operator which maps i/p into o/p

classification of system

↓

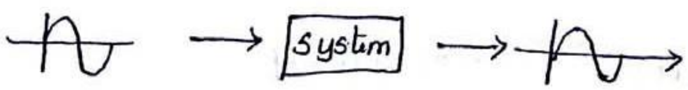
continuous time system — Discrete time system.

↓

1. Static & Dynamic systems
2. Linear & Non-Linear
3. Time invariant & Time variant
4. Linear TIV & LTIV
5. Stable system
6. casual & non-causal systems.

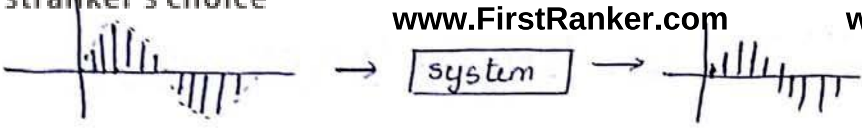
(i) continuous time systems

→ A continuous time system operates on a continuous time i/p signal to produce a continuous time o/p signal



(ii) Discrete time systems:

A discrete time system operates on a discrete time i/p signal to produce a discrete time o/p signal.



classifications :

(i) static and Dynamic Systems:

→ A static system or system is said to be static if its o/p at any instant depends only on present values of i/p.

Ex: $y(t) = ax(t)$ (ii) $y(t) = a^r x(t)$

at $t=0$ $y(0) = ax(0)$ at $t=0$ $y(0) = a^r x(0)$

at $t=1$ $y(1) = ax(1)$ at $t=1$ $y(1) = a^r x(1)$

→ A system is said to be dynamic if its o/p depends on present & past values of i/p.

Ex: $y(t) = x(t-1) + x(t-2) + x(t)$

at $t=2$

$$y(2) = x(2-1) + x(2-2) + x(2) = \underbrace{x(1) + x(0)}_{\text{past}} + \underbrace{x(2)}_{\text{present}}$$

(ii) Linear and Non Linear Systems :

→ A system is said to be linear if its satisfies the superposition principle.

→ It states that the response of the system to a weighted sum of signals be equal to the corresponding weighted sum of o/p's of the system to each of the individual i/p signal.

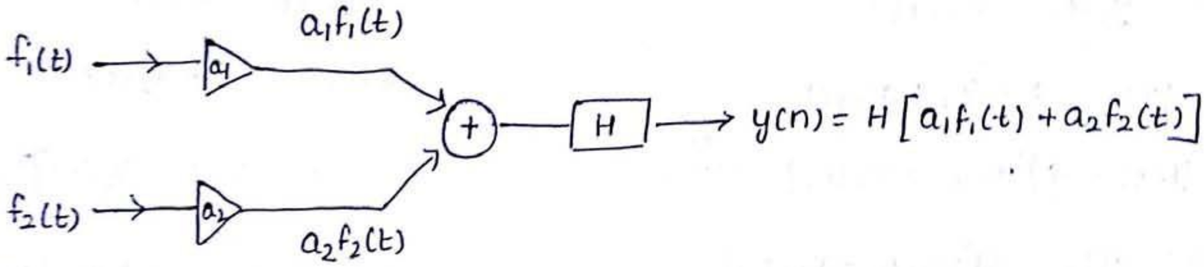
$$H[a_1 f_1(t) + a_2 f_2(t)] = a_1 H[f_1(t)] + a_2 H[f_2(t)]$$

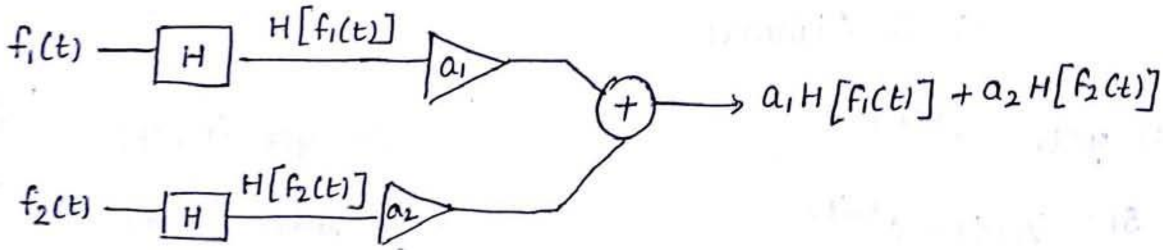
where a_1, a_2 are weighted constants.

$$a_1 f_1(t) \xrightarrow{\text{Response}} a_1 H[f_1(t)]$$

$$a_2 f_2(t) \xrightarrow{\text{Response}} a_2 H[f_2(t)]$$

$H[a_1 f_1(t) + a_2 f_2(t)] = a_1 H[f_1(t)] + a_2 H[f_2(t)]$





Block diagram.

→ Any system which does not obey the above principle is called as non-linear systems.

check for Linearity:

Procedure:

1. Apply different i/p's separately and get the o/p.
2. Apply different i/p's simultaneously and get the output.
3. If both outputs are same it is linear otherwise non-linear.

Ex:

(i) $y(t) = 4 \sin t \, x(t)$

step 1: $y_1(t) = 4 \sin t \, x_1(t)$

$y_2(t) = 4 \sin t \, x_2(t)$

$y_1(t) + y_2(t) = 4 \sin t \, [x_1(t) + x_2(t)]$

step 2: $y(t) = 4 \sin t \, [x_1(t) + x_2(t)]$

$S_1 = S_2$

$y(t) = ax(t)$

Sol S_1: $y_1(t) = a\,x_1(t)$

$y_2(t) = a\,x_2(t)$

$y(t) = a\,x_1(t) + a\,x_2(t)$

$y(t) = a[x_1(t) + x_2(t)]$

S_2: $y(t) = a[x_1(t) + x_2(t)]$

$S_1 = S_2$ (Linear)

③ $y(t) = x^2(t)$

Sol S_1: $y_1(t) = x_1^2(t)$

$y_2(t) = x_2^2(t)$

$y(t) = x_1^2(t) + x_2^2(t)$

S_2: $y(t) = [x_1(t) + x_2(t)]^2 \rightarrow$

$S_1 \neq S_2$ (Non-Linear)

(4) $y(t) = e^{x(t)}$

Sol S_1: $y_1(t) = e^{x_1(t)}$

$y_2(t) = e^{x_2(t)}$

$y(t) = e^{x_1(t)} + e^{x_2(t)}$

$y(t) \doteq e^{[x_1(t) + x_2(t)]}$

S_2: $e^{x_1(t)} \cdot e^{x_2(t)}$

$S_1 \neq S_2$ (Non-Linear)

⑤ $y(t) = t\,x(t)$

Sol S_1: $y_1(t) = t\,x_1(t)$

$y_2(t) = t\,x_2(t)$

$y(t) = t[x_1(t) + x_2(t)]$

S_2: $y(t) = t[x_1(t) + x_2(t)]$

$S_1 = S_2$ (Linear)

⑪ $y(t) = x(t^2)$

Sol S_1: $y_1(t) = x_1(t^2)$

$y_2(t) = x_2(t^2)$

$y(t) = x_1(t^2) + x_2(t^2)$

S_2: $y(t) = x_1(t^2) + x_2(t^2)$

$S_1 = S_2$ (Linear)

⑥ $y(t) = x(t - t_0)$

Sol S_1: $y_1(t) = x_1(t - t_0)$

$y_2(t) = x_2(t - t_0)$

$y(t) = x_1(t - t_0) + x_2(t - t_0)$

S_2: $y(t) = x_1(t - t_0) + x_2(t - t_0)$

$S_1 = S_2$ (Linear)

⑦ $y(t) = 3x(t+3)$

Sol S_1: $y_1(t) = 3x_1(t+3)$

$y_2(t) = 3x_2(t+3)$

$y(t) = 3[x_1(t+3) + x_2(t+3)]$

S_2: $3[x_1(t+3) + x_2(t+3)]$

$S_1 = S_2$ (Linear)

⑧ $y(t) = Ax(t) + B$

Sol $y_1(t) = Ax_1(t) + B$

$y_2(t) = Ax_2(t) + B$

$y(t) = A[x_1(t) + x_2(t)] + 2B$

$y(t) = A[x_1(t) + x_2(t)] + B$

$S_1 \neq S_2$ = Nonlinear

⑧ $y(t) = x(t+1)e^{-t}$

Sol S_1: $y_1(t) = x_1(t+1)e^{-t}$; $y_2(t) = x_2(t+1)e^{-t}$

$y(t) = e^{-t}[x_1(t+1) + x_2(t+1)]$

S_2: $y(t) = e^{-t}[x_1(t+1) + x_2(t+1)]$

$S_1 = S_2$ (Linear)

⑨ $y(t) = \cos[x(t)]$

Sol S_1: $y_1(t) = \cos[x_1(t)]$; $y_2(t) = \cos[x_2(t)]$

$y(t) = \cos[x_1(t)] + \cos[x_2(t)]$

S_2: $\cos[x_1(t) + x_2(t)]$

$S_1 \neq S_2$ (Non-Linear)

⑩ $y(t) = k\Delta x(t)$ where $\Delta x(t) = [x(t+1) - x(t)]$

Sol S_1: $y_1(t) = k[x_1(t+1) - x_1(t)]$; $y_2(t) = k\Delta x_2(t)$

$y(t) = [x_1(t+1) - x_1(t) + x_2(t+1) - x_2(t)].k$

S_2: $y(t) = k[x_1(t+1) - x_1(t) + x_2(t+1) - x_2(t)]$

$S_1 = S_2$ (Linear)

⑫ $y(t) = 4x(t) + 2\dfrac{dx(t)}{dt} \rightarrow$ Linear

Time Variant And Time Invariant Systems

→ A system is said to be time invariant if the system does not depend on time i.e system delay is not function of time.

Ex:

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

$$x(t) \longrightarrow y(t)$$
$$x(t-t_0) \longrightarrow y(t-t_0)$$

→ A time shift t_0 in the input results in the same amount of time shift in the o/p but the waveshape does not change.

i.e the i/p and/ to o/p characteristics does not change with time.

→ Any system which doesnot obey the above principle is called as time varying system.

→ An electrical system is said to be time invariant if its component values (R, L, C) does not change with time.

Check for time Invariant:

1. Shift the i/p only and get the o/p.
2. Shift the entire system and get the o/p.
3. If both steps are identical for o/p then it is time invariant system.

Ex!

(1) $y(t) = 4x(t)$

Sol $S_1: y(t) = 4x(t-1)$
$S_2: y(t-1) = 4x(t-1)$] $\rightarrow S_1 = S_2$ (TIV)

(2) $y(t) = 4t\,x(t)$

Sol $S_1: y(t) = 4t\,x(t-1)$
$S_2: y(t-1) = 4(t-1)(x(t-1))$] $\rightarrow S_1 \neq S_2$ (TV)

(3) $y(t) = a x(t)$

Sol $S_1: y(t) = a x(t-1)$
$S_2: y(t-1) = a x(t-1)$] $\rightarrow S_1 = S_2$ (TIV)

(4) $y(t) = a x(t) + b$

Sol $S_1: y(t) = a x(t-1) + b$
$S_2: y(t-1) = a x(t-1) + b$] $S_1 = S_2$ (TIV)

(5) $y(t) = 5t[x(t)]^2$

Sol $S_1: y(t) = 5t[x(t-1)]^2$
$S_2: y(t-1) = 5(t-1)[x(t-1)]^2$ } $S_1 \neq S_2$ (TV)

(6) $y(t) = x(t+1)e^{-t}$

Sol $S_1: y(t) = x(t+1-1)e^{-t} = x(t)e^{-t}$
$S_2: y(t-1) = x(t+1-1)e^{-(t-1)}$
$= x(t) \cdot e^{-t}\, e$ → constant
$S_1 = S_2$ (TIV)

(7) $y(t) = x(t+3)$

Sol $S1: y(t) = x(t+3-1)$

$S2: y(t-1) = x(t+3-1)$ $\Big] \rightarrow S_1 = S_2$ (TIV)

$= x(t+2)$

(8) $y(t) = x^2(t)$

Sol $S1: y(t) = x^2(t-1)$

$S2: y(t-1) = x^2(t-1)$

$\therefore S_1 = S_2$ (TIV)

(9) $y(t) = e^{x(t)}$

Sol $S1: y(t) = e^{x(t-1)}$

$S2: y(t-1) = e^{x(t-1)}$

$S_1 = S_2$ (TIV)

Linear Time Invariant System (LTI):

→ Any system which obeys the linearity and time invariant property is called as LTI system.

Linear Time Variant System (LTV):

→ Any system which obeys the linearity and does not obey time invariant property is called LTV system.

Ex: $y(t) = a x(t)$

Linearity: $y_1(t) = a x_1(t)$; $y_2(t) = a x_2(t)$

$y(t) = a x_1(t) + a x_2(t)$

$y(t) = a[x_1(t) + x_2(t)]$

$S2: y(t) = a[x_1(t) + x_2(t)]$.

$\therefore S_1 = S_2$.

T.I:

$y(t) = a x(t)$

$S1: y(t) = a x(t-1)$

$S2: y(t-1) = a x(t-1)$ $\Big] \rightarrow S_1 = S_2$ (TIV)

$\therefore$ It is a linear time invariant system (LTI)

$\|^{ly}$:

(2) $y(t) = t x(t) \rightarrow$ LTI

(3) $y(t) = a x(t) + b \rightarrow$ NLTI

(4) $y(t) = a x^2(t) \rightarrow$ NLTI

(5) $y(t) = e^{x(t)} \rightarrow$ NLTI

(6) $y(t) = x(t - t_0) \rightarrow$ LTI

Stable System:

→ System is absolutely integrable

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$$

Causal And Non Causal Systems:

→ A system is said to be causal if o/p $y(t_0)$ depends only on the values of i/p $x(t)$ at $t < t_0$ {present, i/p, past i/ps, past o/ps} {$x(t)=0$, for $t<0$; noncausal $t\leq 0$, or $t\leq 0$ and $t>0$; $x(t)\neq 0$ for $t<0$}

Ex: $y(t) = 4x(t-1)$

$y(2) = 4x(2-1) \Rightarrow 4x(2)$

$y(t) = 4x(t-1) + x(t)$

$y(2) = 4x(1) + x(2)$

→ A system is said to be non-causal if the o/p depends on future values of i/p i.e future i/ps & o/ps.

Ex: $y(t) = 4x(t+1)$

$y(2) = 4x(3)$

Examples whether it is causal & Non causal:

(1) $y(t) = K[x(t+1) - x(t)]$

$y(0) = K[x(1) - x(0)] \rightarrow$ Noncausal

(2) $y(t) = 3x(t+3)$

$y(0) = 3x(3) \longrightarrow$ Non causal

(3) $y(t) = (t+3)x(t-3)$

$y(0) = (0+3)x(0-3)$

$= 3x(-3) \rightarrow$ causal

(4) $y(t) = x(t) + 3x(t+4)$

when $t=0$, $y(0) = x(0) + 3x(4)$

when $t=1$, $y(1) = x(1) + 3x(5)$

So here response at $t=0$, $y(0)$ depends on the present i/p & future i/p

hence system is noncausal.

(5) $y(t) = x(t^2)$

$t=-1$, $y(-1) = x(1) \rightarrow$ future

$t=0$, $y(0) = x(0) \rightarrow$ present

$t=1$, $y(1) = x(1) \rightarrow$ present

$t=2$, $y(2) = x(4) \rightarrow$ future

Noncausal.

Except at $t=0$, $t=1$, the response of any value of t depends on future i/p.

(6) $y(t) = x(2t) \longrightarrow$ Noncausal

(7) $y(t) = x(t) - x(t-1) \rightarrow$ causal

(8) $y(t) = x(t) + \int_0^t x(\lambda)\, d\lambda$

$= x(t) + z(\lambda)]_0^t \Rightarrow$ Causal

At $t=0$, $t=1$, $t=2$

Impulse Response:

The response of a system for an impulse i/p is called a impulse response of the system and it is denoted by h(t)

$$\delta(t) \longrightarrow \boxed{\text{system}} \longrightarrow h(t)$$

$$\delta(t) \rightarrow h(t)$$

→ Every system is characterised by its impulse response.

$$\underset{f(t)}{i/p} \longrightarrow \boxed{h(t)} \longrightarrow \underset{r(t)}{o/p}$$

Response of a System for an arbitary i/p:

Response of $\delta(t) \rightarrow h(t)$

$$\delta(t-t_0) \rightarrow h(t-t_0)$$

$$\delta(t)+\delta(t-t_0) = h(t)+h(t-t_0)$$

The response of a system for a given i/p f(t) is determined by using superposition principle.

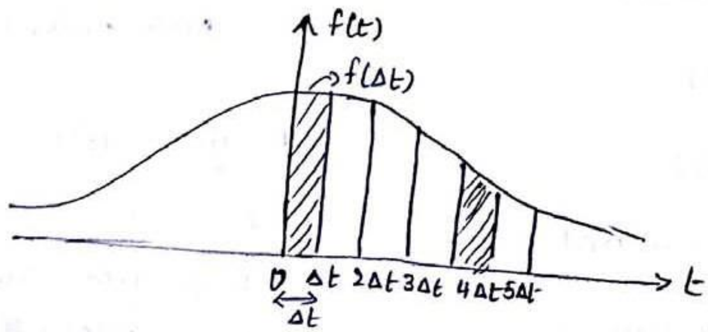
step 1: Resolve the i/p function interms of impulse functions.

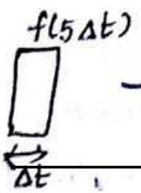
step 2: Determine individually the response of LTI system for impulse function.

step 3: Find the sum of individual responses which will become the overall response r(t).

Representation of a function f(t) in terms of an impulse function:

Here the function f(t) is a impulse train function.





area $f(5\Delta t) \times \Delta t$

$$f(t) = \sum_{n=-\infty}^{\infty} f(n\Delta t)\,\Delta t \cdot \delta(t-n\Delta t)$$

$$f(t) = \underset{\Delta t \to 0}{Lt} \sum_{n=-\infty}^{\infty} f(n\Delta t)\,\Delta t\,\delta(t-n\Delta t)$$

The rectangular of width Δt & height $f(n\Delta t)$ and area under the rectangles is $\Delta t \cdot f(n\Delta t)$ and this n^{th} element approached a delta function of strength $f(n\Delta t)\,\Delta t$ located at $t = n\Delta t$. and this delta function is represented as $f(n\Delta t)\,\Delta t\,\delta(t-n\Delta t)$

$$\boxed{f(t) = \underset{\Delta t \to 0}{Lt} \sum_{n=-\infty}^{\infty} f(n\Delta t) \cdot \Delta t \cdot \delta(t-n\Delta t)}$$

As $\Delta t \to 0$, the n^{th} element may be considered.

2) Determination of $r(t)$ for the input $f(t)$:

Let $h(t)$ be the impulse response of the system.

$$\delta(t) \longrightarrow \boxed{\text{system}} \longrightarrow h(t)$$

then $\delta(t) \longrightarrow h(t)$

$$\delta(t-n\Delta t) \longrightarrow h(t-n\Delta t)$$

$$f(n\Delta t)\,\delta(t-n\Delta t) \longrightarrow f(n\Delta t) \cdot h(t-n\Delta t)$$

$$f(n\Delta t) \cdot \Delta t\,\delta(t-n\Delta t) \longrightarrow f(n\Delta t) \cdot \Delta t\,h(t-n\Delta t)$$

$$\underset{\Delta t \to 0}{Lt} \sum_{n=-\infty}^{\infty} f(n\Delta t) \cdot \Delta t\,\delta(t-n\Delta t) \longrightarrow \underset{\Delta t \to 0}{Lt} \sum_{n=-\infty}^{\infty} f(n\Delta t) \cdot \Delta t\,h(t-n\Delta t)$$

$$f(t) \longrightarrow \boxed{\text{system}} \longrightarrow r(t)$$

$$r(t) = \underset{\Delta t \to 0}{Lt} \sum_{n=-\infty}^{\infty} f(n\Delta t) \cdot \Delta t\,h(t-n\Delta t)$$

$\Delta t \to 0$ means summation becomes integration.

$$r(t) = \int_{-\infty}^{\infty} f(\tau) \cdot h(t-\tau)\,d\tau$$

$$r(t) = f(t) \otimes h(t)$$

$$f(t) \longrightarrow \boxed{h(t)} \longrightarrow f(t) \otimes h(t).$$

of a system is known for an impulse fn, the response then response to any other function can be obtained from the above eqn.

→ An unit impulse function is called as a Test function and it is used to characterise a system.

$$r(t) = f(t) \otimes h(t)$$

In frequency domain $r(t) \xleftrightarrow{FT} R(\omega)$

$f(t) \xleftrightarrow{FT} F(W)$

$h(t) \xleftrightarrow{FT} H(W)$

Using convolution property

$$f(t) \otimes h(t) = F(\omega) \cdot H(\omega)$$

$$R(\omega) = F(\omega) \cdot H(\omega)$$

$$\boxed{H(\omega) = \frac{R(\omega)}{F(\omega)}}$$

→ When $F(\omega) = 1$; i.e i/p is unit impulse $H(\omega) = R(\omega)$

→ Transfer function $H(\omega)$ of a system is defined as the transform of the response of a system where the i/p is unit impulse function.

$$H(\omega) = \underbrace{|H(\omega)|}_{\text{Amplitude response of the system}} \, e^{\overbrace{j\theta(\omega)}^{\text{phase response of the system}}}$$

$$\boxed{\begin{aligned} \ln[H(\omega)] &= \ln[|H(\omega)|] + j\theta(\omega) \\ &= \underbrace{L(\omega)}_{\text{Gain of the system}} + \underbrace{j\,\theta(\omega)}_{\text{phase shift introduced by system}} \end{aligned}}$$

Note: An impulse function contains all frequencies in equal amount so we can use it as a test function.

$$\underbrace{H(\omega)}_{F.T[h(t)]} = \frac{R(\omega)}{F(\omega)} \rightarrow \text{Transfer fn of LTI system.}$$

$$h(t) = I.F.T[H(\omega)]$$

Ms.Rincy Sara luka

FILTER CHARACTERISTICS OF LINEAR SYSTEMS :

IDEAL LOW PASS FILTERS :

→ It transmits all the signals below certain frequency 'B' Hz without any distortion.

→ The range of frequencies from 0Hz to 'B' Hz is called passband of lowpass filter.

→ The frequency 'B' Hz is called cut-off frequency of the ideal lowpass filter.

→ The transfer function of ideal low pass filter can be written as

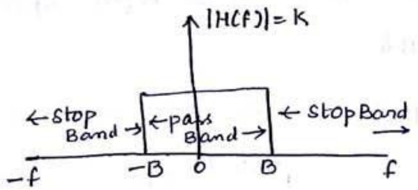
$$H(f) = K e^{-j2\pi f t_0} \quad ; \ -B \leq f < B$$
$$= 0 \quad ; \ |f| > B.$$

K = amplitude is assumed to be unity.

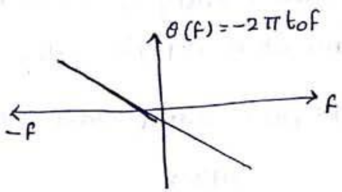
→ By K=1 in above eqn

$$H(f) = e^{-j2\pi f t_0} \quad ; \ -B \leq f < B$$
$$= 0 \quad ; \ |f| > B$$

→ By inverse fourier transform, h(t) can be obtained for ideal LPF



a) magnitude response.



(b) phase response

$$h(t) = \int_{-B}^{B} e^{-j2\pi f t_0} \cdot e^{j2\pi f t} \, df$$

$$= \int_{-B}^{B} \left[e^{j2\pi f (t-t_0)} \right] df = \frac{1}{j2\pi(t-t_0)} \left[e^{j2\pi f(t-t_0)} \right]_{-B}^{B}$$

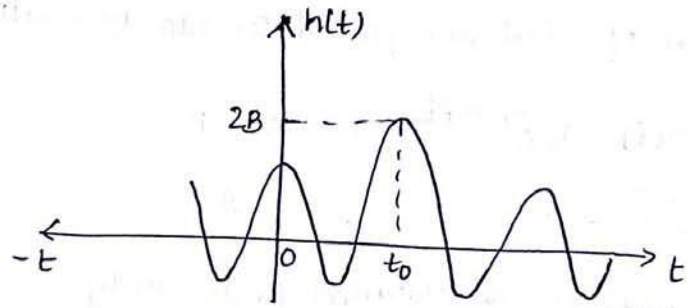
$$= \frac{1}{j2\pi(t-t_0)} \left[e^{j2\pi B(t-t_0)} - e^{-j2\pi B(t-t_0)} \right]$$

$$= \frac{1}{\pi(t-t_0)}\left[\frac{e^{j2\pi B(t-t_0)} - e^{-j2\pi B(t-t_0)}}{2j}\right]$$

$$= \frac{1}{\pi(t-t_0)} \sin\left[2\pi B(t-t_0)\right]$$

$$h(t) = 2B\left(\frac{\sin[2\pi B(t-t_0)]}{2\pi B(t-t_0)}\right) = 2B\,\mathrm{sinc}[2B(t-t_0)]$$

Response



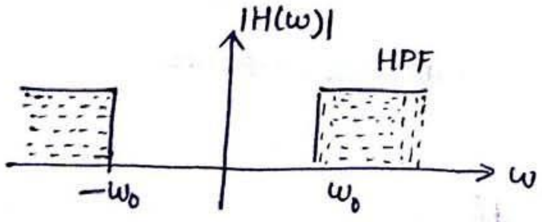
→ Figure shows that impulse response exists for negative values of 't'. But actually unit impulse is applied at t=0 always.

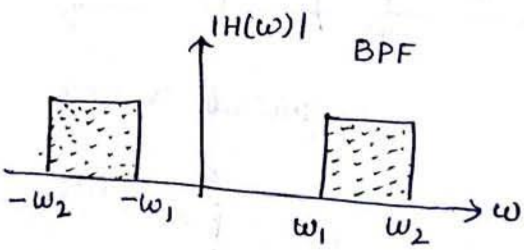
→ Practicaly it is impossible to implement such a system.

OTHER IDEAL FILTERS SUCH AS HPF, BPF etc.,

→ In realizability of ideal LPF its response begins before input is applied and hence it is not physically realizable.

→ $\|^{ly}$ HPF, BPF ideal have frequency response as shown in figure





→ These have sharp transition in frequency response.

→ All ideal filters are physically not realizable since their impulse response is non-causal.

...UCTION FOR FILTER CHARACTERISTICS:

$$f(t) \longrightarrow \boxed{h(t)} \longrightarrow r(t)$$

$F(\omega)$ $\quad$ $H(\omega)$ $\quad$ $R(\omega)$

$$R(\omega) = F(\omega).H(\omega)$$

→ The spectrum of o/p is $F(\omega).H(\omega)$ i.e the system acts as a kind of filter to various frequency components.

→ Some frequency components are boosted in strength and some are attenuated and some remain unaffected.

→ ||ly each freq component undergoes a different amount of phase shift i.e the modification is carried out according to $H(\omega)$.
↳ acts as waiting fn for two different frequencies.

DISTORTIONLESS TRANSMISSION THROUGH SYSTEM:

→ It means output signal is an exact replica of the i/p signal.

$$i/p \longrightarrow \boxed{\text{System}} \longrightarrow o/p$$

→ The difference between i/p and o/p of such system is that

1. Amplitude of the o/p signal may increase or decrease by some factor w.r. to i/p.
2. The o/p sgl may be delayed in time w.r to i/p sgl because of system delay.

→ o/p sgl y(t) can be written in terms of i/p x(t) as

$$y(t) = k\,x(t-t_0)$$

k ↓ constant Represents change in amplitude

t_0 ↳ time delay in transmission of signal through a system.

By taking fourier transform

$$Y(f) = F[y(t)] = F\{k\,x(t-t_0)\}$$

From time shifting property of FT

$$Y(f) = k\,X(f)\,e^{-j2\pi f t_0}$$

~~Transfer fn $H(f) = \frac{Y(f)}{X(f)}$~~

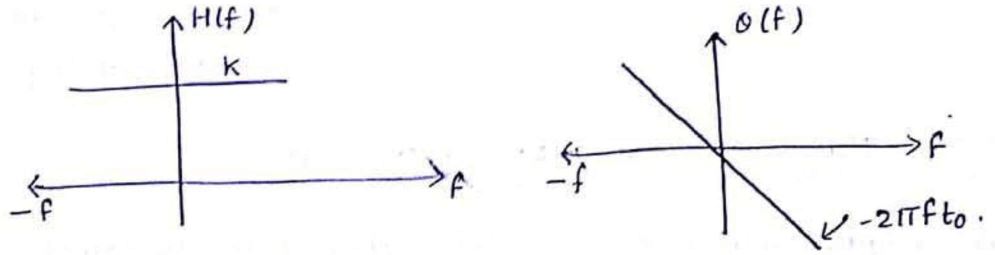
$$\therefore \; H(f) = \frac{Y(f)}{X(f)} = k \cdot e^{-j2\pi f t_0}.$$

Magnitude of transfer fn independent of frequency.

→ Transfer function has constant amplitude at all frequencies. The phase shift is

$$\theta(f) = -2\pi f t_0$$
$$= (-2\pi t_0) f.$$

→ phase shift is linearly proportional to frequency.



(a) Amplitude Spectrum (b) phase spectrum passing through origin

→ By considering simple example

Let there be signal in time domain as

$$x(t) = \cos(2\pi f t)$$

Let the o/p sgl be same in amplitude but shifted in time by t_0 sec

$$y(t) = \cos[2\pi f(t - t_0)]$$

$$\therefore \; y(t) = \cos(2\pi f t - 2\pi f t_0) = \cos(2\pi f t - \theta(f))$$

$\therefore$ phase shift of $y(t)$ is

$$\theta(f) = -2\pi f t_0.$$

which is proportional to frequency 'f'.

Two types :

(i) Amplitude distortion

(ii) phase distortion.

AMPLITUDE DISTORTION

→ This distortion occurs when $|H(\omega)|$ is not constant over frequency band of interest and the frequency components present in I/P sgl are transmitted with different gain and attenuation.

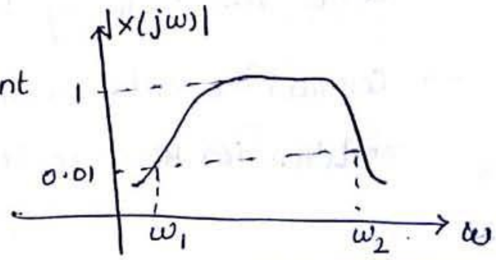
PHASE DISTORTION:

→ This distortion occurs when phase of $H(\omega)$ is not linearly changing with time and different frequency components in I/p are subjected to different time delays during transmission.

SIGNAL BANDWIDTH: The band of frequencies that contains most of signal energy is called B.W of signal denoted by fm.

→ It is the range of significant signal frequencies which are present in the signal.

→ observe in the waveform $x(t)$ has significant frequencies from ω_1 to ω_2.

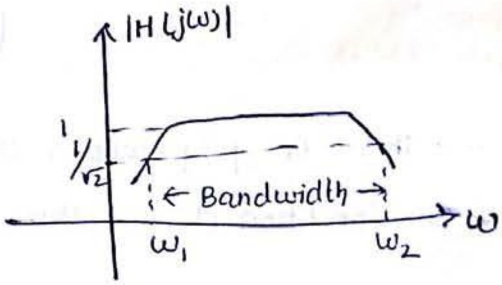


→ The B.W of this signal is $\omega_2-\omega_1$

→ All the physically obtained signals have limited bandwidth.

SYSTEM BANDWIDTH

→ The B.W of a system is defined as range of frequencies over which $|H(\omega)|$ remains with in $1/\sqrt{2}$ times of its mid-band value. for distortionless transmission the System must have infinite B.W but physical system are limited to finite BW.



→ So a system with finite BW can provide distortionless transmission for a band limited signal if $|H(\omega)|$ remains constant over BW of the signal.

→ The range of frequencies for which magnitude $|H(j\Omega)|$ of the systems remains within $\frac{1}{\sqrt{2}}$ of its maximum value

$\rightarrow$ A system is said to be causal if ... if $t<t_0$

$$h(t-t_0)=0 \;;\; t<t_0$$

i.e if i/p is zero for $t<t_0$, then o/p is also zero for $t<t_0$.

$\rightarrow$ Any system which does not obey the above rule is non-causal system.

$\rightarrow$ If two i/p to a causal system are equal upto some time 't_0' then corresponding o/p must be equal upto that time instant.

POLY-WIENER CRITERION

$\rightarrow$ This gives the condition for causality in frequency domain (or) in other words the frequency domain equivalent of causal system i.e $H(\omega)$.

$\rightarrow$ Consider a system with transfer function $H(\omega)$, the necessary and sufficient condition for $H(\omega)$ to be transfer function of causal fn is

$$\int_{-\infty}^{\infty} \frac{|\ln[H(j\omega)]|}{1+\omega^2}\, d\omega < \infty \rightarrow (1)$$

provided $|H(j\omega)|$ is square integral.

$$\int_{-\infty}^{\infty} |H(j\omega)|^2 d\omega < \infty \rightarrow (2)$$

$\rightarrow$ This is poly-wiener criteria. If condition (2) is not satisfied then the condition (1) is neither necessary nor sufficient.

PHYSICAL REALIZABILITY:

$\rightarrow$ A system is said to be physically realizable if it obeys the causal condition.

i.e $h(t)=0$ for $t<0$.

Ex: $H(\omega) = \dfrac{1}{1+j\omega}$

$$h(t) = e^{-t} u(t)$$
$$= 0 \quad \text{for } t<0$$

So the above system for transfer fn is realizable in freq. domain.

$$\int_{-\infty}^{\infty} \frac{|\ln|H(\omega)||}{H(\omega)^2}\, d\omega < \infty$$

→ The frequency domain statement interpreted ... a physically realizable system may be zero for some discrete frequency but it can never be zero for a finite band of frequencies.

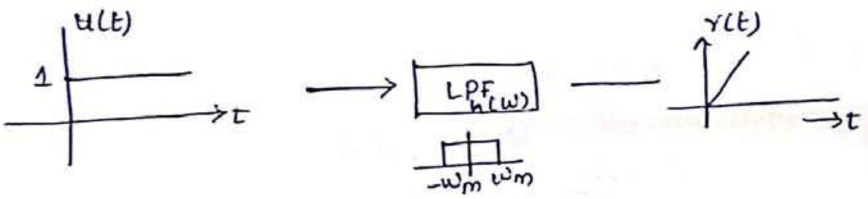
→ $H(\omega)$ for a realisable system cannot decay faster than a function of exponential order.

Ex: A system with T.F $e^{-\omega}$ is realisable whereas $e^{-\omega^2}$ is not as it decay faster.

RELATIONSHIP BETWEEN RISE TIME AND BANDWIDTH:

→ If a unit step fn u(t) is applied to an ideal LPF, the o/p will show a gradual rise instead of a sharp rise in the i/p.

→ The rise time (t_r) is the time required by the response to reach its final value from initial value.



Transfer function of ideal low pass filter is

$$H(\omega) = |H(\omega)| e^{j\theta(\omega)}$$

$$= G(\omega) e^{-j\omega t_0}$$

↓ Rectangular pulse with magnitude K. for $-B < f \leq B$ i.e $-\omega_m \leq \omega \leq \omega_m$ where $\omega_m = 2\pi B$.

and $\theta(\omega) = -2\pi f t_0 = -\omega t_0$.

→ Fourier transform of unit step fn u(t)

$$FT\{u(t)\} \Rightarrow U(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$$

→ Fourier transform of response $R(\omega)$, input and $H(\omega)$ related as

$$R(\omega) = \left[\pi\delta(\omega) + \frac{1}{j\omega}\right] H(\omega) = \pi\delta(\omega).H(\omega) + \frac{1}{j\omega} H(\omega).$$

$\delta(\omega)$ exists only for $\omega = 0$ and $H(\omega)|_{\omega=0}$

$$R(\omega) = \pi\delta(\omega) + \frac{1}{j\omega} H(\omega).$$

By taking IFT for above eqn

$$r(t) = IFT[R(\omega)] = IFT\left\{\pi\delta(\omega) + \frac{1}{j\omega} H(\omega)\right\}$$

$$= IFT\left\{\pi\delta(\omega) + \frac{1}{j\omega} G(\omega) e^{-j\omega t_0}\right\} \quad (\because H(\omega) = G(\omega) e^{-j\omega t_0})$$

Inverse fourier transform of $\pi\delta(\omega)$ is $\frac{1}{2}$.

$$\left\{\because 1 \rightarrow 2\pi\delta(\omega), \quad \tfrac{1}{2} \leftarrow \pi\delta(\omega)\right\}$$

$$r(t) = \frac{1}{2} + IFT\left\{\frac{1}{j\omega} G(\omega) e^{-j\omega t_0}\right\}$$

$$= \frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{j\omega} G(\omega) e^{-j\omega t_0} \cdot e^{j\omega t} d\omega$$

we know $G(\omega) = 1$ for $-\omega_m \le \omega \le \omega_m$

$$= \frac{1}{2} + \frac{1}{2\pi}\int_{-\omega_m}^{\omega_m} \frac{e^{j\omega(t-t_0)}}{j\omega} d\omega$$

$$= \frac{1}{2} + \frac{1}{2\pi}\int_{-\omega_m}^{\omega_m} \frac{\cos\omega(t-t_0) + j\sin\omega(t-t_0)}{j\omega} d\omega$$

$$= \frac{1}{2} + \frac{1}{2\pi}\underbrace{\int_{-\omega_m}^{\omega_m} \frac{\cos\omega(t-t_0)}{j\omega} d\omega}_{=\text{ zero for its odd term}} + \frac{1}{2\pi}\int_{-\omega_m}^{\omega_m} \underbrace{\frac{\sin\omega(t-t_0)}{\omega}}_{\text{even}} d\omega.$$

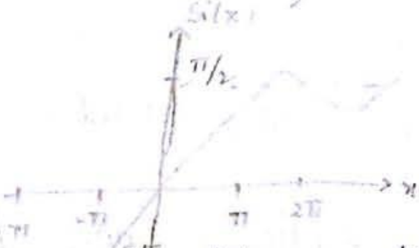
Si(x) is an odd fn

$Si(-x) = -Si(x)$

(ii) $Si(0) = 0$

(iii) $Si(\infty) = \frac{\pi}{2}$

$Si(-\infty) = -\frac{\pi}{2}$



$$r(t) = \frac{1}{2} + \frac{1}{2\pi} \times 2\int_0^{\omega_m} \frac{\sin\omega(t-t_0)}{\omega} d\omega = \frac{1}{2} + \frac{1}{\pi}\int_0^{\omega_m} \frac{\sin\omega(t-t_0)}{\omega} d\omega$$

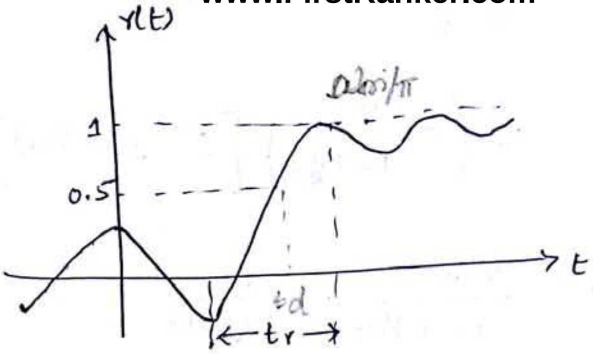
$$= \frac{1}{2} + \frac{1}{\pi}\left[Si\,\omega(t-t_0)\right]_0^{\omega_m}$$

$$= \frac{1}{2} + \frac{1}{\pi} Si\,\omega_m(t-t_0) \longrightarrow \text{sine integral}$$

$$\rightarrow \frac{dr(t)}{dt}\Big|_{t=t_0} = \frac{1}{\pi}\cos[\omega_m(t-t_0)] \cdot \omega_m$$

The rise time is given as $t_r = \frac{2\pi}{\omega_m} = \frac{1}{B}$

cut off frequency of LPF $\quad \frac{1}{t_r} = \frac{\omega_m}{\pi} \Rightarrow t_r = \frac{\omega_m}{\pi}$



$\omega_c \to \infty$

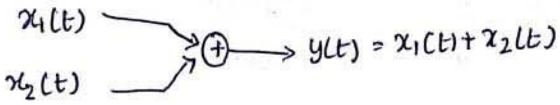
$y(t) = 1$

$\omega_c \to -\infty$

$y(t) = 0$

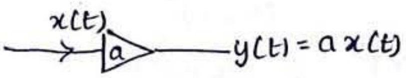
Note : (Elements of block diagram)

① Adder :



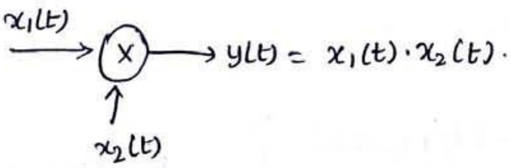
which performs the addition of two signal sequences to form sum

② constant multiplier :



It represents applying a scale factor on i/p $x(t)$.

③ Signal multiplier :



The multiplication of two signal to form product sequence.

PROBLEMS :

① The impulse response of continuous time system is given as

$$h(t) = \frac{1}{RC}\, e^{-t/RC} \cdot u(t)$$

Determine the frequency response & plot the magnitude phase plots.

Sol Take FT

$$H(\omega) = \int_{-\infty}^{\infty} h(t) \cdot e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{RC} \cdot e^{-t/RC} \cdot u(t)\, e^{-j\omega t}\, dt$$

$$= \frac{1}{RC} \int_{0}^{\infty} e^{-t/RC} \cdot e^{-j\omega t}\, dt \qquad \left(\because \text{ step } u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}\right.$$

$$= \frac{1}{RC} \int_0^{\infty} e^{-t\left(j\omega + \frac{1}{RC}\right)} dt$$

$$= \frac{1}{RC} \left(- \frac{1}{j\omega + \frac{1}{RC}} \right) \left[e^{-t(j\omega + 1/RC)} \right]_0^{\infty}$$

$$H(\omega) = \frac{1/RC}{j\omega + 1/RC} = \frac{1}{1 + j\omega RC}.$$

Magnitude & phase

$$H(\omega) = \frac{1}{1 + j\omega RC} \times \frac{1 - j\omega RC}{1 - j\omega RC} = \frac{1 - j\omega RC}{1 + (\omega RC)^2}$$

$$= \frac{1}{1 + (\omega RC)^2} + j \frac{-\omega RC}{1 + (\omega RC)^2}$$

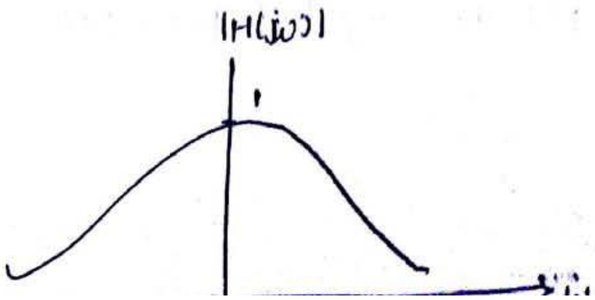
$$|H(\omega)| = \left\{ \frac{1}{[1 + (\omega RC)^2]^2} + \frac{(\omega RC)^2}{[1 + (\omega RC)^2]^2} \right\}^{1/2}$$

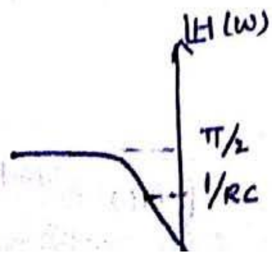
$$= \frac{1}{\sqrt{1 + (\omega RC)^2}}$$

$$\angle H(\omega) = \tan^{-1} \left\{ \frac{(-\omega RC)/1 + (\omega RC)^2}{1 + [1 + (\omega RC)^2]} \right\} = -\tan^{-1}(\omega RC)$$

If $RC = 1$,

$$|H(\omega)| = \frac{1}{\sqrt{1 + \omega^2}} \quad ; \quad \angle H(\omega) = -\tan^{-1}(\omega).$$





② For the system shown find the T.T & impulse response of the system.

$f(t) = e^{-at} \quad t>0$

$\quad\;\; 0 \quad$ elsewhere $\quad ; \quad Y(\omega) = \dfrac{1}{\alpha + j\omega}$

Sol

$$H(\omega) = \frac{R(\omega)}{F(\omega)}$$

$$R(\omega) = Y(\omega)$$

$$F(\omega) \longrightarrow \boxed{} \longrightarrow Y(\omega)$$

$$F(t) = e^{-at}$$

$$F(\omega) = \frac{1}{a+j\omega} \quad ; \quad Y(\omega) = \frac{1}{\alpha + j\omega}$$

$$H(\omega) = \frac{1/\alpha+j\omega}{1/a+j\omega} = \frac{a+j\omega}{\alpha+j\omega}$$

$$F^{-1}\left[\frac{a+j\omega}{\alpha+j\omega}\right] \Rightarrow \frac{a+\alpha-\alpha+j\omega}{\alpha+j\omega} = \frac{a-\alpha}{\alpha+j\omega} + \frac{\alpha+j\omega}{\alpha+j\omega}$$

$$= \frac{a-\alpha}{\alpha+j\omega} + 1$$

$$\boxed{h(t) = (a-\alpha)\, e^{-\alpha t} u(t) + \delta(t)}$$

③ The linear system impulse response is $[e^{-2t} + e^{-3t}]u(t)$ find the excitation to produce an o/p of $t \cdot e^{-2t} u(t)$?

Sol

$$h(t) = [e^{-2t} + e^{-3t}]u(t)$$

$$F(\omega) \longrightarrow \boxed{H(\omega)} \longrightarrow R(\omega)$$

$$r(t) = t \cdot e^{-2t} u(t)$$

$$H(\omega) = \frac{R(\omega)}{F(\omega)}$$

$$F(\omega) = \frac{R(\omega)}{H(\omega)}$$

$$r(t) = t \cdot e^{-2t} u(t) \xleftrightarrow{F.T} \frac{1}{(2+j\omega)^2} \qquad \left(\because t \cdot e^{-at} u(t) \longleftrightarrow \frac{1}{(a+j\omega)^2}\right)$$

$$R(\omega) = \frac{1}{(2+j\omega)^2}$$

$h(t) = e^{-2t} u(t) + e^{-3t} u(t)$

$$H(\omega) = \frac{1}{2+j\omega} + \frac{1}{3+j\omega}$$

$$R(\omega) = \frac{1/(2+j\omega)}{\frac{3+j\omega+2+j\omega}{(2+j\omega)(3+j\omega)}} = \frac{1}{2+j\omega} \times \frac{3+j\omega}{5+2j\omega}$$

$$\frac{3+j\omega}{(2+j\omega)(5+2j\omega)} = \frac{A}{2+j\omega} + \frac{B}{5+2j\omega}$$

$$3+j\omega = A(5+2j\omega) + B(2+j\omega)$$

put $3+j\omega = 5A+2B + j\omega(2A+B)$

put $j\omega = 0$; put $j\omega(-2)$

$(3 = 5A+2B) \times 1$

$(1 = 2A+B) \times 2$

$A = 1, \; B = -1$

$$R(\omega) = \frac{1}{2+j\omega} - \frac{1}{5+2j\omega} = \frac{1}{2+j\omega} - \frac{1}{2[5/2+j\omega]}$$

$$\boxed{r(t) = e^{-2t} u(t) - \frac{1}{2} e^{-5/2 t} u(t)}$$

DIFFERENTIAL EQUATION:

→ To obtain frequency response & impulse response.

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t).$$

differentiation property of FT is

$$\frac{d}{dt} x(t) \xleftrightarrow{FT} j\omega X(\omega).$$

$$\sum_{K=0}^{N} a_K (j\omega)^K Y(\omega) = \sum_{K=0}^{M} b_K (j\omega)^K X(j\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{K=0}^{M} b_K (j\omega)^K}{\sum_{K=0}^{N} a_K (j\omega)^K}.$$

system transfer fn.

PROBLEMS:

① The differential equation of system is given as $\frac{d^2 y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = -\frac{dx(t)}{dt}$

Determine the frequency response & impulse response.

Sol

$$\frac{d^2 y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = -\frac{dx(t)}{dt}$$

Taking F.T

$$(j\omega)^2 Y(\omega) + 5(j\omega)Y(\omega) + 6Y(\omega) = -j\omega X(\omega)$$

$$Y(\omega)\left[(j\omega)^2 + 5j\omega + 6\right] = -j\omega X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{-j\omega}{(j\omega)^2 + 5j\omega + 6}$$

$$H(\omega) = \frac{-j\omega}{(j\omega+2)(j\omega+3)} = \frac{A}{j\omega+2} + \frac{B}{j\omega+3}$$

$$= \frac{2}{j\omega+2} - \frac{3}{j\omega+3}$$

$$h(t) = \left[2 \cdot e^{-2t} - 3e^{-3t}\right] u(t)$$

$$\left\{ \because e^{-at} u(t) \xleftrightarrow{FT} \frac{1}{a+j\omega} \right\}$$

impulse response of the system.

(8) The input voltage to the RC circuit is given by $x(t) = te^{-t/RC}u(t)$ and impulse response of this circuit is given by $h(t) = \frac{1}{RC}e^{-t/RC}u(t)$. Find output $y(t)$

Sol

output $y(t) = x(t) * h(t)$

In frequency domain

$$Y(\omega) = X(\omega)H(\omega)$$

and $H(\omega) = F[h(t)]$

$$H(\omega) = F\left\{\frac{1}{RC}e^{-t/RC}.u(t)\right\}$$

$$= \frac{1}{RC}\cdot\frac{1}{\frac{1}{RC}+j\omega} = \frac{1}{1+j\omega RC}$$

$$X(\omega) = F\left[t\cdot e^{-t/RC}u(t)\right]$$

$$= \int_{-\infty}^{\infty} t\cdot e^{-t/RC}\, e^{-j\omega t}\,dt = \frac{1}{\left(\frac{1}{RC}+j\omega\right)^2} = \frac{(RC)^2}{(1+j\omega RC)^2}$$

$$\left[\because\ te^{-at}u(t) \xleftrightarrow{FT} \left(\frac{1}{a+j\omega}\right)^2\right]$$

$$Y(\omega) = X(\omega)\cdot H(\omega)$$

$$= \frac{(RC)^2}{(1+j\omega RC)^2}\cdot\frac{1}{(1+j\omega RC)} = \frac{(RC)^2}{(1+j\omega RC)^3}$$

$$y(t) = F^{-1}\{Y(\omega)\} = F^{-1}\left\{\frac{(RC)^2}{(1+j\omega RC)^3}\right\} = F^{-1}\left\{\frac{\cancel{(RC)^2}}{\cancel{(RC)^3}\left(\frac{1}{RC}+\right.}\right.$$

$$\boxed{y(t) = \frac{1}{RC}\cdot\frac{t^2\cdot e^{-t/RC}}{2}u(t)}$$

Ms.Rincy Sara luka

1) $h(t) = e^{-5|t|}$

For stability $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$

$$\therefore \int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} |e^{-5|t|}|\, dt = \int_{-\infty}^{\infty} e^{-5|t|}\, dt$$

$$= \int_{-\infty}^{0} e^{5t}\, dt + \int_{0}^{\infty} e^{-5t}\, dt = \left[\frac{e^{5t}}{5}\right]_{-\infty}^{0} + \left[\frac{e^{-5t}}{-5}\right]_{0}^{\infty}$$

$= \frac{2}{5}$ = constant / so system is stable.

2) $h(t) = e^{4t} u(t)$

$$= \int_{-\infty}^{\infty} |e^{4t} u(t)|\, dt = \int_{-\infty}^{\infty} e^{4t} u(t)\, dt$$

$$= \int_{0}^{\infty} e^{4t}\, dt = \frac{e^{\infty}}{4} - \frac{e^{0}}{4} = \infty - \frac{1}{4} = \infty$$ (Unstant.

3) $h(t) = e^{-4t} u(t)$ (stable)

4) $h(t) = t \cos t\, u(t)$ (unstable.

$$\int_{0}^{\infty} t \cos t\, dt$$

5) $h(t) = e^{-t} \sin t\, u(t)$ (stable.

$$= \int_{0}^{\infty} e^{-t} \sin t\, dt$$

→ The system produces the o/p of $y(t) = e^{-t}u(t)$ for an input of $x(t) = e^{-2t}u(t)$. Determine the impulse response and frequency response of the system.

Sol

$$y(t) = e^{-t}u(t)$$

$$x(t) = e^{-2t}u(t)$$

Consider standard fourier transform pair $e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{a+j\omega}$

$$Y(\omega) = \frac{1}{1+j\omega} \; ; \; X(\omega) = \frac{1}{2+j\omega}.$$

From equation

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1/1+j\omega}{1/2+j\omega} = \frac{2+j\omega}{1+j\omega}.$$

Multiply ~~and divide~~ the numerator & denominator by $1-j\omega$

$$H(\omega) = \frac{2+j\omega}{1+j\omega} \times \frac{1-j\omega}{1-j\omega} = \frac{(2+j\omega)(1-j\omega)}{(1)^2-(j\omega)^2} = \frac{2-2j\omega+j\omega+\omega^2}{1+\omega^2}$$

$$= \frac{2+\omega^2-j\omega}{1+\omega^2} \Rightarrow \frac{2+\omega^2}{1+\omega^2} + j\frac{-\omega}{1+\omega^2}.$$

Magnitude $|H(\omega)| = \left\{ \sqrt{\left[\frac{2+\omega^2}{1+\omega^2}\right]^2 + \left[\frac{\omega^2}{1+\omega^2}\right]^2} \right\}$

$$|H(\omega)| = \sqrt{\frac{4+\omega^2}{1+\omega^2}} \; ; \; \angle H(\omega) = \tan^{-1}\left(\frac{\frac{-\omega}{1+\omega^2}}{\frac{2+\omega^2}{1+\omega^2}}\right)$$

$$= -\tan^{-1}\left(\frac{\omega}{2+\omega^2}\right)$$

$$\therefore \; H(\omega) = \frac{2+j\omega}{1+j\omega} \Rightarrow \frac{1+j\omega+1}{1+j\omega} \Rightarrow 1+\frac{1}{1+j\omega}.$$

Inverse fourier transform

$$\boxed{h(t) = IFT\{H(\omega)\} = \delta(t) + e^{-t}u(t)}$$

impulse response.

transfer function of LPF is given by

$$H(\omega) = \begin{cases} (1+k\cos\omega T)e^{-j\omega T} & ; |\omega| < 2\pi B \\ 0 & ; |\omega| > 2\pi B. \end{cases}$$

Determine the output y(t) when a pulse x(t) bandlimited in B is applied at the input.

Sol)

$$Y(\omega) = X(\omega) H(\omega)$$

$$= X(\omega)[1 + k\cos\omega T]e^{-j\omega T}$$

$$= X(\omega)e^{-j\omega T} + k\, X(\omega)\cos\omega T\, e^{-j\omega T}$$

we know

$$x(t-T) + x(t+T) \longleftrightarrow 2X(\omega)\cos\omega T$$

$$x(t-T) \longleftrightarrow X(\omega)e^{-j\omega T}$$

$$y(t) = F^{-1}[Y(\omega)]$$

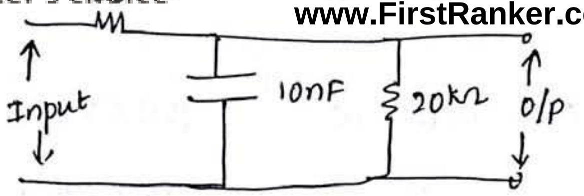
$$= F^{-1}\left[X(\omega)e^{-j\omega T} + k X(\omega)e^{-j\omega T}\cos\omega T\right]$$

$$= x(t-T) + \frac{k}{2}\left[x(t-T-T) + x(t-T+T)\right]$$

$$y(t) = x(t) + \frac{k}{2}\left[x(t-T) + x(t+T)\right]$$

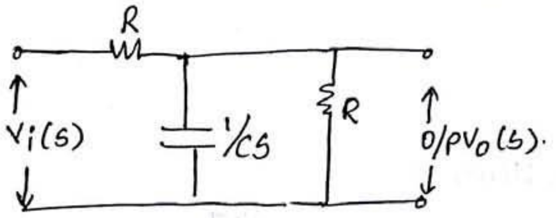
delayed by T.

2) Determine the maximum bandwidth of signals that can be transmitted through low pass RC filter as shown in figure , if over this bandwidth, the gain variation is to be 10% and the phase variation is to be within 7% of ideal characteristics.



Sol

RC network transformed into s-domain representation.



$$H(s) = \frac{V_o(s)}{V_i(s)}$$

$$= \frac{[R \parallel (1/Cs)]}{([R \parallel (1/Cs)] + R]}$$

$$= \frac{(R/Cs) / [R + 1/Cs]}{[R/Cs] / [[R + 1/Cs)] + R}$$

$$= \frac{R/(1+sCR)}{[R/(1+sCR)] + R} = \frac{R}{R + R(1+sCR)} = \frac{\cancel{R}}{\cancel{R}(1+1+sCR)}$$

$$H(s) = \frac{1}{2+sCR}.$$

But given $R = 20k\Omega$
$C = 10nF$.

$$H(s) = \frac{1}{2+s(10\times10^{-9}\times20\times10^{3})} = \frac{1}{2+(2s/10^4)} = \frac{10^4}{2\times10^4+2s} = \frac{1}{2+s(2\times10^{-4})}$$

$$\therefore H(s) = \frac{5000}{s+10000}$$

put $s = j\omega$

$$H(\omega) = \frac{5000}{j\omega + 10000}$$

$$|H(\omega)| = \frac{5000}{\sqrt{\omega^2 + 10000}}$$

$$\phi(\omega) = -\tan^{-1}\left(\frac{\omega}{10000}\right).$$

At $\omega = 0$, $|H(\omega)|_{\omega=0} = \frac{5000}{10000} = 0.5$.

But there is 10% variation in gain over bandwidth B.

$$H(\omega) = 0.5 - 0.5 \times 10\% = 0.45$$

$$|H(\omega)| = \frac{5000}{\sqrt{B^2 + 10^8}}$$

$$B^2 + 10^8 = \left(\frac{5000}{0.45}\right)^2 \Rightarrow B^2 = 23.46 \times 10^6$$

$$B = 4.84\ KHZ$$

But $B = 2\pi f$

$$f = \frac{B}{2\pi} = \frac{4.84 \times 10^3}{2\pi} = 770.8\ HZ$$

phase at frequency, $f = 770.8\ HZ$

$$\phi(\omega) = -\tan^{-1}\left(\frac{4.84}{10}\right) = -25.83\%.$$

(2) There are several possible ways of estimating an essential bandwidth of non-bandlimited signal. For a low pass signal, for example, the essential b.w may

CONVOLUTION AND CORRELATION OF SIGNALS:

→ Convolution is used to find common area between two signals or two fns.

The convolution f(t) of two time functions $f_1(t)$ and $f_2(t)$ is designed or defined as

$$f(t) = f_1(t) \otimes f_2(t)$$

$$= \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau$$

(or)

$$= \int_{-\infty}^{\infty} f_2(\tau)\, f_1(t-\tau)\, d\tau$$

→ Convolution is a mathematical operation and multiplication is one of the form of convolution.

Ex: 25
×25
———
625

In convolution method

(i) $f_1(\tau)$ & $f_2(\tau)$
↓ 25 ↓ 25

(ii) $f_2(-\tau)$
⇒ 52

(iii) $f_2(t-\tau)$
$f_2(-\tau)$ shifted to right side by t seconds.

→ Here 't' is varied from $-\infty$ to ∞

Convolution Procedure:

Step 1: $f_1(t) \otimes f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau$

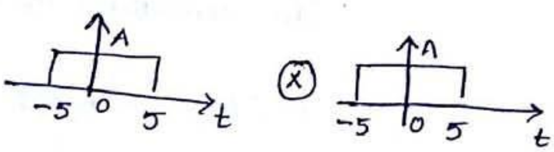
Here the independent variable convolution integral is τ, so replace t by τ to get $f_1(\tau)$ & $f_2(\tau)$

Step 2: $f_1(\tau)$ is first function and $f_2(\tau)$ is the 2nd function. $f_2(-\tau)$ is the mirror image of the $f_2(\tau)$

Step 3: $f_2(t-\tau)$ represents the function $f_2(-\tau)$ shifted to right side by t sec. 't' is varied from $-\infty$ to ∞ and find common area between two functions.

→ The value of convolution obtained at different values of 't' and may be plotted on a graph.

① Find the F.T of



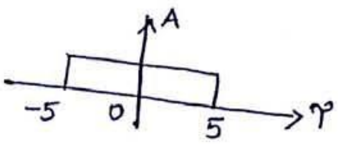
Sol Step 1:

$f_1(t) = A\,G_{10}(t) \longrightarrow f_1(\tau) = A\,G_{10}(\tau)$

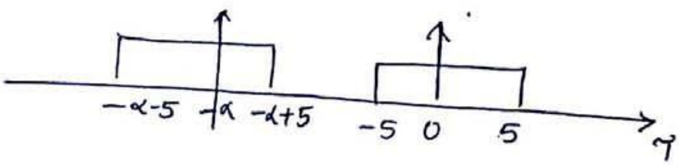
$f_2(t) = A\,G_{10}(t) \longrightarrow f_2(\tau) = A\,G_{10}(\tau)$

Step 2:

$f_2(-\tau) = A\,G_{10}(-\tau) \longrightarrow$ even function

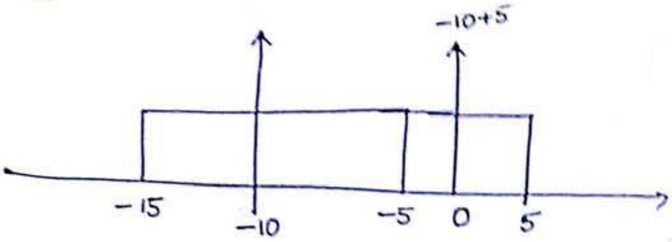


① $f_1(\tau)$ at $t = -\alpha$



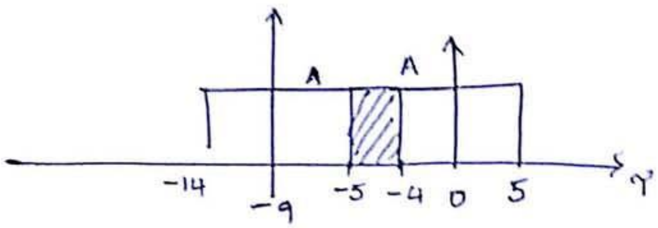
$f_1(t) \otimes f_2(t) = 0$

(2) at $t=-10$



$-10+5=-5$
$-10-5=-15$

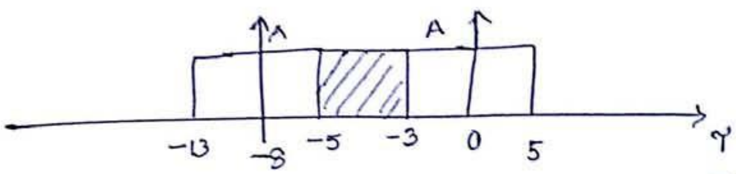
(3) at $t=-9$



$-9-5=-14$
$-9+5=-4$

$$\Rightarrow \int_{-5}^{-4} A\cdot A\, d\gamma = A^2[\gamma]_{-5}^{-4}$$
$$= A^2[-4+5]$$
$$= A^2$$

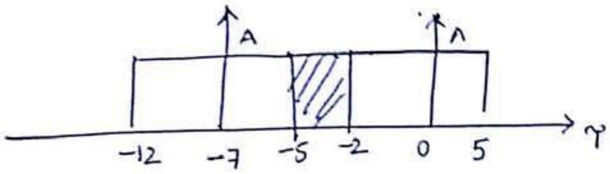
(4) at $t=-8$



$-8+5=-3$
$-8-5=-13$

$$\Rightarrow \int_{-5}^{-3} A\cdot A\, d\gamma = A^2[\gamma]_{-5}^{-3}$$
$$= A^2[-3+5] \Rightarrow 2A^2.$$

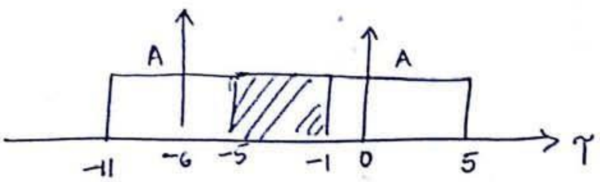
(5) at $t=-7$



$-7+5=-2$
$-7-5=-12$

$$\Rightarrow \int_{-5}^{-2} A\cdot A\, d\gamma = A^2[\gamma]_{-5}^{-2}$$
$$= A^2[-2+5] = 3A^2.$$

(6) at $t=-6$



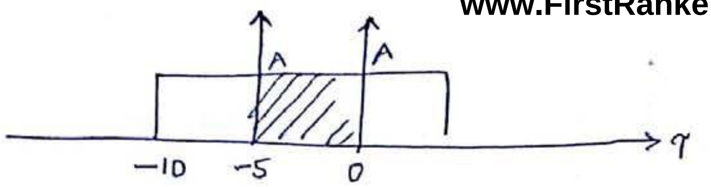
$-6+5=-1$
$-6-5=-11$

$$\Rightarrow \int_{-5}^{-1} A\cdot A\, d\gamma \Rightarrow A^2[-1+5] = 4A^2.$$

at t = -5

$-5-5=-10$

$-5+5=0$

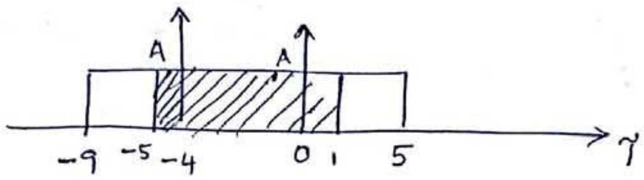


$$\int_{-5}^{0} A\cdot A\, d\gamma \Rightarrow A^2\cdot[\gamma]_{-5}^{0} = A^2\cdot 5$$

(8) at $t=-4$

$-4+5=1$

$-4-5=-9$

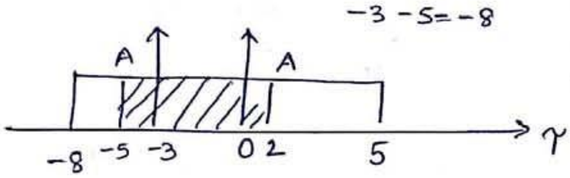


$$\int_{-5}^{1} A\cdot A\, d\gamma \Rightarrow A^2[\gamma]_{-5}^{1} = 6A^2.$$

(9) at $t=-3$

$-3+5=+2$

$-3-5=-8$



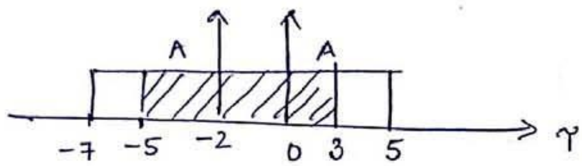
$$\int_{-5}^{2} A\cdot A\, d\gamma \Rightarrow A^2[\gamma]_{-5}^{2}$$

$$\Rightarrow A^2[2+5] = 7A^2.$$

(10) at $t=-2$

$-2+5=+3$

$-2-5=-7$

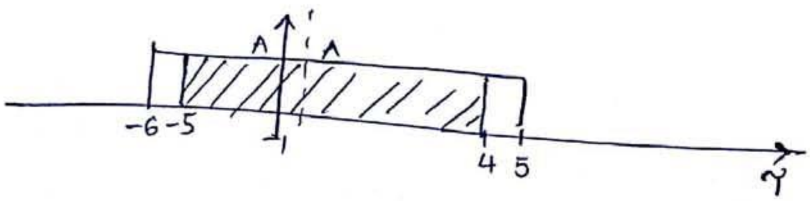


$$\int_{-5}^{3} A\cdot A\, d\gamma \Rightarrow A^2[\gamma]_{-5}^{3} = A^2[3+5] = 8A^2$$

(11) at $t=-1$

$-1+5=4$

$-1-5=-6$



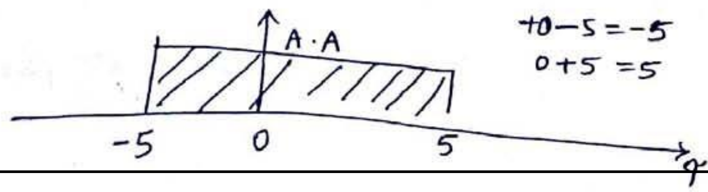
$$\Rightarrow \int_{-5}^{4} A\cdot A\, d\gamma \Rightarrow A^2[\gamma]_{-5}^{-4} = A^2[9]$$

$$= 9A^2$$

(12) at $t=0$

$+0-5=-5$

$0+5=5$

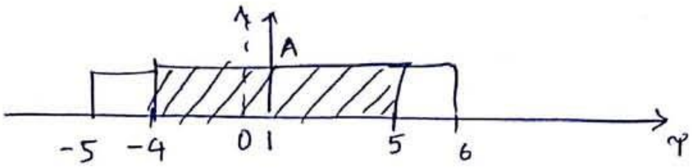


$$\Rightarrow \int_{-5}^{5} A\cdot A\, d\gamma = A^2[\gamma]_{-5}^{5} = 10A^2$$

(13) at $t=1$

$-5+1=-4$

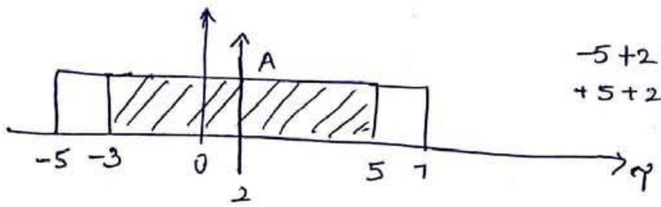
$+5+1=6$



$$\int_{-4}^{5} A\cdot A\, d\gamma \Rightarrow A^2[\gamma]_{-4}^{5}$$

$$\Rightarrow A^2[5+4] = 9A^2$$

(14) at $t=2$



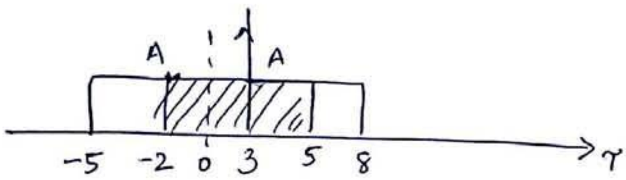
$-5+2=-3$

$+5+2=7$

$$\int_{-3}^{5} A\cdot A\, d\gamma = A^2[\gamma]_{-3}^{5}$$

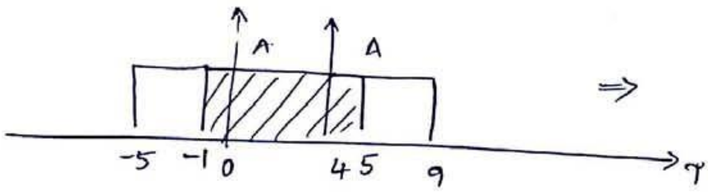
$$= 8A^2$$

(15) at $t=3$



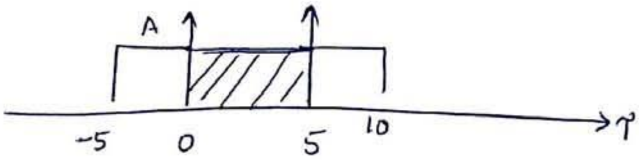
$$\int_{-2}^{5} A\cdot A\, d\gamma \Rightarrow A^2[5+2] = 7A^2$$

(16) $t=4$



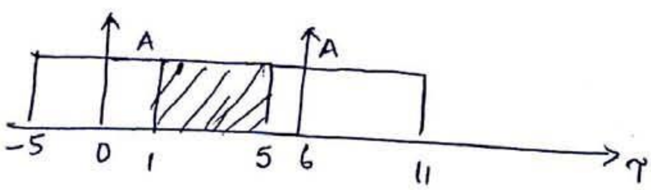
$$\Rightarrow \int_{-1}^{5} A\cdot A\, d\gamma = A^2[5+1] = 6A^2$$

(17) $t=5$



$$\int_{0}^{5} A\cdot A\, d\gamma = A^2[5]$$

(18) $t=6$



$$\int_{1}^{5} A\cdot A\, d\gamma \Rightarrow A^2[\gamma]_{1}^{5} = A^2[4]$$

|| for $t=7 \Rightarrow 3A^2$

$t=8 \Rightarrow 2A^2$

$t=9 \Rightarrow 1A^2$

$t=10 \Rightarrow 0$

$t=-10\,;\,10 = 0$

$t=-9\,;\,9 = 1A^2$

$t=-8\,;\,8 = 2A^2$

$t=-7\,;\,7 = 3A^2$

$t=-6\,;\,6 = 4A^2$

$t=-5\,;\,5 = 5A^2$

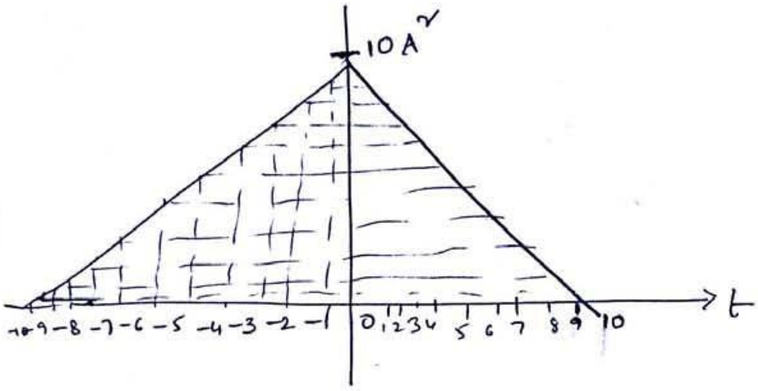
$t=-4\,;\,4 = 6A^2$

$t=-3\,;\,3 = 7A^2$

$t=-2\,;\,2 = 8A^2$

$t=-1\,;\,1 = 9A^2$

$t=0 \Rightarrow 10A^2$



4

Convolution : Mathematical way of combining two signals to form a third signal. i.e input signal, output signal and impulse response.

→ It is used to express the input and output relationship of a LTI system

→ If two functions x(t) and y(t) in time domain are defined then convolution z(t) is

$$z(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau.$$

(or)

$x(t) \otimes y(t)$.

↳ Read as x(t) convolved with y(t).

Convolution in time domain:

→ The convolution in time domain is equivalent to multiplication of their spectra in frequency domain i.e $x(t) \xleftrightarrow{} X(\omega)$ and $y(t) \longleftrightarrow Y(\omega)$ then

$$x(t) * y(t) \longleftrightarrow X(\omega) \cdot Y(\omega).$$

Proof x(t) fourier transform is given by

$$F\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$F\{x(t) * y(t)\} = \int_{-\infty}^{\infty} [x(t) * y(t)]\, e^{-j\omega t}\, dt$$

$$\text{But } x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$$

$$\therefore F\{x(t) * y(t)\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau \right] e^{-j\omega t}\, dt$$

Inter changing the order of integration, we get

$$F\{x(t) * y(t)\} = \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} y(t-\tau)\, e^{-j\omega t}\, dt \right] d\tau$$

Using time shifting property, we get

$$\int_{-\infty}^{\infty} y(t-\tau)\, e^{-j\omega t}\, dt = Y(\omega)\, e^{-j\omega \tau}$$

$$F[x(t)*y(t)] = \int_{-\infty}^{\infty} x(\tau)\, Y(\omega)\, e^{-j\omega\tau}\, d\tau$$

$$= Y(\omega)\int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau = Y(\omega)\, X(\omega).$$

$$x(t)*y(t) \longleftrightarrow X(\omega)\, Y(\omega)$$

$\downarrow$

This is time convolution theorem.

Convolution in frequency domain:

→ Multiplication of two functions in time domain is equivalent to convolution of their spectra in frequency domain.

$x(t) \longleftrightarrow X(\omega)$ and $y(t) \longleftrightarrow Y(\omega)$ then $2\pi\, x(t)\, y(t) \longleftrightarrow X(\omega) * Y(\omega)$ (or)

$x(t)\, y(t) \longleftrightarrow X(f) * Y(f)$.

proof

$$X(\omega) = F\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x(t) = F^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$F\{x(t)\, y(t)\} = \int_{-\infty}^{\infty} x(t)\, y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)\, e^{j\lambda t}\, d\lambda\right] y(t)\, e^{-j\omega t}\, dt$$

Interchanging the order of integration we get

$$F\{x(t)\, y(t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)\left[\int_{-\infty}^{\infty} y(t)\cdot e^{-j\omega t} e^{j\lambda t}\, dt\right] d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)\left[\int_{-\infty}^{\infty} y(t)\cdot e^{-j(\omega-\lambda)t}\, dt\right] d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)\, Y(\omega-\lambda)\, d\lambda = \frac{1}{2\pi}\left[X(\omega) * Y(\omega)\right]$$

$$x(t)\, y(t) \longleftrightarrow \frac{1}{2\pi} X(\omega) * Y(\omega)$$

$$2\pi\, x(t)\, y(t) \longleftrightarrow X(\omega) * Y(\omega)$$

Graphical convolution:

→ The convolution by inspection provides the information needed without complicated calculations. This convolution by inspection procedure is called graphical convolution.

Procedure:

(1) $x(\tau)$ is the first function, where an independent variable 't' is replaced by dummy variable 'τ'.

(2) $y(-\tau)$ is the mirror image of $y(\tau)$ i.e $y(\tau)$ is flipped.

(3) $y(t-\tau)$ represents the function $y(-\tau)$ shifted to right by t seconds.

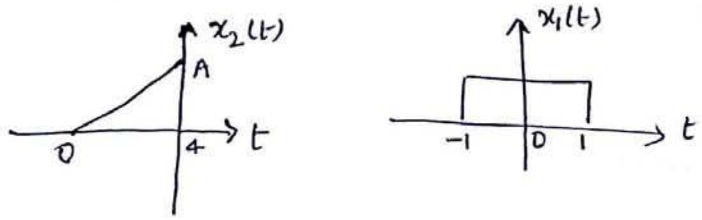
(4) For a particular value of $t=a$, integration of product $x(\tau)\,y(a-\tau)$ represents the area under the (common area) product curve.

$$\int_{-\infty}^{\infty} x(\tau)\,y(a-\tau)\,d\tau = \left[x(t) * y(t)\right]_{t=a}$$

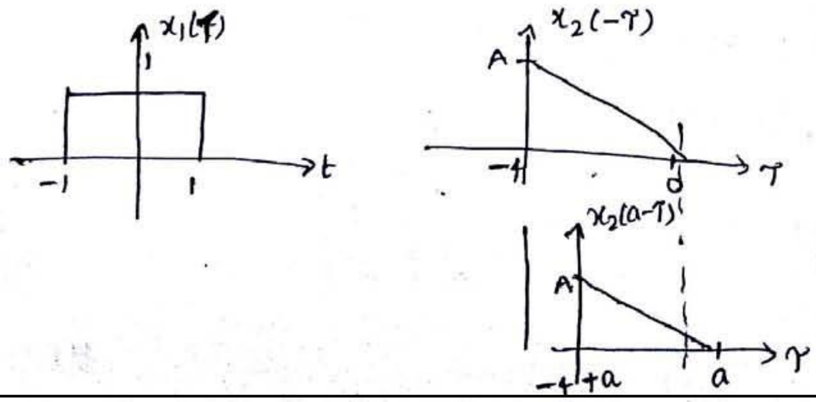
(5) The procedure is repeated for different values of t.
For negative value of t, the function $y(-\tau)$ is shifted left by t seconds.

6) The value of convolution obtained at different values of t (i.e +ve, −ve).

Q. Find the convolution of the functions $x_1(t)$ and $x_2(t)$



Sol



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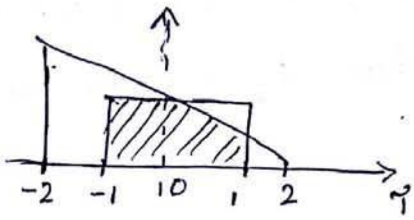
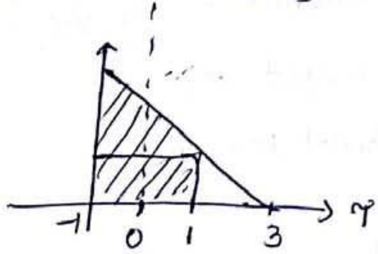
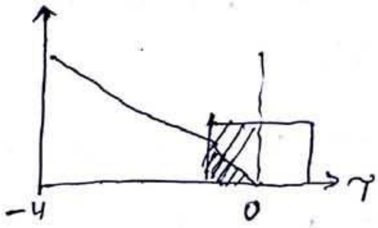
$[x_1(t) * x_2(t)]_{a=2}$



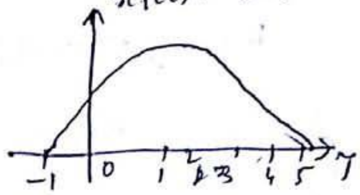
$[x_1(t) * x_2(t)]_{a=3}$



$[x_1(t) * x_2(t)]_{a=0}$



$x_1(t) * x_2(t)$

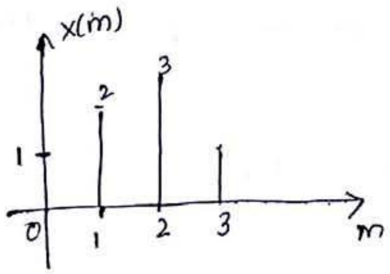


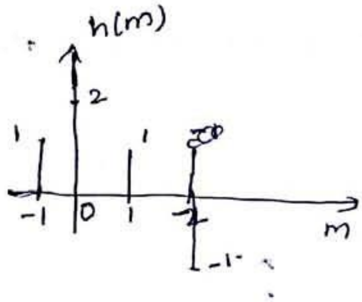
2) Determine the response of LTI system whose input $x(n)$ and impulse response $h(n)$ are given by $x(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$ and $h(n) = \{1, \underset{\uparrow}{2}, 1, -1\}$

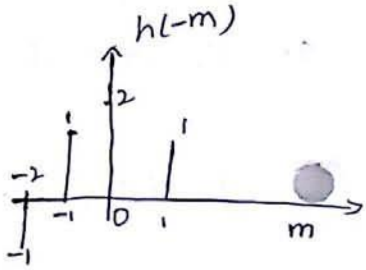
Sol The input sequence starts at $n=0$ and impulse response starts at $n=-1$.

O/p sequence starts at $n = 0+(-1) = -1$

→ i/p & impulse response consists of 4 samples, so o/p consists of $4+4-1 = 7$ sa

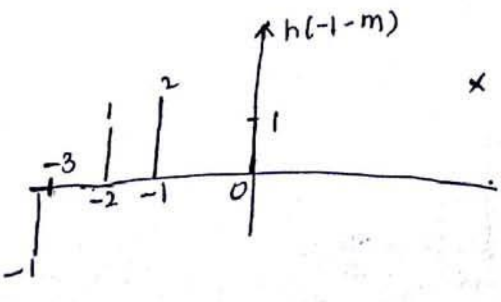






$$y(n) = \sum_{m=-\infty}^{\infty} x(m) h(n-m).$$

when $n = -1$; $y(-1) = \sum_{m=-\infty}^{\infty} x(m)\, h(-1-m)$



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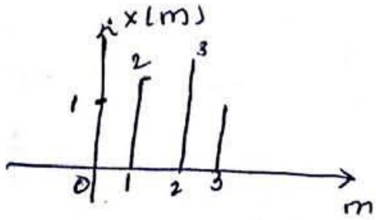

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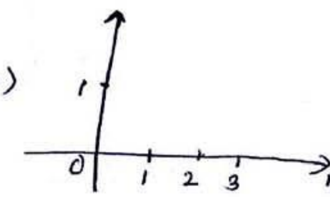
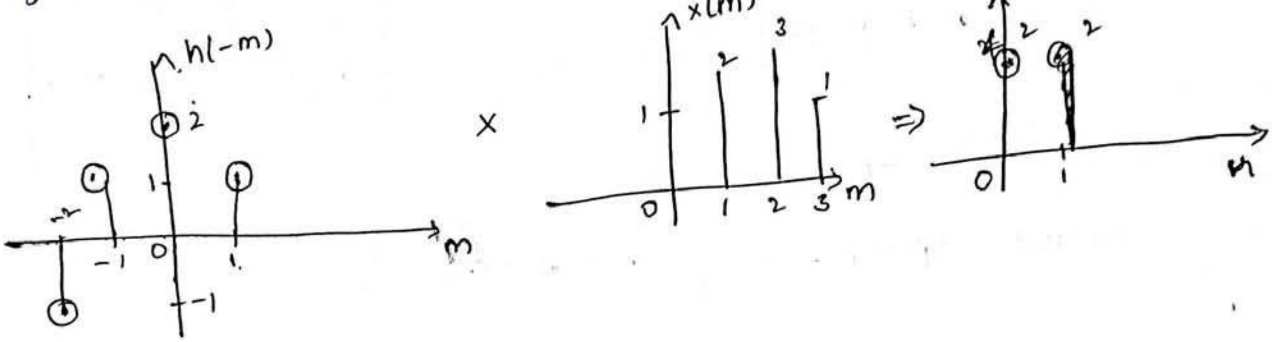


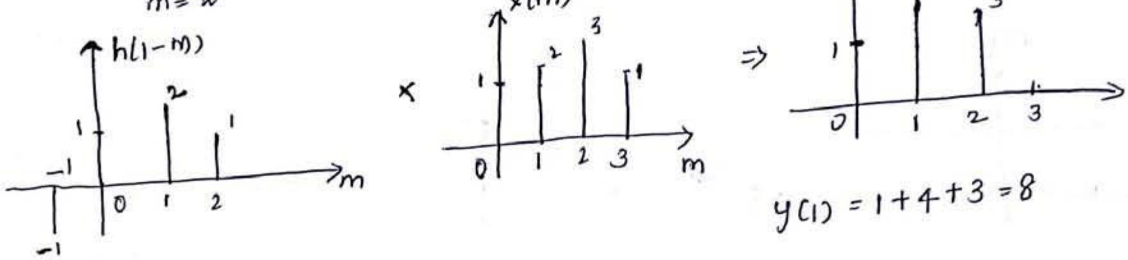
$y(-1) = 1$

when $n=0$, $y(0) = \sum_{m=-\infty}^{\infty} x(m)\, h(0-m)$

$y(0) = 2+2 = 4$

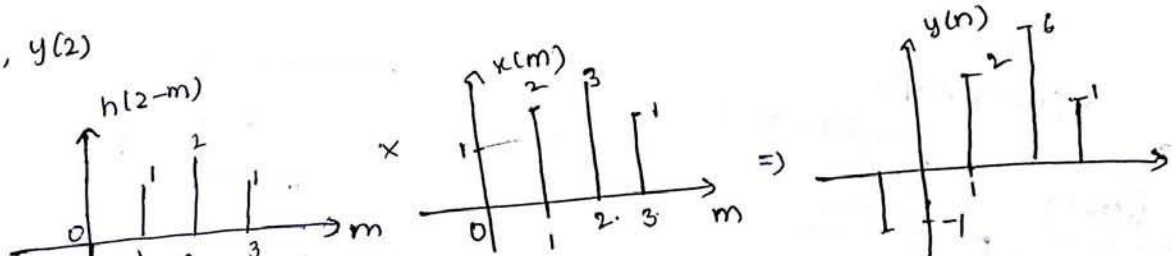


when $n=1$, $y(1) = \sum_{m=-\infty}^{\infty} x(m)\, h(1-m)$



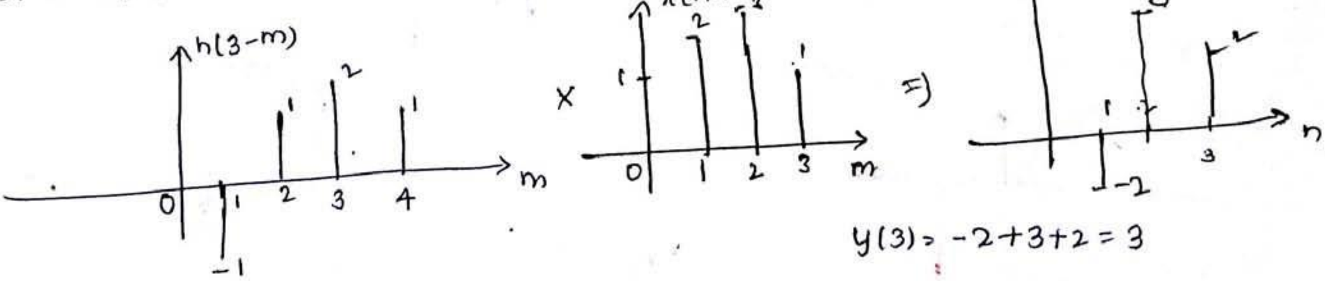
$y(1) = 1+4+3 = 8$

when $n=2$, $y(2)$



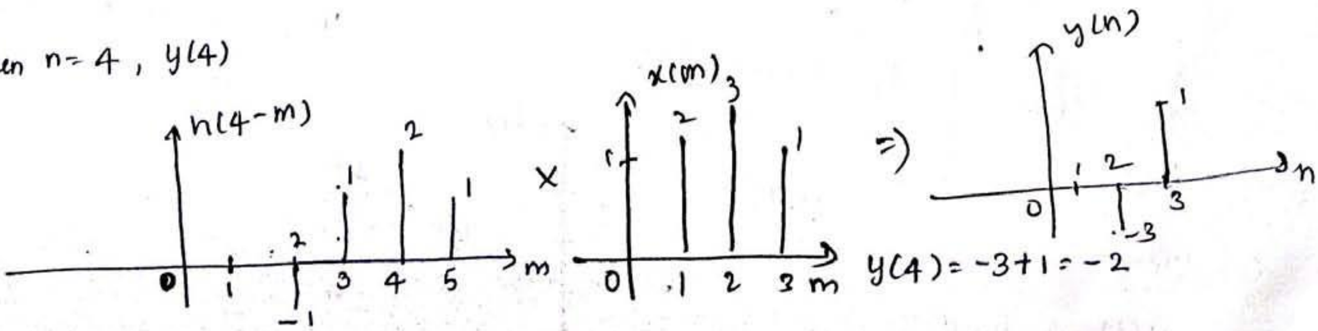
$y(2) = -1+2+6+1 = 8$

when $n=3$, $y(3)$



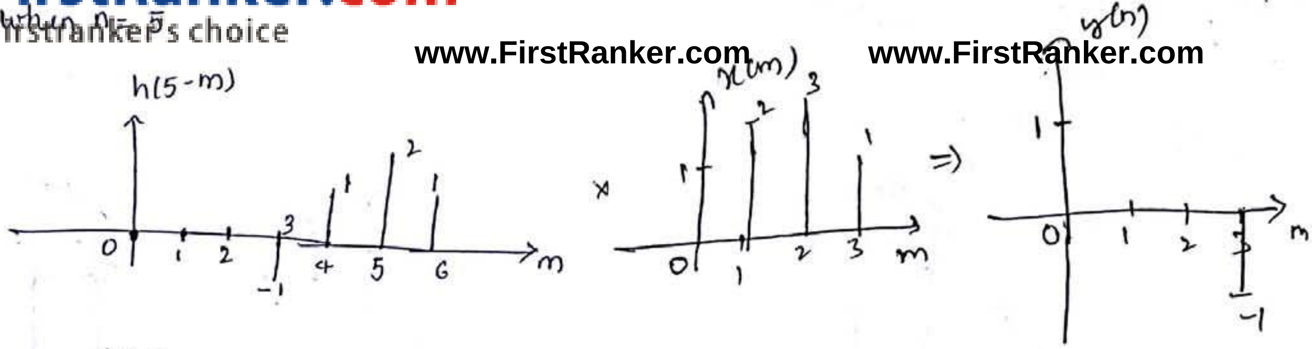
$y(3) = -2+3+2 = 3$

when $n=4$, $y(4)$



$y(4) = -3+1 = -2$

when n = 5

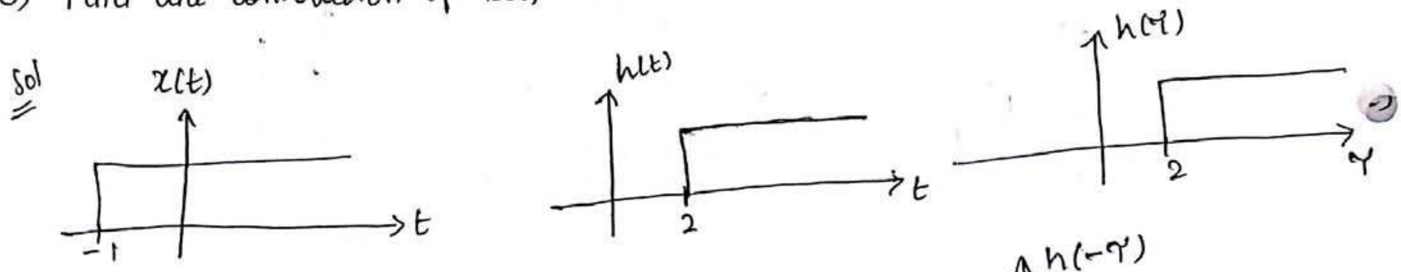


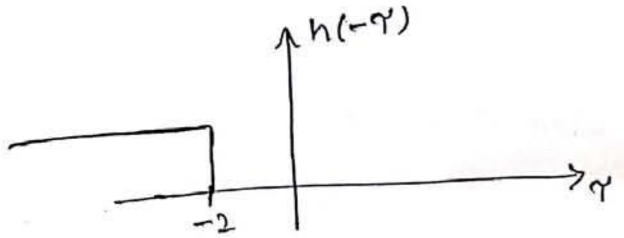
$y(5) = -1$

Output sequence $y(n) = \{1, \underset{\uparrow}{4}, 8, 8, 3, -2, -1\}$

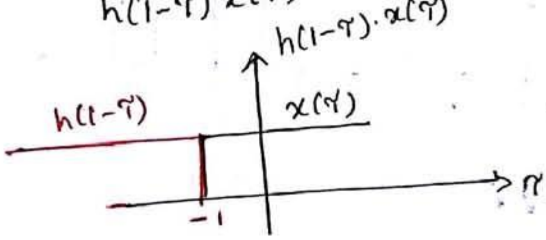
(3) Find the convolution of $x(t) = u(t+1)$ and $h(t) = u(t-2)$

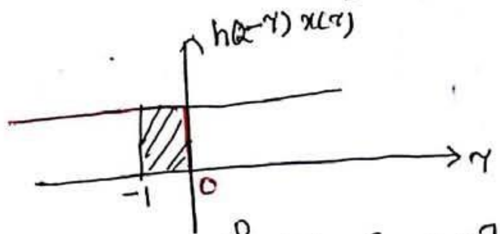
Sol





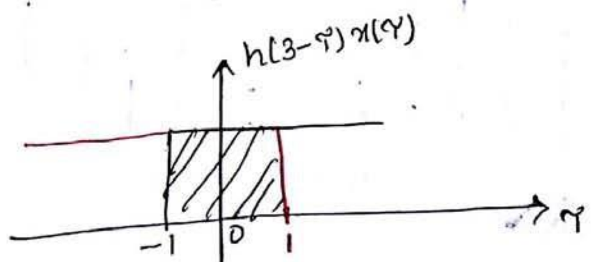
$h(1-\tau)\,x(\tau)$

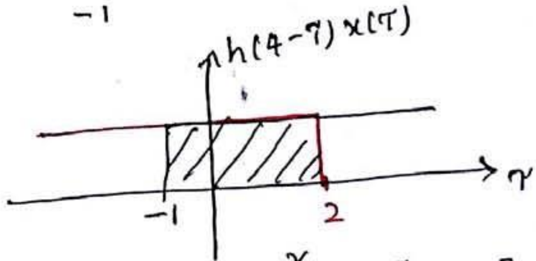




$y(1) =$ no overlap $= 0$

$$y(2) = \int_{-1}^{0} 1 \cdot d\tau = [0-(-1)] = 1$$

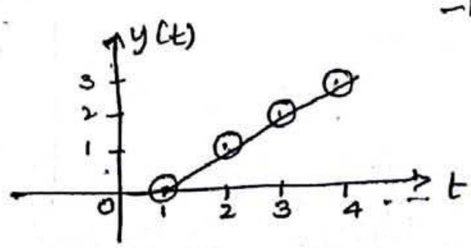




$$y(3) = \int_{-1}^{1} 1 \cdot d\tau = [1-(-1)] = 2$$

$$y(4) = \int_{-1}^{2} 1 \cdot d\tau = [2-(-1)] = 3.$$

$\therefore\ y(t) =$



Condition

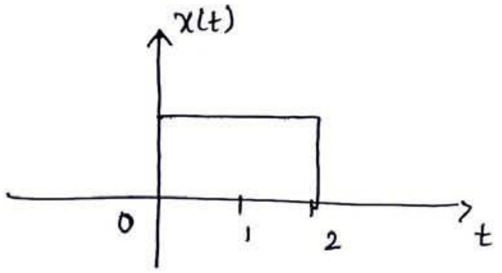
$y(t) = 0$ for $t \leq 0$

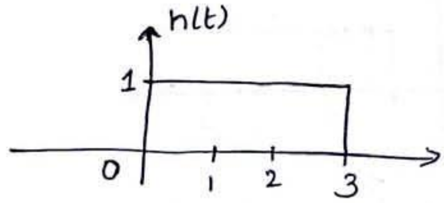
$y(t) = t-1$ for $t > 0$

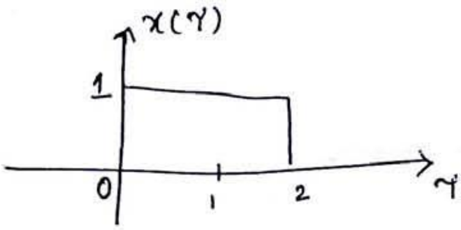
(7) Find the convolution of $x(t)$ and $h(t)$

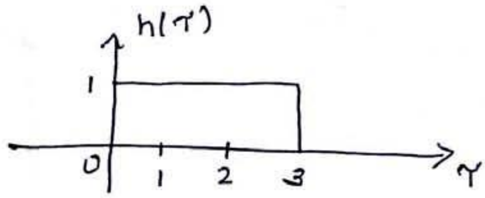
$$x(t) = 1 \quad 0 \le t < 2$$
$$= 0 \quad \text{otherwise}$$
$$h(t) = 1 \quad 0 \le t \le 3$$
$$= 0 \quad \text{otherwise}$$

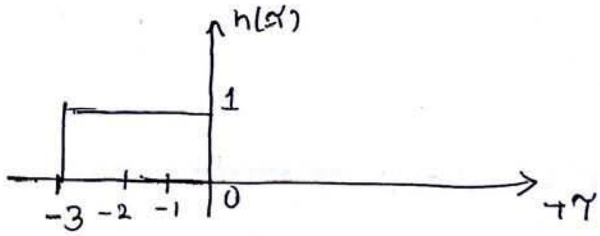
Sol



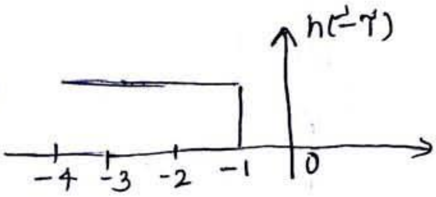


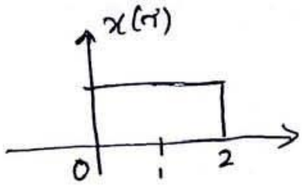






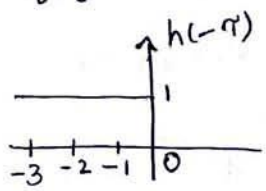
$t = -1$

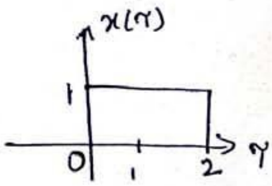




No overlap. $y(t) = 0$

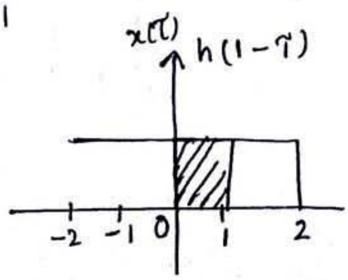
$t = 0$

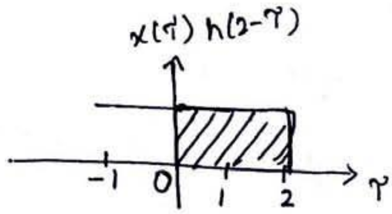




$y(t) = 0$ for $t \le 0$

$t = 1$





for $0 < t \le 1$

$$y(1) = \int_0^1 1 . d\tau = [\tau]_0^1 = 1$$

for $0 < t \le 2$

$$y(2) = \int_0^2 1 . d\tau = [\tau]_0^2 = 2.$$

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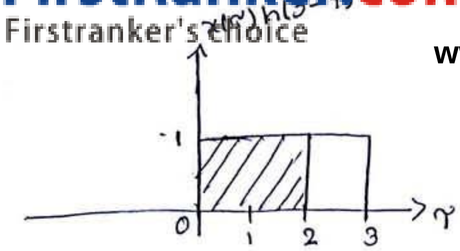
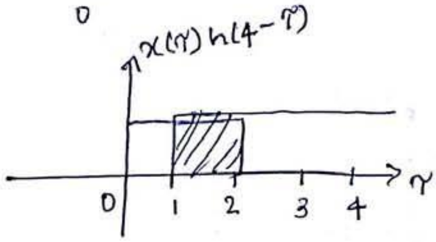
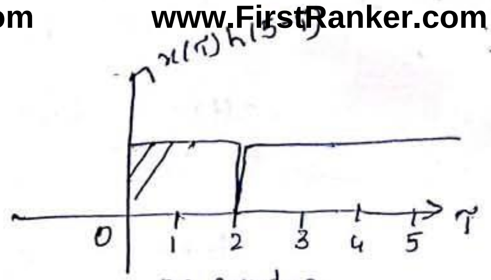


$$y(3) = \int_0^2 d\tau = 2$$

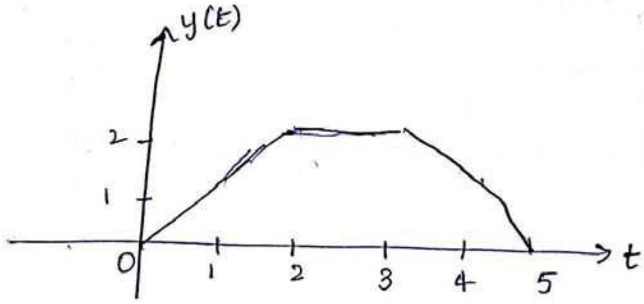


$$y(4) = \int_1^2 d\tau = [2-1] = 1$$



no overlap.

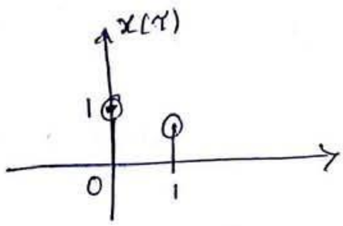
$y(5) = 0$.

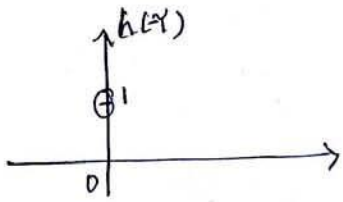


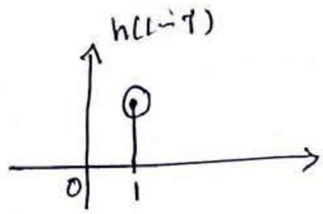
→ (5) Find the linear convolution $x(n) = \{1, 0.5\}$, $h(n) = \{1\}$

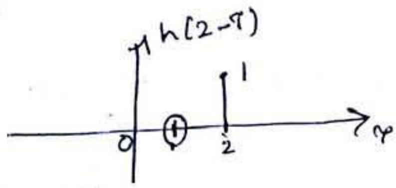
Sol $n_1 = 0$, $n_2 = 0$, so sequence starts at 0

$N_1 = 2$, $N_2 = 1$, $N_1 + N_2 - 1 = 2 + 1 - 1 = 2$





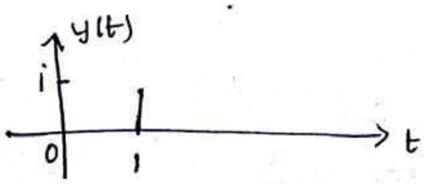




$y(0) = 1$

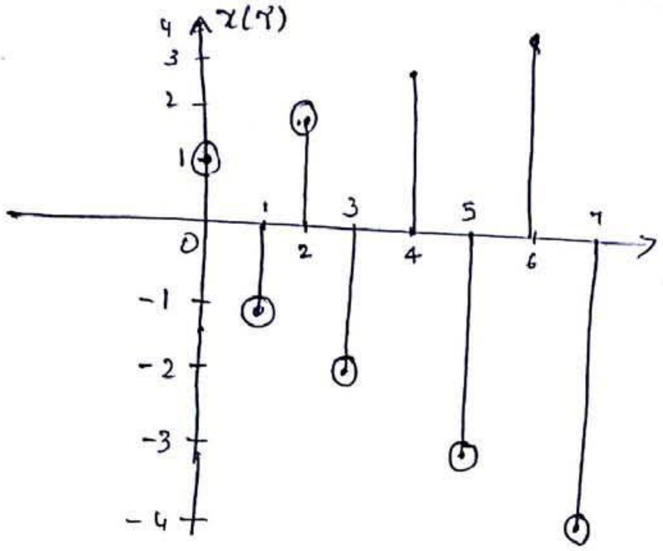
$y(1) = 0 + 0.5 \times 1 = 0.5$

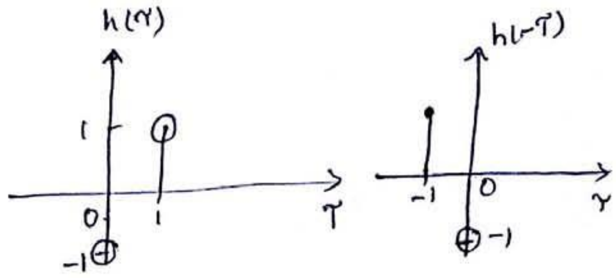
$y(2) = 0$.



6) . Perform the linear convolution of $x(n) = \{1, -1, 2, -2, 3, -3, 4, -4\}$ $h(n) = \{-1, 1\}$

Sol



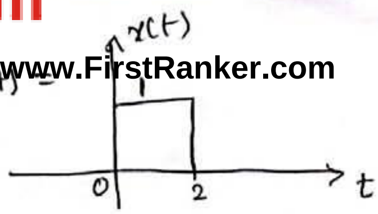


$n_1 = 0$, $n_2 = 0$ O/p sequence starts from 0

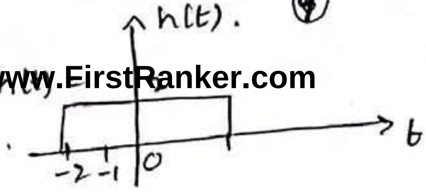
$N_1 = 8$, $N_2 = 2$, $N_1 + N_2 - 1 = 8 + 2 - 1 = 9$

$y(n) = \{-1, 2, -3, 4, -5, 6, -7, 8, 4\}$

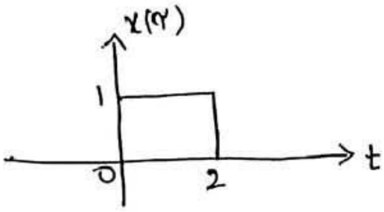
(7) Find the convolution of $x(t)$ =

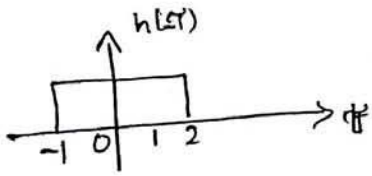


$h(t)$.

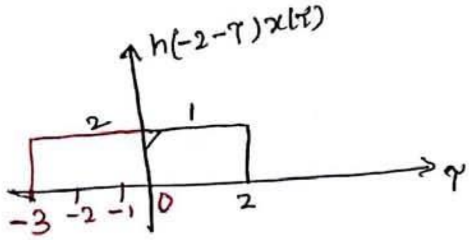


Sol



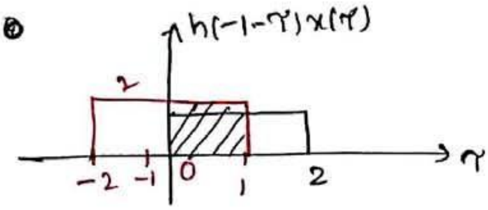


$y(-2) \Rightarrow 0$



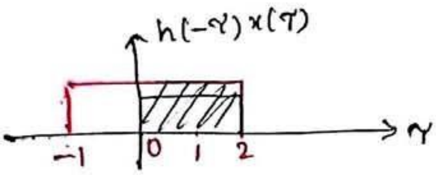
no overlap.

$y(-1) \Rightarrow$



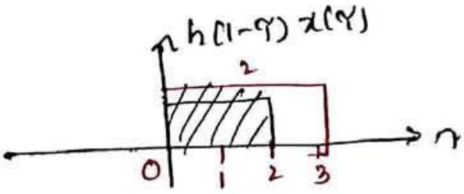
$\Rightarrow \int_0^1 2.1\, dt = 2.$

$y(0)$



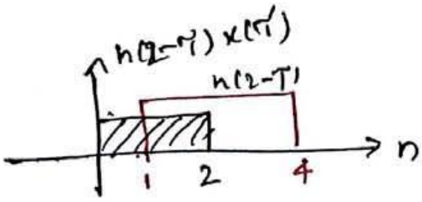
$\Rightarrow \int_0^2 2.1\, dt = 2[2-0] = 4$

$y(1) =$



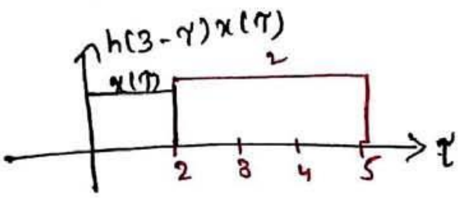
$\Rightarrow \int_0^2 2.1\, dt = 2[2-0] = 4$

$y(2) =$

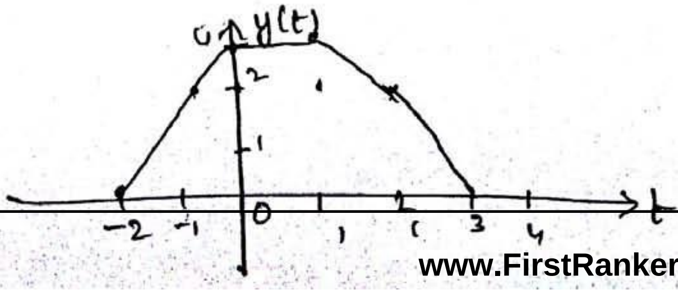


$\Rightarrow \int_1^2 2.1\, dt = 2[2-1] = 2$

$y(3) =$



$= \int$ no overlap $= 0$



PROPERTIES OF CONVOLUTION :

Let us consider two signals $x_1(t)$ and $x_2(t)$. The convolution of two signals is given by eqnation.

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

1. Commutative Property :

Convolution obeys commutative property.

$$x_1(t) * x_2(t) = x_2(t) * x_1(t).$$

Proof

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \rightarrow ①$$

Let $t-\tau = p \Rightarrow \tau = t-p$

$-d\tau = dp$

when $\tau = -\infty$, $p = t+\infty = \infty$

when $\tau = \infty$, $p = t-\infty = -\infty$

Substituting in eq ① we get

$$x_1(t) * x_2(t) = -\int_{+\infty}^{-\infty} x_1(t-p)\, x_2(p)\, dp$$

$$= \int_{-\infty}^{\infty} x_2(p)\, x_1(t-p)\, dp$$

$$= x_2(t) * x_1(t)$$

2. Distributive Property :

$$x_1(t) * [x_2(t) + x_3(t)] = x_1(t) * x_2(t) + x_1(t) * x_3(t).$$

Proof

$$= x_1(t) * [x_2(t) + x_3(t)]$$

{ considering $x_4(t) = x_2(t) + x_3(t)$ }

$$= x_1(t) * x_4(t)$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, x_4(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, [x_2(t-\tau) + x_3(t-\tau)]\, d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau + \int_{-\infty}^{\infty} x_1(\tau)\, x_3(t-\tau)\, d\tau$$

$$= [x_1(t) * x_2(t)] + [x_1(t) * x_3(t)]$$

RHS.

ciative Property

$x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t)$ / $x_1(n) * [x_2(n) * x_3(n)] = [x_1(n) * x_2(n)] * x_3(n)$

Proof

Let $y_1(t) = x_1(t) * x_2(t)$

Let us replace by 'p'

$y_1(p) = x_1(p) * x_2(p)$

$= \sum_{m=-\infty}^{\infty} x_1(m)\, x_2(p-m)$

Let $y_2(t) = x_2(t) * x_3(t)$

$y_2(t) = \sum_{q=-\infty}^{\infty} x_1(q)\, x_2(t-q)$

$y_2(t-m) = \sum_{q=-\infty}^{\infty} x_1(q)\, x_2(t-m-q)$

where p, m, and q are dummy variables.

RHS:

$[x_1(t) * x_2(t)] * x_3(t)$

$= y_1(t) * x_3(t)$

$= \sum_{p=-\infty}^{\infty} y_1(p)$

Proof

Let $y_1(n) = x_1(n) * x_2(n)$

Let us replace n by p

$y_1(p) = x_1(p) * x_2(p)$

$= \sum_{m=-\infty}^{\infty} x_1(m)\, x_2(p-m)$

Let $y_2(n) = x_2(n) * x_3(n)$

$\therefore y_2(n) = \sum_{q=-\infty}^{\infty} x_1(q)\, x_2(n-q)$

$\therefore y_2(n-m) = \sum_{q=-\infty}^{\infty} x_1(q)\, x_2(n-q-m)$

LHS:

$[x_1(n) * x_2(n)] * x_3(n)$

$= y_1(n) * x_3(n)$

$= \sum_{p=-\infty}^{\infty} y_1(p)\, x_3(n-p)$

$= \sum_{p=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x_1(m)\, x_2(p-m)\, x_3(n-p)$

$= \sum_{m=-\infty}^{\infty} x_1(m) \sum_{p=-\infty}^{\infty} x_2(p-m)\, x_3(n-p).$

Let $p-m=q$

$\therefore p = m+q$

when $p=-\infty$, $q = p-m = -\infty - m = -\infty$

when $p=+\infty$, $q = p-m = +\infty - m = +\infty$.

On replacing (p-m) by 'q' and 'p' by q+m'

$= \sum_{m=-\infty}^{\infty} x_1(m) \sum_{q=-\infty}^{\infty} x_2(q)\, x_3(n-q-m)$

$= \sum_{m=-\infty}^{\infty} x_1(m)\, y_2(n-m)$

$= x_1(n) * y_2(n)$

$= x_1(n) * [x_2(n) * x_3(n)]$

$=$ RHS

(4) Shift property :

If $x_1(t) * x_2(t) = z(t)$ then $x_1(t) * x_2(t-T) = z(t-T)$

Proof

$$x_1(t) * x_2(t-T) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-T-\tau)\, d\tau$$

$$= z(t-T)$$

$||^{ly}$ $x_1(t-T) * x_2(t) = z(t-T)$ and $x_1(t-T_1) * x_2(t-T_2) = z(t-T_1-T_2)$.

(5) Convolution with impulse :

Convolution of a signal $x(t)$ with unit impulse is the signal $x(t)$ itself.

$$x(t) * \delta(t) = x(t).$$

Proof

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau \qquad \begin{cases} \delta(t-\tau) = 1 \text{ for } t=\tau \\ \quad = 0 \text{ otherwise} \end{cases}$$

$$= x(t)$$

(6) Convolution with shifted impulse :

Convolution of a signal $x(t)$ with shifted impulse $\delta(t-t_0)$ is equal to $x(t-t_0)$

$$x(t) * \delta(t-t_0) = x(t-t_0)$$

Proof

$$x(t) * \delta(t-t_0) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau-t_0)\, d\tau$$

$$= x(\tau)\Big|_{\tau=t-t_0} = x(t-t_0).$$

(7) Convolution with unit step :

Convolution of a signal $x(t)$ with unit step signal $u(t)$ is given by

$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

Proof

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)\, u(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{t} x(\tau)\, d\tau \qquad \because u(t-\tau) = 1 \text{ for } \tau < t \Rightarrow 0 \text{ for } \tau > t$$

Correlations:

→ Correlation is basically used to compare two signals or it is a measure of the degree to which two signals are similar.

→ Two types

(1) Cross-correlation

(2) Auto-correlation.

Cross Correlation:

The cross correlation between a pair of signals/sequences $x_1(t)$ and $y(t)$ is given by

$$r_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)\, y^*(t-\tau)\,dt \Big/ r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\, y(n-l) \quad \text{where } l = 0, \pm 1, \pm 2, \cdots \rightarrow (1)$$

$$= \int_{-\infty}^{\infty} x(t+\tau)\, y^*(t)\,dt$$

l → shift lag parameter.

→ The subscript xy indicates that $x(n)$ is the reference sequence that remains unshifted in time and $y(n)$ is shifted 'l' units in time w.r.to $x(n)$.

→ If we want to fix $y(n)$ and shift $x(n)$ then

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n)\, x(n-l)$$

$$= \sum_{n=-\infty}^{\infty} y(n+l)\, x(n) \rightarrow (2)$$

→ If time shift $l = 0$ then we get

$$r_{xy}(0) = r_{yx}(0) = \sum_{n=-\infty}^{\infty} x(n)\, y(n).$$

Comparing eq(1) with eq(2) we find that

$$r_{xy}(l) = r_{yx}(-l)$$

↳ folded version of $r_{xy}(l)$ abt $l=0$

We can rewrite the eq(1) as

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\, y[-(l-n)]$$

$$= x(l) * y(-l)$$

Auto Correlation :

It is the correlation of a sequence within itself.

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)x(n-l)$$

(or)

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n+l)\,x(n).$$

If time shift $l=0$, then

$$r_{xx}(0) = \sum_{n=-\infty}^{\infty} x^2(n).$$

Auto Correlation Signal :

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\,x^*(t-\tau)\,dt$$

$$= \int_{-\infty}^{\infty} x(t+\tau)\,x^*(t)\,dt$$

For real values

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\,x(t-\tau)\,dt$$

$$= \int_{-\infty}^{\infty} x(t+\tau)\,x(t)\,dt.$$

Properties of Cross Correlation function for energy signals.

1. For a real valued signals

$$R_{yx}(\tau) = R_{xy}(-\tau)$$

Proof

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t+\tau)\,y(t)\,dt$$

$$R_{yx}(\tau) = \int_{-\infty}^{\infty} y(t+\tau)\,x(t)\,dt = \int_{-\infty}^{\infty} y(t)\,x(t-\tau)\,dt$$

$$= \int_{-\infty}^{\infty} x(t-\tau)\,y(t)\,dt = R_{xy}(-\tau)$$

For complex valued signals $R_{yx}(\tau) = R^*_{xy}(-\tau)$.

(d) If $R_{xy}(0) = 0$ then $\int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt = 0$. Then the sgls are said to be orthogonal over the entire time interval.

Cross correlation of periodic signals:

The cross correlation between two periodic signals $x(t)$ and $y(t)$ is given by

$$R_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, y^*(t-\tau)\, dt \quad \text{or} \quad \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, y^*(t-\tau)\, dt.$$

$\|^{ly}$

$$R_{yx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} y(t)\, x^*(t-\tau)\, dt$$

Properties for periodic signals:

property 1: The fourier transform of cross correlation is equal to multiplication of fourier transform of one signal and complex conjugate of F.T of other signal

$$R_{xy}(\tau) \longleftrightarrow \frac{1}{T_0^2} \sum_{k=-\infty}^{\infty} X_1(kf_0)\, X_2^*(kf_0)\, \delta(f-kf_0).$$

2: If cross correlation is executed at origin $(\tau = 0)$

$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, y^*(t)\, dt = 0$ i.e $R_{xy}(0) = 0$. then sgls are said to be orthogonal

3: Cross correlation exhibits conjugate symmetry

$$R_{xy}(\tau) = R_{yx}^*(-\tau)$$

4. The cross correlation is not commutative

$$R_{xy}(\tau) \neq R_{yx}(\tau)$$

Properties of autocorrelation function: (4)

Property 1: The autocorrelation is an even fn of τ. That is $R_x(\tau) = R_x(-\tau)$

If $x(t)$ is real valued $\rightarrow$ It has even symmetry

Otherwise

$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\,dt$

$R^*_{xx}(\tau) = \int_{-\infty}^{\infty} x^*(t)\, x(t+\tau)\,dt$

$\therefore R^*_{xx}(-\tau) = \int_{-\infty}^{\infty} x^*(t)\, x(t+\tau)\,dt$

$= R_{xx}(\tau)$

$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x(t-\tau)\,dt \rightarrow R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x(t+\tau)\,dt$

$= \int_{-\infty}^{\infty} x(t+\tau)\, x(t)\,dt \Rightarrow$ $t+\tau = x \Rightarrow dt = dx$.

$= \int_{-\infty}^{\infty} f_1(x-\tau)\, f_1(x)\,dx$

For complex $\rightarrow$ It has hermitian symmetry. changing the variable x to t, we

$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\,dt$

$= \int_{-\infty}^{\infty} x(t+\tau)\, x^*(t)\,dt$.

$= \int_{-\infty}^{\infty} f_1(t-\tau)\, f_1(t)\,dt$

$= \phi_{11}(\tau)$

i.e $R_{xx}(-\tau) = R^*_{xx}(\tau)$.

Property 2:

The autocorrelation fn is bounded by its value at the origin That is

$$R_x(0) \geq [R_x(\tau)] \text{ for any } \tau$$

$\rightarrow$ The largest value occurs at $\tau = 0$ of autocorrelation fn.

proof Consider a finite energy signal $x(t)$.

other method:

Consider the fn $x(t)$ & $x(t+\tau)$

$[x(t) + x(t+\tau)]^2 \geq 0$

$x^2(t) + x^2(t+\tau) \pm 2x(t)x(t+\tau) \geq 0$

$x^2(t) + x^2(t+\tau) \geq \pm 2x(t)\,x(t+\tau)$

Integrating on b.s.

$\therefore \int_{-\infty}^{\infty} |x(t)|^2 dt + \int_{-\infty}^{\infty} |x(t+\tau)|^2 dt$

$\geq 2\int_{-\infty}^{\infty} x(t)\, x(t+\tau)\,dt$

$\therefore E + E \geq 2R_{xx}(\tau)$

$E \geq R_{xx}(\tau)$

or $R(0) \geq R_{xx}(\tau)$

$y(t) = \int_{-\infty}^{\infty} |x(t+\tau) - \underset{\text{finite value}}{a}\, x(t)|^2 dt$. Obviously $|y(t)| > 0$

$y(t) = \int_{-\infty}^{\infty} |x(t+\tau) - a\,x(t)|^2 dt$

$= \int_{-\infty}^{\infty} |x(t+\tau)|^2 dt + |a|^2 \int_{-\infty}^{\infty} |x(t)|^2 dt - 2|a| \left| \int_{-\infty}^{\infty} x(t+\tau)\, x(t) \right| dt$

$R_{xx}(0) = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t+\tau)|^2 dt$

$y(t) \geq [1 + |a|^2] R_{xx}(0) - 2|a|\,|R_{xx}(\tau)|$

For $|a| = 1$

$2R_{xx}(0) \geq 2|R_{xx}(\tau)|$ since $|y(t)| > 0$.

$\Rightarrow R_{xx}(0) > R_{xx}(\tau)$

Property

The value of auto correlation fn at $\tau=0$ is equal to energy of the signal

$$E = R_{xx}(0) = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Proof

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau) dt$$

$$R_{xx}(0) = \int_{-\infty}^{\infty} x(t) \cdot x^*(t) dt \Rightarrow \int_{-\infty}^{\infty} |x(t)|^2 dt = E$$

Autocorrelation of power signals:

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x^*(t-\tau) dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t+\tau)\, x^*(t) dt$$

by putting $\tau = 0$

$$R_{xx}(0) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

$$\boxed{R_{xx}(0) = P}$$

$\therefore$ The autocorrelation fn at origin is equal to average power of the signal.

For periodic signals:

(1) If $R_{xx}(\tau)$ and $R^*_{xx}(-\tau)$ are complex conjugate of each other then $R(\tau) = R^*(-\tau)$ called as conjugate property.

Proof

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x^*(t-\tau) dt$$

$$R^*_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^*(t)\, x(t-\tau) dt$$

$$R^*_{xx}(-\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^*(t)\, x(t+\tau) dt = R(\tau)$$

Correlation:

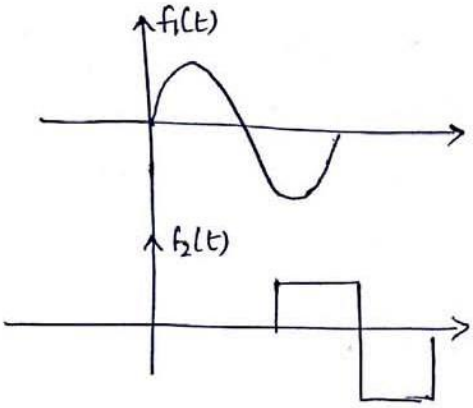
→ correlation means comparison

There are two types of correlations

(1) cross correlation

(2) Auto correlation.

Cross correlation: It is the measure of similarity between one waveform and time delayed version of the another waveform.

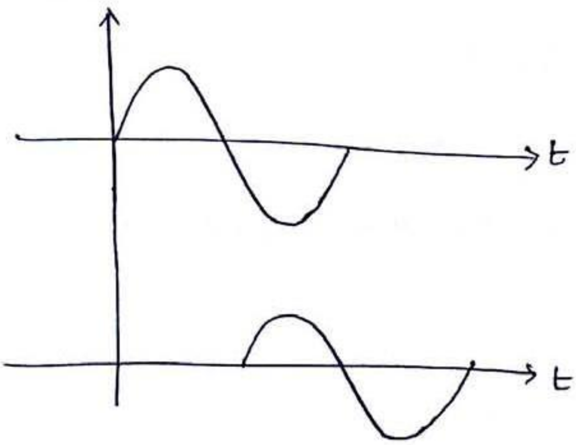


Equation of cross correlation is

$$R_{12}(\tau) = \int_{-\infty}^{\infty} f_1(t)\, f_2^*(t-\tau)\, dt$$

Auto correlation: It is the measure of similarity between one waveform and time delayed version of the same waveform.

Ex:



Equation of auto correlation is

$$R_{11}(\tau) = \int_{-\infty}^{\infty} f_1(t)\, f_1^*(t-\tau)\, dt.$$

APPLICATION OF CORRELATION:

→ It is used in the determination of signal that are contaminated with noise.

CORRELATION FOR ENERGY SIGNALS:

$$E = \int_{-\infty}^{\infty} f^2(t)\,dt \rightarrow \text{cross} \rightarrow \text{two fns}$$

Let $f_1(t)$ & $f_2(t)$ be two energy signals then cross co-relation b/w them is defined as

$$R_{12}(\tau) = \int_{-\infty}^{\infty} f_1(t)\, f_2^*(t-\tau)\,dt$$

$$R_{12}(\tau) = \int_{-\infty}^{\infty} f_1(t+\tau)\, f_2^*(t)\,dt$$

Here τ is delay parameter or scanning parameter or scaling parameter

→ From the above two eqns cross correlation function obtained by shifting $f_2(t)$ in +ve direction by an amount τ is equal to the cross correlation function obtained by shifting $f_1(t)$ in −ve direction by an amount τ.

$$R_{21}(\tau) = \int_{-\infty}^{\infty} f_2(t)\, f_1^*(t-\tau)\,dt$$

$$R_{21}(\tau) = +\int_{-\infty}^{\infty} f_2(t+\tau)\, f_1^*(t)\,dt$$

By comparing with the convolution integral we can define cross correlation function of $f_1(t)$ & $f_2(t)$ as

$$R_{12}(\tau) = f_1(t) \otimes f_2(-t)$$

→ If $f_1(t)$ & $f_2(t)$ are even fns then cross correlation function becomes equal to convolution.

Differences between convolution & correlation

convolution	correlation
→ $f_1(t) \otimes f_2(t) = \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\,d\tau$ τ = dummy variable	→ $R_{12}(\tau) = \int_{-\infty}^{\infty} f_1(t)\, f_2^*(t-\tau)\,dt$ t = dummy variable.
→ convolution is a fn of physical time 't'	→ function of τ
→ It obeys commutation law $f_1(t) \otimes f_2(t) = f_2(t) \otimes f_1(t)$	→ It does not obey commutation Law. $R_{12}(\tau) \neq R_{21}(\tau)$
→ It is used to evaluate the response of the system for arbitary i/p $f_1(t) \otimes f_2(t) = F_1(\omega)\, F_2(\omega)$	→ It is used to eliminate noise from the signals.

PROPERTIES OF CROSS CORRELATION:

Property: The FT of cross correlation function of two signal is equal to product of FT of one signal and complex conjugate F.T of other signal.

$$F[R_{12}(\tau)] = X_1(f)\, X_2^*(f)$$

Proof

$$X_1(f) = \int_{-\infty}^{\infty} x_1(t)\, e^{-j2\pi f t}\, dt \rightarrow (1)$$

$$X_2(f) = \int_{-\infty}^{\infty} x_2(t)\, e^{-j2\pi f t}\, dt \rightarrow (2)$$

put $t = t-\tau$
$dt = -d\tau$

$t = t-\tau$
$t = t+\infty = \infty$
$t = t-\infty = -\infty$

$\frac{dt}{d\tau} = 0 - \frac{d\tau}{d\tau}$
$\frac{dt}{d\tau} = -1$
$\Rightarrow dt = -d\tau$

$$X_2(f) = \int_{+\infty}^{-\infty} x_2(t-\tau)\, e^{-j2\pi f t} \cdot e^{j2\pi f\tau} (-d\tau)$$

$$X_2(f) = \int_{-\infty}^{\infty} x_2(t-\tau) e^{-j2\pi f t} \cdot e^{j2\pi f \tau}\, d\tau$$

Applying conjugate on both sides

$$X_2^*(f) = \int_{-\infty}^{\infty} x_2^*(t-\tau)\, e^{j2\pi f t} \cdot e^{-j2\pi f\tau}\, d\tau \rightarrow (3)$$

Now multiplying the eq(1) & (3) we get

$$X_1(f)\, X_2^*(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t-\tau)\, e^{-j2\pi f\tau}\, dt\, d\tau$$

$$= \int_{-\infty}^{\infty} R_{12}(\tau)\, e^{-j2\pi f\tau}\, d\tau \qquad \left[\because R_{12} = \int_{-\infty}^{\infty} x_1(t)\, x_2^*(t-\tau)\, dt\right]$$

$$\boxed{X_1(f)\, X_2^*(f) = F[R_{12}(\tau)]}$$

(2) The cross correlation fn is zero at origin i.e at $\tau = 0$,

$$R_{12}(\tau) = R_{12}(0) = 0$$

proof: Consider $R_{12}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x_1(t)\, x_2^*(t-\tau)\, dt$

$$R_{12}(0) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x_1(t)\, x_2^*(t)\, dt$$

$= 0 \quad (\because x_1(t)\ \&\ x_2(t)$ are periodic)

3) The cross correlation fn satisfies conjugate symmetry property i.e $R_{12}(\tau) = R_{21}^{*}(-\tau)$

proof

$$R_{12}(\tau) = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x_1(t)\, x_2^{*}(t-\tau)\, dt \rightarrow (1)$$

$$R_{21}(\tau) = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x_2(t)\, x_1^{*}(t-\tau)\, dt \rightarrow (2)$$

Putting $\tau = -\tau$

$$R_{21}(-\tau) = \left\{ \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x_2(t)\, x_1^{*}(t+\tau)\, dt \right\}$$

Applying conjugate on both sides, we get

$$R_{21}^{*}(-\tau) = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x_2^{*}(t)\, x_1(t+\tau)\, dt$$

{Now ~~put~~ substitute $t = t-\tau$, $dt = dt$ }

$$= \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x_2^{*}(t-\tau)\, x_1(t)\, dt$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x_1(t)\, x_2^{*}(t-\tau)\, dt$$

$$= R_{12}(\tau)$$

$$\boxed{\therefore\ R_{21}^{*}(-\tau) = R_{12}(\tau)}$$

(4) The cross correlation fn does not satisfy commutative property i.e $R_{12}(\tau) \neq R_{21}(\tau)$

proof

$$R_{12}(\tau) = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x_1(t)\, x_2^{*}(t-\tau)\, dt$$

$$R_{21}(\tau) = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x_2(t)\, x_1^{*}(t-\tau)\, dt$$

As above eqn are not same

$$R_{12}(\tau) \neq R_{21}(\tau)$$

(5) The cross correlation for energy signals is given by

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2^{*}(t-\tau)\, dt$$

(or)

$$R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(t)\, x_1^{*}(t-\tau)\, dt$$

(8) ~~Consider two energy signals $g_1(t)$ & $g_2(t)$, which may be complex valued. Show that $\int_{-\infty}^{\infty} |g_1(t) - g_2(t)|^2 dt = R_{11}(0) + R_{22}(0) - 2\,Re[R_{12}(0)]$ where $R_{11}(\tau)$ and $R_{22}(\tau)$ are autocorrelation fn of $g_1(t)$ & $g_2(t)$ & $R_{12}(\tau)$ is their cross~~

PROBLEMS ON AUTOCORRELATION AND CROSS CORRELATION

(1) Determine the cross correlation fn $R_{xy}(\lambda) \to R_{12}(\lambda)$ of two signal $g_1(t)$ and $g_2(t)$ defined by

$$g_1(t) = \begin{cases} A\cos(2\pi f_1 t + \theta_1) & 0 \leq t \leq T \\ 0 & \text{elsewhere} \end{cases} \qquad g_2(t) = \begin{cases} A\cos(2\pi f_2 t + \theta_2) & 0 \leq t < T \\ 0 & \text{elsewhere} \end{cases}$$

How does varying the frequency difference $|f_1 - f_2|$ affect this cross correlation fn.

Sol Cross correlation is given as

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2^*(t-\tau)\, dt$$

or

$$R_{12}(\tau) = \int_{-\infty}^{\infty} g_1(t)\, g_2^*(t-\lambda)\, dt$$

$$= \int_0^T A\cos(2\pi f_1 t + \theta_1) \cdot A\cos(2\pi f_2(t-\lambda) + \theta_2)\, dt$$

$$= \int_0^T A\cos(2\pi f_1 t + \theta_1) \cdot A\cos(2\pi f_2 t - 2\pi f_2 \lambda + \theta_2)\, dt$$

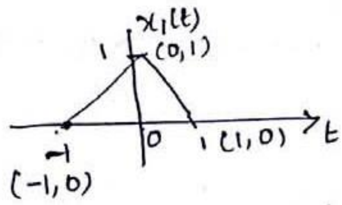
$\{\because \cos(A-B) + \cos(A+B)\}$

$$= \frac{A^2}{2}\int_0^T \{\cos(2\pi f_1 t + \theta_1 - 2\pi f_2 t - \theta_2 + 2\pi f_2 \lambda) + \cos(2\pi f_1 t + \theta_1 + 2\pi f_2 t - 2\pi f_2 \lambda + \theta_2)\}\, dt$$

$$= \frac{A^2}{2}\left\{ \left[\frac{\sin(2\pi f_1 t + \theta_1 - 2\pi f_2 t - \theta_2 + 2\pi f_2 \lambda)}{2\pi(f_1 - f_2)}\right]_0^T + \left[\frac{\sin[2\pi(f_1+f_2)t + \theta_1 + \theta_2 - 2\pi f_2 \lambda]}{2\pi(f_1+f_2)}\right]_0^T \right\}$$

$$= \frac{A^2}{2}\left\{ \frac{\sin(2\pi(f_1-f_2)T + \theta_1 - \theta_2 + 2\pi f_2 \lambda) - \sin[\theta_1 - \theta_2 + 2\pi f_2 \lambda]}{2\pi(f_1-f_2)} + \frac{\sin[(2\pi(f_1+f_2)T + \theta_1 + \theta_2 - 2\pi f_2 \lambda) - \sin[\theta_1 + \theta_2 - 2\pi f_2 \lambda]}{2\pi(f_1+f_2)} \right\}$$

the cross correlation b/w triangular and gate pulse as shown in fig.



Sol

$x_1(t)$ $(x_1, y_1) = (-1,0), (x_2, y_2) = (0,1)$

$y = x_1(t),\ x = t$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$x_1(t) = t + 1$

$$x_1(t) = \begin{cases} 1+t & \text{for } -1 \le t \le 0 \\ 1-t & \text{for } 0 \le t \le 1 \end{cases}$$

$x_2(t) = 1$ for $-1 \le t \le 1$

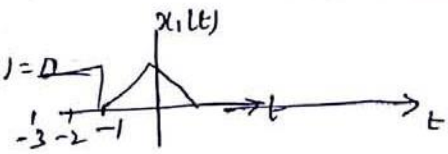
$x_2(t-\tau) = 1$ for $-1 \le t-\tau \le 1$

i.e $\tau - 1 \le t \le \tau + 1$

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2(t-\tau)\, dt.$$

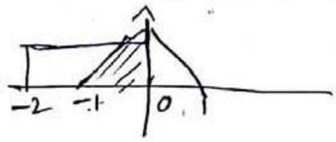
Case i; $\tau < -2$

There is no overlap b/w $x_1(t)$ and $x_2(t)$ $\therefore$ $R_{12}(\tau) = 0$



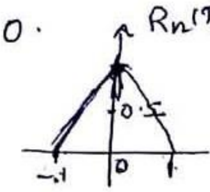
case (ii) $-2 \le \tau \le -1$

$$R_{12}(\tau) = \int_{-1}^{0} (t+1)\, dt$$



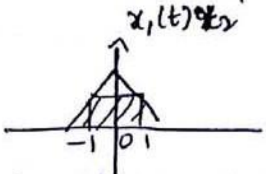
$= \cancel{\left[\frac{t^2}{2}\right]_{-1}^{0} + \left[t\right]_{-1}^{0}}$ $\quad \frac{bh}{2} \Rightarrow \frac{1}{2} \times 1 \times 1 = 0.5$ between -1 to 0.

$= \frac{1}{2} + 1 =$



case (iii)

$R_{12}(\tau) = \frac{1}{2} \times 1 \times 1 +$

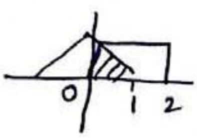


between -1 to 1

$\frac{1}{2} \times 1 \times 1 = 1$

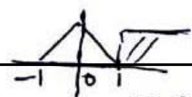
Case (iv) $= \frac{1}{2} \times 1 \times 1$

$= 0.5$



between 0 to 1

Case (v)



$= 0.$

Energy Density Spectrum:

Spectral density: It is the distribution of power or energy of a signal per unit bandwidth as a function of frequency.

Energy and power Signals:

→ Signals with finite energy i.e $0<E<\infty$ and $P=0$ are called energy signals.
  e.g: aperiodic signals like pulse

→ Signals with finite average power i.e $0<P<\infty$ and $E=\infty$ are called power sgls i.e periodic signals.

The energy of a signal $x(t)$ is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt.$$

Parseval's Theorem for energy signal:/ Rayleigh energy theorem:

It defines the energy of a signal in terms of fourier transform.

$$\text{i.e } E = \int_{-\infty}^{\infty} |X(f)|^2 df.$$

Proof: $x(t) \longleftrightarrow X(f)$. Let $x^*(t)$ be conjugate of $x(t)$ such that

$$x^*(t) \longleftrightarrow X^*(-f)$$

Energy of the signal $x(t)$ is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt = \int_{-\infty}^{\infty} x^*(t)\, x(t)\, dt.$$

Replacing $x(t)$ in terms of inverse fourier tranform $X(f)$

$$E = \int_{-\infty}^{\infty} x^*(t) \left\{ \int_{-\infty}^{\infty} X(f)\, e^{j\omega t}\, df \right\} dt$$

Interchanging the order of integration.

$$E = \int_{-\infty}^{\infty} X(f) \left\{ \int_{-\infty}^{\infty} x^*(t)\, e^{j\omega t}\, dt \right\} df$$

$$E = \int_{-\infty}^{\infty} X(f)\, X^*(f)\, df \Rightarrow \int_{-\infty}^{\infty} |X(f)|^2 df.$$

Energy Spectral density: It is the distribution of energy of signal in frequency domain. which is also called as energy density spectrum (ESD or ED) given by

$$\psi(f) = |X(f)|^2 \rightarrow ①$$

Let $x(t)$ and $y(t)$ be the input and output of a linear system. i.e $x(t) \leftrightarrow X(f)$ and $y(t) \leftrightarrow Y(f)$ and $H(f)$ be system transfer function.

$$Y(f) = H(f)X(f) \longrightarrow ②$$

Using Eq (1) we can write as

$$\psi_x(f) = |X(f)|^2$$

$$\psi_y(f) = |Y(f)|^2.$$

$$\psi_y(f) = |Y(f)|^2 = |H(f)|^2 |X(f)|^2 = |H(f)|^2 \psi_x(f).$$

$$\boxed{\psi_y(f) = |H(f)|^2 \psi_x(f)}$$

ESD of the output is the product of ESD of input and square of the magnitude of transfer function.

Power Density Spectrum:

Average power is defined as

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt.$$

But power P is defined as

$$P = \overline{x^2(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt.$$

Parseval's Power Theorem: It defines the power of a signals in terms of its fourier series coefficients.

$$P = \sum_{n=-\infty}^{\infty} |F_n|^2$$

Proof: Consider a function $x(t)$

$$|x(t)|^2 = x(t)\, \underbrace{x^*(t)}_{\text{conjugate of } x(t)} \rightarrow (1)$$

Average power of $x(t)$ for one cycle is

$$P = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, x^*(t)\, dt \rightarrow (2)$$

we have exponential Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \rightarrow (3)$$

Replacing $x(t)$ of eq (2) by eq (3) we get

$$P = \frac{1}{T}\int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} x^*(t)\, dt$$

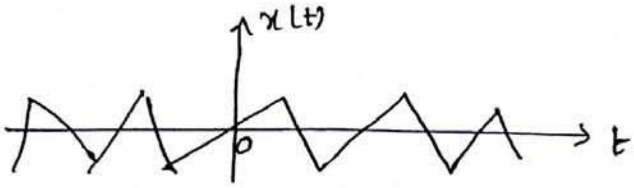
$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} F_n \int_{-T/2}^{T/2} x^*(t)\, e^{jn\omega_0 t} dt = \sum_{n=-\infty}^{\infty} F_n \cdot \frac{1}{T}\int_{-T/2}^{T/2} x^*(t) e^{jn\omega_0 t} dt$$

$$= \sum_{n=-\infty}^{\infty} F_n \cdot F_n^* = \sum_{n=-\infty}^{\infty} |F_n|^2 = P.$$

↳ parsevals power theorem.

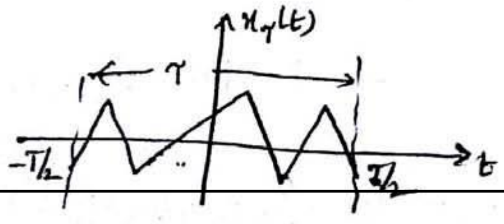
Power Spectral Density (PSD):

The distribution of average power of the signal in frequency domain is called power spectral density or power density spectrum (PSD or PD)



Let us assume that signal is zero outside the interval $|T/2)$.

$$x_T(t) = \begin{cases} x(t) & |t| < T/2 \\ 0 & \text{elsewhere} \end{cases}$$



The signal $x_T(t)$ is of finite duration T and hence it energy signal with energy E given by

$$E = \int_{-\infty}^{\infty} |x_T(t)|^2 dt = \int_{-\infty}^{\infty} |X_T(f)|^2 df$$

where $x_T(t) \longleftrightarrow X_T(f)$.

As $x(t)$ over interval $(-T/2, T/2)$ is same as $x_T(t)$ over the interval $(-\infty, \infty)$

$$\int_{-\infty}^{\infty} |x_T(t)|^2 dt = \int_{-T/2}^{T/2} |x(t)|^2 dt$$

$$\therefore \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 = \frac{1}{T}\int_{-\infty}^{\infty} |X_T(f)|^2 df \rightarrow (1)$$

If $T \to \infty$, the left hand side of eq (1) represents average power P

$$P = \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{|X_T(f)|^2 df}{T}$$

If $T \to \infty$, $\frac{|X_T(f)|^2}{T}$ approaches finite value denoted by $S(f)$ or $S(\omega)$

$$S(f) = \lim_{T\to\infty} \frac{|X_T(f)|^2}{T}$$

Average power

$$P = \overline{x^2(t)} = \int_{-\infty}^{\infty} S(f)\, df = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, d\omega .$$

→ The PSD of periodic function is given by

$$S(f) = \sum_{n=-\infty}^{\infty} |F_n|^2 \delta(f - nf_0)$$

$$\text{|||}^{ly} \quad S(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |F_n|^2 \delta(\omega - n\omega_0).$$

I/p and o/p relation of linear system in terms of PSD is given by

$$S_y(f) = |H(f)|^2 \cdot S_x(f).$$

Comparision of ESD & PSD

ESD	PSD
(1) It gives the distribution of energy of a signal in frequency	(1) It gives the distribution of power of signal in frequency domain
(2) It is given by $\psi(f) = \lvert X(f)\rvert^2$	(2) It is given by $S(f) = \lim_{T\to\infty} \frac{\lvert X_T(f)\rvert^2}{T}$
(3) Total energy is given by $E = \int_{-\infty}^{\infty} \psi(f)\,df$	(3) Total power is given by $P = \int_{-\infty}^{\infty} S(f)\,df$
(4) The auto correlation for an energy signal and its ESD form a Fourier transform pair $R(\tau) \longleftrightarrow \psi(f)$	(4) The auto correlation for a power signal and its PSD form a Fourier transform pair. $R(\tau) \longleftrightarrow S(f)$

RELATION BETWEEN AUTOCORRELATION AND SPECTRAL DENSITIES

1. The autocorrelation function $R(\tau)$ of an energy signal and its energy spectral density (ESD), $\psi(f)$ forms a fourier transform pair,

$$R(\tau) \longleftrightarrow \psi(f)$$

proof: Cross correlation of two energy signals $x(t)$ and $y(t)$ is given as

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} X(f)\, Y^*(f)\, e^{j\omega\tau}\, df.$$

If both functions are same, then auto correlation is given by

$$R(\tau) = \int_{-\infty}^{\infty} X(f)\, X^*(f)\, e^{j\omega\tau} df = \int_{-\infty}^{\infty} \lvert X(f)\rvert^2 e^{j\omega\tau} df$$

$$= F^{-1}\left[\lvert F(f)\rvert^2\right]$$

But $\lvert F(f)\rvert^2 = \psi(f)$

$$\therefore\ R(\tau) = F^{-1}[\psi(f)] \Rightarrow F[R(\tau)] = \psi(f)$$

$$R(\tau) \longleftrightarrow \psi(f)$$

The autocorrelation function $R(\tau)$ and power spectral density (PSD), $S(f)$ of a power signal form a fourier transform pair.

$$R(\tau) \longleftrightarrow S(f)$$

Proof

Autocorrelation function of power $x(t)$ in terms of fourier series coefficients is given as

$$R(\tau) = \sum_{n=-\infty}^{\infty} \underbrace{x_n x_{-n}}_{\text{exponential fourier series coefficients}} e^{jn\omega_0\tau}$$

$$R(\tau) = \sum_{n=-\infty}^{\infty} |x_n|^2 e^{jn\omega_0\tau}$$

Taking Fourier transform.

$$F[R(\tau)] = \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} |x_n|^2 e^{jn\omega_0\tau} \right] e^{-j\omega\tau} d\tau$$

Interchanging the order of integration & summation, we get

$$F[R(\tau)] = \sum_{n=-\infty}^{\infty} |x_n|^2 \int_{-\infty}^{\infty} e^{-j\tau(\omega - n\omega_0)} d\tau$$

$$= 2\pi \sum_{n=-\infty}^{\infty} |x_n|^2 \cdot \delta(\omega - n\omega_0) = \sum_{n=-\infty}^{\infty} |x_n|^2 \delta(f - nf_0).$$

The RHS is the PSD $S(\omega)$ or $S(f)$ of periodic function $x(t)$

$$\therefore F[R(\tau)] = S(f)$$

$$R(\tau) = F^{-1}[S(f)]$$

$$\therefore R(\tau) \longleftrightarrow S(f).$$

RELATION BETWEEN CONVOLUTION AND CORRELATION:

(1) In correlation, physical time 't' plays the role of dummy variable & it appears after solving the integral but in convolution delay parameter τ plays the role of dummy variable.

2) Correlation $R_{xy}(\tau)$ is a function of delay parameter τ, whereas convolution is a function of time.

3) Correlation can be obtained by convolving $x(t)$ & $y^*(t)$

4) Convolution does not depend on which function is being shifted whereas correlation does i.e convolution is commutative.

(1) The convolution and correlation are identical for even signals.

Proof For two signals $x_1(t)$ and $x_2(t)$

The definition for correlation is given by

$$R_{12} = \int_{-\infty}^{\infty} x_1(t)\, x_2(t-\tau)\, dt \longrightarrow (1)$$

$$R_{21} = \int_{-\infty}^{\infty} x_2(t)\, x_1(t-\tau)\, dt \longrightarrow (2)$$

The definition for convolution

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \longrightarrow (3)$$

$$= \int_{-\infty}^{\infty} x_2(\tau)\, x_1(t-\tau)\, d\tau \longrightarrow (4)$$

Taking eq (1) & eq (3)

$$R_{12} = \int_{-\infty}^{\infty} x_1(t)\, x_2(t-\tau)\, dt$$

Replacing dummy variable 't' by 'p' we get

$$= \int_{-\infty}^{\infty} x_1(p)\, x_2(p-\tau)\, dp$$

Since it is a even signal i.e $x(t) = x(-t)$

$$= \int_{-\infty}^{\infty} x_1(p)\, x_2(-(\tau-p))\, dp = S$$

$$S = \int_{-\infty}^{\infty} x_1(p)\, x_2(\tau-p)\, dp$$

Replace dummy variable τ by t

$$S = \int_{-\infty}^{\infty} x_1(p)\, x_2(t-p)\, dp$$

Again replacing the variable p by τ

$$S = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$= x_1(t) * x_2(t)$.

(4) ... of autocorrelation property.

Fourier transform of autocorrelation yields π times the energy density spectrum of $f_1(t)$ $\quad |F(\omega)|^2 = \int_{-\infty}^{\infty} R_{11}(T)\, e^{-j\omega t}\, dT = \pi S_1(\omega)$

proof

From definition of FT, FT of $R_{11}(T)$ is

$$\int_{-\infty}^{\infty} R_{11}(T) e^{-j\omega T} dT = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(t)\cdot f_1(t-T)\, e^{-j\omega T}\, dt\, dT$$

$$= \int_{-\infty}^{\infty} f_1(t) e^{-j\omega t} dt \int_{-\infty}^{\infty} f_1(t-T)\, e^{j\omega(t-T)}\, dT$$

putting $t-T=-x \Rightarrow dT = dx$

$$= F_1(\omega)\int_{-\infty}^{\infty} f_1(-x)\, e^{-j\omega x}\, dx$$

$$= F_1(\omega)\, F_1(-\omega)$$

$$= |F_1(\omega)|^2 = \pi S_1(\omega).$$

(5) Fourier transform of R_{12} (cross correlation fn) is $F_1(\omega)F_2(-\omega) \longleftrightarrow F_1(\omega)F_2(-\omega)$

proof Fourier transform of $f_1(t)$ & $f_2(t)$ are

$f_1(t) \Leftrightarrow F_1(\omega)$, $f_2(t) \longleftrightarrow F_2(\omega)$ $\quad ||^{ly}$ $f_1(-t) \longleftrightarrow F_1(-\omega)$ & $f_2(-t) \longleftrightarrow F_2(-\omega)$

$$R_{12}(T) = f_1(t) * f_2(-t).$$

$\therefore$ Fourier transform of $f_1(t) * f_2(-t)$ is $F_1(\omega)F_2(-\omega)$.

$$\therefore\ R_{12}(T) = f_1(t) * f_2(-t) \longleftrightarrow F_1(\omega)\cdot F_2(-\omega).$$

(6) Graphically $R_{12}(T)$ is same as $R_{21}(T)$ where it is folded back about the vertical axis at $T=0$. $R_{12}(T) = R_{21}(-T)$.

proof

$$R_{12}(T) = \int_{-\infty}^{\infty} f_1(t)\cdot f_2(t-T)\, dt = \int_{-\infty}^{\infty} f_1(t+T)\, f_2(t)\, dt$$

$$R_{21}(T) = \int_{-\infty}^{\infty} f_2(t)\cdot f_1(t-T)\, dt$$

$$R_{21}(-T) = \int_{-\infty}^{\infty} f_2(t)\cdot f_1(t-(-T))\, dt = \int_{-\infty}^{\infty} f_2(t)\, f_1(t+T)\, dt = R_{12}(T)$$

ESD(Energy Spectral Density):

→ Gives the distribution of energy of the signal in the frequency domain.

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

we know

$\omega = 2\pi f$

$d\omega = 2\pi df$

$$\therefore E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(f)|^2 \cdot 2\pi df = \int_{-\infty}^{\infty} |X(f)|^2 df \quad \to (1)$$

coz $|X(2\pi f)|$ is written as $X(f)$.

where $|X(f)| \to$ amplitude Spectrum.

If we denote $|X(f)|^2$ by $\psi(f)$

$$\therefore \text{ESD}: \psi(f) = |X(f)|^2 \to (2)$$

putting (2) in eq (1)

$$E = \int_{-\infty}^{\infty} \psi(f) df.$$

total energy of the sgl. → total area under the curve $\psi(f)$.

→ $\psi(f)$ represents → Energy spectral density of sgl $x(t)$ in joules per hertz.

Effect of Systems on ESD

Let the ESD of $x(t)$ be $\psi_x(f)$ and $y(t)$ be $\psi_y(f)$. The signal $x(t)$ is applied at the input of LTI system and $y(t)$ is obtained at the output.

→ Let LTI system to be an ideal filter which has pass band from f_L to f_H. i.e only signal will be passed without any effect from f_L to f_H.

We know that $E = \int_{-\infty}^{\infty} \psi(f)\, df$

Energy at the output will be

$$E_y = \int_{-\infty}^{\infty} \psi_y(f)\, df.$$

→ If $\psi_y(f)$ is symmetric for positive & negative values of 'f', then

$$E_y = 2\int_{0}^{\infty} \psi_y(f)\, df.$$

$$E_y = 2\int_{f_L}^{f_H} \psi_y(f)\, df$$

$$= 2\int_{f_L}^{f_H} |Y(f)|^2 df.$$

We know $Y(\omega) = H(\omega)X(\omega)$.

$$E_y = 2\int_{f_L}^{f_H} |H(f)X(f)|^2 df$$

$$= 2\int_{f_L}^{f_H} |H(f)|^2 |X(f)|^2 df$$

$$= 2\int_{f_L}^{f_H} |H(f)|^2 \psi_x(f)\, df$$

→ The filter passes all the frequencies b/w f_L & f_H. i.e $H(f) = 1$ for $f_L \le f \le f_H$.

$$E_y = 2\int_{f_L}^{f_H} \psi_x(f)\, df$$

energy of the o/p sgl. in terms of ESD of i/p sgl.

RELATION BETWEEN CORRELATION AND CONVOLUTION:

The convolution of $f_1(t)$ and $f_2(-t)$ by $\rho_{12}(t)$, we have

$$\rho_{12}(t) = f_1(t) * f_2(-t)$$
$$= \int_{-\infty}^{\infty} f_1(\tau)\, f_2(\tau - t)\, d\tau$$

The dummy variable τ in the above integral may be replaced by another variable x.

$$\rho_{12}(t) = \int_{-\infty}^{\infty} f_1(x)\, f_2(x-t)\, dx.$$

Changing the variable from t to τ, we get

$$\rho_{12}(\tau) = \int_{-\infty}^{\infty} f_1(x)\, f_2(x-\tau)\, dx$$
$$= \phi_{12}(\tau)$$

Hence $\phi_{12}(\tau) = f_1(t) * f_2(-t)\big|_{t=\tau} = \rho_{12}(\tau)$

$\|^{ly}$ $\phi_{21}(\tau) = f_1(-t) * f_2(t)\big|_{t=\tau} = \rho_{21}(\tau)$.

and $\phi_{11}(\tau) = f_1(t) * f_1(-t)\big|_{t=\tau} = \rho_{11}(\tau)$

DETECTION OF PERIODIC SIGNALS IN THE PRESENCE OF NOISE BY CORRELATION:

→ Now we consider that periodic signals are affected by noise that finds the applications in the detection of radar and sonar signals, periodic component in brain waves and cyclical component in ocean wave analysis.

→ If $s(t)$ is periodic signal and $n(t)$ represents the noise signal, then

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} s(t)\, n(t-\tau)\, dt = 0 \quad \text{for all } \tau.$$

$\phi_{sn}(\tau) = 0$
↙
crosscorrelation fn

Detection by Autocorrelation:

Let s(t) be a periodic signal mixed with noise signal n(t). Then the received sgl f(t) is $[s(t)+n(t)]$.

→ Let $\bar{\phi}_{ff}(\tau)$, $\bar{\phi}_{ss}(\tau)$, $\bar{\phi}_{nn}(\tau)$ denote the autocorrelation functions of f(t), s(t) and n(t) respectively.

$$\bar{\phi}_{ff}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, f(t-\tau)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} [s(t)+n(t)][s(t-\tau)+n(t-\tau)]\, dt$$

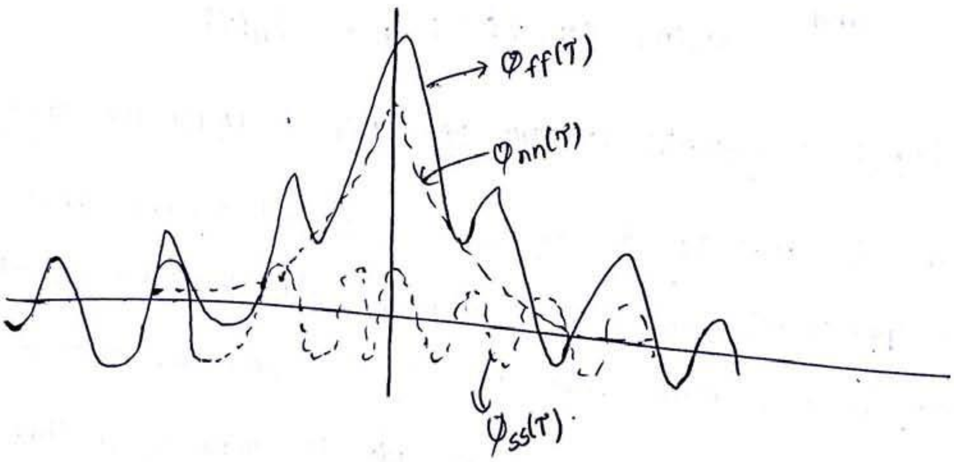
$$= \bar{\phi}_{ss}(\tau) + \bar{\phi}_{nn}(\tau) + \bar{\phi}_{sn}(\tau) + \bar{\phi}_{ns}(\tau)$$

(∵ s(t) & n(t) are uncorrelated)

$$\bar{\phi}_{sn}(\tau) = \bar{\phi}_{ns}(\tau) = 0$$

$$\therefore \quad \bar{\phi}_{ff}(\tau) = \bar{\phi}_{ss}(\tau) + \bar{\phi}_{nn}(\tau).$$

exhibit a periodic nature at larger values of τ.



→ It follows that f(t) contains a periodic signal of frequency displayed by $\bar{\phi}_{ff}(\tau)$.

→ If $\bar{\phi}_{ff}(\tau)$ does exhibit such a periodic nature it is possible to separate $\bar{\phi}_{ss}(\tau)$ and $\bar{\phi}_{nn}(\tau)$ {non periodic component}.

EXTRACTION OF SIGNAL FROM NOISE BY FILTERING:

→ A signal masked by noise can be detected either by correlation techniques or filtering.

→ Correlation technique in time domain and filtering in frequency domain.

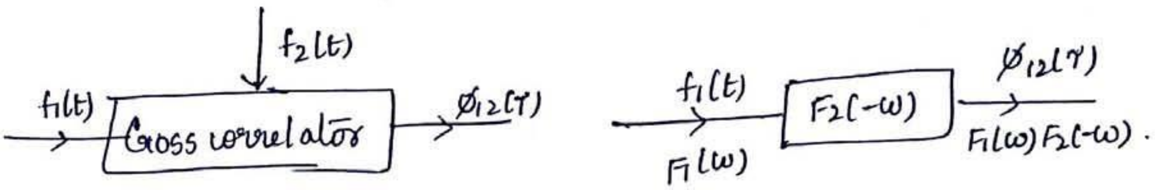


Fig: Crosscorrelation in time & frequency domain.

The impulse response $h(t)$ of a system with a transfer function $F_2(-\omega)$ is given by

$$h(t) = F^{-1}[F_2(-\omega)]$$

But $f_2(t) \longleftrightarrow F_2(\omega)$

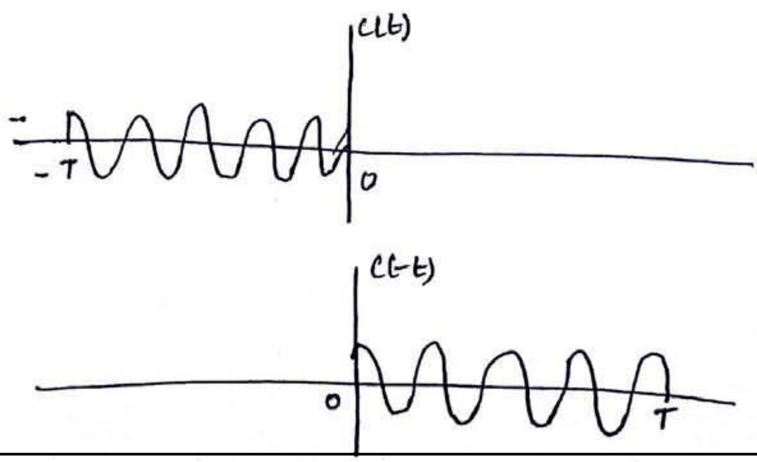
and $f_2(-t) \longleftrightarrow F_2(-\omega)$

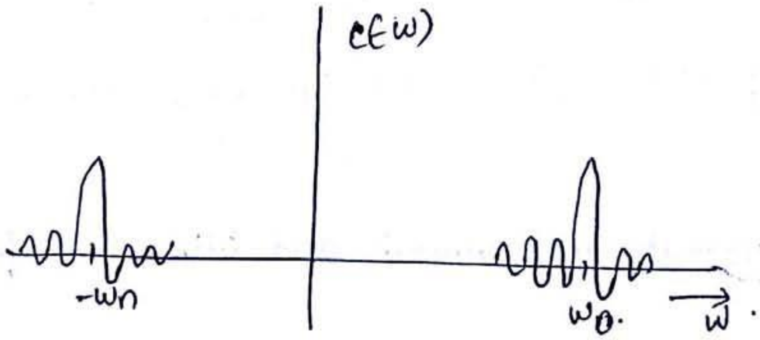
∴ hence $h(t) = f_2(-t)$.

→ The received signal $f(t)$ is

$$f(t) = s(t) + n(t)$$

→ We are filtering out all of the noise signal and extracting the desired periodic signal $s(t)$ by a filter which allows only the frequency components present in $s(t)$ to pass through.

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Topic 12 Notes

Jeremy Orloff

12 Laplace transform

12.1 Introduction

The Laplace transform takes a function of time and transforms it to a function of a complex variable s. Because the transform is invertible, no information is lost and it is reasonable to think of a function $f(t)$ and its Laplace transform $F(s)$ as two views of the same phenomenon. Each view has its uses and some features of the phenomenon are easier to understand in one view or the other.

We can use the Laplace transform to transform a linear time invariant system from the time domain to the s-domain. This leads to the system function $G(s)$ for the system –this is the same system function used in the Nyquist criterion for stability.

One important feature of the Laplace transform is that it can transform analytic problems to algebraic problems. We will see examples of this for differential equations.

12.2 A brief introduction to linear time invariant systems

Let's start by defining our terms.

Signal. A signal is any function of time.

System. A system is some machine or procedure that takes one signal as input does something with it and produces another signal as output.

Linear system. A linear system is one that acts linearly on inputs. That is, $f_1(t)$ and $f_2(t)$ are inputs to the system with outputs $y_1(t)$ and $y_2(t)$ respectively, then the input $f_1 + f_2$ produces the output $y_1 + y_2$ and, for any constant c, the input cf_1 produces output cy_1.

This is often phrased in one sentence as input $c_1f_1 + c_2f_2$ produces output $c_1y_1 + c_2y_2$, i.e. linear combinations of inputs produces a linear combination of the corresponding outputs.

Time invariance. Suppose a system takes input signal $f(t)$ and produces output signal $y(t)$. The system is called time invariant if the input signal $g(t) = f(t-a)$ produces output signal $y(t-a)$.

LTI. We will call a linear time invariant system an LTI system.

Example 12.1. Consider the constant coefficient differential equation

$$3y'' + 8y' + 7y = f(t)$$

This equation models a damped harmonic oscillator, say a mass on a spring with a damper, where $f(t)$ is the force on the mass and $y(t)$ is its displacement from equilibrium. If we consider f to be the input and y the output, then this is a linear time invariant (LTI) system.

Example 12.2. There are many variations on this theme. For example, we might have the LTI system

$$3y'' + 8y' + 7y = f'(t),$$

where we call $f(t)$ the input signal and $y(t)$ the output signal.

12.3 Laplace transform

Definition. The Laplace transform of a function $f(t)$ is defined by the integral

$$\mathcal{L}(f;s) = \int_0^\infty e^{-st} f(t)\,dt,$$

for those s where the integral converges. Here s is allowed to take complex values.

Important note. The Laplace transform is only concerned with $f(t)$ for $t \geq 0$. Generally, speaking we can require $f(t) = 0$ for $t < 0$.

Standard notation. Where the notation is clear, we will use an upper case letter to indicate the Laplace transform, e.g, $\mathcal{L}(f;s) = F(s)$.

The Laplace transform we defined is sometimes called the one-sided Laplace transform. There is a two-sided version where the integral goes from $-\infty$ to ∞.

12.3.1 First examples

Let's compute a few examples. We will also put these results in the Laplace transform table at the end of these notes.

Example 12.3. Let $f(t) = e^{at}$. Compute $F(s) = \mathcal{L}(f;s)$ directly. Give the region in the complex s-plane where the integral converges.

$$\begin{aligned}\mathcal{L}(e^{at};s) &= \int_0^\infty e^{at} e^{-st}\,dt = \int_0^\infty e^{(a-s)t}\,dt = \left.\frac{e^{(a-s)t}}{a-s}\right|_0^\infty \\ &= \begin{cases} \frac{1}{s-a} & \text{if } \mathrm{Re}(s) > \mathrm{Re}(a) \\ \text{divergent} & \text{otherwise} \end{cases}\end{aligned}$$

The last formula comes from plugging ∞ into the exponential. This is 0 if $\mathrm{Re}(a-s) < 0$ and undefined otherwise.

Example 12.4. Let $f(t) = b$. Compute $F(s) = \mathcal{L}(f;s)$ directly. Give the region in the complex s-plane where the integral converges.

$$\begin{aligned}\mathcal{L}(b;s) &= \int_0^\infty b e^{-st}\,dt = \left.\frac{be^{-st}}{-s}\right|_0^\infty \\ &= \begin{cases} \frac{b}{s} & \text{if } \mathrm{Re}(s) > 0 \\ \text{divergent} & \text{otherwise} \end{cases}\end{aligned}$$

The last formula comes from plugging ∞ into the exponential. This is 0 if $\mathrm{Re}(-s) < 0$ and undefined otherwise.

Example 12.5. Let $f(t) = t$. Compute $F(s) = \mathcal{L}(f;s)$ directly. Give the region in the complex s-plane where the integral converges.

$$\mathcal{L}(t; s) = \int_0^\infty t e^{-st}\, dt = \left.\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right|_0^\infty$$
$$= \begin{cases} \frac{1}{s^2} & \text{if } \mathrm{Re}(s) > 0 \\ \text{divergent} & \text{otherwise} \end{cases}$$

Example 12.6. Compute $\mathcal{L}(\cos(\omega t))$

answer: We use the formula $\cos(\omega t) = \dfrac{e^{i\omega t} + e^{-i\omega t}}{2}$. So,

$$\mathcal{L}(\cos(\omega t); s) = \frac{1/(s - i\omega) + 1/(s + i\omega)}{2} = \frac{s}{s^2 + \omega^2}$$

12.3.2 Connection to Fourier transform

The Laplace and Fourier transforms are intimately connected. In fact, the Laplace transform is often called the Fourier-Laplace transform. To see the connection we'll start with the Fourier transform of a function $f(t)$. $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt$. If we assume $f(t) = 0$ for $t < 0$, this becomes

$$\hat{f}(\omega) = \int_0^\infty f(t)e^{-i\omega t}\, dt. \quad (1)$$

Now if $s = i\omega$ then the Laplace transform is

$$\mathcal{L}(f; s) = \mathcal{L}(f; i\omega) = \int_0^\infty f(t)e^{-i\omega t}\, dt \quad (2)$$

Comparing these two equations we see that $\hat{f}(\omega) = \mathcal{L}(f; i\omega)$. We see the transforms are basically the same things using different notation –at least for functions that are 0 for $t < 0$.

12.4 Exponential type

The Laplace transform is defined when the integral for it converges. Functions of exponential type are a class of functions for which the integral converges for all s with $\mathrm{Re}(s)$ large enough.

Definition. We say that $f(t)$ has exponential type a if there exists an M such that $|f(t)| < Me^{at}$ for all $t \geq 0$.

Note. As we've defined it, the exponential type of a function is not unique. For example, a function of exponential type 2 is clearly also of exponential type 3. It's nice, but not always necessary, to find the smallest exponential type for a function.

Theorem. If f has exponential type a then $\mathcal{L}(f)$ converges absolutely for $\mathrm{Re}(s) > a$.

Proof. We prove absolute convergence by bounding $|f(t)e^{-st}|$. The key here is that $\mathrm{Re}(s) > a$ implies $\mathrm{Re}(a - s) < 0$. So, we can write

$$\int_0^\infty |f(t)e^{-st}|\, dt \leq \int_0^\infty |Me^{(a-s)t}|\, dt = \int_0^\infty Me^{\mathrm{Re}(a-s)t}\, dt$$

The last integral clearly converges when $\text{Re}(a-s) < 0$. QED

Example 12.7. Here is a list of some functions of exponential type.

$$\begin{aligned} f(t) = e^{at}: &\quad |f(t)| < 2e^{\text{Re}(a)t} && \text{(exponential type Re}(a)) \\ f(t) = 1: &\quad |f(t)| < 2 = 2e^{0\cdot t} && \text{(exponential type 0)} \\ f(t) = \cos(\omega t): &\quad |f(t)| \leq 1 && \text{(exponential type 0)} \end{aligned}$$

In the above, all of the inequalities are for $t \geq 0$.

For $f(t) = t$, it is clear that for any $a > 0$ there is an M depending on a such that $|f(t)| \leq Me^{at}$ for $t \geq 0$. In fact, it is a simple calculus exercise to show $M = 1/(ae)$ works. So, $f(t) = t$ has exponential type a for any $a > 0$.

The same is true of t^n. It's worth pointing out that this follows because, if f has exponential type a and g has exponential type b then fg has exponential type $a+b$. So, if t has exponential type a then t^n has exponential type na.

12.5 Properties of Laplace transform

We have already used the linearity of Laplace transform when we computed $\mathcal{L}(\cos(\omega t))$. Let's officially record it as a property.

Property 1. The Laplace transform is linear. That is, if a and b are constants and f and g are functions then

$$\mathcal{L}(af + bg) = a\mathcal{L}(f) + b\mathcal{L}(g). \tag{3}$$

(The proof is trivial –integration is linear.)

Property 2. A key property of the Laplace transform is that, with some technical details,

Laplace transform transforms derivatives in t to multiplication by s (plus some details).

This is proved in the following theorem.

Theorem. If $f(t)$ has exponential type a and Laplace transform $F(s)$ then

$$\mathcal{L}(f'(t); s) = sF(s) - f(0), \text{ valid for } \text{Re}(s) > a. \tag{4}$$

Proof. We prove this using integration by parts.

$$\mathcal{L}(f'; s) = \int_0^\infty f'(t)e^{-st}\,dt = f(t)e^{-st}\Big|_0^\infty + \int_0^\infty sf(t)e^{-st}\,dt = -f(0) + sF(s).$$

In the last step we used the fact that at $t = \infty$, $f(t)e^{-st} = 0$, which follows from the assumption about exponential type.

Equation 4 gives us formulas for all derivatives of f.

$$\mathcal{L}(f''; s) = s^2F(s) - sf(0) - f'(0) \tag{5}$$

$$\mathcal{L}(f'''; s) = s^3F(s) - s^2f(0) - sf'(0) - f''(0) \tag{6}$$

Proof. For Equation 5:

$\mathcal{L}(f'';s) = \mathcal{L}((f')';s) = s\mathcal{L}(f';s) - f'(0) = s\,(sF(s) - f(0)) - f'(0) = s^2F(s) - sf(0) - f'(0)$. QED

The proof Equation 6 is similar. Also, similar statements hold for higher order derivatives.

Note. There is a further complication if we want to consider functions that are discontinuous at the origin or if we want to allow $f(t)$ to be a generalized function like $\delta(t)$. In these cases $f(0)$ is not defined, so our formulas are undefined. The technical fix is to replace 0 by 0^- in the definition and all of the formulas for Laplace transform. You can learn more about this by taking 18.031.

Property 3. Theorem. If $f(t)$ has exponential type a, then $F(s)$ is an analytic function for $\text{Re}(s) > a$ and

$$F'(s) = -\mathcal{L}(tf(t);s). \tag{7}$$

Proof. We take the derivative of $F(s)$. The absolute convergence for $\text{Re}(s)$ large guarantees that we can interchange the order of integration and taking the derivative.

$$F'(s) = \frac{d}{ds}\int_0^\infty f(t)\mathrm{e}^{-st}\,dt = \int_0^\infty -tf(t)\mathrm{e}^{-st}\,dt = \mathcal{L}(-tf(t);s).$$

This proves Equation 7.

Equation 7 is called the s-derivative rule. We can extend it to more derivatives in s: Suppose $\mathcal{L}(f;s) = F(s)$. Then,

$$\mathcal{L}(tf(t);s) = -F'(s) \tag{8}$$

$$\mathcal{L}(t^nf(t);s) = (-1)^nF^{(n)}(s) \tag{9}$$

Equation 8 is the same as Equation 7 above. Equation 9 follows from this.

Example 12.8. Use the s-derivative rule and the formula $\mathcal{L}(1;s) = 1/s$ to compute the Laplace transform of t^n for n a positive integer.

<u>**answer:**</u> Let $f(t) = 1$ and $F(s) = \mathcal{L}(f;s)$. Using the s-derivative rule we get

$$\begin{aligned}
\mathcal{L}(t;s) &= \mathcal{L}(tf;s) = -F'(s) = \frac{1}{s^2} \\
\mathcal{L}(t^2;s) &= \mathcal{L}(t^2f;s) = (-1)^2F''(s) = \frac{2}{s^3} \\
\mathcal{L}(t^n;s) &= \mathcal{L}(t^nf;s) = (-1)^nF^n(s) = \frac{n!}{s^{n+1}}
\end{aligned}$$

Property 4. t-shift rule. As usual, assume $f(t) = 0$ for $t < 0$. Suppose $a > 0$. Then,

$$\mathcal{L}(f(t-a);s) = \mathrm{e}^{-as}F(s) \tag{10}$$

Proof. We go back to the definition of the Laplace transform and make the change of variables $\tau = t - a$.

$$\begin{aligned}
\mathcal{L}(f(t-a);s) &= \int_0^\infty f(t-a)\mathrm{e}^{-st}\,dt = \int_a^\infty f(t-a)\mathrm{e}^{-st}\,dt \\
&= \int_0^\infty f(\tau)\mathrm{e}^{-s(\tau+a)}\,d\tau = \mathrm{e}^{-sa}\int_0^\infty f(\tau)\mathrm{e}^{-s\tau}\,d\tau = \mathrm{e}^{-sa}F(s).
\end{aligned}$$

The properties in Equations 3-10 will be used in examples below. They are also in the table at the end of these notes.

12.6 Differential equations

Coverup method. We are going to use partial fractions and the coverup method. We will assume you have seen partial fractions. If you don't remember them well or have never seen the coverup method, you should read the note *Partial fractions and the coverup method* posted with the class notes.

Example 12.9. Solve $y'' - y = \mathrm{e}^{2t}$, $y(0) = 1$, $y'(0) = 1$ using Laplace transform.

answer: Call $\mathcal{L}(y) = Y$. Apply the Laplace transform to the equation:

$$(s^2Y - sy(0) - y'(0)) - Y = \frac{1}{s-2}$$

Algebra: $(s^2 - 1)Y = \dfrac{1}{s-2} + s + 1$, so

$$Y = \frac{1}{(s-2)(s^2-1)} + \frac{s+1}{s^2-1} = \frac{1}{(s-2)(s^2-1)} + \frac{1}{s-1}$$

Use partial fractions to write

$$Y = \frac{A}{s-2} + \frac{B}{s-1} + \frac{C}{s+1} + \frac{1}{s-1}.$$

The coverup method gives $A = 1/3$, $B = -1/2$, $C = 1/6$.

We recognize $\dfrac{1}{s-a}$ as the Laplace transform of e^{at}, so

$$y(t) = A\mathrm{e}^{2t} + B\mathrm{e}^{t} + C\mathrm{e}^{-t} + \mathrm{e}^{t} = \frac{1}{3}\mathrm{e}^{2t} - \frac{1}{2}\mathrm{e}^{t} + \frac{1}{6}\mathrm{e}^{-t} + \mathrm{e}^{t}.$$

Example 12.10. Solve $y'' - y = 1$, $y(0) = 0$, $y'(0) = 0$.

answer: The rest (zero) initial conditions are nice because they will not add any terms to the algebra. As in the previous example we apply the Laplace transform to the entire equation.

$$s^2Y - Y = \frac{1}{s}, \quad \text{so} \quad Y = \frac{1}{s(s^2-1)} = \frac{1}{s(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1}$$

The coverup method gives $A = -1$, $B = 1/2$, $C = 1/2$. So,

$$y = A + B\mathrm{e}^{t} + C\mathrm{e}^{-t} = -1 + \frac{1}{2}\mathrm{e}^{t} + \frac{1}{2}\mathrm{e}^{-t}.$$

12.7 System functions and the Laplace transform

When we introduced the Nyquist criterion for stability we stated without any justification that the system was stable if all the poles of the system function $G(s)$were in the left half-plane. We also asserted that the poles corresponded to exponential modes of the system. In this section we'll use the Laplace transform to more fully develop these ideas for differential equations.

12.7.1 Lightning review of 18.03

Definitions.
1. $D = \frac{d}{dt}$ is called a differential operator. Applied to a function $f(t)$ we have $Df = \frac{df}{dt}$. We read Df as 'D applied to f.'

Example 12.11. If $f(t) = t^3 + 2$ then $Df = 3t^2$, $D^2 f = 6t$.

2. If $P(s)$ is a polynomial then $P(D)$ is called a polynomial differential operator.

Example 12.12. Suppose $P(s) = s^2 + 8s + 7$. What is $P(D)$? Compute $P(D)$ applied to $f(t) = t^3 + 2t + 5$. Compute $P(D)$ applied to $g(t) = e^{2t}$.

answer: $P(D) = D^2 + 8D + 7I$. (The I in $7I$ is the identity operator.) To compute $P(D)f$ we do the following.

$$\begin{aligned} f(t) &= t^3 + 2t + 5 \\ Df(t) &= 3t^2 + 2 \\ D^2 f(t) &= 6t \end{aligned}$$

Therefore,

$$(D^2 + 8D + 7I)f = 6t + 8(3t^2 + 2) + 7(t^3 + 2t + 5) = 7t^3 + 24t^2 + 20t + 51.$$

$$\begin{aligned} g(t) &= e^{2t} \\ Dg(t) &= 2e^{2t} \\ D^2 g(t) &= 4e^{2t} \end{aligned}$$

Therefore,

$$(D^2 + 8D + 7I)g = 4e^{2t} + 8(2)e^{2t} + 7e^{2t} = (4 + 16 + 7)e^{2t} = P(2)e^{2t}.$$

The substitution rule is a straightforward statement about the derivatives of exponentials.
Substitution rule:

$$P(D)e^{st} = P(s)e^{st}. \tag{11}$$

Proof. This is obvious. We 'prove it' by example. Let $P(D) = D^2 + 8D + 7I$. Then

$$P(D)e^{at} = a^2 e^{at} + 8ae^{at} + 7e^{at} = (a^2 + 8a + 7)e^{at} = P(a)e^{at}.$$

Let's continue to work from this specific example. From it we'll be able to remind you of the general approach to solving constant coefficient differential equations.

Example 12.13. Suppose $P(s) = s^2 + 8s + 7$. Find the exponential modes of the equation $P(D)y = 0$.

answer: The exponential modes are solutions of the form $y(t) = e^{s_0 t}$. Using the substititution rule

$$P(D)e^{s_0 t} = 0 \;\Leftrightarrow\; P(s_0) = 0.$$

That is, $y(t) = e^{s_0 t}$ is a mode exactly when s_0 is a root of $P(s)$. The roots of $P(s)$ are -1, -7. So the modal solutions are $y_1(t) = e^{-t}$ and $y_2(t) = e^{-7t}$.

Example 12.14. Redo the previous example using the Laplace transform.

answer: For this we solve the differential equation with arbitrary initial conditions:

$$P(D)y = y'' + 8y' + 7y = 0; \quad y(0) = c_1,\, y'(0) = c_2.$$

Let $Y(s) = \mathcal{L}(y; s)$. Applying the Laplace transform to the equation we get

$$(s^2 Y(s) - sy(0) - y'(0)) + 8(sY(s) - y(0)) + 7Y(s) = 0$$

Algebra:

$$(s^2 + 8s + 7)Y(s) - sc_1 - c_2 - 8c_1 = 0 \;\Leftrightarrow\; Y = \frac{sc_1 + 8c_1 + c_2}{s^2 + 8s + 7}$$

Factoring the denominator and using partial fractions, we get

$$Y(s) = \frac{sc_1 + 8c_1 + c_2}{s^2 + 8s + 7} = \frac{sc_1 + 8c_1 + c_2}{(s+1)(s+7)} = \frac{A}{s+1} + \frac{B}{s+7}.$$

We are unconcerned with the exact values of A and B. Taking the Laplace inverse we get

$$y(t) = Ae^{-t} + Be^{-7t}.$$

That is, $y(t)$ is a linear combination of the exponential modes.

You should notice that the denominator in the expression for $Y(s)$ is none other than the characteristic polynomial $P(s)$.

12.7.2 The system function

Example 12.15. With the same $P(s)$ as in Example 12.12 solve the inhomogeneous DE with rest initial conditions: $P(D)y = f(t), \quad y(0) = 0,\, y'(0) = 0$.

answer: Taking the Laplace transform of the equation we get

$$P(s)Y(s) = F(s).$$

Therefore

$$Y(s) = \frac{1}{P(s)} F(s)$$

We can't find $y(t)$ explicitly because $f(t)$ isn't specified.

But, we can make the following definitions and observations. Let $G(s) = 1/P(s)$. If we declare f to be the input and y the output of this linear time invariant system, then $G(s)$ is called the system function. So, we have

$$Y(s) = G(s) \cdot F(s). \tag{12}$$

The formula $Y = G \cdot F$ can be phrased as

$$\text{output} = \text{system function} \times \text{input}.$$

Note well, the roots of $P(s)$ correspond to the exponential modes of the system, i.e. the poles of $G(s)$ correspond to the exponential modes.

The system is called stable if the modes all decay to 0 as t goes to infinity. That is, if all the poles have negative real part.

Example 12.16. This example is to emphasize that not all system functions are of the form $1/P(s)$. Consider the system modeled by the differential equation

$$P(D)x = Q(D)f,$$

where P and Q are polynomials. Suppose we consider f to be the input and x to be the ouput. Find the system function.

answer: If we start with rest initial conditions for x and f then the Laplace transform gives $P(s)X(s) = Q(s)F(s)$ or

$$X(s) = \frac{Q(s)}{P(s)} \cdot F(s)$$

Using the formulation

$$\text{output} = \text{system function} \times \text{input},$$

we see that the system function is $G(s) = \dfrac{Q(s)}{P(s)}$.

Note that when $f(t) = 0$ the differential equation becomes $P(D)x = 0$. If we make the assumption that the $Q(s)/P(s)$ is in reduced form, i.e. P and Q have no common zeros, then the modes of the system (which correspond to the roots of $P(s)$) are still the poles of the system function.

Comments. All LTI systems have system functions. They are not even all of the form $Q(s)/P(s)$. But, in the s-domain, the output is always the system function times the input. If the system function is not rational then it may have an infinite number of poles. Stability is harder to characterize, but under some reasonable assumptions the system will be stable if all the poles are in the left half-plane.

The system function is also called the transfer function. You can think of it as describing how the system transfers the input to the output.

12.8 Laplace inverse

Up to now we have computed the inverse Laplace transform by table lookup. For example, $\mathcal{L}^{-1}(1/(s-a)) = e^{at}$. To do this properly we should first check that the Laplace transform has an inverse.

We start with the bad news: Unfortunately this is not strictly true. There are many functions with the same Laplace transform. We list some of the ways this can happen.

1. If $f(t) = g(t)$ for $t \geq 0$, then clearly $F(s) = G(s)$. Since the Laplace transform only concerns $t \geq 0$, the functions can differ completely for $t < 0$.

2. Suppose $f(t) = e^{at}$ and

$$g(t) = \begin{cases} f(t) & \text{for } t \neq 1 \\ 0 & \text{for } t = 1. \end{cases}.$$

That is, f and g are the same except we arbitrarily assigned them different values at $t = 1$. Then, since the integrals won't notice the difference at one point, $F(s) = G(s) = 1/(s - a)$. In this sense it is impossible to define $\mathcal{L}^{-1}(F)$ uniquely.

The good news is that the inverse exists as long as we consider two functions that only differ on a negligible set of points the same. In particular, we can make the following claim.

Theorem. Suppose f and g are continuous and $F(s) = G(s)$ for all s with $\mathrm{Re}(s) > a$ for some a. Then $f(t) = g(t)$ for $t \geq 0$.

This theorem can be stated in a way that includes piecewise continuous functions. Such a statement takes more care, which would obscure the basic point that the Laplace transform has a unique inverse up to some, for us, trivial differences.

We start with a few examples that we can compute directly.

Example 12.17. Let $f(t) = \mathrm{e}^{at}$. So, $F(s) = \dfrac{1}{s-a}$. Show

$$f(t) = \sum \mathrm{Res}(F(s)\mathrm{e}^{st}) \tag{13}$$

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\mathrm{e}^{st}\, ds \tag{14}$$

The sum is over all poles of $\mathrm{e}^{st}/(s-a)$. As usual, we only consider $t > 0$.

Here, $c > \mathrm{Re}(a)$ and the integral means the path integral along the vertical line $x = c$.

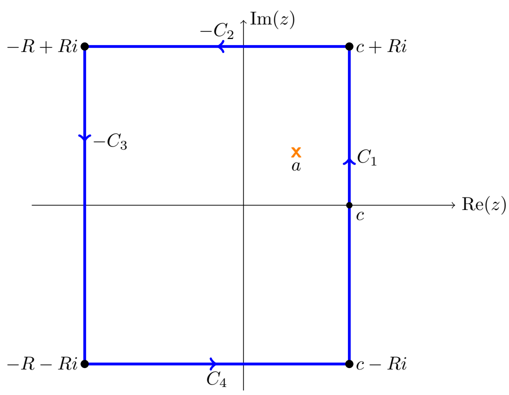
<u>**answer:**</u> Proving Equation 13 is straightforward: It is clear that $\dfrac{\mathrm{e}^{st}}{s-a}$ has only one pole which is at $s = a$. Since, $\sum \mathrm{Res}\left(\dfrac{\mathrm{e}^{st}}{s-a}, a\right) = \mathrm{e}^{at}$ we have proved Equation 13.

Proving Equation 14 is more involved. We should first check the convergence of the integral. In this case, $s = c + iy$, so the integral is

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\mathrm{e}^{st}\, ds = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{(c+iy)t}}{c+iy-a}\, i\, dy = \frac{\mathrm{e}^{ct}}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{iyt}}{c+iy-a}\, dy.$$

The (conditional) convergence of this integral follows using exactly the same argument as in the example near the end of Topic 9 on the Fourier inversion formula for $f(t) = \mathrm{e}^{at}$. That is, the integrand is a decaying oscillation, around 0, so its integral is also a decaying oscillation around some limiting value.

Now we use the contour shown below.



We will let R go to infinity and use the following steps to prove Equation 14.

1. The residue theorem guarantees that if the curve is large enough to contain a then

$$\frac{1}{2\pi i}\int_{C_1-C_2-C_3+C_4}\frac{e^{st}}{s-a}\,ds = \sum \mathrm{Res}\left(\frac{e^{st}}{s-a},a\right) = e^{at}.$$

2. In a moment we will show that the integrals over C_2, C_3, C_4 all go to 0 as $R \to \infty$.

3. Clearly as R goes to infinity, the integral over C_1 goes to the integral in Equation 14

Putting these steps together we have

$$e^{at} = \lim_{R\to\infty}\int_{C_1-C_2-C_3+C_4}\frac{e^{st}}{s-a}\,ds = \int_{c-i\infty}^{c+i\infty}\frac{e^{st}}{s-a}\,ds.$$

Except for proving the claims in step 2, this proves Equation 14.

To verify step 2 we look at one side at a time.

C_2: C_2 is parametrized by $s = \gamma(u) = u + iR$, with $-R \le u \le c$. So,

$$\left|\int_{C_2}\frac{e^{st}}{s-a}\,ds\right| = \int_{-R}^{c}\left|\frac{e^{(u+iR)t}}{u+iR-a}\right| \le \int_{-R}^{c}\frac{e^{ut}}{R}\,du = \frac{e^{ct}-e^{-Rt}}{tR}.$$

Since c and t are fixed, it's clear this goes to 0 as R goes to infinity.

The bottom C_4 is handled in exactly the same manner as the top C_2.

C_3: C_3 is parametrized by $s = \gamma(u) = -R + iu$, with $-R \le u \le R$. So,

$$\left|\int_{C_3}\frac{e^{st}}{s-a}\,ds\right| = \int_{-R}^{R}\left|\frac{e^{(-R+iu)t}}{-R+iu-a}\right| \le \int_{-R}^{R}\frac{e^{-Rt}}{R+a}\,du = \frac{e^{-Rt}}{R+a}\int_{-R}^{R}du = \frac{2Re^{-Rt}}{R+a}.$$

Since a and $t > 0$ are fixed, it's clear this goes to 0 as R goes to infinity.

Example 12.18. Repeat the previous example with $f(t) = t$ for $t > 0$, $F(s) = 1/s^2$.

This is similar to the previous example. Since F decays like $1/s^2$ we can actually allow $t \ge 0$

Theorem 12.19. Laplace inversion 1. Assume f is continuous and of exponential type a. Then for $c > a$ we have

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(s)e^{st}\,ds. \tag{15}$$

As usual, this formula holds for $t > 0$.

Proof. The proof uses the Fourier inversion formula. We will just accept this theorem for now. Example 12.17 above illustrates the theorem.

Theorem 12.20. Laplace inversion 2. Suppose $F(s)$ has a finite number of poles and decays like $1/s$ (or faster). Define

$$f(t) = \sum \mathrm{Res}(F(s)e^{st}, p_k),\ \text{where the sum is over all the poles } p_k. \tag{16}$$

Then $\mathcal{L}(f;s) = F(s)$

Proof. Proof to be added. The basic ideas are present in the examples above, though it requires a fairly clever choice of contours.

The integral inversion formula in Equation 15 can be viewed as writing $f(t)$ as a 'sum' of exponentials. This is extremely useful. For example, for a linear system if we know how the system responds to input $f(t) = e^{at}$ for all a, then we know how it responds to any input by writing it as a 'sum' of exponentials.

12.9 Delay and feedback.

Let $f(t) = 0$ for $t < 0$. Fix $a > 0$ and let $h(t) = f(t - a)$. So, $h(t)$ is a delayed version of the signal $f(t)$. The Laplace property Equation 10 says $H(s) = e^{-as}F(s)$, where H and F are the Laplace transforms of h and f respectively.

Now, suppose we have a system with system function $G(s)$. (Again, called the open loop system.) As before, can feed the output back through the system. But, instead of just multiplying the output by a scalar we can delay it also. This is captured by the feedback factor ke^{-as}.

The system function for the closed loop system is

$$G_{CL}(s) = \frac{G}{1 + ke^{-as}G}$$

Note even if you start with a rational function the system function of the closed loop with delay is not rational. Usually it has an infinite number of poles.

Example 12.21. Suppose $G(s) = 1$, $a = 1$ and $k = 1$ find the poles of $G_{CL}(s)$.

<u>**answer:**</u> $G_{CL}(s) = \dfrac{1}{1 + e^{-s}}$. So the poles occur where $e^{-s} = -1$, i.e. at $in\pi$, where n is an odd integer. There are an infinite number of poles on the imaginary axis.

Example 12.22. Suppose $G(s) = 1$, $a = 1$ and $k = 1/2$ find the poles of $G_{CL}(s)$. Is the closed loop system stable?

<u>**answer:**</u> $G_{CL}(s) = \dfrac{1}{1 + e^{-s}/2}$. So the poles occur where $e^{-s} = -2$, i.e. at $-\log(2) + in\pi$, where n is an odd integer. Since $-\log(2) < 0$, there are an infinite number of poles in the left half-plane. With all poles in the left half-plane, the system is stable.

Example 12.23. Suppose $G(s) = 1$, $a = 1$ and $k = 2$ find the poles of $G_{CL}(s)$. Is the closed loop system stable?

<u>**answer:**</u> $G_{CL}(s) = \dfrac{1}{1 + 2e^{-s}}$. So the poles occur where $e^{-s} = -1/2$, i.e. at $\log(2) + in\pi$, where n is an odd integer. Since $\log(2) > 0$, there are an infinite number of poles in the right half-plane. With poles in the right half-plane, the system is not stable.

Remark. If $\mathrm{Re}(s)$ is large enough we can express the system function $G(s) = \dfrac{1}{1 + ke^{-as}}$ as a geometric series

$$\frac{1}{1 + ke^{-as}} = 1 - ke^{-as} + k^2e^{-2as} - k^3e^{-3as} + \ldots$$

So, for input $F(s)$, we have output

$$X(s) = G(s)F(s) = F(s) - ke^{-as}F(s) + k^2e^{-2as}F(s) - k^3e^{-3as}F(s) + \ldots$$

Using the shift formula Equation 10, we have

$$x(t) = f(t) - kf(t-a) + k^2 f(t-2a) - k^3 f(t-3a) + \dots$$

(This is not really an infinite series because $f(t) = 0$ for $t < 0$.) If the input is bounded and $k < 1$ then even for large t the series is bounded. So bounded input produces bounded output –this is also what is meant by stability. On the other hand if $k > 1$, then bounded input can lead to unbounded output –this is instability.

12.10 Table of Laplace transforms

Properties and Rules

We assume that $f(t) = 0$ for $t < 0$.

Function	Transform	
$f(t)$	$F(s) = \int_0^\infty f(t)e^{-st}\,dt$	(Definition)
$a\,f(t) + b\,g(t)$	$a\,F(s) + b\,G(s)$	(Linearity)
$e^{at} f(t)$	$F(s-a)$	(s-shift)
$f'(t)$	$sF(s) - f(0)$	
$f''(t)$	$s^2 F(s) - sf(0) - f'(0)$	
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$	
$tf(t)$	$-F'(s)$	
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	
$f(t-a)$	$e^{-as} F(s)$	(t-translation or t-shift)
$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$	(integration rule)
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	

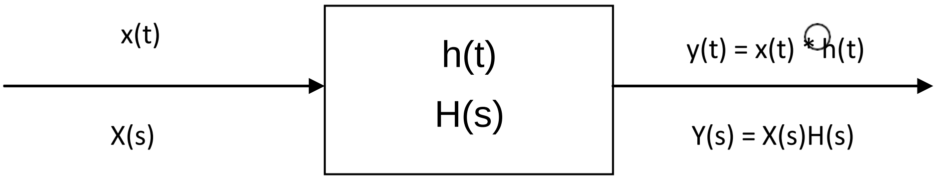
Function Table

Function	Transform	Region of convergence
1	$1/s$	$\mathrm{Re}(s) > 0$
e^{at}	$1/(s-a)$	$\mathrm{Re}(s) > \mathrm{Re}(a)$
t	$1/s^2$	$\mathrm{Re}(s) > 0$
t^n	$n!/s^{n+1}$	$\mathrm{Re}(s) > 0$
$\cos(\omega t)$	$s/(s^2+\omega^2)$	$\mathrm{Re}(s) > 0$
$\sin(\omega t)$	$\omega/(s^2+\omega^2)$	$\mathrm{Re}(s) > 0$
$\mathrm{e}^{at}\cos(\omega t)$	$(s-a)/((s-a)^2+\omega^2)$	$\mathrm{Re}(s) > \mathrm{Re}(a)$
$\mathrm{e}^{at}\sin(\omega t)$	$\omega/((s-a)^2+\omega^2)$	$\mathrm{Re}(s) > \mathrm{Re}(a)$
$\delta(t)$	1	all s
$\delta(t-a)$	e^{-as}	all s
$\cosh(kt) = \dfrac{\mathrm{e}^{kt}+\mathrm{e}^{-kt}}{2}$	$s/(s^2-k^2)$	$\mathrm{Re}(s) > k$
$\sinh(kt) = \dfrac{\mathrm{e}^{kt}-\mathrm{e}^{-kt}}{2}$	$k/(s^2-k^2)$	$\mathrm{Re}(s) > k$
$\dfrac{1}{2\omega^3}(\sin(\omega t) - \omega t\cos(\omega t))$	$\dfrac{1}{(s^2+\omega^2)^2}$	$\mathrm{Re}(s) > 0$
$\dfrac{t}{2\omega}\sin(\omega t)$	$\dfrac{s}{(s^2+\omega^2)^2}$	$\mathrm{Re}(s) > 0$
$\dfrac{1}{2\omega}(\sin(\omega t) + \omega t\cos(\omega t))$	$\dfrac{s^2}{(s^2+\omega^2)^2}$	$\mathrm{Re}(s) > 0$
$t^n\mathrm{e}^{at}$	$n!/(s-a)^{n+1}$	$\mathrm{Re}(s) > \mathrm{Re}(a)$
$\dfrac{1}{\sqrt{\pi\, t}}$	$\dfrac{1}{\sqrt{s}}$	$\mathrm{Re}(s) > 0$
t^a	$\dfrac{\Gamma(a+1)}{s^{a+1}}$	$\mathrm{Re}(s) > 0$

UNIT-6
Z-Transform

Introduction

A linear system can be represented in the complex frequency domain (s-domain here s = σ + jω) using the LaPlace Transform.



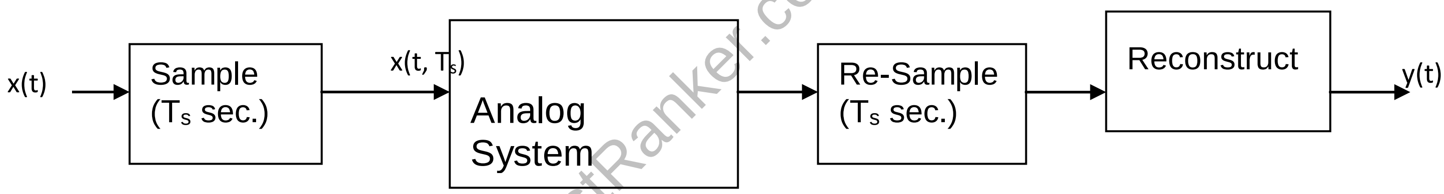
Where the direct transform is:

$$L\{x(t)\} = X(s) = \int_{t=0}^{\infty} x(t)\varepsilon^{-st}dt$$

And x(t) is assumed zero for t ≤ 0. The Inversion integral is a contour integral in the complex plane (seldom used, tables are used instead)

$$L^{-1}\{X(s)\} = x(t) = \frac{1}{2\pi j}\int_{s=\sigma-j\infty}^{\sigma+j\infty} X(s)\varepsilon^{st}ds$$

Where σ is chosen such that the contour integral converges. If we now assume that x(t) is ideally sampled as in:



Where: $x_n = x(n*T_s) = x(t)|_{t=n*T_s}$ and $y_n = y(n*T_s) = y(t)|_{t=n*T_s}$

Analyzing this equivalent system using standard analog tools will establish the z-Transform.

Sampling

Substituting the Sampled version of x(t) into the definition of the LaPlace Transform we get

$$L\{x(t,T_s)\} = X_T(s) = \int_{t=0}^{\infty} x(t,T_s)\varepsilon^{-st}dt$$

But

$$x(t,T_s) = \sum_{n=0}^{\infty} x(t) * p(t - n*T_s)$$ (For x(t) = 0 when t < 0)

Therefore

$$X_T(s) = \int_{t=0}^{\infty}\left[\sum_{n=0}^{\infty} x(n*T_s)*\delta(t-n*T_s)\right]\varepsilon^{-st}dt$$

Now interchanging the order of integration and summation and using the sifting property of δ-functions

$$X_T(s)=\sum_{n=0}^{\infty} x(n*T_s)\int_{t=0}^{\infty}\delta(t-n*T_s)\varepsilon^{-st}dt$$

$$X_T(s)=\sum_{n=0}^{\infty} x(n*T_s)\varepsilon^{-nT_s s}$$ (We are assuming that the first sample occurs at t = 0+)

if we now adjust our nomenclature by letting:

$z = \varepsilon^{sT}$, $x(n*Ts) = x_n$, and $X(z) = X_T(s)\big|_{z=\varepsilon^{sT}}$

$$\boxed{X(z)=\sum_{n=0}^{\infty} x_n z^{-n}}$$

Which is the direct z-transform (one-sided; it assumes $x_n = 0$ for $n < 0$).

The inversion integral is:

$$\boxed{x_n=\frac{1}{2\pi j}\oint_c X(z)z^{n-1}dz}$$ (This is a contour integral in the complex z-plane)

(The use of this integral can be avoided as tables can be used to invert the transform.)

To prove that these form a transform pair we can substitute one into the other.

$$x_k = \frac{1}{2\pi j}\oint_c\left[\sum_{n=0}^{\infty} x_n z^{-n}\right]z^{k-1}dz$$

Now interchanging the order of summation and integration (valid if the contour followed stays in the region of convergence):

$$x_k = \frac{1}{2\pi j}\sum_{n=0}^{\infty} x_n\oint_c z^{k-n-1}dz$$

If "C" encloses the origin (that"s where the pole is), the Cauchy Integral theorem says:

$\oint_c z^{k-n-1} dz = \begin{cases} 0 & \text{for } n \neq k \\ 2\pi j & \text{for } n = k \end{cases}$

Properties of the *z* transform

For the following

$$Z\{f[n]\} = \sum_{n=0}^{n=\infty} f[n] z^{-n} = F(z) \quad Z\{g_n\} = \sum_{n=0}^{n=\infty} g_n z^{-n} = G(z)$$

- **Linearity:**

$Z\{af_n + bg_n\} = aF(z) + bG(z)$. and ROC is $R_f \cap R_g$

which follows from definition of *z*-transform.

- **Time Shifting**

If we have $f[n] \Leftrightarrow F(z)$ then $f[n-n_0] \Leftrightarrow z^{-n_0} F(z)$

The ROC of $Y(z)$ is the same as $F(z)$ except that there are possible pole additions or deletions at $z = 0$ or $z = \infty$.

Proof:

Let $y[n] = f[n-n_0]$ then

$$Y(z) = \sum_{n=-\infty}^{\infty} f[n-n_0] z^{-n}$$

Assume $k = n - n_0$ then $n = k + n_0$, substituting in the above equation we have:

$$Y(z) = \sum_{k=-\infty}^{\infty} f[k] z^{-k-n_0} = z^{-n_0} F[z]$$

- **Multiplication by an Exponential Sequence**

Let $y[n] = z_0^n f[n]$ then $Y(z) = X\left(\frac{z}{z_0}\right)$

The consequence is pole and zero locations are scaled by z_0. If the ROC of $FX(z)$ is $r_R < |z| < r_L$, then the ROC of $Y(z)$ is

$r_R < |z/z_0| < r_L$, *i.e.*, $|z_0| r_R < |z| < |z_0| r_L$

Proof:

$$Y(z)=\sum_{n=-\infty}^{\infty} z_0^n x[n]z^{-n} = \sum_{n=-\infty}^{\infty} x[n]\left(\frac{z}{z_0}\right)^{-n} = X\left(\frac{z}{z_0}\right)$$

The consequence is pole and zero locations are scaled by z_0. If the ROC of $X(z)$ is $rR<|z|<rL$, then the ROC of $Y(z)$ is

$rR < |z/z_0| < rL$, *i.e.*, $|z_0|rR < |z| < |z_0|rL$

- **Differentiation of *X*(*z*)**

If we have $f[n] \Leftrightarrow F(z)$ then $nf[n] \xleftrightarrow{Z} -z\frac{dF(z)}{z}$ and ROC = R_f

Proof:

$$F(z)=\sum_{n=-\infty}^{\infty} f[n]z^{-n}$$

$$-z\frac{dF(z)}{dz}=-z\sum_{n=-\infty}^{\infty} -n f[n]z^{-n-1} = \sum_{n=-\infty}^{\infty} -n f[n]z^{-n}$$

$$-z\frac{dF(z)}{dz} \xleftrightarrow{Z} nf[n]$$

- **Conjugation of a Complex Sequence**

If we have $f[n] \Leftrightarrow F(z)$ then $f^*[n] \xleftrightarrow{Z} F^*(z^*)$ and ROC = R_f

Proof:

Let $y[n] = f^*[n]$, then

$$Y(z) = \sum_{n=-\infty}^{\infty} f^*[n]z^{-n} = \left(\sum_{n=-\infty}^{\infty} f[n][z^*]^{-n}\right)^* = F^*(z^*)$$

- **Time Reversal**

If we have $f[n] \Leftrightarrow F(z)$ then $f^*[-n] \xleftrightarrow{z} F^*\left(1/z^*\right)$

Let $y[n] = f^*[-n]$, then

$$Y(z) = \sum_{n=-\infty}^{\infty} f^*[-n] z^{-n} = \left(\sum_{n=-\infty}^{\infty} f[-n] \left[z^*\right]^{-n} \right)^* = \left(\sum_{k=-\infty}^{\infty} f[k] \left(1/z^*\right)^{-k} \right)^* = F^*\left(1/z^*\right)$$

If the ROC of $F(z)$ is $r_R < |z| < r_L$, then the ROC of $Y(z)$ is

$$r_R < \left|1/z^*\right| < r_L \; i.e., \qquad \frac{1}{r_R} > |z| > \frac{1}{r_L}$$

When the time reversal is without conjugation, it is easy to show

$$f[-n] \xleftrightarrow{z} F(1/z) \quad \text{and ROC is } \frac{1}{r_R} > |z| > \frac{1}{r_L}$$

A comprehensive summery for the z-transform properties is shown in Table 2

Table 2 Summery of z-transform properties

Property	Sequence	z-Transform	Region of Convergence
Linearity	$ax(n) + by(n)$	$aX(z) + bY(z)$	Contains $R_x \cap R_y$
Shift	$x(n - n_0)$	$z^{-n_0} X(z)$	R_x
Time reversal	$x(-n)$	$X(z^{-1})$	$1/R_x$
Exponentiation	$\alpha^n x(n)$	$X(\alpha^{-1} z)$	$\|\alpha\| R_x$
Convolution	$x(n) * y(n)$	$X(z)Y(z)$	Contains $R_x \cap R_y$
Conjugation	$x^*(n)$	$X^*(z^*)$	R_x
Derivative	$nx(n)$	$-z\dfrac{dX(z)}{dz}$	R_x

Note: Given the z-transforms $X(z)$ and $Y(z)$ of $x(n)$ and $y(n)$, with regions of convergence R_x and R_y, respectively, this table lists the z-transforms of sequences that are formed from $x(n)$ and $y(n)$.

Example 3: Find the z transform of $3n + 2 \times 3^n$.

Solution From the linearity property

$$Z\{3n + 2 \times 3^n\} = 3Z\{n\} + 2Z\{3^n\}$$

and from the Table 1

$$Z\{n\} = \frac{z}{(z-1)^2} \quad \text{and} \quad Z\{3^n\} = \frac{z}{(z-3)}$$

(r^n with $r = 3$). Therefore

$$Z\{3n + 2 \times 3^n\} = \frac{3z}{(z-1)^2} + \frac{2z}{(z-3)}$$

Example 4: Find the z-transform of each of the following sequences:

(a) $\boldsymbol{x(n) = 2^n u(n) + 3(½)^n u(n)}$
(b) $\boldsymbol{x(n) = \cos(n\omega_0)u(n)}$.

Solution:

(a) Because $x(n)$ is a sum of two sequences of the form $\alpha^n u(n)$, using the linearity property of the z-transform, and referring to Table 1, the z-transform pair

$$X(z) = \frac{1}{1-2z^{-1}} + \frac{3}{1-\frac{1}{2}z^{-1}} = \frac{4-\frac{13}{2}z^{-1}}{(1-2z^{-1})\left(1-\frac{1}{2}z^{-1}\right)}$$

(b) For this sequence we write

$x(n) = \cos(n\omega_0)\, u(n) = ½(e^{jn\omega_0} + e^{-jn\omega_0})\, u(n)$

Therefore, the z-transform is

$$X(z) = \frac{1}{2}\frac{1}{1-e^{jn\omega_0}z^{-1}} + \frac{1}{2}\frac{1}{1-e^{-jn\omega_0}z^{-1}}$$

with a region of convergence $|z| > 1$. Combining the two terms together, we have

$$X(z) = \frac{1-(\cos\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$$

The Inverse *z*-Transform

The *z*-transform is a useful tool in linear systems analysis. However, just as important as techniques for finding the z-transform of a sequence are methods that may be used to invert the z-transform and recover the sequence ***x*(*n*)**from ***X*(*z*)**. Three possible approaches are described below.

- **Partial Fraction Expansion**

For *z*-transforms that are rational functions of *z*,

$$X(z) = \frac{\sum_{k=0}^{q} b(k)z^{-k}}{\sum_{k=0}^{p} a(k)z^{-k}} = C\frac{\prod_{k=1}^{q}(1-\beta_k z^{-1})}{\prod_{k=1}^{p}(1-\alpha_k z^{-1})}$$

a simple and straightforward approach to find the inverse *z*-transform is to perform a partial fraction expansion of $X(z)$. Assuming that p >q, and that all of the roots in the denominator are simple, $\alpha_i \neq \alpha_k$ for $i \neq k$, $X(z)$ may be expanded as follows:

$$X(z) = \sum_{k=1}^{p} \frac{A_k}{1-\alpha_k z^{-1}} \qquad \text{Eq(3)}$$

for some constants A_k for $k = 1,2, \ldots, p$. The coefficients A_k may be found by multiplying both sides of Eq. (3) by $(1 - \alpha_k z^{-1})$ and setting $z = \alpha_k$. The result is

$$A_k = \left[(1-\alpha_k z^{-1})X(z)\right]_{z=\alpha_k}$$

If $p \leq q$, the partial fraction expansion must include a polynomial in $z^{-1}$of order (p-q). The coefficients of this polynomial may be found by long division (i.e., by dividing the numerator polynomial by the denominator). For multiple-order poles, the expansion must be modified. For example, if $X(z)$ has a second-order pole at z = α_k, the expansion will include two terms,

$$\frac{B_1}{1-\alpha_k z^{-1}} + \frac{B_2}{(1-\alpha_k z^{-1})^2}$$

where B_1,and $B_2$are given by

$$B_1 = \alpha_k \left[\frac{d}{dz}(1-\alpha_k z^{-1})^2 X(z) \right]_{z=\alpha_k}$$

$$B_2 = \left[(1-\alpha_k z^{-1})^2 X(z) \right]_{z=\alpha_k}$$

Example 5: Suppose that a sequence $x(n)$has a z-transform

$$X(z) = \frac{4 - \frac{7}{4}z^{-1} + \frac{1}{4}z^{-2}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = \frac{4 - \frac{7}{4}z^{-1} + \frac{1}{4}z^{-2}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

Solution:

With a region of convergence $|z| > ½$. Because $p = q = 2$, and the two poles are simple, the partial fraction expansion has the form

$$X(z) = C + \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-\frac{1}{4}z^{-1}}$$

The constant C is found by long division:

$$\begin{array}{r|l} & \quad 2 \\ \cline{2-2} \frac{1}{8}z^{-2} - \frac{3}{4}z^{-1} + 1 & \frac{1}{4}z^{-2} - \frac{7}{4}z^{-1} + 4 \\ & \frac{1}{4}z^{-2} - \frac{3}{2}z^{-1} + 2 \\ \cline{2-2} & \quad -\frac{1}{4}z^{-1} + 2 \end{array}$$

Therefore, $C = 2$ and we may write $X(z)$ as follows:

$$X(z) = 2 + \frac{2 - \frac{1}{4}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

Next, for the coefficients $A_1$and A_2we have

$$A_1 = \left[\left(1-\frac{1}{2}z^{-1}\right)X(z)\right]_{z^{-1}=2} = \left.\frac{4 - \frac{7}{4}z^{-1} + \frac{1}{4}z^{-2}}{1-\frac{1}{4}z^{-1}}\right|_{z^{-1}=2} = 3$$

and

$$A_2 = \left[\left(1 - \tfrac{1}{4}z^{-1}\right)X(z)\right]_{z^{-1}=4} = \left.\frac{4 - \frac{7}{4}z^{-1} + \frac{1}{4}z^{-2}}{1 - \frac{1}{2}z^{-1}}\right|_{z^{-1}=4} = -1$$

Thus, the complete partial fraction expansion becomes

$$X(z) = 2 + \frac{3}{1 - \frac{1}{2}z^{-1}} - \frac{1}{1 - \frac{1}{4}z^{-1}}$$

Finally, because the region of convergence is the exterior of the circle $|z| > 1$, $x(n)$ is the right-sided sequence

$$x(n) = 2\delta(n) + 3\left(\tfrac{1}{2}\right)^n u(n) - \left(\tfrac{1}{4}\right)^n u(n)$$

- **Power Series**

The z-transform is a power series expansion,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \cdots + x(-2)\,z^2 + x(-1)\,z + x(0) + x(1)\,z^{-1} + x(2)\,z^{-2} + \cdots$$

where the sequence values $x(n)$are the coefficients of z^{-n} in the expansion. Therefore, if we can find the power series expansion for $X(z)$, the sequence values $x(n)$may be found by simply picking off the coefficients of z^{-n}.

Example 6: Consider the z-transform

$$X(z) = \log(1 + az^{-1}) \qquad |z| > |a|$$

Solution:

The power series expansion of this function is

$$\log(1 + az^{-1}) = \sum_{n=1}^{\infty} \frac{1}{n}(-1)^{n+1}a^n z^{-n}$$

Therefore, the sequence x(n) having this z-transform is

$$x(n) = \begin{cases} \frac{1}{n}(-1)^{n+1}a^n & n > 0 \\ 0 & n \leq 0 \end{cases}$$

- **Contour Integration**

Another approach that may be used to find the inverse *z*-transform of *X*(*z*) is to use contour integration. This procedure relies on Cauchy's integral theorem, which states that if *C* is a closed contour that encircles the origin in a counterclockwise direction,

$$\frac{1}{2\pi j}\oint_C z^{-k}dz = \begin{cases} 1 & k = 1 \\ 0 & k \neq 1 \end{cases}$$

With

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Cauchy's integral theorem may be used to show that the coefficients *x*(*n*) may be found from *X*(*z*) as follows:

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

where *C*is a closed contour within the region of convergence of *X*(*z*) that encircles the origin in a counterclockwise direction. Contour integrals of this form may often by evaluated with the help of Cauchy's residue theorem,

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}\,dz = \sum\left[\text{residues of } X(z)z^{n-1}\text{at the poles inside } C\right]$$

If *X*(*z*) is a rational function of z with a first-order pole at $z = \alpha_k$,

$$\text{Res}\left[X(z)z^{n-1} \text{ at } z = \alpha_k\right] = \left[(1-\alpha_k z^{-1})X(z)z^{n-1}\right]_{z=\alpha_k}$$

Contour integration is particularly useful if only a few values of *x*(*n*) are needed.

Example 7:

Find the inverse of each of the following *z*-transforms:

$$(a)\quad X(z) = 4 + 3(z^2 + z^{-2}) \qquad 0 < |z| < \infty$$

$$(b)\quad X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{3}{1 - \frac{1}{3}z^{-1}} \qquad |z| > \frac{1}{2}$$

$$(c)\quad X(z) = \frac{1}{1 + 3z^{-1} + 2z^{-2}} \qquad |z| > 2$$

$$(d)\quad X(z) = \frac{1}{(1 - z^{-1})(1 - z^{-2})} \qquad |z| > 1$$

Solution:

a) Because $X(z)$ is a finite-order polynomial, $x(n)$ is a finite-length sequence. Therefore, $x(n)$ is the coefficient that multiplies z^{-1} in $X(z)$. Thus, $x(0) = 4$ and $x(2) = x(-2) = 3$.

b) This *z*-transform is a sum of two first-order rational functions of *z*. Because the region of convergence of $X(z)$ is the exterior of a circle, $x(n)$ is a right-sided sequence. Using the *z*-transform pair for a right-sided exponential, we may invert $X(z)$ easily as follows:

$$x(n) = \left(\tfrac{1}{2}\right)^n u(n) + 3\left(\tfrac{1}{3}\right)^n u(n)$$

c) Here we have a rational function of *z* with a denominator that is a quadratic in *z*. Before we can find the inverse z-transform, we need to factor the denominator and perform a partial fraction expansion:

$$X(z) = \frac{1}{1 + 3z^{-1} + 2z^{-2}} = \frac{1}{(1 + 2z^{-1})(1 + z^{-1})}$$

$$= \frac{2}{1 + 2z^{-1}} - \frac{1}{1 + z^{-1}}$$

Because $x(n)$ is right-sided, the inverse *z*-transform is

$$x(n) = 2(-2)^n u(n) - (-1)^n u(n)$$

d) One way to invert this *z*-transform is to perform a partial fraction expansion. With

$$X(z) = \frac{1}{(1-z^{-1})(1-z^{-2})} = \frac{1}{(1-z^{-1})^2(1+z^{-1})}$$
$$= \frac{A}{1+z^{-1}} + \frac{B_1}{1-z^{-1}} + \frac{B_2}{(1-z^{-1})^2}$$

the constants A, B_1, and B_2 are as follows:

$$A = [(1+z^{-1})X(z)]_{z=-1} = \tfrac{1}{4}$$
$$B_1 = \left[\frac{d}{dz}(1-z^{-1})^2 X(z)\right]_{z=1} = \left[\frac{z^{-2}}{(1+z^{-1})^2}\right]_{z=1} = \tfrac{1}{4}$$
$$B_2 = [(1-z^{-1})^2 X(z)]_{z=1} = \tfrac{1}{2}$$

Inverse transforming each term, we have

$$x(n) = \tfrac{1}{4}[(-1)^n + 1 + 2(n+1)]u(n)$$

Example 7:

Find the inverse z-transform of the second-order system

$$X(z) = \frac{1+\frac{1}{4}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2} \qquad |z| > \tfrac{1}{2}$$

Here we have a second-order pole at $z = ½$. The partial fraction expansion for $X(z)$ is

$$X(z) = \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{\left(1-\frac{1}{2}z^{-1}\right)^2}$$

The constant A_1 is

$$A_1 = \tfrac{1}{2}\left[\frac{d}{dz}\left(1-\tfrac{1}{2}z^{-1}\right)^2 X(z)\right]_{z=1/2} = \tfrac{1}{2}\left[-\tfrac{1}{4}z^{-2}\right]_{z=1/2} = -\tfrac{1}{2}$$

and the constant A_2 is

$$A_2 = \left[\left(1-\tfrac{1}{2}z^{-1}\right)^2 X(z)\right]_{z=1/2} = \tfrac{3}{2}$$

Therefore,

$$X(z) = -\frac{\frac{1}{2}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{3}{2}}{\left(1-\frac{1}{2}z^{-1}\right)^2}$$

and

$$x(n) = -\left(\tfrac{1}{2}\right)^{n+1}u(n) + 3(n+1)\left(\tfrac{1}{2}\right)^{n+1}u(n)$$

Example 8:

Find the inverse *z*-transform of $X(z) = \sin z$.

Solution

To find the inverse *z*-transform of $X(z) = \sin z$, we expand $X(z)$ in a Taylor series about $z = 0$ as follows:

$$X(z) = X(z)\Big|_{z=0} + z\frac{dX(z)}{dz}\Big|_{z=0} + \frac{z^2}{2!}\frac{d^2X(z)}{dz^2}\Big|_{z=0} + \cdots + \frac{z^n}{n!}\frac{d^nX(z)}{dz^n}\Big|_{z=0} + \cdots$$

$$= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}$$

Because

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

we may associate the coefficients in the Taylor series expansion with the sequence values $x(n)$. Thus, we have

$$x(n) = (-1)^n\frac{1}{(2|n|+1)!} \qquad n = -1, -3, -5, \ldots$$

Example 8:

Evaluate the following integral:

$$\frac{1}{2\pi j}\oint_C \frac{1+2z^{-1}-z^{-2}}{(1-\frac{1}{2}z^{-1})(1-\frac{2}{3}z^{-1})}z^3 dz$$

where the contour of integration C is the unit circle.

Solution:

Recall that for a sequence $x(n)$ that has a z-transform $X(z)$, the sequence may be recovered using contour integration as follows:

$$x(n) = \frac{1}{2\pi j}\oint_c X(z)z^{n-1}dz$$

Therefore, the integral that is to be evaluated corresponds to the value of the sequence $x(n)$ at $n = 4$ that has a z-transform

$$X(z) = \frac{1+2z^{-1}-z^{-2}}{(1-\frac{1}{2}z^{-1})(1-\frac{2}{3}z^{-1})}$$

Thus, we may find $x(n)$ using a partial fraction expansion of $X(z)$ and then evaluate the sequence at $n = 4$. With this approach, however, we are finding the values of $x(n)$ for all n. Alternatively, we could perform long division and divide the numerator of $X(z)$ by the denominator. The coefficient multiplying z^{-4} would then be the value of $x(n)$ at $n = 4$, and the value of the integral. However, because we are only interested in the value of the sequence at $n = 4$, the easiest approach is to evaluate the integral directly using the Cauchy integral theorem. The value of the integral is equal to the sum of the residues of the poles of $X(z)z^3$ inside the unit circle. Because

$$X(z)z^3 = z^3\frac{z^2+2z-1}{(z-\frac{1}{2})(z-\frac{2}{3})}$$

has poles at z =1/2 and z =2/3,

$$\text{Res}[X(z)z^3]_{z=\frac{1}{2}} = \left[z^3\frac{z^2+2z-1}{z-\frac{2}{3}}\right]_{z=\frac{1}{2}} = -\frac{3}{16}$$

and