

VECTOR ANALYSIS**INTRODUCTION:**

Electromagnetics is a branch of Physics (or) electrical engineering which is used to study the electric and magnetic Phenomena.

What is a field?

Consider a magnet. It has its own effect in a region surrounding it. The effect can be placed by placing another magnet near the first magnet. Such an effect can be defined by a particular physical function.

In the region surrounding the magnet, there exists a particular value for that physical function, at every point, describing the effect of magnet.

So field can be defined as the region in which, at each point there exists a corresponding value of some physical function.

If the field is produced is due to magnetic effects, it is called MAGNETIC FIELD.

There are two types of electric charges, Positive and negative. Such an electric charge produces a field around it which is called an ELECTRIC FIELD.

Moving charges produces current and current carrying conductor produces a magnetic field. In such case electric and magnetic fields are related to each other. Such a field is called ELECTROMAGNETIC FIELD. Such fields may be time varying or time independent.

It is seen that distribution of a quantity in a space is defined by a field. Hence to quantify the field, three dimensional representation plays an important ~~role~~ role. Such three dimensional representation can be made easy by the use of vector analysis.

SCALARS & VECTORS:

The various quantities involved in the study of engineering electromagnetics can be classified as

1. scalars &
2. vectors.

SCALAR:

The scalar is a quantity whose value may be represented by a single real number, which may be +ve (or) -ve. The direction is not at all required in describing a scalar. Thus

A scalar is a quantity which is wholly characterized by its magnitude.

eg: temperature, mass, volume, density, speed, electric charge etc.

VECTOR:

A quantity which has both, a magnitude and a specific direction in space is called a vector.

In electromagnetics vectors defined in two and three dimensional spaces are required but vectors may be defined in n-dimensional space.

A vector is a quantity which is characterized by both, a magnitude and a direction.

eg: force, velocity, displacement, electric field intensity, magnetic field intensity, acceleration etc. (2)

VECTOR FIELD**SCALAR FIELD:**

The distribution of a scalar quantity with a definite position in a space is called SCALAR FIELD.

eg: 1. Temperature of atmosphere.

(It has a definite value in the atmosphere but no need of direction to specify).

2. Height of surface of earth above sea level
3. Sound intensity in an auditorium.
4. Light intensity in a room
5. Atmospheric pressure in a given region etc.

VECTOR FIELD:

If a quantity which is specified in a region to define a field is a vector then the corresponding field is called a vector field.

eg: 1. Gravitational force on a mass in a space is a vector field. [This force has a value at various points in a space and always has a specific direction].

2. Velocity of particles in a moving fluid
3. Wind velocity of atmosphere
4. Voltage gradient in a cable
5. Displacement of a flying bird in a space.
6. Magnetic field existing from north to south poles.

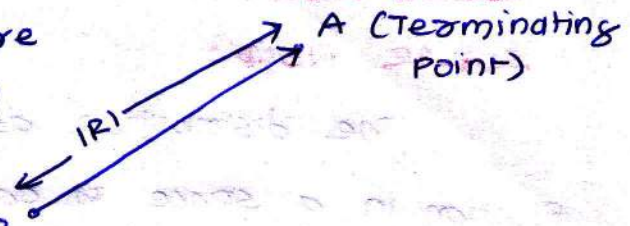
REPRESENTATION OF A VECTOR:

In two dimensional, a vector can be represented by a straight line with an arrow in a plane. The length of the ~~vector~~ segment is the magnitude of a vector while the

The vector shown in figure is symbolically denoted as $\vec{OA}$. Its length is called as magnitude, which is R for the vector OA .

(Starting Point)

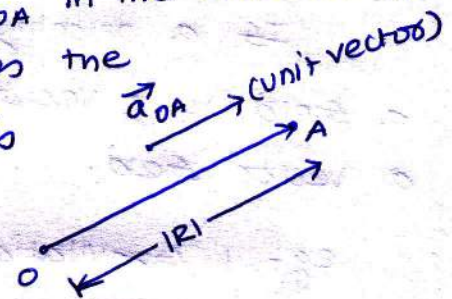
It is represented as $|\vec{OA}| = R$



UNIT VECTOR:

A unit vector has a function to indicate the direction. Its magnitude is always unity, irrespective of its direction. Thus for any vector, to indicate its direction a unit vector can be used.

Consider a unit vector $\vec{a}_{OA}$ in the direction of $\vec{OA}$ as shown in fig. This indicates the direction of $\vec{OA}$ but its magnitude is unity.



So vector $\vec{OA}$ can be represented completely as its magnitude R and the direction as indicated by the unit vector along its direction.

$$\vec{OA} = |\vec{OA}| \vec{a}_{OA} = R \vec{a}_{OA}$$



letter $\vec{a}$ is used to indicate the unit vector

unit vector along the direction OA and $|\vec{a}_{OA}| = 1$

2 MARK QUESTION:

1. Mention the purpose of unit vector in vector algebra.

In case if a vector is known then the unit vector along that vector can be obtained by dividing the vector by its magnitude. Thus unit vector can be expressed as,

$$\text{Unit Vector } \vec{a}_{OA} = \frac{\vec{OA}}{|\vec{OA}|}$$

2 mark question.

1. Express unit vector in terms of a vector and its magnitude.

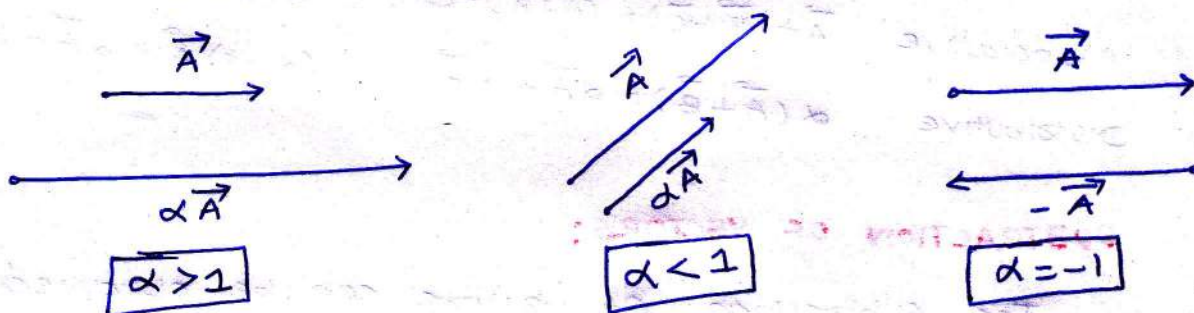
VECTOR ALGEBRA: [Scaling, Addition, subtraction].

SCALING OF VECTOR

→ This is multiplication by a scalar to a vector

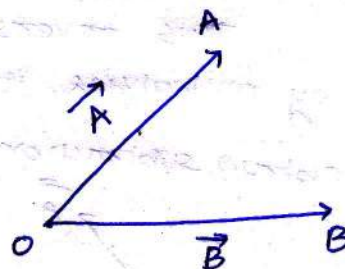
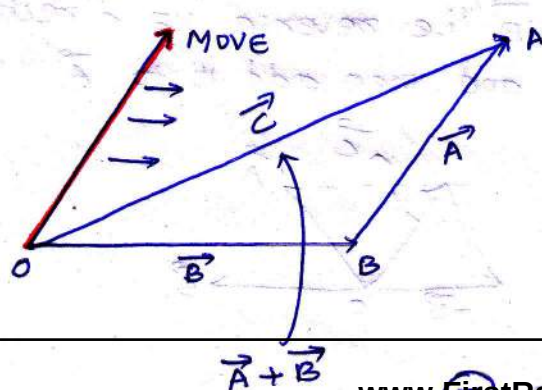
→ This changes the magnitude (length) of a vector but not its direction, when scalar is positive

→ when scalar = -1, the magnitude remains same but direction of the vector reverses.

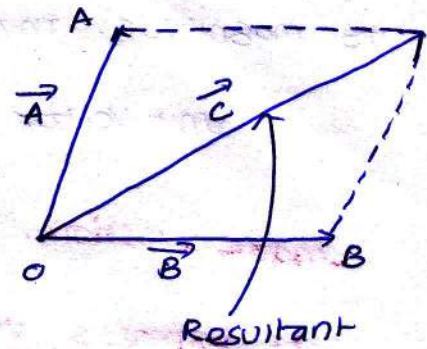


ADDITION OF VECTORS:

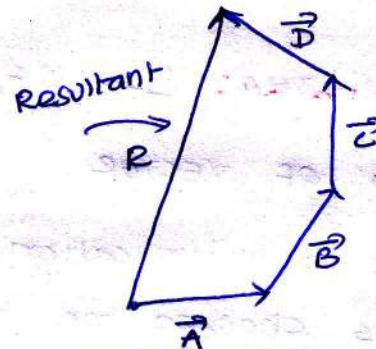
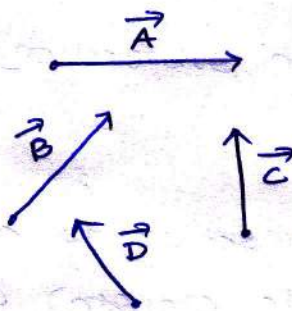
→ The vectors which lie on the same plane are called coplanar vectors.



Complete The Parallelogram as shown in fig. Then the diagonal of the parallelogram represents the addition of the two vectors.



HEAD TO TRIAL RULE:



Law
commutative
Associative
Distributive

Addition

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

$$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$$

$$\alpha(\vec{A} + \vec{B}) = \alpha\vec{A} + \alpha\vec{B}$$

Multiplication by scalar

$$\alpha\vec{A} = \vec{A}\alpha$$

$$B(\alpha\vec{A}) = (\alpha B)\vec{A}$$

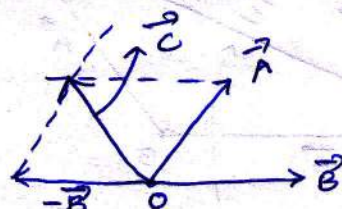
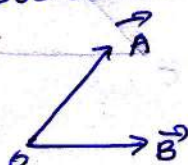
$$(\alpha + \beta)\vec{A} = \alpha\vec{A} + \beta\vec{A}$$

SUBTRACTION OF VECTORS:

The subtraction of vectors can be obtained from the rules of addition. If $\vec{B}$ is to be subtracted from $\vec{A}$ then based on addition it can be represented as

$$\vec{C} = \vec{A} + (-\vec{B})$$

thus reverse the sign $\vec{B}$ i.e reverse its direction by multiplying it with (-1) . and then add it to $\vec{A}$ to obtain subtraction.



Two vectors are said to be identical if their difference is zero.

eg: $\vec{A} - \vec{B} = \vec{0}$
 $\Rightarrow \vec{A} = \vec{B}$ $\vec{A}$ & $\vec{B}$ are identical.

VECTOR MULTIPLICATION:

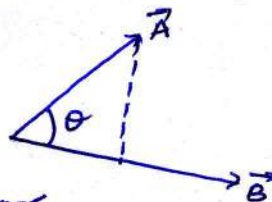
consider two vectors $\vec{A}$ and $\vec{B}$. There are two types of products existing depending upon the result of the multiplication. These two types of products are

1. Scalar (or) DOT Product
2. Vector (or) CROSS Product

SCALAR (OR) DOT PRODUCT OF VECTORS:

- It is denoted by $\vec{A} \cdot \vec{B}$
- It is defined as the product of the magnitude of $\vec{A}$, the magnitude of $\vec{B}$ and the cosine of smaller angle b/w them.
- It also can be defined as the product of magnitude of $\vec{B}$ and the projection of $\vec{A}$ onto $\vec{B}$ or vice versa

$$\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta_{AB}$$



The result of such a dot product is scalar hence it is also called as scalar product.

PROPERTIES OF DOT PRODUCT

1. If the two vectors are $\parallel$ to each other i.e. $\theta = 0$ then $\cos \theta_{AB} = 1$ thus $\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}|$ for $\parallel$ vectors.
2. If two vectors are $\perp$ to each other i.e. $\theta = 90^\circ$ then $\cos \theta_{AB} = 0$ thus $\vec{A} \cdot \vec{B} = 0$ for $\perp$ vectors.

3. If the dot product of vector with itself is performed, the result is square of the magnitude of that vector.

$$\vec{A} \cdot \vec{A} = |\vec{A}| |\vec{A}| \cos 0^\circ = |\vec{A}|^2$$

4. Any unit vector dotted with itself is unity.

$$\vec{a}_x \cdot \vec{a}_x = 1 = \vec{a}_y \cdot \vec{a}_y = \vec{a}_z \cdot \vec{a}_z$$

5. The dot product obeys commutative, & distributive law

$$(ie) \vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$$

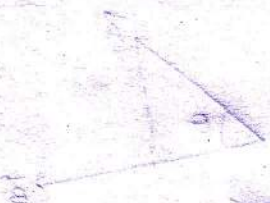
2 mark question.

1. Given two vectors, How to identify whether they are $\perp$ or $\parallel$ to each other.

APPLICATION OF DOT PRODUCT:

1. To determine the angle b/w the two vectors.

$$\theta = \cos^{-1} \left\{ \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| |\vec{B}|} \right\}$$



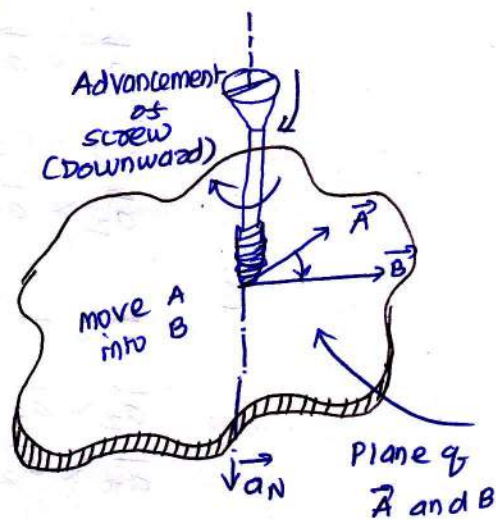
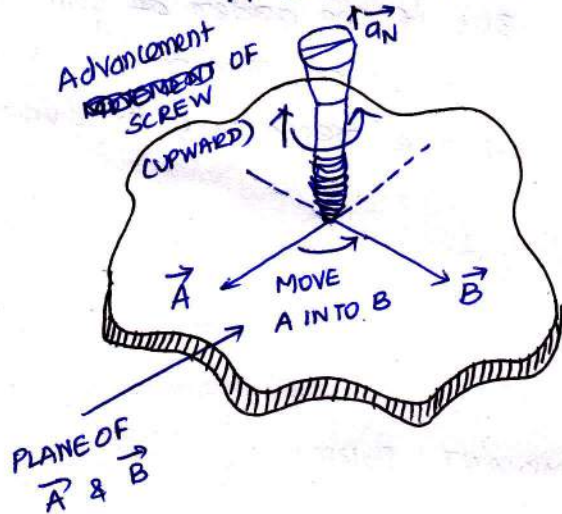
$$\theta = \cos^{-1} \left\{ \frac{\vec{A} \cdot \vec{A}}{|\vec{A}| |\vec{A}|} \right\} = \cos^{-1} \left\{ \frac{\vec{A} \cdot \vec{A}}{|\vec{A}|^2} \right\}$$

consider two vectors $\vec{A}$ & $\vec{B}$ then their cross product is denoted as $\vec{A} \times \vec{B}$ and defined as the product of the magnitudes of $\vec{A}$ & $\vec{B}$ and the sine of the smaller angle between $\vec{A}$ and $\vec{B}$.

CROSS PRODUCT is a vector quantity and has a direction $\perp$ to the plane, containing the two vectors $\vec{A}$ and $\vec{B}$.

Mathematically cross product is expressed as

$$\vec{A} \times \vec{B} = |\vec{A}| |\vec{B}| \sin \theta_{AB} \vec{a}_N$$



PROPERTIES OF CROSS PRODUCT:

1. The commutative law is not applicable to the cross product thus

$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$$

2. Reversing the order of the vectors $\vec{A}$ and $\vec{B}$, a unit vector $\vec{a}_N$ reverses its direction hence we can write

$$\vec{A} \times \vec{B} = -[\vec{B} \times \vec{A}] \quad \text{anticommutative}$$

3. The cross product is not associative, thus

$$\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$$

4. With respect to addition cross product is distributive, thus

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

5. If two vectors are $\parallel$ to each other, (ie) they are in same direction then $\theta = 0^\circ$ & hence cross product of such two vectors is zero.

7. CROSS PRODUCT OF UNIT VECTORS.

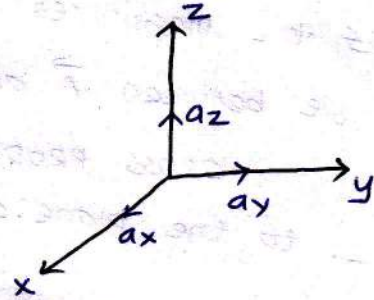
Consider the unit vectors $\vec{a}_x$, $\vec{a}_y$ and $\vec{a}_z$ which are mutually $\perp$ to each other as shown in fig.

Then

$$\vec{a}_x \times \vec{a}_y = |\vec{a}_x| |\vec{a}_y| \sin(90^\circ) \vec{a}_N$$

In this case $\vec{a}_N = \vec{a}_z$

and $|\vec{a}_x| = |\vec{a}_y| = \sin(90^\circ) = 1$



$$\begin{aligned} \therefore \vec{a}_x \times \vec{a}_y &= \vec{a}_z \\ \vec{a}_y \times \vec{a}_z &= \vec{a}_x \\ \vec{a}_z \times \vec{a}_x &= \vec{a}_y \end{aligned}$$

But if the order of unit vectors is reversed, the result is -ve of the remaining third unit vector

Thus

$$\begin{aligned} \vec{a}_y \times \vec{a}_x &= -\vec{a}_z \\ \vec{a}_z \times \vec{a}_y &= -\vec{a}_x \\ \vec{a}_x \times \vec{a}_z &= -\vec{a}_y \end{aligned}$$

CROSS PRODUCT IN DETERMINANT FORM:

Consider two vectors

$$\vec{A} = A_x \vec{a}_x + A_y \vec{a}_y + A_z \vec{a}_z$$

$$\vec{B} = B_x \vec{a}_x + B_y \vec{a}_y + B_z \vec{a}_z$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\vec{A} \times \vec{B} = [A_y B_z - B_y A_z] \vec{a}_x + [A_z B_x - A_x B_z] \vec{a}_y + [A_x B_y - A_y B_x] \vec{a}_z$$

Let $\vec{A}$, $\vec{B}$ and $\vec{C}$ are the three given vectors. Then the product of these three vectors is classified into two ways called,

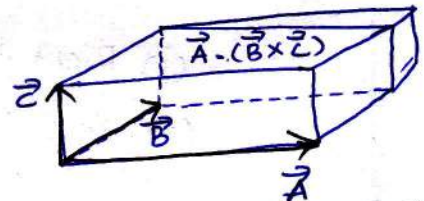
1. Scalar triple Product
2. Vector triple Product

SCALAR TRIPLE PRODUCT: (Scalar triple Product of 3 vectors $\vec{A}$, $\vec{B}$ & $\vec{C}$)

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

1. It represents the volume of parallelepiped.



3. cyclic order a, b, c is to be followed. If the order is changed, the sign is reversed.

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = -\vec{B} \cdot (\vec{A} \times \vec{C})$$

2. If two (or) three vectors are equal then the result of the scalar triple product is zero.

VECTOR TRIPLE PRODUCT:

The vector triple product of the three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ is mathematically defined as

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

"bac-cab" RULE

Problem

1. Three fields are given by $\vec{A} = 2\vec{a}_x - \vec{a}_z$, $\vec{B} = 2\vec{a}_x - \vec{a}_y + 2\vec{a}_z$
 $\vec{C} = 2\vec{a}_x - 3\vec{a}_y + \vec{a}_z$

Find the scalar and vector triple product.

Scalar triple Product

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} 2 & 0 & -1 \\ 2 & -1 & 2 \\ 2 & -3 & 1 \end{vmatrix}$$

$$= 14$$

Vector triple product

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

$$\vec{A} \cdot \vec{C} = (2)(2) + (0)(-3) + (-1)(1) = 3$$

$$\vec{A} \cdot \vec{B} = (2)(2) + (0)(-1) + (-1)(2) = 2$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = 3\vec{B} - 2\vec{C}$$

$$= 3[2\vec{a}_x - \vec{a}_y + 2\vec{a}_z] - 2[2\vec{a}_x - 3\vec{a}_y + \vec{a}_z]$$

$$= 2\vec{a}_x + 3\vec{a}_y + 4\vec{a}_z$$

Three types of co-ordinate systems are

- (i) Cartesian (or) Rectangular co-ordinate system
- (ii) cylindrical co-ordinate system
- (iii) spherical co-ordinate system.

CARTESIAN CO-ORDINATE SYSTEM:

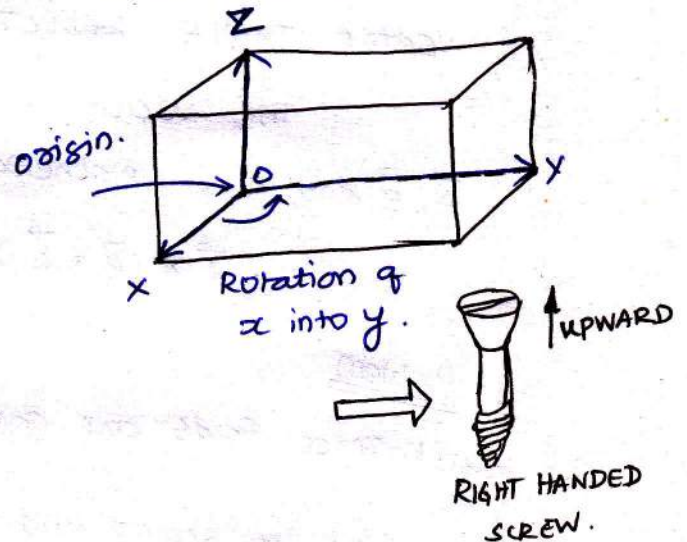
- Also called Rectangular co-ordinate system
- Three co-ordinates x, y, z mutually $\perp$ to each other.
- Intersection of x, y, z is called origin.

There are two types of such systems, they are

- (i) Right handed system
- (ii) Left handed system.

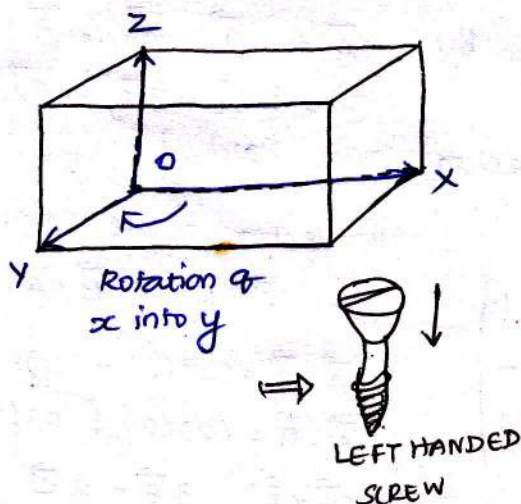
RIGHT HANDED SYSTEM:

If x axis is rotated towards y axis through a smaller angle, thus this rotation causes the upward movement of right handed screw in the z axis direction.



LEFT HANDED SYSTEM:

Downward movement of screw.



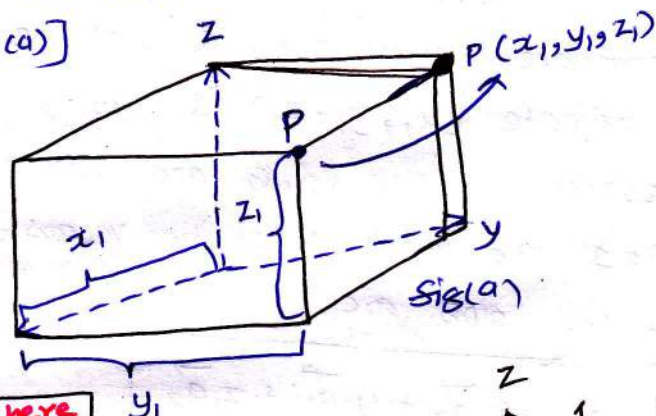
NOTE: RIGHT HANDED SYSTEM IS COMMONLY USED

→ A Point in rectangular co-ordinate system located by three coordinates namely x , y and z co-ordinates.

→ The Point can be reached by moving from origin, The distance x in x direction, Then the distance y in y direction and finally distance z in z direction.

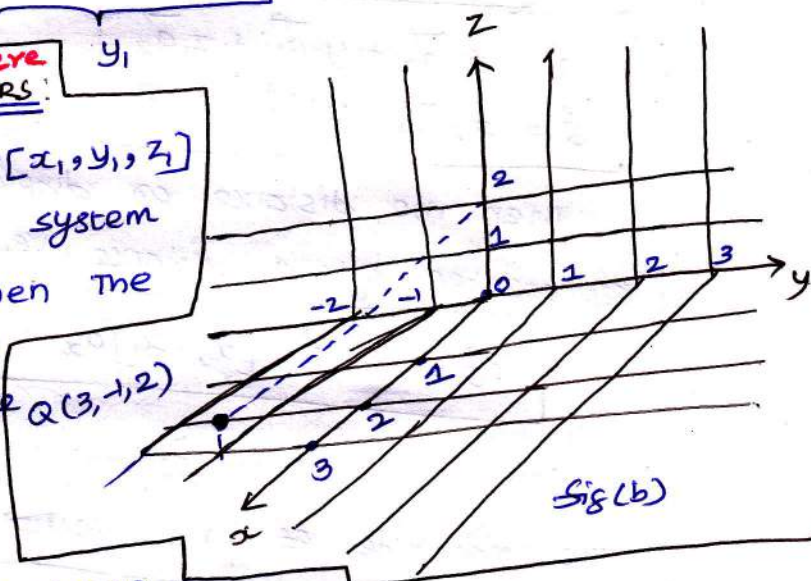
→ Consider a Point P having co-ordinates x_1, y_1 and z_1 . It is represented as $P(x_1, y_1, z_1)$. The co-ordinates x_1, y_1, z_1 may be +ve or -ve [sig(a)]

→ The Point $Q(3, -1, 2)$ can be shown in this system as shown in fig(b)



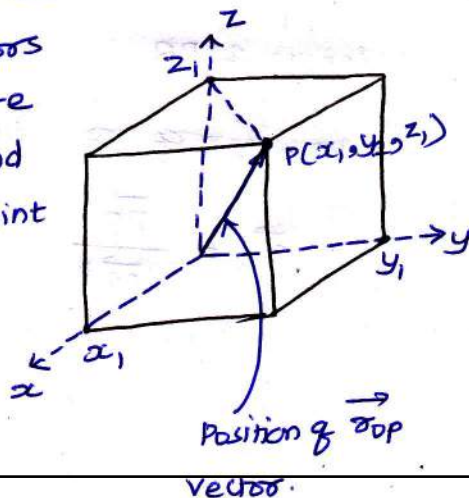
FOLLOW (13A) & (13B) sheets & come here.
POSITION & DISTANCE VECTORS:

Consider a Point $P(x_1, y_1, z_1)$ in cartesian co-ordinate system as shown in fig (c). Then the Position vector of Point 'P' is represented by the distance $Q(3, -1, 2)$ of Point P from the origin directed from origin to point P . This is also called RADIUS VECTOR.



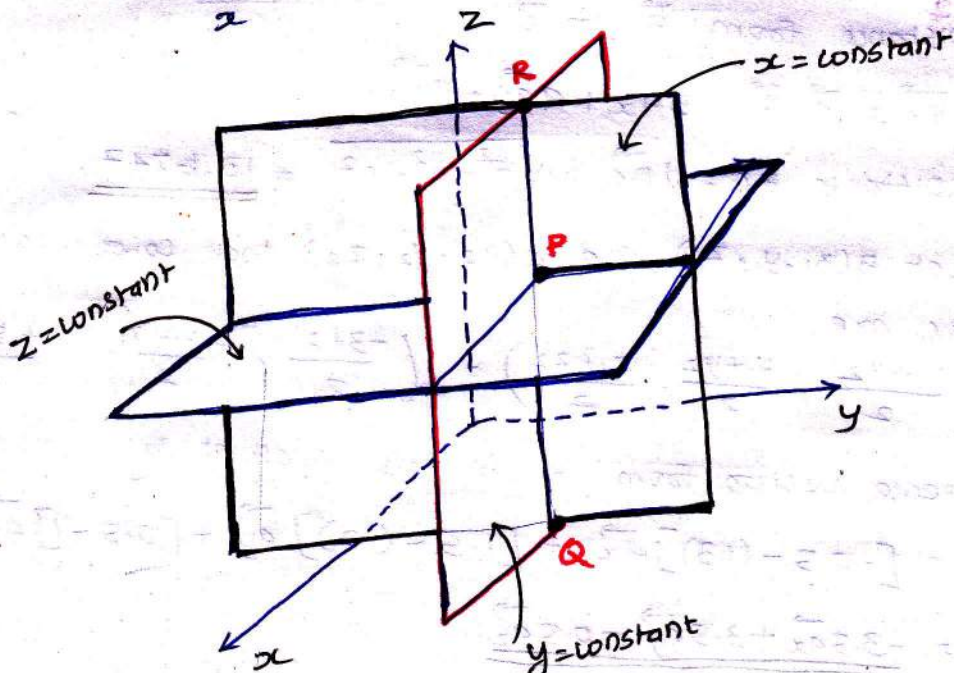
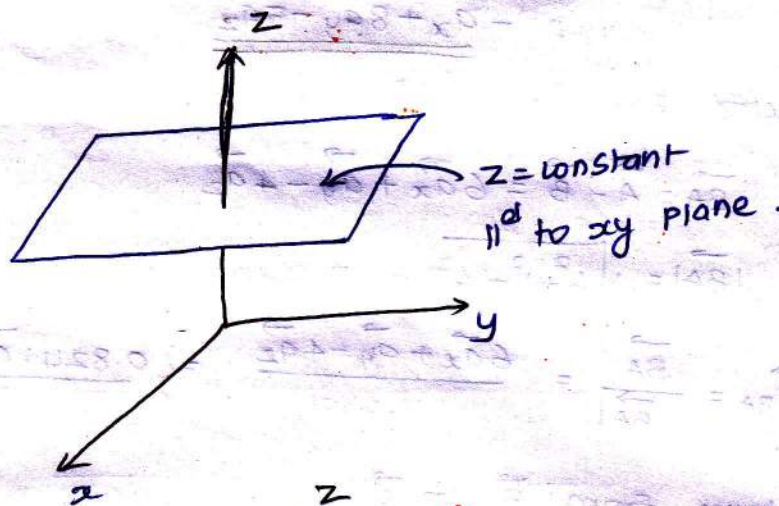
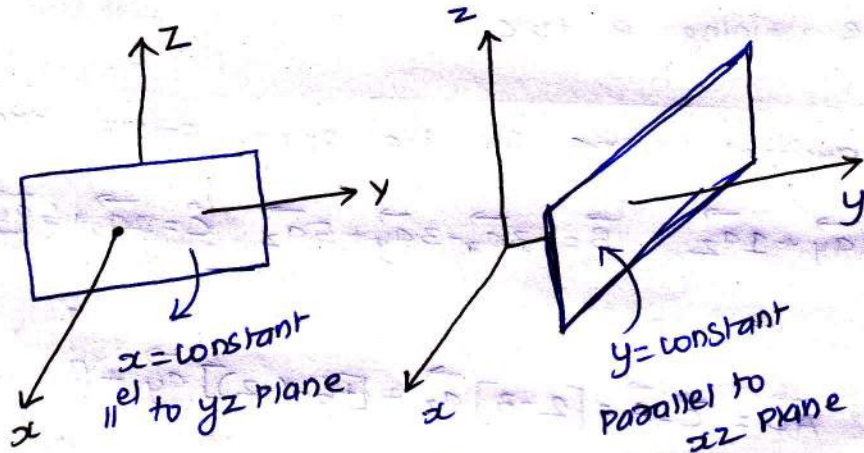
- The three components of the Position vector $\vec{r}_{op}$ are three vectors oriented along the three co-ordinate axes with the magnitudes x_1, y_1 and z_1 . Thus the position vector of Point P can be represented as

$$\vec{r}_{op} = x_1 \vec{a}_x + y_1 \vec{a}_y + z_1 \vec{a}_z$$



Alternate method to consider three surfaces

namely $x = \text{constant}$, $y = \text{constant}$ and $z = \text{constant}$. The common intersection of all these three surfaces is the point to be defined and the constants indicate the co-ordinates of that point. as shown in figures given below.



Given three points in Cartesian co-ordinate system as

$$A(3, -2, 1), B(-3, -3, 5) \text{ and } C(2, 6, -4)$$

Find (i) The vector from A to C

(ii) The unit vector from B to A

(iii) The distance from B to C.

(iv) The vector from A to the midpoint of the straight line joining B to C.

Solution:

The position vectors for the given points are

$$\vec{A} = 3\vec{a}_x - 2\vec{a}_y + 1\vec{a}_z \quad \vec{B} = -3\vec{a}_x - 3\vec{a}_y + 5\vec{a}_z \quad \vec{C} = 2\vec{a}_x + 6\vec{a}_y - 4\vec{a}_z$$

vector

(i) from ~~vector~~ A to C

^

$$\begin{aligned} \vec{AC} &= \vec{C} - \vec{A} = [2-3]\vec{a}_x + [6-(-2)]\vec{a}_y + [-4-(+1)]\vec{a}_z \\ &= -\vec{a}_x + 8\vec{a}_y - 5\vec{a}_z \end{aligned}$$

unit

(ii) Vector from B to A

^

$$\vec{BA} = \vec{A} - \vec{B} = 6\vec{a}_x + \vec{a}_y - 4\vec{a}_z$$

$$|\vec{BA}| = \sqrt{6^2 + 1^2 + 4^2} = 7.2801$$

$$\vec{a}_{BA} = \frac{\vec{BA}}{|\vec{BA}|} = \frac{6\vec{a}_x + \vec{a}_y - 4\vec{a}_z}{7.2801} = 0.8241\vec{a}_x + 0.1373\vec{a}_y - 0.5494\vec{a}_z$$

(iii) Distance from B to C

$$\vec{BC} = \vec{C} - \vec{B} = 5\vec{a}_x + 9\vec{a}_y - 9\vec{a}_z$$

$$\text{Distance } BC = |\vec{BC}| = \sqrt{5^2 + 9^2 + 9^2} = 13.6747$$

(iv) Let $B(x_1, y_1, z_1)$ and $C(x_2, y_2, z_2)$ then co-ordinates of midpoint of BC are

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2} \right) = \left(\frac{-3+2}{2}, \frac{-3+6}{2}, \frac{+5-4}{2} \right) = (-0.5, 1.5, 0.5)$$

Hence vector from A to this midpoint is

$$= [-0.5-(+3)]\vec{a}_x + [1.5-(-2)]\vec{a}_y + [0.5-1]\vec{a}_z$$

$$= -3.5\vec{a}_x + 3.5\vec{a}_y - 0.5\vec{a}_z$$

The magnitude of vectors in terms of three mutually $\perp$ components are given by www.FirstRanker.com www.FirstRanker.com

$$|\vec{r}_{OP}| = \sqrt{(x_1)^2 + (y_1)^2 + (z_1)^2}$$

If point P has co-ordinates (1, 2, 3) then its position vector is

$$\vec{r}_{OP} = 1\vec{a}_x + 2\vec{a}_y + 3\vec{a}_z \quad |\vec{r}_{OP}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14} = 3.7416$$

Now consider two points in a Cartesian coordinate system, P and Q with the co-ordinate (x_1, y_1, z_1) & (x_2, y_2, z_2) respectively. The points are shown in Fig 1. The individual position vectors of the points are

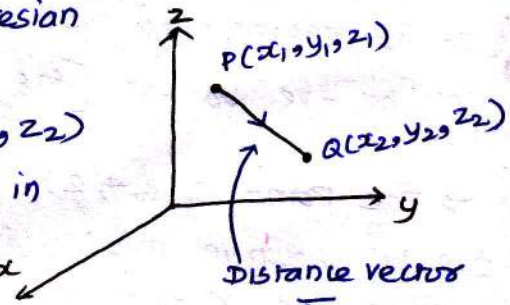


Fig 1

$$\begin{aligned} \vec{P} &= x_1\vec{a}_x + y_1\vec{a}_y + z_1\vec{a}_z \\ \vec{Q} &= x_2\vec{a}_x + y_2\vec{a}_y + z_2\vec{a}_z \end{aligned}$$

Then the distance or displacement from P to Q is represented by a distance vector $\vec{PQ}$ and is given by

$$\vec{PQ} = \vec{Q} - \vec{P} = [x_2 - x_1]\vec{a}_x + [y_2 - y_1]\vec{a}_y + [z_2 - z_1]\vec{a}_z$$

This is also called separation vectors.

The magnitude of this vector is given by

distance formula $\rightarrow$ $|\vec{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
 $\uparrow$
 length of PQ

~~direction of~~ unit vector along direction of PQ is

$$\vec{a}_{PQ} = \frac{\vec{PQ}}{|\vec{PQ}|}$$

PROBLEM:

1. Obtain the unit vector in the direction from the origin towards the point $P(3, -3, 2)$

Solution:

The origin $O(0, 0, 0)$ while $P(3, -3, 2)$ hence the distance vector $\vec{OP}$ is

$$\vec{OP} = (3-0)\vec{a}_x + (-3-0)\vec{a}_y + (2-0)\vec{a}_z$$

$$= 3\vec{a}_x - 3\vec{a}_y + 2\vec{a}_z$$

$$|\vec{OP}| = \sqrt{3^2 + (-3)^2 + 2^2} = 4.6904$$

Hence the unit vector along the direction OP is

$$\vec{a}_{OP} = \frac{\vec{OP}}{|\vec{OP}|} = \frac{3\vec{a}_x - 3\vec{a}_y + 2\vec{a}_z}{4.6904}$$

$$= 0.6396\vec{a}_x - 0.6396\vec{a}_y + 0.4264\vec{a}_z$$

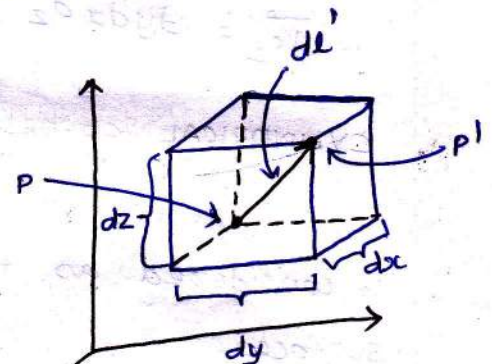
DIFFERENTIAL ELEMENTS IN CARTESIAN CO-ORDINATOR SYSTEM:

Consider a point $P(x, y, z)$ in the rectangular coordinate system. Let us increase each co-ordinate by differential amount. A new point P' will be obtained, having co-ordinates $(x+dx, y+dy, z+dz)$

dx = Differential length in x dir.

dy = Differential length in y dir

dz = Differential length in z dir.



Hence differential vector length

also called elementary vector length can be represented as

$$\vec{dL} = dx\vec{a}_x + dy\vec{a}_y + dz\vec{a}_z$$

→ This is the vector joining P to new point P' .

The distance as P' from P is given by magnitude of the differential vector length.

$$|\vec{dL}| = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

Hence the differential volume of the rectangular parallelepiped is given by,

$$dv = dx dy dz$$

Note: $\vec{d\mathbf{r}}$ is a vector but dv is a scalar.

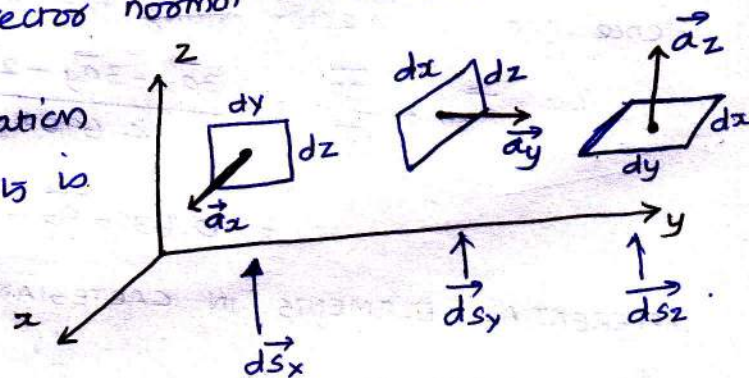
Let us define differential surface areas, the differential surface element $\vec{ds}$ is represented as

$$\vec{ds} = ds \vec{a}_n$$

where ds = Differential surface area of the element.

$\vec{a}_n$ = Unit vector normal to surface ds .

The vector representation of these three elements is given as,



$$\vec{ds}_x = dy dz \vec{a}_x$$

$$\vec{ds}_y = dx dz \vec{a}_y$$

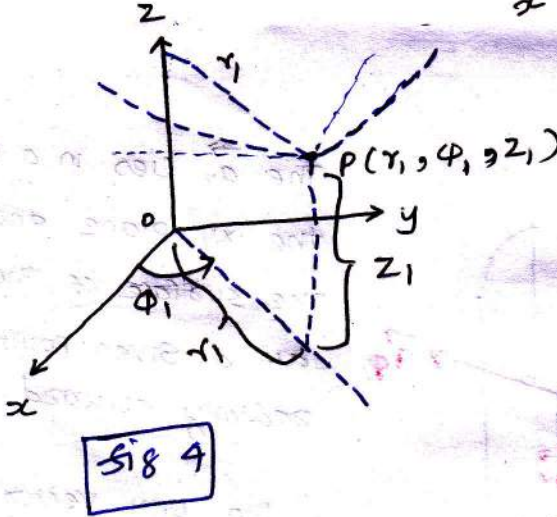
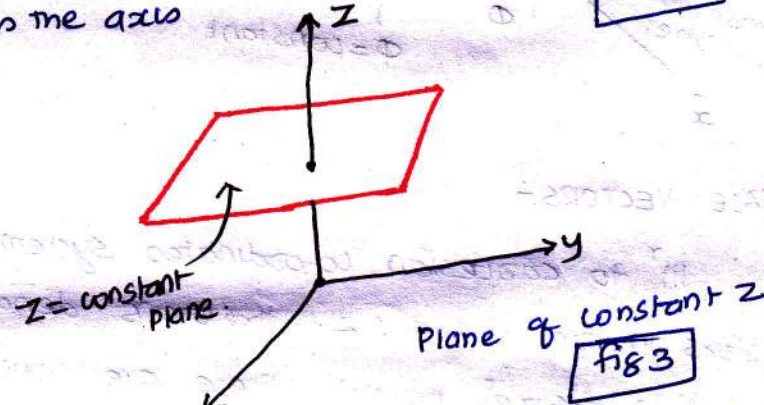
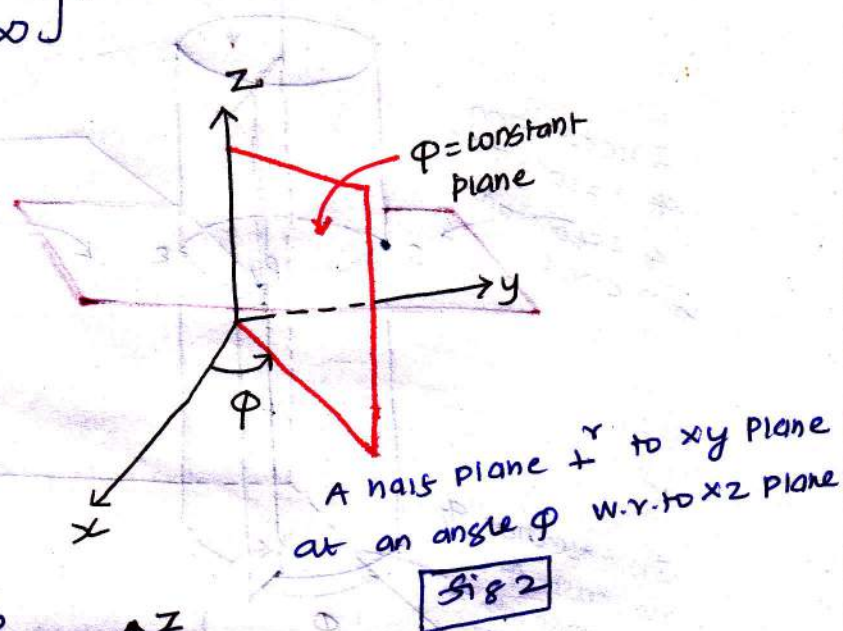
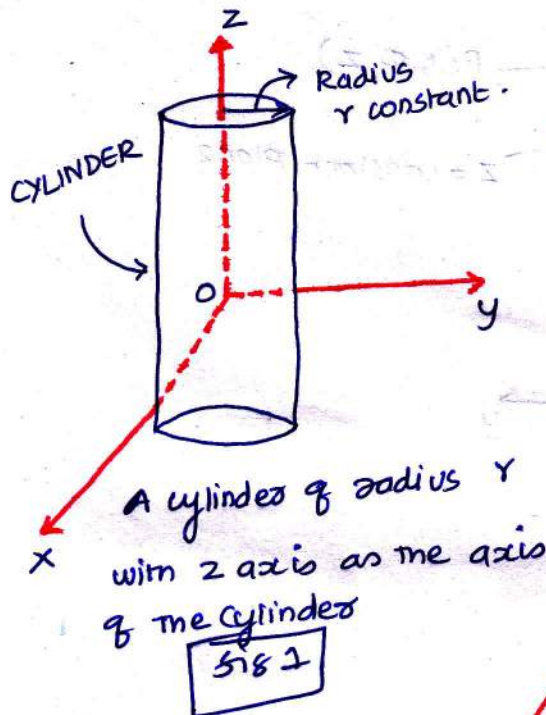
$$\vec{ds}_z = dy dx \vec{a}_z$$

CYLINDRICAL CO-ORDINATE SYSTEM:

In this system of co-ordinates, any point in a space is considered as the point of intersection of the following surfaces.

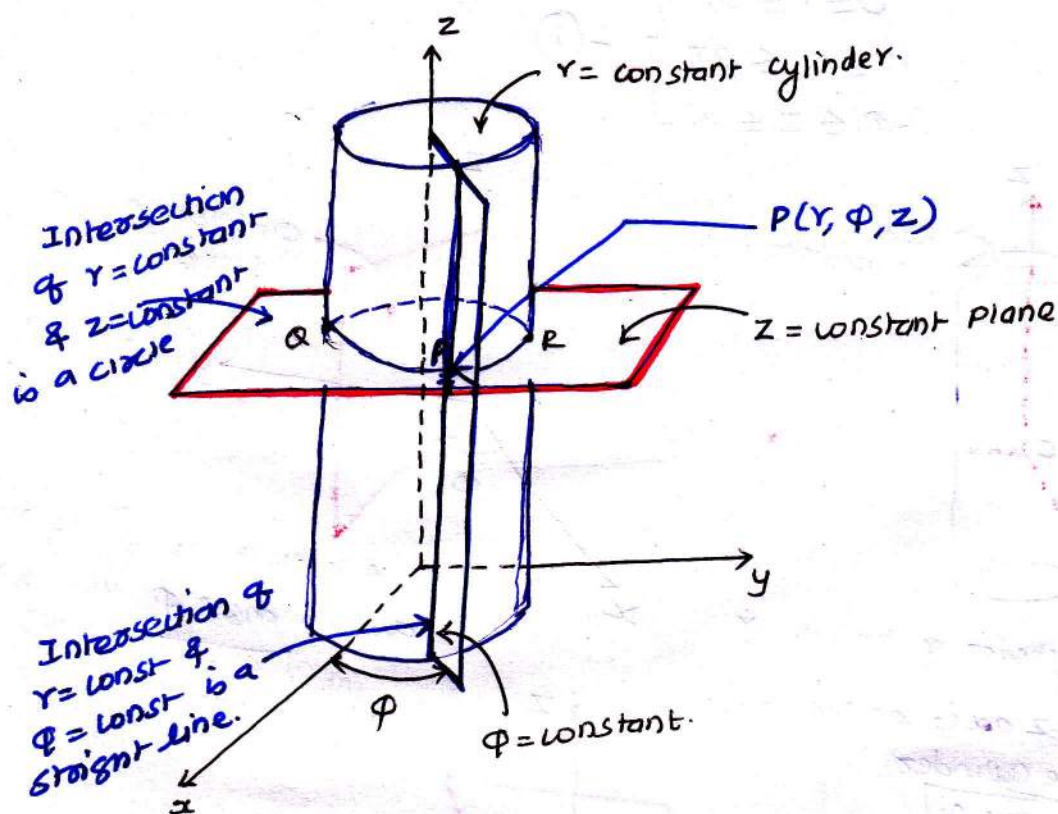
1. Plane of constant z which is $\parallel$ to xy plane
2. A cylinder of radius r with z axis as the axis of the cylinder
3. A half plane $\perp$ to xy plane and at an angle ϕ w.r. to xz plane. The angle ϕ is called azimuthal angle

$$\left. \begin{aligned} 0 \leq r \leq \infty \\ 0 \leq \phi \leq 2\pi \\ -\infty \leq z \leq \infty \end{aligned} \right\} \text{--- (1)}$$



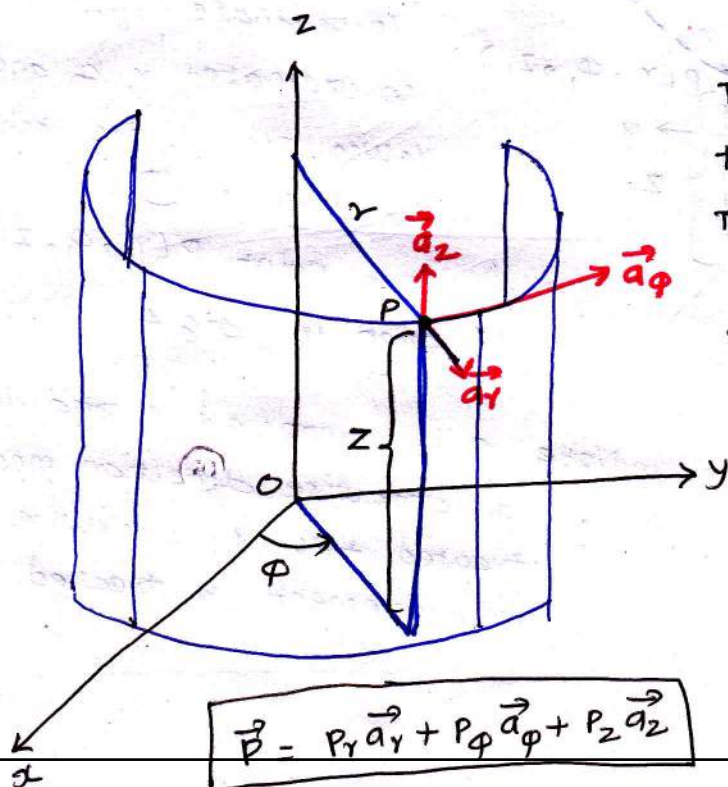
The Point P in cylindrical co-ordinate system has three co-ordinates r , ϕ and z whose values lie in the respective ranges as given in (1).
The point $P(r, \phi, z)$ can be seen in Fig 4

Note: ϕ is expressed in radians and for ϕ , anticlockwise direction measurement is treated +ve & clockwise direction measurement is treated -ve.



BASE VECTORS:-

In r to cartesian co-ordinates system, there are three unit vectors in the r, ϕ and z directions denoted as $\vec{a}_r, \vec{a}_\phi$ and $\vec{a}_z$ as shown in figure below. These are mutually $\perp$ to each other.



The $\vec{a}_r$ lies in a plane $\parallel$ to the xy plane and is $\perp$ to the surface of the cylinder at a given point pointing radially outward.

The unit vector $\vec{a}_\phi$ also lies in a plane $\parallel$ to xy plane but it is tangent to the cylinder, pointing in the direction of increasing ϕ , at the given point.

The unit vector $\vec{a}_z$ is $\parallel$ to z axis and directed towards increasing z .

$$\vec{r} = P_r \vec{a}_r + P_\phi \vec{a}_\phi + P_z \vec{a}_z$$

where P_r is radius r

P_ϕ is angle ϕ

P_z is co-ordinate of point P .

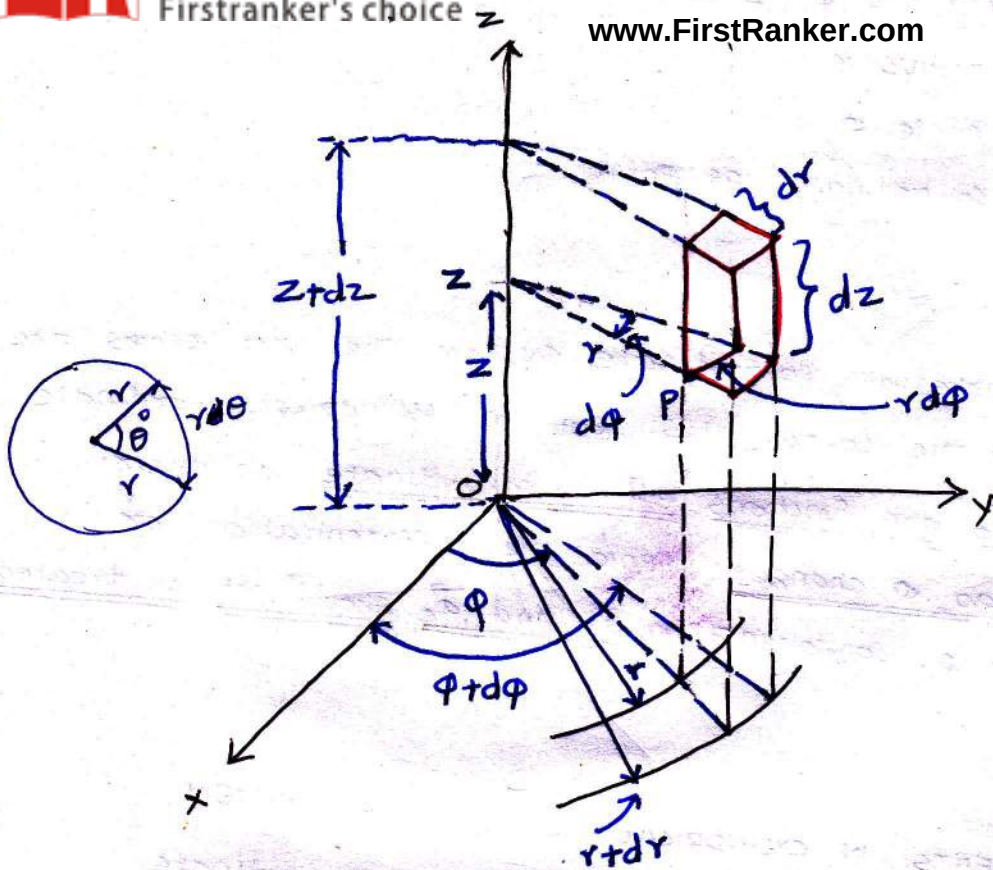
Key note:

In the cartesian co-ordinate system, the unit vectors are not dependent on the co-ordinates. But in cylindrical co-ordinate system $\vec{a}_r$ and $\vec{a}_\phi$ are functions as ϕ co-ordinate as their direction changes as ϕ changes. Hence the differentiation or integration w.r. to ϕ components in $\vec{a}_r$ and $\vec{a}_\phi$ should not be treated as constants.

DIFFERENTIAL ELEMENTS IN CYLINDRICAL CO-ORDINATE SYSTEM:

consider a point $P(r, \phi, z)$ in a cylindrical co-ordinate system. Let each co-ordinate is increased by the differential amount. The differential increments in r, ϕ, z are $dr, d\phi$ and dz respectively.

- Now there are two cylinders of radius r and $r+dr$
- There are two radial planes at the angles ϕ and $\phi+d\phi$
- Two horizontal planes at the heights z and $z+dz$.
- Differential lengths in r and z directions are dr and dz respectively.
- In ϕ direction, $d\phi$ is the change in angle ϕ and is not the differential length.
- Due to this change $d\phi$, there exists a differential arc length in ϕ direction. This differential length, due to $d\phi$, in ϕ direction is $r d\phi$ as shown in fig.



Hence the differential vector length in cylindrical co-ordinate system is given by,

$$d\vec{l} = dr\vec{a}_r + r d\phi \vec{a}_\phi + dz\vec{a}_z$$

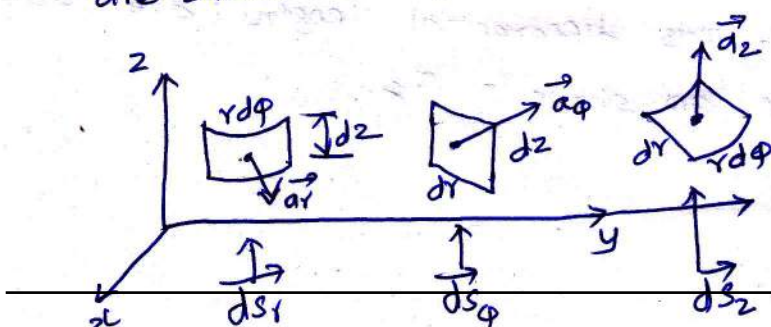
The magnitude of the differential length vector is given by,

$$|d\vec{l}| = \sqrt{(dr)^2 + (r d\phi)^2 + (dz)^2}$$

The differential volume of the differential element formed is given by,

$$dv = dr \times r d\phi \times dz$$

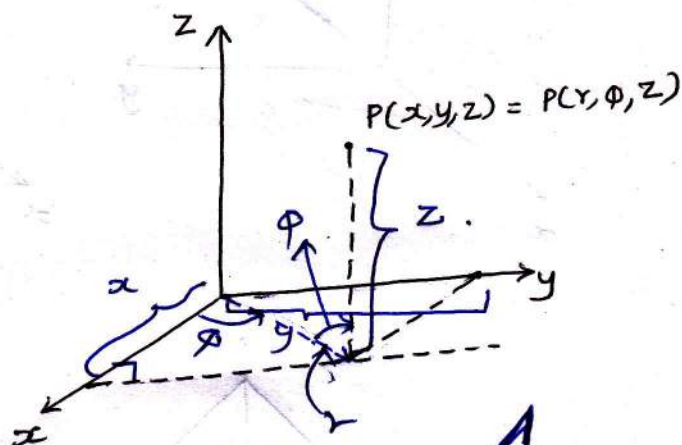
The differential surface areas in the three directions are shown in fig below.



$$\begin{aligned} d\vec{s}_r &= r d\phi dz \vec{a}_r \\ d\vec{s}_\phi &= dr dz \vec{a}_\phi \\ d\vec{s}_z &= r d\phi dr \vec{a}_z \end{aligned}$$

Consider a point 'P' whose cartesian co-ordinates are x, y and z while the cylindrical co-ordinates are r, ϕ and z as shown in fig below.

$$\begin{aligned} x &= r \cos \phi \\ y &= r \sin \phi \\ z &= z \end{aligned}$$

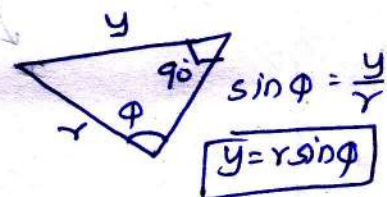
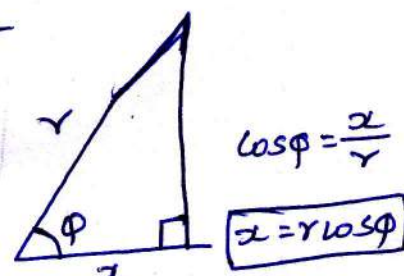


It can be seen that, r can be expressed in terms of x and y as.

$$r = \sqrt{x^2 + y^2}$$

while $\tan \phi = \frac{y}{x}$

From The
Planes



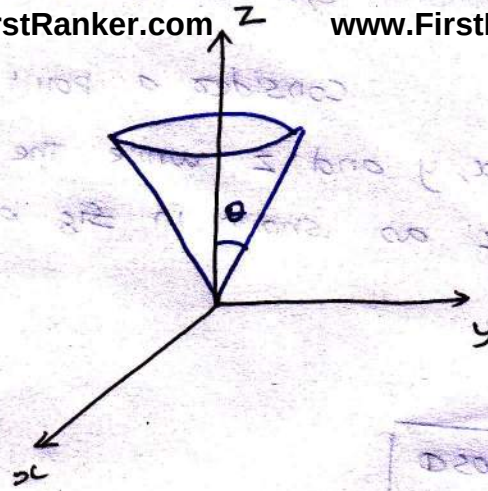
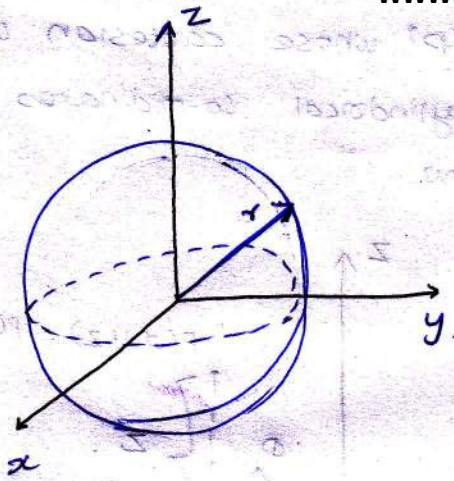
Thus the transformation from cartesian to cylindrical can be obtained from the equations

$$r = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1} \frac{y}{x} \quad \text{and} \quad z = z$$

SPHERICAL CO-ORDINATE SYSTEM:

The surfaces which are used to define the spherical co-ordinate system on the three cartesian axes are

- (i) Sphere of radius r , origin as the centre of the sphere
- (ii) A right circular cone with its apex at the origin and its axis as z axis. Its half angle is θ . It rotates about z axis and θ varies from 0° to 180°
- (iii) A half plane $\perp$ to xy plane containing z axis, making an angle ϕ with the xz plane.



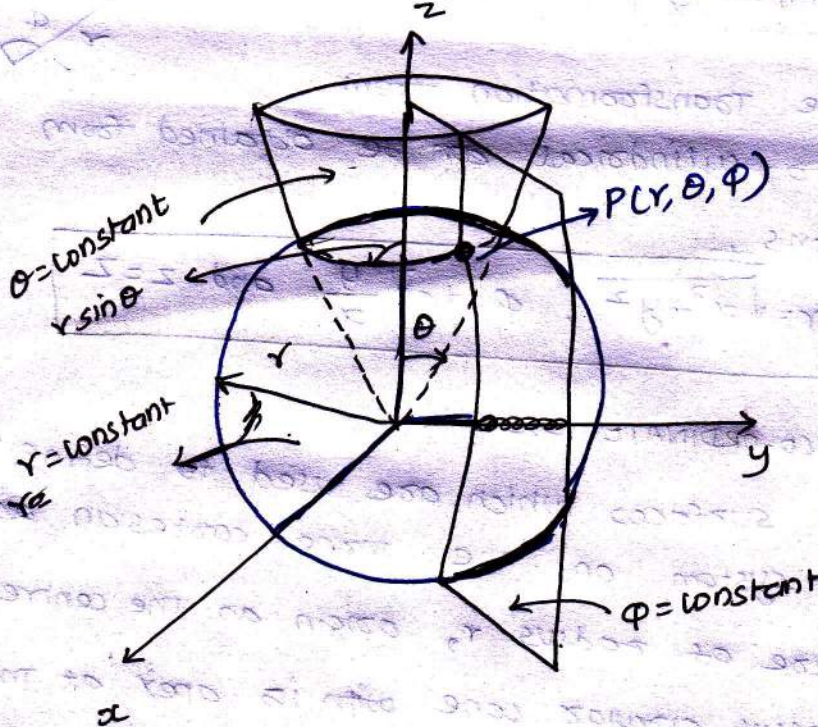
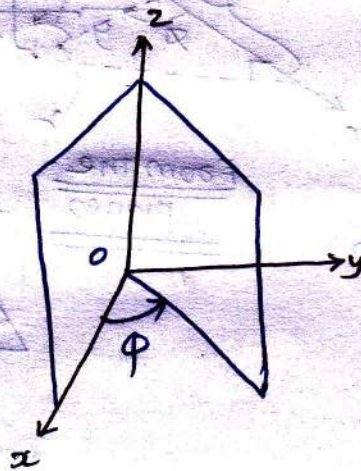
$$\begin{aligned} x &= r \sin \theta \cos \phi \\ y &= r \sin \theta \sin \phi \\ z &= r \cos \theta \end{aligned}$$

The range of the variables are

$$0 \leq r < \infty$$

$$0 \leq \phi < 2\pi$$

$$0 \leq \theta \leq \pi \text{ as } \frac{1}{2} \text{ angle.}$$



BASE VECTORS :

→ The unit vector $\vec{a}_r$ is directed from the centre of the sphere i.e. origin to the given point P. It is directed radially outward, normal to the sphere.

It lies in the cone $\theta = \text{const}$ and plane $\phi = \text{constant}$.

→ The unit vector $\vec{a}_\theta$ is tangent to the sphere and oriented in the direction of increasing θ . It is normal to the conical surface.

→ The 3rd unit vector $\vec{a}_\phi$ is tangent to the sphere and also tangent to the conical surface. It is oriented in the direction of increasing ϕ . It is same as defined in the cylindrical co-ordinate system.

Here the vector of point 'P' can be represented as

$$\vec{P} = r_r \vec{a}_r + r_\theta \vec{a}_\theta + r_\phi \vec{a}_\phi$$

DIFFERENTIAL ELEMENTS IN SPHERICAL CO-ORDINATE SYSTEMS:

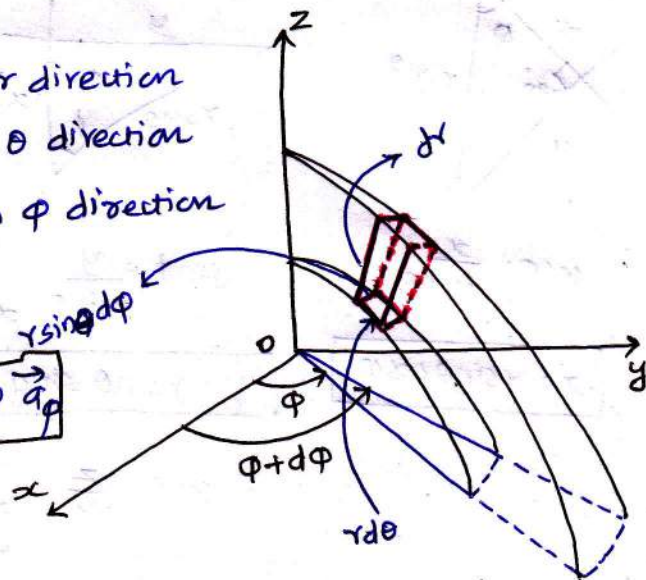
dr = Differential length in r direction

$r d\theta$ = Differential length in θ direction

$r \sin\theta d\phi$ = Differential length in ϕ direction

∴ differential vector length

$$d\vec{L} = dr \vec{a}_r + r d\theta \vec{a}_\theta + r \sin\theta d\phi \vec{a}_\phi$$



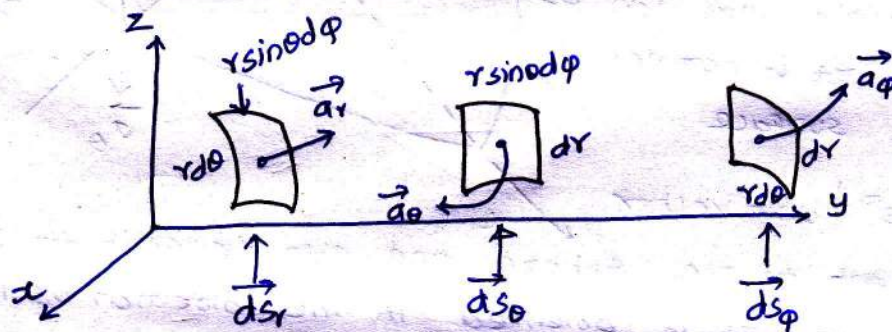
Hence the magnitude of the differential length vector is given by,

$$|\vec{dl}| = \sqrt{(dr)^2 + (r d\theta)^2 + (r \sin\theta d\phi)^2}$$

Hence the differential volume of the differential element formed, in spherical co-ordinate system is given by,

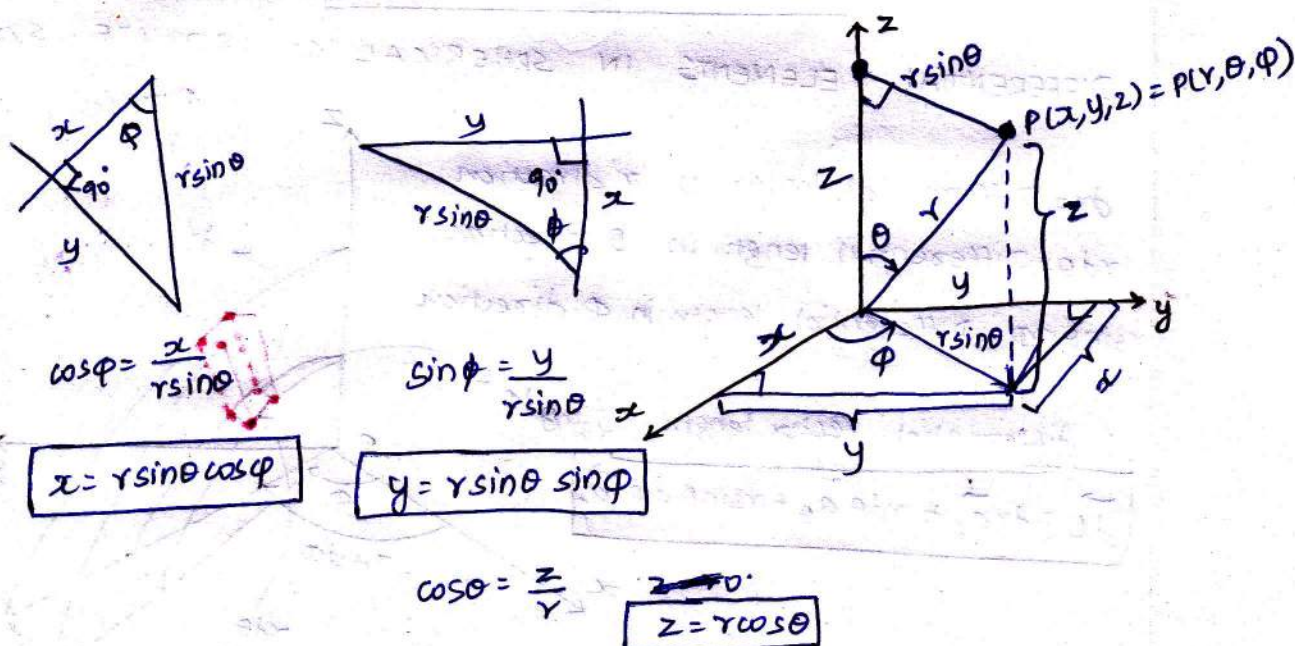
$$dV = r^2 \sin\theta dr d\theta d\phi$$

The differential surface areas in the three directions are shown in fig below.



$$\vec{ds}_r = r^2 \sin\theta d\theta d\phi \quad \vec{ds}_\theta = r \sin\theta dr d\phi \quad \vec{ds}_\phi = r dr d\theta$$

RELATIONSHIP B/W CARTESIAN & SPHERICAL SYSTEMS:



$$\begin{aligned} x &= r \sin \theta \cos \phi \\ y &= r \sin \theta \sin \phi \\ z &= r \cos \theta \end{aligned}$$

Now r can be expressed as,

$$\begin{aligned} x^2 + y^2 + z^2 &= r^2 \sin^2 \theta \cos^2 \phi + r^2 \sin^2 \theta \sin^2 \phi + r^2 \cos^2 \theta \\ &= r^2 \sin^2 \theta [\cos^2 \phi + \sin^2 \phi] + r^2 \cos^2 \theta \\ &= r^2 [\sin^2 \theta + \cos^2 \theta] \\ &= r^2 \end{aligned}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

While $\tan \phi = \frac{y}{x}$ and $\cos \theta = \frac{z}{r}$.

As r is known, θ can be obtained.

Thus the Transformation from cartesian to spherical coordinates system can be obtained from the following equations.

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \cos^{-1} \left[\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right] \text{ and } \phi = \tan^{-1} \left(\frac{y}{x} \right)$$

TRANSFORMATION OF VECTORS:

TRANSFORMATION OF VECTORS FROM CARTESIAN TO CYLINDRICAL

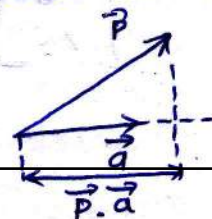
Consider a vector $\vec{A}$ in cartesian co-ordinate system as,

$$\vec{A} = A_x \vec{a}_x + A_y \vec{a}_y + A_z \vec{a}_z$$

While the same vector in cylindrical co-ordinate system can be represented as

$$\vec{A} = A_r \vec{a}_r + A_\phi \vec{a}_\phi + A_z \vec{a}_z$$

From the dot product it is known that the component of vector in the direction of unit vector is its dot product with that unit vector.

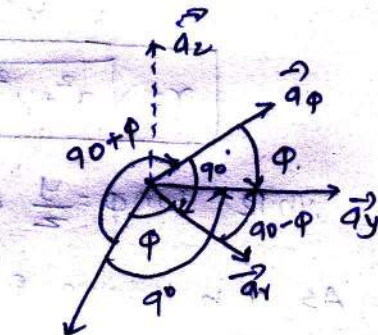
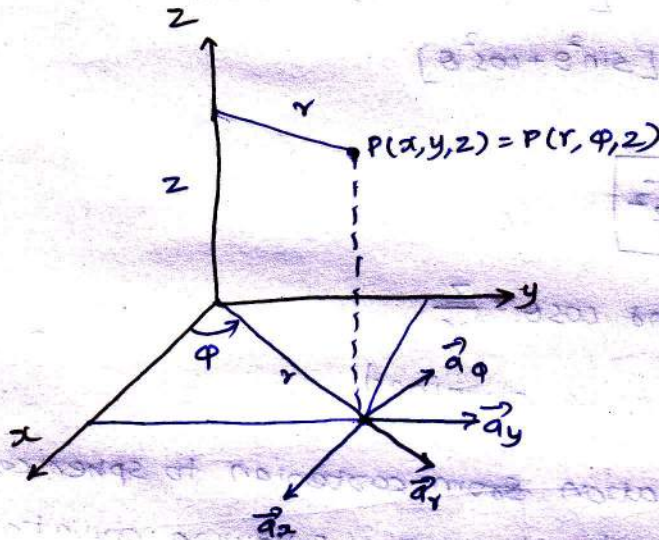


Hence the component of $\vec{A}$ in the direction of $\vec{a}_r$ is the dot product of $\vec{A}$ with $\vec{a}_r$. This component is nothing but A_r .

$$\therefore A_r = \vec{A} \cdot \vec{a}_r$$

$$= [A_x \vec{a}_x + A_y \vec{a}_y + A_z \vec{a}_z] \cdot \vec{a}_r$$

$$= A_x \vec{a}_x \cdot \vec{a}_r + A_y \vec{a}_y \cdot \vec{a}_r + A_z \vec{a}_z \cdot \vec{a}_r$$



$$\vec{a}_x \cdot \vec{a}_r = (1)(1) \cos \phi = (1)(1) \cos \phi = \cos \phi.$$

$$\vec{a}_x \cdot \vec{a}_\phi = (1)(1) \cos(90 + \phi) = -\sin \phi$$

$$\vec{a}_y \cdot \vec{a}_r = (1)(1) \cos(90 - \phi) = \sin \phi$$

$$\vec{a}_y \cdot \vec{a}_\phi = (1)(1) \cos \phi = \cos \phi.$$

$$\vec{a}_z \cdot \vec{a}_r = \vec{a}_z \cdot \vec{a}_\phi = 0$$

$$\vec{a}_z \cdot \vec{a}_z = 1 \Rightarrow A_r = A_x \cos \phi + A_y \sin \phi.$$

iii)

$$A_\phi = \vec{A} \cdot \vec{a}_\phi \Rightarrow A_\phi = -A_x \sin \phi + A_y \cos \phi$$

$$A_z = \vec{A} \cdot \vec{a}_z = A_z \Rightarrow A_z = A_z$$

$$\therefore \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Now it is necessary to find the transformation from cylindrical to cartesian. Hence we assume $\vec{A}$ is known in cylindrical system.

Thus component of $\vec{A}$ in ax direction is given by

$$\begin{aligned} A_x &= \vec{A} \cdot \vec{a}_x = [A_r \vec{a}_r + A_\phi \vec{a}_\phi + A_z \vec{a}_z] \cdot \vec{a}_x \\ &= A_r \vec{a}_r \cdot \vec{a}_x + A_\phi \vec{a}_\phi \cdot \vec{a}_x + A_z \vec{a}_z \cdot \vec{a}_x \end{aligned}$$

As dot product is commutative

$$\vec{a}_r \cdot \vec{a}_x = \vec{a}_x \cdot \vec{a}_r = \cos \phi$$

Hence

$$A_x = A_r \cos \phi + (-\sin \phi) A_\phi$$

$$A_x = A_r \cos \phi - A_\phi \sin \phi$$

iii) y

$$A_y = \vec{A} \cdot \vec{a}_y = \sin \phi A_r + \cos \phi A_\phi$$

$$A_z = A_z$$

$$\therefore \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}$$

TRANSFORMATION OF VECTORS FROM CARTESIAN TO SPHERICAL

Let

$$\vec{A} = A_x \vec{a}_x + A_y \vec{a}_y + A_z \vec{a}_z$$

$$A_r = \vec{A} \cdot \vec{a}_r = [A_x \vec{a}_x + A_y \vec{a}_y + A_z \vec{a}_z] \cdot \vec{a}_r$$

$$A_r = A_x \vec{a}_x \cdot \vec{a}_r + A_y \vec{a}_y \cdot \vec{a}_r + A_z \vec{a}_z \cdot \vec{a}_r$$

$$A_\theta = \vec{A} \cdot \vec{a}_\theta = [A_x \vec{a}_x + A_y \vec{a}_y + A_z \vec{a}_z] \cdot \vec{a}_\theta$$

$$A_\theta = A_x \vec{a}_x \cdot \vec{a}_\theta + A_y \vec{a}_y \cdot \vec{a}_\theta + A_z \vec{a}_z \cdot \vec{a}_\theta$$

$$\vec{A} = A_x \vec{a}_x + A_y \vec{a}_y + A_z \vec{a}_z$$

$$\vec{A} \cdot \vec{a}_\phi = [A_x \vec{a}_x + A_y \vec{a}_y + A_z \vec{a}_z] \cdot \vec{a}_\phi$$

$$= A_x \vec{a}_x \cdot \vec{a}_\phi + A_y \vec{a}_y \cdot \vec{a}_\phi + A_z \vec{a}_z \cdot \vec{a}_\phi$$

The dot product of spherical unit vectors are given below.

	$\vec{a}_r$	$\vec{a}_\theta$	$\vec{a}_\phi$
$\vec{a}_x$	$\sin\theta \cos\phi$	$\cos\theta \cos\phi$	$-\sin\phi$
$\vec{a}_y$	$\sin\theta \sin\phi$	$\cos\theta \sin\phi$	$\cos\phi$
$\vec{a}_z$	$\cos\theta$	$-\sin\theta$	0

$$\therefore \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta \cos\phi & \sin\theta \sin\phi & \cos\theta \\ \cos\theta \cos\phi & \cos\theta \sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

TRANSFORMATION OF VECTORS FROM SPHERICAL TO CARTESIAN

$$\vec{A} = A_r \vec{a}_r + A_\theta \vec{a}_\theta + A_\phi \vec{a}_\phi$$

$$A_x = \vec{A} \cdot \vec{a}_x = A_r \vec{a}_r \cdot \vec{a}_x + A_\theta \vec{a}_\theta \cdot \vec{a}_x + A_\phi \vec{a}_\phi \cdot \vec{a}_x$$

$$A_y = \vec{A} \cdot \vec{a}_y = A_r \vec{a}_r \cdot \vec{a}_y + A_\theta \vec{a}_\theta \cdot \vec{a}_y + A_\phi \vec{a}_\phi \cdot \vec{a}_y$$

$$A_z = \vec{A} \cdot \vec{a}_z = A_r \vec{a}_r \cdot \vec{a}_z + A_\theta \vec{a}_\theta \cdot \vec{a}_z + A_\phi \vec{a}_\phi \cdot \vec{a}_z$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta \cos\phi & \cos\theta \cos\phi & -\sin\phi \\ \sin\theta \sin\phi & \cos\theta \sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

DISTANCE OF ALL CO-ORDINATE SYSTEMS

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \quad - \text{cartesian}$$

$$d = \sqrt{r_2^2 + r_1^2 - 2r_1 r_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2} \quad - \text{cylindrical}$$

$$d = \sqrt{r_2^2 + r_1^2 - 2r_1 r_2 \cos\theta_2 \cos\theta_1 - 2r_1 r_2 \sin\theta_2 \sin\theta_1 \cos(\phi_2 - \phi_1)}$$

- spherical.

Let

$$\vec{A} = A_r \vec{a}_r + A_\theta \vec{a}_\theta + A_\phi \vec{a}_\phi$$

$$A_r = A_r \vec{a}_r \cdot \vec{a}_r + A_\theta \vec{a}_\theta \cdot \vec{a}_r + A_\phi \vec{a}_\phi \cdot \vec{a}_r$$

$$A_\theta = A_r \vec{a}_r \cdot \vec{a}_\theta + A_\theta \vec{a}_\theta \cdot \vec{a}_\theta + A_\phi \vec{a}_\phi \cdot \vec{a}_\theta$$

$$A_z = A_r \vec{a}_r \cdot \vec{a}_z + A_\theta \vec{a}_\theta \cdot \vec{a}_z + A_\phi \vec{a}_\phi \cdot \vec{a}_z$$

$$\vec{a}_r \cdot \vec{a}_r = \sin \theta$$

$$\vec{a}_\theta \cdot \vec{a}_r = \cos \theta$$

$$\vec{a}_\phi \cdot \vec{a}_r = 0$$

$$\vec{a}_r \cdot \vec{a}_\theta = 0$$

$$\vec{a}_\theta \cdot \vec{a}_\theta = 1$$

$$\vec{a}_\phi \cdot \vec{a}_\theta = 0$$

$$\vec{a}_r \cdot \vec{a}_z = \cos \theta$$

$$\vec{a}_\theta \cdot \vec{a}_z = -\sin \theta$$

$$\vec{a}_\phi \cdot \vec{a}_z = 0$$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_z \end{bmatrix} = \begin{bmatrix} \sin \theta & \cos \theta & 0 \\ 0 & 1 & 0 \\ \cos \theta & -\sin \theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

TRANSFORMATION OF VECTORS FROM CYLINDRICAL TO SPHERICAL

Let

$$\vec{A} = A_r \vec{a}_r + A_\theta \vec{a}_\theta + A_z \vec{a}_z$$

$$A_r = A_r \vec{a}_r \cdot \vec{a}_r + A_\theta \vec{a}_\theta \cdot \vec{a}_r + A_z \vec{a}_z \cdot \vec{a}_r$$

$$A_\theta = A_r \vec{a}_r \cdot \vec{a}_\theta + A_\theta \vec{a}_\theta \cdot \vec{a}_\theta + A_z \vec{a}_z \cdot \vec{a}_\theta$$

$$A_z = A_r \vec{a}_r \cdot \vec{a}_z + A_\theta \vec{a}_\theta \cdot \vec{a}_z + A_z \vec{a}_z \cdot \vec{a}_z$$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_z \end{bmatrix} = \begin{bmatrix} \sin \theta & 0 & \cos \theta \\ \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_z \end{bmatrix}$$

In electromagnetic theory a charge can exist in

Point form, line form, surface form ~~and~~ or volume form.

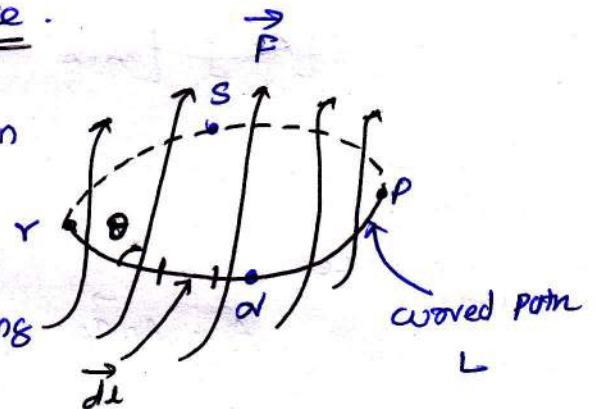
Hence for charge distribution analysis, the following types of integrals are required.

1. Line integral
2. Surface integral
3. Volume integral.

LINE INTEGRAL

- A line can exist as a straight line or it can be distance travelled along a wave
- ~~The~~ From mathematical point of view, a line is a curved path in a space.

Consider a vector field $\vec{F}$ shown in fig. The curved path shown in the field is P-Q. This is called Path of integration and corresponding integral can be defined as



$$\int_L \vec{F} \cdot d\vec{l} = \int_P^Q |\vec{F}| dl \cos \theta \quad [\text{using dot product definition}]$$

where

$dl \rightarrow$ Elementary length.

This is called line integral of $\vec{F}$ around the curved path ~~for~~.

The curved path can be of two types.

(i) open path as P-Q shown in fig

(ii) closed path as P-Q-R-S-P.

The closed path is also called contour. The corresponding integral is called contour integral, closed integral (or) circular integral, and mathematically defined as.

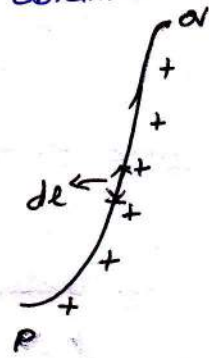
$$\oint \vec{F} \cdot d\vec{l} = \text{circular integral}$$

If there exists a charge along a line as

shown in figure, then the total charge is obtained by calculating a line integral.

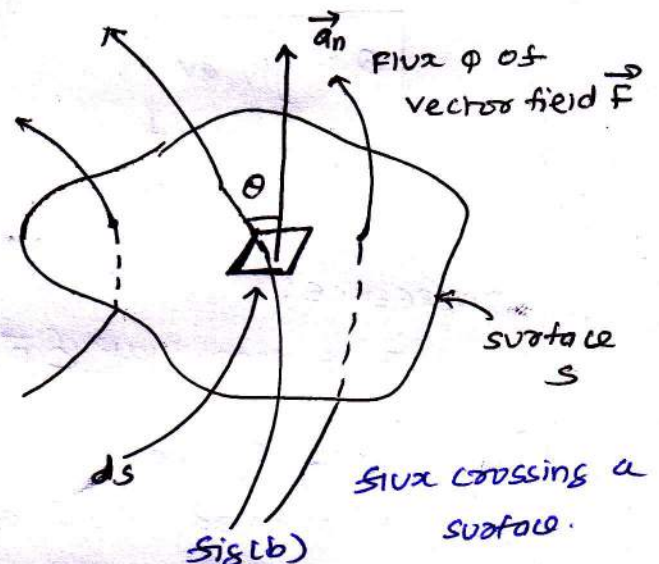
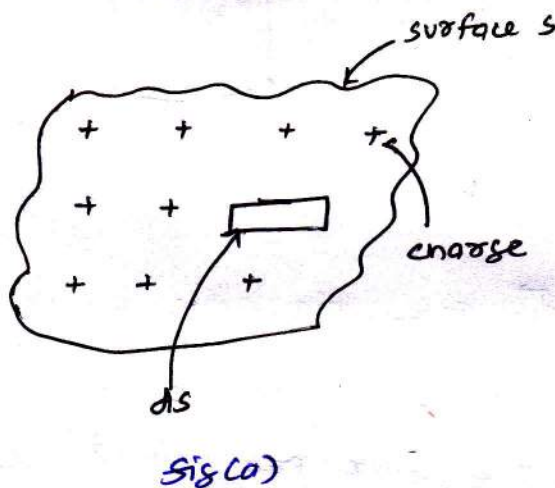
$$Q = \int_L \rho_L \cdot dL$$

ρ_L = Line charge density (or)
charge per unit length (C/m).



SURFACE INTEGRAL:

In electromagnetic theory a charge may exist in a distributed form. It may be spreaded over a surface as shown in figure(a) below.



Similarly a flux Φ may pass through a surface as shown in fig(b). While doing analysis of such cases an integral is required called SURFACE INTEGRAL, to be carried out over a surface related to a vector field.

For a charge distribution shown in fig(a), we can write total charge existing on the surface as

$$Q = \int_S \rho_s \cdot ds$$

$\rho_s \rightarrow$ surface charge density in C/m^2
 $ds \rightarrow$ Elementary surface area.

IIIrd from fig(b), the total flux crossing the surface S can be expressed as

$$\Phi = \int_S \vec{F} \cdot d\vec{s} = \int_S |\vec{F}| ds \cos \theta$$

$$\Phi = \oint_S \vec{F} \cdot d\vec{s}$$

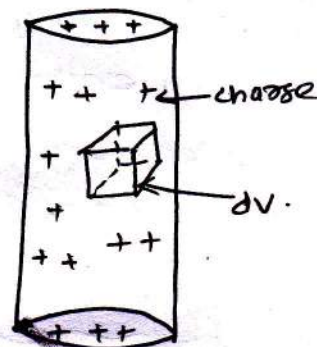
VOLUME INTEGRAL

If the charge distribution exists in a three dimensional volume form as shown in figure below, then a volume integral is required to calculate the total charge.

thus if ρ_v is the volume charge density over volume V . then the volume integral is defined as

$$Q = \int_V \rho_v \cdot dv$$

dv = Elementary volume.



DIVERGENCE:

It is seen that $\oint_S \vec{F} \cdot d\vec{s}$ gives the flux flowing across the surface S . Then mathematically divergence is defined as the net outward flow of the flux per unit volume over a closed incremental surface. It is denoted as $\text{div } \vec{F}$ and given by

$$\text{div } \vec{F} = \lim_{\Delta V \rightarrow 0} \frac{\oint_S \vec{F} \cdot d\vec{s}}{\Delta V} = \text{Divergence of } \vec{F}$$

symbolically it is denoted as

$$\nabla \cdot \vec{F} = \text{Divergence of } \vec{F}$$

Where $\nabla = \text{Vector operator} = \frac{\partial}{\partial x} \vec{a}_x + \frac{\partial}{\partial y} \vec{a}_y + \frac{\partial}{\partial z} \vec{a}_z$

But $\vec{F} = F_x \vec{a}_x + F_y \vec{a}_y + F_z \vec{a}_z$

$$\therefore \nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \rightarrow \text{In cartesian form}$$

$$\nabla \cdot \vec{F} = \frac{1}{r} \frac{\partial}{\partial r} (r F_r) + \frac{1}{r} \frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z} \quad \text{cylindrical}$$

$$\nabla \cdot \vec{F} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 F_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta F_\theta) + \frac{1}{r \sin \theta} \frac{\partial F_\phi}{\partial \phi} \quad \text{spherical}$$

(X) The vector field having its divergence zero is called solenoidal field

$$\nabla \cdot \vec{A} = 0 \quad \text{for } \vec{A} \text{ to be solenoidal}$$

GRADIENT OF A SCALAR:

consider that in space let w be the univariate function of x, y and z co-ordinates. in the cartesian system. This is the scalar function and denoted as $w(x, y, z)$. Consider a vector operator in cartesian system denoted as ∇ (called del). It is defined as.

$$\nabla (\text{del}) = \frac{\partial}{\partial x} \vec{a}_x + \frac{\partial}{\partial y} \vec{a}_y + \frac{\partial}{\partial z} \vec{a}_z$$

The operation of the vector operator $\text{del}(\nabla)$ on a scalar function is called gradient of a scalar.

$$\text{Grad } w = \nabla w = \left(\frac{\partial}{\partial x} \vec{a}_x + \frac{\partial}{\partial y} \vec{a}_y + \frac{\partial}{\partial z} \vec{a}_z \right) w$$

$$\nabla w = \frac{\partial w}{\partial x} \vec{a}_x + \frac{\partial w}{\partial y} \vec{a}_y + \frac{\partial w}{\partial z} \vec{a}_z \quad \text{cartesian}$$

In cylindrical co-ordinates

$$\nabla w = \frac{\partial w}{\partial r} \vec{a}_r + \frac{1}{r} \frac{\partial w}{\partial \phi} \vec{a}_\phi + \frac{\partial w}{\partial z} \vec{a}_z$$

spherical co-ordinates

$$\nabla w = \frac{\partial w}{\partial r} \vec{a}_r + \frac{1}{r} \frac{\partial w}{\partial \theta} \vec{a}_\theta + \frac{1}{r \sin \theta} \frac{\partial w}{\partial \phi} \vec{a}_\phi$$

$$\nabla \times \vec{F} = \text{curl of } \vec{F}$$

curl indicates the rotational property of vector field.

If curl of vector $\vec{F}$ is zero, the vector field is irrotational.

$$\nabla \times \vec{F} = \left[\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right] \vec{a}_x + \left[\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right] \vec{a}_y + \left[\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right] \vec{a}_z$$

$$\nabla \times \vec{F} = \begin{bmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{bmatrix} \quad \text{CARTESIAN}$$

$$\nabla \times \vec{F} = \begin{bmatrix} \vec{a}_r & r\vec{a}_\phi & \vec{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_r & F_\phi & F_z \end{bmatrix} \quad \text{CYLINDRICAL}$$

$$\nabla \times \vec{F} = \begin{bmatrix} \vec{a}_r & r\vec{a}_\theta & r\sin\theta\vec{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{bmatrix} \quad \text{SPHERICAL}$$

→ Electrostatics is a branch related to the electric charges which are static i.e. are at rest.

→ An Electric charge has an effect in a region or a space around it. This region is called an electric field of that charge.

→ Such an Electric field produced due to stationary electric charge does not vary with time. It is time invariant and called static electric field.

COULOMB'S LAW:

→ Study of electrostatics started from French army engineer, Colonel Charles Coulomb.

→ The experiments are related to the force exerted b/w the two point charges, which are placed near each other.

→ The force exerted is due to the electric fields produced by the point charges.

POINT CHARGE:

- A Point charge means that the electric charge which is spreaded on a surface or space whose geometrical dimensions are very very small compared to the other dimensions, in which the effect of its electric field is to be studied. Thus a point charge has a location but not the dimensions.

- A charge can be +ve or -ve

- A charge is actually the deficiency or excess of electrons in the atoms of a particle.

- An electron possesses a -ve charge. So the deficiency of an electron produces +ve charge while excess of an electron produces -ve charge.

~~charge is measured in coulombs (C)~~

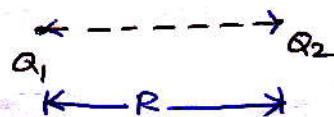
STATEMENT OF COULOMB'S LAW:

The Coulomb's law states that force b/w the two point charges Q_1 and Q_2

1. Acts along the line joining the two point charges.
2. Is directly proportional to the product (Q_1, Q_2) of the two charges
3. Is inversely proportional to the square of distance b/w them.

— Consider two point charges Q_1 & Q_2 separated by distance R .

→ The charge Q_1 exerts a force on Q_2 while Q_2 exerts a force on Q_1 .



→ The force acts along the line joining Q_1 and Q_2

→ The force exerted b/w them is repulsive if the charges are of same polarity, while it is attractive if charges are of different polarity.

Force F b/w two charges is expressed as

$$F \propto \frac{Q_1 Q_2}{R^2}$$

The Coulomb's law also states that this force depends on the medium in which the point charge are located. This effect is included as constant k proportionality

$$\therefore F = k \frac{Q_1 Q_2}{R^2}$$

$$k = \frac{1}{4\pi\epsilon}$$

$\epsilon \rightarrow$ Permittivity of the medium in which charges are located.

units of ϵ Farads/meter (F/m)

$$\epsilon = \epsilon_0 \epsilon_r$$

$\epsilon_0 \rightarrow$ Permittivity of free space or vacuum.

$\epsilon_r \rightarrow$ Relative Permittivity (or) dielectric constant of the medium w.r. to free space.

$\epsilon \rightarrow$ Absolute permittivity

For the free space or vacuum, $\epsilon_r = 1$

$$\therefore \epsilon = \epsilon_0$$

$$\therefore F = \frac{1}{4\pi\epsilon_0} \frac{Q_1 Q_2}{R^2}$$

$$\epsilon_0 = \frac{1}{36\pi} \times 10^{-9} = 8.854 \times 10^{-12} \text{ F/m.}$$

$$k = \frac{1}{4\pi\epsilon_0} = \frac{1}{4\pi \times 8.854 \times 10^{-12}} = 8.98 \times 10^9 \approx 9 \times 10^9 \text{ N/F.}$$

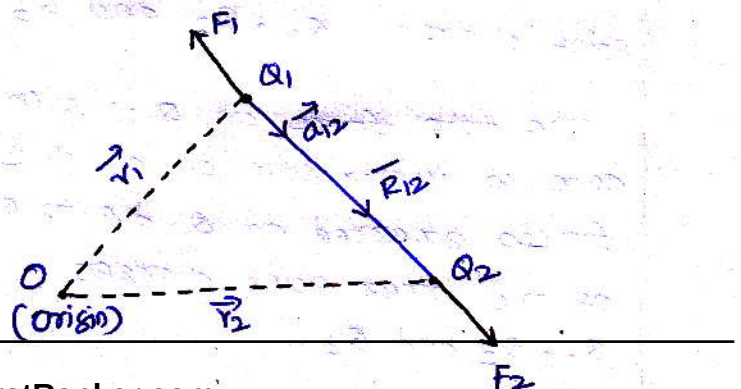
$$F = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2}$$

← This is the force b/w two Point charges located in free space or vacuum.

VECTOR FORM OF COULOMB'S LAW:

The force exerted b/w the two Point charges has a fixed direction which is a straight line joining the two charges. Hence the force exerted b/w the two charges can be expressed in a vector form.

Consider two Point charges Q_1 and Q_2 located at the points having position vectors $\vec{r}_1$ and $\vec{r}_2$ as shown in fig below



then the force exerted by Q_1 on Q_2 acts along the direction $\vec{R}_{12}$ where $\vec{R}_{12}$ is the vector from Q_1 to Q_2 . Hence the force in the vector form can be expressed as.

$$\vec{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2} \vec{a}_{12}$$

$$\vec{a}_{12} = \text{unit vector along } \vec{R}_{12} = \frac{\text{Vector}}{\text{Magnitude of vector}}$$

$$\vec{a}_{12} = \frac{\vec{R}_{12}}{|\vec{R}_{12}|} = \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|} = \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|}$$

where $|\vec{R}_{12}|$ is the distance b/w the two charges.

IIIrd the force F_1 is exerted on Q_1 due to Q_2 . It can be expressed as

$$\vec{F}_1 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{21}^2} \vec{a}_{21}$$

$$\vec{a}_{21} = \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|} \quad \text{But } \vec{r}_1 - \vec{r}_2 = -[\vec{r}_2 - \vec{r}_1]$$

$$\vec{a}_{21} = -\vec{a}_{12}$$

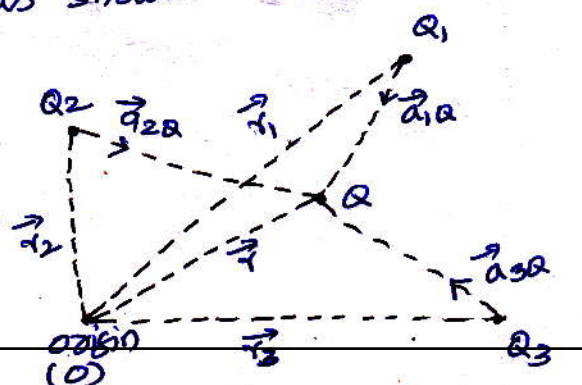
$$\therefore \vec{F}_1 = -\vec{F}_2$$

PRINCIPLE OF SUPERPOSITION:

If there are more than two point charges, then each will exert on the other, then the net force on any charge can be obtained by the principle of superposition.

Consider a point charge Q surrounded by three other point charges Q_1 , Q_2 and Q_3 as shown in figure below.

The total force on Q in such a case is vector sum of all the forces exerted on Q due to each of the other point charges Q_1 , Q_2 and Q_3 .



considered force exerted on Q due to Q_1 . At this time, according to principle of superposition effect of Q_2 and Q_3 are to be suppressed.

$$\vec{F}_{Q_1Q} = \frac{Q_1Q}{4\pi\epsilon_0 R_{1Q}^2} \vec{a}_{1Q} \quad \text{where} \quad \vec{a}_{1Q} = \frac{\vec{r} - \vec{r}_1}{|\vec{r} - \vec{r}_1|}$$

force exerted due to Q_2 on Q is

$$\vec{F}_{Q_2Q} = \frac{Q_2Q}{4\pi\epsilon_0 R_{2Q}^2} \vec{a}_{2Q} \quad \text{where} \quad \vec{a}_{2Q} = \frac{\vec{r} - \vec{r}_2}{|\vec{r} - \vec{r}_2|}$$

and force exerted due to Q_3 on Q is

$$\vec{F}_{Q_3Q} = \frac{Q_3Q}{4\pi\epsilon_0 R_{3Q}^2} \vec{a}_{3Q} \quad \text{where} \quad \vec{a}_{3Q} = \frac{\vec{r} - \vec{r}_3}{|\vec{r} - \vec{r}_3|}$$

Hence the total force on Q is

$$\vec{F}_Q = \vec{F}_{Q_1Q} + \vec{F}_{Q_2Q} + \vec{F}_{Q_3Q}$$

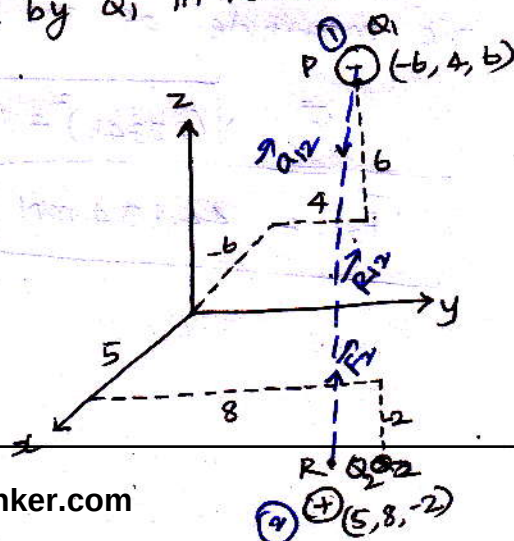
In general if there are n other charges then force exerted on Q due to all other charges is,

$$\vec{F}_Q = \vec{F}_{Q_1Q} + \vec{F}_{Q_2Q} + \dots + \vec{F}_{Q_nQ}$$

$$\vec{F}_Q = \frac{Q}{4\pi\epsilon_0} \sum_{i=1}^n \frac{Q_i}{R_{iQ}^2} \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|}$$

Problems:

A charge $Q_1 = -20\mu C$ is located at $P(-6, 4, 6)$ and a charge $Q_2 = 50\mu C$ is located at $R(5, 8, -2)$ in a free space. Find the force exerted on Q_2 by Q_1 in vector form. The distance given are in meters.



$$\vec{P} = -6\vec{a}_x + 4\vec{a}_y + 6\vec{a}_z \quad (Q_1)$$

and

$$\vec{R} = 5\vec{a}_x + 8\vec{a}_y - 2\vec{a}_z \quad (Q_2)$$

The force on Q_2 is given by,

$$\vec{F}_{Q_2} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2} \vec{a}_{12}$$

$$\vec{R}_{12} = \vec{R}_{PR} = \vec{R} - \vec{P} = 11\vec{a}_x + 4\vec{a}_y - 8\vec{a}_z$$

$$|\vec{R}_{12}| = \sqrt{11^2 + 4^2 + 8^2} = 14.1774$$

$$\therefore \vec{a}_{12} = \frac{\vec{R}_{12}}{|\vec{R}_{12}|} = \frac{11\vec{a}_x + 4\vec{a}_y - 8\vec{a}_z}{14.1774}$$

$$\vec{a}_{12} = 0.7758\vec{a}_x + 0.2821\vec{a}_y - 0.5642\vec{a}_z$$

$$\therefore \vec{F}_2 = \frac{-20 \times 10^{-6} \times 50 \times 10^{-6}}{4\pi \times 8.854 \times 10^{-12} (14.1774)^2} [\vec{a}_{12}]$$

$$= -0.0447 [0.7758\vec{a}_x + 0.2821\vec{a}_y - 0.5642\vec{a}_z]$$

$$\vec{F}_2 = -0.0346\vec{a}_x - 0.01261\vec{a}_y + 0.02522\vec{a}_z \text{ N}$$

This is the required force exerted on Q_2 by Q_1 .

The magnitude of the force is,

$$|\vec{F}_2| = \sqrt{(0.0346)^2 + (0.01261)^2 + (-0.02522)^2}$$

$$|\vec{F}_2| = 44.634 \text{ mN}$$

Four point charges each of 10 nC are placed in free space at the points $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 1, 0)$ and $(0, -1, 0)\text{ m}$ respectively. Determine the force on a point charge of 30 nC located at a point $(0, 0, 1)\text{ m}$.

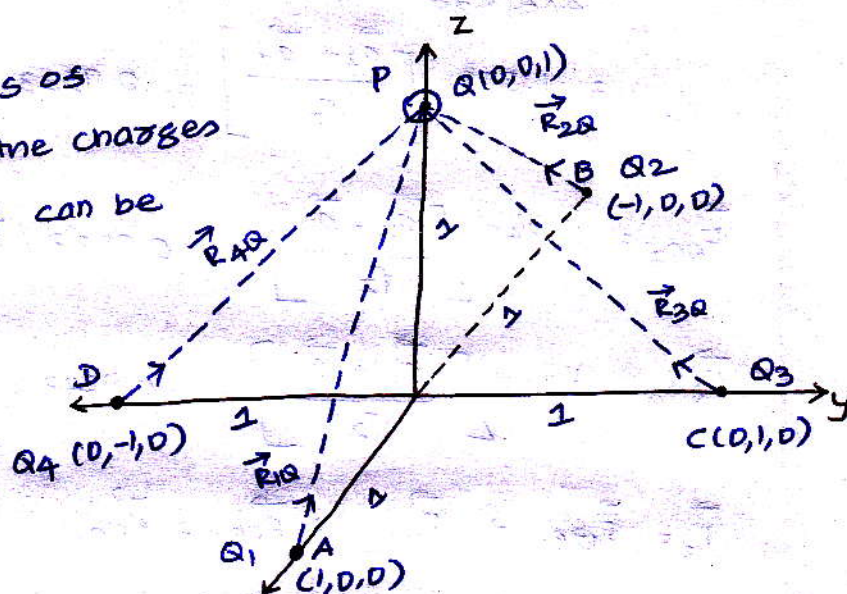
The position vectors of 4 points at which the charges Q_1 to Q_4 are located can be obtained as,

$$\vec{A} = \vec{a}_x$$

$$\vec{B} = -\vec{a}_x$$

$$\vec{C} = \vec{a}_y$$

$$\vec{D} = -\vec{a}_y$$



While the position vector of point P where charge of 30 nC is situated is

$$\vec{P} = \vec{a}_z$$

Consider force on Q due to Q_1 is

$$\vec{F}_{QQ_1} = \frac{Q Q_1}{4\pi\epsilon_0 R_{1Q}^2} \vec{a}_{1Q}$$

$$\vec{R}_{1Q} = \vec{R}_{AP} = \vec{P} - \vec{A} = \vec{a}_z - (\vec{a}_x) = \vec{a}_z - \vec{a}_x$$

$$|\vec{R}_{1Q}| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\therefore \vec{F}_{QQ_1} = \frac{10 \times 10^{-9} \times 30 \times 10^{-9}}{4\pi \times 8.854 \times 10^{-12} \times (\sqrt{2})^2} \left[\frac{\vec{a}_z - \vec{a}_x}{\sqrt{2}} \right]$$

$$\boxed{\vec{F}_{QQ_1} = 0.9533(\vec{a}_z - \vec{a}_x)}$$

Due to symmetry from the figure

$$|\vec{R}_{2Q}| = |\vec{R}_{3Q}| = |\vec{R}_{4Q}| = |\vec{R}_{1Q}| = \sqrt{2}$$

$$\vec{R}_{2Q} = \vec{P} - \vec{B} = \vec{a}_z + \vec{a}_x \quad \therefore \vec{a}_{2Q} = \frac{\vec{a}_z + \vec{a}_x}{\sqrt{2}}$$

$$\vec{R}_{4Q} = \vec{P} - \vec{D} = \vec{a}_2 + \vec{a}_y \quad \vec{a}_{4Q} = \frac{\vec{a}_2 + \vec{a}_y}{\sqrt{2}}$$

$$\begin{aligned} \vec{F}_{QQ2} &= \frac{Q Q_2}{4\pi\epsilon_0 R_{2Q}^2} \vec{a}_{2Q} \\ &= 1.3481 \left[\frac{\vec{a}_2 + \vec{a}_x}{\sqrt{2}} \right] = 0.9533 (\vec{a}_2 + \vec{a}_x) \end{aligned}$$

$$\vec{F}_{QQ3} = 1.3481 \left[\frac{\vec{a}_2 - \vec{a}_y}{\sqrt{2}} \right] = 0.9533 (\vec{a}_2 - \vec{a}_y)$$

$$\vec{F}_{QQ4} = 1.3481 \left[\frac{\vec{a}_2 + \vec{a}_y}{\sqrt{2}} \right] = 0.9533 (\vec{a}_2 + \vec{a}_y)$$

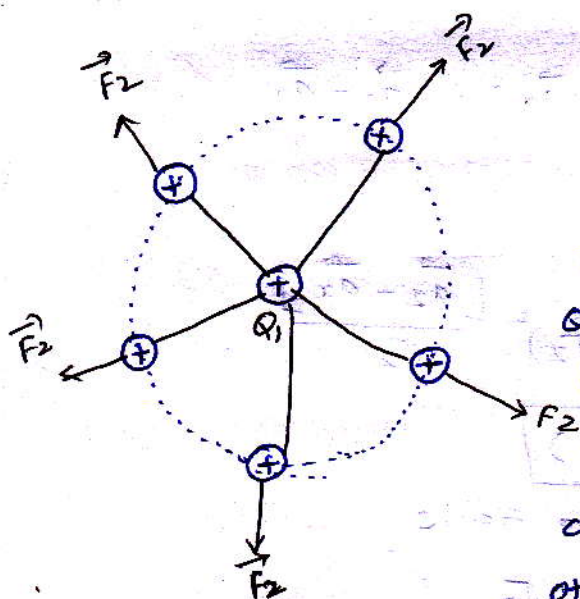
$$\vec{F}_E = \vec{F}_{QQ1} + \vec{F}_{QQ2} + \vec{F}_{QQ3} + \vec{F}_{QQ4}$$

$$= 0.9533 [\vec{a}_2 - \vec{a}_x + \vec{a}_2 + \vec{a}_x + \vec{a}_2 - \vec{a}_y + \vec{a}_2 + \vec{a}_y]$$

$$\boxed{\vec{F}_E = 3.813 \vec{a}_2 \text{ N}}$$

ELECTRIC FIELD INTENSITY:

consider a point charge Q_1 as shown in fig.

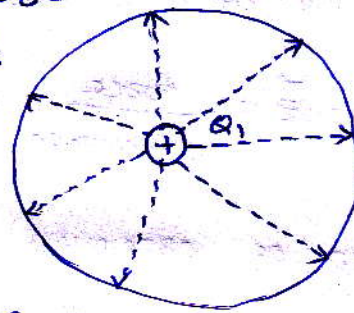


If any other similar charge Q_2 is brought near it, Q_2 experiences a force. In fact if Q_2 is moved around Q_1 , still Q_2 experiences a force as shown in fig.

Thus there exists a region around a charge in which it exerts a force on any other charge. This region where a particular charge exerts a force on any other charge located in that region is called ELECTRIC FIELD of that charge.

The force experienced by the charge Q₂ due to Q₁ is given by Coulomb's law as

$$\vec{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2} \vec{a}_{12}$$



Thus force per unit charge can be written as

$$\frac{\vec{F}_2}{Q_2} = \frac{Q_1}{4\pi\epsilon_0 R_{12}^2} \vec{a}_{12}$$

*** 2 marks *** (This force exerted by unit charge is called ELECTRIC FIELD INTENSITY. (or) ELECTRIC STRENGTH. It is a vector quantity and is directed along a segment from the charge Q₁ to the position of any other charge. It is denoted by E.

$$\vec{E} = \frac{Q_1}{4\pi\epsilon_0 R_{1P}^2} \vec{a}_{1P}$$

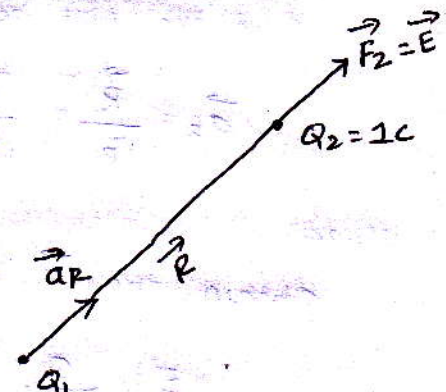
where P = position of any other charge around Q₁)

Consider a charge Q₁ as shown in figure below. The unit positive charge Q₂ = 1C is placed at a distance of R from Q₁. Then the force acting on Q₂ due to Q₁ is along the unit vector $\vec{a}_R$. As the charge Q₂ is unit charge, the force exerted on Q₂ is nothing but electric field intensity $\vec{E}$ of Q₁ at the point where unit charge is placed.

$$\vec{E} = \frac{Q_1}{4\pi\epsilon_0 R^2} \vec{a}_R$$

$$E = \frac{Q_1}{4\pi\epsilon_0 r^2} \vec{a}_r \text{ - spherical system.}$$

r → radius of sphere 'r'.



The definition of Electric field intensity E ,

$$\vec{E} = \frac{\text{Force}}{\text{unit charge}} = \frac{N \text{ (Newtons)}}{C \text{ (Coulomb)}}$$

METHOD OF OBTAINING $\vec{E}$ in CARTESIAN SYSTEM:

consider a charge Q_1 located at point $A(x_1, y_1, z_1)$ as shown in figure. It is required to obtain $\vec{E}$ at any point $B(x, y, z)$ in the cartesian system. then $\vec{E}$ at point B can be obtained using following steps:

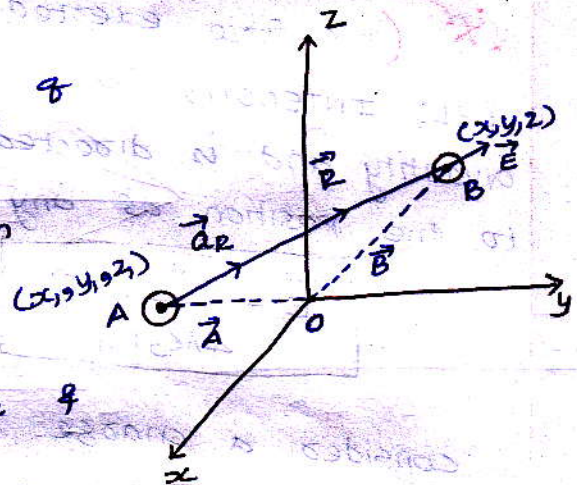
STEP 1:

Obtain the position vectors $\vec{A}$ & $\vec{B}$

Points A & B

$$\therefore \vec{r}_A = \vec{A} \text{ while } \vec{r}_B = \vec{B} \text{ from}$$

their co-ordinates.



$$\vec{A} = x_1 \vec{a}_x + y_1 \vec{a}_y + z_1 \vec{a}_z \text{ \& }$$

$$\vec{B} = x \vec{a}_x + y \vec{a}_y + z \vec{a}_z$$

STEP 2: Find the distance vector $\vec{R}$ directed from A to B .

$$\vec{R} = \vec{B} - \vec{A} = (x - x_1) \vec{a}_x + (y - y_1) \vec{a}_y + (z - z_1) \vec{a}_z$$

STEP 3:

Find the unit vector $\vec{a}_R$ along the direction from A to B .

$$\vec{a}_R = \frac{\vec{R}}{|\vec{R}|} = \frac{\vec{B} - \vec{A}}{|\vec{B} - \vec{A}|}$$

Step 4:

Obtain the $\vec{E}$ at the point B as,

$$E = \frac{Q}{4\pi\epsilon_0 R^2} \vec{a}_R = \frac{Q}{4\pi\epsilon_0 R^2} \frac{\vec{R}}{|\vec{R}|} \text{ V/m}$$

where $R^2 = |\vec{R}|^2 = \text{distance b/w the points } A \text{ \& } B.$

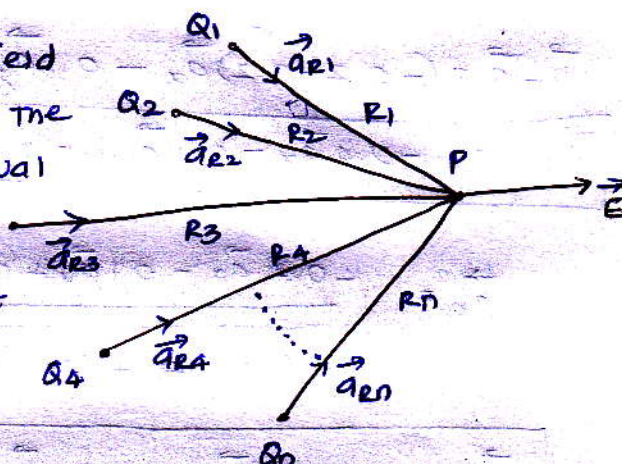
Magnitude of E is given by,

$$|\vec{E}| = \frac{Q}{4\pi\epsilon_0 R^2} \text{ V/m.}$$

ELECTRIC FIELD INTENSITY DUE TO DISCRETE CHARGES:

consider 'n' charges $Q_1, Q_2, \dots, Q_n$ as shown in figure given below. The combined electric field intensity is to be obtained at point P. The distances of point P from $Q_1, Q_2, \dots, Q_n$ are $R_1, R_2, \dots, R_n$ respectively. The unit vectors along these directions are $\vec{a}_{R1}, \vec{a}_{R2}, \dots, \vec{a}_{Rn}$ respectively.

Then the total electric field intensity at point P is the vector sum of the individual field intensities produced Q_3 by the various charges at the point P.



$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \dots + \vec{E}_n$$

$$= \frac{Q_1}{4\pi\epsilon_0 R_1^2} \vec{a}_{R1} + \frac{Q_2}{4\pi\epsilon_0 R_2^2} \vec{a}_{R2} + \dots + \frac{Q_n}{4\pi\epsilon_0 R_n^2} \vec{a}_{Rn}$$

$$\therefore \vec{E} = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^n \frac{Q_i}{R_i^2} \vec{a}_{Ri}$$

Each unit vector can be obtained by using the method discussed earlier

$$\vec{a}_{Ri} = \frac{\vec{r}_p - \vec{r}_i}{|\vec{r}_p - \vec{r}_i|}$$

$\vec{r}_p \rightarrow$ Position vector of point P
 $\vec{r}_i \rightarrow$ Position vector of point where charge Q_i is placed.

1. Determine the electric field intensity at $P(-0.2, 0, -2.3)m$ due to a point charge $q = +5nC$ at $Q(0.2, 0.1, -2.5)m$ in air.

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2} \vec{a}_r$$

$$\vec{a}_r = \frac{\vec{R}_{QP}}{|\vec{R}_{QP}|} = \frac{\vec{P} - \vec{Q}}{|\vec{P} - \vec{Q}|}$$

$$\vec{P} = -0.2\vec{a}_x + 0\vec{a}_y - 2.3\vec{a}_z$$

$$\vec{Q} = 0.2\vec{a}_x + 0.1\vec{a}_y - 2.5\vec{a}_z$$

$$\vec{P} - \vec{Q} = (-0.2 - 0.2)\vec{a}_x - 0.1\vec{a}_y + (-2.3 + 2.5)\vec{a}_z$$

$$\vec{R}_{QP} = -0.4\vec{a}_x - 0.1\vec{a}_y + 0.2\vec{a}_z$$

$$\vec{a}_r = \frac{-0.4\vec{a}_x - 0.1\vec{a}_y + 0.2\vec{a}_z}{\sqrt{(-0.4)^2 + (-0.1)^2 + (0.2)^2}} = \frac{-0.4\vec{a}_x - 0.1\vec{a}_y + 0.2\vec{a}_z}{0.45825}$$

$$\vec{a}_r = -0.8728\vec{a}_x - 0.2182\vec{a}_y + 0.4364\vec{a}_z$$

$$E = \frac{5 \times 10^{-9}}{4\pi \times 8.854 \times 10^{-12} \times (0.45825)^2} [-0.8728\vec{a}_x - 0.2182\vec{a}_y + 0.4364\vec{a}_z]$$

$$\vec{E} = -186.779\vec{a}_x - 46.694\vec{a}_y + 93.389\vec{a}_z \text{ V/m}$$

2. A charge $q = 1C$ is at $(2, 0, 0)$. What charge must be placed at $(-2, 0, 0)$ which will make y component of total $\vec{E}$ zero at the point $(1, 2, 2)$?

The position vectors of points A, B and P are

$$\vec{A} = 2\vec{a}_x$$

$$\vec{B} = -2\vec{a}_y$$

$$\vec{P} = \vec{a}_x + 2\vec{a}_y + 2\vec{a}_z$$

$\vec{E}_A$ is field at P due to Q_1 and will act along $\vec{a}_{AP}$.

$\vec{E}_B$ is field at P due to Q_2 and will act along $\vec{a}_{BP}$.

$$\therefore \vec{E}_A = \frac{Q_1}{4\pi\epsilon_0 R_{AP}^2} \vec{a}_{AP}$$

$$= \frac{Q_1}{4\pi\epsilon_0 R_{AP}^2} \left[\frac{\vec{P}-\vec{A}}{|\vec{P}-\vec{A}|} \right]$$

$$\vec{E}_B = \frac{Q_2}{4\pi\epsilon_0 R_{BP}^2} \vec{a}_{BP} = \frac{Q_2}{4\pi\epsilon_0 R_{BP}^2} \left[\frac{\vec{P}-\vec{B}}{|\vec{P}-\vec{B}|} \right]$$

$$\vec{E} \text{ at } P = \vec{E}_A + \vec{E}_B$$

$$= \frac{1}{4\pi\epsilon_0} \left[\frac{Q_1}{R_{AP}^2} \frac{\vec{P}-\vec{A}}{|\vec{P}-\vec{A}|} + \frac{Q_2}{R_{BP}^2} \frac{\vec{P}-\vec{B}}{|\vec{P}-\vec{B}|} \right]$$

$$= \frac{1}{4\pi\epsilon_0} \left[\frac{1[-\vec{a}_x + 2\vec{a}_y + 2\vec{a}_z]}{(\sqrt{9})^2 \sqrt{9}} + \frac{Q_2(3\vec{a}_x + 2\vec{a}_y + 2\vec{a}_z)}{(\sqrt{17})^2 (\sqrt{17})} \right]$$

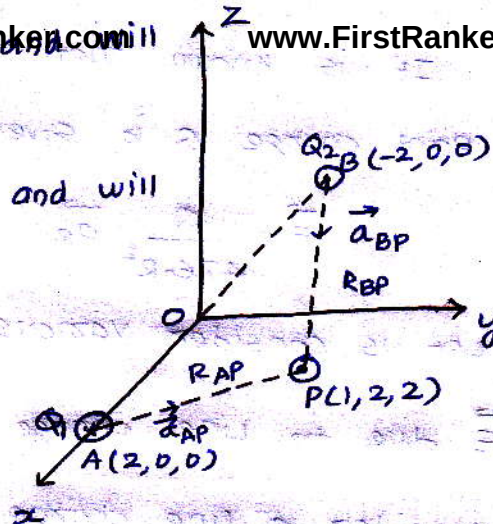
$$= \frac{1}{4\pi\epsilon_0} \left[\frac{-\vec{a}_x + 2\vec{a}_y + 2\vec{a}_z}{27} + \frac{Q_2[3\vec{a}_x + 2\vec{a}_y + 2\vec{a}_z]}{70.0927} \right]$$

The y component of $\vec{E}$ must be zero

$$\therefore \frac{2}{27} + \frac{2Q_2}{70.0927} = 0$$

$$Q_2 = \frac{-2}{27} \times \frac{70.0927}{2} = -2.596 \text{ C}$$

This is required charge Q_2 to be placed at $(-2, 0, 0)$ which will make y component of $\vec{E}$ zero at Point P.



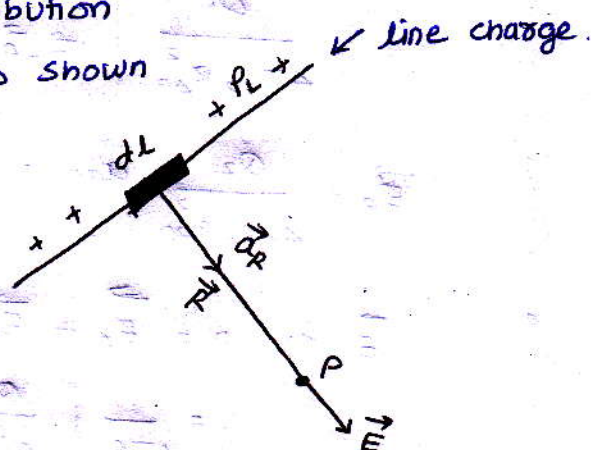
It is known that the electric field intensity due to a point charge Q is given by

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 R^2} \vec{a}_R$$

Let us consider various charge distributions.

$\vec{E}$ due to line charge:

consider a line charge distribution having a charge density ρ_L as shown in figure.



The charge dQ on the differential length dL is

$$dQ = \rho_L dL$$

Hence the differential electric field $d\vec{E}$ at point P due to dQ is given by.

$$d\vec{E} = \frac{dQ}{4\pi\epsilon_0 R^2} \vec{a}_R = \frac{\rho_L dL}{4\pi\epsilon_0 R^2} \vec{a}_R$$

Hence the total $\vec{E}$ at a point P due to line charge can be obtained by integrating $d\vec{E}$ over the length of the charge

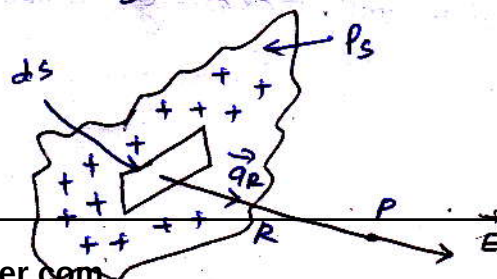
$$\vec{E} = \int_L \frac{\rho_L dL}{4\pi\epsilon_0 R^2} \vec{a}_R$$

$\vec{E}$ due to SURFACE CHARGE:

consider a surface charge density distribution having a charge density ρ_s as shown in figure.

The charge dQ on the differential surface area ds is

$$dQ = \rho_s ds$$



Hence the differential electric field $d\vec{E}$ at point P due to dQ is given by

$$\vec{dE} = \frac{dQ}{4\pi\epsilon_0 R^2} \vec{a_R} = \frac{\rho_s ds}{4\pi\epsilon_0 R^2} \vec{a_R}$$

Hence the total $\vec{E}$ at a point P is to be obtained by integrating $\vec{dE}$ over the surface area on which charge is distributed. Note that this will be a double integration.

$$\therefore \vec{E} = \int_S \frac{\rho_s ds}{4\pi\epsilon_0 R^2} \vec{a_R}$$

The $\vec{a_R}$ and ds to be obtained according to the position of the sheet of charge and the co-ordinate system used.

$\vec{E}$ due to VOLUME CHARGE:

consider a volume charge distribution having a charge density ρ_v as shown in figure.

The charge dQ on differential volume dv is

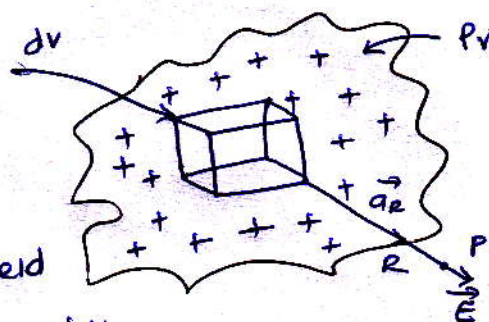
$$dQ = \rho_v dv$$

Hence the differential electric field $\vec{dE}$ at a point P due to dQ is given by

$$\vec{dE} = \frac{dQ}{4\pi\epsilon_0 R^2} \vec{a_R} = \frac{\rho_v dv}{4\pi\epsilon_0 R^2} \vec{a_R}$$

Hence the total $\vec{E}$ at a point P is to be obtained by integrating $\vec{dE}$ over the volume in which charge is accumulated. Note that this integration will be a triple integration.

$$\vec{E} = \int_{Vol} \frac{\rho_v dv}{4\pi\epsilon_0 R^2} \vec{a_R} \quad \vec{a_R} \text{ \& } dv \text{ must be obtained according to the co-ordinate system used.}$$



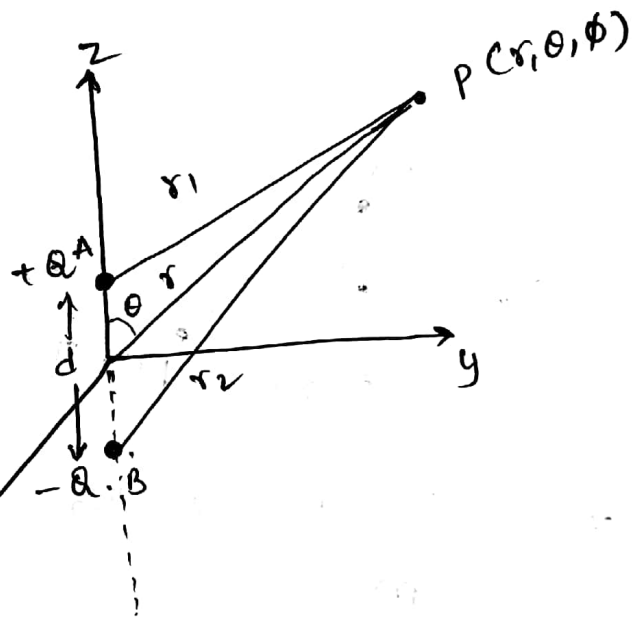
An Electric dipole :-

Two point charges Equal in magnitude but opposite in Sign, Separated by a Very Small distance is called an Electric dipole.

⇒ Electric field Intensity due to an Electric dipole.

Consider an Electric dipole.
The two point charges are Separated by a distance 'd'.

Consider a point $P(r, \theta, \phi)$
in Spherical co-ordinate
System. Let's find out $\vec{E}$



due to dipole. The co-ordinates of A are $[0, 0, +d/2]$ & B are $[0, 0, -d/2]$.

In Spherical co-ordinates ~~system~~. The potential at point P due to the charge +Q is given by

$$V_1 = \frac{+Q}{4\pi\epsilon_0 r_1} \longrightarrow (1)$$

The potential at point P due to -Q is given by

$$V_2 = \frac{-Q}{4\pi\epsilon_0 r_2} \longrightarrow (2)$$

The total potential at point p due to the two point charges is the sum of V_1 & V_2

$$\therefore V = V_1 + V_2$$

$$= \frac{+Q_1}{4\pi\epsilon_0 r_1} + \frac{-Q_2}{4\pi\epsilon_0 r_2}$$

$$\Rightarrow V = \frac{Q}{4\pi\epsilon_0} \left[\frac{1}{r_1} - \frac{1}{r_2} \right] = \frac{Q}{4\pi\epsilon_0} \left[\frac{r_2 - r_1}{r_1 r_2} \right]$$

→ (3)

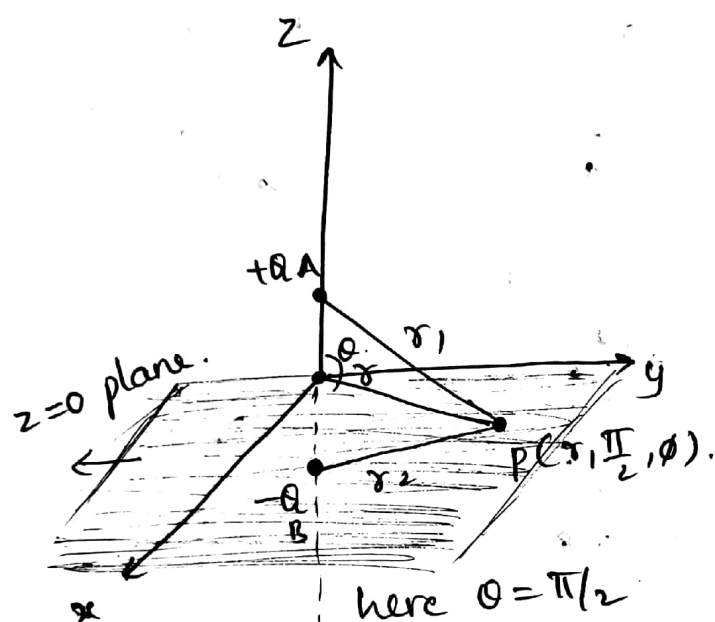


fig (a)

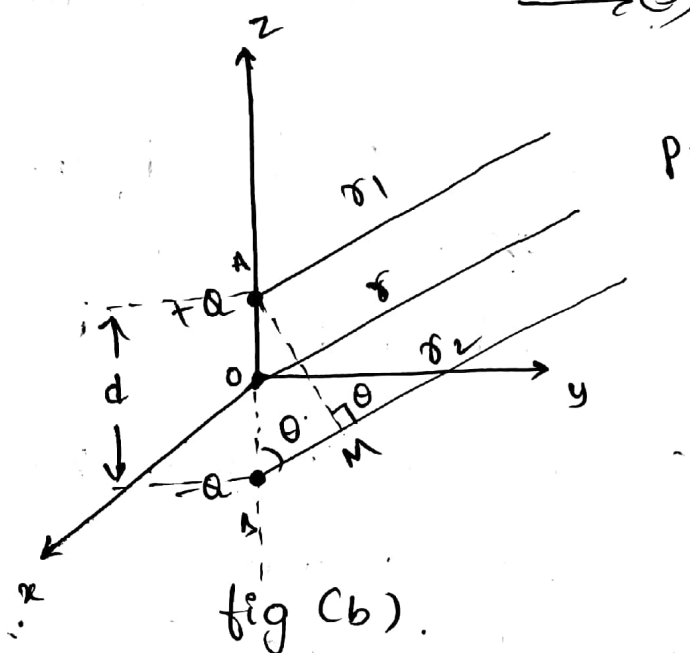


fig (b)

If we consider any point on the xy plane, as shown in fig (a). Then $r_2 = r_1$, hence $V = 0$. Thus $z = 0$ plane is a zero potential plane.

All the points in $z = 0$ plane behaves similar to the points at infinity as all are at zero potential.

Now consider a point p located far away from dipole. Thus r_1, r_2 & r can be assumed to be parallel.

Let the distance vector between the charges $+q$ and $-q$ is $\vec{d}$.

$$\therefore \vec{d} = d\vec{a}_z$$

Its Component along $\vec{a}_r$ can be obtained as

$$d_r = \vec{d} \cdot \vec{a}_r = d\vec{a}_z \cdot \vec{a}_r = d \cos \theta$$

$$\therefore \vec{d} = d \cos \theta \vec{a}_r$$

The product $q\vec{d}$ is called dipole moment and denoted as $\vec{p}$.

$$\therefore \vec{p} = q\vec{d}$$

C-m (Coulomb-meter).

$$\text{Now } \vec{p} \cdot \vec{a}_r = q\vec{d} \cdot \vec{a}_r = qd \cos \theta$$

$\therefore$ Hence V can be written in terms of dipole moment

$$\text{as } \boxed{V = \frac{qd \cos \theta}{4\pi\epsilon_0 r^2} = \frac{\vec{p} \cdot \vec{a}_r}{4\pi\epsilon_0 r^2} \text{ volts.}}$$

& $\vec{E}$ can be written as

$$\left\{ \begin{aligned} C \vec{p} = q\vec{d} &\Rightarrow |\vec{p}| = q|\vec{d}| \\ &\Rightarrow p = qd \end{aligned} \right.$$

$$\vec{E} = \frac{qd}{4\pi\epsilon_0 r^3} [2 \cos \theta \vec{a}_r + \sin \theta \vec{a}_\theta]$$

$$\Rightarrow \boxed{\vec{E} = \frac{p}{4\pi\epsilon_0 r^3} [2 \cos \theta \vec{a}_r + \sin \theta \vec{a}_\theta] \text{ V/m}}$$

Note:- A Single point charge is called monopole.

for which $V \propto \frac{1}{r}$ & $E \propto \frac{1}{r^2}$

The Arrangement of two point charges is called dipole in which $V \propto \frac{1}{r^2}$ & $E \propto \frac{1}{r^3}$.

$$BM = d \cos \theta$$

and $PB = PM + BM$

$$\Rightarrow BM = PB - PM = r_2 - r_1 \quad (\because PM = PA)$$

$$\Rightarrow d \cos \theta = r_2 - r_1 \rightarrow (4)$$

As d is very small, $r_1 \cong r_2 \cong r$ then $r_1 r_2 = r^2 \rightarrow (5)$.

Substitute (4) & (5) in (3)

$$\Rightarrow V = \frac{Q}{4\pi\epsilon_0} \left(\frac{d \cos \theta}{r^2} \right)$$

$$\Rightarrow \boxed{V = \frac{Qd \cos \theta}{4\pi\epsilon_0 r^2} \text{ volts}} \rightarrow \text{potential due to dipole.}$$

Now $\vec{E} = -\nabla V = - \left[\frac{\partial V}{\partial r} \vec{a}_r + \frac{1}{r} \frac{\partial V}{\partial \theta} \vec{a}_\theta + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \vec{a}_\phi \right]$

$$\begin{aligned} \Rightarrow \frac{\partial V}{\partial r} &= \frac{\partial}{\partial r} \left[\frac{Qd \cos \theta}{4\pi\epsilon_0 r^2} \right] = \frac{Qd \cos \theta}{4\pi\epsilon_0} \frac{\partial}{\partial r} \left[\frac{1}{r^2} \right] \\ &= \frac{Qd \cos \theta}{4\pi\epsilon_0} \left[\frac{-2}{r^3} \right] = \frac{-2Qd \cos \theta}{4\pi\epsilon_0 r^3} \end{aligned}$$

$$\Rightarrow \frac{\partial V}{\partial \theta} = \frac{Qd}{4\pi\epsilon_0 r^2} \frac{\partial}{\partial \theta} [\cos \theta] = \frac{-Qd \sin \theta}{4\pi\epsilon_0 r^2}$$

$$\Rightarrow \frac{\partial V}{\partial \phi} = 0$$

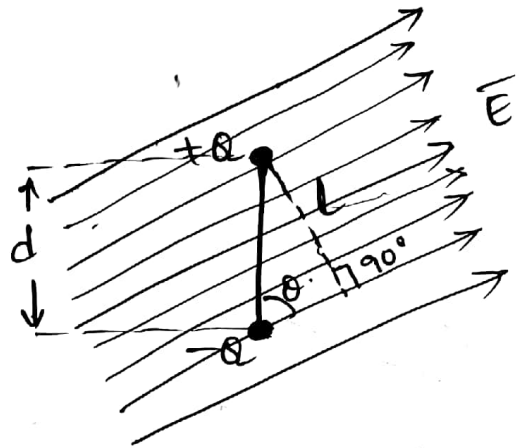
$$\therefore \vec{E} = - \left[\frac{-2Qd \cos \theta}{4\pi\epsilon_0 r^3} \vec{a}_r - \frac{Qd \sin \theta}{4\pi\epsilon_0 r^3} \vec{a}_\theta \right]$$

$$\boxed{\vec{E} = \frac{Qd}{4\pi\epsilon_0 r^3} [2 \cos \theta \vec{a}_r + \sin \theta \vec{a}_\theta]}$$

This is the $\vec{E}$ due to an Electric dipole.

field :-

Consider a dipole is placed in an uniform Electric field $\vec{E}$ making an angle θ with dipole axis.



The charges $+q$ and $-q$ experience a force due to $\vec{E}$ which is equal in magnitude opposite in direction

$$\therefore \text{Force on } +q \Rightarrow \vec{F}_1 = q\vec{E}$$

$$\text{Force on } -q \Rightarrow \vec{F}_2 = -q\vec{E}$$

$$\therefore |\vec{F}_1| = |\vec{F}_2| = qE$$

$\therefore \text{Torque} = \text{Force} \times \text{perpendicular distance of separation}$

$$T = F \times l = qEl$$

where $l = \text{perpendicular distance}$. & from the figure. $\sin\theta = l/d \Rightarrow l = d\sin\theta$

$$\Rightarrow \text{Torque } (T) = qEd\sin\theta$$

$$= qd \cdot E \sin\theta$$

$$T = p \cdot E \sin\theta$$

$$(\because p = qd)$$

$$\therefore AB \sin\theta = \vec{A} \times \vec{B}$$

$$\therefore \text{In vector form } \boxed{\vec{T} = \vec{p} \times \vec{E}}$$

The flow of charges per unit time or rate of flow of charges at a specified point or across a specified surface is called an Electric Current.

$$I = \frac{dq}{dt} \quad \text{C/s or Amperes.}$$

There are two types of Current flows.

- Conventional or drift Current.
- Convection or displacement Current.

1, Conventional Current :-

The Current which exists in the conductors due to the drifting of electrons, under the influence of applied voltage is called drift Current.

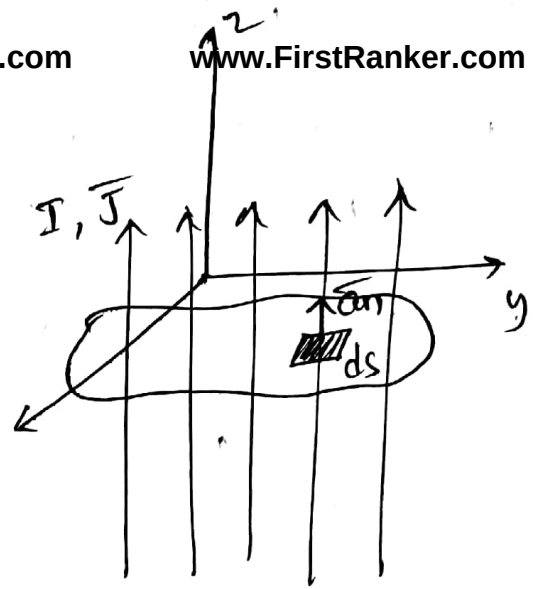
2, Convection Current :-

In dielectrics there can be flow of charges, under the influence of Electric field Intensity. Such a current is called displacement Current or Convection Current. Ex:- Current flowing through Capacitor.

⇒ Current density :-

The Current density is defined as the current passing through the unit surface area, when the surface is held normal to the direction of current. It is measured in Ampere per Square meters (A/m²).

Consider a current I ,
passing through a surface S .
The direction of Current is
normal to the surface therefore
direction of $\vec{J}$ is also normal to
the plane.



To define Current density consider an incremental
Surface ds & its direction $\vec{n}$

$$\therefore d\vec{s} = ds \vec{n} \quad \& \quad \vec{J} = J \vec{n}$$

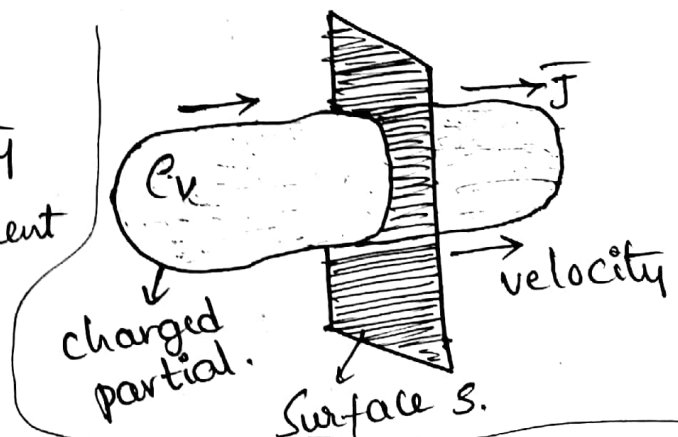
Then the differential Current passing through ds is
given by $dI = \vec{J} \cdot d\vec{s}$

$\therefore$ The total Current

$$I = \int_S \vec{J} \cdot d\vec{s} \quad \text{Amps}$$

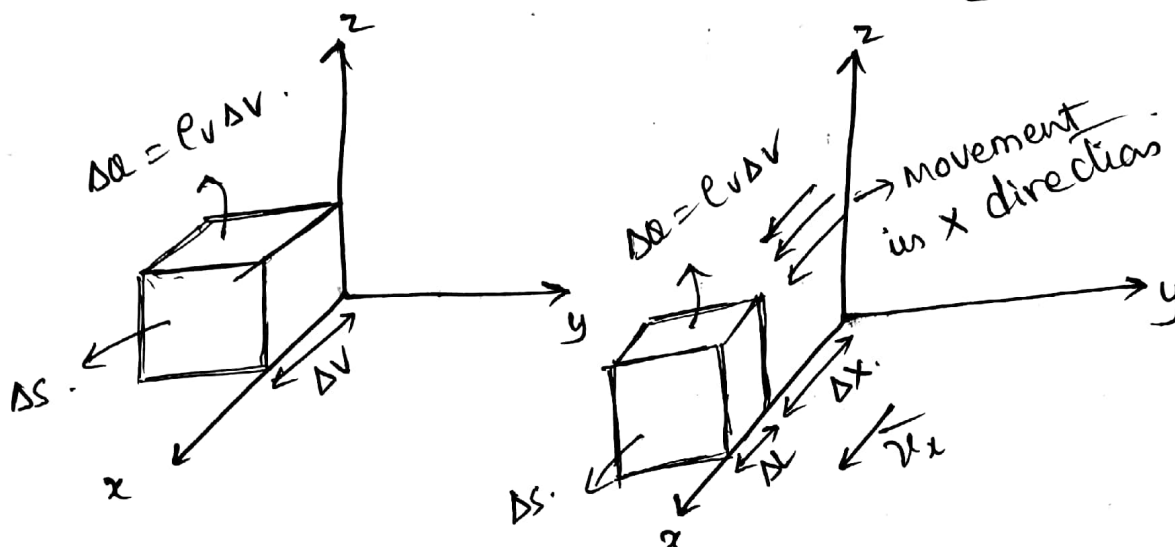
$\Rightarrow$ Convection / displacement Current density

Set of charged particles
give rise to a charge density
 ρ_v in a volume v . The current
density $\vec{J}$ can be related to
the velocity with which volume
charged particles moving in volume v crosses surface S .



To derive Relation b/w $\vec{J}$ & e_v . Consider differential volume ΔV having charge density e_v .

$\therefore$ The charge on ΔV is given by $\boxed{\Delta Q = e_v \Delta V}$



let ΔL is the incremental length while ΔS is the incremental surface $\therefore \Delta V = \Delta S \Delta L$.

$$\& \Delta Q = e_v \Delta S \Delta L$$

let the charge is moving in x direction with velocity $\vec{v}$. Thus velocity has only Component in x direction.

$$\therefore \Delta I = \frac{\Delta Q}{\Delta t}$$

Now $\Delta Q = e_v \Delta S \Delta x$ as the Charge moved a length Δx & it is Responsible for the Current

$$\Rightarrow \Delta I = e_v \Delta S \frac{\Delta x}{\Delta t}$$

$$\therefore \Delta I = e_v \Delta S \cdot v_x$$

$$\Rightarrow \vec{J} = \frac{\Delta I}{\Delta S} = e_v v_x \Rightarrow$$

$$\therefore \frac{\Delta x}{\Delta t} = v_x$$

$$\boxed{I = \oint_s \vec{J} \cdot d\vec{s} = -\frac{dQ}{dt}}$$

This is Integral form of Continuity Equation of Current.
Similarly If the action is reversed then

$$-I = \oint_s \vec{J} \cdot d\vec{s} = \frac{dQ}{dt}$$

using divergence theorem. $\oint_s \vec{J} \cdot d\vec{s} = \int_{vol} (\nabla \cdot \vec{J}) dv$

$$\Rightarrow -\frac{dQ}{dt} = \int_{vol} (\nabla \cdot \vec{J}) dv$$

But $Q = \int_V \rho_v \cdot dv$

$$\Rightarrow -\frac{d}{dt} \left[\int_V \rho_v dv \right] = \int_V (\nabla \cdot \vec{J}) dv$$

$$\therefore \int_V (\nabla \cdot \vec{J}) dv = -\int \frac{\partial \rho_v}{\partial t} \cdot dv$$

If This relation is true for any volume, it must be true even for incremental volume Δv

$$\therefore (\nabla \cdot \vec{J}) \Delta v = -\frac{\partial \rho_v}{\partial t} \Delta v$$

$$\Rightarrow \boxed{\nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t}}$$

This is the point form or differential form of the Continuity Equation.

For steady current ρ_v is not function of time $\Rightarrow \frac{\partial \rho_v}{\partial t} = 0$

$$\Rightarrow \boxed{\nabla \cdot \vec{J} = 0} \text{ for steady current.}$$

$$J = I / \Delta v \sqrt{v_x}$$

$$\vec{J} = J \cdot \vec{a_x}$$

$$\vec{v} = v_x \vec{a_x}$$

$$\boxed{\vec{J} = \rho_v \vec{v}} \quad A/m^2$$

This is the Equation for Convection Current density.

⇒ Continuity Equation :

This the Continuity Equation of the Current is based on the principle of Conservation of Charge.

ie. The charge Can neither be Created nor be destroyed

Consider a closed Surface S with a Current density $\vec{J}$, Then the total Current passing through Surface

$$I = \oint \vec{J} \cdot d\vec{s}$$

The Current flow outward from the closed Surface. Hence the Current I is Constituted due to outward flow of Charges from the closed Surface - S .

According to principle of Conservation of charge there must be decrease of an Equal amount of charges inside the closed Surface.

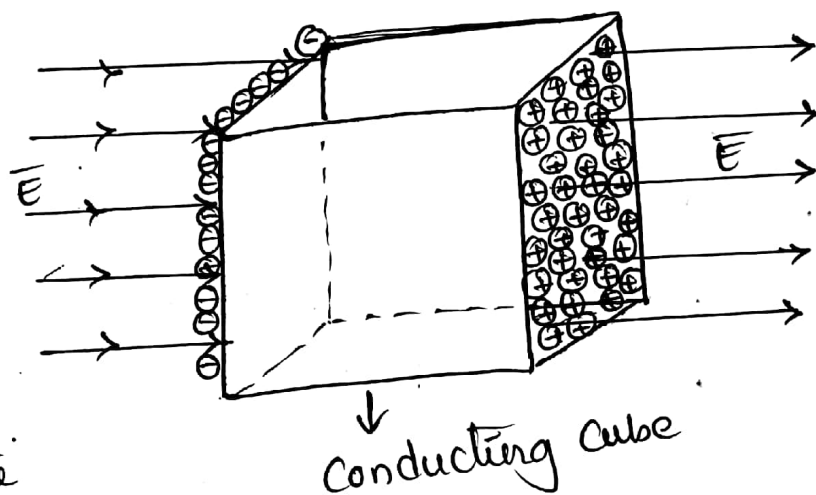
Hence outward ^{rate of} flow of charges get balanced by the rate of decrease of charge inside closed Surface. let Q = charge with in closed Surface

$$-\frac{dQ}{dt} = \text{Rate of decrease of charge with in the Surface}$$

Conductors have large number of free electrons, such that they contribute electrons in the conduction process.

Behaviour of conductors placed in Electric field $\vec{E}$

When the conductor is placed in Electric field $\vec{E}$, the free electrons in the conductor migrate towards the surface of the



conductor and appears as the surface charge, thereby contributing to the surface charge density ' ρ_s '.

In the presence of Electric field the positive charges migrate towards the surface, where the Electric field lines are leaving the conductor, and the negative charges migrate towards the surface at which Electric field lines are entering the conductor (cube).

⇒ properties Related to behaviour of conductor in an Electric field.

→ As All the charges move to the surface and exist as the surface charge, the surface of the conductor seems to be an Equipotential surface.

The migration of free charges towards the surface of a conductor leads to surface charge density. This produces a secondary field inside the conductor which is equal in magnitude to the external field $\vec{E}$ but opposite in direction.

- Therefore electric field inside the conductor is zero
- The electric field intensity $\vec{E}$ at any point on the surface is equal to $\frac{1}{\epsilon_0}$ times the surface charge density at that point.
- As all the charges move to the surface the total inside the conductor is zero i.e. $Q=0$, therefore $\rho_v=0$ inside conductor.

⇒ Conventional / conduction current density :-

Under the influence of applied voltage free electrons in conductor attain average velocity called drift velocity $\vec{v}_d$.

The drift velocity is directly proportional to applied electric field.

$$\vec{v}_d \propto \vec{E}$$

The constant of proportionality is called mobility of electrons & is denoted as ' μ_e '.

$$\Rightarrow \vec{v}_d = -\mu_e \vec{E} \quad \longrightarrow (1)$$

-ve sign indicates that the velocity of electrons is against the direction of $\vec{E}$.

$$\therefore \mu = \frac{\text{velocity}}{\text{Field}} = \frac{\text{m/s}}{\text{V/m}} = \frac{\text{m}^2}{\text{V-s}}$$

According to relation between $\bar{J}$ & $\bar{v}$ we have

$$\bar{J} = \rho_v \bar{v}$$

But in materials protons & electrons are same $\therefore \rho_v = 0$
The drift velocity is the velocity of free electrons hence
above relation can be expressed as

$$\bar{J} = \rho_e \bar{v}_d \longrightarrow (1)$$

ρ_e = charge density due to electrons

$\bar{v}_d$ = drift velocity of electrons

Substitute (1) in (2)

$$\therefore \bar{J} = -\mu_e \rho_e \bar{E}$$

$$\boxed{\therefore \bar{J} = -\mu_e \rho_e \bar{E}} \longrightarrow (3)$$

This is the Equation for Conduction current density.

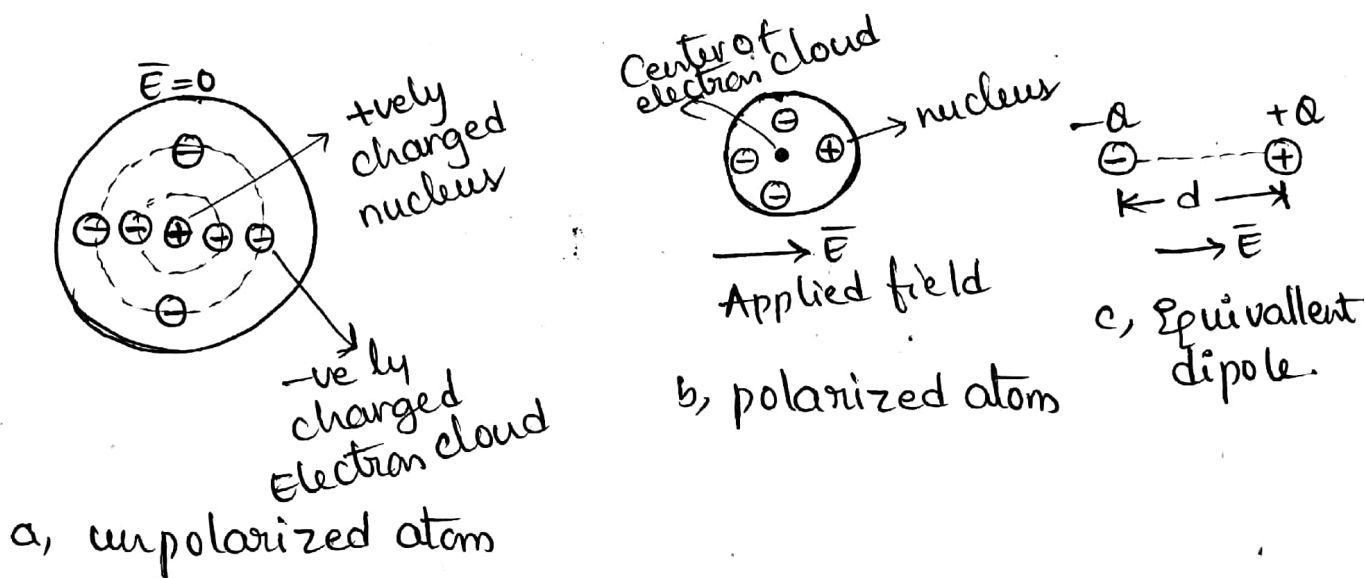
The Relation between $\bar{J}$ & $\bar{E}$ can also be expressed in terms
of conductivity as

$$\boxed{\bar{J} = \sigma \bar{E}} \longrightarrow (4)$$

$$\therefore \text{from (3) \& (4)} \quad \boxed{\sigma = -\mu_e \rho_e} \text{ } \Omega/\text{m}$$

Equation (4) is called point form of ohm's law.

- Dielectric materials do not have free electrons & free charges. The charges in dielectrics are bound by the finite forces and hence called bound charges. As they are bound and not free, they can not contribute to conduction process.
- If they are subjected to $\vec{E}$, they shift their relative positions against normal molecular and atomic forces. This causes the dielectric to store the energy.



- The shift in +ve and -ve charges are opposite directions and under the influence of applied $\vec{E}$ such charges act like small electric dipole.

polarization :-

The process, due to which separation of bound charges results to produce electric dipoles, under influence of electric field $\vec{E}$, is called polarization.

When the dipole results from the displacement of bound charges, the dielectric is said to be polarized.

1. Non polar
2. polar.

In non polar molecules, the dipole arrangement is totally absent, in absence of electric field $\vec{E}$. It results only when an externally field $\vec{E}$ is applied to it.

In polar molecules, the permanent displacement b/w centres of +ve & -ve charges exist. Thus dipole arrangement exist without application of $\vec{E}$.

⇒ Mathematical Expression for polarization :-

when dipole is formed due to polarization, there exist an Electric dipole moment $\vec{p}$

$$\vec{p} = q\vec{d}$$

q = magnitude of one of the two charges

$\vec{d}$ = distance vector b/w - q to + q .

let n = no. of dipoles per unit volume

ΔV = Total volume of dielectric

$N = n\Delta V$ = Total dipoles.

∴ Total dipole moment is to be obtained by using Super position principle as

$$\vec{p}_{\text{total}} = q_1\vec{d}_1 + q_2\vec{d}_2 + \dots + q_n\vec{d}_n$$

$$\boxed{\vec{p}_{\text{total}} = \sum_{i=1}^{n\Delta V} q_i\vec{d}_i}$$

per unit volume.

$$\bar{P} = \lim_{\Delta V \rightarrow 0} \frac{\sum_{i=1}^{n\Delta V} q_i \bar{d}_i}{\Delta V} \quad \text{C/m}^2$$

The units of $\bar{P}$ is C/m^2 & is same as units of flux density. $\therefore$ polarization increases flux density in dielectric medium.

$$\therefore \bar{D} = \epsilon_0 \bar{E} + \bar{P}$$

For isotropic & linear medium, $\bar{P}$ and $\bar{E}$ are parallel

$$\therefore \bar{P} = \chi_e \epsilon_0 \bar{E} \quad \chi_e = \text{Susceptibility of material.}$$

$$\therefore \bar{D} = \epsilon_0 \bar{E} + \chi_e \epsilon_0 \bar{E}$$

$$\bar{D} = (\chi_e + 1) \epsilon_0 \bar{E}$$

$$\Rightarrow \boxed{\bar{D} = \epsilon \bar{E}}$$

The Quantity $\chi_e + 1$ is defined as relative permittivity or dielectric Constant

$$\boxed{\epsilon_r = \chi_e + 1}$$

$\Rightarrow$ Dielectric strength

An Ideal dielectric is non conducting but practically no dielectric can be ideal. As the Electric field applied to dielectric increases sufficiently, due to forces exerted on the molecules, electrons in dielectric become free

Such large electric field, the dielectric becomes conducting due to presence of large number of free electrons. This condition of dielectric is called Dielectric breakdown.

⇒ Dielectric strength :-

The minimum value of applied Electric field at which the dielectric breakdown is called dielectric strength of that dielectric. units are V/m or KV/cm.

⇒ Boundary Conditions :-

When an Electric field passes from one medium to another medium, we need to know the conditions at the boundary b/w the two media. These conditions at the boundary are called boundary conditions.

Depending on the nature of media, there are two types of boundary conditions.

1. Boundary b/w Conductor & free Space (dielectric)
2. Boundary between two dielectrics with different properties.

For studying Boundary Conditions the maxwells Equations for Electrostatic fields are Required.

$$\Rightarrow \oint \vec{E} \cdot d\vec{l} = 0 \quad \& \quad \oint_S \vec{D} \cdot d\vec{s} = Q.$$

Early field Intensity E is decomposed into two components:

- i, tangential to boundary ($E_{\tan}$)
- ii, normal to boundary (E_N).

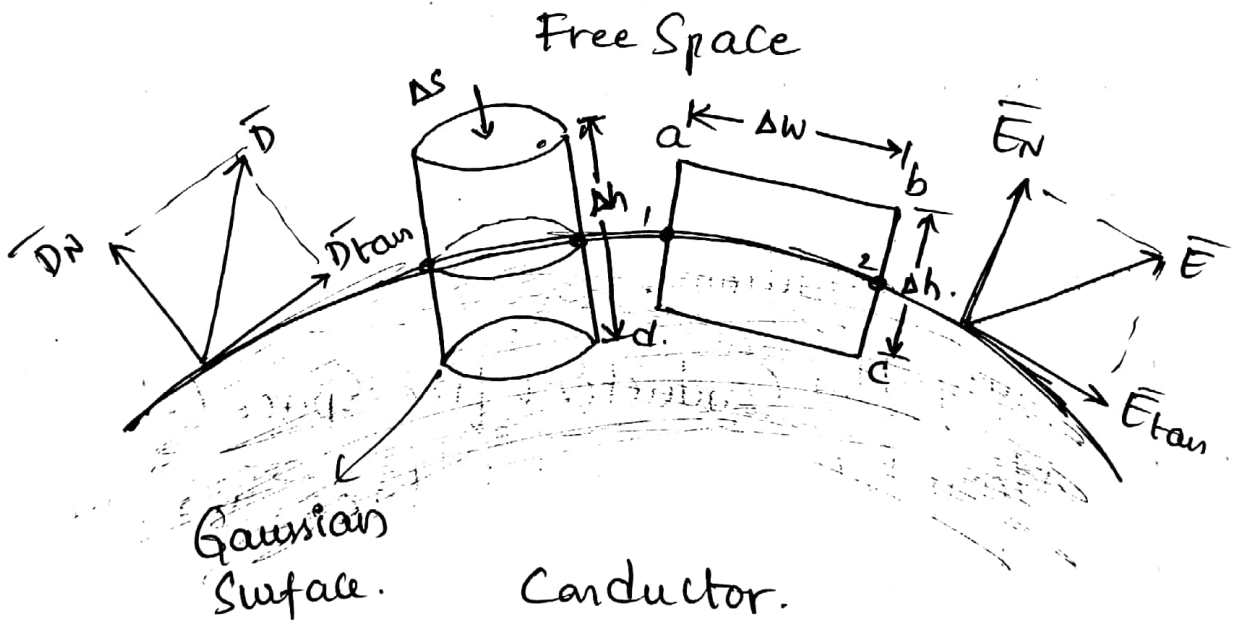
$$\therefore \vec{E} = \vec{E}_{\text{tan}} + \vec{E}_N \quad \text{Similarly } \vec{D}$$

⇒ Boundary Conditions between conductor & free space

For Ideal Conductor we know

- field Intensity E inside Conductor is zero
- Total charge inside the Conductor is zero i.e. $Q=0$
hence D inside Conductor is also zero.

$\therefore \vec{E}, \vec{D}$ & ρ_v within conductor are zero



To determine boundary Conditions let us use the closed path & the Gaussian Surface.

i, E at The Boundary

let E at the boundary making some angle with the boundary. This E can be resolved into two components

1, The Component tangential to the Surface ($\vec{E}_{tan}$).
2, The Component normal to the Surface ($\vec{E}_N$).

Consider a rectangular closed path abcd, such that line integral around a closed path is equals to zero.

$$\therefore \oint \vec{E} \cdot d\vec{l} = 0$$

The Integral over a closed path abcd can be divided into four parts.

$$\therefore \oint \vec{E} \cdot d\vec{l} = \int_a^b \vec{E} \cdot d\vec{l} + \int_b^c \vec{E} \cdot d\vec{l} + \int_c^d \vec{E} \cdot d\vec{l} + \int_d^a \vec{E} \cdot d\vec{l} = 0 \quad \rightarrow (1)$$

portion cd :-

portion c-d is in the conductor where $\vec{E} = 0$ hence corresponding integral is zero

$$\therefore \int_a^b \vec{E} \cdot d\vec{l} + \int_b^c \vec{E} \cdot d\vec{l} + \int_c^d \vec{E} \cdot d\vec{l} = 0$$

portion a-b :- As width Δw is very small, $\vec{E}$ over it can be assumed to be constant hence

$$\int_a^b \vec{E} \cdot d\vec{l} = |\vec{E}| \int_a^b |d\vec{l}| = |\vec{E}| \Delta w$$

but Δw is along tangential direction to the boundary

$$|\vec{E}| = E_{tan} \Rightarrow \int_a^b \vec{E} \cdot d\vec{l} = E_{tan} \Delta w \quad \rightarrow (2)$$

portion b-c :- over a small height Δh , $\vec{E}$ is normal to the boundary & integration can be divided into

$$2 \text{ parts} \quad \int_b^c \vec{E} \cdot d\vec{l} = \int_b^2 \vec{E} \cdot d\vec{l} + \int_2^c \vec{E} \cdot d\vec{l} \quad \rightarrow (3)$$

portion 2-c is in conductor where $\vec{E} = 0$ inside conductor & in portion b-2 $\vec{E}$ has normal component to boundary i.e. $\vec{E}_N$

$$\int_b^2 \vec{E} \cdot d\vec{l} = \int_b^2 E_N \Delta h / 2$$

Similarly for path d-a

$$\therefore \int_d^a \vec{E} \cdot d\vec{l} = -E_N \Delta h / 2 \rightarrow (4)$$

$\therefore$ Substituting (2)(3) & (4) in (1)

$$\Rightarrow E_{tan} \Delta w + E_N \frac{\Delta h}{2} - E_N \frac{\Delta h}{2} = 0$$

$$\Rightarrow E_{tan} \Delta w = 0 \Rightarrow \boxed{E_{tan} = 0}$$

Similarly $\boxed{D_{tan} = \epsilon_0 E_{tan} = 0}$ ($\because \vec{D} = \epsilon_0 \vec{E}$)

Thus the tangential component of the Electric field Intensity and Electric flux density is zero at the boundary between conductor and free space.

D_N at The boundary :-

To find normal component of $\vec{D}$, select a closed Gaussian Surface in the form of right circular cylinder.

$\therefore$ According to Gauss law $\oint \vec{D} \cdot d\vec{s} = Q$.

The Surface integral must be evaluated over three Surfaces

$$\therefore \oint \vec{D} \cdot d\vec{s} = \int_{top} \vec{D} \cdot d\vec{s} + \int_{bottom} \vec{D} \cdot d\vec{s} + \int_{side} \vec{D} \cdot d\vec{s}$$

$$\int_{top} \vec{D} \cdot d\vec{s} + \int_{bottom} \vec{D} \cdot d\vec{s} + \int_{side} \vec{D} \cdot d\vec{s} = Q$$

The bottom Surface is in conductor where $\vec{D} = 0$ inside conductor. Top Surface is in free space & shift the Top Surface to the boundary hence $\Delta h \rightarrow 0$

$$\therefore \int_{top} \vec{D} \cdot d\vec{s} + \int_{side} \vec{D} \cdot d\vec{s} = Q$$

$\Delta h \rightarrow 0$ side integral is also zero. $\therefore \int \vec{D} \cdot d\vec{s} = 0$

hence, $\int_{\text{top}} \vec{D} \cdot d\vec{s} = Q$

when surface shifted to boundary $\vec{D}$ on the surface has only normal component $\therefore |\vec{D}| = D_N$.

$$\Rightarrow |\vec{D}| \int_{\text{top}} |d\vec{s}| = Q$$

$$\Rightarrow \boxed{D_N \cdot \Delta S = Q} \quad \& \quad Q = \rho_s \cdot \Delta S.$$

$$\Rightarrow D_N \Delta S = \rho_s \Delta S \Rightarrow \boxed{D_N = \rho_s}$$

$\therefore$ Normal Component of flux density at the boundary is equals to the surface charge density

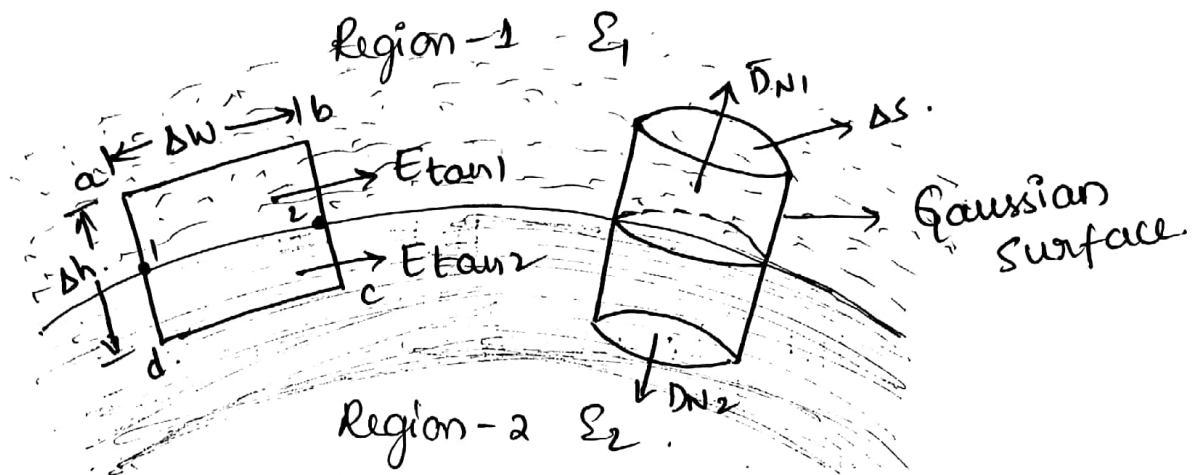
$$\& \quad D_N = \epsilon_0 E_N = \rho_s$$

$$\Rightarrow \boxed{E_N = \frac{\rho_s}{\epsilon_0}}$$

If the free space is replaced by a dielectric medium with permittivity $\epsilon = \epsilon_0 \epsilon_r$

Then boundary Conditions are

$$\boxed{\begin{aligned} E_{\text{tan}} &= D_{\text{tan}} = 0 \\ D_N &= \rho_s \\ E_N &= \frac{\rho_s}{\epsilon} = \frac{\rho_s}{\epsilon_0 \epsilon_r} \end{aligned}}$$



Consider boundary between two dielectrics with permittivities ϵ_1 & ϵ_2 .

Consider a closed path ~~acba~~ abcd as rectangular path with elementary width Δw & elementary height Δh . $\vec{E}$ over closed path can be resolved into four components:

$$\therefore \oint_L \vec{E} \cdot d\vec{l} = \int_a^b \vec{E} \cdot d\vec{l} + \int_b^c \vec{E} \cdot d\vec{l} + \int_c^d \vec{E} \cdot d\vec{l} + \int_d^a \vec{E} \cdot d\vec{l} = 0. \quad \rightarrow (1)$$

$$\therefore \text{NOW } \vec{E}_1 = \vec{E}_{1t} + \vec{E}_{1n}$$

$$\vec{E}_2 = \vec{E}_{2t} + \vec{E}_{2n}$$

$$|\vec{E}_{1t}| = E_{tan1}, \quad |\vec{E}_{2t}| = E_{tan2}$$

$$|\vec{E}_{1n}| = E_{1n}$$

$$|\vec{E}_{2n}| = E_{2n}$$

Now for the rectangular closed path to be reduced at the surface to analyse boundary conditions, $\Delta h \rightarrow 0$

As $\Delta h \rightarrow 0$, $\int_b^c$ and $\int_d^a$ becomes zero hence Equation (1)

becomes

$$\int_a^b \vec{E} \cdot d\vec{l} + \int_c^d \vec{E} \cdot d\vec{l} = 0 \quad \rightarrow (2)$$

Component

portion a-b. is in dielectric 1. & $\vec{E}$ has tangential (12)

$$\int_a^b \vec{E} \cdot d\vec{l} = E_{tan1} \int_a^b |d\vec{l}| = E_{tan1} \Delta W \rightarrow (3)$$

while for c-d.

$$\int_c^d \vec{E} \cdot d\vec{l} = -E_{tan2} \Delta W \rightarrow (4)$$

Substitute (3) and (4) in (2)

$$\Rightarrow E_{tan1} \Delta W - E_{tan2} \Delta W = 0$$

$$\Rightarrow \boxed{E_{tan1} = E_{tan2}}$$

hence $D_{tan1} = \epsilon_1 E_{tan1}$
 $D_{tan2} = \epsilon_2 E_{tan2}$

$$\Rightarrow \boxed{\frac{D_{tan1}}{D_{tan2}} = \frac{\epsilon_1}{\epsilon_2} = \frac{\epsilon_{r1}}{\epsilon_{r2}}}$$

To find the normal components consider a Gaussian Surface. let $\Delta h \rightarrow 0$ hence. (shift the surface of the Gaussian. surface. to the boundary).

$$\therefore \oint_S \vec{D} \cdot d\vec{s} = Q.$$

$$\therefore \int_{top} \vec{D} \cdot d\vec{s} + \int_{bottom} \vec{D} \cdot d\vec{s} + \int_{sides} \vec{D} \cdot d\vec{s} = Q.$$

But $\int_{side} \vec{D} \cdot d\vec{s} = 0$ as $\Delta h \rightarrow 0$

$$\therefore \int_{top} \vec{D} \cdot d\vec{s} + \int_{bottom} \vec{D} \cdot d\vec{s} = Q. \rightarrow (5)$$

at the boundary $\vec{D}$ has normal components

$$\therefore \int_{top} \vec{D} \cdot d\vec{s} = |\vec{D}| \int_{top} |d\vec{s}| = D_{N1} \Delta S. \rightarrow (6)$$

$$\& \int_{bottom} \vec{D} \cdot d\vec{s} = |\vec{D}| \int_{bottom} |d\vec{s}| = -D_{N2} \Delta S. \rightarrow (7)$$

$$D_{N1} \Delta S - D_{N2} \Delta S = Q$$

$$\Rightarrow D_{N1} - D_{N2} = \frac{Q}{\Delta S} = \rho_s.$$

there is no free charge available in perfect dielectric
 $\therefore \rho_s = 0.$

$$\Rightarrow D_{N1} - D_{N2} = 0$$

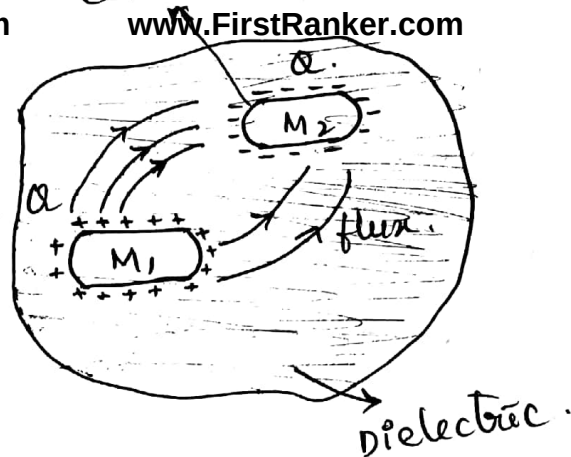
$$\Rightarrow \boxed{D_{N1} = D_{N2}}$$

$$6 \quad D_{N1} = \epsilon_1 E_{N1}, \quad D_{N2} = \epsilon_2 E_{N2} \Rightarrow \frac{D_{N1}}{D_{N2}} = \frac{\epsilon_1 E_{N1}}{\epsilon_2 E_{N2}} = 1$$

$$\Rightarrow \boxed{\frac{E_{N1}}{E_{N2}} = \frac{\epsilon_2}{\epsilon_1} = \frac{\epsilon_{r2}}{\epsilon_{r1}}}$$

Consider two Conducting materials M_1 & M_2 which are placed in a dielectric medium having permittivity ϵ .

They are carrying Equal and Opposite charge.



Such a System which has two Conducting Surfaces carrying Equal and Opposite charges Separated by dielectric is called Capacitive System giving rise to a Capacitance.

The Ratio of magnitudes of the total charge on any one of the two Conductors and potential difference between the conductors is called Capacitance of the two conductor system denoted as C .

$$\Rightarrow C = \frac{Q}{V}$$

Q = charge in Coulombs
 V = potential difference

As charge Q resides only on the surface of the conductor, hence from Gauss law

$$Q = \oint_S \vec{D} \cdot d\vec{s} = \oint_S \epsilon_0 \epsilon_r \vec{E} \cdot d\vec{s} = \oint_S \epsilon \vec{E} \cdot d\vec{s}$$

& potential difference between conductors is given by

$$V = - \int_L \vec{E} \cdot d\vec{l} = - \int^+ \vec{E} \cdot d\vec{l}$$

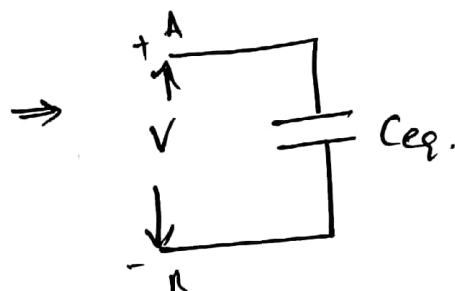
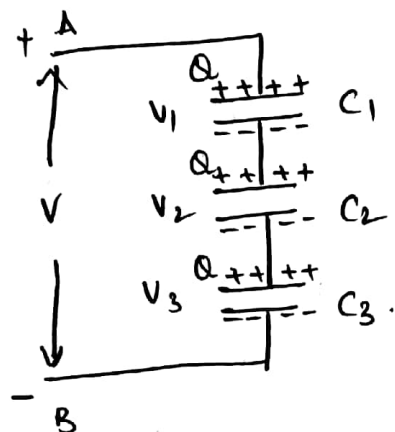
Hence Capacitance can be expressed as

$$C = \frac{Q}{V} = \frac{\oint_S \epsilon \vec{E} \cdot d\vec{s}}{- \int^+ \vec{E} \cdot d\vec{l}} \quad \text{Farads.}$$

Q is increased then E , D and V increases correspondingly hence Capacitance is independent of E , D , V and charge.

⇒ Capacitance in Series :-

Consider three Capacitors in Series connected across the applied voltage V .



Equivalent circuit.

The voltage pushes charge Q on C_1 , then opposite plate of C_1 must have the same charge. $\therefore$ all three capacitors have the same charge Q .

$$\therefore Q = C_1 V_1 = C_2 V_2 = C_3 V_3$$

$$\text{and } V = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3}$$

$\therefore$ From Equivalent circuit

$$C_{eq} = \frac{Q}{V} \text{ or } V = \frac{Q}{C_{eq}}$$

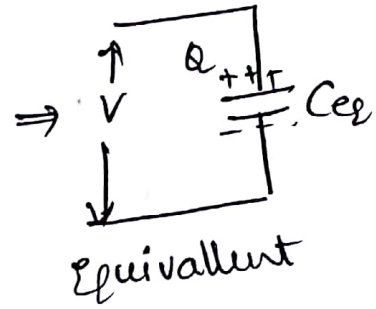
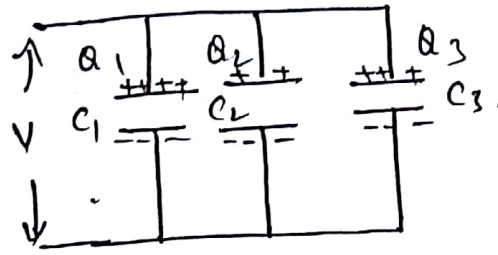
$$\therefore \text{Total voltage } V = V_1 + V_2 + V_3$$

$$\Rightarrow \frac{Q}{C_{eq}} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3}$$

$$\Rightarrow \boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}}$$

$$\text{If } n \text{ Capacitors in Series } \Rightarrow \boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \dots + \frac{1}{C_n}}$$

when Capacitors are in parallel Same voltage exists across them but charges are different.



$$\Rightarrow Q_1 = C_1 V, Q_2 = C_2 V, Q_3 = C_3 V$$

The total charge

$$Q = Q_1 + Q_2 + Q_3$$

From Equivalent circuit $Q = C_{eq} V$

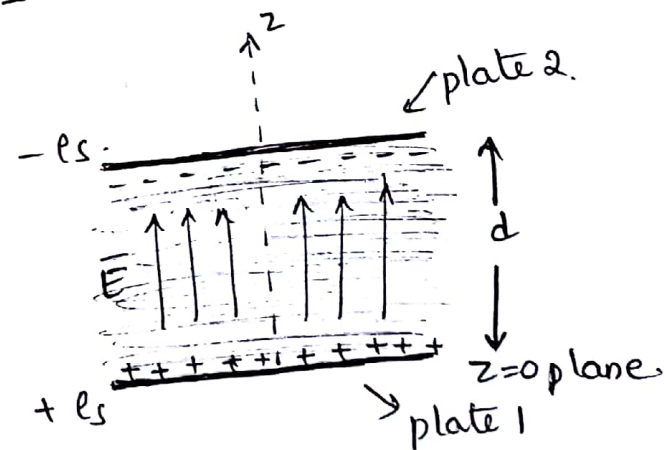
$$\Rightarrow C_{eq} V = C_1 V + C_2 V + C_3 V$$

$$\Rightarrow \boxed{C_{eq} = C_1 + C_2 + C_3} \text{ Farads}$$

For n Series Connected Capacitors $C_{eq} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \dots + \frac{1}{C_n}}$

parallel plate Capacitor :-

It consists of two parallel metallic plates separated by distance 'd'. The space between plates is filled with dielectric of permittivity ϵ .



lower plate carries +ve charge with charge density $+\epsilon_s$. & upper plate carries -ve charge with charge density $-\epsilon_s$. plate 1 is plate at $z=0$ plane & plate 2 is placed at $z=d$ plane.

Total charge on any one of the plate is
 $\therefore Q = \rho_s A$ Coulombs.

where $A =$ area of plates.

To find V let us find out $\vec{E}$ between plates.

Assuming plate 1 to be infinite sheet of charge.

$$\vec{E}_1 = \frac{\rho_s}{2\epsilon} \vec{a}_n = \frac{\rho_s}{2\epsilon} \vec{a}_z \text{ V/m}$$

$\vec{E}_1$ is normal at the boundary between conductor & dielectric without any tangential component.

Similarly for plate 2. $\vec{E}_2 = \frac{-\rho_s}{2\epsilon} \vec{a}_n = \frac{-\rho_s}{2\epsilon} (-\vec{a}_z)$

$$\Rightarrow \vec{E}_2 = \frac{\rho_s}{2\epsilon} \vec{a}_z \text{ V/m.}$$

$\therefore$ Total $\vec{E} = \vec{E}_1 + \vec{E}_2$

$$\vec{E} = \frac{\rho_s}{\epsilon} \vec{a}_z$$

potential difference is given by

$$V = - \int_{-}^{+} \vec{E} \cdot d\vec{l} = - \int_{\text{upper}}^{\text{lower}} \frac{\rho_s}{\epsilon} \vec{a}_z \cdot d\vec{l}$$

where $d\vec{l} = dx \vec{a}_x + dy \vec{a}_y + dz \vec{a}_z$

$$\therefore V = - \int_{z=d}^{z=0} \frac{\rho_s}{\epsilon} \vec{a}_z \cdot (dx \vec{a}_x + dy \vec{a}_y + dz \vec{a}_z)$$

$$= - \int_{z=d}^{z=0} \frac{\rho_s}{\epsilon} dz = - \frac{\rho_s}{\epsilon} [z]_d^0 = - \frac{\rho_s (-d)}{\epsilon}$$

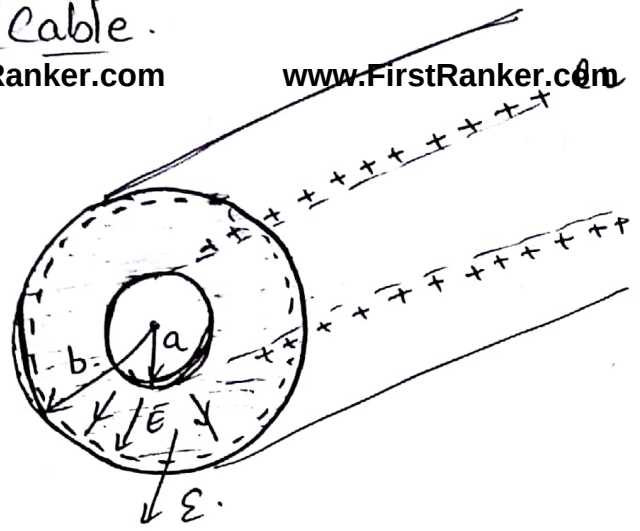
$$V = \frac{\rho_s d}{\epsilon} \text{ volts}$$

$\therefore$ Capacitance $C = \frac{Q}{V} = \frac{\rho_s \cdot A}{\frac{\rho_s \cdot d}{\epsilon}} = \frac{\epsilon A}{d}$

$$\Rightarrow \boxed{C = \frac{\epsilon_0 \epsilon_r A}{d}} \text{ Farads.}$$

$\therefore$ Capacitance depends on permittivity, Area of plate & distance of separation between plates.

Let a = inner radius
 b = outer radius of
 Co-axial Cable & gap
 between conductors is filled
 with dielectric of permittivity
 ϵ . & length of cable is L .



Inner conductor carries charge density $+e_L$ C/m & outer conductor carries charge density $-e_L$ C/m.

$$\therefore Q = e_L \times L.$$

Assume Infinite line charge hence.

$$\vec{E} = \frac{e_L}{2\pi\epsilon r} \vec{a}_r.$$

$\vec{E}$ is directed from inner conductor to the outer conductor

$$\begin{aligned} \therefore V &= - \int_{-}^{+} \vec{E} \cdot d\vec{l} = - \int_{r=b}^{r=a} \frac{e_L}{2\pi\epsilon r} \vec{a}_r \cdot dr \vec{a}_r \\ &= - \frac{e_L}{2\pi\epsilon} [\ln r]_b^a = - \frac{e_L}{2\pi\epsilon} \ln\left(\frac{a}{b}\right) \end{aligned}$$

$$\Rightarrow V = \frac{e_L}{2\pi\epsilon} \ln\left(\frac{b}{a}\right) \text{ volts}$$

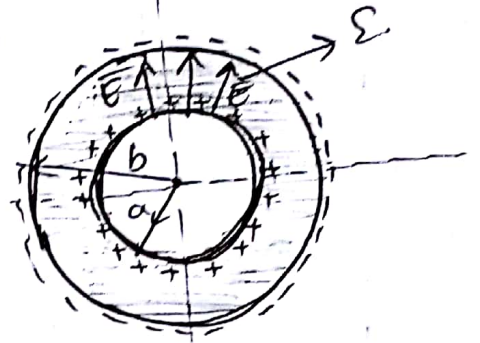
$$\Rightarrow \text{Capacitance } C = \frac{Q}{V} = \frac{e_L \times L}{\frac{e_L}{2\pi\epsilon} \ln\left(\frac{b}{a}\right)}$$

$$\Rightarrow \boxed{C = \frac{2\pi\epsilon L}{\ln\left(\frac{b}{a}\right)} \text{ Farads}}$$

Consider a Spherical Capacitor formed of two Concentric Spherical Conducting shells. of radius a and b .

where $b > a$. The region between shells is filled with permittivity of ϵ .

The inner Sphere Carries +ve charge $+Q$.
and inner Sphere Carries -ve charge $-Q$.



Consider a Gaussian Surface as a Sphere of radius r , hence

$$\vec{E} = \frac{Q}{4\pi\epsilon r^2} \vec{a}_r \quad \text{V/m}$$

$$\therefore \text{potential difference } V = - \int^+ \vec{E} \cdot d\vec{l}$$

$$\Rightarrow V = - \int_{r=b}^{r=a} \frac{Q}{4\pi\epsilon r^2} \vec{a}_r \cdot d\vec{l} \quad \Rightarrow d\vec{l} = dr \vec{a}_r$$

$$\Rightarrow V = - \int_{r=b}^{r=a} \frac{Q}{4\pi\epsilon r^2} \vec{a}_r \cdot dr \vec{a}_r$$

$$= - \int_{r=b}^{r=a} \frac{Q}{4\pi\epsilon r^2} dr$$

$$= - \frac{Q}{4\pi\epsilon} \left[-\frac{1}{r} \right]_{r=b}^{r=a} = \frac{Q}{4\pi\epsilon} \left[\frac{1}{r} \right]_{r=b}^{r=a}$$

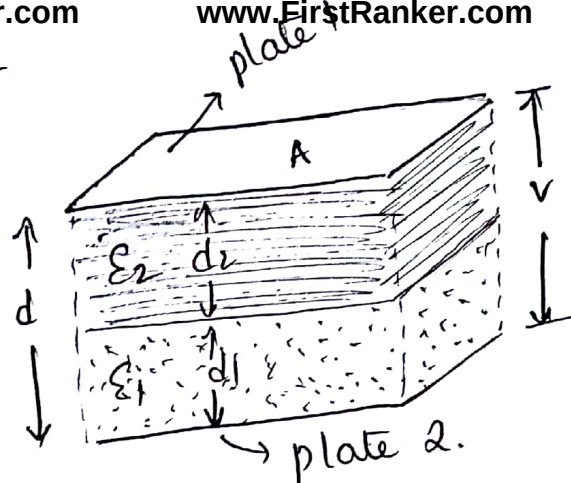
$$\Rightarrow V = \frac{Q}{4\pi\epsilon} \left[\frac{1}{a} - \frac{1}{b} \right] \text{ volts}$$

$$\Rightarrow \text{Capacitance } C = \frac{Q}{V} = \frac{Q}{\frac{Q}{4\pi\epsilon} \left[\frac{1}{a} - \frac{1}{b} \right]}$$

$$\Rightarrow \boxed{C = \frac{4\pi\epsilon}{\left[\frac{1}{a} - \frac{1}{b} \right]} \text{ Farads.}}$$

Composite parallel plate Capacitor is one in which free Space is filled more than one dielectric.

Consider a Composite Capacitor with Space filled with two separate dielectrics for distances d_1 & d_2 .



The dielectric Interface is parallel to Conducting plates. The Space d_1 is filled with ϵ_1 & d_2 filled with ϵ_2 .

Let Q = charge on each plate

E_1 = field Intensity in region d_1

E_2 = " " " d_2 .

Both Intensities are uniform. $V_1 = E_1 d_1$
 $V_2 = E_2 d_2$

$\therefore$ Total voltage $V = V_1 + V_2$

$$\Rightarrow V = E_1 d_1 + E_2 d_2$$

At the dielectric - dielectric interface, The normal Components of flux densities are Equal i.e. $D_{N1} = D_{N2}$

$$\therefore D_1 = \epsilon_1 E_1 \text{ \& } D_2 = \epsilon_2 E_2$$

$$\therefore V = \frac{D_1}{\epsilon_1} d_1 + \frac{D_2}{\epsilon_2} d_2$$

The magnitude of Surface Charge is same on each plate hence $e_s = D_1 = D_2$

$$\Rightarrow V = e_s \left[\frac{d_1}{\epsilon_1} + \frac{d_2}{\epsilon_2} \right]$$

$$\text{Now } C = \frac{Q}{V} = \frac{Q}{e_s \left[\frac{d_1}{\epsilon_1} + \frac{d_2}{\epsilon_2} \right]}$$

$$\text{here } Q = e_s \times A$$

$$C = \frac{Q}{V} = \frac{\epsilon_s A \left[\frac{d_1}{\epsilon_1} + \frac{d_2}{\epsilon_2} \right]}{\frac{d_1}{\epsilon_1} + \frac{d_2}{\epsilon_2}}$$

$$C = \frac{1}{\frac{d_1}{\epsilon_1 A} + \frac{d_2}{\epsilon_2 A}} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$

$$\Rightarrow \boxed{C = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}}$$

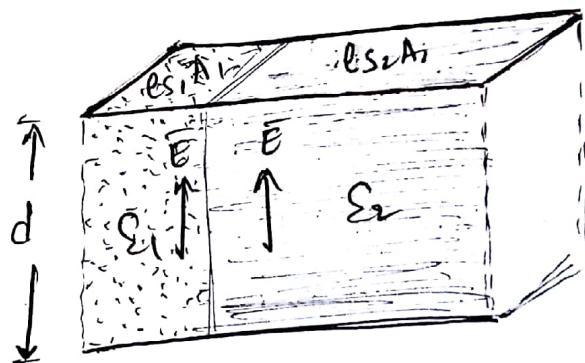
where $C_1 = \frac{\epsilon_1 A}{d_1}$
 $C_2 = \frac{\epsilon_2 A}{d_2}$

If there are n dielectrics between plates

$$\boxed{C = \frac{A}{\frac{d_1}{\epsilon_1} + \frac{d_2}{\epsilon_2} + \dots + \frac{d_n}{\epsilon_n}}}$$

$\Rightarrow$ Dielectric boundary normal to the plates

Consider Composite Capacitor in which dielectric boundary is normal to the conducting plates. The dielectric ϵ_1 occupying area A_1 & ϵ_2 occupying area A_2 .



Total potential across plates is V and distance d .

$$\therefore E = \frac{V}{d}$$

At boundary both $\vec{E}_1$ & $\vec{E}_2$ are tangential and for dielectric-dielectric interface tangential components are equal.

$$\therefore E_{\tan 1} = E_{\tan 2} = E_1 = E_2 = \frac{V}{d}$$

Now $D_1 = \epsilon_1 E_1$ & $D_2 = \epsilon_2 E_2$ $\Rightarrow$ $D_1 = \frac{\epsilon_1 V}{d}$
 $D_2 = \frac{\epsilon_2 V}{d}$

the plate charge is divided into two parts.

one area A_1 charge $Q_1 = \epsilon_{s1} A_1$ & on A_2 $Q_2 = \epsilon_{s2} A_2$

$$\therefore Q = Q_1 + Q_2 = \epsilon_{s1} A_1 + \epsilon_{s2} A_2 = D_1 A_1 + D_2 A_2$$

because on the surface $D_1 = \epsilon_{s1}$ & $D_2 = \epsilon_{s2}$

$$\Rightarrow Q = \frac{\epsilon_1 V A_1}{d} + \frac{\epsilon_2 V A_2}{d}$$

$$\Rightarrow \text{Capacitance } C = \frac{Q}{V} = \frac{\frac{\epsilon_1 V A_1}{d} + \frac{\epsilon_2 V A_2}{d}}{V}$$

$$\Rightarrow C = \frac{\epsilon_1 A_1}{d} + \frac{\epsilon_2 A_2}{d}$$

$$\Rightarrow \boxed{C = C_1 + C_2}$$

Energy stored in a capacitor

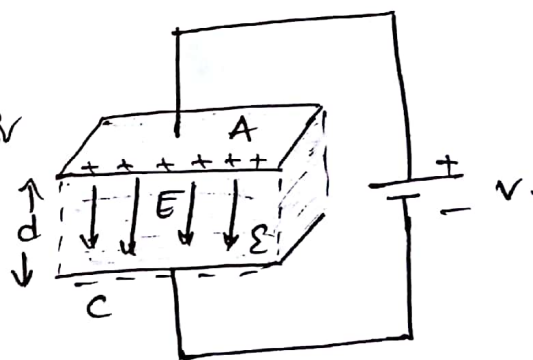
Capacitor Can store the Energy

Consider a parallel plate capacitor

It is supplied with voltage V .

Let $\bar{a}_N$ is the direction normal to

plates $\bar{E} = \frac{V}{d} \bar{a}_N \rightarrow (1)$



Energy stored is given by $W_E = \frac{1}{2} \int_{Vol} \bar{D} \cdot \bar{E} dv$

$$\Rightarrow W_E = \frac{1}{2} \int_{Vol} \epsilon \bar{E} \cdot \bar{E} dv \quad \text{but } \bar{E} \cdot \bar{E} = |\bar{E}|^2$$

$$= \frac{1}{2} \int_{Vol} \epsilon |\bar{E}|^2 dv \quad \text{but } |\bar{E}| = V/d$$

$$= \frac{1}{2} \epsilon \frac{V^2}{d^2} \int_{Vol} dv \quad \text{but } \int_{Vol} dv = A \times d = \text{Volume}$$

$$W_E = \frac{1}{2} \epsilon \frac{V^2 A d}{d^2}$$
$$= \frac{1}{2} \frac{\epsilon A}{d} V^2$$
$$\Rightarrow \boxed{W_E = \frac{1}{2} C V^2 \text{ Joules}}$$

Energy density :-

Energy density is the energy stored per unit volume as volume tends to zero.

$$\therefore W_E = \frac{1}{2} \epsilon \int_{\text{Vol}} |\vec{E}|^2 dV$$

$$\boxed{W_E = \frac{1}{2} \epsilon |\vec{E}|^2 \text{ J/m}^3 = \text{Energy density}}$$

using $|\vec{D}| = \epsilon |\vec{E}|$ we can write

$$W_E = \frac{1}{2} \frac{|\vec{D}|^2}{\epsilon} = \frac{1}{2} |\vec{D}| |\vec{E}| \text{ J/m}^3$$

End of The Unit - II
Sachinval

Magneto statics and Ampere's law

Magnetostatics :-

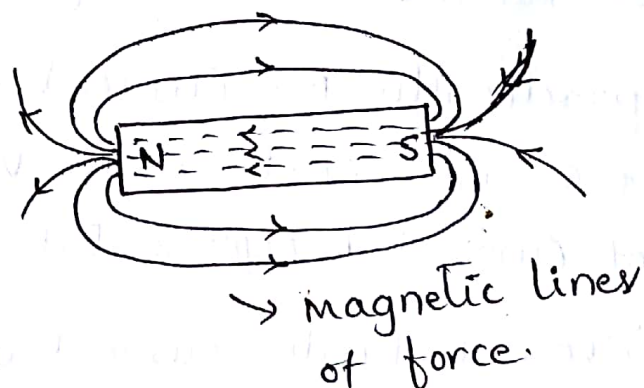
A Current Carrying conductor is always Surrounded by a magnetic field. If the current is steady current (ie A dc. current) Then the magnetic field produced is steady magnetic field ie time invariant,

The study of steady magnetic field, existing in a given Space produced due to flow of steady current through a conductor is called Magneto statics.

Magnetic field and its properties :-

The Region around a magnet within which the influence of magnet can be experienced is called magnetic field. Such a field is represented by imaginary lines around the magnet which are called magnetic lines of force.

Every magnetic flux line starting from North pole must end at South pole & Complete the path from South to North internal to the magnet.

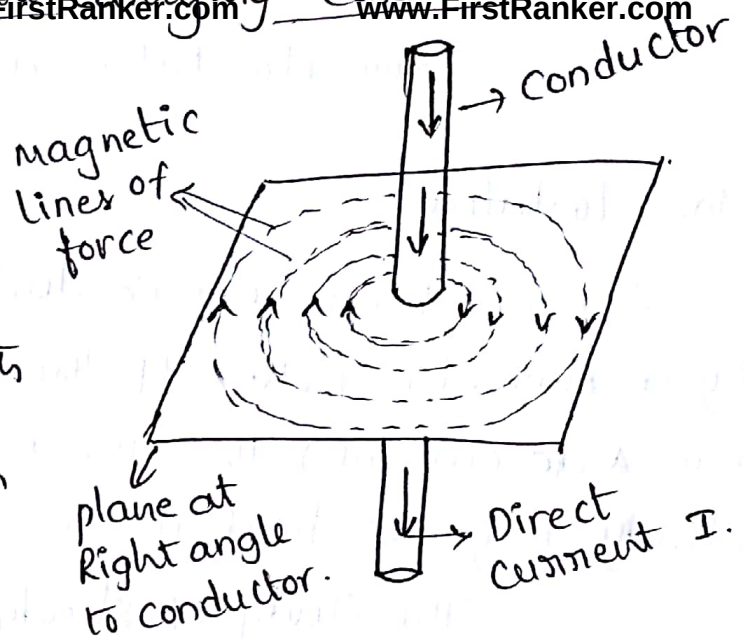


Thus magnetic flux lines exist in the form of closed loop.

Note :- An Isolated magnetic pole can not ~~exist~~ exist.

Magnetic Field due to Current Carrying Conductor :-

straight when a conductor carries a direct current, it produces a magnetic field around it, all along its length



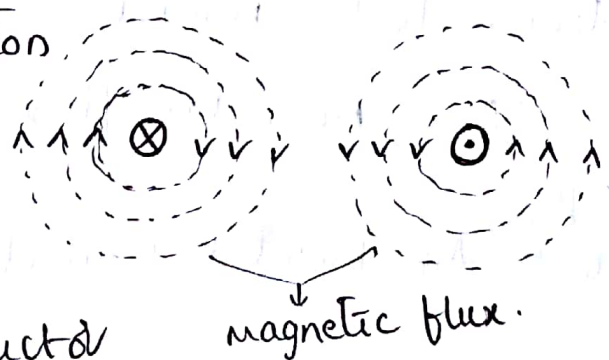
The lines of force in such a case are in the form of concentric circles in the plane at right angle to the conductor.

The direction of such magnetic flux can be experienced using a compass needle. A right hand thumb rule can be used to determine the direction of magnetic field around a conductor carrying a direct current. It states that, hold the current carrying conductor in right hand such that the thumb positioning in the direction parallel to conductor is then curled fingers point in the direction of magnetic lines of flux around it.

practically the current carrying conductor is represented by a small circle i.e. top view of conductor. while the direction of current is represented by "cross" or "dot".

"Cross" Indicates current direction is going into the plane of paper away from the observer

"dot" Indicates current direction is coming out of the conductor i.e. towards the observer



Number of magnetic lines of force is called a magnetic flux denoted as ϕ . It is measured in weber (wb), one weber means 10^8 lines of force. (2)

Magnetic Field Intensity :- ($\vec{H}$)

The Quantative measure of strongness or weakness of magnetic field is given by magnetic field Intensity or magnetic field strength.

magnetic field Intensity at any point in magnetic field is defined as the force experienced by a unit North pole of '1' wb strength. It is measured in N/wb or A/m. It is denoted as $\vec{H}$.

Magnetic flux density :- ($\vec{B}$).

The total magnetic lines of force ie magnetic flux crossing a unit area in a plane at right angles to the direction of flux is called magnetic flux density. It is denoted as $\vec{B}$ and measured in wb/m² or Tesla (T).

Relation between $\vec{B}$ and $\vec{H}$:-

$\vec{B}$ and $\vec{H}$ are related to each other through property of the region in which current carrying conductor is placed. It is called permeability denoted as μ .

$\therefore \vec{B}$ & $\vec{H}$ are related as $\vec{B} = \mu \vec{H} = \mu_0 \mu_r \vec{H}$

μ_0 = permeability of free space. = $4\pi \times 10^{-7}$.

μ_r = Relative permeability = 1 for free space.

units of μ are Henry's/meter or H/m.

Consider a current carrying conductor and place a compass needle under this conductor.

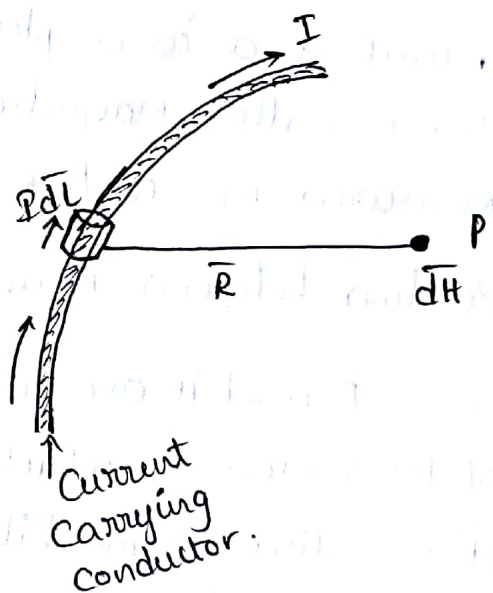
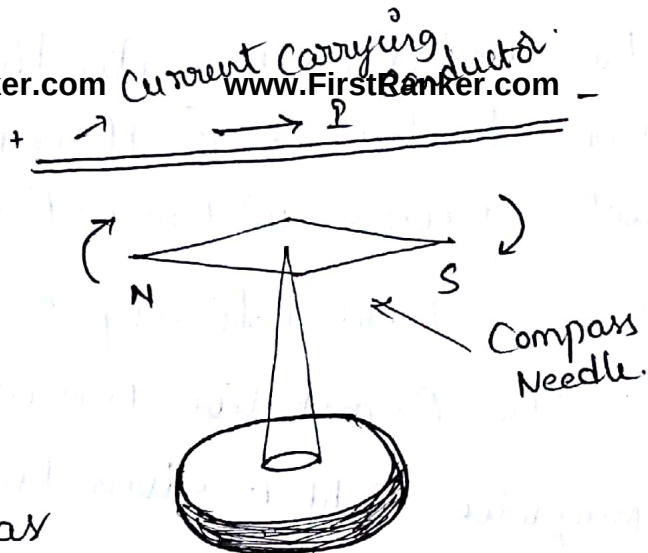
When there was no current through conductor, the needle was pointing along North & South of the earth.

But when the conductor carries current, then the needle was neither attracted to conductor nor repelled from it but it moved and tended to stand at right angles to the conductor.

From this experiment, Oersted proved that an electric current produces a magnetic field.

⇒ Biot Savart Law :-

Consider a differential length dl hence the differential current element is $I dl$. This is very small part of current carrying conductor. The point P is at a distance R from differential current element. The θ is the angle b/w differential current element & the line joining point P to differential current element.



The magnetic field Intensity $\vec{dH}$ produced at a point P due to differential current element $I dL$ is

1. proportional to the product of the current and differential length dL
2. The Sine of the angle between the element and line joining point P to the element
3. Inversely proportional to the Square of the distance R between point P and the element.

$$\therefore \vec{dH} \propto \frac{I dL \sin \theta}{R^2}$$

$$\Rightarrow \vec{dH} = \frac{k I dL \sin \theta}{R^2} \quad \text{where } k = \text{Const of proportionality}$$

$$\Rightarrow \vec{dH} = \frac{I dL \sin \theta}{4\pi R^2}$$

dL = magnitude of vector length $\vec{dL}$

$\vec{a}_R$ = unit vector in the direction of $\vec{R}$

$$\vec{dL} \times \vec{a}_R = |\vec{dL}| |\vec{a}_R| \sin \theta = dL \sin \theta$$

$$\therefore \vec{dH} = \frac{I \vec{dL} \times \vec{a}_R}{4\pi R^2} \quad \text{A/m.} \quad \text{but } \vec{a}_R = \frac{\vec{R}}{|\vec{R}|} = \frac{\vec{R}}{R}$$

$\therefore$ Total magnetic field Intensity

$$\boxed{\vec{H} = \oint \vec{dH} = \oint \frac{I \vec{dL} \times \vec{a}_R}{4\pi R^2}} \quad \text{A/m.}$$

This is called Integral form of Biot-Savart law.

$$\text{(or)} \quad \vec{H} = \oint \frac{I \vec{dL} \times \vec{R}}{4\pi R^3} \quad \text{A/m} \quad \text{since } \vec{a}_R = \frac{\vec{R}}{R}$$

Current density $\vec{K}$ A/m

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Then $I d\vec{l} = \vec{K} ds$

$$\therefore \vec{H} = \int_S \frac{\vec{K} \times \vec{a}_R ds}{4\pi R^2} \quad A/m$$

If the current is flowing through the volume, leads to current density $\vec{J}$ A/m^2

Then $I d\vec{l} = \vec{J} dv$

$$\therefore \vec{H} = \int_V \frac{\vec{J} \times \vec{a}_R dv}{4\pi R^2} \quad A/m^2$$

Note:- Biot-Savart law is also called Ampere's law for the current element.

$\Rightarrow$ H due to Infinitely long straight Conductor :-

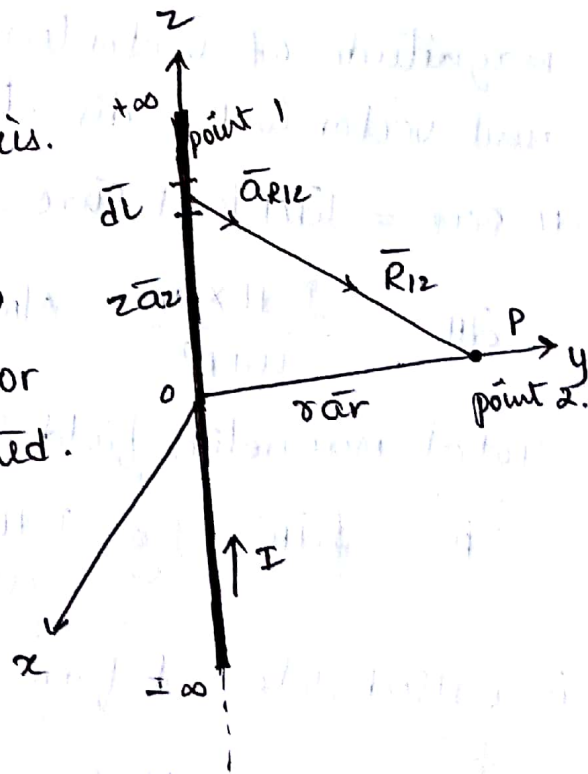
Consider an infinitely long straight conductor, along z-axis. I is the current through the conductor. & Consider a point P at a distance r from conductor at which $\vec{H}$ is to be calculated.

Consider a small element $d\vec{l}$

hence $I d\vec{l} = I dz \vec{a}_z$

From Biot-Savart law

$$d\vec{H} = \frac{I d\vec{l} \times \vec{a}_{R12}}{4\pi R_{12}^2} \quad A/m$$



$$\vec{a}_{R12} = \frac{\vec{R}_{12}}{R_{12}} = \frac{r\vec{a}_r - z\vec{a}_z}{\sqrt{r^2 + z^2}}$$

$$\frac{r\vec{a}_r - z\vec{a}_z}{\sqrt{r^2 + z^2}}$$

$$|\vec{R}_{12}| = R_{12} = \sqrt{r^2 + z^2}$$

$$\therefore d\vec{l} \times \vec{a}_{R12} = dz\vec{a}_z \times \left[\frac{r\vec{a}_r - z\vec{a}_z}{\sqrt{r^2 + z^2}} \right]$$

$$= \frac{1}{\sqrt{r^2 + z^2}} [dz\vec{a}_z \times r\vec{a}_r - dz\vec{a}_z \times z\vec{a}_z]$$

$$= \frac{1}{\sqrt{r^2 + z^2}} [r dz \vec{a}_\phi - 0]$$

$$d\vec{l} \times \vec{a}_{R12} = \frac{r dz \vec{a}_\phi}{\sqrt{r^2 + z^2}}$$

$$\therefore d\vec{H} = \frac{I d\vec{l} \times \vec{a}_{R12}}{u\pi R_{12}^2} = \frac{I r dz \vec{a}_\phi}{u\pi \sqrt{(r^2 + z^2)^2} \times \sqrt{r^2 + z^2}}$$

$$d\vec{H} = \frac{I r dz \vec{a}_\phi}{u\pi (r^2 + z^2)^{3/2}} \quad \text{A/m}$$

$$\therefore \vec{H} = \int_{z=-\infty}^{\infty} d\vec{H} = \int_{z=-\infty}^{\infty} \frac{I r dz \vec{a}_\phi}{u\pi (r^2 + z^2)^{3/2}}$$

$$\text{let } z = r \tan \theta, \quad dz = r \sec^2 \theta d\theta$$

$$\text{If } z = -\infty, \quad \theta = -\pi/2$$

$$z = +\infty, \quad \theta = \pi/2$$

$$\therefore \vec{H} = \int_{-\pi/2}^{\pi/2} \frac{I r \cdot r \sec^2 \theta d\theta \vec{a}_\phi}{u\pi (r^2 + r^2 \tan^2 \theta)^{3/2}}$$

$$= \int_{-\pi/2}^{\pi/2} \frac{I r^2 \sec^2 \theta d\theta \vec{a}_\phi}{u\pi r^3 \sec^3 \theta}$$

$$\because 1 + \tan^2 \theta = \sec^2 \theta$$

$$= \frac{I \vec{a}_\phi}{u\pi r} \int_{-\pi/2}^{\pi/2} \frac{1}{\sec \theta} d\theta = \frac{I \vec{a}_\phi}{u\pi r} \int_{-\pi/2}^{\pi/2} \cos \theta d\theta$$

$$= \frac{I \vec{a}_\phi}{u\pi r} [\sin \theta]_{-\pi/2}^{\pi/2} = \frac{I \vec{a}_\phi}{u\pi r} [1 + 1]$$

$$\boxed{\bar{H} = \frac{I}{2\pi r} \bar{a}_\phi \quad \text{A/m.}}$$

$$\therefore \bar{B} = \mu \bar{H} = \mu_0 \mu_r \bar{H} = \frac{\mu_0 \mu_r I}{2\pi r} \bar{a}_\phi \quad \text{wb/m}^2$$

$\Rightarrow$ $\bar{H}$ due to straight Conductor of finite length :-

Consider a Conductor of finite length placed along z-axis carrying a direct Current I .

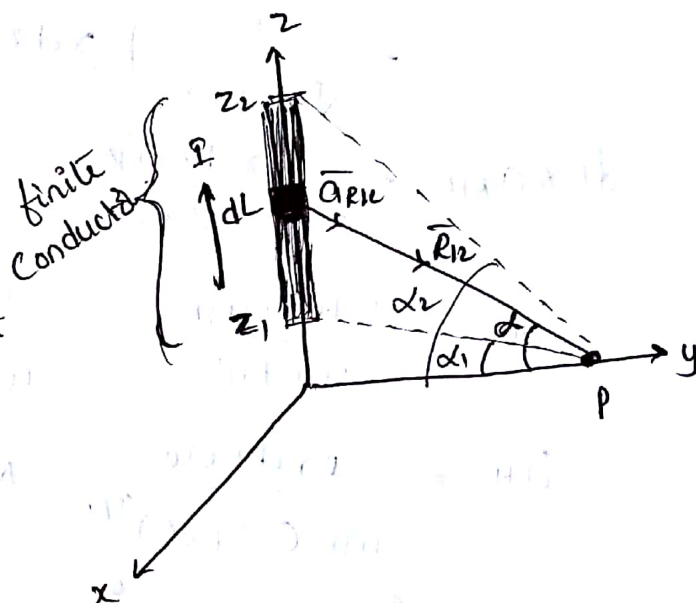
Consider a differential element dL

$$\Rightarrow d\bar{L} = dz \bar{a}_z$$

$$\bar{R}_{12} = r\bar{a}_r - z\bar{a}_z$$

$$|\bar{R}_{12}| = R_{12} = \sqrt{r^2 + z^2}$$

$$\bar{a}_{R12} = \frac{\bar{R}_{12}}{|\bar{R}_{12}|} = \frac{r\bar{a}_r - z\bar{a}_z}{\sqrt{r^2 + z^2}}$$



$$d\bar{H} = \frac{I r dz \bar{a}_\phi}{\sqrt{r^2 + z^2}}$$

According to Biot-Savart Law $d\bar{H}$ at point P is

$$d\bar{H} = \frac{I r dz \bar{a}_\phi}{\mu_0 (r^2 + z^2)^{3/2}} = \frac{I r dz \bar{a}_\phi}{\mu_0 (r^2 + z^2)^{3/2}}$$

$\therefore$ Total magnetic field Intensity

$$\bar{H} = \int_{z_1}^{z_2} d\bar{H} = \int_{z_1}^{z_2} \frac{I r dz \bar{a}_\phi}{\mu_0 (r^2 + z^2)^{3/2}}$$

Let $z = r \tan \alpha$, $dz = r \sec^2 \alpha d\alpha$.

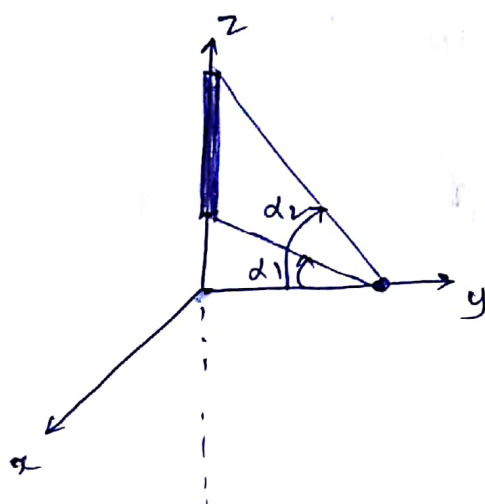
$$\text{If } z = z_1, \alpha_1 = \tan^{-1}\left(\frac{z_1}{r}\right)$$

$$\text{If } z = z_2, \alpha_2 = \tan^{-1}\left(\frac{z_2}{r}\right)$$

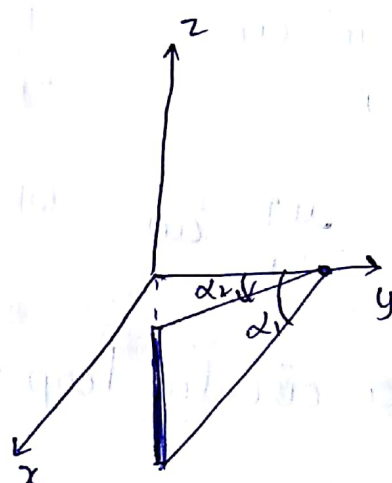
$$\begin{aligned} \vec{H} &= \int_{\alpha_1}^{\alpha_2} \frac{I r \sec^2 \alpha d\alpha \vec{a}_\phi}{4\pi [r^2 + r^2 \tan^2 \alpha]^{3/2}} \\ &= \int_{\alpha_1}^{\alpha_2} \frac{I d\alpha \vec{a}_\phi}{4\pi (\sec \alpha) r} = \frac{I \vec{a}_\phi}{4\pi r} \int_{\alpha_1}^{\alpha_2} \cos \alpha d\alpha \\ &= \frac{I \vec{a}_\phi}{4\pi r} [\sin \alpha]_{\alpha_1}^{\alpha_2} \\ \vec{H} &= \frac{I}{4\pi r} [\sin \alpha_2 - \sin \alpha_1] \vec{a}_\phi \text{ A/m} \end{aligned}$$

$$\vec{B} = \mu \vec{H} = \frac{\mu I}{4\pi r} [\sin \alpha_2 - \sin \alpha_1] \vec{a}_\phi \text{ Wb/m}^2$$

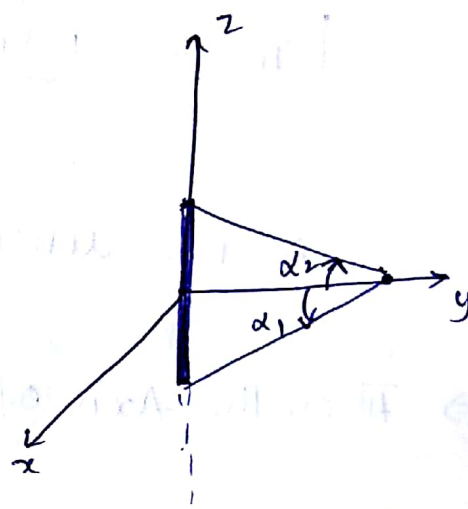
let us see the sign convention for α_1 & α_2



a, Both α_1, α_2 are +ve



b, both α_1, α_2 -ve



c, α_1 negative, α_2 positive.

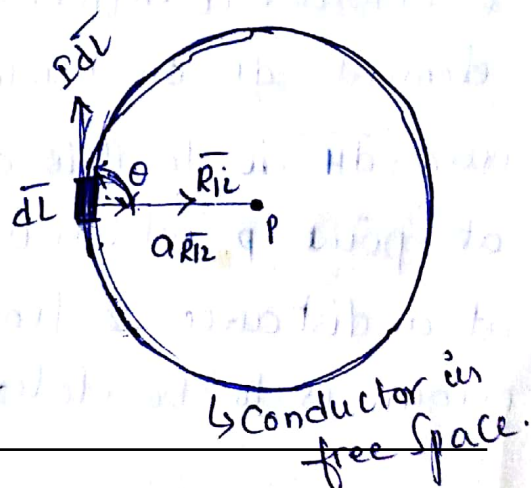
$\Rightarrow$ $\vec{H}$ at the centre of a circular conductor :-

Consider current carrying conductor arranged in a circular form.

$$\begin{aligned} I d\vec{L} \times \vec{a}_{R_{12}} &= I |d\vec{L}| |\vec{a}_{R_{12}}| \sin \theta \vec{a}_n \\ &= I dL \sin \theta \vec{a}_n \end{aligned}$$

$\vec{a}_n$ = unit vector normal to plane.

containing $d\vec{L}$ and $\vec{a}_{R_{12}}$



$$dH = \frac{I dL \times \vec{a}_{R12}}{4\pi R_{12}^2} = \frac{I dL \sin\theta \vec{a}_N}{4\pi R^2}$$

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 $R = R_{12} = \text{radius}$

$$\Rightarrow H = \oint dH = \oint \frac{I dL \sin\theta \vec{a}_N}{4\pi R^2}$$

$$= \frac{I \sin\theta \vec{a}_N}{4\pi R^2} \oint dL$$

$$\because \oint dL = 2\pi R$$

$$= \frac{I \sin\theta \vec{a}_N}{4\pi R^2} \cdot 2\pi R$$

In This case angle $\theta = 90^\circ$ is angle b/w current element and line joining point P and Current element.

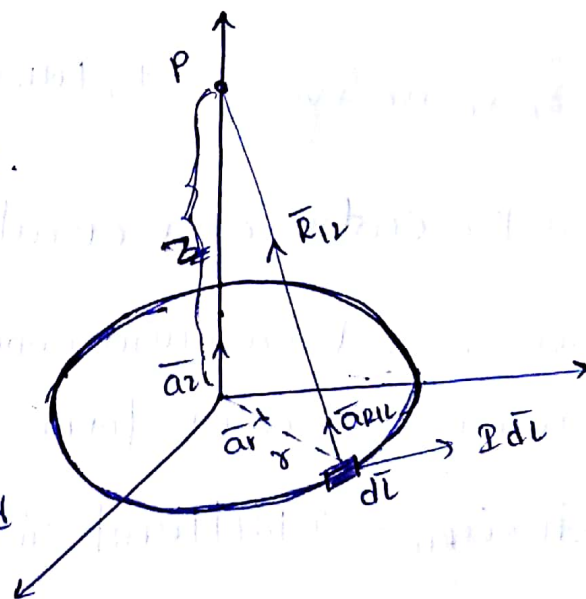
$$\boxed{H = \frac{I \sin 90^\circ \vec{a}_N}{2R} = \frac{I}{2R} \vec{a}_N}$$

$$\& \vec{B} = \mu H = \frac{\mu I}{2R} \vec{a}_N \quad \text{wb/m}^2$$

$\Rightarrow H$ on The Axis of a circular loop :-

Consider a circular loop carrying a direct current I .

& Consider a differential element dL on conductor and dH due to this dL at point P which is located at a distance z from origin. is to be determined.



$$d\vec{l} = r d\phi \vec{a}_\phi$$

$$\vec{R}_{12} = -r\vec{a}_r + z\vec{a}_z \Rightarrow |\vec{R}_{12}| = R_{12} = \sqrt{r^2 + z^2}$$

$$\vec{a}_{R_{12}} = \frac{\vec{R}_{12}}{|\vec{R}_{12}|} = \frac{-r\vec{a}_r + z\vec{a}_z}{\sqrt{r^2 + z^2}}$$

$$\begin{aligned} d\vec{l} \times \vec{a}_{R_{12}} &= r d\phi \vec{a}_\phi \times \left[\frac{-r\vec{a}_r + z\vec{a}_z}{\sqrt{r^2 + z^2}} \right] \\ &= \frac{r d\phi}{\sqrt{r^2 + z^2}} \left[-r \vec{a}_\phi \times \vec{a}_r + z \vec{a}_\phi \times \vec{a}_z \right] \\ &= \frac{r d\phi}{\sqrt{r^2 + z^2}} \left[-r [-\vec{a}_z] + z \vec{a}_r \right] \\ &= \frac{[r^2 d\phi \vec{a}_z + rz d\phi \vec{a}_r]}{\sqrt{r^2 + z^2}} \end{aligned}$$

$$\therefore d\vec{H} = \frac{I d\vec{l} \times \vec{a}_{R_{12}}}{4\pi R_{12}^2} = \frac{I [rz d\phi \vec{a}_r + r^2 d\phi \vec{a}_z]}{4\pi (\sqrt{r^2 + z^2})^2 \times (\sqrt{r^2 + z^2})}$$

$$\begin{aligned} \therefore \vec{H} &= \int d\vec{H} = \int_{\phi=0}^{2\pi} \frac{I [rz d\phi \vec{a}_r + r^2 d\phi \vec{a}_z]}{4\pi (r^2 + z^2)^{3/2}} \\ &= \frac{I}{4\pi} \left\{ \int_{\phi=0}^{2\pi} \frac{z r d\phi}{(r^2 + z^2)^{3/2}} \cdot \vec{a}_r + \int_{\phi=0}^{2\pi} \frac{r^2 \vec{a}_z d\phi}{(r^2 + z^2)^{3/2}} \right\} \end{aligned}$$

magnetic field has no components in the direction of $\vec{r}$ hence $\vec{a}_r = 0$

$$\Rightarrow \vec{H} = \frac{I}{4\pi} \int_{\phi=0}^{2\pi} \frac{r^2 \vec{a}_z}{(r^2 + z^2)^{3/2}} d\phi$$

$$\Rightarrow \vec{H} = \frac{I \vec{a}_z}{4\pi (r^2 + z^2)^{3/2}} \int_0^{2\pi} d\phi$$

$$= \frac{I r^2 \cdot \vec{a}_z}{4\pi (r^2 + z^2)^{3/2}} \cdot [2\pi]$$

$$\boxed{\begin{aligned} \vec{H} &= \frac{I r^2}{2(r^2 + z^2)^{3/2}} \cdot \vec{a}_z \quad \text{A/m.} \\ \vec{B} &= \mu \vec{H} = \frac{\mu I r^2}{2(r^2 + z^2)^{3/2}} \quad \text{A/m} \end{aligned}}$$

Note & If point p is shifted to the origin or center of the conductor

Then $z = 0$.

$$\Rightarrow \boxed{\vec{H} = \frac{I r^2}{2 \cdot r^3} \vec{a}_z = \frac{I}{2r} \vec{a}_z \quad \text{A/m}}$$

This is same as $\vec{H}$ at the center of ~~closed~~ circular conductor.

Ampere's circuital law :-

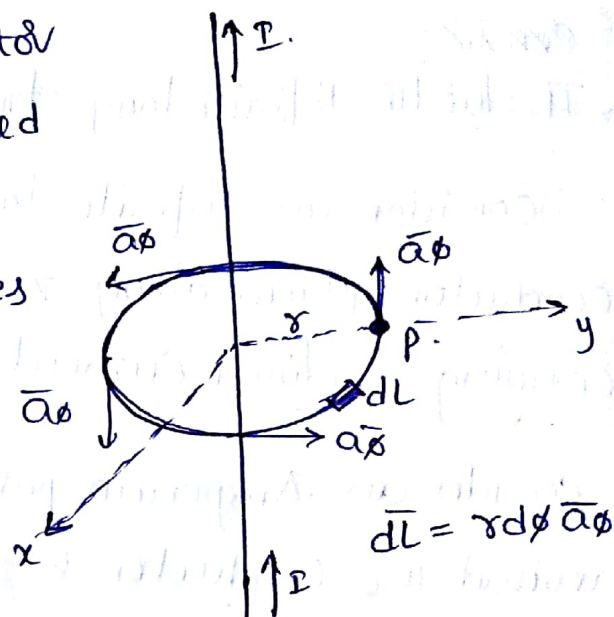
In magnetostatics, the complex problems can be solved using Ampere's circuital law.

Statement :-

The line integral of magnetic field intensity $\vec{H}$ around a closed path is exactly equal to the direct current enclosed by the path.

$$\therefore \boxed{\oint \vec{H} \cdot d\vec{l} = I}$$

Consider long straight conductor carrying direct current I placed along z axis. Consider a closed path of radius r which encloses the straight conductor carrying direct current I . Consider point p where $\vec{H}$ is to be determined.



We know that $\vec{H}$ due to a straight conductor at point p is

$$\vec{H} = \frac{I}{2\pi r} \vec{a}_\phi$$

$$\therefore \vec{H} \cdot d\vec{L} = \frac{I}{2\pi r} \vec{a}_\phi \cdot r d\phi \vec{a}_\phi$$

$$= \frac{I r d\phi}{2\pi r} = \frac{I d\phi}{2\pi}$$

$$\Rightarrow \oint \vec{H} \cdot d\vec{L} = \oint \frac{I d\phi}{2\pi} = \frac{I}{2\pi} \oint d\phi$$

$$= \frac{I}{2\pi} \times 2\pi$$

$$\boxed{\oint \vec{H} \cdot d\vec{L} = I}$$

Hence it is proved that Integral of magnetic field intensity around a closed path is equal to the total current enclosed by that closed path.

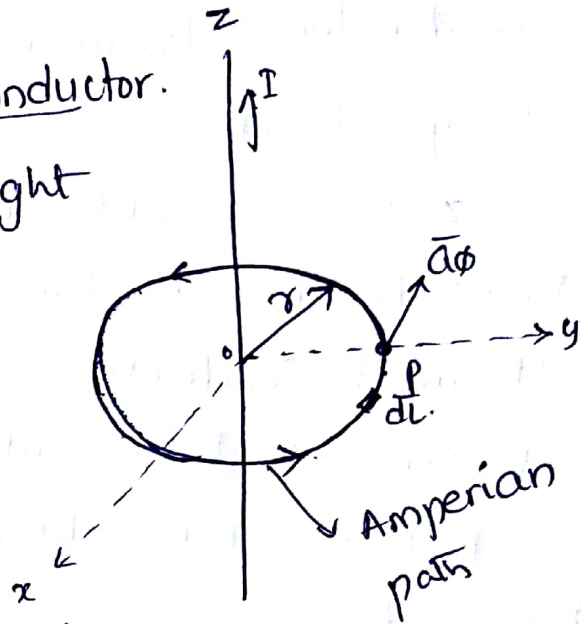
Consider

→ H due to Infinite long straight Conductor.

Consider an infinite long straight conductor placed along z-axis carrying a direct current I .

Consider an Amperian path around the Conductor &

Consider point P , where H to be determined



$$\vec{H} = H_\phi \vec{a}_\phi \quad \text{at point } P.$$

$$d\vec{l} = r d\phi \vec{a}_\phi$$

$$\Rightarrow \vec{H} \cdot d\vec{l} = H_\phi \vec{a}_\phi \cdot r d\phi \vec{a}_\phi$$

$$= H_\phi r d\phi$$

According to Ampere's Circuital law.

$$\oint \vec{H} \cdot d\vec{l} = I$$

$$\Rightarrow \int_0^{2\pi} H_\phi r \cdot d\phi = I$$

$$\Rightarrow H_\phi r \int_0^{2\pi} d\phi = I$$

$$\Rightarrow H_\phi r \cdot 2\pi = I$$

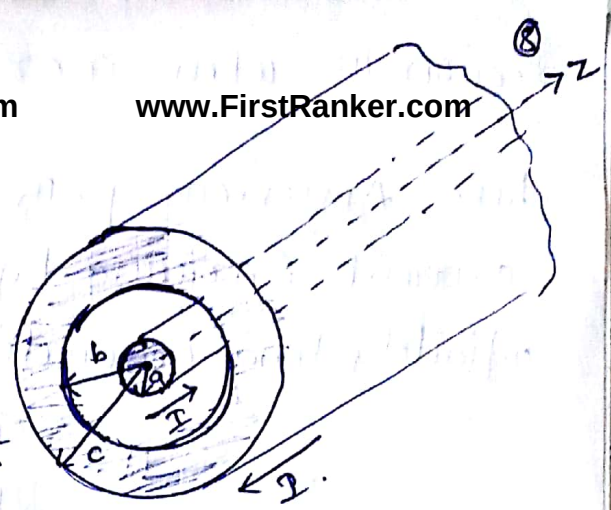
$$\Rightarrow H_\phi = \frac{I}{2\pi r}$$

$$\Rightarrow \boxed{\vec{H} = H_\phi \cdot \vec{a}_\phi = \frac{I}{2\pi r} \vec{a}_\phi} \quad \text{A/m}$$

$$\boxed{\vec{B} = \mu \vec{H} = \frac{\mu I}{2\pi r} \vec{a}_\phi}$$

Consider Co-axial Cable. The inner conductor of radius a carrying a direct current in $+z$ direction.

& outer conductor carrying current in $-z$ direction

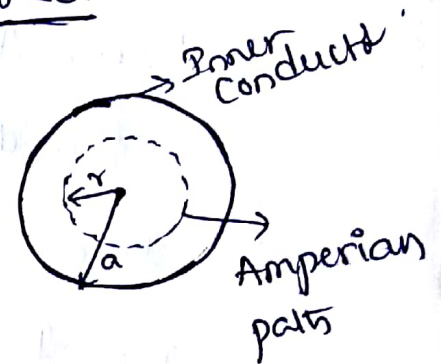


~~we~~ we have four regions to find $\vec{H}$.

Region I :- If Amperian path considered inside inner conductor. i.e. radius of Amperian path $r < a$

Total current enclosed by amperian path

$$I_{enc} = I \times \frac{\pi r^2}{\pi a^2} = I \frac{r^2}{a^2}$$



$$\vec{H} = H_\phi \vec{a}_\phi, \quad d\vec{l} = r d\phi \vec{a}_\phi$$

$$\begin{aligned} \Rightarrow \oint \vec{H} \cdot d\vec{l} &= \int H_\phi \vec{a}_\phi \cdot r d\phi \vec{a}_\phi \\ &= \int_0^{2\pi} H_\phi \cdot r d\phi = H_\phi r \int_0^{2\pi} d\phi \end{aligned}$$

$$\oint \vec{H} \cdot d\vec{l} = H_\phi r \cdot 2\pi$$

$\therefore$ From Ampere's Circuital law

$$\oint \vec{H} \cdot d\vec{l} = I_{enc} \Rightarrow H_\phi 2\pi r = I \frac{r^2}{a^2}$$

$$\Rightarrow H_\phi = \frac{I r}{2\pi a^2} \Rightarrow \vec{H} = H_\phi \vec{a}_\phi$$

$$\Rightarrow \vec{H} = \frac{I r}{2\pi a^2} \vec{a}_\phi \quad \text{A/m}$$

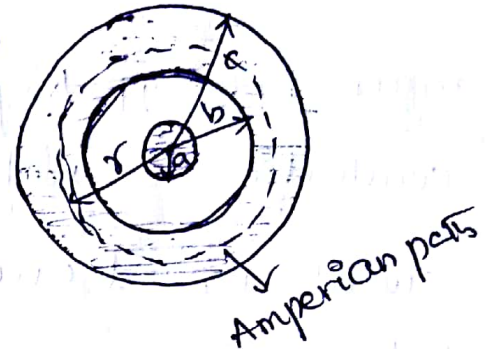
$$\vec{B} = \mu \vec{H} = \frac{\mu I r}{2\pi a^2} \vec{a}_\phi \quad \text{wb/m}^2$$

here Amperian path completely encloses the total current carrying by the conductor hence H due to infinitely long conductor is given by

$$\vec{H} = \frac{I}{2\pi r} \vec{a}_\phi \quad \text{A/m}$$

Region II :- when $b < r < c$.

Consider an Amperian path as shown with radius $b < r < c$



$\therefore$ From Ampere's law

$$\oint \vec{H} \cdot d\vec{l} = I_{enc}$$

$$\therefore I_{enc} = I' + I''$$

$$I' = I \quad \& \quad I'' = (-I) \cdot \frac{\pi(r^2 - b^2)}{\pi(c^2 - b^2)}$$

$$\Rightarrow I'' = - \frac{(r^2 - b^2)}{(c^2 - b^2)} I$$

$$\therefore I_{enc} = I' + I'' = I - \frac{(r^2 - b^2)}{(c^2 - b^2)} I$$

$$= I \left[\frac{c^2 - b^2 - r^2 + b^2}{c^2 - b^2} \right]$$

$$I_{enc} = I \cdot \left[\frac{c^2 - r^2}{c^2 - b^2} \right]$$

$$\therefore H \phi \cdot 2\pi r = I \left[\frac{c^2 - r^2}{c^2 - b^2} \right]$$

$$\Rightarrow H \phi = \frac{I}{2\pi r} \left[\frac{c^2 - r^2}{c^2 - b^2} \right]$$

$$\Rightarrow \vec{H} = H \phi \vec{a}_\phi = \frac{I}{2\pi r} \left[\frac{c^2 - r^2}{c^2 - b^2} \right] \text{ A/m}$$

Region - IV : outside the cable i.e. $r > c$

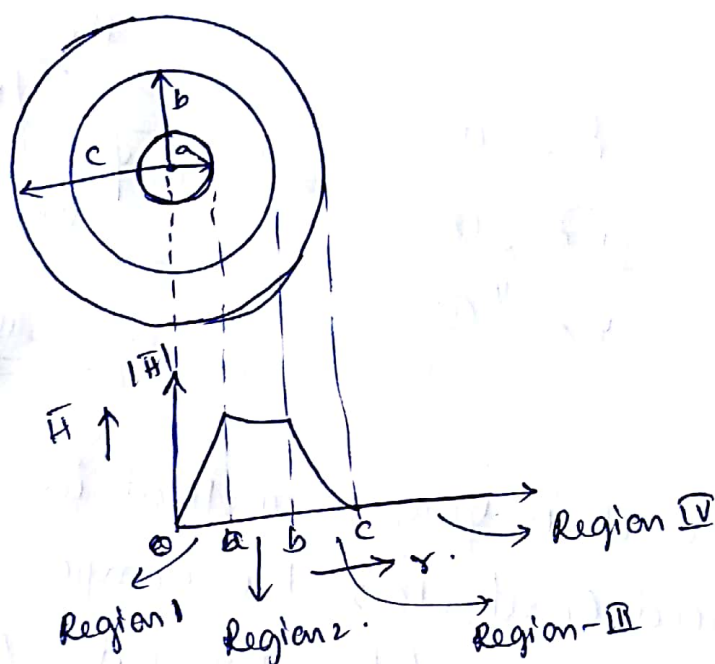
Consider an amperian path with radius $r > c$ $\therefore$

it encloses the total current i.e. $+I$ & $-I$ $\therefore I_{enc} = 0$

$$\therefore I_{enc} = I - I = 0$$

$$\therefore \oint \vec{H} \cdot d\vec{l} = I_{enc} \Rightarrow \boxed{\vec{H} = 0} \quad \text{A/m.}$$

Variation of H with r

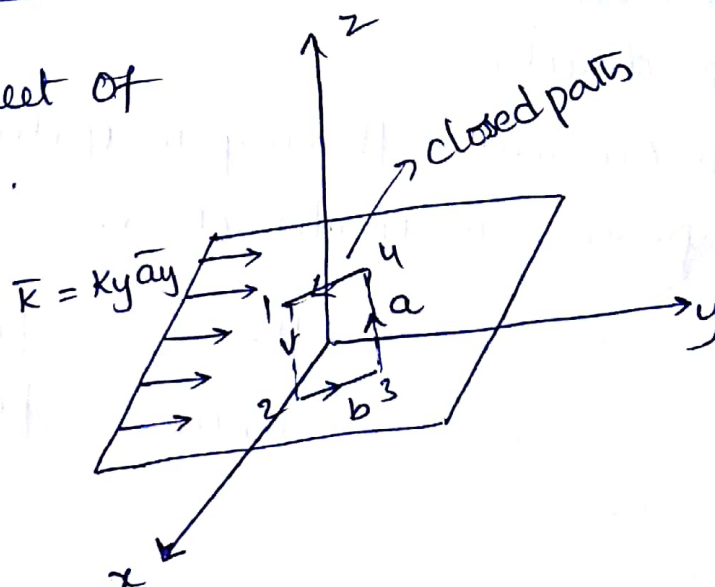


$\Rightarrow$ H due to Infinite sheet of Current

Consider an infinite sheet of current in $z=0$ plane.

The Surface Current density is $\vec{K}$.

$$\vec{K} = K_y \hat{a}_y$$

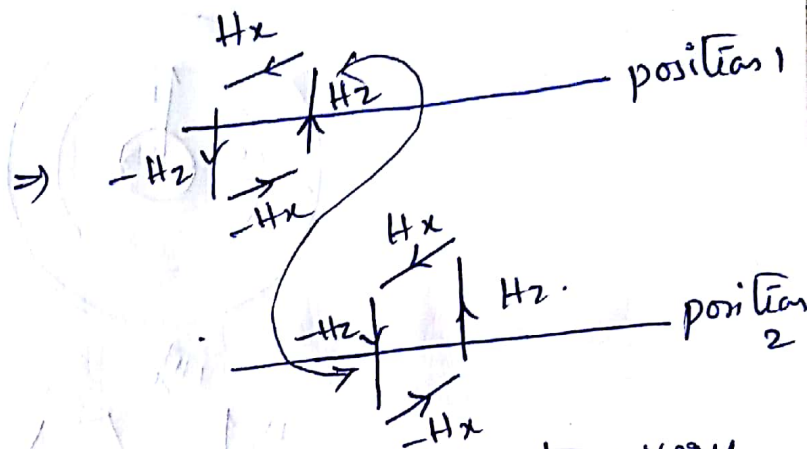
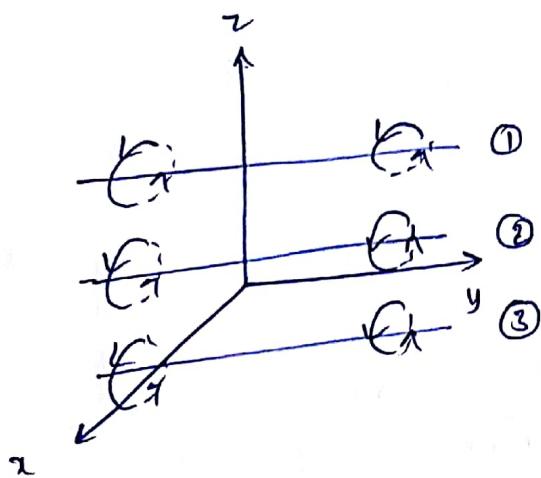


Consider a closed path 1-2-3-4. The width of closed loop b and height is a . It is perpendicular to the direction of current hence in xz plane.

The Current across distance b is given by

$$I_{enc} = Kyb.$$

Consider magnetic lines of force due to current in y direction, according to right hand Thumb rule



From figure it is very clear that in between two very closely spaced conductors, the Component of H in z direction are oppositely directed. All Such Components cancel each other hence H cannot have any Component in z direction.

As Current is flowing in y direction, H cannot have Component in y direction.

$$\begin{aligned} \vec{H} &= H_x \vec{a}_x & \text{for } z > 0 \\ &= -H_x \vec{a}_x & \text{for } z < 0 \end{aligned}$$

$$\oint \vec{H} \cdot d\vec{l} = I_{enc} \quad \text{--- (1)}$$

$$\Rightarrow \oint \vec{H} \cdot d\vec{l} = \int_1^2 \vec{H} \cdot d\vec{l} + \int_2^3 \vec{H} \cdot d\vec{l} + \int_3^4 \vec{H} \cdot d\vec{l} + \int_4^1 \vec{H} \cdot d\vec{l}$$

here paths 1-2 & 3-4 the integral $\oint \vec{H} \cdot d\vec{l} = 0$
(since they are directed in z direction)

paths 2-3 $\vec{r} \cdot d\vec{l} = -dx \vec{a}_x$

$$\therefore \int_2^3 \vec{H} \cdot d\vec{l} = \int_2^3 (-H_x \vec{a}_x) \cdot (-dx \vec{a}_x) = H_x \int_2^3 dx$$

$$\int_2^3 \vec{H} \cdot d\vec{l} = H_x \cdot b$$

paths 4-1 $\vec{r} \cdot d\vec{l} = dx \vec{a}_x$

$$\therefore \int_4^1 \vec{H} \cdot d\vec{l} = \int_4^1 H_x \vec{a}_x \cdot dx \vec{a}_x = H_x \int_4^1 dx = H_x b$$

$$\therefore \oint \vec{H} \cdot d\vec{l} = 0 + H_x b + 0 + H_x b$$

$$= 2 H_x b$$

Equating this to I_{enc} in Equation (1)

$$2b H_x = K_y b \Rightarrow H_x = \frac{1}{2} K_y$$

Hence $\vec{H} = \frac{1}{2} K_y \vec{a}_x$ for $z > 0$
 $= -\frac{1}{2} K_y \vec{a}_x$ for $z < 0$

In general for an infinite sheet of current with density K A/m we can write

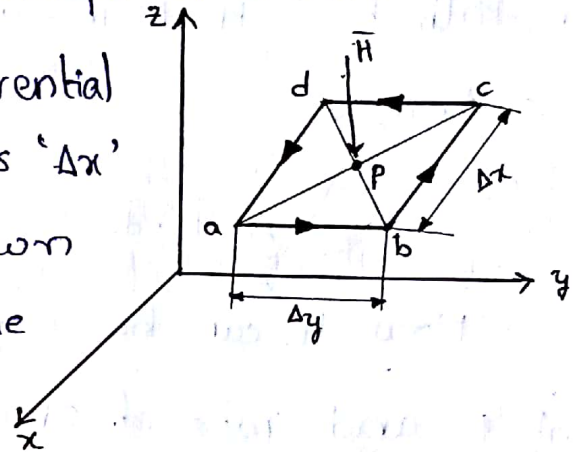
$$\boxed{\vec{H} = \frac{1}{2} K \times \vec{a}_n}$$

$\vec{a}_n$ = normal direction to surface

* Curl :-

In electrostatics, the Gauss's law is applied to the differential volume element to develop the concept of divergence. Similarly in magnetostatics, the Ampere's circuit law is to be applied to the differential surface element to develop the concept of a curl.

Consider the differential surface element having sides ' Δx ' and ' Δy ' plane, The unknown current has produced $\vec{H}$ at the centre of this incremental closed path.



The total magnetic field intensity at the point 'p' which is centre of the small rectangle is,

$$\therefore \vec{H} = H_x \vec{a}_x + H_y \vec{a}_y + H_z \vec{a}_z \longrightarrow \textcircled{1}$$

While the total current density is given by,

$$\therefore \vec{J} = J_x \vec{a}_x + J_y \vec{a}_y + J_z \vec{a}_z \longrightarrow \textcircled{2}$$

To apply Ampere's circuital law to this closed path, let us evaluate the closed line integral of $\vec{H}$ about this path in the direction abcda. According to right hand thumb rule the current is in $\vec{a}_z$ direction.

Along Path a-b, $\vec{H} = H_y \vec{a}_y$ and $d\vec{L} = \Delta y \vec{a}_y$

$$\therefore \vec{H} \cdot d\vec{L} = H_y \vec{a}_y \cdot \Delta y \vec{a}_y = H_y \Delta y \longrightarrow \textcircled{3}$$

The intensity H_y along a-b can be expressed in terms of H_{y_0} , existing at P and the rate of change of H_y in the x -direction with ' x '.

$$\therefore (\vec{H} \cdot d\vec{L})_{a-b} = \left[H_{y_0} + \frac{\Delta x}{2} \frac{\partial H_y}{\partial x} \right] \Delta y \longrightarrow (4)$$

For Path b-c, $\vec{H}$ is in $-\vec{a}_x$ direction hence $-H_x \vec{a}_x$ and $d\vec{L} = \Delta x \vec{a}_x$.

$$\therefore \vec{H} \cdot d\vec{L} = -H_x \Delta x \longrightarrow (5)$$

Now H_x can be expressed in terms of H_{x_0} at Point 'P' and rate of change of H_x in y -direction with ' y '.

$$\therefore H_x = H_{x_0} + \frac{\Delta y}{2} \frac{\partial H_x}{\partial y}$$

The distance of bc from 'P' is " $\Delta y/2$ ".

$$\therefore (\vec{H} \cdot d\vec{L})_{b-c} = - \left[H_{x_0} + \frac{\Delta y}{2} \frac{\partial H_x}{\partial y} \right] \Delta x \longrightarrow (6)$$

For Path c-d, $\vec{H}$ is in $-\vec{a}_y$ direction hence $-H_y \vec{a}_y$ and $d\vec{L} = \Delta y \vec{a}_y$

$$\therefore \vec{H} \cdot d\vec{L} = -H_y \Delta y \longrightarrow (7)$$

But H_y can be expressed in terms of H_{y_0} and rate of change of H_y in negative x -direction. The distance of cd from point 'P' is $(\Delta x/2)$ in negative x -direction

$$\therefore H_y = H_{y_0} - \frac{\Delta x}{2} \frac{\partial H_y}{\partial x}$$

$$\therefore (\vec{H} \cdot d\vec{L})_{c-d} = - \left[H_{y_0} - \frac{\Delta x}{2} \frac{\partial H_y}{\partial x} \right] \Delta y \rightarrow (8)$$

For the path d-a, $\vec{H}$ is in $+\vec{a}_x$ direction hence $H_x \vec{a}_x$ and $d\vec{L} = \Delta x \vec{a}_x$

$$\therefore \vec{H} \cdot d\vec{L} = H_x \Delta x \rightarrow (9)$$

But H_x can be expressed in terms of H_{x_0} and rate of change of H_x in negative y -direction. The distance of da from point 'p' is $(\Delta y/2)$ in negative y -direction.

$$\therefore H_x = H_{x_0} - \frac{\Delta y}{2} \frac{\partial H_x}{\partial y}$$

$$\therefore (\vec{H} \cdot d\vec{L})_{d-a} = \left[H_{x_0} - \frac{\Delta y}{2} \frac{\partial H_x}{\partial y} \right] \Delta x \rightarrow (10)$$

Total $\vec{H} \cdot d\vec{L}$ can be obtained by adding the equation (4), (6), (8) and (10).

$$\begin{aligned} \vec{H} \cdot d\vec{L} = & H_{y_0} \Delta y + \frac{\Delta x \Delta y}{2} \frac{\partial H_y}{\partial x} - H_{x_0} \Delta x - \frac{\Delta x \Delta y}{2} \frac{\partial H_x}{\partial y} \\ & - H_{y_0} \Delta y + \frac{\Delta x \Delta y}{2} \frac{\partial H_y}{\partial x} + H_{x_0} \Delta x - \frac{\Delta x \Delta y}{2} \frac{\partial H_x}{\partial y} \end{aligned}$$

$$\therefore \oint \vec{H} \cdot d\vec{L} = \Delta x \Delta y \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right] \rightarrow (11)$$

According to Ampere's circuital law, this integral must be current enclosed by the differential element.

current enclosed = Current density normal to closed path $\times$
Area of the closed path

$$\therefore I_{enc} = J_z \Delta x \Delta y \rightarrow (12)$$

where, J_z = current density in $\bar{a}_z$ direction as the current enclosed is in $\bar{a}_z$ direction.

From equations (11) and (12) we can write,

$$\frac{\oint \bar{H} \cdot d\bar{L}}{\Delta x \Delta y} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z \rightarrow (13)$$

This gives accurate result as the closed path shrinks to a point i.e., $\Delta x \Delta y$ area tends to zero.

$$\therefore \lim_{\Delta x \Delta y \rightarrow 0} \frac{\oint \bar{H} \cdot d\bar{L}}{\Delta x \Delta y} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z \rightarrow (14)$$

$$\text{Similarly, } \lim_{\Delta y \Delta z \rightarrow 0} \frac{\oint \bar{H} \cdot d\bar{L}}{\Delta y \Delta z} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x \rightarrow (15)$$

$$\text{and } \lim_{\Delta z \Delta x \rightarrow 0} \frac{\oint \bar{H} \cdot d\bar{L}}{\Delta z \Delta x} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y \rightarrow (16)$$

In general we can write,

$$\therefore \lim_{\Delta S_N \rightarrow 0} \frac{\oint \bar{H} \cdot d\bar{L}}{\Delta S_N} = J_N \rightarrow (17)$$

The total $\bar{J}$ now can be obtained by adding (14), (15) & (16)

$$\bar{J} = J_x \bar{a}_x + J_y \bar{a}_y + J_z \bar{a}_z$$

$$\bar{J} = \left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right] \bar{a}_x + \left[\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right] \bar{a}_y + \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right] \bar{a}_z$$

The mathematical operation on 'H' is called curl

the

$$\therefore \vec{J} = \text{curl } \vec{H} = \nabla \times \vec{H} \rightarrow (18)$$

The curl $\vec{H}$ is indicated by $\nabla \times \vec{H}$ which is cross product of operator 'del' and $\vec{H}$.

The equation (18) is called the point form of Ampere's circuital law.

$$\therefore \text{curl } \vec{H} = \nabla \times \vec{H} = \vec{J} \rightarrow (19)$$

This is one of the Maxwell's equations.

* curl in various co-ordinate systems:-

As the curl of $\vec{H}$ i.e., $\nabla \times \vec{H}$ is a cross Product it can be expressed in determinant form in Various co-ordinate systems.

1. Cartesian Co-ordinate system:-

$$\begin{aligned} \nabla \times \vec{H} &= \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} \begin{matrix} \leftarrow \nabla \\ \leftarrow \vec{H} \end{matrix} \\ &= \left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right] \vec{a}_x + \left[\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right] \vec{a}_y + \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right] \vec{a}_z \end{aligned}$$

2. Cylindrical co-ordinate system:-

$$\begin{aligned} \nabla \times \vec{H} &= \frac{1}{r} \begin{vmatrix} \vec{a}_r & r\vec{a}_\phi & \vec{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ H_r & rH_\phi & H_z \end{vmatrix} \\ &= \left[\frac{1}{r} \frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z} \right] \vec{a}_r + \left[\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} \right] \vec{a}_\phi + \left[\frac{1}{r} \frac{\partial (rH_\phi)}{\partial r} - \frac{1}{r} \frac{\partial H_r}{\partial \phi} \right] \vec{a}_z \end{aligned}$$

3. Spherical co-ordinate system:-

$$\nabla \times \vec{H} = \frac{1}{r \sin \theta} \begin{vmatrix} \vec{a}_r & r\vec{a}_\theta & r\sin\theta\vec{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ H_r & rH_\theta & r\sin\theta H_\phi \end{vmatrix}$$

$$\nabla \times \vec{H} = \frac{1}{r \sin \theta} \left[\frac{\partial H_\phi \sin \theta}{\partial \theta} - \frac{\partial H_\theta}{\partial \phi} \right] \vec{a}_r + \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial H_r}{\partial \phi} - \frac{\partial (r H_\phi)}{\partial r} \right] \vec{a}_\theta + \frac{1}{r} \left[\frac{\partial (r H_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta} \right] \vec{a}_\phi$$

* Properties of curl:-

The various properties of curl are,

① The curl of a vector is a vector quantity.

② $\nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$

③ $\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$

④ The divergence of a curl is zero.

$$\nabla \cdot (\nabla \times \vec{A}) = 0$$

⑤ The curl of a scalar makes no sense

$$\nabla \times \alpha = \text{NO sense if } \alpha \text{ is scalar.}$$

⑥ The curl of gradient of a vector is zero.

$$\nabla \times (\nabla \psi) = 0$$

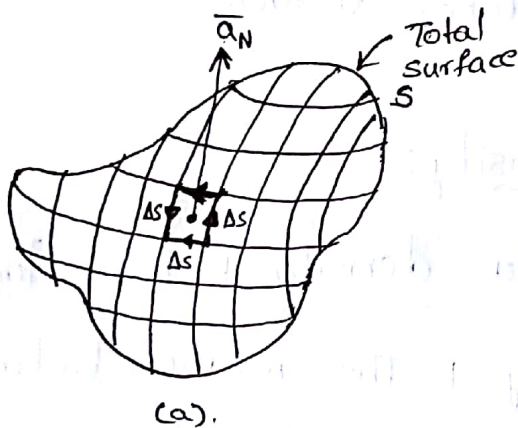
⑦ $\nabla \times \vec{A} \times \vec{B} = \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla) \vec{A} - (\vec{A} \cdot \nabla) \vec{B}$

* Stoke's Theorem:-

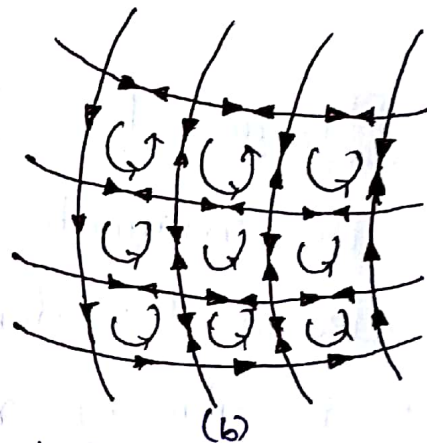
The stoke's theorem states that, "The line integral of a vector $\vec{A}$ around a closed path 'L' is equal to the integral of curl of $\vec{A}$ over the open surface 's' enclosed by the closed path 'L'".

Proof of stoke's Theorem:-

Consider a surface 's' which is splitted into number of incremental surfaces. Each Incremental Surface is having area ' Δs ' as shown in the Figure.



(a).



(b)

Applying definition of the curl to any of these incremental surfaces we can write,

$$\therefore (\nabla \times \vec{H})_N = \frac{\oint \vec{H} \cdot d\vec{L}_{\Delta s}}{\Delta s} \rightarrow \textcircled{1}$$

where, N = Normal to Δs according to right hand rule
 $dL_{\Delta s}$ = Perimeter of the incremental surface Δs

Now the curl of $\vec{H}$ in the normal direction is the dot product of curl of $\vec{H}$ with $\vec{a}_n$ where $\vec{a}_n$ is unit vector, normal to the surface Δs , according to right hand rule.

$$\therefore (\nabla \times \vec{H})_N = (\nabla \times \vec{H}) \cdot \vec{a}_N$$

Using in ① we get

$$\therefore \oint \vec{H} \cdot d\vec{L}_{\Delta s} = (\nabla \times \vec{H}) \cdot \vec{a}_N \Delta s$$

$$\therefore \oint \vec{H} \cdot d\vec{L}_{\Delta s} = (\nabla \times \vec{H}) \cdot \vec{\Delta s} \rightarrow ②$$

$$\boxed{\therefore \oint_L \vec{H} \cdot d\vec{L} = \int_S (\nabla \times \vec{H}) \cdot \vec{\Delta s}}$$

where, dL = Perimeter of the total surface's.

Thus line integral can be expressed as a surface integral which proves the stoke's theorem.

* Magnetic Flux and Flux Density:-

The magnetic flux density $\vec{B}$ is analogous to the electric flux density $\vec{D}$. The relation between $\vec{B}$ and $\vec{H}$ is already mentioned, which is through the property of medium called Permeability ' μ '. The relation is given by,

$$\boxed{\therefore \vec{B} = \mu \vec{H}} \rightarrow ①$$

For the free space, $\mu = \mu_0 = 4\pi \times 10^{-7} \text{ H/m}$ hence,

$$\boxed{\therefore \vec{B} = \mu_0 \vec{H}} \text{ for free space } \rightarrow ②$$

The magnetic flux density has units wb/m^2 and hence it can be defined as the flux in webers passing through

unit area in a plane at right angles to the direction of flux. If the flux passing through the unit area is not exactly at right angles to the plane,

$$\therefore \phi = \int_S \vec{B} \cdot d\vec{S} \text{ webers (wb)} \rightarrow (3)$$

where, ϕ = Magnetic flux in webers

$\vec{B}$ = Magnetic flux density in wb/m² or Tesla (T)

$d\vec{S}$ = open surface through which flux is passing

Now Consider a closed surface which is defining a certain volume. The magnetic flux lines always exist in the form of closed loop. Thus for a closed surface the number of magnetic flux lines entering must be equal to the number of magnetic flux lines leaving. Hence the integral $\vec{B} \cdot d\vec{S}$ evaluated over a closed surface is always zero.

$$\therefore \oint_S \vec{B} \cdot d\vec{S} = 0 \rightarrow (4)$$

This is called law of conservation of magnetic flux or Gauss's law in integral form for magnetic fields.

Applying divergence theorem to equation (4),

$$\therefore \oint_S \vec{B} \cdot d\vec{S} = \int_{Vol} \nabla \cdot \vec{B} dV = 0 \rightarrow (5)$$

where, dv = volume enclosed by the closed surface.
But as dv is not zero, we can write,

$$\therefore \nabla \cdot \vec{B} = 0$$

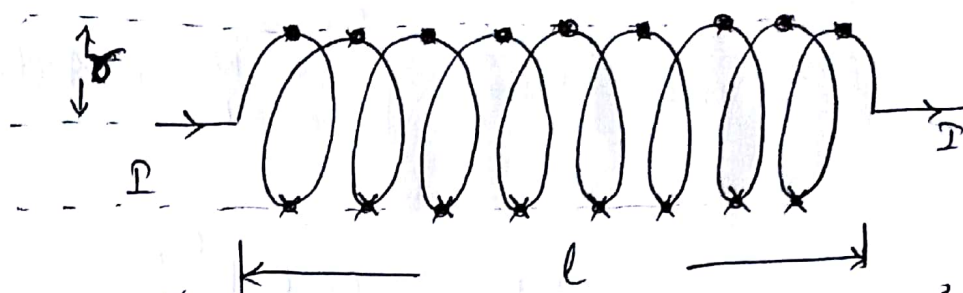
The divergence of magnetic flux density is always zero. This is called Gauss's law in differential form for magnetic fields. This is another Maxwell's equation.

* Maxwell's Equations for static Electromagnetic Fields:-
Let us summarize the Maxwell's equations for static electric and magnetic fields.

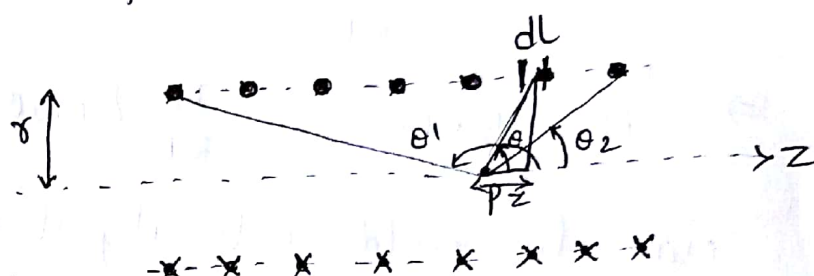
Maxwell's equations in differential or point form		
1.	$\nabla \cdot \vec{D} = \rho_v$	Gauss's law
2.	$\nabla \times \vec{E} = 0$	Conservation of electric field
3.	$\nabla \times \vec{H} = \vec{J} = \vec{j}$	Ampere's circuital law
4.	$\nabla \cdot \vec{B} = 0$	Single magnetic pole cannot exist, conservation of magnetic flux.

The Maxwell's equations in integral form can be summarized as,

Maxwell's equations in integral form	
1.	$\oint \vec{D} \cdot d\vec{s} = \int \rho_v dv = 0$
2.	$\oint \vec{E} \cdot d\vec{l} = 0$
3.	$\oint \vec{H} \cdot d\vec{l} = \oint \vec{J} \cdot d\vec{s} = I$
4.	$\oint \vec{B} \cdot d\vec{s} = 0.$



Consider a solenoid of length l & radius r carrying a direct current I ampere's.



Now let us consider a differential element on solenoid $\therefore d\vec{l} = n dz \vec{a}_z$

where $n = \frac{N}{l}$

Since Solenoid consists of circular loops, and we know that $\vec{H}$ due to circular conductor on its axis as

$$d\vec{H} = \frac{I dl r^2}{2 (\tau^2 + z^2)^{3/2}} = \frac{I n dz r^2}{2 (\tau^2 + z^2)^{3/2}}$$

where $dl = n dz = \left(\frac{N}{l}\right) dz$ & let

& let $\tan \theta = \frac{r}{z} \Rightarrow z = r \cot \theta$ from diagram

$\Rightarrow dz = -r \operatorname{cosec}^2 \theta d\theta$

$$\Rightarrow dH = \frac{I \cdot \frac{N}{l} dz r^2}{2 (\tau^2 + z^2)^{3/2}} = \frac{I \frac{N}{l} (-r \operatorname{cosec}^2 \theta) d\theta r^2}{2 r^3 \operatorname{cosec}^3 \theta}$$

$$dH = - \frac{I \frac{N}{l} \cdot d\theta}{2 \operatorname{cosec} \theta}$$

$$\therefore dH = -\frac{\mu_0 N}{2L} \sin \theta d\theta$$

$$\therefore H = \int dH = \int_{\theta_1}^{\theta_2} -\frac{\mu_0 N}{2L} \sin \theta d\theta$$

$$= -\frac{\mu_0 N}{2L} [-\cos \theta]_{\theta_1}^{\theta_2}$$

$$H = \frac{\mu_0 N}{2L} [\cos \theta_2 - \cos \theta_1]$$

$$\Rightarrow \boxed{\vec{H} = H_z \vec{a}_z = \frac{\mu_0 N}{2L} [\cos \theta_2 - \cos \theta_1] \vec{a}_z}$$

magnetic field Intensity at any point p on the axis of the Solenoid

Case i: at the Center of Solenoid.

at the Center ~~$\theta_1 = \theta_2$~~

$$\therefore \cos \theta_1 = -\cos \theta_2$$

$$\Rightarrow \vec{H} = \frac{\mu_0 N}{2L} [\cos \theta_2 - (-\cos \theta_2)] \vec{a}_z$$

$$= \frac{\mu_0 N}{2L} \cdot 2 \cos \theta_2 \vec{a}_z = \frac{\mu_0 N}{L} \cos \theta_2 \vec{a}_z$$

$$\cos \theta_2 = \frac{l/2}{(\sigma^2 + (l/2)^2)^{1/2}} = -\cos \theta_1$$

$$\Rightarrow \vec{H}_{\text{center}} = \frac{\mu_0 N}{L} \cdot \frac{l/2}{(\sigma^2 + (l/2)^2)^{1/2}} \cdot \vec{a}_z$$

$$= \frac{\mu_0 N}{2\sqrt{\sigma^2 + (l/2)^2}} \vec{a}_z \quad \text{If } l \gg \sigma$$

$$\text{then } \boxed{\vec{H}_{\text{center}} = \frac{\mu_0 N}{2l} \vec{a}_z = \frac{\mu_0 N}{L} \vec{a}_z}$$

$$\bar{H} = \frac{IN}{2L} [\cos\theta_2 - \cos\theta_1] \bar{a}_z$$

If $L \gg a$ then $\theta_1 = 90^\circ$ & $\theta_2 = 0^\circ$

$$\Rightarrow \bar{H} = \frac{IN}{2L} [\cos 0 - \cos 90^\circ] \bar{a}_z$$

$$\boxed{\bar{H}_{\text{left}} = \frac{IN}{2L} \bar{a}_z}$$

Case iii) at right end.

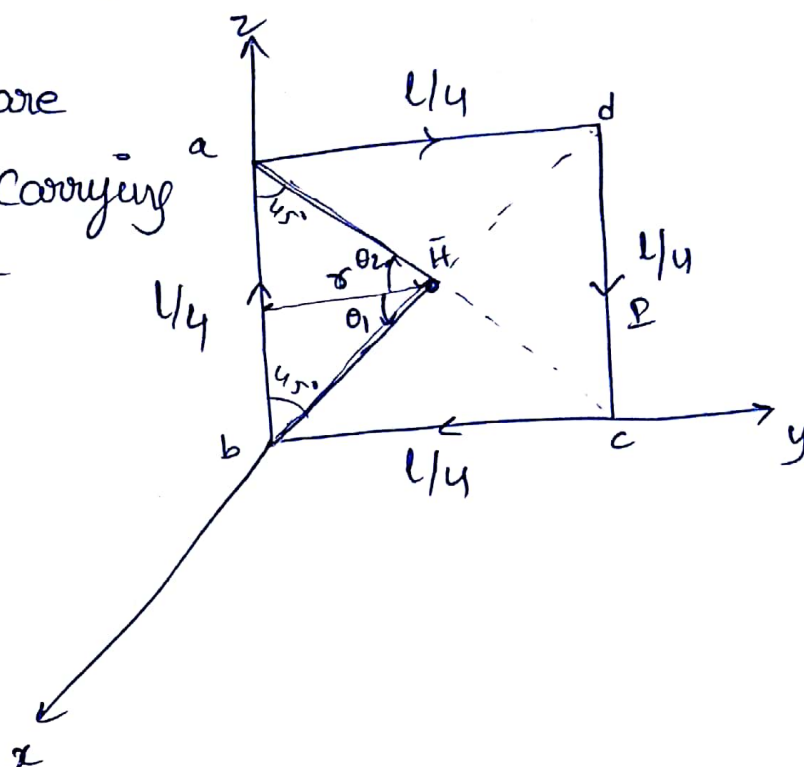
$$\bar{H} = \frac{IN}{2L} [\cos\theta_2 - \cos\theta_1] \bar{a}_z$$

If $L \gg a$ then $\theta_2 = 90^\circ$, $\theta_1 = 180^\circ$

$$\begin{aligned} \Rightarrow \bar{H}_{\text{right}} &= \frac{IN}{2L} [\cos 90^\circ - \cos 180^\circ] \bar{a}_z \\ &= \frac{IN}{2L} [0 - (-1)] \bar{a}_z \end{aligned}$$

$$\boxed{\bar{H}_{\text{right}} = \frac{IN}{2L} \bar{a}_z}$$

Consider a Square Loop of length 'l' carrying a direct-current I Amps.



We know that magnetic field intensity due to a finite conductor is given by

$$\vec{H}_{ab} = \frac{I}{4\pi r} (\sin\theta_2 + \sin\theta_1) \vec{a}_\phi$$

$\therefore$ where $\theta_2 = 45^\circ = \theta_1$ & $r = l/8$

$$\Rightarrow \vec{H}_{ab} = \frac{I}{4\pi l/8} (\sin 45^\circ + \sin 45^\circ) \vec{a}_\phi$$

$$= \frac{2I}{\pi l} \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right] \vec{a}_\phi$$

$$\vec{H}_{ab} = \frac{2\sqrt{2} I}{\pi l} \vec{a}_\phi$$

$\therefore$ Total magnetic field intensity

$$\vec{H} = 4 \vec{H}_{ab} = \frac{4 \times 2\sqrt{2} I}{\pi l} \vec{a}_\phi$$

$$\boxed{\vec{H} = \frac{8\sqrt{2} I}{\pi l} \vec{a}_\phi \text{ A/m}}$$

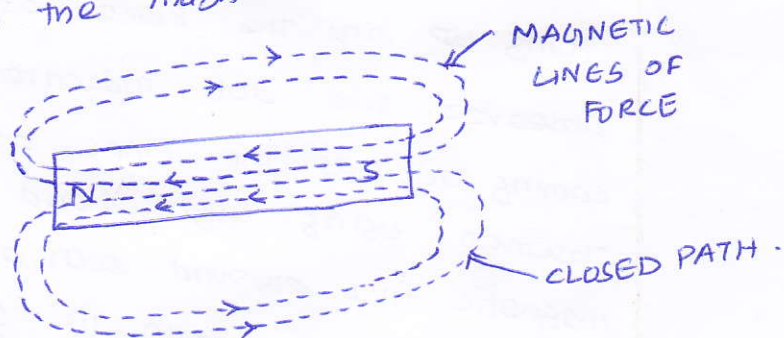
MAGNETIC FIELD & ITS PROPERTIES:

Consider a Permanent magnet, It has two poles North (N) and South (S). The region around a magnet within which the influence of the magnet can be experienced is called magnetic field.

Such a field is represented by imaginary lines around the magnet which are called magnetic lines of force. These lines of force are also called magnetic lines of flux or magnetic flux lines.

An important difference b/w electric flux lines and magnetic flux lines can be observed here. In case of electric flux, the flux lines originate from an isolated positive charge and diverge to terminate at infinity. While for a -ve charge, electric flux lines converge on a charge, starting from infinity. But in case of magnetic flux, the poles exist in pairs only.

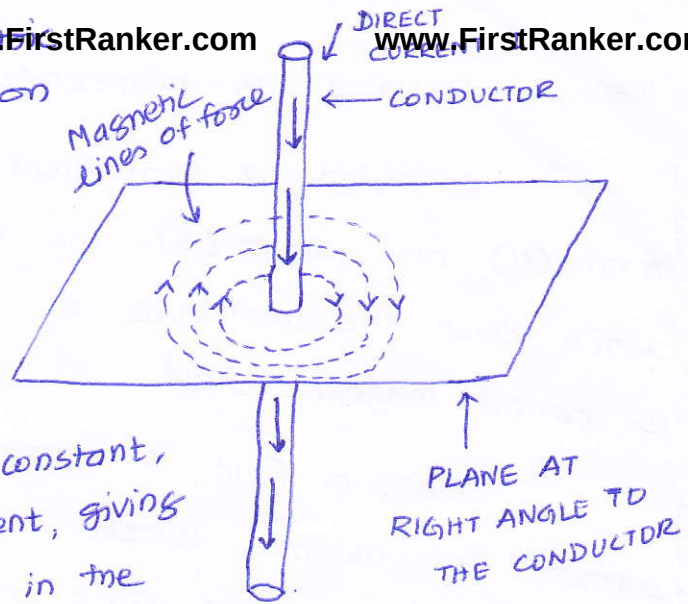
Hence every magnetic flux line starting from North Pole must end at south pole and complete the path from south to north internal to the magnet.



MAGNETIC FIELD DUE TO CURRENT CARRYING CONDUCTOR:

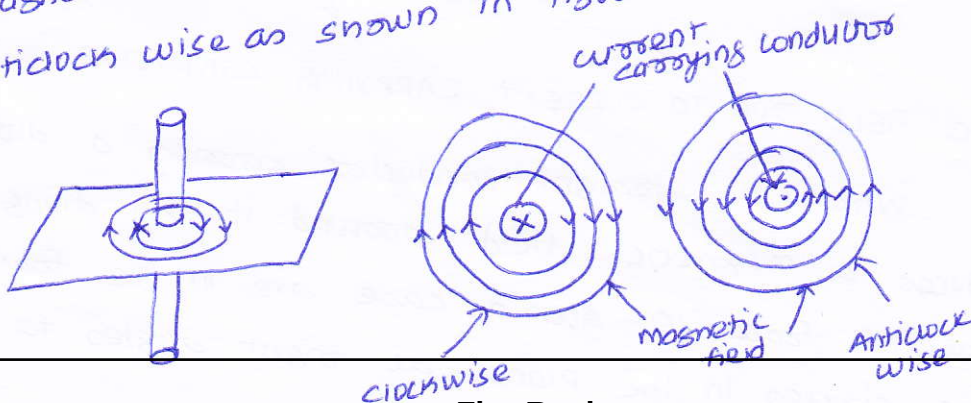
When a straight conductor carries a direct current, it produces a magnetic field around it, all along its length. The lines of force in such a case are in the form of concentric circles in the planes at right angles to the conductor.

The direction of magnetic field lines around depends on the direction of current through the conductor. As long as the current is constant and current is time independent, magnetic lines of force are also constant, static and time independent, giving a steady magnetic field in the space around the conductor.



A right hand thumb rule is used to determine the direction of magnetic field around a conductor carrying a direct current. It states that, hold the current carrying conductor in right hand such that thumb pointing in the direction of current and fingers point in the direction of the magnetic lines of flux around it.

The cross indicates that the current direction is going ~~up~~ into the plane of the paper away from the observer. The dot indicates that the current direction is coming out of the plane of the paper, towards the observer. Using right hand thumb rule, the direction of magnetic flux around such a conductor is either clockwise (or) anticlockwise as shown in figure.



the magnetic lines of force i.e magnetic flux lines always form closed loops and exist in the form of concentric circles, around a current carrying conductor. The total number of magnetic lines of force is called a magnetic flux denoted as ϕ . It is measured in weber (Wb). one Wb means 10^8 lines of force.

MAGNETIC FIELD INTENSITY: ($\vec{H}$)

The quantitative measure of strongness or weakness of the magnetic field is given by magnetic field intensity or magnetic field strength. The magnetic field intensity at any point in the magnetic field is defined as the force experienced by a unit north pole of one weber strength, when placed at that point.

The magnetic flux lines are measured in webers (Wb) while magnetic field intensity is measured in Newtons/weber [N/Wb] or amperes per meter [A/m] or ampere turns/meter [AT/m]. It is denoted by $\vec{H}$.

MAGNETIC FLUX DENSITY: ($\vec{B}$)

The total magnetic lines of force i.e magnetic flux crossing a unit area in a plane at right angles to the direction of flux is called magnetic flux density. It is denoted by $\vec{B}$. It is measured in Wb/m^2 which is also called Tesla (T).

RELATION BETWEEN $\vec{B}$ & $\vec{H}$.

In magnetostatics, the $\vec{B}$ and $\vec{H}$ are related to each other thro' the property of the region in which current carrying conductor is placed. It is called permeability denoted as μ .

For any other region

a relative permeability is specified as μ_r

and $\mu = \mu_0 \mu_r$.

The $\vec{B}$ and $\vec{H}$ are related as

$$\vec{B} = \mu \vec{H} = \mu_0 \mu_r \vec{H}$$

For free space

$$\vec{B} = \mu_0 \vec{H}$$

For nonmagnetic media

$$\mu_r = 1$$

For magnetic materials

$$\mu_r > 1$$

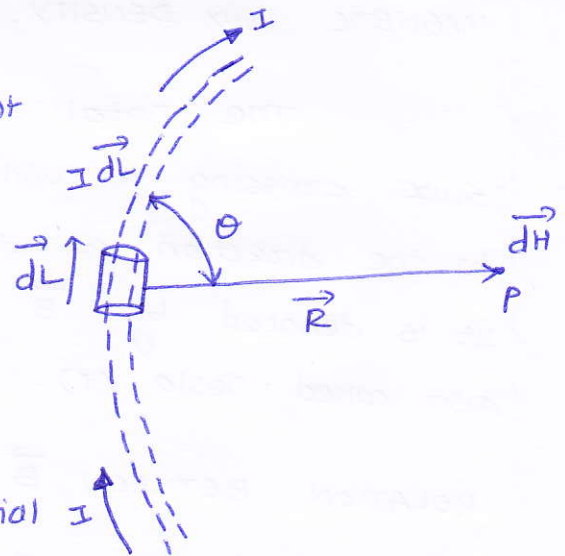
BIOT-SAVART LAW:

Consider a conductor carrying a direct current I and a steady magnetic field produced around it. The Biot-Savart law allows us to obtain the differential magnetic field intensity $d\vec{H}$, produced at point P , due to a differential current element $I dL$.

consider a differential length dL hence the differential current element is $I dL$. This is a very small part of the current carrying conductor. The point P is at a distance R from the differential current element. The

θ is the angle b/w the differential I current element and the line

joining point P to the differential current element.



The magnetic field intensity dH produced at a point P due to a differential current element IdL is,

1. Proportional to the product of current I and differential length dL .
2. The sine of the angle b/w the element and the line joining point P to the element
3. And Inversely proportional to the square of the distance R b/w point P and the element.

Mathematically, the Biot-Savart's law can be stated as,

$$\vec{dH} \propto \frac{IdL \sin \theta}{R^2} \text{ ---- (1)}$$

$$\vec{dH} = \frac{kIdL \sin \theta}{R^2} \text{ ---- (2)}$$

$k \rightarrow$ constant of proportionality

$$k = \frac{1}{4\pi}$$

$$\therefore \vec{dH} = \frac{IdL \sin \theta}{4\pi R^2} \text{ ---- (3)}$$

Let dL = Magnitude of vector length $\vec{dL}$ and

$\vec{a_R}$ = unit vector in the direction from differential current element to point P.

Then from rule of cross product

$$\vec{dL} \times \vec{a_R} = dL |\vec{a_R}| \sin \theta = dL \sin \theta \text{ ---- (4)}$$

using (4) in (3) we get

$$\vec{dH} = \frac{I \vec{dL} \times \vec{a_R}}{4\pi R^2} \text{ A/m. ---- (5)}$$

$$\text{But } \vec{a_R} = \frac{\vec{R}}{|\vec{R}|} = \frac{\vec{R}}{R} \therefore$$

$$\vec{dH} = \frac{I \vec{dL} \times \vec{R}}{4\pi R^3} \text{ A/m ---- (6)}$$

Equations (5) and (6) is the mathematical form of Biot-Savart's law.

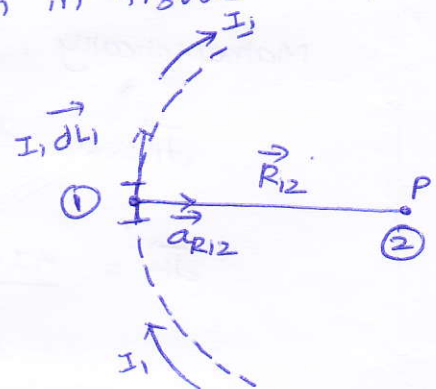
field intensity $\vec{H}$, the equation (5) takes the integral form as,

$$\vec{H} = \oint \frac{I d\vec{L} \times \vec{a}_R}{4\pi R^2} \quad \text{--- (7)}$$

The closed line integral is required to ensure that all the current elements are considered. This is because current can flow only in the closed path, provided by the closed circuit.

If the current element is considered at Point 1 and point P at Point 2, as shown in figure then,

$$d\vec{H}_2 = \frac{I_1 d\vec{L}_1 \times \vec{a}_{R12}}{4\pi R_{12}^2} \text{ A/m.}$$



$I_1 \rightarrow$ current flowing thro' dL_1
at Point 1

$d\vec{L}_1 \rightarrow$ Differential vector length at
Point 1

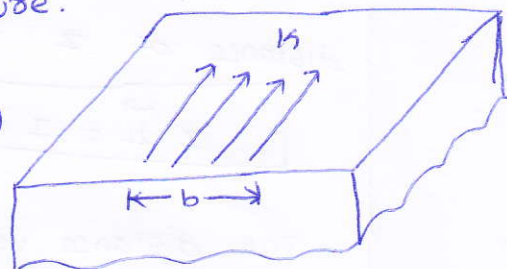
$\vec{a}_{R12} \rightarrow$ Unit vector in the direction from element at
Point 1 to the Point P at Point 2

$$\vec{a}_{R12} = \frac{\vec{R}_{12}}{|\vec{R}_{12}|} = \frac{\vec{R}_{12}}{R_{12}}$$

$$\therefore H = \oint \frac{I_1 d\vec{L}_1 \times \vec{a}_{R12}}{4\pi R_{12}^2} \text{ A/m.} \quad \text{--- (8)}$$

This is called Integral form of Biot-Savart Law.

considers a surface carrying a uniform current over its surface as shown in figure. Then the surface current density is denoted as $\vec{K}$ and is measured in (A/m)



Thus for uniform current density, current I in any width b is given by $I = Kb$, where width b is $\perp$ to the direction of current flow.

Thus if ds is the differential surface area considered of a surface having current density $\vec{K}$ then

$$I d\vec{L} = \vec{K} ds \text{ ---- (1)}$$

If the current density in a volume of a given conductor is $\vec{J}$ measured in A/m^2 then the differential volume dv we can write

$$I d\vec{L} = \vec{J} dv \text{ ---- (2)}$$

Hence biot-savart's law can be expressed for surface current considering $\vec{K} ds$ while for volume current considering $\vec{J} dv$.

$$\begin{aligned} \therefore \vec{H} &= \oint_S \frac{\vec{K} \times \vec{r}}{4\pi r^2} ds \text{ A/m} \\ \vec{H} &= \oint_{Vol} \frac{\vec{J} \times \vec{r}}{4\pi r^2} dv \text{ A/m} \end{aligned} \text{ ---- (3)}$$

The Biot-savart's law is also called Ampere's law for the current element.

$\vec{H}$ due to infinitely long straight conductor:

consider an infinitely long straight conductor, along z-axis the current passing through the conductor is a direct current of I Amp. The field intensity $\vec{H}$ at a point P is to be calculated, which is at a distance 'r' from the z-axis.

at point 1, along the z-axis at a distance of z from origin.

$$\therefore I d\vec{L} = I dz \vec{a}_z \quad \text{--- (1)}$$

The distance vector joining Point 1 to Point 2 is $\vec{R}_{12}$ and can be written as

$$\begin{aligned} \vec{R}_{12} &= (r-0)\vec{a}_r + (0-0)\vec{a}_\phi + (0-z)\vec{a}_z \\ &= r\vec{a}_r + 0\vec{a}_\phi - z\vec{a}_z \end{aligned}$$

$$\therefore \vec{R}_{12} = r\vec{a}_r - z\vec{a}_z \quad \text{--- (2)}$$

$$\vec{a}_{R12} = \frac{\vec{R}_{12}}{|\vec{R}_{12}|} = \frac{-z\vec{a}_z + r\vec{a}_r}{\sqrt{r^2 + z^2}} = \frac{r\vec{a}_r - z\vec{a}_z}{\sqrt{r^2 + z^2}}$$

$$d\vec{L} \times \vec{a}_{R12} = \begin{bmatrix} \vec{a}_r & \vec{a}_\phi & \vec{a}_z \\ 0 & 0 & dz \\ r & 0 & -z \end{bmatrix} = r dz \vec{a}_\phi$$

Note: While taking cross product, $|\vec{R}_{12}|$ is neglected for convenience and must be considered for further calculations.

$$\therefore I d\vec{L} \times \vec{a}_{R12} = \frac{I r dz \vec{a}_\phi}{\sqrt{r^2 + z^2}}$$

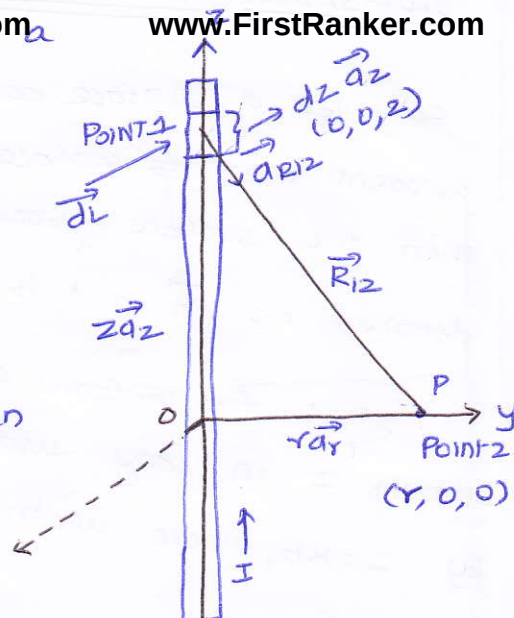
According to Biot-Savart's law, $d\vec{H}$ at point 2 is

$$d\vec{H} = \frac{I d\vec{L} \times \vec{a}_{R12}}{4\pi R_{12}^2} = \frac{I r dz \vec{a}_\phi}{\sqrt{r^2 + z^2}} \cdot \frac{1}{4\pi (r^2 + z^2)^{3/2}}$$

$$\frac{I dz \vec{a}_z}{4\pi (r^2 + z^2)^{3/2}} \times \left(\frac{r\vec{a}_r - z\vec{a}_z}{\sqrt{r^2 + z^2}} \right) d\vec{H} = \frac{I r dz \vec{a}_\phi}{4\pi (r^2 + z^2)^{3/2}}$$

$$a_z \times a_r = a_\phi \quad a_z \times a_\phi = 0$$

$$d\vec{H} = \frac{I r dz}{4\pi (r^2 + z^2)^{3/2}} \vec{a}_\phi$$



Integrating $d\vec{H}$ over the entire length as $\vec{H} = \frac{I r d\vec{a}_\phi}{4\pi (r^2 + z^2)^{3/2}}$

$$\therefore \vec{H} = \int_{z=-\infty}^{\infty} d\vec{H} = \int_{z=-\infty}^{\infty} \frac{I r d\vec{a}_\phi}{4\pi (r^2 + z^2)^{3/2}}$$

Put $z = r \tan \theta$ ← can be obtained by using $z = r \tan \theta$.
 $z^2 = r^2 \tan^2 \theta$
 and $dz = r \sec^2 \theta d\theta$, $z = -\infty$, $\theta = -\pi/2$ & $z = +\infty$, $\theta = +\pi/2$

$$\therefore \vec{H} = \int_{\theta=-\pi/2}^{\theta=\pi/2} \frac{I r r \sec^2 \theta d\theta \vec{a}_\phi}{4\pi (r^2 + r^2 \tan^2 \theta)^{3/2}}$$

$$= \int_{-\pi/2}^{\pi/2} \frac{I r^2 \sec^2 \theta d\theta \vec{a}_\phi}{4\pi r^3 (\sec^3 \theta)}$$

$$\left[\because (r^2 + r^2 \tan^2 \theta)^{3/2} = [r^2 (1 + \tan^2 \theta)]^{3/2} = r^3 \sec^3 \theta \right] \cdot 1 + \tan^2 \theta = \sec^2 \theta$$

$$\vec{H} = \int_{-\pi/2}^{\pi/2} \frac{I}{4\pi r} \cdot \frac{1}{\sec \theta} \cdot d\theta \cdot \vec{a}_\phi \Rightarrow \frac{I}{4\pi r} \int_{-\pi/2}^{\pi/2} \cos \theta d\theta \vec{a}_\phi$$

$$\vec{H} = \frac{I}{4\pi r} [\sin \theta]_{-\pi/2}^{\pi/2} \vec{a}_\phi = \frac{I}{4\pi r} [\sin \pi/2 - \sin(-\pi/2)] \vec{a}_\phi$$

$$\vec{H} = \frac{I}{4\pi r} [1 - (-1)] \vec{a}_\phi = \frac{2I}{4\pi r} \vec{a}_\phi$$

$$\boxed{\vec{H} = \frac{I}{2\pi r} \vec{a}_\phi} \text{ A/m}$$

$$\boxed{\vec{B} = \mu \vec{H} = \frac{\mu I}{2\pi r} \vec{a}_\phi \text{ Wb/m}^2}$$

AMPERE'S CIRCUITAL LAW:

In electrostatics, The Gauss's law is useful to obtain the $\vec{E}$ in case of complex problems. Similarly in the magneto-statics, The complex problems can be solved using a law called Ampere's circuital law (or) Ampere's work law.

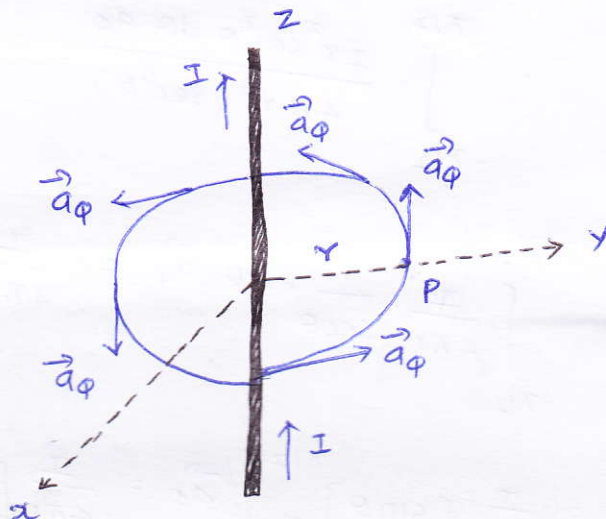
The line integral of magnetic field intensity $\vec{H}$ around a closed path is exactly equal to the direct current enclosed by that path.

The mathematical representation of Ampere's circuital law is,

$$\oint \vec{H} \cdot d\vec{L} = I$$

PROOF FOR AMPERE'S CIRCUITAL LAW:

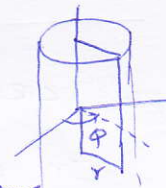
Consider a long straight conductor carrying direct current I placed along z -axis as shown in figure below.



consider a closed circular path of radius r which encloses the straight conductor carrying direct current I . The point P is at a distance r from the conductor. consider $d\vec{L}$ at point P which is in $\vec{a}_\phi$ direction, tangential to circular path at point P .

$$\therefore d\vec{L} = r d\phi \vec{a}_\phi \quad \text{--- (1)}$$

While $\vec{H}$ obtained at point P , from Biot-savart's law due to infinitely long conductor



is
$$\vec{H} = \frac{I}{2\pi r} \vec{a}_\phi \quad \left[\text{from } \textcircled{5} \text{ of unit IV} \right]$$

--- (2)

$$\vec{H} \cdot d\vec{L} = \frac{I r d\phi}{2\pi r}$$

$$\vec{H} \cdot d\vec{L} = \frac{I d\phi}{2\pi} \quad \text{--- (3)}$$

Integrating (3) over the entire closed path,

$$\oint \vec{H} \cdot d\vec{L} = \int_{\phi=0}^{2\pi} \frac{I d\phi}{2\pi} = \frac{I}{2\pi} [\phi]_0^{2\pi} = I$$

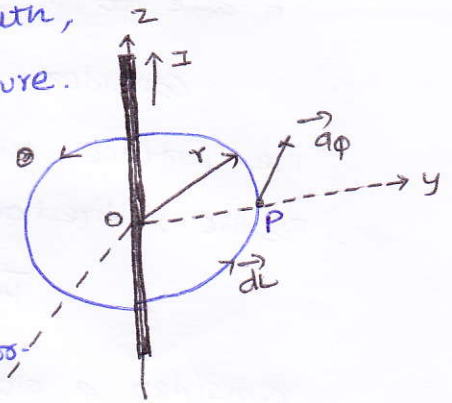
= current carried by conductor.

This proves that the integral $\vec{H} \cdot d\vec{L}$ along closed path gives direct current enclosed by that closed path.

APPLICATION OF AMPERE'S CIRCUITAL LAW:

$\vec{H}$ due to infinitely straight conductor
consider an infinitely long straight conductor placed along z-axis, carrying a direct current I as shown in the figure. consider the Amperian closed path, enclosing the conductor as shown in figure.

consider point P on the closed path at which $\vec{H}$ is to be obtained. The radius of the path is r and hence P is at a $\perp^r$ distance from the conductor.



The magnitude of $\vec{H}$ depends on r and the direction is always tangential to the closed path i.e. $\vec{a}_\phi$. so $\vec{H}$ has only component in $\vec{a}_\phi$ direction say H_ϕ .

consider elementary length $d\vec{L}$ at a point P and in cylindrical co-ordinates it is $r d\phi$ in $\vec{a}_\phi$ direction.

$$\therefore \vec{H} = H_\phi \vec{a}_\phi \quad \text{and} \quad d\vec{L} = r d\phi \vec{a}_\phi$$

$$\therefore \vec{H} \cdot d\vec{L} = H_\phi \vec{a}_\phi \cdot r d\phi \vec{a}_\phi = H_\phi r d\phi$$

$$\oint \vec{H} \cdot d\vec{L} = I$$

$$\int_{\phi=0}^{2\pi} H_{\phi} r d\phi = I$$

$$H_{\phi} r [\phi]_0^{2\pi} = I$$

$$H_{\phi} r 2\pi = I$$

$$\Rightarrow H_{\phi} = \frac{I}{2\pi r}$$

Hence $\vec{H}$ at Point P is given by

$$\vec{H} = H_{\phi} \vec{a}_{\phi} = \frac{I}{2\pi r} \vec{a}_{\phi} \text{ A/m}$$

$$\therefore \vec{B} = \mu \vec{H} = \frac{\mu I}{2\pi r} \vec{a}_{\phi}$$

$\vec{H}$ due to infinite sheet of current:

Consider an infinite sheet of current in the $z=0$ plane. The surface current density is $\vec{K}$. The current is flowing in Positive y direction hence

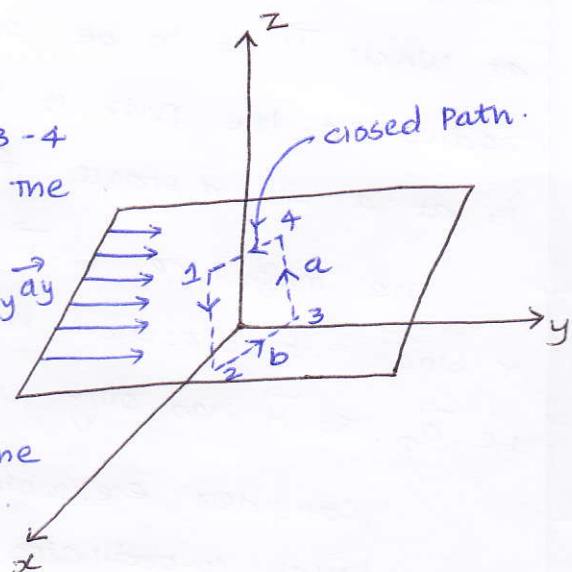
$$\vec{K} = K_y \vec{a}_y$$

consider a closed path 1-2-3-4 as shown in fig. The width of the path is 'b' while height is 'a'.

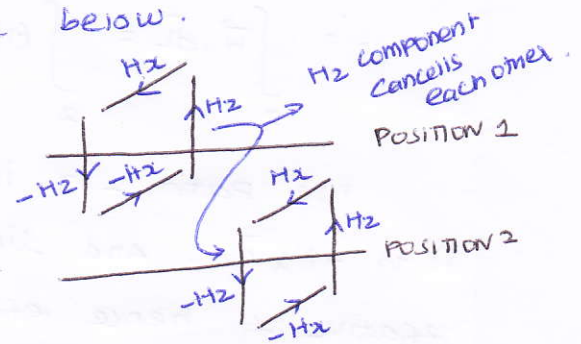
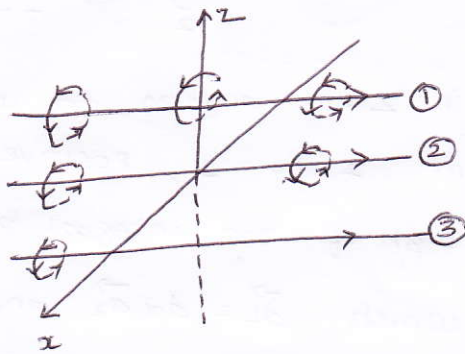
It is $\perp^r$ to the direction of current hence in xz plane.

The current flowing across the distance 'b' is given by $K_y b$.

$$\therefore I_{enc} = K_y b \text{ ---- (1)}$$



considered the magnetic lines of force due to the current in $\vec{a}_y$ direction, according to the right hand thumb rule. These are shown in figure below.



It is clear that in b/w two very closely spaced conductors, the components of $\vec{H}$ in z direction are oppositely directed $[-H_z$ for position 1 and $+H_z$ for position 2 b/w the two opposite positions]. All such components cancel each other and hence $\vec{H}$ cannot have any component in z -direction.

As current is flowing in y direction, $\vec{H}$ cannot have component in y direction.

so $\vec{H}$ has only component in x direction.

$$\vec{H} = H_x \vec{a}_x \quad \text{for } z > 0$$

$$= -H_x \vec{a}_x \quad \text{for } z < 0$$

Applying Ampere's circuital law.

$$\oint \vec{H} \cdot d\vec{L} = I_{enc} \quad \text{--- (3)} = \int_1^2 + \int_2^3 + \int_3^4 + \int_4^1 [\vec{H} \cdot d\vec{L}]$$

Evaluate the integral along the path 1-2-3-4-1.

For path 1-2, $d\vec{L} = dz \vec{a}_z$

3-4, $d\vec{L} = -dz \vec{a}_z$

But $\vec{H}$ is in x -direction while $\vec{a}_z \cdot \vec{a}_z = 0$.

Hence along the paths 1-2 and 3-4, the integral

$$\oint \vec{H} \cdot d\vec{L} = 0$$

$$\therefore \int_2^3 \vec{H} \cdot d\vec{L} = \int_2^3 (H_x \vec{a}_x) \cdot (dx \vec{a}_x) = -H_x \int_2^3 dx = b H_x \quad \text{--- (4)}$$

The path 2-3 is lying in $z < 0$ region for which $\vec{H}$ is $-H_x \vec{a}_x$. And limits from 2 to 3, positive x to negative x hence effective sign of the integral is +ve

considered path 4-1 along which $d\vec{L} = dx \vec{a}_x$ and it is in the region $z > 0$ hence $\vec{H} = H_x \vec{a}_x$

$$\therefore \int_4^1 \vec{H} \cdot d\vec{L} = \int_4^1 (H_x \vec{a}_x) \cdot (dx \vec{a}_x) = H_x \int_4^1 dx = b H_x \quad \text{--- (5)}$$

$\therefore$ sub (4) and (5) in (3) we get

$$\oint \vec{H} \cdot d\vec{L} = b H_x + b H_x = 2b H_x \quad \text{--- (6)}$$

using (4) in eqn (6) we get

$$\oint \vec{H} \cdot d\vec{L} = 2b H_x = I_{enc} = \mu_0 y b$$

$$\therefore 2b H_x = \mu_0 y b$$

$$H_x = \frac{1}{2} \mu_0 y$$

Hence $\vec{H} = \frac{1}{2} \mu_0 y \vec{a}_x$ for $z > 0$

$$= -\frac{1}{2} \mu_0 y \vec{a}_x \quad \text{for } z < 0$$

In general for an infinite sheet of current density $\vec{K}$ A/m we can write

$$\vec{H} = \frac{1}{2} \vec{K} \times \vec{a}_N$$

Where

$\vec{a}_N$ = unit vector normal to the current sheet to the point at which $\vec{H}$ is to be obtained.

consider the differential surface element in the xy plane, as shown in figure below. The unknown current has produced $\vec{H}$ at the centre of the incremental closed path.

The total magnetic field at point P which is at the centre of the small rectangle is,

$$\vec{H} = H_{x0} \vec{a}_x + H_{y0} \vec{a}_y + H_{z0} \vec{a}_z \quad \text{--- (1)}$$

while the total current density is given by,

$$\vec{J} = J_x \vec{a}_x + J_y \vec{a}_y + J_z \vec{a}_z \quad \text{--- (2)}$$

To apply Ampere's circuital law to this closed path, let us evaluate the closed line integral of $\vec{H}$ about this path in the direction abcda. According to right hand thumb rule the current is in $\vec{a}_z$ direction.

Along path a-b,

$$\vec{H} = H_y \vec{a}_y \text{ \& } d\vec{L} = \Delta y \vec{a}_y$$

$$\therefore \vec{H} \cdot d\vec{L} = H_y \Delta y \quad \text{--- (3)}$$

The intensity H_y along a-b can be expressed in terms of H_{y0} existing at P and the rate of change of H_y in the x -direction with x .

The distance in x direction of a-b from point P is $(\frac{\Delta x}{2})$. Hence $\vec{H} \cdot d\vec{L}$ along a-b can be expressed as

$$(\vec{H} \cdot d\vec{L})_{ab} = \left[H_{y0} + \frac{\partial H_y}{\partial x} \left(\frac{\Delta x}{2} \right) \right] \Delta y \quad \text{--- (4)}$$

For path b-c, $\vec{H}$ is in $-\vec{a}_x$ direction hence $-H_x \vec{a}_x$ and

$$d\vec{L} = \Delta x \vec{a}_x$$

$$\therefore \vec{H} \cdot d\vec{L} = -H_x \Delta x \quad \text{--- (5)}$$

$$H_x = H_{x0} + \frac{\Delta y}{2} \frac{\partial H_x}{\partial y}$$

The distance of bc from P is $\Delta y/2$.

$$\therefore (\vec{H} \cdot d\vec{L})_{b-c} = - \left[H_{x0} + \frac{\partial H_x}{\partial y} \frac{\Delta y}{2} \right] \Delta x \quad \text{--- (6)}$$

For path $c-d$ $\vec{H}$ is in $-\vec{a}_y$ direction hence $-H_y \vec{a}_y$ and $d\vec{L} = \Delta y \vec{a}_y$

$$\therefore \vec{H} \cdot d\vec{L} = -H_y \Delta y \quad \text{--- (7)}$$

BUT H_y can be expressed in terms of H_{y0} and rate of change of H_y in negative x direction. The distance of cd from point P is $(\Delta x/2)$ in negative x direction

$$\therefore H_y = H_{y0} - \frac{\Delta x}{2} \frac{\partial H_y}{\partial x}$$

$$\therefore (\vec{H} \cdot d\vec{L})_{c-d} = - \left[H_{y0} - \frac{\Delta x}{2} \frac{\partial H_y}{\partial x} \right] \Delta y \quad \text{--- (8)}$$

For path $d-a$, $\vec{H}$ is in $+\vec{a}_x$ direction hence $H_x \vec{a}_x$ and $d\vec{L} = \Delta x \vec{a}_x$

$$\therefore \vec{H} \cdot d\vec{L} = H_x \Delta x \quad \text{--- (9)}$$

BUT H_x can be expressed in terms of H_{x0} and rate of change of H_x in negative y direction. The distance of da from point P is $(\frac{\Delta y}{2})$ in negative y direction

$$\therefore H_x = \left[H_{x0} - \frac{\Delta y}{2} \frac{\partial H_x}{\partial y} \right] \quad \text{--- (10)}$$

$$\therefore (\vec{H} \cdot d\vec{L})_{d-a} = \left[H_{x0} - \frac{\Delta y}{2} \frac{\partial H_x}{\partial y} \right] \Delta x \quad \text{--- (10)}$$

$$\vec{H} \cdot d\vec{L} = H_{y0} \Delta y + \frac{\Delta x \Delta y}{2} \frac{\partial H_y}{\partial x} - H_{x0} \Delta x - \frac{\Delta x \Delta y}{2} \frac{\partial H_x}{\partial y} \\ - H_{y0} \Delta y + \frac{\Delta x \Delta y}{2} \frac{\partial H_y}{\partial x} + H_{x0} \Delta x - \frac{\Delta x \Delta y}{2} \frac{\partial H_x}{\partial y}$$

$$\oint \vec{H} \cdot d\vec{L} = \Delta x \Delta y \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right] \text{----- (11)}$$

According to Ampere's circuital law, this integral must be current enclosed by the differential element.

current enclosed = current density normal to closed path } x Area of that closed path

$$I_{\text{enc}} = J_z \Delta x \Delta y \text{--- (12)}$$

where J_z = current density in $\vec{a}_z$ direction as the current enclosed is in $\vec{a}_z$ direction.

From eqn (11) & (12)

$$\oint \vec{H} \cdot d\vec{L} = I_{\text{enc}}$$

$$\oint \vec{H} \cdot d\vec{L} = \Delta x \Delta y \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right] = J_z \Delta x \Delta y$$

$\div \Delta x \Delta y$

$$\frac{\oint \vec{H} \cdot d\vec{L}}{\Delta x \Delta y} = \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right] = J_z \text{--- (13)}$$

This gives accurate result as the closed path shrinks to a point (ie) $\Delta x \Delta y$ area tends to zero.

$$\lim_{\Delta x \Delta y \rightarrow 0} \frac{\oint \vec{H} \cdot d\vec{L}}{\Delta x \Delta y} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z \text{--- (14)}$$

we can write,

$$\lim_{\Delta y \Delta z \rightarrow 0} \frac{\oint \vec{H} \cdot d\vec{L}}{\Delta y \Delta z} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x \quad \text{--- (15)}$$

and $\lim_{\Delta z \Delta x \rightarrow 0} \frac{\oint \vec{H} \cdot d\vec{L}}{\Delta z \Delta x} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y \quad \text{--- (16)}$

In general we can write,

$$\lim_{\Delta S_N \rightarrow 0} \frac{\oint \vec{H} \cdot d\vec{L}}{\Delta S_N} = J_N \quad \text{--- (17)}$$

$\rightarrow \text{curl}$

where J_N = current density normal to the surface ΔS .

The term on left hand side of the equation is called curl $\vec{H}$. The ΔS_N is area enclosed by the closed line integral.

the total $\vec{J}$ now can be obtained by adding eqn (14), (15) and (16).

$$\begin{aligned} \vec{J} &= J_x \vec{a}_x + J_y \vec{a}_y + J_z \vec{a}_z \\ &= \left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right] \vec{a}_x + \left[\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right] \vec{a}_y + \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right] \vec{a}_z \end{aligned}$$

$$\boxed{\vec{J} = \text{curl } \vec{H} = \nabla \times \vec{H}} \quad \text{--- (18)}$$

The curl $\vec{H}$ is indicated by $\nabla \times \vec{H}$ which is cross product of operators 'del' and $\vec{H}$.

The equation (18) is called the point form of

Ampere's circuital law.

$$\boxed{\text{curl } \vec{H} = \nabla \times \vec{H} = \vec{J}}$$

This is one of the Maxwell's equations.

The curl of a vector in the direction of the unit vector is the ratio of the line integral of the vector around a closed contour, to the enclosed area bounded by the contour, as the enclosed area diminished to zero.

1. The curl of a vector is a vector quantity
2. $\nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$
3. $\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$
4. The divergence of a curl is zero
 $\nabla \cdot (\nabla \times \vec{A}) = 0$
5. The curl of a gradient of a vector is zero
 $\nabla \times \nabla V = 0$

STROKE'S THEOREM:

Analogous to the divergence theorem in electrostatics, there exists Stokes's theorem in magnetostatics. The Stokes's theorem relates the line integral to a surface integral.

The Stokes's theorem states that,

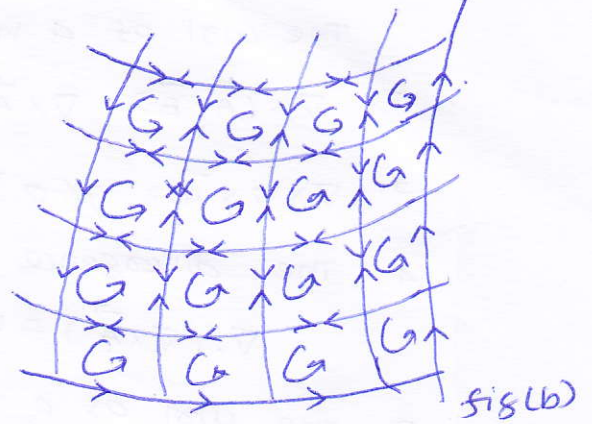
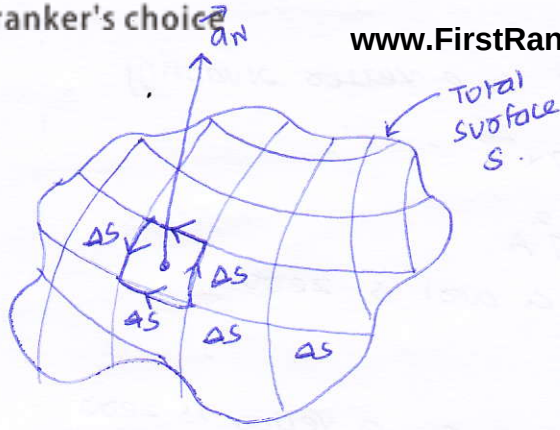
"The line integral of vector $\vec{A}$ around a closed path L is equal to the integral of curl of $\vec{A}$ over the open surface S enclosed by the closed path L ".

The theorem is applicable only when $\vec{A}$ and $\nabla \times \vec{A}$ are continuous on the surface S .

$$\oint_L \vec{H} \cdot d\vec{L} = \int_S (\nabla \times \vec{H}) \cdot d\vec{S}$$

PROOF'S OF STROKE'S THEOREM:

Consider a surface S which is splitted into number of incremental surfaces. Each incremental surface is having area Δs as shown in figure.



Applying by definition of the curl to any of these incremental surfaces we can write

$$(\nabla \times \vec{H})_N = \frac{\oint \vec{H} \cdot d\vec{L}_{\Delta S}}{\Delta S} \quad \text{--- (1)}$$

where

$N \rightarrow$ Normal to ΔS according to right hand rule

$dL_{\Delta S} \rightarrow$ Perimeter of the incremental surface ΔS .

Now the curl of $\vec{H}$ in the normal direction is the dot product of curl of $\vec{H}$ with $\vec{a}_N$, where $\vec{a}_N$ is unit vector, normal to the surface ΔS , according to right hand rule,

$$\therefore (\nabla \times \vec{H})_N = (\nabla \times \vec{H}) \cdot \vec{a}_N$$

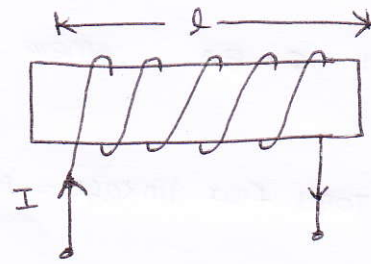
$$\therefore \oint \vec{H} \cdot d\vec{L}_{\Delta S} = (\nabla \times \vec{H}) \cdot \vec{a}_N \Delta S$$

$$\therefore \oint \vec{H} \cdot d\vec{L}_{\Delta S} = (\nabla \times \vec{H}) \cdot \vec{\Delta S}$$

To obtain total curl for every incremental surface, add the closed line integrals for each ΔS . From fig(b), it can be seen that at a common boundary b/w the two incremental surfaces, the line integral is getting cancelled as the boundary is getting traced in two opposite directions.

Consider a solenoid of N turns as shown in fig.

Let the current flowing through the solenoid be I Amps. Let the length of the solenoid be l and the cross-section area be A .



Magnetic field intensity H inside the solenoid is given by,

$$H = \frac{NI}{l} \text{ (A/m)} \quad \text{--- ①}$$

Total flux linkage is given by

$$\text{Total flux linkage} = N\phi = N(B)(A)$$

$$[\because \phi = BA]$$

$$W_b = \frac{W_b}{m^2} \times m^2$$

$$N\phi = NBA$$

$$\text{But } B = \mu H$$

$$\therefore N\phi = N(\mu H)A$$

$$= \mu N^2 H A$$

$$= \mu N \left[\frac{NI}{l} \right] A \Rightarrow N\phi = \frac{\mu N^2 I A}{l}$$

Inductance of a solenoid is given by

$$L = \frac{\text{Total flux linkage}}{\text{Total current}} = \frac{\mu N^2 I A / l}{I}$$

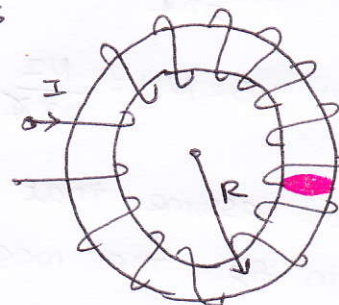
$$L = \frac{\mu N^2 A}{l}$$

INDUCTANCE OF A TOROID:

Consider a toroid ring with N turns and carrying current I . Let the radius of the toroid be R as shown in fig.

The magnetic flux density inside a toroidal ring is given by

$$B = \frac{\mu NI}{2\pi R} \quad \text{--- ①}$$



$$[\because B = \mu H = \mu \cdot \left(\frac{NI}{2\pi R} \right)]$$

Perimeter of circle

given by,

$$\text{Total flux linkage} = N\Phi$$

But $\Phi = BA$ where $A \rightarrow$ Area of cross-section of a toroidal ring.

$$\therefore \text{Total flux linkage} = N(B)(A) = N \left[\frac{NNI}{2\pi R} \right] (A) = \frac{NN^2IA}{(2\pi R)}$$

The inductance of a toroid is given by,

$$L = \frac{\text{Total flux linkage}}{\text{Total current}}$$

$$= \frac{NN^2IA}{(2\pi R)I} = \frac{NN^2A}{2\pi R} \text{ H.}$$

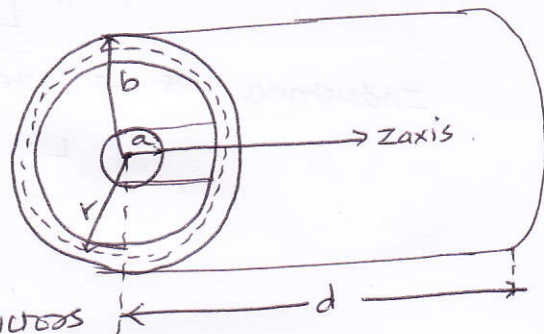
$$L = \frac{NN^2A}{2\pi R} \text{ H}$$

where $A \rightarrow$ Area of cross section of toroidal ring $= \pi r^2 \text{ m}^2$

INDUCTANCE OF A CO-AXIAL CABLE:

Consider a co-axial cable with inner conductor radius 'a' and outer conductor radius 'b'.

Let the current through the co-axial cable be I .



For the co-axial cable the magnetic field intensity at any point b/w inner and outer conductors is given by

$$H = \frac{I}{2\pi r} \quad \text{where } a < r < b \quad \text{--- (1)}$$

$$\text{But } B = \mu H = \frac{\mu I}{2\pi r} \quad \text{--- (2)}$$

The total magnetic flux is given by

$$\Phi = \int_S \vec{B} \cdot d\vec{s}$$

Now $d\vec{s} = dr dz \vec{a}_\phi$ [from cylindrical co-ordinate system].

$$\therefore \Phi = \int_{z=0}^{z=d} \int_{r=a}^{r=b} \frac{NI}{2\pi r} \vec{a}_\phi \cdot dr dz \vec{a}_\phi$$

$$= \frac{NI}{2\pi} \int_{z=0}^{z=d} dz \int_{r=a}^{r=b} \frac{1}{r} dr$$

$$= \frac{NI}{2\pi} [z]_0^d [\ln r]_a^b$$

$$\boxed{\Phi = \frac{NI}{2\pi} d \ln\left(\frac{b}{a}\right)} \quad \text{--- (3)}$$

The inductance of a co-axial cable is given by

$$L = \frac{\text{Total flux linkage}}{\text{Total current}}$$

$$\boxed{L = \frac{\frac{NI d \ln(\frac{b}{a})}{2\pi}}{I} = \frac{Nd}{2\pi} \ln\left(\frac{b}{a}\right) \text{ H.}}$$

The Inductance of a co-axial cable may be expressed per unit length as

$$\boxed{\frac{L}{d} = \frac{N}{2\pi} \ln\left(\frac{b}{a}\right) \text{ H/m.}}$$

MAGNETIC ENERGY - ENERGY STORED IN A MAGNETIC FIELD

Energy stored in a inductor is given by

$$W_m = \frac{1}{2} LI^2$$

$$\Delta L = \frac{\Delta \Phi}{\Delta I} = \frac{B \Delta S}{\Delta I}$$

ΔS = differential surface area
 $= \Delta x \Delta z$

$$\therefore \Delta L = \frac{B (\Delta x \Delta z)}{\Delta I} \quad \text{--- (1)}$$

$$B = \mu H$$

$$\therefore \Delta L = \frac{\mu H \Delta x \Delta z}{\Delta I} \quad \text{--- (2)}$$

$\Delta I = H (\Delta y)$ ($\because$ conducting sheet current is in y direction).
 --- (2)

$\therefore$ Energy stored

$$\Delta W_m = \frac{1}{2} \Delta L \Delta I^2 \quad \text{--- (3)}$$

sub (1) & (2) in (3) we get

$$\begin{aligned} \Delta W_m &= \frac{1}{2} \left[\frac{\mu H \Delta x \Delta z}{\Delta I} \right] \left[H (\Delta y) \right]^2 \\ &= \frac{1}{2} \mu H^2 (\Delta x \Delta y \Delta z) \end{aligned}$$

$$\text{But } \Delta V = \Delta x \Delta y \Delta z$$

$$\therefore \Delta W_m = \frac{1}{2} \mu H^2 \Delta V$$

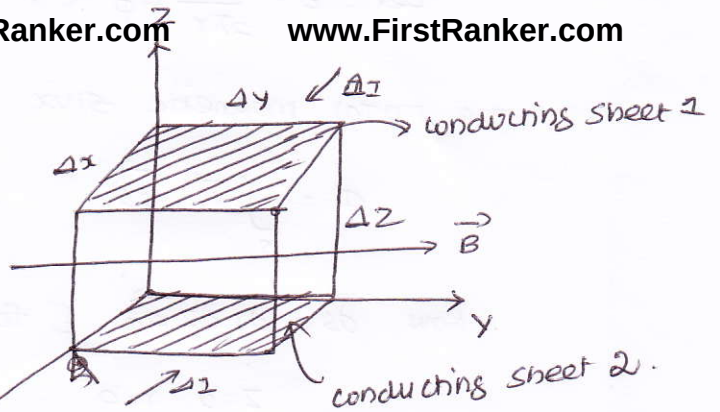
magnetostatics energy density function

$$W_m = \lim_{\Delta V \rightarrow 0} \frac{\Delta W_m}{\Delta V} = \frac{1}{2} \mu H^2 \quad (\text{J/m}^3)$$

different forms

$$W_m = \frac{1}{2} (\mu H) H = \frac{1}{2} B H$$

$$W_m = \frac{1}{2} B (B/\mu) = \frac{B^2}{2\mu}$$



Self and Mutual Inductance

Syllabus :

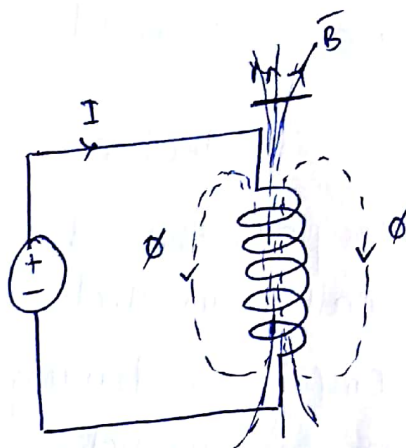
Self and mutual Inductance - determination of Self-inductance of solenoid and toroid - mutual inductance b/w a straight long wire and Square loop wire in same plane - energy stored and density in a magnetic field.

Electric circuit	magnetic circuit
1. The path traced by the current is called an electric circuit	1. path traced by magnetic flux is called magnetic circuit
2. Emf is the driving force measured in volts	2. mmf is the driving force measured in Ampere-turns
3. Resistance $R = \frac{\text{Emf}}{\text{current}} = \frac{l}{\sigma s}$	3. Reluctance $\mathcal{R} = \frac{\text{mmf}}{\text{flux}} = \frac{l}{\mu s} \frac{\text{AT}}{\text{wb}}$
4. Conductivity σ	4. permeability μ .
5. field intensity E	5. Field intensity H
6. Current density $J = \frac{I}{s} = \sigma E \text{ A/m}^2$	6. Flux density $B = \frac{\phi}{s} = \mu H \text{ wb/m}^2$
7. Reciprocal of resistance is Conductance (G).	7. Reciprocal of reluctance is permeance (P).
8. ohm's law $e = IR$	8. ohm's law $e_m = \mathcal{R} \phi$
9. Kirchhoff's laws $\sum I = 0$ $\sum \text{emf} = 0$	9. Kirchhoff's laws $\sum \phi = 0$ $\sum \text{MMF} = \sum \phi s = \sum H l$

A wire or conductor of certain length, when twisted into coil becomes a basic inductor. For every conductor carrying current I and producing magnetic field $\vec{B}$, there exists a self inductance. When two such coils are placed very close to each other there exist a mutual inductance between coils.

Self inductance :-

When a closed conducting path or a circuit carries current I , a magnetic field $\vec{B}$ is produced. This causes a magnetic flux



$$\phi = \int_s \vec{B} \cdot d\vec{s}$$

In closed path or circuit consists of N turns then the flux produced ϕ again links with N turns. gives rise to flux linkage. λ .

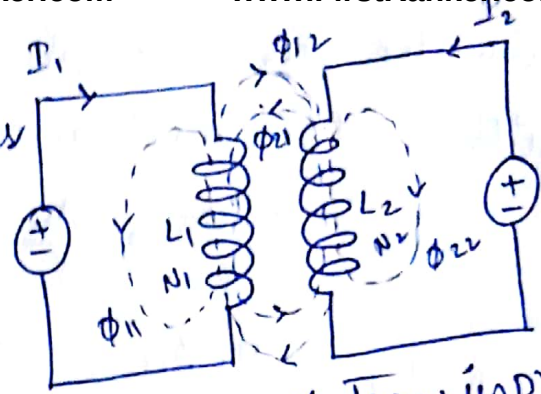
$$\lambda = N\phi \text{ wb.t.}$$

The ratio of total flux linkage to the current flowing through the circuit is called inductance and is given by

$$L = \frac{N\phi}{I} = \frac{\lambda}{I} \text{ H.}$$

The inductance is also known as self-inductance as the flux produced by current through circuit links with the same circuit.

Consider two different circuits with self inductances L_1 and L_2 are kept close to each other.



As two circuits are placed very close to each other. These two circuits are magnetically coupled.

N_1 = No. of Turns in primary
 N_2 = " " " " Secondary
 I_1 = Current in primary
 I_2 = Current in Secondary

Let Φ_{11} = flux produced by circuit 1 due to I_1
 Φ_{12} = Some part of Φ_{11} flowing or links with circuit 2
 Φ_{22} = flux produced by circuit 2 due to I_2
 Φ_{21} = Some part of Φ_{22} links with circuit 1.

The mutual inductance between two coils is defined as The flux linkage of one circuit to the current in other circuit.

$$M_{12} = \frac{\text{flux linkage of circuit 1}}{\text{Current in circuit 2}} = \frac{N_1 \Phi_{21}}{I_2}$$

$$\text{Similarly } M_{21} = \frac{\text{flux linkage of circuit 2}}{\text{Current in circuit 1}} = \frac{N_2 \Phi_{12}}{I_1}$$

If medium b/w circuits is linear without any ferromagnetic material then

$$M_{12} = M_{21} = M$$

efficient of coupling b/w circuits :-
Consider an Example of magnetically coupled circuits. Self inductances of circuits are

$$L_1 = \frac{N_1 \phi_{11}}{I_1} = \frac{\lambda_{11}}{I_1}$$

$$L_2 = \frac{N_2 \phi_{22}}{I_2} = \frac{\lambda_{22}}{I_2}$$

ϕ_{12} = flux linking with circuit 2 due to current I_1 in circuit 1.

$$\Rightarrow \phi_{12} = k_1 \phi_{11}$$

ϕ_{21} = flux linking with circuit 1 due to I_2 in circuit 2.

$$\Rightarrow \phi_{21} = k_2 \phi_{22}$$

$$\therefore M_{12} = \frac{N_1 \phi_{21}}{I_2} = \frac{N_1 (k_2 \phi_{22})}{I_2}$$

$$M_{12} = \frac{N_2 \phi_{12}}{I_1} = \frac{N_2 (k_1 \phi_{11})}{I_1}$$

Assuming The medium Surrounding 2 coils then.

$$M_{12} = M_{21} = M$$

$$\Rightarrow M^2 = M_{12} M_{21} = \frac{N_1 (k_2 \phi_{22})}{I_2} \times \frac{N_2 (k_1 \phi_{11})}{I_1}$$

$$= k_1 k_2 \frac{N_1 \phi_{11}}{I_1} \cdot \frac{N_2 \phi_{22}}{I_2}$$

$$M^2 = k_1 k_2 \cdot L_1 L_2 \Rightarrow M = \sqrt{k_1 k_2} \sqrt{L_1 L_2}$$

Let $\sqrt{k_1 k_2} = K$. (Co-efficient of coupling)

$$\Rightarrow M = K \sqrt{L_1 L_2} \Rightarrow \boxed{K = \frac{M}{\sqrt{L_1 L_2}}}$$

→ Series aiding $\Rightarrow L_{eq} = (L_1 + L_2 + 2M)$ H

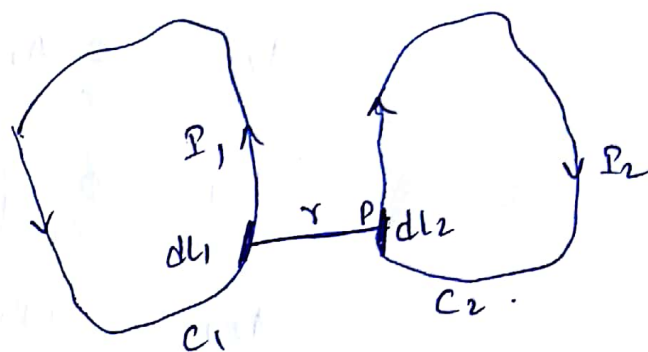
→ Series Opposing $\Rightarrow L_{eq} = (L_1 + L_2 - 2M)$ H

→ parallel aiding $\Rightarrow L_{eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$

→ parallel Opposing $\Rightarrow L_{eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}$

Neuman's Formula

Consider two closed loops C_1 & C_2 of any random shape placed in linear medium carrying currents I_1 & I_2 respectively.



Let r be the distance of separation between C_1 & C_2 .
Let S_1 & S_2 be the surfaces of loop 1 & loop 2, and p located along surface S_2 of loop 2.

The vector magnetic potential at point p due to loop 1 is given by.

$$A_1 = \frac{\mu}{4\pi} \oint_{C_1} \frac{I_1 d\vec{l}_1}{r} = \frac{\mu I_1}{4\pi} \oint \frac{d\vec{l}_1}{r}$$

$\vec{B}_1$ is magnetic flux density expressed in terms of vector magnetic potential $\vec{A}_1$ is

$$\vec{B}_1 = \nabla \times \vec{A}_1$$

Let ϕ_{12} be the flux linking coil 2 due to current in coil 1.

$$\lambda_{21} = N_2 \phi_{12} = \int \vec{B}_1 \cdot d\vec{S}_2$$

$$\lambda_{21} = \Phi_{12} = \int_{S_2} \vec{B}_1 \cdot d\vec{S}_2$$

$$= \int_{S_2} (\nabla \times \vec{A}_1) \cdot d\vec{S}_2$$

By applying Stokes Theorem, convert Surface integral to line integral.

$$\int_{S_2} (\nabla \times \vec{A}_1) \cdot d\vec{S}_2 = \oint_{C_2} \vec{A}_1 \cdot d\vec{L}_2$$

$$\lambda_{21} = \oint_{C_2} \vec{A}_1 \cdot d\vec{L}_2$$

$$\lambda_{21} = \oint_{C_2} \left[\frac{\mu}{4\pi} \oint_{C_1} \frac{I_1 d\vec{L}_1}{r} \cdot d\vec{L}_2 \right]$$

$$\lambda_{21} = \frac{\mu I_1}{4\pi} \oint_{C_2} \oint_{C_1} \frac{d\vec{L}_1 \cdot d\vec{L}_2}{r}$$

The mutual inductance is defined as

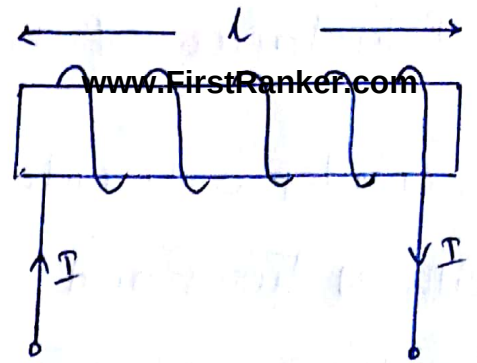
$$M = \frac{\lambda_{21}}{I_1} = \frac{\lambda_{12}}{I_2}$$

$$\Rightarrow M_{21} = \frac{\mu}{4\pi} \oint_{C_2} \oint_{C_1} \frac{d\vec{L}_1 \cdot d\vec{L}_2}{r} \quad \rightarrow (1)$$

$$\text{Similarly } M_{12} = \frac{\mu}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\vec{L}_1 \cdot d\vec{L}_2}{r} \quad \rightarrow (2)$$

Equations (1) & (2) are called Neuman's integrals or Neuman's formulae.

Consider a Solenoid of N turns.
Let I is the current through Solenoid & l is the length.



We know that magnetic field intensity inside the Solenoid is given by

$$H = \frac{NI}{l} \text{ A/m.}$$

$$\text{Total flux linkage } \lambda = N\phi$$

$$= NBA$$

$\therefore A = \text{Area of cross section of Core}$

$$= N(\mu H)A$$

$$\Rightarrow \text{flux linkage } \lambda = N\mu \left(\frac{NI}{l} \right) A$$

$$\lambda = \frac{\mu N^2 I A}{l}$$

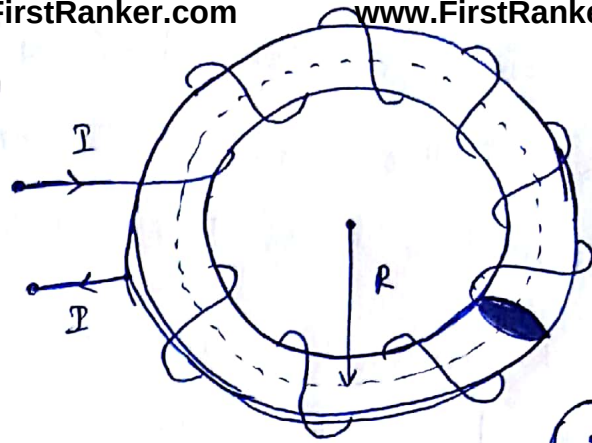
Thus the inductance of solenoid is given by

$$L = \frac{\text{Total flux linkage}}{\text{Current}} = \frac{\lambda}{I}$$

$$L = \frac{\frac{\mu N^2 I A}{l}}{I} = \frac{\mu N^2 A}{l}$$

$$\Rightarrow \boxed{L = \frac{\mu N^2 A}{l} = \frac{\mu_0 \mu_r N^2 A}{l} \text{ H}}$$

Consider a toroidal ring with N turns and carrying current I . Let mean radius of the toroid is R .



Area of Cross-section of ring.

magnetic field intensity inside the toroid is given by

$$H = \frac{NI}{2\pi R}$$

$$\Rightarrow B = \mu H = \frac{\mu NI}{2\pi R}$$

Total flux linkage of toroidal ring having N turns is given by

$$\text{Total flux linkage} = N\Phi$$

$$\lambda = N\Phi = NBA$$

$$\therefore A = \pi r^2$$

$$= N\mu H A$$

$$= N\mu \left(\frac{NI}{2\pi R} \right) A$$

$$\lambda = \frac{\mu N^2 I A}{2\pi R}$$

Inductance of toroid is given by

$$L = \frac{\text{Total flux linkage}}{\text{Total current}}$$

$$L = \frac{\lambda}{I}$$

$$\frac{\mu N^2 I A}{2\pi R} \quad \frac{\mu N^2 A}{2\pi R}$$

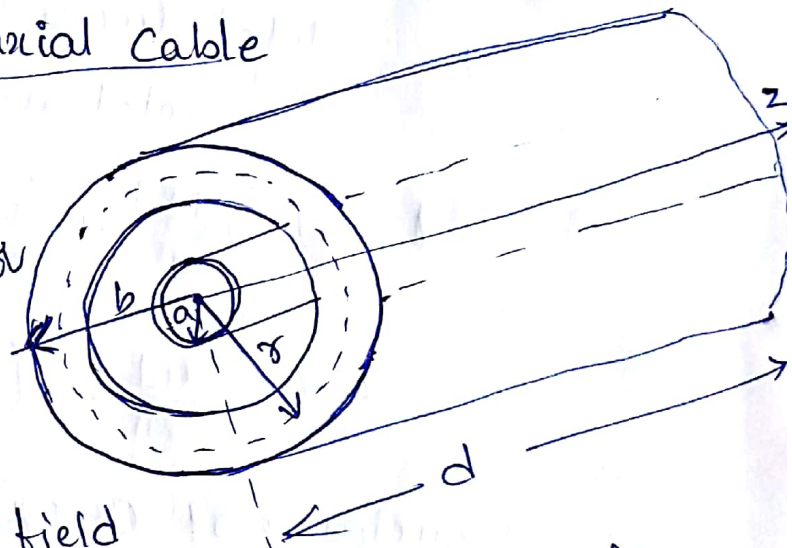
$$\Rightarrow \boxed{L = \frac{\mu N^2 A}{2\pi R} H}$$

For a toroid with N number of turns, h as height of toroid with r_1 as inner radius and r_2 as outer radius then inductance is given by

$$L = \frac{\mu N^2 h}{2\pi} \ln\left(\frac{r_2}{r_1}\right) H$$

$\Rightarrow$ Inductance of Co-axial Cable

Consider a Coaxial Cable of inner Conductor radius 'a' and outer Conductor radius 'b'



For coaxial Cable the field intensity at any point between inner and outer Conductor is given by

$$H = \frac{I}{2\pi r} \quad \text{where } a < r < b$$

$$\text{But } B = \mu H = \frac{\mu I}{2\pi r}$$

The magnetic flux density will be in radial plane extending from $r=a$ to $r=b$ & $z=0$ to $z=d$.

$$\text{let } \overline{B} = \frac{\mu I}{2\pi r} \overline{a_\phi} \quad T$$

total magnetic flux

$$\phi = \int_s \vec{B} \cdot d\vec{s} \quad \text{here } d\vec{s} = dr dz \vec{a}_\phi$$

$$\phi = \int_{z=0}^{z=d} \int_{r=a}^{r=b} \left[\frac{\mu I}{2\pi r} \vec{a}_\phi \cdot dr dz \vec{a}_\phi \right]$$

$$\phi = \frac{\mu I}{2\pi} [z]_0^d [\ln r]_a^b$$

$$= \frac{\mu I d}{2\pi} \ln\left(\frac{b}{a}\right)$$

∴ Inductance of Co-axial Cable is

$$L = \frac{\text{Total flux linkage}}{\text{Total Current}}$$

$$L = \frac{\frac{\mu I d}{2\pi} \ln\left(\frac{b}{a}\right)}{I}$$

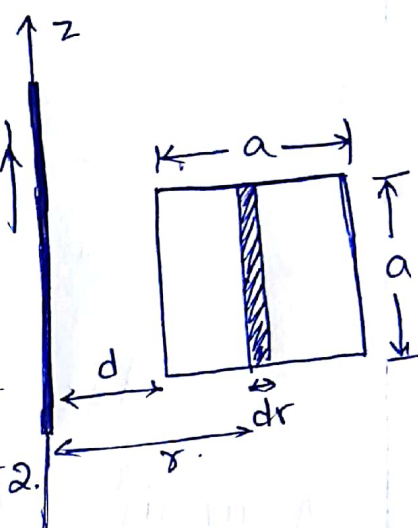
$$\therefore L = \frac{\mu d}{2\pi} \ln\left(\frac{b}{a}\right) \text{ H}$$

∴ Inductance of Co-axial Cable may be expressed per unit length as

$$\frac{L}{d} = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right) \text{ H/m.}$$

mutual Inductance between a long straight wire and Square loop lying in same plane.

Consider Square loop with sides a . A straight long Conductor I_1 is kept parallel to longer side of the loop, along z -axis.



Consider long straight wire is circuit 1, while a Square loop is circuit 2.

The magnetic field intensity at a distance d from long Conductor, due to current I_1 in it can be expressed using Ampere's law.

$$\oint \vec{H} \cdot d\vec{l} = I_{enc}$$

$$\therefore H_{\phi}(2\pi r) = I_1$$

$$\Rightarrow H_{\phi} = \frac{I_1}{2\pi r} \Rightarrow \vec{H} = \frac{I_1}{2\pi r} \vec{a}_{\phi} \text{ A/m}$$

$\therefore$ flux density $\vec{B}$ is given by

$$\vec{B}_1 = B_{\phi} \vec{a}_{\phi} = \mu_0 H_{\phi} \vec{a}_{\phi} = \frac{\mu_0 I_1}{2\pi r} \vec{a}_{\phi}$$

The flux linkage in circuit 2 due to current in circuit 1 is

$$\lambda_{21} = \int_S \vec{B}_1 \cdot d\vec{s}_2$$

here $d\vec{s}_2 = a dr \vec{a}_{\phi}$

$$= \int_d^{d+a} \left(\frac{\mu_0 I_1}{2\pi r} \right) \vec{a}_{\phi} \cdot (a dr) \vec{a}_{\phi}$$

$$\Rightarrow \frac{\mu_0 I_1 a}{2\pi} \int_d^{d+a} \frac{1}{r} dr$$

$$= \frac{\mu_0 I_1 a}{2\pi} \ln r \Big|_d^{d+a} = \frac{\mu_0 I_1}{2\pi} [\ln(d+a) - \ln d]$$

$$= \frac{\mu_0 I_1 a}{2\pi} \ln \left[\frac{d+a}{d} \right]$$

$$\therefore \lambda_{21} = \frac{\mu_0 I_1 a}{2\pi} \ln \left[1 + \frac{a}{d} \right]$$

Assuming single turn circuit, mutual inductance is given by

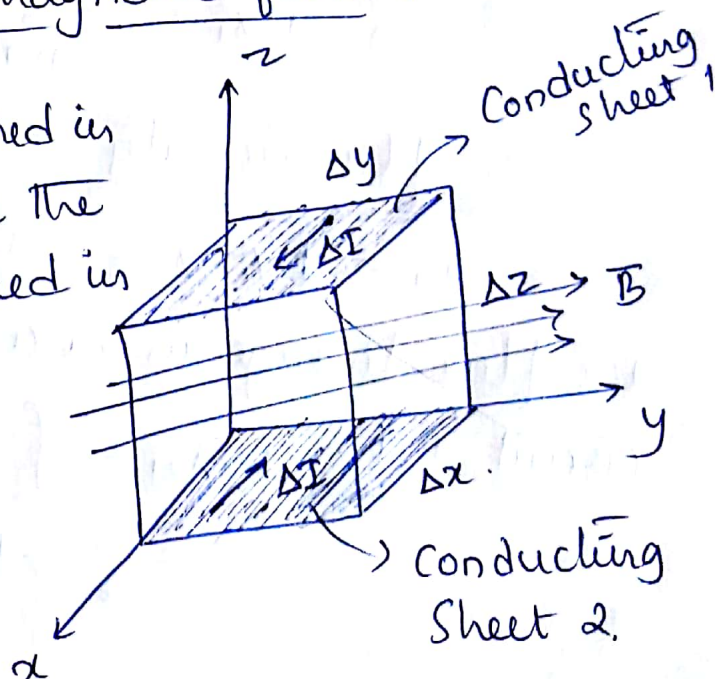
$$M_{21} = \frac{\lambda_{21}}{I_1} = \frac{\mu_0 I_1 a}{2\pi} \ln \left[1 + \frac{a}{d} \right] \frac{1}{I_1}$$

$$\Rightarrow \boxed{M_{21} = \frac{\mu_0 a}{2\pi} \ln \left[1 + \frac{a}{d} \right] \text{ H.}}$$

⇒ Energy stored in magnetic field :-

In Capacitor energy stored in electrostatic field, while the inductor energy stored in magnetic field.

$$\therefore W_m = \frac{1}{2} L I^2$$



as shown here.

Consider that at the top and bottom surfaces of a differential volume, conducting sheets with current ΔI present.

From definition of an inductance, we can write inductance, ΔL , of a differential volume as.

$$\Delta L = \frac{\Delta \Phi}{\Delta I} = \frac{B \Delta S}{\Delta I}$$

ΔS = differential surface area

$$= \Delta x \Delta z$$

$$\Delta L = \frac{B (\Delta x \Delta y)}{\Delta I}$$

But $B = \mu H$

$$\Rightarrow \Delta L = \frac{\mu H \Delta x \Delta y}{\Delta I}$$

Now differential current ΔI can be expressed in terms of magnetic field intensity H . The current through the conducting sheet present at top and bottom is in y direction

$$\Delta I = H (\Delta y)$$

$\therefore$ energy stored due to differential volume

$$\Delta W_m = \frac{1}{2} \Delta L \Delta I^2$$

$$\Rightarrow \Delta W_m = \frac{1}{2} \left[\frac{\mu H \Delta x \Delta y}{H \Delta y} \right] [H \Delta y]^2$$

$$\Delta W_m = \frac{1}{2} \mu H^2 (\Delta x \Delta y \Delta z)$$

Hence energy stored in a inductor is

$$\Delta W_m = \frac{1}{2} \mu H^2 \Delta V$$

The magnetic energy density function is defined

as

$$W_m = \lim_{\Delta V \rightarrow 0} \frac{\Delta W_m}{\Delta V} = \frac{1}{2} \mu H^2 \quad \text{J/m}^3$$

$$\therefore W_m = \frac{1}{2} (\mu H) H = \frac{1}{2} B H$$

$$\rightarrow W_m = \frac{1}{2} B \cdot \frac{B}{\mu} = \frac{B^2}{2\mu}$$

$\therefore$ In a linear medium the energy in a magneto-static field is given by

$$W_m = \int w_m dv = \frac{1}{2} \int (\vec{B} \cdot \vec{H}) dv$$

$$= \frac{1}{2} \int \mu H^2 dv = \frac{1}{2} \int \frac{B^2}{\mu} dv$$

Time Varying Fields

The Equations describing relationships between Time Varying Electric & magnetic fields are known as maxwells Equations. Time Varying fields or dynamic fields are produced due to Time Varying Currents.

⇒ Faraday's law & lenz's law :-

According to faradays law, a static magnetic field Can not produce any Current flow. But with Time Varying fields, an electromotive force induces which may drive a current in a closed path or circuit.

Statement of Faradays law :-

"The electro motive force induced in a closed path or circuit is proportional to rate of change of magnetic flux enclosed by closed path".

He observed that when a closed path moves in a magnetic field, Current is generated and hence $\mathcal{E}$ mf. The same observations he made with closed path kept fixed and magnetic field was varied. The effect is Commonly called electro magnetic Induction.

$$\therefore e = -N \frac{d\phi}{dt}$$

→ (1)

N = No. of turns in circuit
 e = induced $\mathcal{E}$ mf

Assume Single turn circuit. i.e $N = 1$ Then

$$e = - \frac{d\phi}{dt}$$

statement of lenz's law :- The -ve sign in the induced emf according to faraday was explained by lenz's law

"The direction of induced emf is such that it opposes the cause producing it i.e. changes in magnetic flux".

$$\therefore e = -N \frac{d\phi}{dt}$$

where -ve sign indicates induced emf opposes the cause.

Let us consider faradays law, The induced emf is a scalar quantity measured in volts & it is given by

$$e = \oint \vec{E} \cdot d\vec{L} \quad \longrightarrow (2)$$

The induced emf in equation (2) indicates a voltage about a closed path i.e. any part of the closed path changes emf also changes".

Let the magnetic flux passing through a specified area is given by $\phi = \int_s \vec{B} \cdot d\vec{s}$ $\vec{B}$ = flux density

Substitute in equation of faradays law

$$\Rightarrow e = -\frac{d\phi}{dt} = -\frac{d}{dt} \int_s \vec{B} \cdot d\vec{s} \quad \longrightarrow (3)$$

$\therefore$ From (2) and (3)

$$e = \oint \vec{E} \cdot d\vec{L} = -\frac{d}{dt} \int_s \vec{B} \cdot d\vec{s}$$

variation of flux ϕ with time t can be caused due to any of the following conditions.

- i, By having stationary closed paths in time varying field $\vec{B}$
- ii, By having a time varying closed paths in a static field $\vec{B}$
- iii, By having time varying closed paths in time varying field $\vec{B}$

In the first case induced $\mathcal{E}$ mf is called statically induced $\mathcal{E}$ mf, in second case $\mathcal{E}$ mf induced is called dynamically induced $\mathcal{E}$ mf

⇒ statically Induced $\mathcal{E}$ mf / Transformer $\mathcal{E}$ mf

A statically closed path in a time varying $\vec{B}$ field.

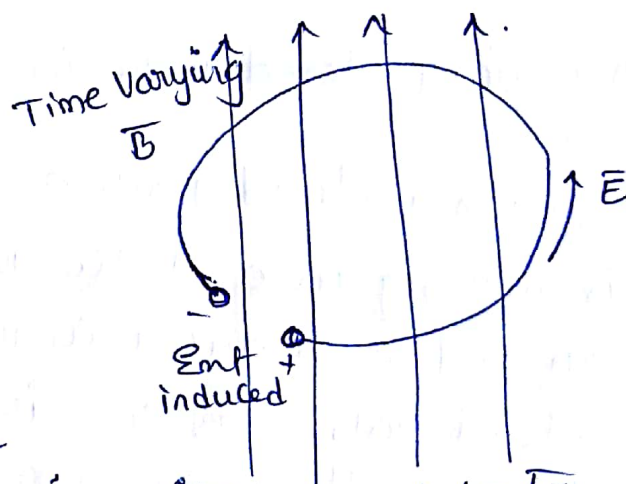
According Faradays law we have

$$\mathcal{E} = \oint \vec{E} \cdot d\vec{l} = - \frac{d}{dt} \int_s \vec{B} \cdot d\vec{s}$$

In this case conductor is time varying and the flux is varying with respect to time hence we can write

$$\oint \vec{E} \cdot d\vec{l} = - \int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$$

This is similar to Transformer action & $\mathcal{E}$ mf is called Transformer $\mathcal{E}$ mf.



$$\oint_L \vec{E} \cdot d\vec{l} = \int_S (\nabla \times \vec{E}) \cdot d\vec{S}$$

$$\therefore \int_S (\nabla \times \vec{E}) \cdot d\vec{S} = - \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

Assume both surface integrals taken over identical surfaces then

$$\boxed{\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}}$$

This equation is called Maxwell's 4th equation.

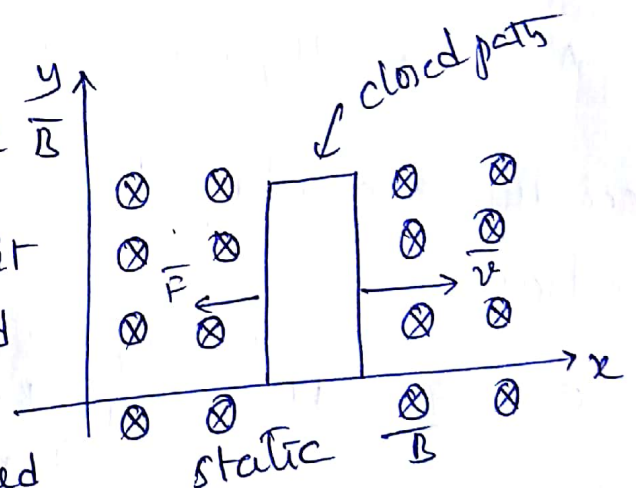
If $\vec{B}$ is not varying then $\frac{\partial \vec{B}}{\partial t} = 0$

$$\boxed{\nabla \times \vec{E} = 0}$$

⇒ Dynamically induced emf / motional emf / Generator emf etc.

A moving closed path in static $\vec{B}$

when a closed path or circuit is moving in a static $\vec{B}$ field an emf is induced in the closed path. This emf is called dynamically induced emf



Consider that a charge q is moved in a magnetic field $\vec{B}$ at a velocity $\vec{v}$. Then the force on charge is given by $\vec{F} = q \vec{v} \times \vec{B}$

but motional electric field is given by $\vec{E}_m = \frac{\vec{F}}{q} = \vec{v} \times \vec{B}$

the induced emf is given by.

$$\oint \vec{E} \cdot d\vec{l} = \oint (\vec{v} \times \vec{B}) \cdot d\vec{l}$$

This is emf induced when a closed path moving in field $\vec{B}$.

⇒ when moving closed path is placed in a time varying $\vec{B}$:

when a moving closed path placed in a time varying $\vec{B}$,
Then both statically & dynamically induced emf are present.
∴ The induced emf in this case is sum of transformer emf & motional emf

$$\oint \vec{E} \cdot d\vec{l} = \int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} + \oint (\vec{v} \times \vec{B}) \cdot d\vec{l}$$

⇒ Displacement Current density

For static electromagnetic fields, according to Ampere's law, we can write

$$\nabla \times \vec{H} = \vec{J} \quad \rightarrow (1)$$

Taking divergence on both sides

$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J}$$

But according to vector identity "divergence of the curl of any vector field is zero".

$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J} = 0 \quad \rightarrow (2)$$

But from the equation of continuity

$$\nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t} \quad \rightarrow (3)$$

$\frac{\partial e_v}{\partial t} = 0$, Then only Equation (2) becomes true.

Thus Equations (2) & (3) are not compatible for time varying fields. we must modify Equation (1) by adding one unknown term say $\bar{N}$

$\therefore$ Equation becomes $\nabla \times \bar{H} = \bar{J} + \bar{N}$

Again apply divergence on both sides

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J} + \nabla \cdot \bar{N} = 0$$

$$\text{As } \nabla \cdot \bar{J} = -\frac{\partial e_v}{\partial t} \quad \therefore \quad \nabla \cdot \bar{N} = -\nabla \cdot \bar{J}$$

$$\boxed{\nabla \cdot \bar{N} = \frac{\partial e_v}{\partial t}}$$

According to Gauss law

$$e_v = \nabla \cdot \bar{D}$$

$$\Rightarrow \nabla \cdot \bar{N} = \frac{\partial}{\partial t} (\nabla \cdot \bar{D}) = \nabla \cdot \frac{\partial \bar{D}}{\partial t}$$

Compare two sides

$$\bar{N} = \frac{\partial \bar{D}}{\partial t}$$

$\therefore$ Now we can write Ampere's circuital law

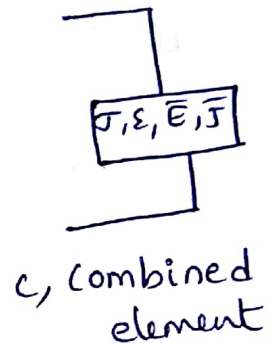
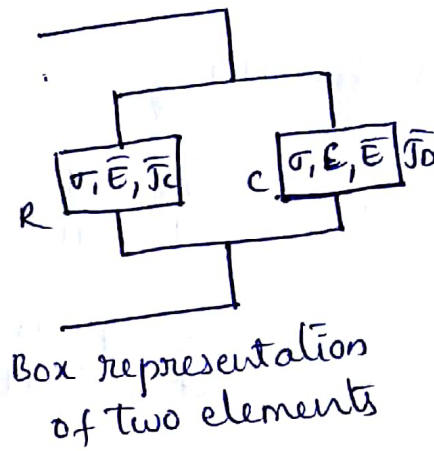
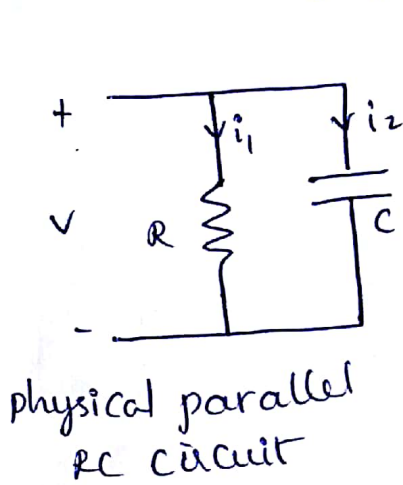
$$\nabla \times \bar{H} = \bar{J}_c + \frac{\partial \bar{D}}{\partial t}$$

$\bar{J}_c$ = conduction current density

$\frac{\partial \bar{D}}{\partial t} = \bar{J}_D$ = displacement current density

$$\Rightarrow \boxed{\nabla \times \bar{H} = \bar{J}_c + \bar{J}_D}$$

Consider that the parallel RC combination is driven by the time varying i.e. sinusoidal voltage V .



Current through resistor is $i_1 = \frac{V}{R}$

This current is called conduction current, because it ~~is~~ is due to the flow of electrons & is indicated by i_c .

Let A be the cross-sectional area of resistor.

$$\therefore J_c = \frac{i_c}{A} = \sigma E$$

Now Assume initial charge on a capacitor is zero. Then for Time Varying ~~fields~~ voltage applied across parallel plate Capacitor. Then

$$i_2 = C \frac{dV}{dt}$$

Let two plates of area A are Separated by distance d with dielectric having permittivity ϵ in between plates

$$i_2 = \frac{\epsilon A}{d} \frac{dV}{dt} \rightarrow (1)$$

is called displacement current denoted by i_D & E produced by voltage applied is

$$E = \frac{V}{d} \Rightarrow V = Ed.$$

Substitute V in (1)

$$\Rightarrow i_D = i_2 = \frac{\Sigma A}{d} \cdot \frac{d}{dt} (dE)$$

$$= \frac{\Sigma A}{d} \cdot \frac{dE}{dt} \quad \& \quad E = \frac{D}{\Sigma}$$

$$\Rightarrow i_D = \Sigma A \cdot \frac{d}{dt} \left(\frac{D}{\Sigma} \right)$$

$$= \Sigma A \cdot \frac{1}{\Sigma} \cdot \frac{dD}{dt}$$

$$i_D = A \cdot \frac{dD}{dt}$$

Now displacement current density

$$\bar{J}_D = \frac{i_D}{A}$$

$$\bar{J}_D = \frac{A \frac{dD}{dt}}{A}$$

$$\Rightarrow \bar{J}_D = \frac{\partial D}{\partial t}$$

$$\therefore \text{Total } \bar{J} = \bar{J}_C + \bar{J}_D$$

we have

$$\vec{J} = \vec{J}_c + \vec{J}_D$$

$$= \sigma \vec{E} + \frac{\partial \vec{D}}{\partial t}$$

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for electric field $\vec{E}$, let the time dependence be given by $e^{j\omega t}$. Then

$$\vec{J} = \sigma \vec{E} + \frac{\partial}{\partial t} (\epsilon \vec{E})$$

$$\vec{J} = \sigma \vec{E} + j\omega \epsilon \vec{E}$$

Then The ratio of conduction current density magnitude to the magnitude of displacement current density is given by.

$$\frac{|\vec{J}_c|}{|\vec{J}_D|} = \frac{\sigma}{\omega \epsilon}$$

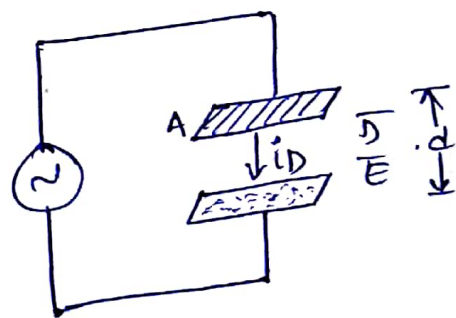
if $\frac{\sigma}{\omega \epsilon} \gg 1$ medium is conductor

$\frac{\sigma}{\omega \epsilon} \ll 1$ medium is dielectric

⇒ prove that the displacement current in dielectric of parallel plate capacitor is equal to the conduction current in the leads

Consider parallel plate capacitor is connected to time varying voltage V .

$$V = V_m \sin \omega t$$



The current through the leads the capacitor is conduction current & is given by

$$i_c = C \frac{dV}{dt} = C \frac{d}{dt} (V_m \sin \omega t)$$

$$i_c = (C\omega) V_m \sin \omega t \longrightarrow (1)$$

displacement current is the current flowing through the dielectric between parallel plates.

$$i_D = \int_S \vec{J}_D \cdot d\vec{s}$$

$$i_D = \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{s}$$

$\vec{D}$ & $d\vec{s}$ are ~~per~~ perpendicular to plates and hence in same direction.

$$\therefore i_D = \int_S \frac{\partial D}{\partial t} ds.$$

$$= \int_S \frac{\partial \epsilon E}{\partial t} ds$$

$$= \epsilon \int_S \frac{\partial E}{\partial t} ds$$

$$= \epsilon \int_S \frac{\partial}{\partial t} \left(\frac{V}{d} \right) ds$$

$$\downarrow \because E = \frac{V}{d}$$

$$= \frac{\epsilon}{d} \int_S \frac{\partial V}{\partial t} ds$$

$$= \frac{\epsilon}{d} \frac{\partial V}{\partial t} \int_S ds.$$

$$= \frac{\epsilon}{d} \cdot \frac{\partial V}{\partial t} \cdot A$$

$$\Rightarrow i_D = \frac{\epsilon A}{d} \cdot \frac{\partial V}{\partial t}$$

$$i_D = \epsilon \cdot \frac{\partial V}{\partial t}$$

$$\because C = \frac{\epsilon A}{d}$$

$$\Rightarrow i_D = C \cdot \frac{\partial}{\partial t} (V_m \sin \omega t) = C \cdot V_m \omega \cos \omega t$$

$$i_D = (C\omega) V_m \cos \omega t \rightarrow (2)$$

From (1) & (2) it is clear that current through dielectric of capacitor is equals to the current through the leads.

Maxwell's Equations for static fields

Maxwell's Equations are derived from Faraday's law, Ampere's circuit law, Gauss's law for electrostatic fields & Gauss's law for magnetostatic fields.

a. Maxwell's Equation derived from Faraday's law :-

For electrostatic fields, the work done over a closed path is always zero.

$$\therefore \oint \vec{E} \cdot d\vec{l} = 0 \rightarrow (1)$$

Equation (1) is called integral form of Maxwell's Equation derived from Faraday's law for static field.

Using Stokes theorem converting the closed line integral into surface integral.

$$\oint \vec{E} \cdot d\vec{l} = \int_S (\nabla \times \vec{E}) \cdot d\vec{s} = 0$$

$$\therefore \int_S (\nabla \times \vec{E}) \cdot d\vec{s} = 0 \quad d\vec{s} \neq 0 \text{ hence}$$

$$\boxed{\nabla \times \vec{E} = 0} \rightarrow (2)$$

This Equation is called point or differential form of Maxwell's Equation derived from Faraday's law for static fields.

b. Maxwell's Equation derived from Ampere's circuit law

For magnetostatics an Ampere's circuital law states that line integral of field intensity H around a closed path is exactly equal to current enclosed by that path.

above Equation is called

$$\Rightarrow \text{we know } I = \int_S \vec{J} \cdot d\vec{s}$$

$$\Rightarrow \oint \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{s}$$

above Equation is called integral form of maxwell's Equation derived from Ampere's circuit law for static field.

apply Stokes Theorem to LHS

$$\Rightarrow \oint \vec{H} \cdot d\vec{l} = \int_S (\nabla \times \vec{H}) \cdot d\vec{s} = \int_S \vec{J} \cdot d\vec{s}$$

$$\text{hence } \nabla \times \vec{H} = \vec{J}$$

This Equation is called point or differential form of maxwell's Equation derived from Ampere's law for static field.

c, maxwell's Equation derived from Gauss's law for Electrostatic fields :-

According to Gauss law for electrostatic field, the electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

$$\psi = \oint \vec{D} \cdot d\vec{s} = Q_{\text{enclosed}}$$

The most common form to represent Gauss law, is with volume charge density ρ_v

$$\oint \vec{D} \cdot d\vec{s} = \int_V \rho_v \cdot dV \longrightarrow (4)$$

Equation (1) is called integral form of Maxwell's Equation derived from Gauss's law for static electric field.

Apply divergence theorem on ~~RHS~~ LHS.

$$\oint_S \vec{D} \cdot \vec{dS} = \int_V \rho_v dV$$

$$\oint_S \vec{D} \cdot \vec{dS} = \int_V (\nabla \cdot \vec{D}) dV = \int_V \rho_v dV$$

$$\Rightarrow \int_V (\nabla \cdot \vec{D}) dV = \int_V \rho_v dV$$

If integration is applied over identical volume elements then $\boxed{\nabla \cdot \vec{D} = \rho_v}$

This Equation is called point or differential form of Maxwell's Equation derived from electrostatic fields.

2. Maxwell's Equation derived from Gauss law for magnetostatic fields :-

According to Gauss law for magnetostatic field we have. magnetic flux cannot reside in a closed surface due to non existence of single magnetic pole.

$$\oint_S \vec{B} \cdot \vec{dS} = 0$$

This is called integral form of Maxwell's Equation derived from Gauss law for static magnetic field.

Apply divergence theorem

$$\oint_S \vec{B} \cdot \vec{dS} = \int_V (\nabla \cdot \vec{B}) dV = 0$$

$$\Rightarrow \boxed{\nabla \cdot \vec{B} = 0}$$

This is point form of Maxwell Equation derived from Gauss law.

Maxwell's Equations for Time Varying Fields

Maxwell's Equations are derived from Faraday's law, Ampere's circuit law, Gauss's law for electric field and Gauss law for magnetic field.

A, Maxwell's Equation derived from Faraday's law

From Faraday's law which relates EMF induced in a circuit to the time rate of decrease of total magnetic flux linking the circuit

$$\oint \vec{E} \cdot d\vec{l} = - \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$$

This is the Maxwell's Equation derived from Faraday's law. Expressed in integral form.

Statement :- "The total electromotive force (EMF) induced in a closed path is equal to negative surface integral of the rate of change of flux density with respect to time over the entire surface bounded by same closed path".

using Stokes Theorem, Convert line integral to surface integral $\Rightarrow \oint \vec{E} \cdot d\vec{l} = \int_S (\nabla \times \vec{E}) \cdot d\vec{s} = - \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$

Assuming that integration is taken over identical surface then $\boxed{\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}}$

This is Maxwell's Equation derived from Faraday's law expressed in point form or differential form.

law :

According to Ampere's law, The line integral of magnetic field intensity $\vec{H}$ around a closed path is equal to current enclosed by the path.

$$\oint \vec{H} \cdot d\vec{l} = I_{enc}$$

Replacing current in terms of current density

$$I_{enc} = \int_S \vec{J} \cdot d\vec{s}$$

$$\Rightarrow \oint \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{s}$$

Above expression can be made further general by adding displacement current density

$$\oint \vec{H} \cdot d\vec{l} = \int_S \left[\vec{J} + \frac{\partial \vec{D}}{\partial t} \right] \cdot d\vec{s}$$

This is Maxwell's Equation derived from Ampere's circuit law. & it is called integral form of Maxwell Equation

Statement : "The total magnetomotive force around any closed path is equal to the surface integral of the conduction and displacement current densities over the entire surface bounded by the same closed path."

Applying Stokes Theorem on LHS

$$\oint \vec{H} \cdot d\vec{l} = \int_S (\nabla \times \vec{H}) \cdot d\vec{s} = \int_S \left[\vec{J} + \frac{\partial \vec{D}}{\partial t} \right] \cdot d\vec{s}$$

the integration is applied over identical surface
then we can write

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

above Equation is called point form of maxwell's Equation derived from Ampere's circuit law.

c, maxwell's Equation derived from Gauss law for electric fields

According to Gauss law, the total flux out of the closed surface is equal to the net charge within the surface

$$\oint_S \vec{D} \cdot d\vec{s} = Q_{enc}$$

we can write Q_{enc} in terms of volume charge density as

$$Q_{enc} = \int_V \rho_v \cdot dV$$

$$\Rightarrow \boxed{\oint_S \vec{D} \cdot d\vec{s} = \int_V \rho_v \cdot dV}$$

This Equation is called maxwell's Equation for electric fields derived from Gauss's law expressed in integral form ~~and~~

Statement : "The Total flux leaving out of a closed surface is equal to total charge enclosed by a finite volume."

using divergence theorem. $\int_V (\nabla \cdot \vec{D}) dV = \int_V \rho_v dV$

$$\Rightarrow \oint_S \vec{D} \cdot d\vec{s} = \int_V (\nabla \cdot \vec{D}) dV$$

$$\Rightarrow \int_V (\nabla \cdot \vec{D}) dV = \int_V \rho_v dV$$

then we can write

$$\nabla \cdot \vec{D} = \rho_v$$

above Equation is called point form of maxwell's Equation derived from Gauss law.

2. maxwell's Equation derived from Gauss's Law magnetic fields

for magnetic fields, the Surface integral of $\vec{B}$ over a closed Surface S is always zero due to non existence of monopole.

$$\oint_S \vec{B} \cdot d\vec{s} = 0$$

This is maxwell's Equation expressed in integral form.

statement :- "The Surface integral of magnetic flux density over closed Surface is always equals to zero."

using divergence theorem Convert Surface integral to Volume integral

$$\oint_S \vec{B} \cdot d\vec{s} = \int_V (\nabla \cdot \vec{B}) dV = 0.$$

$$\Rightarrow \int_V \nabla \cdot \vec{B} dV = 0 \quad \text{as } dV \neq 0$$

$$\Rightarrow \boxed{\nabla \cdot \vec{B} = 0}$$

This is point form of maxwell's Equation derived from Gauss law for magnetic fields.

~~For magnetic fields, the surface integral of $\vec{B}$ over a closed surface is always zero, due to no~~

Differential form	Integral form.	Significance
$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\oint \vec{E} \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$	Faraday's law
$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$	$\oint \vec{H} \cdot d\vec{l} = \int_S (\vec{J} + \frac{\partial \vec{D}}{\partial t}) \cdot d\vec{s}$	Ampere's circuital law
$\nabla \cdot \vec{D} = \rho_v$	$\oint_S \vec{D} \cdot d\vec{s} = \int_V \rho_v dV$	Gauss's Law
$\nabla \cdot \vec{B} = 0$	$\oint_S \vec{B} \cdot d\vec{s} = 0$	No isolated magnetic charges

Poynting Theorem and Poynting vector :

Time varying fields or dynamic fields constitute the electro magnetic waves (like electro magnetic waves in radio waves), they travel through the free space or a dielectric. In electro magnetic waves, the power and energy relationships can be explained in terms of amplitudes of electric and magnetic fields.

The resulting theorem is the most fundamental relationships of the electromagnetic theory which is known as Poynting theorem.

By means of electromagnetic waves, an energy can be transported from transmitter to receiver. The energy stored in an electric field & magnetic field is transmitted at a certain rate of energy flow, which can be calculated with the help of Poynting theorem.

Electric field expressed in V/m

magnetic field expressed in A/m .

As the product of these two fields gives $(\frac{V}{m} \times \frac{A}{m} = \frac{watt}{m^2})$ a new vector called as power density & it is the product of $\vec{E}$ and $\vec{H}$ and power density itself is a vector quantity as it has particular direction.

$$\therefore \vec{P} = \vec{E} \times \vec{H} \quad \text{watt/m}^2$$

where $\vec{P}$ is called Poynting vector, & it is the instantaneous power density vector associated with electro-magnetic field (EM) at a given point.

The Poynting theorem is based on law of conservation of energy in electromagnetism.

Poynting theorem can be stated as

"The net power flowing out of a given volume 'V' is equal to the time rate of decrease in energy stored within volume 'V' minus the ohmic power dissipated."

$$\text{Suppose } \vec{E} = E_x \vec{a}_x, \vec{H} = H_y \vec{a}_y$$

$$\therefore \vec{P} = \vec{E} \times \vec{H} = E_x \vec{a}_x \times H_y \vec{a}_y = P_z \vec{a}_z$$

$\therefore \vec{E}, \vec{H} \text{ \& } \vec{P}$ are perpendicular to each other

Consider electric field propagate
in free space.

$$\vec{E} = [E_m \cos(\omega t - \beta z)] \vec{a}_x$$

magnetic field

$$\vec{H} = (H_m \cos(\omega t - \beta z)) \vec{a}_y$$

In the medium, the ratio of magnitudes of $\vec{E}$ & $\vec{H}$ depends on intrinsic impedance η

$$\eta = \eta_0 = \frac{E_m}{H_m} = 120\pi = 377\Omega$$

more over in free space, electromagnetic wave travels at a speed of light

$$\therefore \vec{H} = [H_m \cos(\omega t - \beta z)] \vec{a}_y$$

$$\vec{H} = \left[\frac{E_m}{\eta_0} \cos(\omega t - \beta z) \right] \vec{a}_y$$

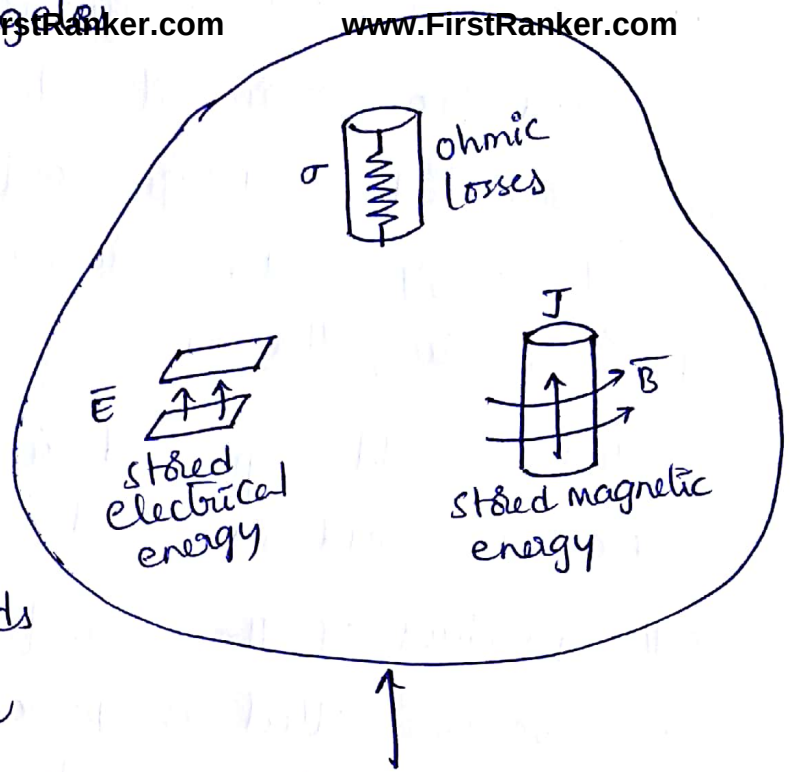
According to Poynting theorem.

$$\vec{P} = \vec{E} \times \vec{H} = E_m \cos(\omega t - \beta z) \vec{a}_x \times \left[\frac{E_m}{\eta_0} \cos(\omega t - \beta z) \right] \vec{a}_y$$

$$\boxed{\vec{P} = \frac{E_m^2}{\eta_0} \cos^2(\omega t - \beta z) \vec{a}_z \text{ W/m}^2}$$

This is power density measured in watt/m². Thus power passing through area is given by

$$\text{power} = \text{power density} \times \text{Area}$$



$$\begin{aligned}
 P_{avg} &= \frac{1}{T} \int_0^T \frac{E_m^2}{\eta} \cos^2(\omega t - \beta z) dt \\
 &= \frac{E_m^2}{T\eta} \int_0^T \frac{1 + \cos 2(\omega t - \beta z)}{2} dt \\
 &= \frac{E_m^2}{T\eta} \left[\frac{t}{2} + \frac{\sin 2(\omega t - \beta z)}{2(2\omega)} \right]_0^T \\
 &= \frac{E_m^2}{T\eta} \left[\frac{T}{2} + \frac{\sin(2\omega T - 2\beta z)}{4\omega} \right]_0^T \\
 &= \frac{E_m^2}{T\eta} \left[\frac{T}{2} + \frac{\sin(2\omega T - 2\beta z)}{4\omega} - \frac{\sin(-2\beta z)}{4\omega} \right]
 \end{aligned}$$

But $\omega T = 2\pi$

$$\begin{aligned}
 \therefore P_{avg} &= \frac{E_m^2}{T\eta} \left[\frac{T}{2} + \frac{\sin(4\pi - 2\beta z)}{4\omega} + \frac{\sin 2\beta z}{4\omega} \right] \\
 &= \frac{E_m^2}{T\eta} \left[\frac{T}{2} - \frac{\sin 2\beta z}{4\omega} + \frac{\sin 2\beta z}{4\omega} \right] \\
 &= \frac{E_m^2}{2\eta}
 \end{aligned}$$

Hence average power is given by

$$P_{avg} = \frac{1}{2} \frac{E_m^2}{\eta} \text{ W/m}^2$$