

Simple Stresses and Strains

* A material when subjected to an external load system undergoes a deformation. Against this deformation the material will offer a resistance which tends to prevent the deformation. This resistance is offered by the material by virtue of its strength.

Stress :: The force of resistance offered by a body against the deformation is called the stress. The external force acting on the body is called load.

(a)
The resistance force offered by a body per unit area

$$\boxed{\text{Intensity of stress } p = \frac{P}{A}}$$

Let the due to the application of load the length of the member change from l to $l + \Delta l$. The ratio of change in length to the original length of the member is called strain.

$$\boxed{\text{Strain } e = \frac{\Delta l}{l}}$$

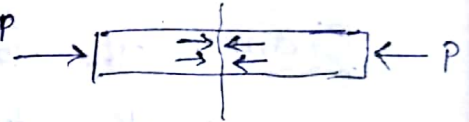
(i) Tensile Stress: When the resistance offered by a section of member is against an increase in length, the section is said to offer a tensile stress.



(ii) Compressive Stress:

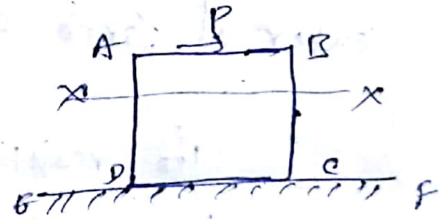
Ex. the bar ABC is subjected to pushing axial loads, the resistance is set up by any section such as xx against decrease length. This resistance

is called Compressive stress



(iii) Shear Stress:

The force applied tangentially on the top face of the block is called shear force.



$$\text{Shear Stress} = \tau = \frac{R}{P} = \frac{\text{shear resistance}}{\text{shear area}}$$

Elastic limit:

Whenever we are applying an external force on a body and the body undergoes deformation and whenever we release the load, the body gets to original shape and size, this property is known as elasticity.

It is observed that when a material is loaded such that the intensity of stress within a certain limit, the ratio of the intensity of stress to the corresponding strain is a constant which is characteristic of that material.

$$\frac{\text{Intensity of stress}}{\text{strain}} = \text{constant}$$

$$\frac{P}{e} = E$$

E - Young's modulus or modulus of elasticity.

$$\frac{\text{shear load}}{\text{shear strain}} = G$$

G → modulus of rigidity

Note: Units N/mm^2 (or) N/m^2 $1 \text{ N/m}^2 = 10^{-6} \text{ N/mm}^2 = 1 \text{ Pa}$

Ex: An elastic rod 25 mm in diameter, 200 mm long extends by 0.25 mm under a tensile load of 40 kN. Find the intensity of stress, the strain and the elastic modulus.

sd $d = 25 \text{ mm}$, $\lambda = 200 \text{ mm}$, $P = 40 \text{ kN}$

$$\text{Area} = \frac{\pi}{4} (d)^2 = \frac{\pi}{4} (25)^2 = 490.87 \text{ mm}^2$$

$$\text{Intensity of stress } f = \frac{P}{A} = \frac{40 \times 1000}{490.87} = 81.49 \text{ N/mm}^2$$

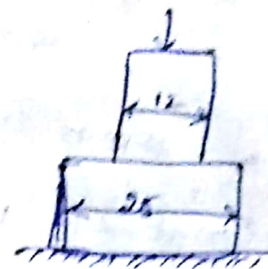
$$\text{Strain } e = \frac{\Delta l}{\lambda} = \frac{0.25}{200} = 0.00125$$

$$E = \frac{f}{e} = \frac{81.49}{0.00125} = 65192 \text{ N/mm}^2$$

the maximum and minimum stress produced in the stepped bar due to an axially applied compressive load of 12 kN

sol $A_1 = \frac{\pi}{4} (12)^2 = 113.09 \text{ mm}^2$

$$A_2 = \frac{\pi}{4} (25)^2 = 490.87 \text{ mm}^2$$



Maximum stress $f_{\max} = \frac{P}{A_1} = \frac{12 \times 10^3}{113.09} = 105.10 \text{ N/mm}^2$

Minimum stress $f_{\min} = \frac{P}{A_2} = \frac{12 \times 10^3}{490.87} = 24.45 \text{ N/mm}^2$

Ex: A hollow steel column of external diameter 250 mm has to support an axial load of 2000 kN. If the ultimate stress for the steel column is 480 N/mm². Find the internal diameter of the column allowing a load factor of 4.

sol $D = 250 \text{ mm}, P = 2000 \times 1000 \text{ N}, f = 480 \text{ N/mm}^2, \text{FoS} = 4$

$\therefore \text{Safe stress} = f = \frac{480}{4} = 120 \text{ N/mm}^2$

Sectional area required $A = \frac{P}{f} = \frac{2000 \times 1000}{120} = 16667 \text{ mm}^2$

$$\frac{\pi}{4} (D^2 - d^2) = 16667$$

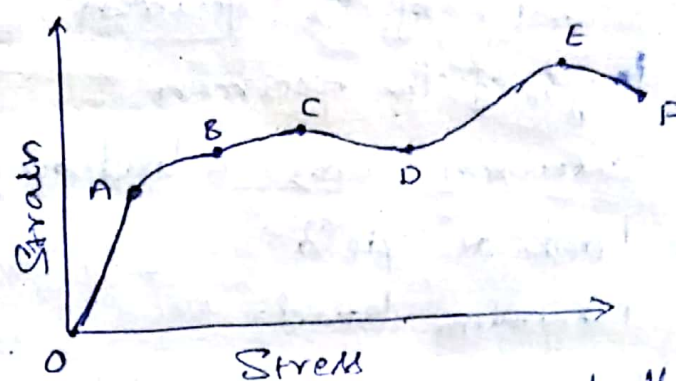
$$\frac{\pi}{4} (250^2 - d^2) = 16667$$

$$62500 - d^2 = 21221.8$$

$$d^2 = 41279$$

$$d = 203.17 \text{ mm}$$

By doing a tensile test we will obtain this plot



- * The plot from O to A is straight line and this stress corresponding to point A is called proportionality limit. In this range stress proportional to strain.
- * If the specimen is extended beyond the limit of proportionality, upto the condition shown at B, the material still remains elastic. The stress at B is called the Elastic limit.
- * If the specimen is extended by the elastic limit, plastic deformation takes place. In the range B to C the strain increases with almost constant stress. The stress at C is called upper yield limit.
- * At the condition shown at D the material again offers resistance to greater extension. The stress at D is called the Lower yield Point.
- * As the load increased, the extension increases and at the condition shown at E a necking of the specimen is developed and called ultimate tensile stress.
- * As the extension increased, the load required is decreases and the specimen breaks and called stress of failure.

tested in a laboratory:

- (i) Diameter of the specimen = 25 mm
- (ii) length of the specimen = 300 mm
- (iii) extension under a load of 15 kN = 0.045 mm
- (iv) Load at yield = 127.65 kN
- (v) Maximum load = 208.60 kN
- (vi) length of specimen after failure = 375 mm
- (vii) Neck diameter = 17.25 mm

Determine

- (i) Young's Modulus (ii) yield point (iii) ultimate stress
- (iv) % of elongation (v) percentage of reduction in area
- (vi) Safe stress adopting a factor of safety of 2.

Sol

$$\text{Area} = \frac{\pi (25)^2}{4} = 490.87 \text{ mm}^2$$

$$\text{At Load 15 kN, stress} = \frac{15 \times 1000}{490.87} = 30.56 \text{ N/mm}^2$$

$$\text{Strain } \epsilon = \frac{0.045}{300} = 1.5 \times 10^{-4}$$

$$\text{Young's Modulus } E = \frac{30.56}{1.5 \times 10^{-4}} = 2.037 \times 10^5 \text{ N/mm}^2$$

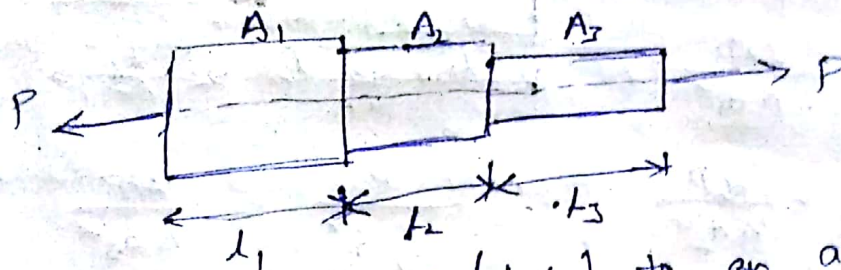
$$\text{Stress at yield point} = \frac{127.65 \times 10^3}{490.87} = 260.05 \text{ N/mm}^2$$

$$\text{Ultimate stress} = \frac{208.6 \times 10^3}{490.87} = 424.96 \text{ N/mm}^2$$

$$\% \text{ elongation} = \frac{375 - 300}{300} \times 100 = 25 \%$$

$$\% \text{ reduction in area} = \frac{d^2 - d_1^2}{d^2} \times 100 = \frac{25^2 - 17.25^2}{25^2} \times 100 = 49.59 \%$$

$$\text{Safe stress} = \frac{\text{Yield stress}}{FOS} = \frac{260.05}{2} = 130.025 \text{ N/mm}^2$$



The section is subjected to an axial load P . Even though the total force on each section is the same, the intensities of f will be different for the three sections.

Intensity of stress for the portion $AB = f_1 = \frac{P}{A_1}$

$$BC = f_2 = \frac{P}{A_2}$$

$$CD = f_3 = \frac{P}{A_3}$$

Let E be the Young's Modulus

Strain of the part $AB = e_1 = \frac{f_1}{E}$

$$BC = e_2 = \frac{f_2}{E}$$

$$CD = e_3 = \frac{f_3}{E}$$

Change in length of the part $AB = \Delta l_1$

$$BC = \Delta l_2$$

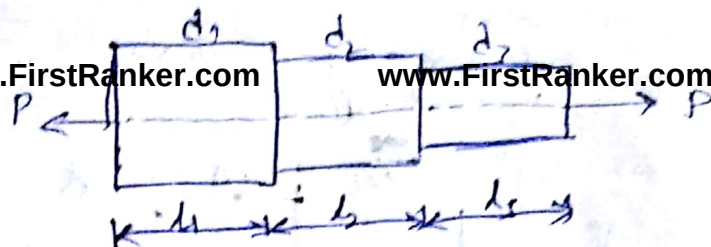
$$CD = \Delta l_3$$

$\therefore$ Total change in length of the bar $= \Delta l = \Delta l_1 + \Delta l_2 + \Delta l_3$

Ex: A bar consists of three lengths l_1, l_2, l_3 and of diameters d_1, d_2, d_3 and is subjected to an axial pull P . Show that the extension of the rod is given by

$$\Delta l = \frac{4P}{\pi E} \left[\frac{l_1}{d_1^2} + \frac{l_2}{d_2^2} + \frac{l_3}{d_3^2} \right]$$

Stresses in 3 parts



$$f_1 = \frac{P}{\frac{\pi}{4} d_1^2} = \frac{4P}{\pi d_1^2}$$

$$f_2 = \frac{P}{\frac{\pi}{4} d_2^2} = \frac{4P}{\pi d_2^2}, \quad f_3 = \frac{P}{\frac{\pi}{4} d_3^2} = \frac{4P}{\pi d_3^2}$$

Extensions of the 3 parts

$$\Delta l_1 = \frac{f_1 l_1}{E} = \frac{4P l_1}{\pi d_1^2 E}$$

$$\left[\because e = \frac{\Delta l}{l}, E = \frac{f}{e} \right]$$

$$\left[\Delta l = \frac{P L}{A E} \right]$$

$$\Delta l_2 = \frac{f_2 l_2}{E} = \frac{4P l_2}{\pi d_2^2 E}, \quad \Delta l_3 = \frac{f_3 l_3}{E} = \frac{4P l_3}{\pi d_3^2 E}$$

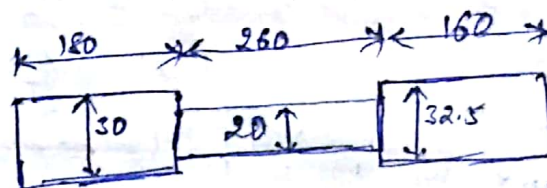
$$\therefore \text{Total extension } \Delta L = \Delta l_1 + \Delta l_2 + \Delta l_3$$

$$\Delta L = \frac{4P l_1}{\pi d_1^2 E} + \frac{4P l_2}{\pi d_2^2 E} + \frac{4P l_3}{\pi d_3^2 E}$$

$$\Delta L = \frac{4P}{\pi E} \left[\frac{l_1}{d_1^2} + \frac{l_2}{d_2^2} + \frac{l_3}{d_3^2} \right]$$

Ex: Find stresses in the three parts and total extension of the bar for an axial pull of 40 kN. $E = 2 \times 10^5 \text{ N/mm}^2$

sol



$$f_1 = \frac{P}{A_1} = \frac{40 \times 10^3}{\frac{\pi}{4} (30)^2} = 56.58 \text{ N/mm}^2$$

$$f_2 = \frac{P}{A_2} = \frac{40 \times 10^3}{\frac{\pi}{4} (20)^2} = 127.4 \text{ N/mm}^2$$

$$f_3 = \frac{P}{A_3} = \frac{40 \times 10^3}{\frac{\pi}{4} (32.5)^2} = 48.21 \text{ N/mm}^2$$

$$\begin{aligned} \Delta L &= \Delta L_1 + \Delta L_2 + \Delta L_3 \\ &= \frac{F_1 L_1}{E} + \frac{F_2 L_2}{E} + \frac{F_3 L_3}{E} \\ &= \frac{1}{2 \times 10^5} \left[(56.58 \times 180) + (27.4 \times 200) + (48.21 \times 160) \right] \text{mm} \\ &= 0.255 \text{ mm} \end{aligned}$$

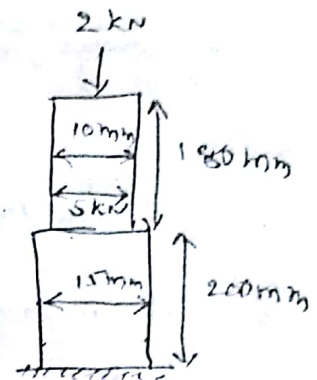
Ex: Find the decrease in length of the steel bar loaded. Take $E = 2 \times 10^5 \text{ N/mm}^2$

Sol Stress in the upper part

$$f_1 = \frac{2000}{\frac{\pi}{4} (10)^2} = 25.46 \text{ N/mm}^2$$

Stress in the lower part $f_2 = \frac{7000}{\frac{\pi}{4} (15)^2}$

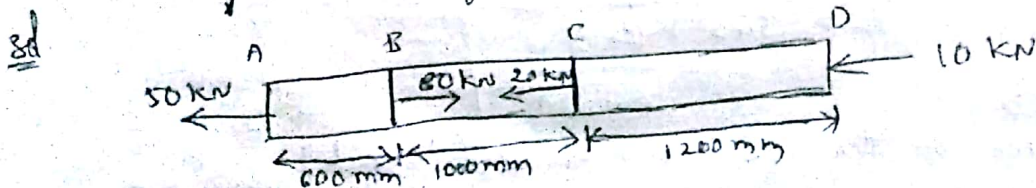
$$= 39.61 \text{ N/mm}^2$$



Decrease in length of the bar

$$\begin{aligned} \Delta L &= \frac{f_1 L_1}{E} + \frac{f_2 L_2}{E} \\ &= \frac{1}{2 \times 10^5} (25.46 \times 180 + 39.61 \times 200) \\ &= 0.0625 \text{ mm} \end{aligned}$$

Ex A brass bar having a cross-sectional area of 1000 mm^2 is subjected axial forces. Find the total change in length of the bar. $E = 1.05 \times 10^5 \text{ N/mm}^2$



part AB:

Extension of AB $= \frac{PL}{AE} = \frac{50 \times 1000 \times 600}{1000 \times 1.05 \times 10^5} = 0.285 \text{ mm}$ (extension)

$$80 - 50 = 30 \text{ kN}$$

$$\text{Contraction of ISC} = \frac{P_2 l_2}{AE} = \frac{30 \times 1000 \times 1000}{1000 \times 10.5 \times 10^5} = 0.2857 \text{ mm (Contraction)}$$

part CD - the section of the bar subjected compression of 10kN

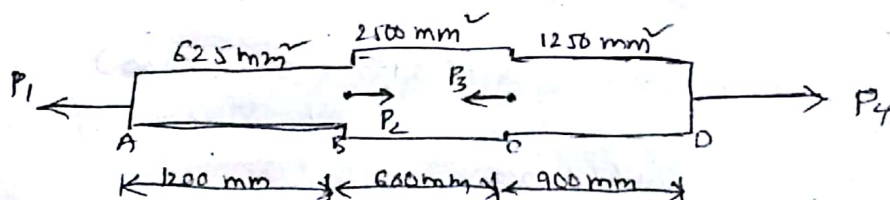
$$\text{Contraction of CD} = \frac{P_3 l_3}{AE} = \frac{10 \times 1000 \times 1200}{1000 \times 10.5 \times 10^5} = 0.1143 \text{ mm (Contraction)}$$

$$\therefore \text{change in length of the bar} = 0.2857 - 0.2857 - 0.1143 = -0.1143 \text{ mm}$$

$$\text{decrease in length of the bar} = 0.1143 \text{ mm}$$

Ex: A member ABCD is subjected to point loads P_1, P_2, P_3 and P_4 . Calculate the force P_2 necessary for equilibrium if $P_1 = 45 \text{ kN}$, $P_3 = 450 \text{ kN}$, $P_4 = 130 \text{ kN}$. Determine the total elongation of the member, assuming $E = 2.1 \times 10^5$

Sol



Resolving the forces on the rod along its axis.

$$P_1 + P_3 = P_2 + P_4$$

$$P_2 = 45 + 450 - 130$$

$$P_2 = 365 \text{ kN}$$

Part AB

force on the c/s = $P_1 = 45 \text{ kN}$ (Tensile)

$$\text{Extension of AB} = \frac{P_1 l_1}{AE} = \frac{45 \times 1000}{625 E} = \frac{86400}{E} \text{ mm (extension)}$$

force on the c/s $\frac{P}{A} = \frac{320000}{\pi \times 25^2} = -320 \text{ MPa (compressive)}$

$$\text{contraction of BC} = \frac{-320000 \times 900}{2500E} = \frac{76800}{E} \text{ mm (contraction)}$$

Part CD:

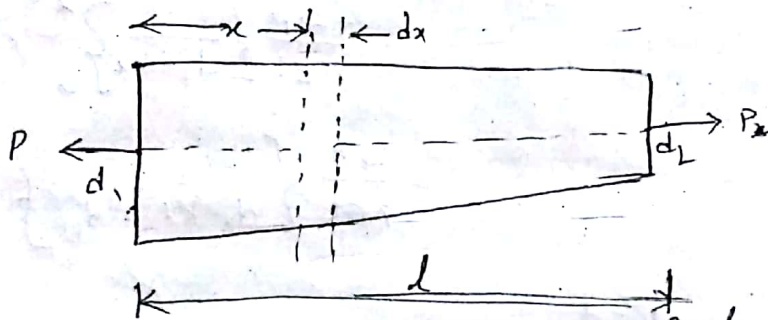
force on the c/s $= P_4 = 130 \text{ kN (tensile)}$

$$\text{Extension of CD} = \frac{R_4 L_4}{AE} = \frac{130000 \times 900}{1250E} = \frac{93600}{E} \text{ mm (extension)}$$

$$\therefore \text{Total change in length} = \frac{86400}{E} - \frac{76800}{E} + \frac{93600}{E}$$

$$= \frac{103200}{E} = \frac{103200}{2.1 \times 10^5} = 0.4914 \text{ mm (extension)}$$

Extension of A Tapering ROD



Above figure shows a bar uniformly tapering from a diameter d_1 to diameter d_2 . Let the section subjected to an axial tensile load P .

Consider an elemental length dx of the bar at a distance x from the larger end. Let the diameter of the bar be d' at distance x from larger end.

$$\therefore d' = d_1 - \left(\frac{d_1 - d_2}{l} \right) x$$

$$\text{Let } \left(\frac{d_1 - d_2}{l} \right) = k \quad \therefore d' = d_1 - kx$$

$\therefore$ C/s Area at distance x from larger end

Intensity of stress of the section

$$f' = \frac{P}{A'} = \frac{4P}{\pi(d_1 - kx)^2}$$

$$\therefore \text{Strain } e' = \frac{f'}{E} = \frac{4P}{\pi E (d_1 - kx)^2}$$

$$\therefore \text{Extension of the elemental length } dx = e' dx = \frac{4P}{\pi E (d_1 - kx)^2} \cdot dx$$

$$\therefore \text{Total extension of the bar} = \delta = \frac{4P}{\pi E} \int_0^l \frac{dx}{(d_1 - kx)^2}$$

$$= \frac{-4P}{\pi E k} \left\{ \frac{1}{d_1 - kx} \right\}_0^l$$

$$= \frac{4P}{\pi E k} \left\{ \frac{1}{d_1 - kx} - \frac{1}{d_1} \right\}$$

$$\text{But } k = \frac{d_1 - d_2}{l}$$

$$\therefore \delta = \frac{4Pl}{\pi E (d_1 - d_2)} \left\{ \frac{1}{d_1 - d_1 + d_2} - \frac{1}{d_1} \right\}$$

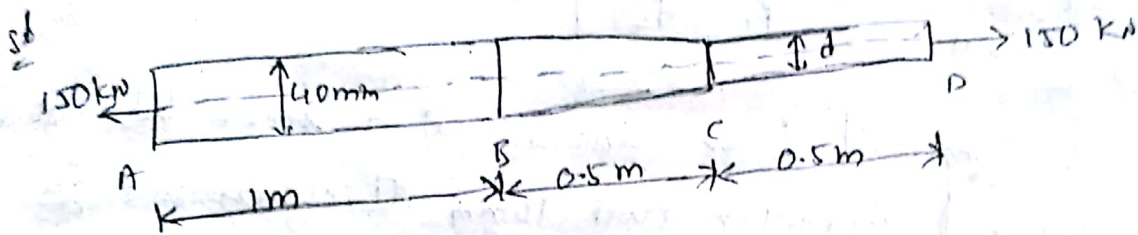
$$= \frac{4Pl}{\pi E (d_1 - d_2)} \left(\frac{1}{d_2} - \frac{1}{d_1} \right) = \frac{4Pl}{\pi E (d_1 - d_2)} \cdot \frac{d_1 - d_2}{d_1 d_2}$$

$$\therefore \delta = \frac{4Pl}{\pi E d_1 d_2}$$

for the particular case when rod is uniform dia
 $d_1 = d_2 = d$

$$\therefore \delta = \frac{4Pl}{\pi E d^2}$$

A bar of length 1m has a uniform diameter of 40mm. The diameter decreases uniformly to d . For the next remaining 0.5m length it has a uniform diameter of d mm. When a load of 150 kN is applied, the observed extension is 2.4 mm. Determine the diameter d . Take modulus of elasticity for steel equal to 200 kN/mm².



$$\text{Extension AB} = \frac{4PL}{\pi E d^3} = \frac{4 \times 150 \times 1000}{\pi \times 40^3 \times 200} = 0.5968 \text{ mm}$$

$$\text{Extension BC} = \frac{150 \times 4 \times 500}{\pi \times 40 d \times 200} = \frac{11.93662}{d} \text{ mm}$$

$$\text{Extension of CD} = \frac{150 \times 4 \times 500}{\pi d^3 \times 200} = \frac{477.465}{d^3} \text{ mm}$$

$$\therefore \text{Total extension} = 0.5968 + \frac{11.93662}{d} + \frac{477.465}{d^3} = 2.40$$

$$\frac{11.93662}{d} + \frac{477.465}{d^3} = 1.8032$$

$$\frac{11.93662 d + 477.465}{d^3} = 1.8032$$

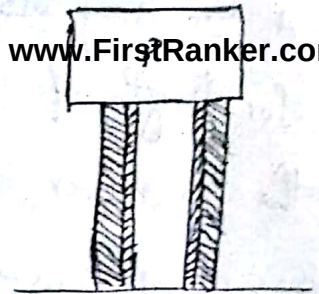
$$11.93662 d + 477.465 = 1.8032 d^3$$

$$d^3 - 6.6197 d - 264.7876 = 0$$

$$d = 13.913 \text{ mm}$$

$$ax^2 + bx + c = 0$$

Load on Outer tube + Load on the inner tube
= Total load on the column



$$\therefore f_1 A_1 + f_2 A_2 = P$$

Let Δl be the decrease in length of the column

$$\therefore \text{Strain of each tube} = e = \frac{\Delta l}{l}$$

$$\text{But } e = \frac{f_1}{E_1} = \frac{f_2}{E_2}$$

Ex: A compound tube consists of a steel tube 150mm internal diameter and 10mm thickness and an outer brass tube 170mm internal diameter and 10mm thickness. The two tubes are of the same length. The compound tube carries an axial load of 1000kN. Find the stresses and the load carried by each tube and the amount it shortens. Length of each tube is 150mm. $E_s = 2 \times 10^5 \text{ N/mm}^2$ & $E_b = 1 \times 10^5 \text{ N/mm}^2$

Sol Area of steel tube $A_s = \frac{\pi}{4} [170^2 - 150^2] = 5026.50 \text{ mm}^2$

Area of brass tube $A_b = \frac{\pi}{4} [190^2 - 170^2] = 5654.82 \text{ mm}^2$

Strain in steel = Strain in brass

$$\frac{f_s}{E_s} = \frac{f_b}{E_b} \quad \therefore f_s = \frac{E_s}{E_b} f_b$$

$$f_s = \frac{2 \times 10^5}{1 \times 10^5} f_b \quad \therefore f_s = 2 f_b$$

$\therefore$ Load on steel + Load on brass = Total load

$$15707.82 f_b = 1 \times 10^6$$

$$f_b = \frac{1 \times 10^6}{15707.82} = 63.66 \text{ N/mm}^2$$

$$f_s = 2f_b = 2 \times 63.66 = 127.32 \text{ N/mm}^2$$

$$\text{Load on brass tube} = P_b = f_b A_b = 63.66 \times 5684.82$$

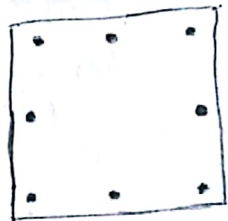
$$P_b = 360046 \text{ N say } 360 \text{ kN}$$

$$\therefore \text{load on steel tube} = 1000 - 360 = 640 \text{ kN}$$

Decrease in length of the compound tube = Decrease in length of either of the tubes = Decrease in length of the brass tube

$$= \frac{f_b l}{E_b} = \frac{63.66}{1 \times 10^5} \times 110 = 0.095 \text{ mm}$$

Ex: A reinforced concrete column is 300mm x 300mm in section. The column is provided with 8 bars of 20mm diameter. The column carries a load of 360 kN. Find the stresses in concrete and the steel bars.



$E_s = 2.1 \times 10^5 \text{ N/mm}^2$ and $E_c = 0.14 \times 10^5 \text{ N/mm}^2$

Sol Area of steel bars $A_s = 8 \times 314.2 = 2513.6 \text{ mm}^2$ say 2514

Area of concrete $A_c = 300^2 - 2514 = 87486 \text{ mm}^2$

$$\frac{f_c}{E_c} = \frac{f_s}{E_s}$$

$$f_s = \frac{E_s f_c}{E_c} \Rightarrow f_s = \frac{2.1 \times 10^5}{0.14 \times 10^5} f_c$$

$$f_s = 15 f_c$$

$$f_s A_s + f_c A_c = P$$

$$15 f_c \times 2514 + f_c 87486 = 360000$$

$$125196 f_c = 360000$$

$$f_c = 2.87 \text{ N/mm}^2$$

$$f_s = 15 \times 2.87 = 43.05 \text{ N/mm}^2$$

Temperature Stresses:

When the temperature of a material changes there will be corresponding change in its dimensions. When a member is free to expand or contract due to temperature changes, no stresses will induced. If we prevent the change in length due to temperature, stresses will developed.



Let a rod of AB of length 'l' and temperature rise by T. If the member expansion would be \$BB' = \alpha T l\$, where \$\alpha\$ is the coefficient of linear expansion. If the member is prevented from expansion, compressive stress will be induced.

If the fixture at B removed the rod freely expands

$$\text{so } \alpha T l = BB'.$$

Let an external force applied at B so that the rod is decreased in its length from \$(l + \alpha T l)\$ to \$l\$.

Let \$a\$ be the sectional area of the rod

$$\therefore \text{Compressive stress } f = \frac{P}{a}$$

with l .

$$\text{Stress} = E \text{ (Young's modulus)}$$

Strain

$$\frac{f}{\alpha T} = E$$

$$f = \alpha T E$$

The thrust (force) on the rod section

$$P = f a = \alpha T E a$$

In general, the temp strain = $\frac{\text{Expansion (or) Contraction prevented}}{\text{Original length}}$

Suppose a rod of length l , when subjected to a rise of temperature is permitted to expand only by ' δ ', the temp strain

$$e = \frac{\text{Expansion prevented}}{\text{Original length}} = \frac{\alpha T l - \delta}{l}$$

$$\therefore \text{Temperature stress } f = E e = \frac{E (\alpha T l - \delta)}{l}$$

Ex: A 15 mm diameter steel rod passes centrally through a copper tube 50 mm external diameter and 40 mm internal diameter. The tube is closed at each end by rigid plates of negligible thickness. The nuts are tightened lightly home on the projecting parts of the rod. If the temperature of the assembly is raised by 60°C , calculate the stresses developed in copper and steel. $E_s = 2.10 \times 10^5 \text{ N/mm}^2$; $E_c = 1.05 \times 10^5 \text{ N/mm}^2$

$$\alpha_s = 12 \times 10^{-6} \text{ per } ^\circ\text{C}; \alpha_c = 17.5 \times 10^{-6} \text{ per } ^\circ\text{C}$$

Area of steel rod $A_s = \frac{\pi}{4} (15)^2 = 56.25 \pi \text{ mm}^2$

Area of copper $A_c = \frac{\pi}{4} (50-40)^2 = 225 \pi \text{ mm}^2$

Free expansion of steel $= \alpha_s T l$

Free expansion of copper $= \alpha_c T l$

Let the actual expansion of each component be 'x'

$\alpha_c T l > x > \alpha_s T l$

$\therefore$ Steel is in tension and copper is in compression for the equilibrium of the system

Tension in steel = Compression in copper

$f_s A_s = f_c A_c$

$f_s = \left[\frac{A_c}{A_s} \right] f_c$

$f_s = \left[\frac{225 \pi}{56.25 \pi} \right] f_c$

$f_s = 4 f_c$

Actual expansion of steel = Actual ^{Compression} expansion of copper

$\alpha_s T l + \frac{f_s}{E_s} l = \alpha_c T l - \frac{f_c}{E_c} l$

$\therefore \alpha_s T + \frac{f_s}{E_s} = \alpha_c T - \frac{f_c}{E_c}$

But $f_s = 4 f_c$

$12 \times 10^{-6} \times 60 + \frac{4 f_c}{2.10 \times 10^5} = 17.5 \times 10^{-6} \times 60 - \frac{f_c}{1.05 \times 10^5}$

$72 + 1.905 f_c = 105 - 0.952 f_c$

$f_c = 11.55 \text{ N/mm}^2 \text{ (Compressive)}$

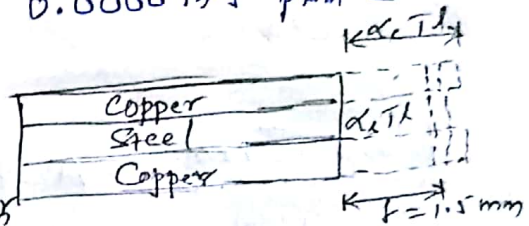
$f_s = 4 \times 11.55 = 46.20 \text{ N/mm}^2$

Two copper bars each having the same area and length as the steel bar at 15°C . At this stage they are rigidly connected together at both the ends. When the temperature is raised to 315°C , the length of the bars increases by 1.5 mm . Determine the original length and final stresses in the bars. Take the original length and final stresses in the bars. Take

$$E_s = 2.1 \times 10^5 \text{ N/mm}^2, E_c = 1 \times 10^5 \text{ N/mm}^2$$

$$\alpha_s = 0.000012 \text{ per } ^\circ\text{C}, \alpha_c = 0.0000175 \text{ per } ^\circ\text{C}$$

Let A is area of steel



$\therefore$ area of Copper components = $2A \text{ mm}^2$

Free expansion of steel component = $\alpha_s T L$
copper " = $\alpha_c T L$

Let δ be the actual expansion of each bar

$$\alpha_s T L < \delta < \alpha_c T L$$

$\therefore$ Steel is in tension and copper is in compression

Tension in steel = Compression in copper

$$f_s A = f_c (2A)$$

$$f_s = 2 f_c$$

Actual expansion of steel = Actual expansion of copper

$$\alpha_s T L + \frac{f_s}{E_s} L = \alpha_c T L - \frac{f_c}{E_c} L$$

$$\alpha_s T + \frac{f_s}{E_s} = \alpha_c T - \frac{f_c}{E_c}$$

$$(T = 315 - 15 = 300^\circ\text{C})$$

$$0.000012 \times 300 + \frac{2 f_c}{2.1 \times 10^5} = 0.0000175 \times 300 - \frac{f_c}{1 \times 10^5}$$

$$360 + 0.952 f_c = 525 - f_c$$

$$1.952 f_c = 165$$

$$f_c = 84.52 \text{ N/mm}^2 \text{ (compression)}$$

Actual expansion of steel = Actual expansion of copper

$$= \alpha_s T l + \frac{\Delta s}{E_s} l = 1.50$$

$$0.000012 \times 300 l + \frac{169.04}{2.1 \times 10^5} l = 1.50$$

$$360 l + 86.495 l = 150000$$

$$l = 340.526 \text{ mm}$$

Hoop Stress: A thin steel tyre of internal diameter d . Such a tyre can be shrunk on to a wheel of slightly bigger diameter D . The steel tyre is heated so that its diameter exceeds D . In this stage the steel tyre is slipped on to the wheel. If the now the tyre be cooled it is prevented from assuming its original diameter d . Hence it will grip the wheel.



Hence a tensile stress is induced circumferentially along the tyre, such stress is called hoop stress

$$\text{Temperature strain} = e = \frac{\text{Contraction prevented}}{\text{original length}} = \frac{\pi D - \pi d}{\pi d}$$

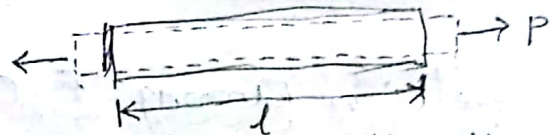
$$e = \frac{D-d}{d}$$

$$\text{Stress } f = eE = \left(\frac{D-d}{d} \right) E$$

Suppose a rectangular bar of width 'b', depth 'd' and length 'l' be subjected to an axial tensile load. the deformation of the member will take place such that the length of the member will increase the lateral dimensions namely the length, breadth and the depth will decrease.

The ratio $\frac{\delta l}{l}$ is called longitudinal strain

$\frac{\delta b}{b}$ and $\frac{\delta d}{d}$ is called the lateral strain



when the deformation of the member is within the elastic limit it is found that the ratio of the lateral strain to the longitudinal strain is a constant for a given material. This ratio is called Poisson's ratio

$$\mu = \frac{1}{m} = \text{poisson's ratio} = \frac{\text{lateral strain}}{\text{longitudinal strain}}$$

Volumetric Strain:

When a member is subjected to forces deforming it, it undergoes changes in its dimensions and hence its volume will be subjected to a change. The ratio of the change in volume to the original volume is called the volumetric strain.

$$e_v = \frac{\text{change in volume}}{\text{original volume}}$$

* Volumetric Strain of rectangular bar

$$e_v = \frac{\delta l}{l} + \frac{\delta d}{d} + \frac{\delta b}{b} = e_l + e_d + e_b$$

* Volumetric Strain of a cylindrical rod

$$e_v = \frac{\delta l}{l} + 2 \frac{\delta d}{d}$$

* Volumetric Strain of sphere

$$e_v = 3 \frac{\delta d}{d} = 3e_d$$

Relation Among E, C & K

$$* E = 2C(1 + \mu)$$

E - Young's modulus

C - modulus of rigidity

$$C = \frac{\text{Shear stress}}{\text{Shear strain}}$$

$$E = \frac{\text{Stress}}{\text{Strain}}$$

$$* E = 3K(1 - 2\mu)$$

K - Bulk modulus

$$K = \frac{\text{Stress}}{\text{volumetric strain}}$$

$$1 + \mu = \frac{E}{2C} \quad \text{--- (1)}$$

$$1 - 2\mu = \frac{E}{3K} \quad \text{--- (2)}$$

multiply (1) by 2

$$2(1 + \mu) = \frac{E}{C} \quad \text{--- (3)}$$

$$1 - 2\mu = \frac{E}{3k}$$

$$2\mu + \frac{E}{3k} = \frac{E}{c}$$

$$3 = \frac{E}{c} + \frac{E}{3k} \Rightarrow \frac{3kE + EC}{3kc}$$

$$3 = \frac{E(3k + c)}{3kc} \quad \therefore E = \frac{3kc}{3k + c}$$

Ex: In tensile test, a test piece 30mm in diameter, 250mm gauge length stretched 0.106mm under a pull of 60kN.

In torsion test, the same rod twisted 0.017 radian over a length of 250mm, when a torque of 450 Nm was applied. Find the three elastic moduli and the Poisson's ratio for the material of the test piece.

Sol

$$d = 30\text{mm}, \quad l = 250\text{mm}; \quad \text{pull} = 60\text{kN} = 60000\text{N}$$

$$\text{Tensile stress } f = \frac{60000}{\frac{\pi}{4}(30)^2} = 84.88 \text{ N/mm}^2$$

$$\text{Tensile strain } e = \frac{0.106}{250} = 4.24 \times 10^{-4}$$

$$\text{Modulus of Elasticity } E = \frac{f}{e} = \frac{84.88}{4.24 \times 10^{-4}} = 2.002 \times 10^5 \text{ N/mm}^2$$

$$\text{Modulus of rigidity } c = \frac{TL}{\phi J} = \frac{450 \times 10^3 \times 250}{\frac{\pi}{32}(30)^4 \times 0.017} = 0.852 \times 10^5$$

$$E = 2C(1 + \mu)$$

$$\frac{E}{2C} = 1 + \mu = \frac{2.002 \times 10^5}{2 \times 0.852 \times 10^5} = 1.203$$

$$\therefore \text{poisson's Ratio } \mu = 1.203 - 1$$

$$\mu = 0.203$$

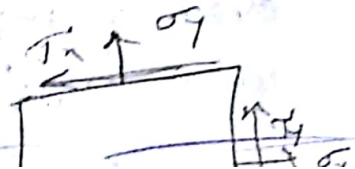
$$E = 3k(1 - 2\mu)$$

$$\text{Bulk modulus } k = \frac{E}{3(1 - 2\mu)}$$

$$= \frac{2.002 \times 10^5}{3(1 - 2 \times 0.203)} = 1.123 \times 10^5 \text{ N/mm}^2$$

Principal Planes & Principal Stresses

The planes which are subjected to only normal stresses i.e. on which tangential stresses are zero are called principal planes and the normal stresses acting on them are called principal stresses.



The work done on the member will be stored by the member as energy and is called strain energy or resilience.

1. Proof Resilience: The maximum energy stored within the elastic limit is called proof resilience. It is the strain energy corresponding to stress at elastic limit.
2. Modulus of Resilience: The proof resilience per unit volume of a body is called modulus of resilience.

$$\text{Strain energy stored in the member} = \frac{f^2 (A) L}{2E}$$

$$\text{Strain energy stored by the member per unit volume} = \frac{f^2}{2E}$$

Conditions of Loading

1. Gradually applied load

$$\text{Stress } (f) = \frac{P}{A}$$

$$\text{Strain energy } U = \frac{f^2 A L}{2E}$$

2. Suddenly applied load

$$\text{Stress } (f) = 2 \frac{P}{A}$$

Hence the maximum stress intensity due to a suddenly applied load is twice the stress intensity produced by the load of the same magnitude applied gradually.

Stress (f) = $\sqrt{\frac{2PhE}{Al}}$

COM of $\frac{PE}{L} = \frac{Ph}{L}$

$$f = \frac{P}{A} + \sqrt{\frac{P^2}{A^2} + \frac{2PhE}{Al}}$$

Ex: A steel bar 18mm in diameter is pulled axially by a force of 10kN. If the bar is 250mm long calculate the strain energy per unit volume of the bar and the total strain energy stored by the bar. $E = 2 \times 10^5 \text{ N/mm}^2$

Sol Area of bar $A = \frac{\pi}{4} (15)^2 = 176.71 \text{ mm}^2$

Stress in the bar $= f = \frac{10 \times 10^3}{176.71} = 56.59 \text{ N/mm}^2$

Strain energy stored per unit volume

$$\frac{f^2}{2E} = \frac{56.59^2}{2 \times 2 \times 10^5} = 8.006 \times 10^{-3} \text{ N/mm per mm}^3$$

Total strain energy stored $= \frac{f^2 A L}{2E}$

$$= 8.006 \times 10^{-3} \times 176.71 \times 250$$

$$= 353.688 \text{ Nmm.}$$

Ex: A steel rod 50 mm in diameter is 3m long. Find the maximum instantaneous stress induced when a pull of 100kN is suddenly applied to it. Find also the instantaneous elongation. Take $E = 2 \times 10^5 \text{ N/mm}^2$

Sol $A = \frac{\pi}{4} (50)^2 = 625 \pi \text{ mm}^2$

max stress $f = \frac{2P}{A} = \frac{2 \times 100 \times 1000}{625 \pi} = 101.86 \text{ N/mm}^2$

max. instantaneous elongation $\Delta l = \frac{f l}{E}$

$$= \frac{101.86 \times 3000}{2 \times 10^5}$$

$$= 1.528 \text{ mm.}$$

A load of 100N falls by gravity a vertically distance of 3000mm, when it is suddenly stopped by a collar at the end of a vertical rod of length 6m and dia 20mm. The top of the bar is rigidly fixed to a ceiling. Calculate the maximum stress and strain induced in the bar. $E = 1.96 \times 10^5 \text{ N/mm}^2$

Ex Falling weight $W = 100\text{N}$

$$h = 3000\text{mm}$$

$$L = 6000\text{mm}$$

$$d = 20\text{mm}$$

$$E = 1.96 \times 10^5 \text{ N/mm}^2$$

$$\text{max. stress } f = \frac{W}{A} + \sqrt{\frac{W^2}{A^2} + \frac{2EWh}{AL}}$$

$$= \frac{W}{A} + \sqrt{\left(\frac{W}{A}\right)^2 + 2E\left(\frac{W}{A}\right)\left(\frac{h}{L}\right)}$$

$$\frac{W}{A} = \frac{100}{\frac{\pi}{4}(20)^2} = 0.3183 \text{ N/mm}^2$$

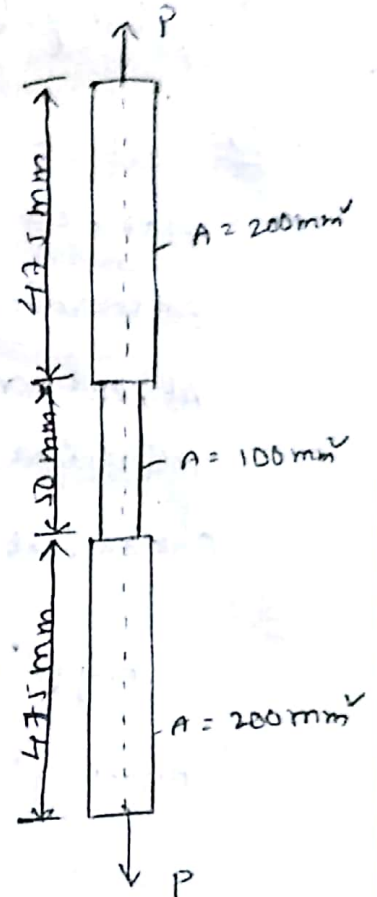
$$\left(\frac{W}{A}\right)^2 = (0.3183)^2 = 0.1013$$

$$\frac{h}{L} = \frac{3000}{6000} = 0.5$$

$$f = 0.3183 + \sqrt{0.1013 + 2 \times 1.96 \times 10^5 \times 0.3183 \times 0.5}$$

$$= 249.77 \text{ N/mm}^2$$

$$\text{maximum strain } e = \frac{f}{E} = \frac{249.77}{1.96 \times 10^5} = 0.001274 \text{ mm}$$



* The planes which have no shear stress, are known as principal stresses. Hence principal planes are the planes of zero shear stress. These planes carry only normal stresses. The normal stresses acting on the principal plane, are known as principal stresses.

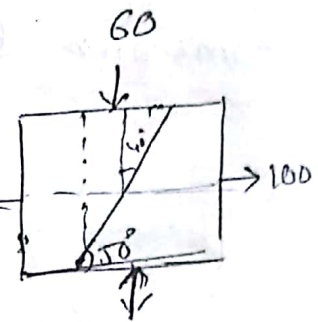
Ex: At a point in a strained material the principal stresses are 100 N/mm^2 (tensile), 60 N/mm^2 (compressive). Determine the normal stress, shear stress and resultant stress on a plane inclined at 50° to the axis of major principal stress. Also determine the maximum shear stress at that point.

Sol

major principal stress (σ_1) = 100 N/mm^2

minor principal stress (σ_2) = -60 N/mm^2

(-ve sign due to compressive)



Angle of inclined plane with the axis of major principal stress = 50°

Angle of the inclined plane with the axis of minor principal stress $\theta = 90^\circ - 50^\circ = 40^\circ$

$$\text{Normal stress } (\sigma_n) = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta$$

$$= \frac{100 + (-60)}{2} + \frac{100 - (-60)}{2} \cos(2 \times 40^\circ)$$

$$= \frac{100 - 60}{2} + \frac{100 + 60}{2} \cos 80^\circ$$

$$= 20 + 80 \times \cos 80^\circ$$

$$= 33.89 \text{ N/mm}^2$$

$$= \frac{100 - (-60)}{2} \sin 80^\circ$$

$$= 78.785 \text{ N/mm}^2$$

$$\text{Resultant stress } (\sigma_R) = \sqrt{\sigma_n^2 + \sigma_T^2}$$

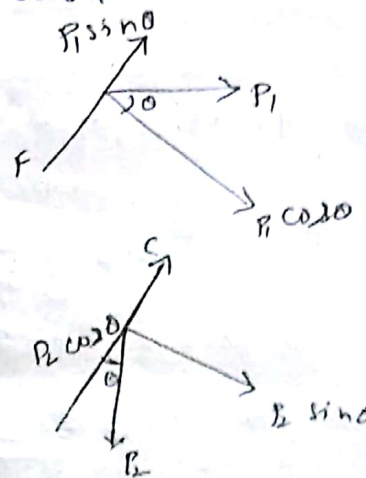
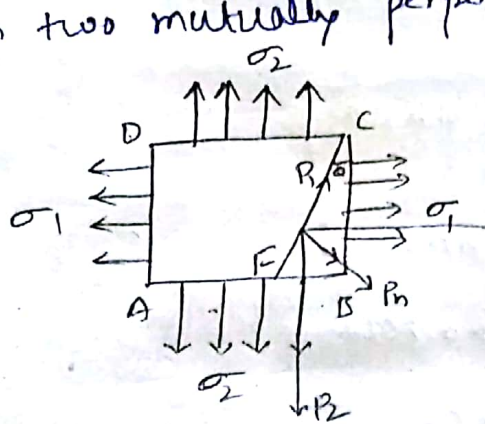
$$= \sqrt{33.89^2 + 78.785^2}$$

$$= 85.765 \text{ N/mm}^2$$

$$\text{Maximum shear stress } (\sigma_T)_{\max} = \frac{\sigma_1 - \sigma_2}{2}$$

$$= \frac{100 - (-60)}{2} = 80 \text{ N/mm}^2$$

Note: A member subjected to like Direct stresses in two mutually perpendicular directions.



Ex: A rectangular bar of c/s area of 11000 mm^2 is subjected to a tensile load P . The permissible normal and shear stresses on the oblique plane BC are given as 7 N/mm^2 and 3.5 N/mm^2 . Determine the safe value of P .

Normal stress $\sigma_n = 7 \text{ N/mm}^2$

Shear stress $\sigma_T = 3.5 \text{ N/mm}^2$

Angle of oblique plane with the axis of bar = 60°

$\therefore$ Angle of oblique plane ^{BC} with normal c/s of the

bar $\theta = 90^\circ - 60^\circ = 30^\circ$

let P - safe load

σ - safe stress

$$\boxed{\text{Normal stress } \sigma_n = \sigma \cos^2 \theta}$$

$$\sigma = \frac{\sigma_n}{\cos^2 \theta} = \frac{7}{(0.866)^2}$$

$$= 9.334 \text{ N/mm}^2$$

$$\boxed{\text{Shear stress } \sigma_T = \frac{\sigma}{2} \sin 2\theta}$$

$$3.5 = \frac{\sigma}{2} \sin 2 \times 30^\circ$$

$$= \frac{\sigma}{2} \times 0.866$$

$$\sigma = \frac{3.5 \times 2}{0.866} = 8.083 \text{ N/mm}^2$$

The safe stress is the least of the two i.e., 8.083 N/mm^2

$\therefore$ Safe load $P = \text{Safe stress} \times \text{area}$

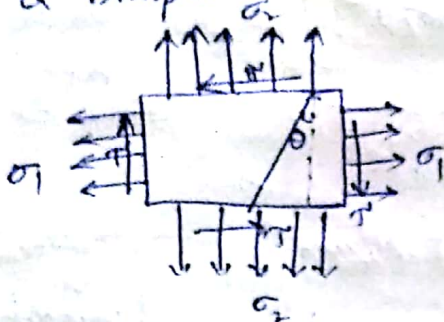
$$= 8.083 \times 11000$$

$$= 88.913 \text{ kN}$$

Note: A member is subjected to a Direct stress in one plane

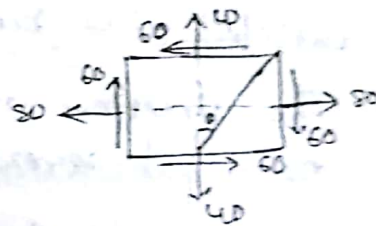


Note: A member subjected to direct stresses in two mutually perpendicular directions, accompanied by a simple shear stress



Ex: At a point within a body subjected to two mutually perpendicular directions, the stresses are 80 N/mm^2 tensile and 40 N/mm^2 tensile. Each of the above stresses is accompanied by a shear stress of 60 N/mm^2 . Determine the normal stress, shear stress and resultant stress on an oblique plane inclined at 45° with the axis of minor stress.

sd major tensile stress $\sigma_1 = 80 \text{ N/mm}^2$
 minor " " $\sigma_2 = 40 \text{ N/mm}^2$
 shear stress $\tau = 60 \text{ N/mm}^2$



Angle of oblique plane, with axis of minor tensile stress $\theta = 45^\circ$

$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta + \tau \sin 2\theta$$

$$= \frac{80 + 40}{2} + \frac{80 - 40}{2} \cos (2 \times 45^\circ) + 60 \sin (2 \times 45^\circ)$$

$$= 60 + 20 \cos 90^\circ + 60 \sin 90^\circ$$

$$= 120 \text{ N/mm}^2$$

$$\text{shear stress } \tau_r = \frac{\sigma_1 - \sigma_2}{2} \sin 2\theta - \tau \cos 2\theta$$

$$= \frac{80 - 40}{2} \sin (2 \times 45^\circ) - 60 \cos (2 \times 45^\circ)$$

$$= 20 \text{ N/mm}^2$$

$$\boxed{\text{Resultant } (\sigma_R) = \sqrt{(\sigma_n)^2 + (\sigma_t)^2}}$$

$$= \sqrt{(120)^2 + (20)^2}$$

$$= 121.655 \text{ N/mm}^2$$

Ex: A rectangular block of material is subjected to a tensile stress of 110 N/mm^2 on one plane and a tensile stress of 47 N/mm^2 on the plane at right angles to the former. Each of the above stresses is accompanied by a shear stress of 63 N/mm^2 and that associated with the former tensile stress tends to rotate the block anticlockwise find:

- The direction and magnitude of each of the principal stress and
- magnitude of the greatest shear stress

Sol

Major, Minor Principal Stresses

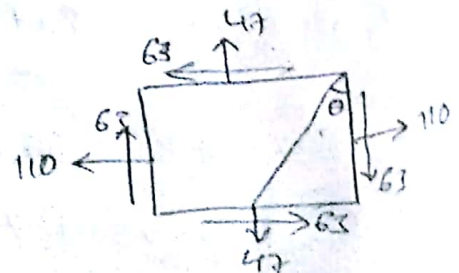
$$= \frac{\sigma_1 + \sigma_2}{2} \pm \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2}$$

Major principal stress

$$= \frac{\sigma_1 + \sigma_2}{2} + \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2}$$

$$= \frac{110 + 47}{2} + \sqrt{\left(\frac{110 - 47}{2}\right)^2 + (63)^2}$$

$$= 148.936 \text{ N/mm}^2$$



$$\begin{aligned} \text{Principal stress} &= \frac{\sigma_1 + \sigma_2}{2} \pm \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2} \\ &= \frac{110 + 47}{2} \pm \sqrt{\left(\frac{110 - 47}{2}\right)^2 + 63^2} \\ &= 8.064 \text{ N/mm}^2 \end{aligned}$$

The direction of principal stresses

$$\tan 2\theta = \frac{2\tau}{\sigma_1 - \sigma_2} = \frac{2 \times 63}{110 - 47}$$

$$2\theta = \tan^{-1}(2)$$

$$2\theta = 63^\circ 26' \quad (\text{or}) \quad 243^\circ 26'$$

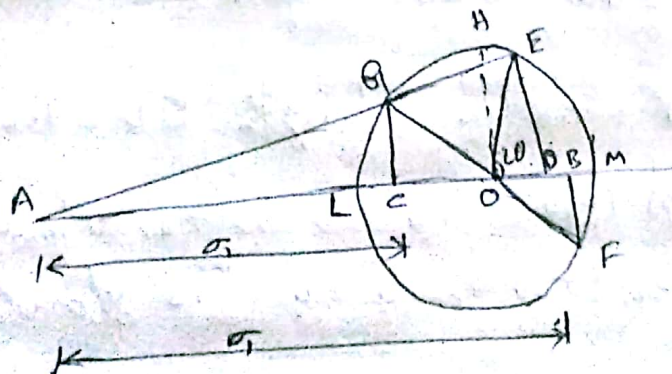
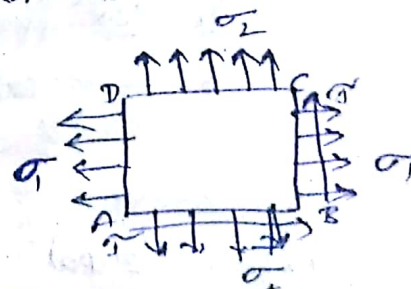
$$\theta = 31^\circ 43' \quad (\text{or}) \quad 121^\circ 43'$$

(ii) Magnitude of the greatest shear stress

$$\begin{aligned} (\sigma_r)_{\max} &= \frac{1}{2} \sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2} \\ &= \frac{1}{2} \sqrt{(110 - 47)^2 + (4 \times 63)^2} \\ &= 70.436 \text{ N/mm}^2 \end{aligned}$$

Mohr's Circle: (Graphical Method)

* When a body is subjected to two mutually Perpendicular Principal Tensile stresses accompanied by a simple shear



Ex. A point in a strained material is subjected to stresses. Using Mohr's circle method, find the normal and tangential stresses across the oblique plane. Check the answer analytically.

sd Major principal stress $\sigma_1 = 65 \text{ N/mm}^2$

Minor " " $\sigma_2 = 35 \text{ N/mm}^2$

Shear stress $\tau = 25 \text{ N/mm}^2$

Angle of oblique plane $\theta = 45^\circ$

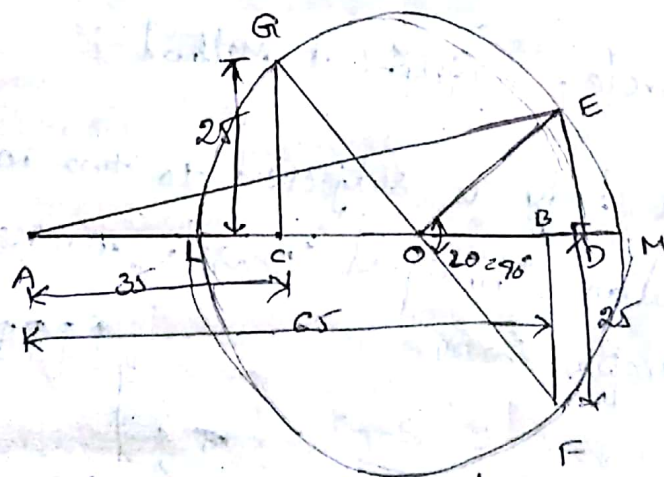
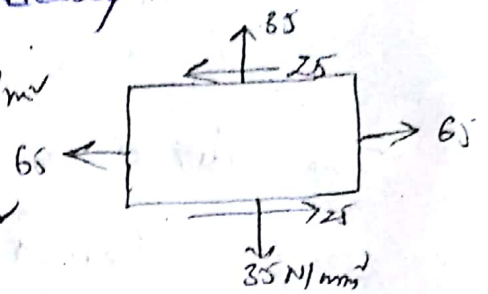
Mohr's circle method

Let $1 \text{ cm} = 10 \text{ N/mm}^2$

$$\sigma_1 = \frac{65}{10} = 6.5 \text{ cm}$$

$$\sigma_2 = \frac{35}{10} = 3.5 \text{ cm}$$

$$\text{and } \tau = \frac{25}{10} = 2.5 \text{ cm}$$



AD - Normal stress and ED - shear stress

length AD = 7.5 cm, length ED = 1.5 cm

Normal stress (σ_n) = $7.5 \times 10 = 75 \text{ N/mm}^2$

Shear stress (σ_r) = $1.5 \times 10 = 15 \text{ N/mm}^2$

Normal stress (σ_n) = $\frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta + \tau \sin 2\theta$

$$= \frac{65 + 35}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 180 + 25 \sin 180$$

$$= 50 + 0 + 25 = 75 \text{ N/mm}^2$$

shear stress (τ_r) = $\frac{\sigma_1 - \sigma_2}{2} \sin 2\theta - \tau \cos 2\theta$

$$= \frac{65 - 35}{2} \sin 180 - 25 \cos 180$$

$$= 15 \text{ N/mm}^2$$

Shear Forces & Bending Moments

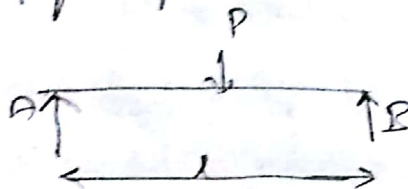
Beam: A beam is a structural member subjected to a system of external forces at right angles to its axis

Types of beams

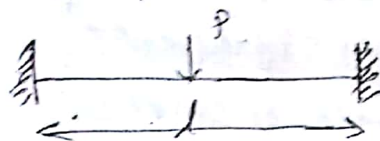
1. Cantilever beam



2. Simply Supported beam



3. Fixed Beam



4. Continuous beam



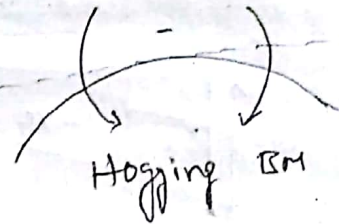
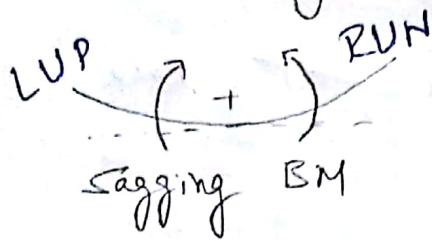
Types of loadings on Beams:

1. Concentrated load (point load)
2. Uniformly distributed load
3. Uniformly varying load.

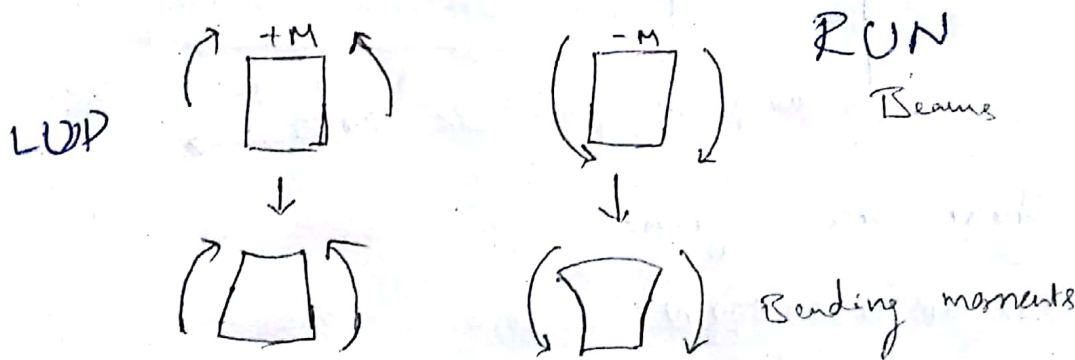
Shear Force: Shear forces are unaligned forces pushing one part of body in one direction, and another part the body in the opposite direction. Shear force acting on a substance is a direction perpendicular to

The shearing forces at the end of the beam is equal to the vertical reaction at the support.

Bending Moment: A bending moment is the reaction induced in a structural element when an external force causing the element to bend.



* We will consider sagging moments as +ve and hogging moments as -ve

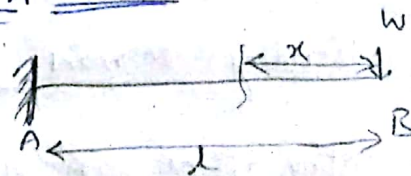


Shear force and Bending Moment Diagrams

for cantilever with point load

shear force F_x at x

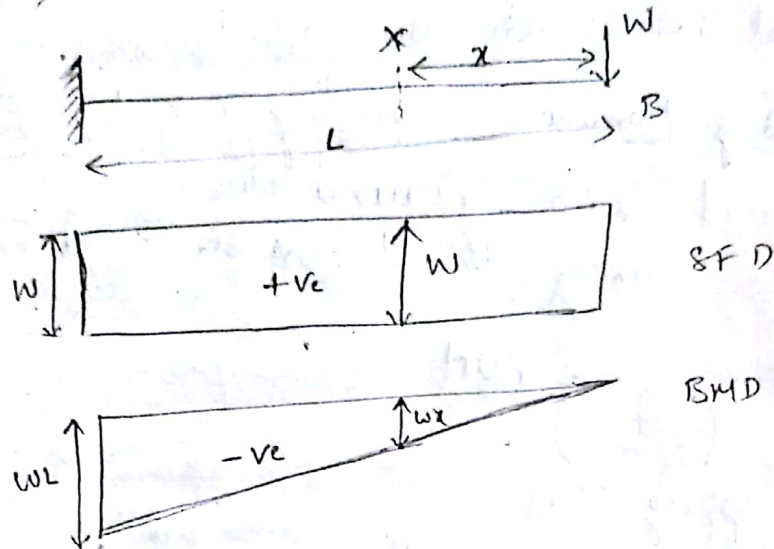
Bending moment M_x at x



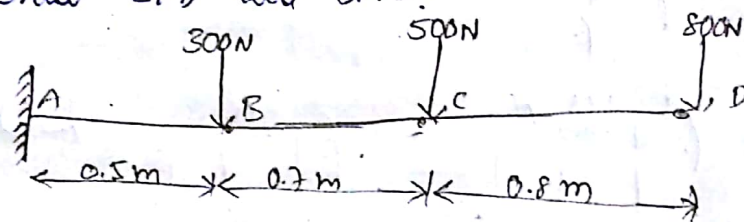
$F_x = +w$ since the force acting downwards at right side

At $x=0$ i.e., at B B.M = 0

At $x=L$ i.e., at A B.M = $w \times L$



Ex: A cantilever beam of length 2m carries the point loads. Draw SFD and BMD.



Shear Force Diagram:

$$\text{SF at D} = +800 \text{ N}$$

$$\text{SF at C } F_c = +800 + 500 = +1300 \text{ N}$$

$$\text{SF at B } F_b = +800 + 500 + 300 = +1600 \text{ N}$$

$$\text{SF at A } F_a = +1600 \text{ N}$$

Bending moment Diagram:

$$\text{The B.M at D} = 0$$

(i) The BM at any section b/w C and D at distance x and D is given

$$M_x = -800 \times x$$

∴ BM at C, $M_C = -800 \times 0.8 = -640 \text{ Nm}$

(ii) The BM between B + C at a distance x from D

(At C, $x = 0.8$ & at B, $x = 0.8 + 0.7 = 1.5 \text{ m}$) Hence here x varies from 0.8 to 1.5)

$$M_x = -800x - 500(1.5 - 0.8)$$

B.M at B is obtained by substituting $x = 1.5 \text{ m}$

$$\begin{aligned} M_B &= -800 \times 1.5 - 500(1.5 - 0.8) \\ &= -1200 - 350 = -1550 \text{ Nm} \end{aligned}$$

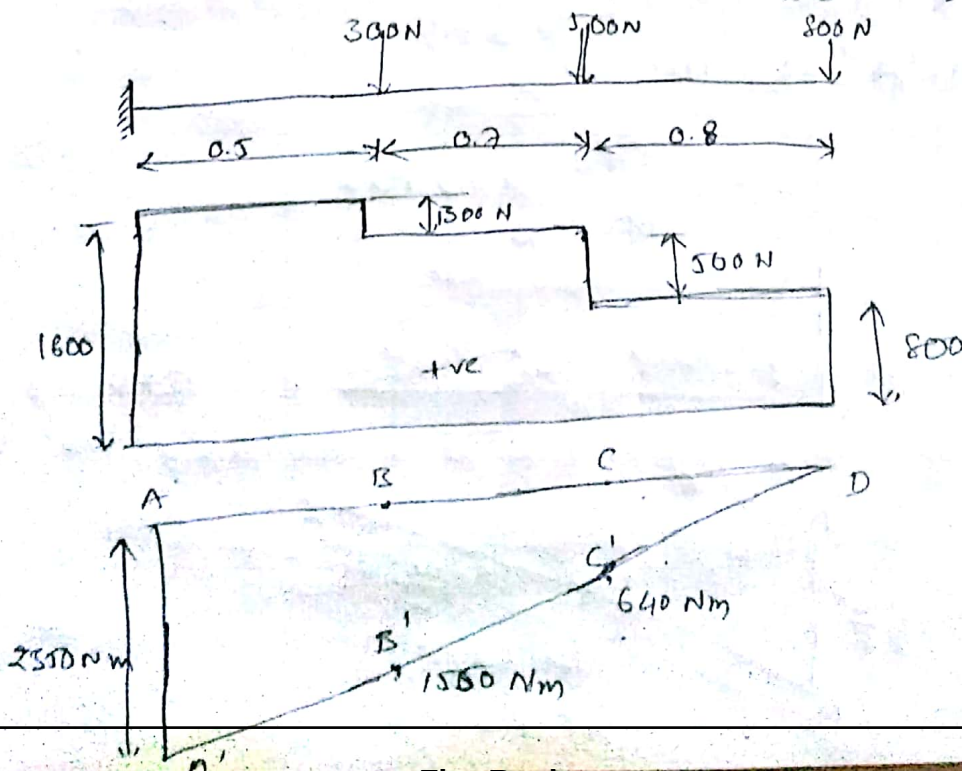
(iii) The BM between A + B at a distance x from D

(At B $x = 1.5$ & at A $x = 2 \text{ m}$. Hence x varies from 1.5 to 2 m)

$$M_x = -800x - 500(x - 0.8) - 300(x - 1.5)$$

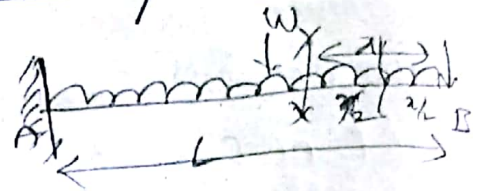
B.M at A obtained by $x = 2 \text{ m}$

$$\begin{aligned} M_A &= -800 \times 2 - 500(2 - 0.8) - 300(2 - 1.5) \\ &= -1600 - 600 - 150 = -2350 \text{ Nm} \end{aligned}$$



Shear Forces and Bending Moment Diagrams for Cantilever Beam with a uniformly distributed load

Take a section X at a distance of x from B .



$$F_x = w \cdot x \quad (\text{length of right portion})$$

This resultant force is acting downwards. But the resultant force on the right portion acting downwards is considered +ve. Hence shear force at x is +ve

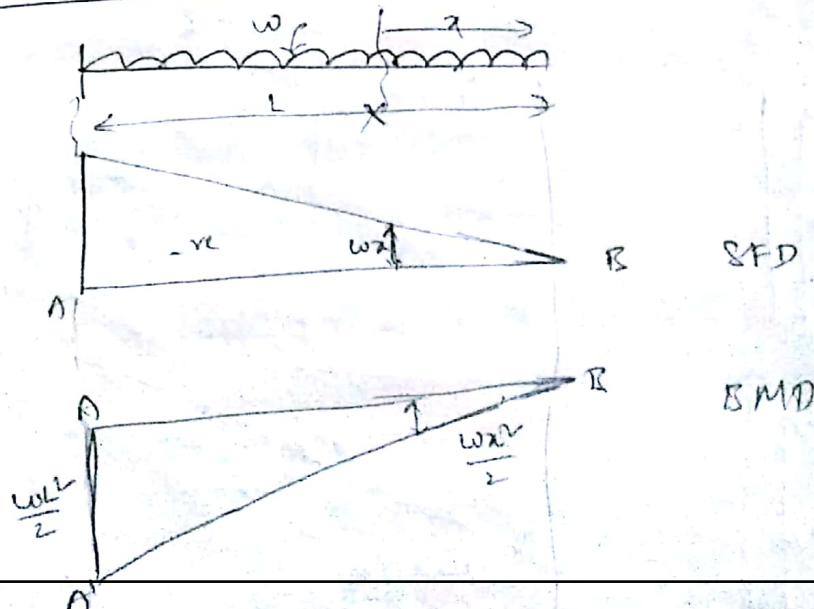
$$\begin{aligned} \text{At } B \quad F_x &= 0 \text{ at } x=0 \\ \text{At } A \quad F_x &= +w \cdot L \text{ at } x=L \end{aligned}$$

The B.M at the section x is given by

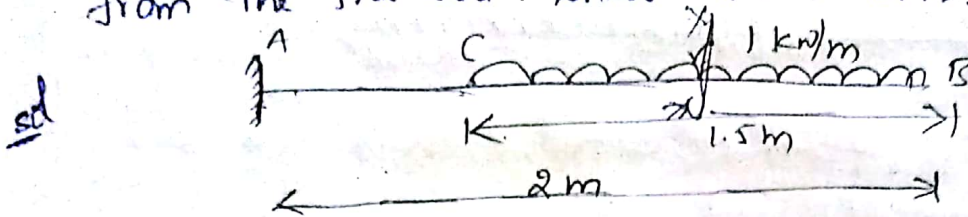
$$M_x = -(\text{Total load right portion}) \times \text{Distance of C.G. of right portion from } x$$

$$= -(w \cdot x) \cdot \frac{x}{2} = -w \frac{x^2}{2}$$

At $B \quad x=0 \quad M_x=0$
At $A \quad x=L, \quad M_x = -w \frac{L^2}{2}$



A cantilever beam of length 2m carries a uniformly distributed load of 1kN/m run over a length of 1.5m from the free end. Draw SFD & BMD.



SFD

$$F_x = wx$$

$$= 1.0 \times x$$

At B $x=0$ hence $F_x=0$

At C, $x=1.5$ hence $F_x = 1.0 \times 1.5 = 1.5 \text{ kN}$

As no load b/w A & C, the SF remains constant.

$$F_B = 0, F_C = 1.5 \text{ kN and } F_A = 1.5 \text{ kN}$$

BMD

(i) The BM at any section b/w B and C is

$$M_x = -\frac{wx^2}{2}$$

$$\text{At B } x=0 \quad M_B = -\frac{1 \times 0^2}{2} = 0$$

$$\text{At C } x=1.5 \text{ m} \quad M_C = -\frac{1 \times 1.5^2}{2} = -1.125 \text{ Nm}$$

(ii) The BM at any section b/w A & C at a distance x from the free end (x varies from 1.5m to 2m)

$$\text{Total load due to UDL} = w \times 1.5 = 1.5 \text{ kN}$$

This load is acting at a distance of $\frac{1.5}{2} = 0.75 \text{ m}$ from the free end B or at a distance of $(x - 0.75)$ from any section b/w A and C

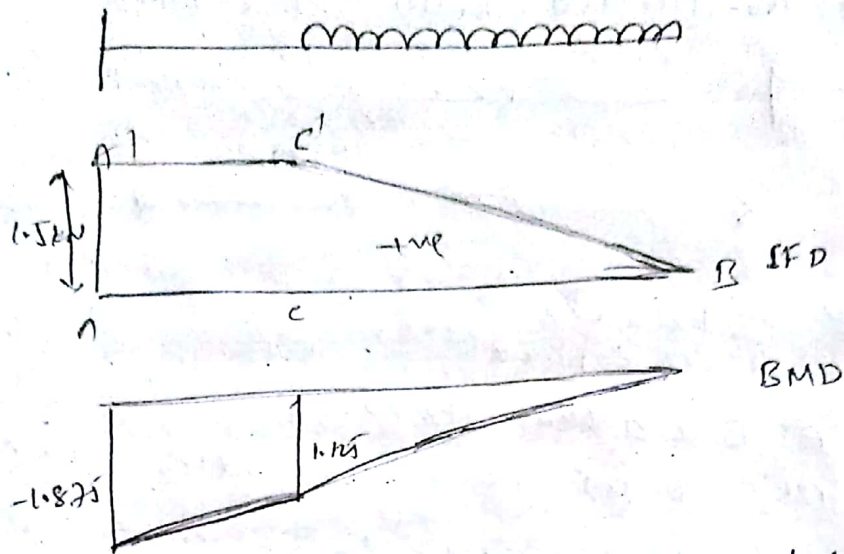
$$M_x = (\text{load due to UDL}) (x - 0.75)$$

$$= -1.5 \times (x - 0.75)$$

$$A) A \quad x = 8 \text{ m}$$

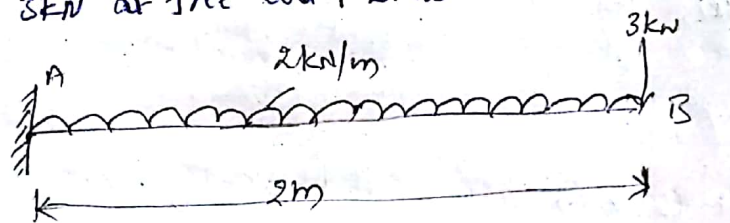
$$M_c = -1.5(1.5 - 0.75) = -1.125 \text{ Nm}$$

$$M_A = -1.5(2 - 0.75) = -1.875 \text{ Nm}$$



Ex: A cantilever of length 2m carries a uniformly distributed load of 2 kN/m length over the whole length and a point load of 3 kN at free end. Draw SFD and BMD.

Sol



SFD

SFD

shear Force at B = 3 kN

At any section b/w A and B

$$F_x = 3 + w \cdot x$$

$$= 3 + 2x$$

At B, $x = 0$ hence $F_B = 3 \text{ kN}$

At A, $x = 2$ hence $F_B = 7 \text{ kN}$

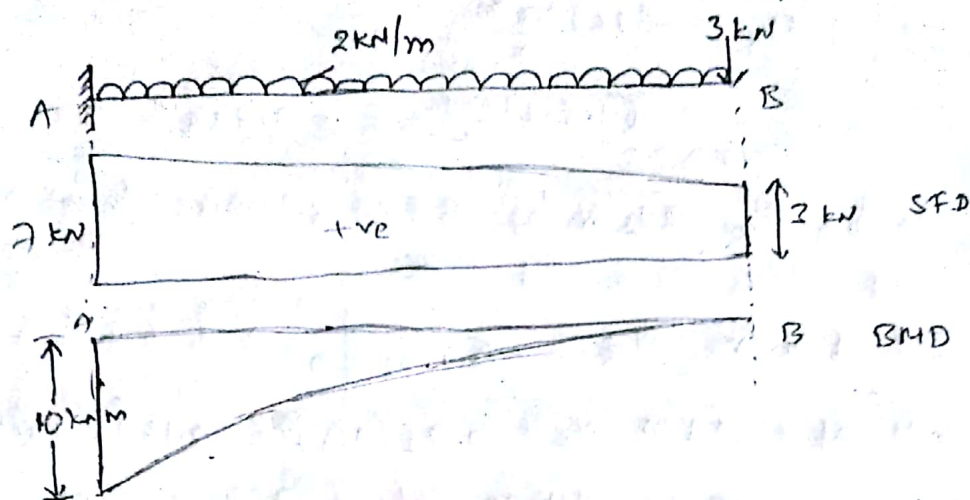
BMD BM at any section at a distance x from free end

$$M_x = -\left(3x + w \cdot x \cdot \frac{x}{2}\right)$$

$$= -\left(3x + \frac{2x^2}{2}\right)$$

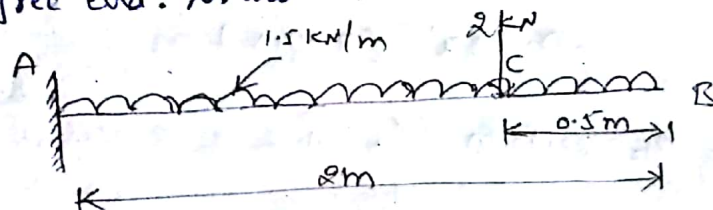
$$= -(3x + x^2)$$

BM at $x=0$ $M_B = (3 \times 0 + 0) = 0$
BM at $x=2$ $M_A = -(3 \times 2 + 2) = -10 \text{ kNm}$



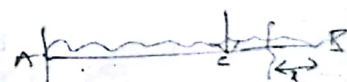
Ex: A cantilever of length 2m carries a uniformly distributed load of 1.5 kN/m runs over the whole length and a point load of 2 kN at a distance of 0.5m from the free end. Draw SFD & BMD.

Sol



SFD: (i) Any section C & B at distance x

$$F_x = wx = 1.5x$$



$$F_x \text{ at } x=0 \quad F_B = 0$$

$$F_x \text{ at } x=0.5 \quad F_C = 1.5 \times 0.5 = 0.75 \text{ kN}$$

(ii) Any section b/t A & C



$$F_x = wx + 2 \text{ kN}$$

$$= 1.5x + 2$$

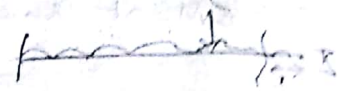
(x varies from 0.5 to 2m because section b/t A & C)

$$F_x \text{ at } x=0.5 \text{ m} \quad F_C = 1.5(0.5) + 2 = 2.75 \text{ kN}$$

$$F_x \text{ at } x=2 \text{ m} \quad F_A = 1.5(2) + 2 = 5 \text{ kN}$$

$$M_x = -(wx) \cdot \frac{x}{2}$$

$$= -(1.5 \cdot 2) \frac{x}{2} = -0.75x$$



In the section b/t B & C, x varies from 0 to 0.5

$$\text{BM at } x=0 \quad M_B = -0.7(0) = 0$$

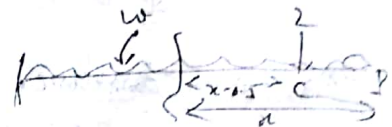
$$\text{BM at } x=0.5 \text{ m} \quad M_C = 0.75(0.5) = 0.1875 \text{ kNm}$$

(ii) BM at any section b/t A & C

$$M_x = -(wx) \cdot \frac{x}{2} - 2(x-0.5)$$

$$= -(1.5x) \cdot \frac{x}{2} - 2(x-0.5)$$

$$= -0.75x^2 - 2(x-0.5)$$



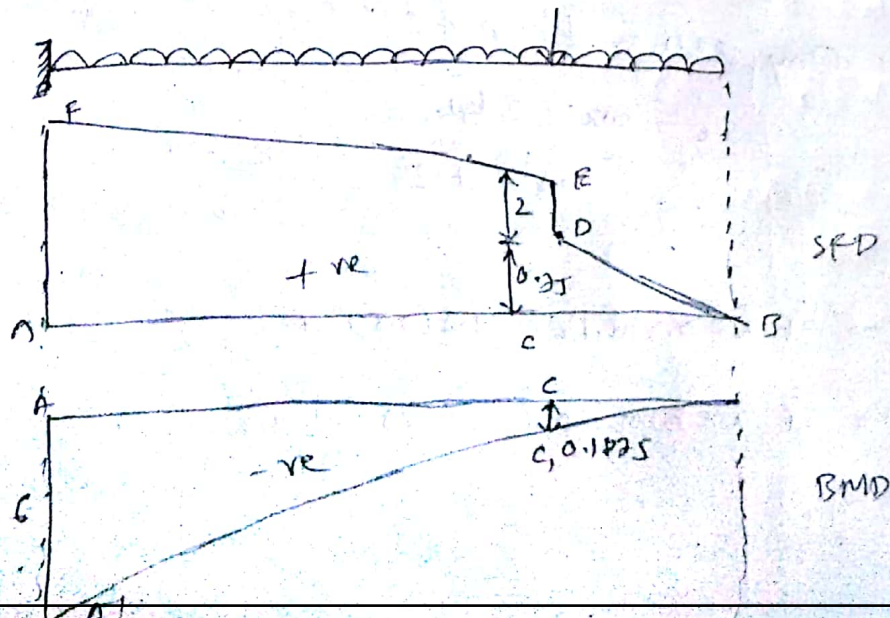
In the section b/t A & C x varies from 0.5 m to 2 m

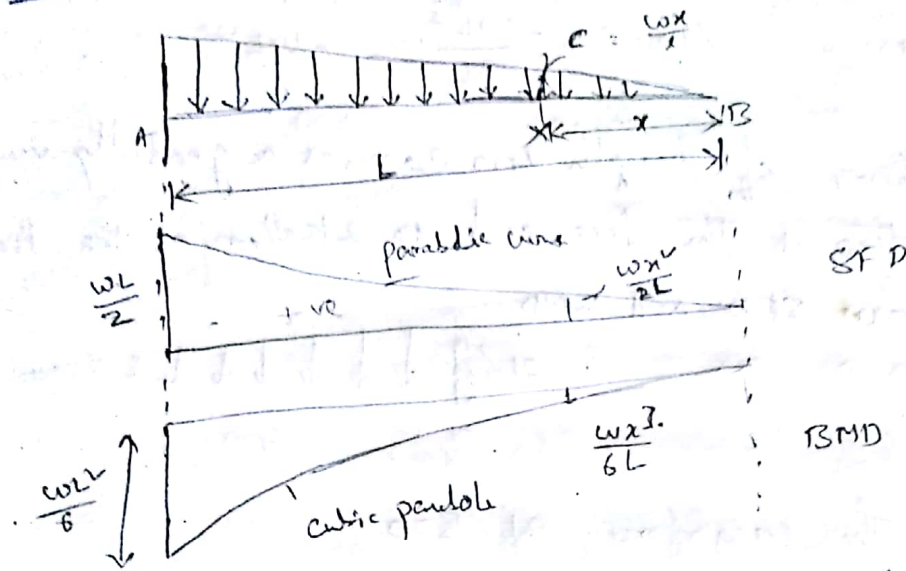
$$\text{BM at } x=0.5, \quad M_C = -0.75(0.5) - 2(0.5-0.5)$$

$$= -0.1875 \text{ kNm}$$

$$\text{BM at } x=2 \text{ m}, \quad M_A = -0.75(2)^2 - 2(2-0.5)$$

$$= -6 \text{ kNm}$$





Take a section X at a distance x from the free end B
At x distance load will be $\frac{w}{L}x$ per unit length

$$\text{Hence } CX = \frac{w \cdot w}{L}$$

The shear force at x is

$F_x = \text{Total load on the cantilever for a length } x \text{ from B}$

$= \text{Area of triangle BCX}$

$$= \frac{XB \cdot XC}{2} = \frac{x \left(\frac{w \cdot x}{L} \right)}{2}$$

$$= \frac{wx^2}{2L}$$

$$\text{SF at } x=0, F_B = \frac{w(0)^2}{2L} = 0$$

$$\text{SF at } x=L, F_A = \frac{wL^2}{2L} = \frac{w \cdot L}{2}$$

The BM at the section X at distance x from B

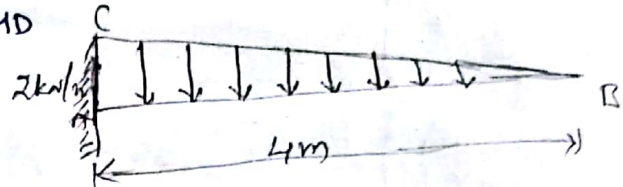
$M_x = -(\text{Total load for a length } x) \times \text{Distance of load from X}$

$= -(\text{Area of } \Delta^b \text{ BCX}) \times (\text{Distance of C.G. of } \Delta^b \text{ from X})$

$$= -\left(\frac{wx^2}{2L} \right) \times \frac{x}{3} = -\frac{wx^3}{6L}$$

BM at $x = L$ $M_A = -\frac{WL^3}{6L} = -\frac{WL^2}{6}$

Ex: Cantilever of length 4m carries a gradually varying load, zero at the free end to 2kN/m at the fixed end. Draw SFD and BMD



Sd

SFD

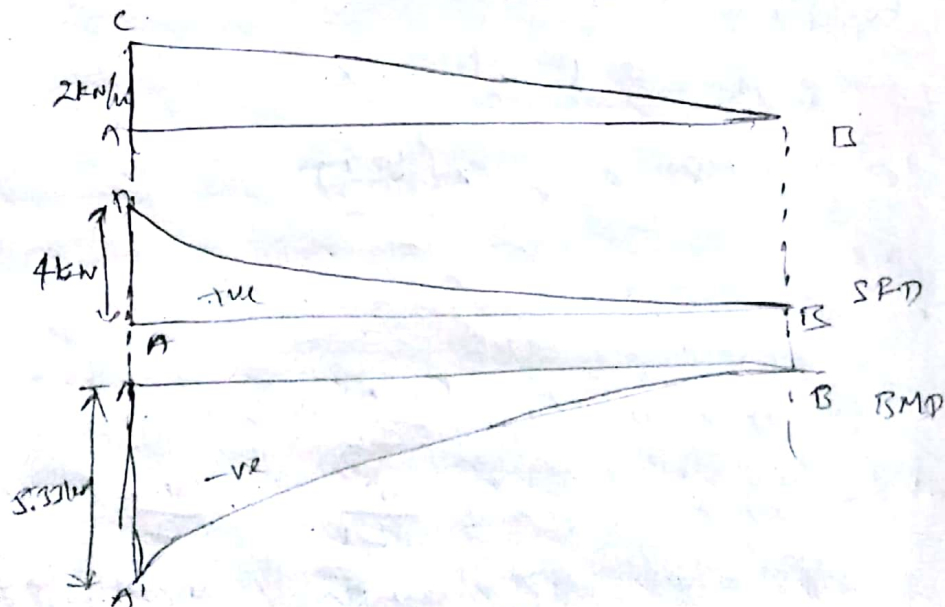
The shear force at B = 0

The shear force at C = $\frac{WL}{2} = \frac{4 \times 2}{2} = 4 \text{ kN}$

BMD

The BM at B is zero

The BM at A = $-\frac{WL^2}{6} = -\frac{2 \times (4)^2}{6}$
 $= -5.33 \text{ kNm}$



at mid-point

SFD

Reactions at supports

will be equal to $\frac{w}{2}$ the $R_A = R_B = \frac{wL}{2}$

Take a section X b/w A & C from A

$$R_A = \frac{w}{2}$$

(Left portion acting upwards +ve)

now consider section b/w C & B from A

$$\left(\frac{w}{2} - w\right) = -\frac{w}{2}$$

At the section shear force changes to $+\frac{w}{2}$ to $-\frac{w}{2}$

BMD (i) BM at any section b/w A & C from A

$$M_x = R_A \cdot x \quad \text{or} \quad M_x = +\frac{w}{2} \cdot x$$

$$\text{At } x=0 \quad M_A = \frac{w}{2} (0) = 0$$

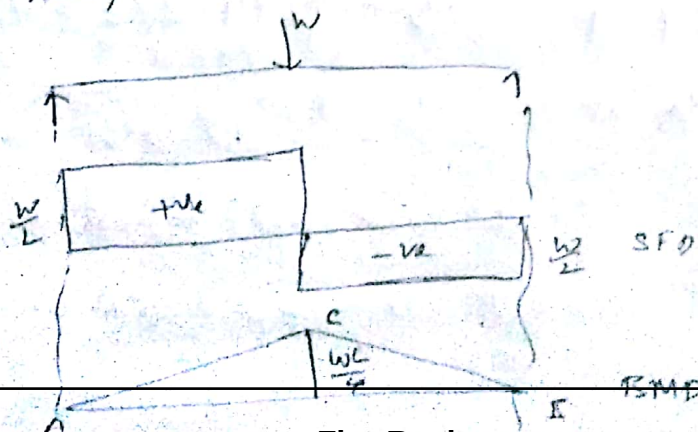
$$\text{At } x = \frac{L}{2} \quad M_C = \frac{w}{2} \times \frac{L}{2} = \frac{wL}{4}$$

(ii) BM at any section b/w C & B from A

$$M_x = R_A \cdot x - w \left(x - \frac{L}{2}\right) = \frac{w}{2} x - wx + wx \frac{L}{2} = \frac{wL}{2} - \frac{wx}{2}$$

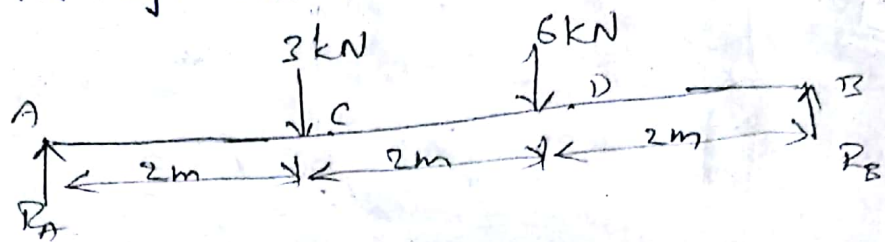
$$\text{At C } x = \frac{L}{2} \quad M_C = \frac{wL}{2} - \frac{w \times L}{2 \times 2} = \frac{wL}{4}$$

$$\text{At B } x = L, \quad M_B = \frac{wL}{2} - \frac{w}{2} \times L = 0$$



Ex. A simply supported beam of length 6m, carries point load of 3kN and 6kN at distances of 2m & 4m from the left end. Draw SFD & BMD.

Sol.



Reactions R_A & R_B

$$R_B \times 6 = 3 \times 2 + 6 \times 4 = 30$$

$$R_B = \frac{30}{6} = 5 \text{ kN}$$

$$R_A = \text{Total load on beam} - R_B$$

$$= (3 + 6) - 5 = 4 \text{ kN}$$

SFD

$$\text{SF at A, } F_A = R_A = 4 \text{ kN}$$

$$\text{SF at C, } F_C = 4 - 3 = 1 \text{ kN}$$

$$\text{SF at D, } F_D = 4 - 3 - 6 = -5 \text{ kN}$$

$$(\text{SF at B, } F_B = \cancel{4 - 3 - 6} - 5 = -5 \text{ kN})$$

The shear force b/w D & B is constant & equal -5 kN

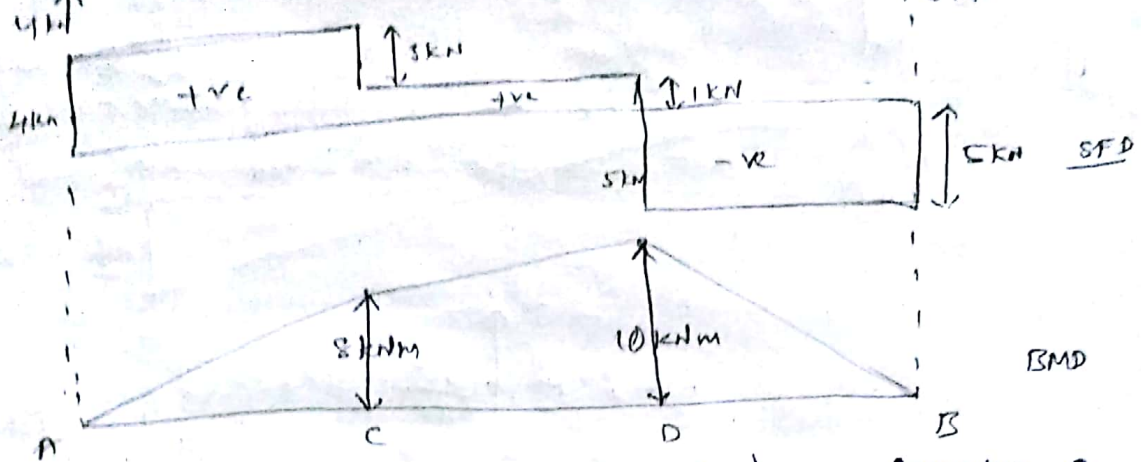
BMD

$$\text{BM at A, } M_A = 0$$

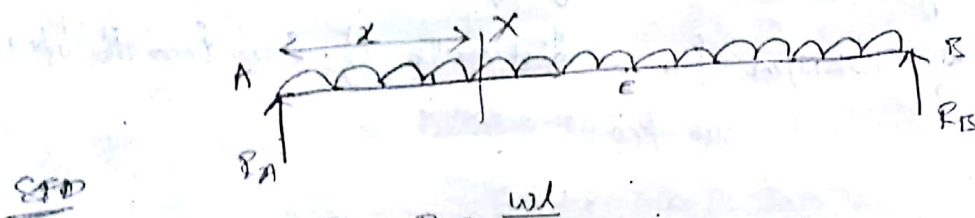
$$\text{BM at C, } M_C = R_A \times 2 = 4 \times 2 = 8 \text{ kNm}$$

$$\text{BM at D, } M_D = R_A \times 4 - 3 \times 2 = 4 \times 4 - 3 \times 2 = 10 \text{ kNm}$$

$$\text{MB at B, } M_B = 0$$



SFD & BMD for simply supported beam carrying a uniformly Distributed load



Reactions $R_A = R_B = \frac{wL}{2}$
Consider any section X at a distance x from left end A.

$$F_x = R_A - wx = \frac{wL}{2} - wx \quad (\because x \text{ varies from } 0 \text{ to } L)$$

At A, $x=0$ hence $F_A = \frac{wL}{2} - w \cdot 0 = \frac{wL}{2}$

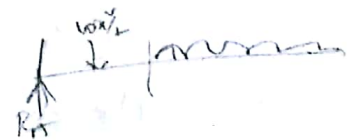
At B, $x=L$ hence $F_B = \frac{wL}{2} - wL = \frac{wL}{2} - wL = -\frac{wL}{2}$

At C, $x = \frac{L}{2}$ hence $F_C = \frac{wL}{2} - w \cdot \frac{L}{2} = \frac{wL}{2} - \frac{wL}{2} = 0$

BMD BM at the section X at a distance x from A

$$M_x = R_A \cdot x - w \cdot x \cdot \frac{x}{2}$$

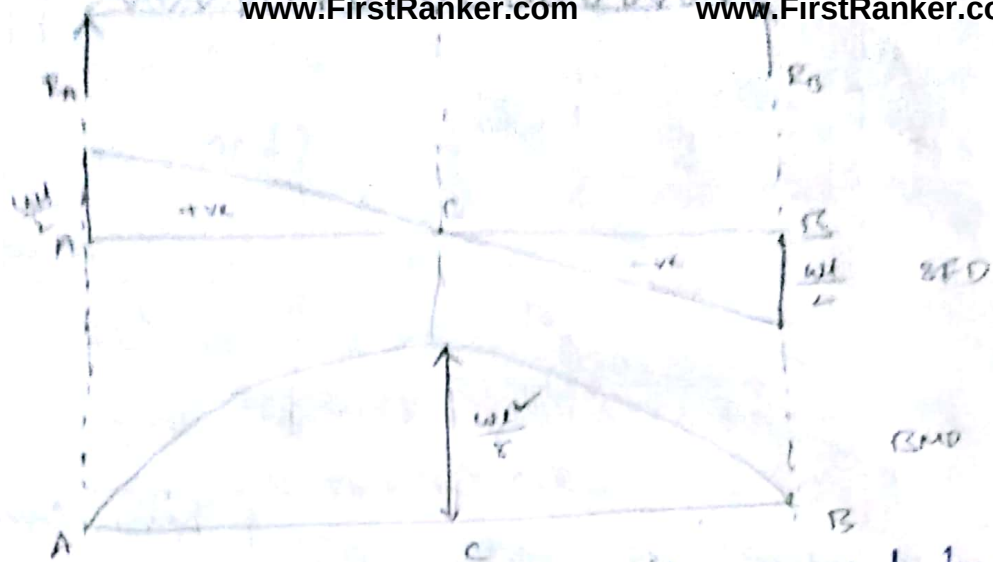
$$= \frac{w \cdot L \cdot x}{2} - \frac{w \cdot x^2}{2}$$



BM at A, $x=0$, $M_A = \frac{w \cdot L \cdot 0}{2} - \frac{w(0)^2}{2} = 0$

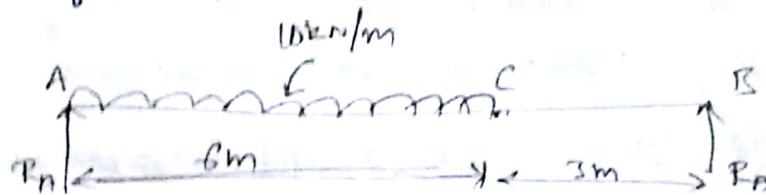
BM at B, $x=L$, $M_B = \frac{w \cdot L \cdot L}{2} - \frac{wL^2}{2} = 0$

BM at C, $x = \frac{L}{2}$, $M_C = \frac{w \cdot L \cdot \frac{L}{2}}{2} - \frac{w(\frac{L}{2})^2}{2} = \frac{wL^2}{8}$



Q.1 Draw the SFD and BMD for a simply supported beam of length 9m and carrying a uniformly distributed load of 10kN/m for a distance of 6m from the left end.

sol



$$R_B \times 9 = 10 \times 6 \times \frac{6}{2} = 180$$

$$R_B = \frac{180}{9} = 20 \text{ kN}$$

$$R_A + R_B = 60 \Rightarrow R_A = 60 - 20 = 40 \text{ kN}$$

SFD

Any section between A & C

$$F_x = R_A - 10x$$

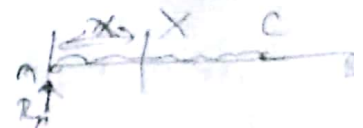
$$= 40 - 10x$$

x varies from 0 to 6m

$$\text{At } A, x=0 \text{ hence } F_A = 40 \text{ kN}$$

$$\text{At } C, x=6 \text{ hence } F_C = 40 - 10(6) = -20 \text{ kN}$$

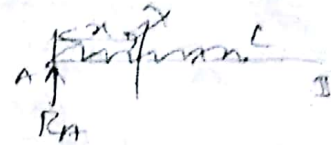
So changes from +ve to -ve, when moving left to right, so somewhere B.P will be seen



BMD B.M at any section $\frac{3}{4}$ of A to C.

$$M_x = R_A x - 10x \cdot \frac{x}{2}$$

$$= 40x - 5x^2$$



Here x varies from 0 to 6m.

At A $x=0$, $M_A = 40(0) - 5(0)^2 = 0$

At C, $x=6$, $M_C = 40(6) - 5(6)^2 = 60 \text{ kNm}$

At D, $x=4$, $M_D = 40(4) - 5(4)^2 = 80 \text{ kNm}$

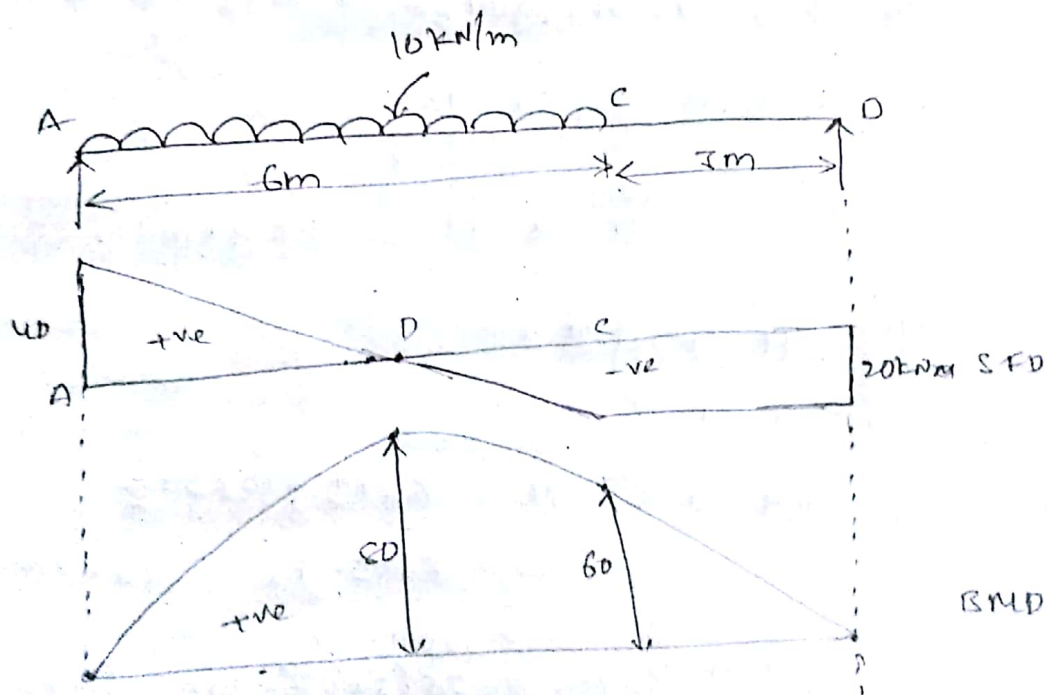
At B, BM is zero

Maximum B.M

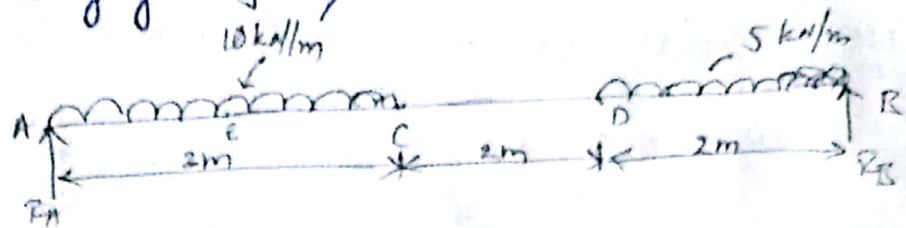
The B.M is +ve where SF becomes zero.

From SFD, at point D SF=0 after changing its sign

So Maximum BM = 80 kN.



of simply supported beam of length 7 m carrying uniformly distributed loads.



$$R_A + R_B = 30 + 10 = 40$$

$$R_B \times 7 = 10 \times 3 \times \frac{3}{2} + 5 \times 2 \times \left[3 + 2 + \frac{2}{2} \right]$$

$$= 45 + 60 = 105$$

$$R_B = 15 \text{ kN} \quad R_A = 25 \text{ kN}$$

SFD

$$\text{SF at A } F_A = 25 \text{ kN}$$

$$\text{SF at C } F_C = R_A - 3 \times 10 = 25 - 30 = -5 \text{ kNm}$$

$$\text{SF at D } F_D = -5 \text{ kNm}$$

$$\text{SF at B } F_B = -15 \text{ kN}$$

The SF is zero at point E between A & C. Let us find E

$$\text{The SF at E} = R_A - 10x$$

$$= 25 - 10x$$

$$25 - 10x = 0 \Rightarrow x = 2.5 \text{ m}$$

BMD

$$\text{B.M. at A is 0}$$

$$\text{B.M. at B is 0}$$

$$\text{B.M. at C } M_C = R_A \times 3 - 10 \times 3 \times \frac{3}{2}$$

$$= 25 \times 3 - 45 = 30 \text{ kNm}$$

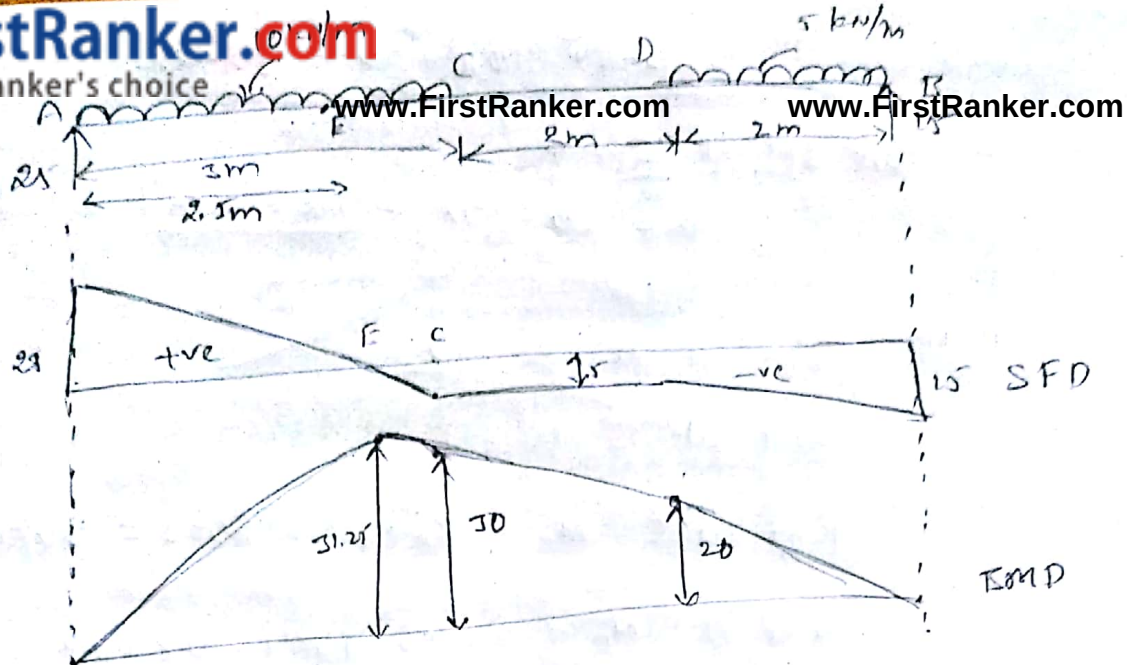
$$\text{B.M. at D } M_D = 25(3+2) - 10 \times 5 \times \left(\frac{3}{2} + 2 \right)$$

$$= 20 \text{ kNm}$$

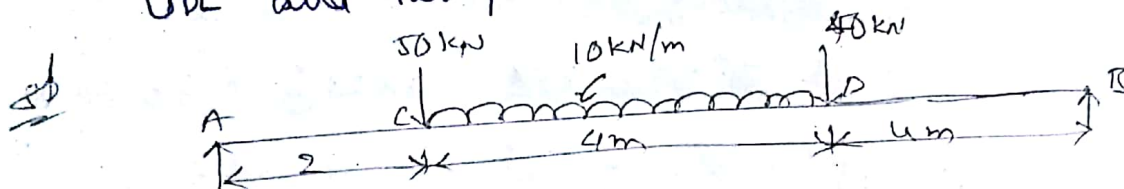
$$\text{B.M. at E } M_E = R_A \times 2.5 - 10 \times 2.5 \times \frac{2.5}{2}$$

$$= 25 \times 2.5 - 5 \times 6.25$$

$$= 31.25 \text{ kNm}$$



Ex: A simply supported beam of length 10m, carries the UDL and two point loads. Draw SFD & BMD.



$$R_A + R_B = 50 + 40 + (4 \times 10) = 130$$

$$R_A \times 10 = 50 \times 2 + 40 \times 6 + 10 \times 4 \times (2 + \frac{4}{2})$$

$$\boxed{R_B = 50 \text{ kN}} \quad \boxed{R_A = 80 \text{ kN}}$$

SFD

$$\text{SF at A } F_A = 80 \text{ kN}$$

$$\text{SF at C } F_C = 80 - 50 = 30 \text{ kN}$$

$$\text{SF at D, } F_D = 80 - 50 - 40 - 40 = -50 \text{ kN (R.H.S)}$$

$$\text{SF at D, } F_D = 80 - 50 - 40 = -10 \text{ kN (L.H.S)}$$

$$\text{SF at B, } F_B = -50 \text{ kN}$$

The S.F is '0' at E b/w C & D

Let the distance of E from point A is x

$$\text{SF at E } F_E = R_A - 50 - 10(x - 2)$$

Sum SF at E = 0

$$0 = 50 - 10x$$

$$x = 5m$$

BMD

BM at A is '0'

BM at B is '0'

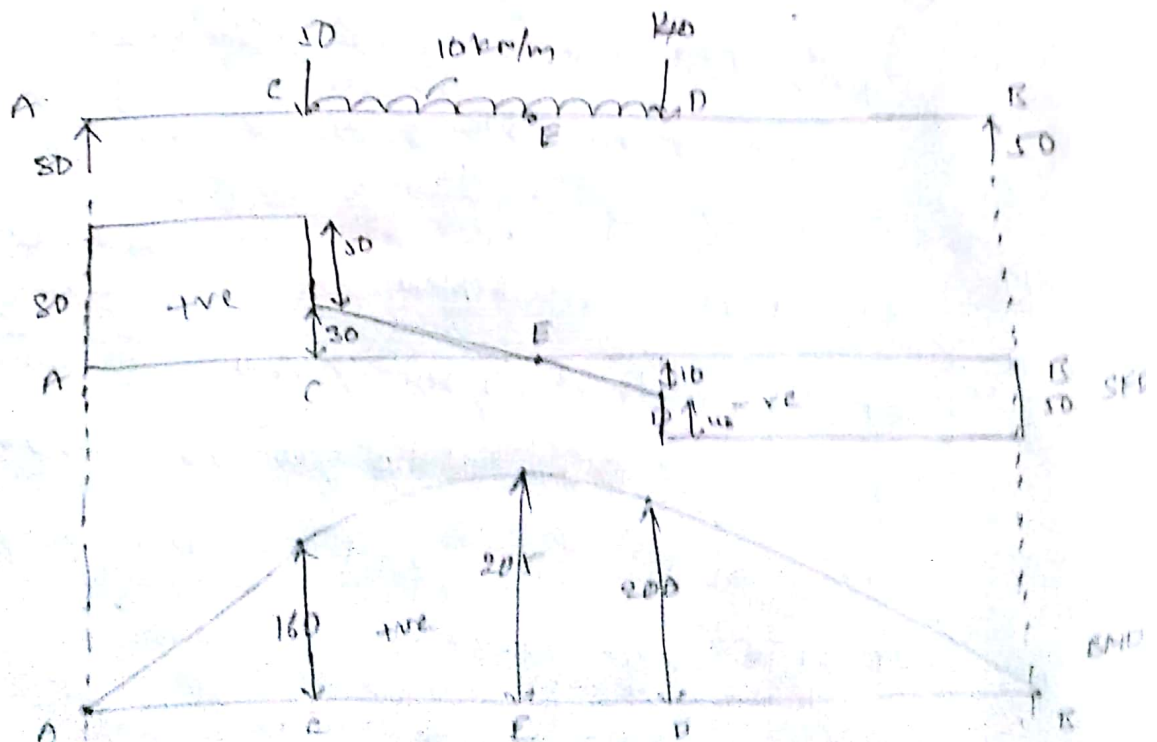
$$BM \text{ at C } M_C = R_A \times 2 = 50 \times 2 = 100 \text{ kNm}$$

$$\begin{aligned} BM \text{ at D } M_D &= R_A \times 6 - 50 \times 4 - 10 \times 4 \times \frac{4}{2} \\ &= 50 \times 6 - 200 - 80 = 200 \text{ kNm} \end{aligned}$$

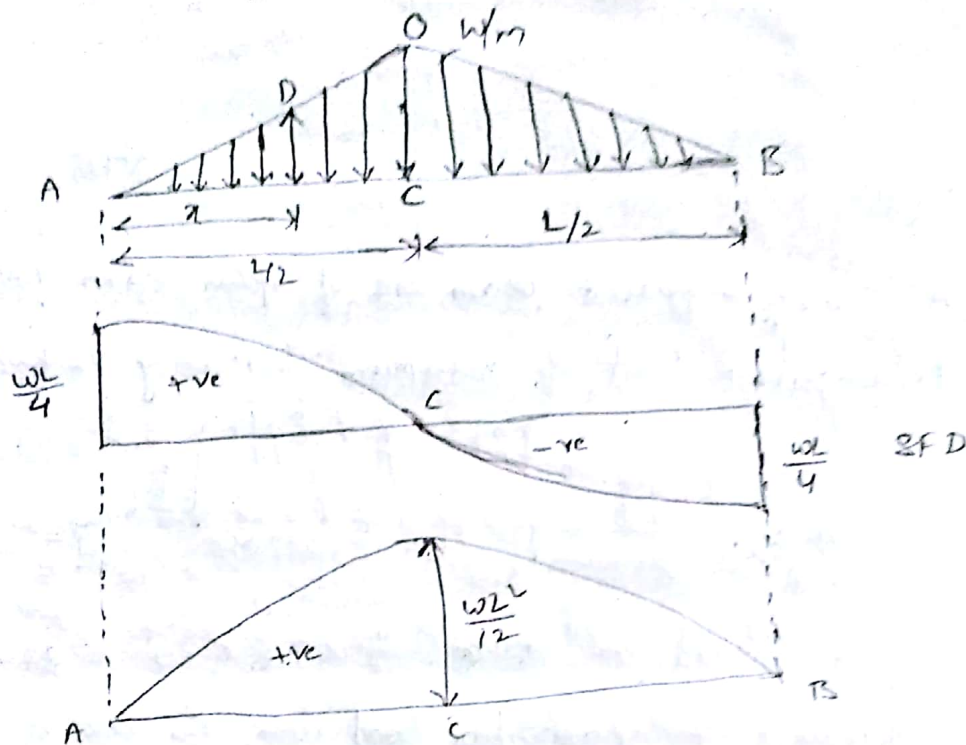
here $x = 5m$

$$\begin{aligned} BM \text{ at E } M_E &= R_A \times 5 - 50(5-2) - 10 \times (5-2) \times \left(\frac{5-2}{2}\right) \\ &= 50 \times 5 - 50 \times 3 - 10 \times 3 \times \frac{3}{2} = 205 \text{ kNm} \end{aligned}$$

Max BM is 205 because at E SF = 0



Bending moment diagram for a simply supported beam carrying a UDL from zero at each end to w per unit length at centre

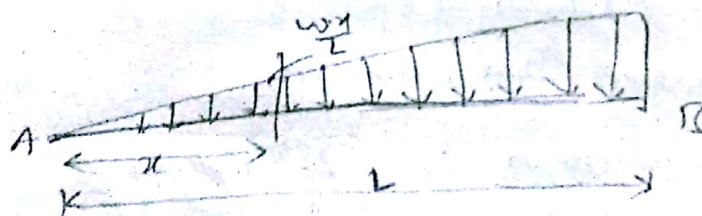


SF at A = $F_A = \frac{wL}{4}$, SF at B is $F_B = -\frac{wL}{4}$

BM at C is $M_C = \frac{wL^2}{12}$, $M_A = 0 = M_B$ (supports)

Maximum BM = $\frac{w \cdot L^2}{12}$

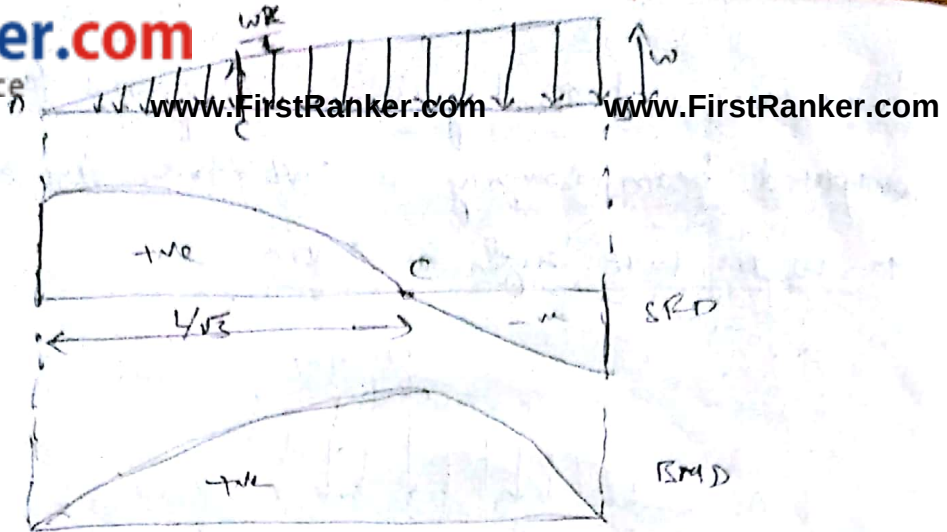
SFD & BMD for a simply supported beam carrying a UDL from '0' at one end to w per unit length at other end



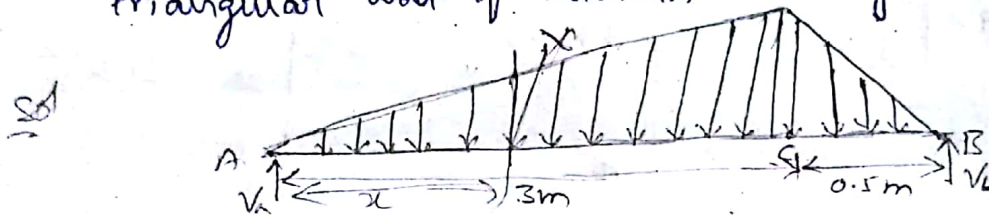
SF at A = $F_A = \frac{wL}{6}$, SF at B = $F_B = -\frac{wL}{3}$

BM at C $M_C = \frac{wL}{6} \times \frac{wx}{6}$

Maximum BM is $\frac{wL^2}{18}$



Ex: A simply supported beam AB of span 3.5m carries a triangular load of maximum intensity 7.2 kN/m.



$$\text{Total load on beam} = \frac{1}{2} \times 3.5 \times 7.2 = 12.6 \text{ kN}$$

Distance of triangular load from the end A

$$= \frac{l+a}{3} = \frac{3.5+3}{3} = \frac{6.5}{3}$$

$$V_b \times 3.5 = 12.6 \times \frac{6.5}{3} = 7.80 \text{ kN}$$

$$V_a = 12.6 - 7.8 = 4.80 \text{ kN}$$

At any section b/w A & C load = $\frac{x}{3} \times 7.2 = 2.4x$

SFD $SF = 4.80 - \frac{1}{2} \times x \cdot (2.4x)$

$$= 4.80 - 1.2x^2$$

x varies from 0 to 3m

At $x=0$ SF at A $F_a = 4.8$

At $x=3$ SF at C $F_c = 4.8 - (1.2)(3)^2$
 $= -6 \text{ kN}$

$$x^2 = \frac{7.2}{12}$$

$$\therefore x = 2.55 \text{ m from A}$$

BMD BM at any section b/w A & B

$$M = 4.8x - \frac{1}{2} \times (2.4x) \times \frac{x}{3}$$

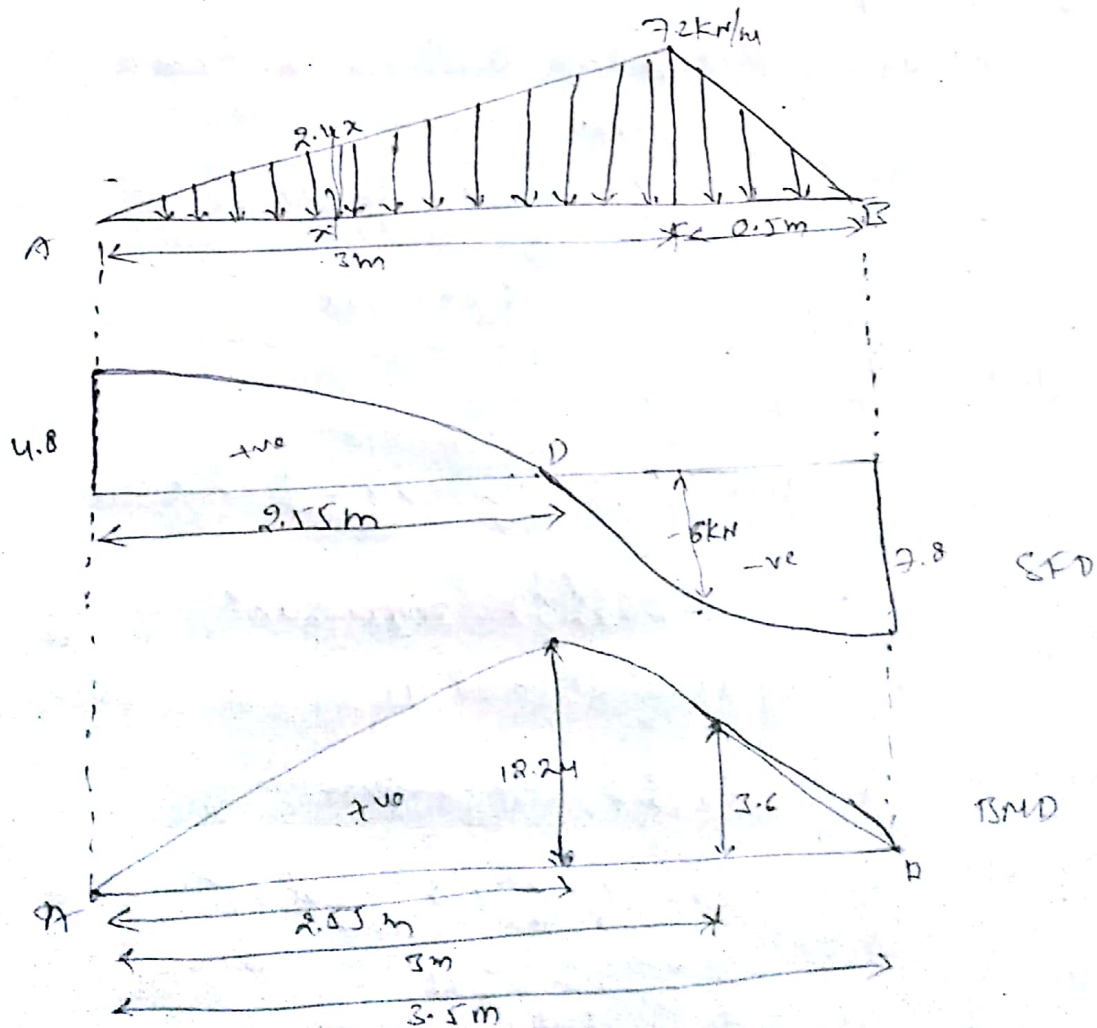
x varies from 0 to 3m

BM at $x=0$ i.e., $M_a = 0$

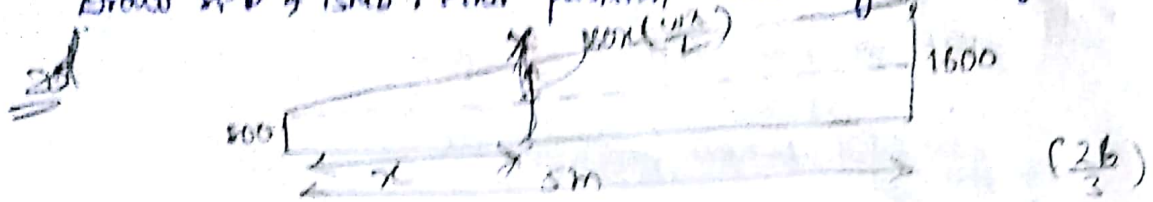
$$\begin{aligned} \text{BM at } x=3 \text{ i.e., } M_c &= 4.8(3) - \frac{1}{2} \times 3(2.4 \times 3) \times \frac{3}{3} \\ &= 14.4 - 10.8 = 3.6 \text{ kNm} \end{aligned}$$

maximum B.M occurs at $x = 2.55 \text{ m}$

$$B.M_{\max} = 4.8 \times 2.55 = 12.24 \text{ kNm.}$$



Ex. A beam of length 5m carries a uniformly increasing load of 200N/m run at one end to 1600N/m run at the other end. Draw SFD & BMD. Find position and magnitude of Max BM.



$$UDL = 200 \times 5 = 1000 \text{ N}$$

$$\text{Triangular load} = \frac{1}{2} \times 5 \times 800 = 2000 \text{ N}$$

Reactions

$$R_{AS} = 1000 \times \frac{5}{2} + 2000 \left(\frac{2}{3} \times 5 \right)$$

$$R_B = 2000 + 1333.33 = 3333.33 \text{ N}$$

$$R_A = 6000 - 3333.33 = 2666.67 \text{ N}$$

Consider a section at distance x from A
or section $x-x$

$$\begin{aligned} \text{Rate of loading} &= 200 + \frac{x}{5} \times 800 \left(\frac{Wx}{L} \right) \\ &= 200 + 160x \end{aligned}$$

SFD: SF at section $x-x$

$$F = 2666.67 - 200x - \frac{1}{2} \times x \times 160x$$

$$= 2666.67 - 800x - 80x^2$$

x varies from 0 to 5m

$$\text{At } x=0, F_A = 2666.67$$

$$\text{At } x=5, F_B = 2666.67 - 800(5) - 80(5)^2$$

$$= -3333.33 \text{ N}$$

position of x

$$2666.67 - 800x - 80x^2 = 0 \Rightarrow x^2 + 10x - 33.33 = 0$$

$$x = 2.637 \text{ m}$$

BMD The BM at any section

$$M_x = R_A x - 800x \times \frac{x}{2} - \frac{1}{2} \cdot x \cdot 160x \cdot \frac{x}{3}$$

$$= 2666.67x - 400x^2 - \frac{80}{3}x^3$$

At $x=0$, $M_A=0$

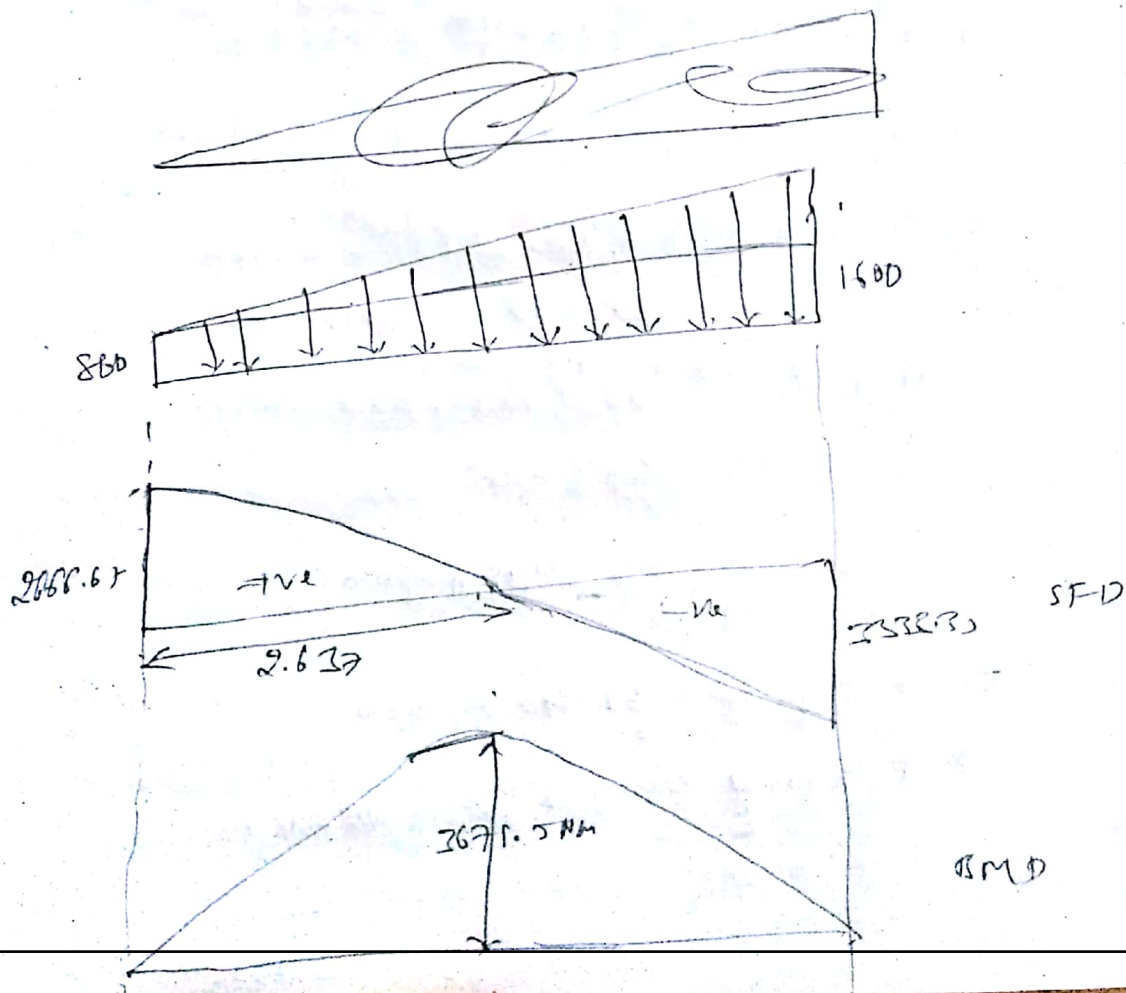
At $x=5$, $M_B=0$

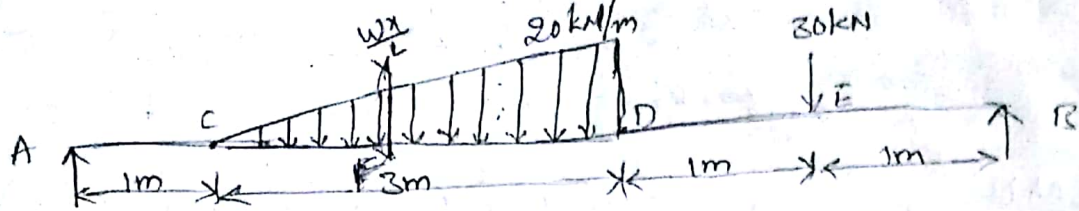
Maximum BM is at $x = 2.637 \text{ m}$.

~~B.M. = 2666.67 \times 2.637~~

$$B.M._{max} = 2666.67(2.637) - 400(2.637)^2 - \frac{80}{3}(2.637)^3$$

$$= 3761.5 \text{ Nm}$$





Sd

$$V_B \times 6 = 30 \times 5 + \frac{1}{2} \times 3 \times 20 \times 3, \quad \boxed{V_B = 40 \text{ kN}}$$

$$\boxed{\therefore R_A = 60 - 40 = 20 \text{ kN}}$$

$$R_A + R_B = 30 + \frac{1}{2} \times 3 \times 20 = 60$$

SFD

SF at any section in AC = 20 kN

consider a section b/w C + D, load at section = $\frac{wx}{3}$

$$\begin{aligned} SF &= 20 - \frac{1}{2} \times 20 \times \frac{x}{3} \\ &= 20 - \frac{10}{3} x \end{aligned}$$

x varies from 0 to 3m

$$\text{SF at } x=0, \quad F_C = 20 \text{ kN}$$

$$\text{SF at } x=1\text{m}, \quad F_F = 20 - \frac{10}{3} = 16.67 \text{ kN}$$

$$\text{SF at } x=2\text{m}, \quad F = 20 - \frac{10}{3}(2) = 6.67 \text{ kN}$$

$$\text{SF at } x=3\text{m} \quad F_D = 20 - \frac{10}{3}(3) = -10 \text{ kN}$$

Section of shear force zero

$$20 - \frac{10}{3} x = 0$$

$$x = 2.449 \text{ m from C}$$

$$\text{SF at E is } F_E = 30 - 40 = -10 \text{ kN}$$

$$\text{SF at in any section EB} = -40 \text{ kN}$$

BMD at C = 20 kNm = 20 kNm

At any section b/w C & D from C

$$\begin{aligned} \text{BMD} = M_x &= 20(1+x) - \frac{1}{2}x \cdot \frac{20x}{3} \cdot \frac{x}{3} \\ &= 20(1+x) - \frac{10}{9}x^3 \end{aligned}$$

At $x=0$, BMD = 20 kNm

At $x=1$, BMD = $20(1+1) - \frac{10}{9}(1)^3 = 38.39 \text{ kNm}$

At $x=2$, BMD = $20(1+2) - \frac{10}{9}(2)^3 = 51.4 \text{ kNm}$

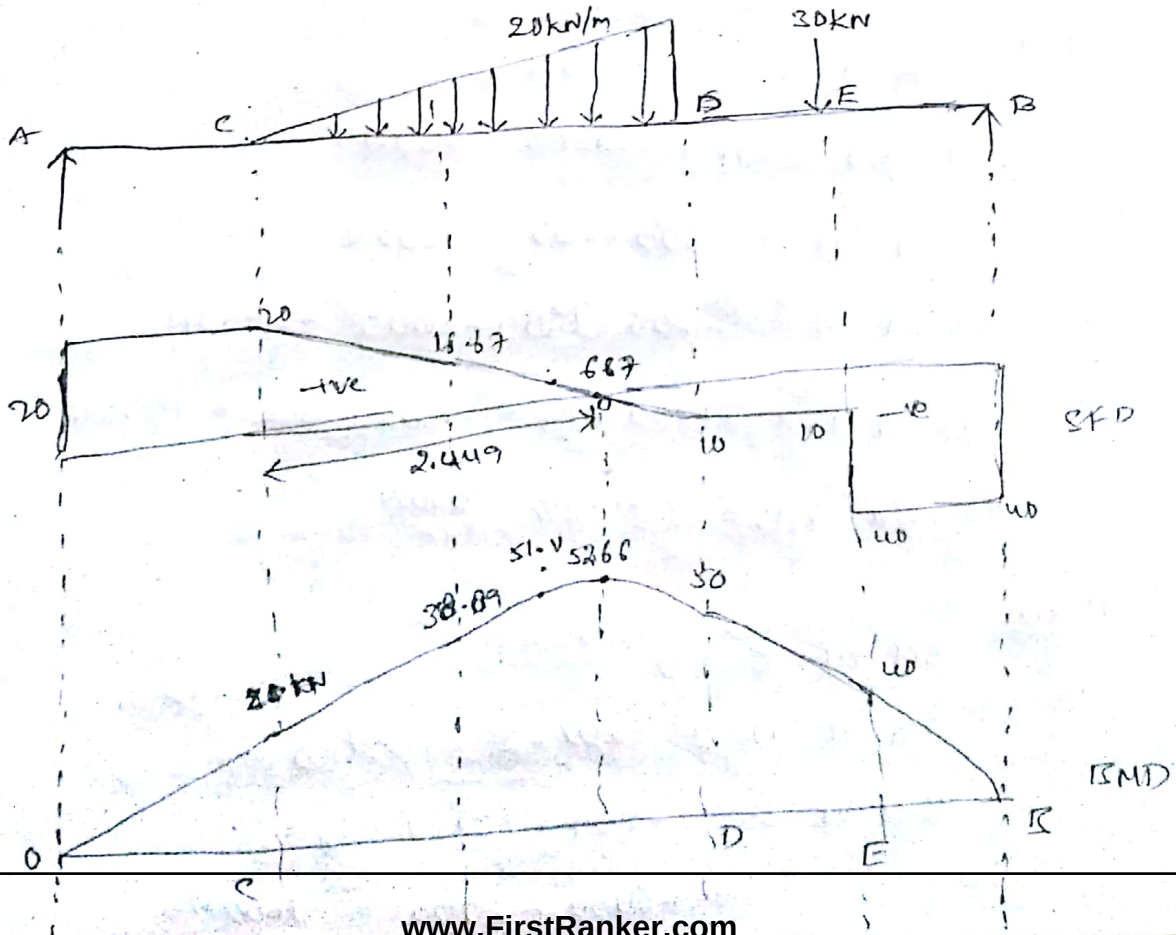
At $x=3$, BMD = $20(1+3) - \frac{10}{9}(3)^3 = 50 \text{ kNm}$

At the section where shear force = 0 $x = 2.449 \text{ m}$

$$\begin{aligned} M_{\text{max}} &= 20(1+2.449) - \frac{10}{9}(2.449)^3 \\ &= 52.66 \text{ kNm} \end{aligned}$$

BMD at F = $40 \times 1 = 40 \text{ kNm}$

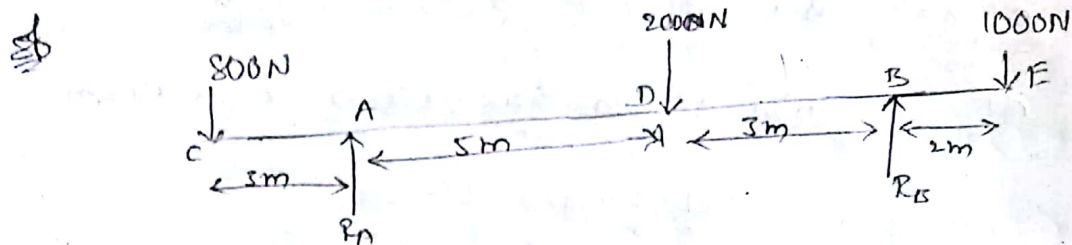
BMD at B = 0.



Over Hanging Beams
If the end portion of a beam is extended beyond the support, such beam is known as over hanging beam.

Point of Contraflexure: It is the point where the B.M is zero after changing its sign from +ve to -ve or vice versa.

Exs. Draw the SFD & BMD, and determine the point of contraflexure for the beam given below.



$$R_A + R_B = 3800 \text{ N}$$

$$R_B \times 8 + 800 \times 3 = 2000 \times 5 + 1000(8+2)$$

$$\boxed{R_B = 2200 \text{ N}}$$

$$R_A = 3800 - 2200 = 1600 \text{ N}$$

SFD

$$\text{SF at C} = -800 \text{ N}$$

$$\text{SF b/w C \& A remains } -800 \text{ N}$$

$$\text{SF at A} = 1600 - 800 = 800 \text{ N}$$

$$\text{SF at D} = -800 + 1600 - 2000 = -1200 \text{ N}$$

$$\text{SF at B} = -1200 + R_B = -1200 + 2200 = 1000 \text{ N}$$

$$\text{SF b/w B \& E remains } 1000 \text{ N}$$

BMD

$$\text{BM at C} = 0$$

$$\text{BM at A} = -800 \times 3 = -2400 \text{ Nm}$$

$$\text{BM at D} = -800 \times (3+5) + R_A \times 5$$

$$= -6400 + 8000 = 1600 \text{ Nm}$$

B.M at E = 0

point of contraflexure:

Two point of contraflexures O_1 & O_2 (A+D, D+B)

(i) let the point O_1 is x m from A

$$\text{Then B.M at } O_1 = -800(3+x) + R_A x \\ = -800(3+x) + 1600x$$

But B.M at $O_1 = 0$

$$0 = -800(3+x) + 1600x$$

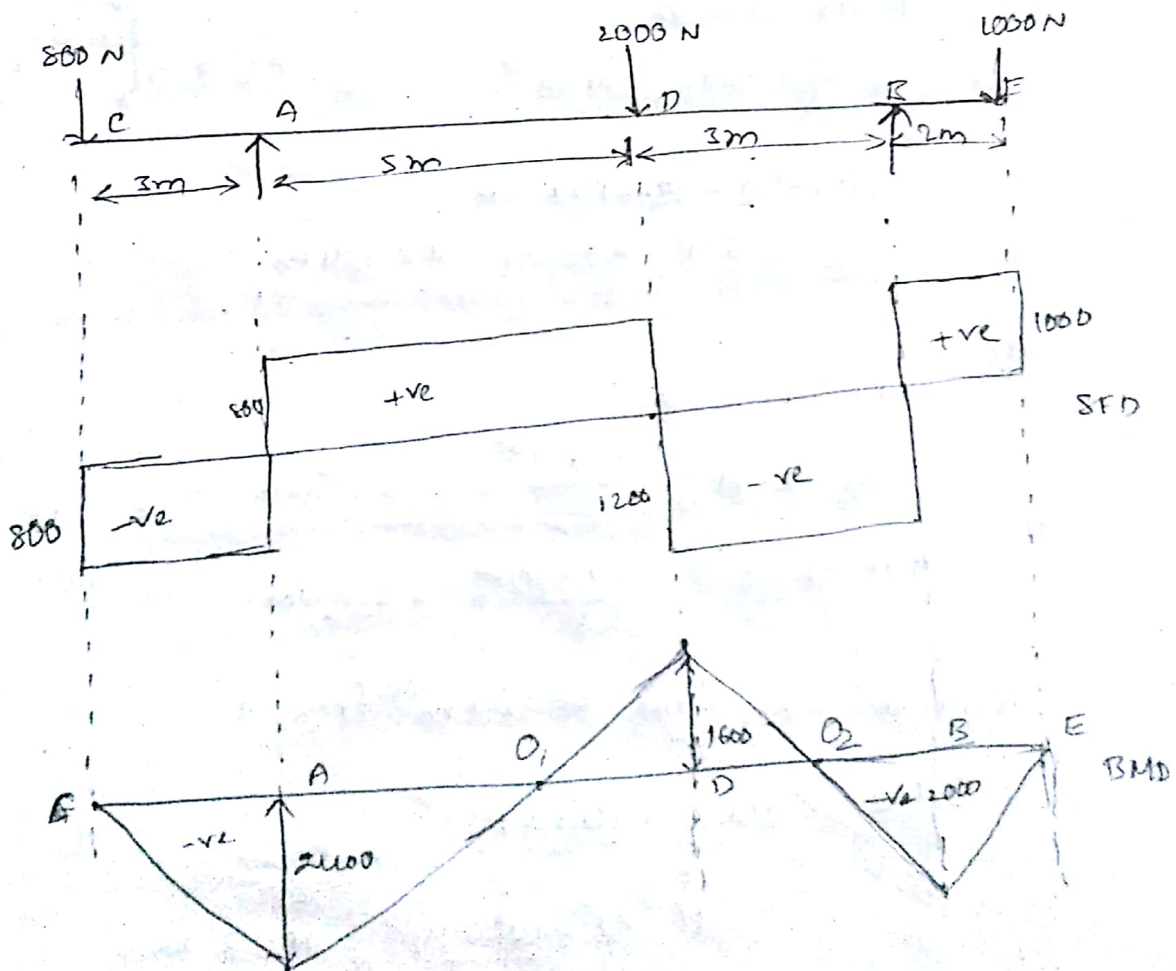
$x = 3$ m. from A point O_2 x m from B.

$$(ii) \text{ B.M at } O_2 = 1000(2+x) - R_B x \\ = 1000(2+x) - 2200x$$

B.M at $O_2 = 0$

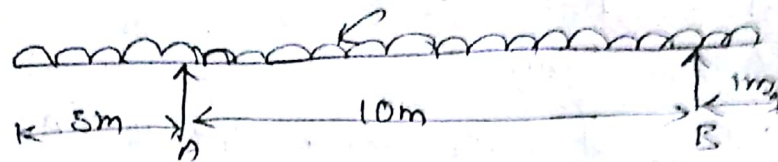
$$1000(2+x) - 2200x = 0$$

$x = 1.67$ m from B



50

100



$$V_B \times 10 = 10 \times 10.5 + 10 \times 5 - 3 \times 10$$

$$V_B = 56 \text{ kN}$$

$$V_A + V_B = 100 \Rightarrow V_A = 84 \text{ kN}$$

SFD

$$\text{SF at C} = 0$$

$$\text{SF at L.H.S of A} = -30$$

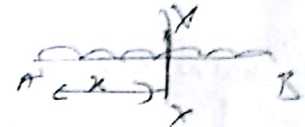
$$\text{SF at R.H.S of A} = 84 - 30 = 54 \text{ kN}$$

$$\text{SF at L.H.S of B} = -30 + 84 - 100 = -46 \text{ kN}$$

$$\text{SF at R.H.S of B} = -30 + 84 - 100 + 56 = 10 \text{ kN}$$

$$\text{SF at D} = 0$$

Section of zero shear



$$84 - 30 - 10x = 0$$

$$54 = 10x \Rightarrow x = 5.4 \text{ m}$$

BMD

$$\text{BM at C} = 0$$

$$\text{BM at A} = \frac{10 \times 3^2}{2} = -45 \text{ kNm}$$

$$\text{BM at B} = \frac{-10 \times 10^2}{2} = -5 \text{ kNm}$$

B.M at zero shear of 5.4 m from A

$$B_{\text{Max}} = 84x - \frac{10(x+3)^2}{2} = 0$$

$$= 84(5.4) - \frac{10(8.4)^2}{2} = 100.8 \text{ kNm}$$

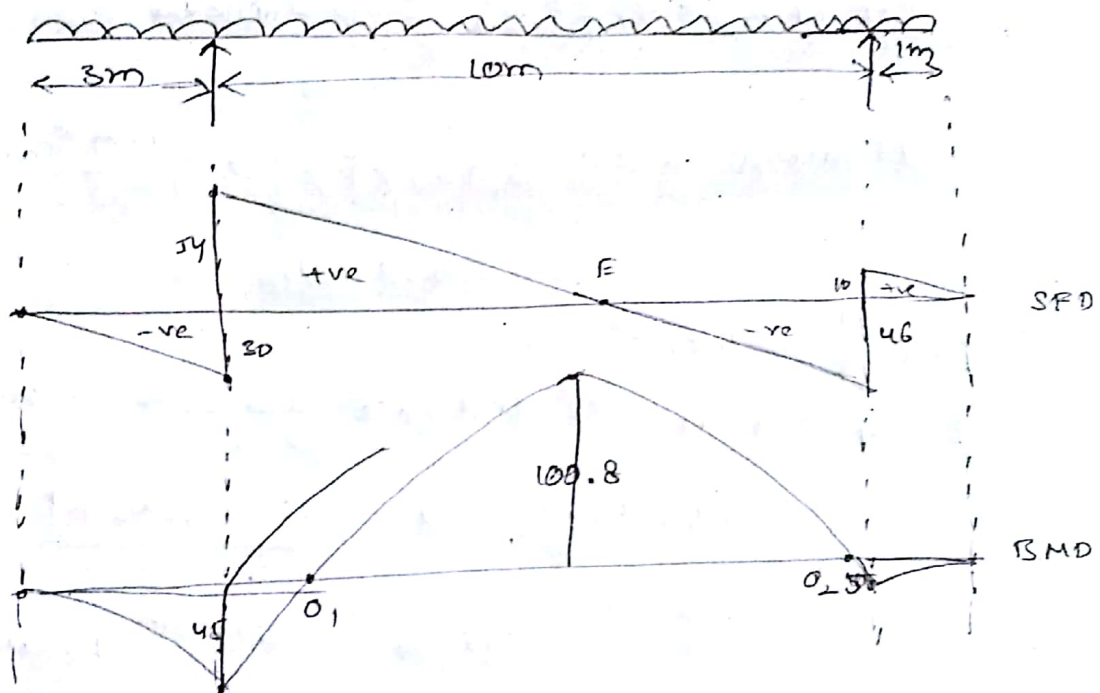
$$\frac{8x - 10(x+3)}{2} = 0$$

$$x^2 - 10.8x + 9 = 0 \quad \text{by solving}$$

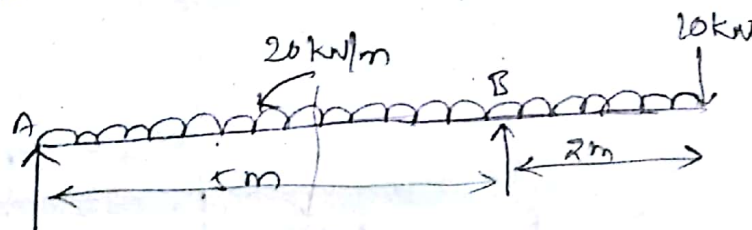
$$x = 0.91 \text{ m and } x = 9.89 \text{ m}$$

Distance of point of Contraflexure O_1 from A = 0.91 m

$$O_2 \text{ from B} = 10 - 9.89 = 0.11 \text{ m}$$



Ex: Draw SFD + BMD and locate the POC.



$$V_B \times 5 = 20 \times 7 \times \frac{7}{2} + 10 \times 7$$

$$\boxed{V_B = 112 \text{ kN}} \quad \boxed{V_A = 38 \text{ kN}}$$

SFD SF at A = 38 kN

$$\text{SF at L.H.S of B} = 38 - 100 = -62 \text{ kN}$$

$$\text{SF at R.H.S of B} = 38 - 100 + 112 = 50 \text{ kN}$$

SF just L.H.S. of $C = 10 \text{ kN}$

Section $\frac{1}{2}$ of $A+B$

$$38 - 20x = 0 \Rightarrow x = 1.9 \text{ m}$$

BMD

B.M at $A = 0$

B.M at $C = 0$

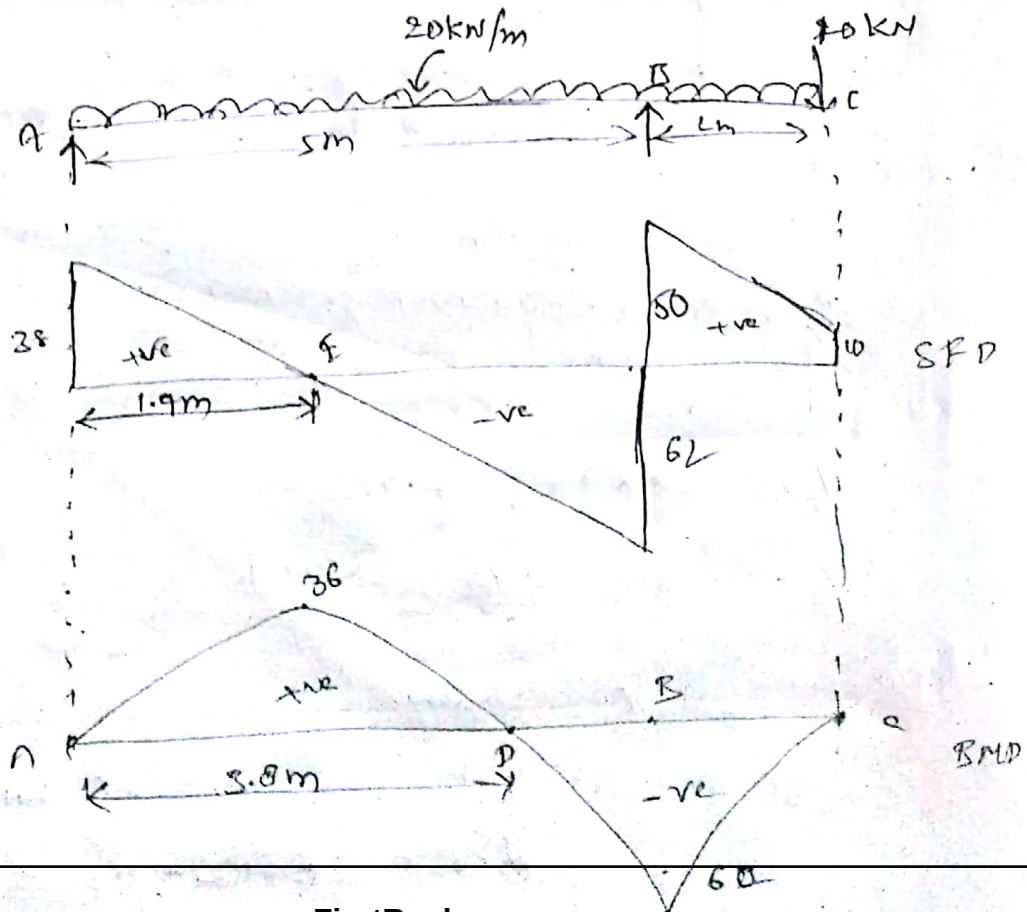
$$\text{B.M at } B = -10 \times 2 - 20 \times \frac{2^2}{2} = -60 \text{ kNm}$$

$$\begin{aligned} \text{Maximum B.M} &= 38 \times 1.9 - 20 \times \frac{1.9^2}{2} \\ &= 36.20 \text{ kNm} \end{aligned}$$

Point of Contraflexure

Let the B.M at the distance $x \text{ m}$ from A be zero

$$38x - 20 \frac{x^2}{2} = 0 \Rightarrow x = 3.80 \text{ m}$$



SFD & BMD for beams subjected to couples
When a beam is subjected to a couple at a section, only the bending moment at the section of the couple changes ~~suddenly~~ suddenly in magnitude equal to that of the couple as there is no change in load due to couple at the section. But while calculating the reactions, the magnitude of the couple is considered.

Ex: Draw SFD & BMD for the beam given



Taking moments about A.

(i) $10 \times 5 \times \frac{5}{2} = 12500 \text{ Nm}$ (clockwise)

(ii) Couple = 15000 Nm (anticlockwise)

Net moment = 2500 Nm (anticlockwise)

$V_6 \times 10 = 25 \text{ kNm} \Rightarrow V_6 = 2.5 \text{ kN}$ (upwards)

$V_6 = 10 \times 5 + 2.5 = 52.5 \text{ kN}$ (upwards)

The moment must be balanced by the moments due to reaction at B. Hence the moment about A due to reaction at B should be equal to 2500 Nm (clockwise), this only possible when R_B is acting downwards

$R_B \times 10 = 2500$

$R_B = 250 \text{ N}$

$R_A = \text{Total load on beam} + R_B$

$= 1000 \times 5 + 250$

$= 5250 \text{ N}$ ($\because R_B$ is +ve as acting downwards)

At $A = R_A = 5250 \text{ N}$

SF at $C = 5250 - 5 \times 1000 = 250 \text{ N}$

SF b/w B & C remains const and equal 250 N

BMD

BM at $A = 0$

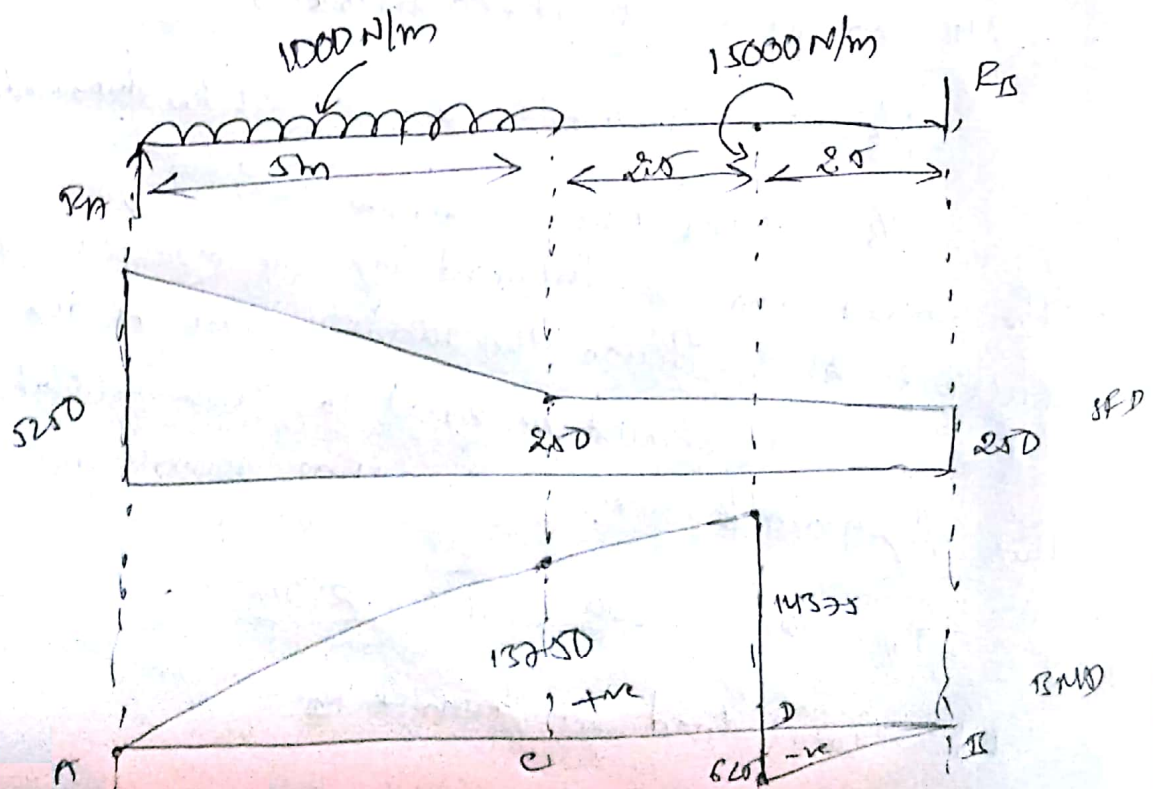
BM at $C = R_A \times 5 - 1000 \times 5 \times \frac{5}{2}$
 $= 13750 \text{ Nm}$

~~BM just L.H.S of $D = 5250 \times 7.5 - 1000 \times 5 \times \frac{5}{2}$
 $- 1000 \times 2.5 \times \frac{(2.5)^2}{2}$~~

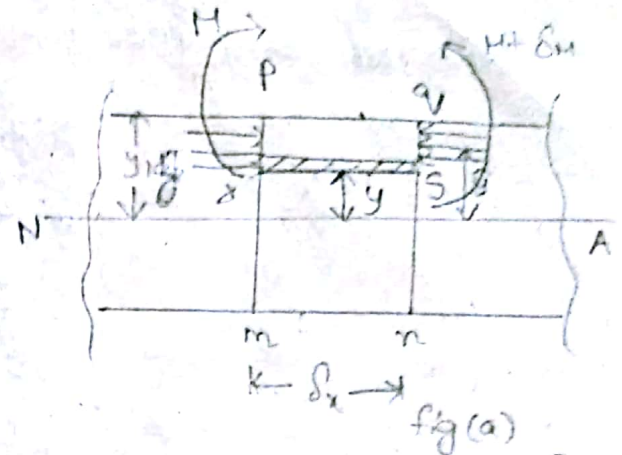
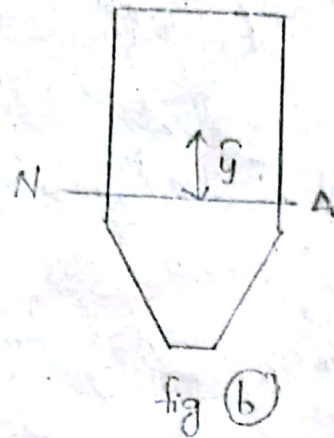
BM just L.H.S of $D = 5250 \times 7.5 - 1000 \times 5 \left(\frac{5}{2} + 2.5 \right)$
 $= 14375 \text{ Nm}$

BM just R.H.S of $D = -R_B \times 2.5$
 $= -250 \times 2.5 = -625 \text{ Nm}$

BM at $B = 0$



THE END



let us consider a short slice of the beam of length δx with a variation of bending moment over its length from M to $M + \delta M$. Take a layer of δy at a distance y from the neutral axis. let the width of this layer be ' b ' then. and its area $\delta x b$ between the successive faces of the slice there will be an excess forces. since the stress at the $p-x$ will be less than that of the face $q-y$. This excess force will be balanced by the shearing stress/acting along the layer δy .

$\therefore$ The total force on the face ($p-x$)

$$P = \sigma \times A$$

$$= \frac{M}{I} \times \delta y \times b$$

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$\sigma = \frac{My}{I}$$

$\therefore$ The total force = $\int_y \frac{My}{I} b dy$

$$\int_y \frac{M + \delta M y}{I} b dy$$

The Excess force between sq $\phi \delta p$

$$= \int_y \frac{M y}{I} b dy - \int_y \frac{M y}{I} b dy$$

$$= \int_y \frac{\delta M y}{I} b dy$$

But $\int_y y b dy$ is moment of area of the face $p \delta$ about the neutral axis

$$\int_y y b dy = A \bar{y}$$

$\therefore$ This excess force is balanced by a shearing stress τ acting along $x \delta$. The force due to this stress is $\tau x (b \times \delta x)$

$$\therefore \tau x b \times \delta x = \frac{\delta M}{I} \cdot A \bar{y}$$

$$\tau = \left(\frac{\delta M}{\delta x} \right) \cdot \frac{A \bar{y}}{I b}$$

$$\therefore \frac{\delta M}{\delta x} = \text{shear force} = S$$

$$\therefore \boxed{\tau = \frac{S}{I b} \cdot A \bar{y}}$$

* Shear stress variation in rectangular beams:

Let 'b' be the width and 'd' the depth of rectangular section. Let τ be the shear stress at a layer. at a distance 'y' from neutral axis.

$$\tau = \frac{S}{Ib} (Ay)$$

$$I = \frac{bd^3}{12} \quad \dots b=b$$

$$A = b \times \left(\frac{d}{2} - y\right)$$

$$\bar{y} = y + \frac{\left(\frac{d}{2} - y\right)}{2}$$

$$= \frac{2y + \frac{d}{2} - y}{2}$$

$$= \frac{\frac{d}{2} + y}{2}$$

$$= \frac{1}{2} \left(\frac{d}{2} + y\right)$$

$$\therefore \tau = \frac{S}{\frac{bd^3}{12} \times b} \times b \left(\frac{d}{2} - y\right) \frac{1}{2} \left(\frac{d}{2} + y\right)$$

$$= \frac{6S}{bd^3} \left(\frac{d}{2} - y\right) \left(\frac{d}{2} + y\right)$$

$$\boxed{\tau = \frac{6S}{bd^3} \left(\frac{d^2}{4} - y^2\right)}$$

$$\tau = \frac{6S}{bd^3} \times \frac{d^2}{4}$$

($\because$ Neutral axis)

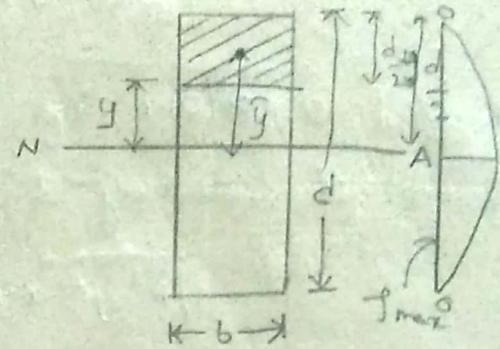
$$= \frac{3S}{2bd}$$

$$\tau_{max} = \frac{3}{2} \left(\frac{S}{bd}\right)$$

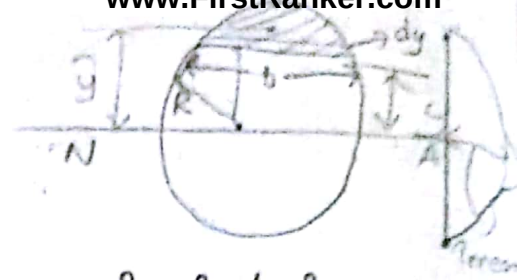
$$= \frac{3}{2} (\tau_{mean})$$

* shear stress variation in circular solid circular section:

A solid circular section of radius 'R' is shown in fig. Consider an elementary strip of thickness 'dy' at a distance 'y' from neutral axis.



$$I = \frac{S}{Ib} (\Delta y)$$



Area of elementary ring = $b \times dy$

Moment of elementary strip = $(b \times dy) y$ $R^2 = y^2 + \left(\frac{b}{2}\right)^2$

$$= 2 \sqrt{R^2 - y^2} \times y \times dy$$

$$\frac{b}{2} = \sqrt{R^2 - y^2}$$

$$b = 2 \sqrt{R^2 - y^2}$$

∴ Moment of total area R

$$A\bar{y} = \int_y 2 \sqrt{R^2 - y^2} \times y \times dy$$

$$= \int_y^R b \times y \times dy = \int_b^0 b \times \frac{-b}{4} db$$

$$\therefore b = 2 \sqrt{R^2 - y^2}$$

$$b^2 = 4 (R^2 - y^2)$$

Diff.

$$2b \times db = 2(-2y \times dy)$$

$$b \times db = -4y \times dy$$

$$y \times dy = \frac{-b}{4} db$$

$$y = y ; b = b$$

$$y = R ; b = 0$$

$$= - \int_b^0 \frac{b^2}{4} db$$

$$= \int_0^b \frac{b^2}{4} db$$

$$= \frac{1}{4} \left(\frac{b^3}{3} \right)_0^b$$

$$= \frac{b^3}{12}$$

$$= \frac{4S}{\pi R^4} \times \frac{636^2}{123}$$

$$= \frac{S}{3\pi R^4} \times 4(R^2 - y^2)$$

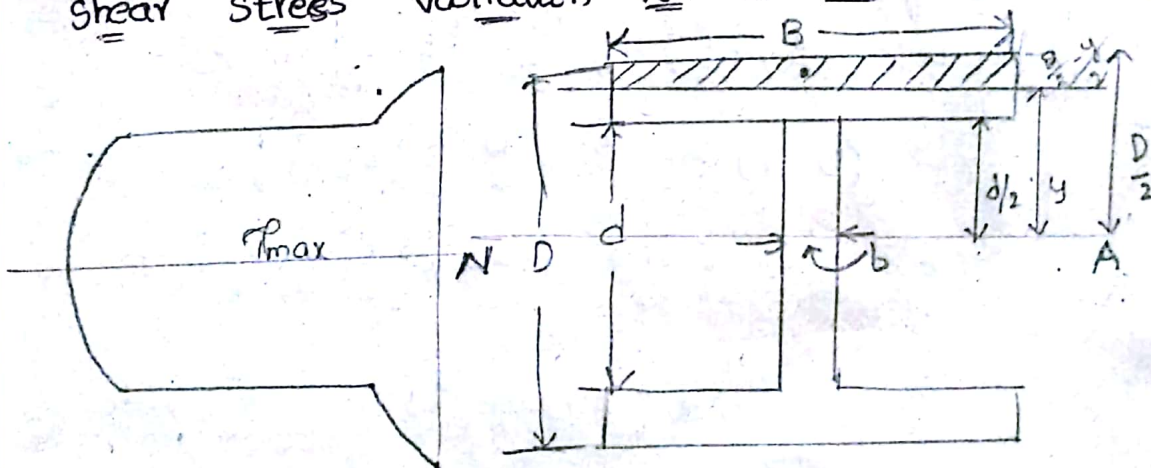
$$= \frac{4}{3} \left(\frac{S}{\pi R^4} \right) (R^2 - y^2)$$

$$\therefore \tau_{\max} = \frac{4}{3} \left(\frac{S}{\pi R^4} \right) (R^2)$$

$$= \frac{4}{3} \left(\frac{S}{\pi R^2} \right)$$

$$= \frac{4}{3} \tau_{\text{mean}}$$

Shear stress variation for I-section:-



Shear stress variation in top flange:-

$$\tau = \frac{S}{I_b} (A \bar{y})$$

$$A = B \times \left(\frac{D}{2} - y \right)$$

$$\bar{y} = y + \frac{\left(\frac{D}{2} - y \right)}{2}$$

$$= y + \frac{D}{4} + \frac{y}{2}$$

$$= \frac{D}{4} + \frac{y}{2} = \frac{1}{2} \left(\frac{D}{2} + y \right)$$

$$= \frac{S}{I} \left(b \times \left(\frac{D}{2} - y \right) \times \frac{1}{2} \left(\frac{D}{2} + y \right) \right)$$

$$\tau = \frac{S}{2I} (D^2 - y^2)$$

At $y = \frac{D}{2}$: Top flange (or) top fibre

$$\therefore \tau = \frac{S}{2I} \left(\frac{D^2}{4} - \frac{D^2}{4} \right)$$

$$= 0$$

Shear stress distribution in web:

$$\tau = \frac{S}{Ib} (A \bar{y})$$

$$A = b \left(\frac{D}{2} - \frac{d}{2} \right) \times \left(\frac{d}{2} + \frac{\left(\frac{D}{2} - \frac{d}{2} \right)}{2} \right) = b \times \left(\frac{d}{2} - y \right) \times \left(y + \frac{\left(\frac{d}{2} - y \right)}{2} \right)$$

$$= \frac{b}{2} (D - d) \times \frac{1}{2} \left(d + \frac{D}{2} - \frac{d}{2} \right) + b \times \left(\frac{d}{2} - y \right) \times \frac{\left(2y + \frac{d}{2} - y \right)}{2}$$

$$A \bar{y} = \frac{b}{4} (D - d) \left(\frac{D + d}{2} \right) + \frac{b}{2} \times \left(\frac{d}{2} - y \right) \left(\frac{d}{2} + y \right)$$

$$= \frac{b}{8} (D^2 - d^2) + \frac{b}{2} \left(\frac{d^2}{4} - y^2 \right)$$

$$\therefore \tau = \frac{S}{Ib} (A \bar{y})$$

$$= \frac{S}{Ib} \left[\frac{b}{8} (D^2 - d^2) + \frac{b}{2} \left(\frac{d^2}{4} - y^2 \right) \right]$$

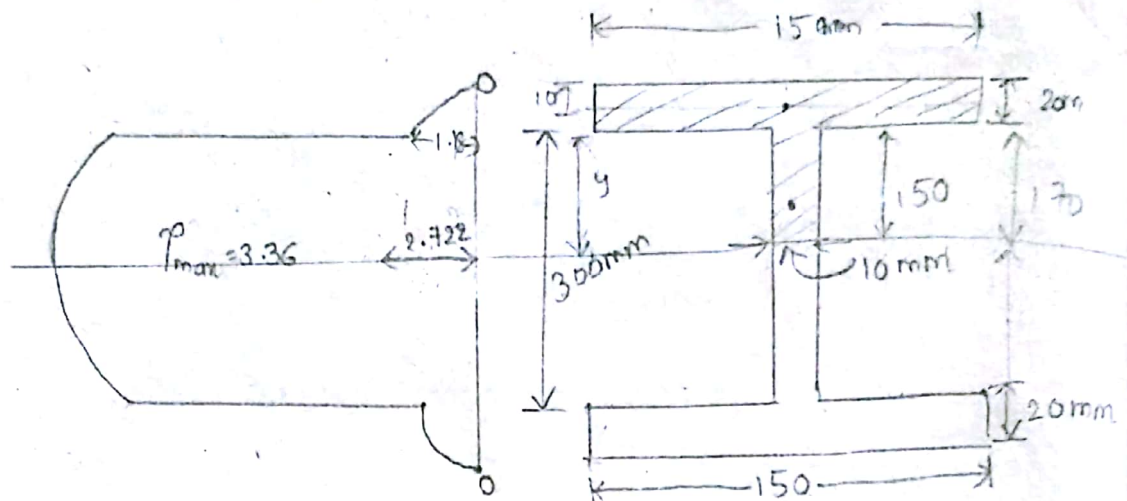
$\therefore$ Max shear stress developed at the Neutral axis

$$y = 0$$

$$\tau = \frac{S}{Ib} \left[\frac{b}{8} (D^2 - d^2) + \frac{bd^2}{8} \right]$$

$$\tau_{\max} = \frac{S}{8Ib} [b(D^2 - d^2) + bd^2]$$

* An I-section with rectangular ends as the following dimensions. Flanges 15cm x 2cm ; web = 30cm x 1cm . find the max. shearing stress develop in the beam for a shearing force of 10 kN. Sketch the shear stress distribution.



Given data:-

Shearing stress $S = 10 \times 10^3 \text{ N}$

Moment of Inertia $I = \frac{BD^3}{12} - \frac{bd^3}{12}$

$$= \frac{150 \times 340^3}{12} - \frac{(150-10) (300)^3}{12}$$

$$= 491.3 \times 10^6 - 315 \times 10^6$$

$$= 176.3 \times 10^6 \text{ mm}^4$$

Max. Shear stress developed at the neutral-axis

$$\therefore y = 0$$

$$\therefore \tau = \frac{S}{Ib} (A\bar{y})$$

$$A\bar{y} = (150 \times 20)(170-10) + (10 \times 150) \left(\frac{150}{2}\right)$$

$$= 592.5 \times 10^3 \text{ mm}^3$$

$$= 3.36 \text{ N/mm}^2$$

$$\tau = \frac{S}{I_b} [B(D^2 - d^2) + bd^2] \Rightarrow \frac{10 \times 10^3}{8 \times 176.3 \times 10} [150(340^2 - 300^2) + 10(300^2)]$$

$$= 3.36 \text{ N/mm}^2$$

* shear stress distribution in top flange:-

$$\tau = \frac{S}{I_b} (A\bar{y})$$

$$A\bar{y} = (150 \times (170 - y)) \times \left(y + \frac{(170 - y)}{2}\right)$$

$$\tau = \frac{10 \times 10^3}{176.3 \times 10^6 \times 150} (150(170 - y)) \times \left(\frac{2y + 170 - y}{2}\right)$$

$$\tau_{y=170} = 0$$

$$\tau_{y=150} = \frac{10 \times 10^3}{176.3 \times 10^6 \times 150} (150(170 - 150)) \times \left(\frac{2(150) + 170 - 150}{2}\right)$$

$$= 0.18 \text{ N/mm}^2$$

* Shear stress distribution in web:-

$$\tau = \frac{S}{I_b} (A\bar{y})$$

$$A\bar{y} = (10(150 - y)) \times \left(y + \frac{(150 - y)}{2}\right) + (150 \times 20)(60)$$

$$\tau = \frac{10 \times 10^3}{176.3 \times 10^6 \times 10} ((480 \times 10^3) + 10(150 - y) \left(\frac{2y + 150 - y}{2}\right))$$

$$\tau_{y=0} = 3.36$$

$$\tau_{y=150} = 2.722$$

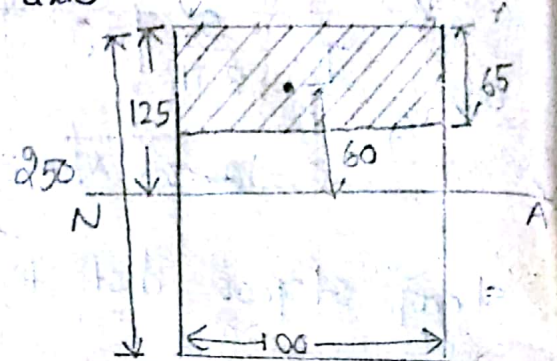
A rectangular beam 100 mm wide & 250 mm deep is subjected to a shear force of 50 kN. Determine average shear stress, Max. shear stress, shear stress at a distance 65 mm from the neutral axis.

Given data:

$$\text{Shear force } S = 50 \text{ kN} \\ = 50 \times 10^3 \text{ N}$$

$$\text{width } b = 100 \text{ mm}$$

$$\text{depth } d = 250 \text{ mm}$$



$$(i) \text{ Average shear stress } = \frac{S}{bd} = \frac{50 \times 10^3}{100 \times 250} = 2 \text{ N/mm}^2$$

$$(ii) \text{ Max. shear stress } \tau_{\max} = \frac{3}{2} \tau_{\text{mean}} = \frac{3}{2} \times 2 = 3 \text{ N/mm}^2$$

(iii) Shear stress at a distance 65 mm from the neutral axis.

$$I = \frac{bd^3}{12} = \frac{100 \times 250^3}{12} = 130.208 \times 10^6 \text{ mm}^4$$

$$A\bar{y} = 100 \times (125 - 65) = 6000 \text{ mm}^2$$

$$\bar{y} = 65 + \frac{60}{2} = 95$$

$$\tau = \frac{50 \times 10^3}{130.208 \times 10^6 \times 100} (6000 \times 95)$$

$$= 2.188 \text{ N/mm}^2$$

A timber beam of rectangular section is S.S at the ends and carries a point load at the center of the beam the max. bending stress is 12 N/mm^2 and Max. shearing stress 1 N/mm^2 . Find the ratio of the span to the depth.

Given data:

Max. bending moment in case of S.S.B with point load at the center.

$$M_{\max} = \frac{\omega l}{4}$$

$$\text{Max. shear force } S = \frac{\omega}{2}$$

Let,
 b = breadth of beam
 d = depth of beam
 l = span of beam.

(i) Max. shear stress $\tau_{\max} = 1 \text{ N/mm}^2$

$$\tau_{\max} = \frac{3}{2} \tau_{\text{mean}}$$

$$= \frac{3}{2} \cdot \frac{S}{bd}$$

$$= \frac{3\omega}{4 \times bd}$$

$$= \frac{3\omega}{4bd}$$

$$\therefore 1 = \frac{3\omega}{4bd}$$

$$\frac{4}{3} = \frac{\omega}{bd}$$

(ii) Max. bending stress $\sigma_{\max} = 12 \text{ N/mm}^2$ using bending Equation

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$\begin{aligned}\sigma_{\max} &= \frac{M_{\max} \cdot y}{I} \\ &= \frac{\frac{wl}{4} \cdot \frac{d}{2}}{\frac{bd^3}{12}} \\ &= \frac{\frac{wl}{2}}{\frac{bd^3}{12}} \\ &= \frac{3wl}{2bd^2}\end{aligned}$$

$$\therefore 12 = \frac{3wl}{2bd^2}$$

$$\frac{l}{d} = \frac{12 \times 2 \times bd}{3w}$$

$$\frac{l}{d} = \frac{8}{w/bd}$$

$$\therefore \frac{w}{bd} = \frac{4}{3}$$

$$= \frac{8}{4/3}$$

$$= 8 \times \frac{3}{4}$$

$$\therefore \frac{l}{d} = 6$$

$\therefore$ Ratio of span to the depth = 6

A simply supported beam of span 1.3 m having a c/s 150 mm wide 250 mm deep carries a point load w at its center. The permissible stresses are 7 N/mm^2 in bending and 1 N/mm^2 in shearing. Calculate safe load.

Given data:

S.S.B with point load max. bending moment $wl/4$

$$\text{Max. shear force} = \frac{w}{2}$$

$$\text{Bending stress } \sigma_{\max} = 7 \text{ N/mm}^2$$

$$\text{Shearing stress } \tau_{\max} = 1 \text{ N/mm}^2$$

$$\text{width } b = 150 \text{ mm}$$

$$\text{depth } d = 250 \text{ mm}$$

$$I = \frac{bd^3}{12} = \frac{150 \times 250^3}{12} = 195.312 \times 10^6 \text{ mm}^4$$

Let point load at the center = $w \text{ kN}$

Max. safe load considering flexural stress

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$\text{Here } M = \frac{wl}{4} = \frac{w \times 1.3}{4} = 0.325 \text{ kN-m}$$

$$\frac{0.325w}{195.312 \times 10^6} = \frac{7}{125}$$

$$\frac{0.325w \times 125}{195.312 \times 10^6 \times 7} = w$$

$$w = \frac{7 \times 195.312 \times 10^6}{125 \times 0.325 \times 10^6}$$

$$\tau_{\max} = 1 \text{ N/mm}^2$$

$$\tau_{\max} = \frac{3}{2} \tau_{\text{mean}}$$

$$= \frac{3}{2} \frac{S}{bd}$$

$$= \frac{3}{2} \times \frac{\frac{w}{2}}{150 \times 250}$$

$$= \frac{3}{2} \times \frac{w \times 10^3}{2 \times 150 \times 250}$$

$$= \frac{3w \times 10^3}{4 \times 150 \times 250}$$

$$1 = 0.02w$$

$$1 = 0.02w \text{ N/mm}^2$$

$$\frac{1}{0.02} = w$$

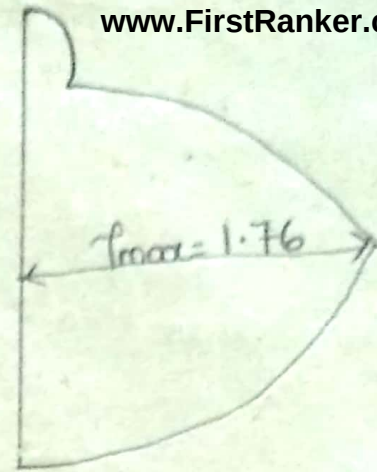
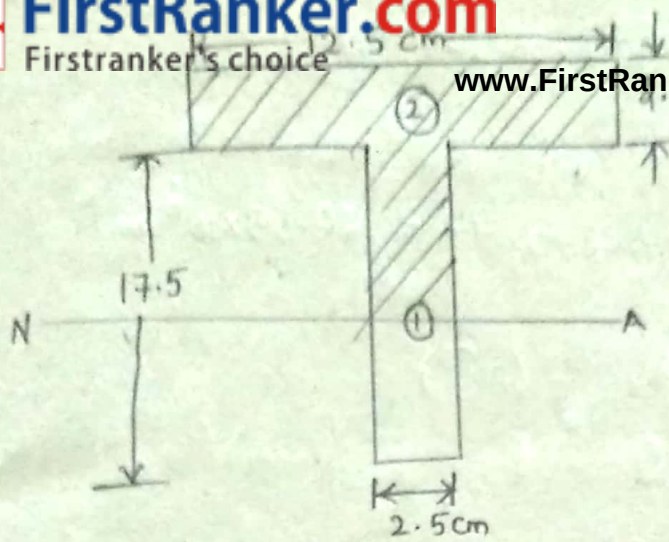
$$50 = w$$

$$\therefore w = 50 \text{ kN}$$

$\therefore$ Max. safe load = Minimum of above 2

$\therefore$ The safe load = 33.67 kN.

* A S.S.B carries a u.d.l of intensity 2.5 kN/m over entire span of 5m. The c/s of the beam is a I-section having the dimensions as shown in figure. Calculate the Max. shear stresses for the section of the beam.



Given data:-

Let $\bar{y}$ = center of gravity of I-section from the bottom

$$\bar{y} = \frac{A_1 y_1 + A_2 y_2}{A_1 + A_2}$$

Here A_1 = area of web ; A_2 = Area of flange

$$A_1 = 25 \times 175 = 4375 \text{ mm}^2$$

$$A_2 = 125 \times 25 = 3125 \text{ mm}^2$$

y_1 = centre of gravity of web from the bottom

y_2 = centre of gravity of top flange from the bottom

$$y_1 = \frac{175}{2} = 87.5 \text{ mm}$$

$$y_2 = 175 + \frac{25}{2} = 187.5 \text{ mm}$$

$$\therefore \bar{y} = \frac{4375 \times 87.5 + 3125 \times 187.5}{4375 + 3125} = 129.16 \text{ mm}$$

$\therefore$ Moment of Inertia $I = I_1 + I_2$

$$I_1 = \frac{bd^3}{12} + \sum A_i (y_i - \bar{y})^2$$

$$= \frac{25 \times 175^3}{12} + 4375 (87.5 - 129.16)^2$$

$$18.76 \times 10^6 \text{ mm}^4$$

$$= 10.79 \times 10^6 \text{ mm}^4$$

$$\therefore I = I_1 + I_2 = 18.76 + 10.79 = 29.55 \times 10^6 \text{ mm}^4$$

$$\therefore \text{Max. shear stress } \tau = \frac{S}{Ib} (A\bar{y})$$

$$\text{Here } S = \frac{2.5 \times 5}{2} = 6.25 \text{ kN}$$

$$(\because \text{In S.S.B udl over the span } S = \frac{wl}{2})$$

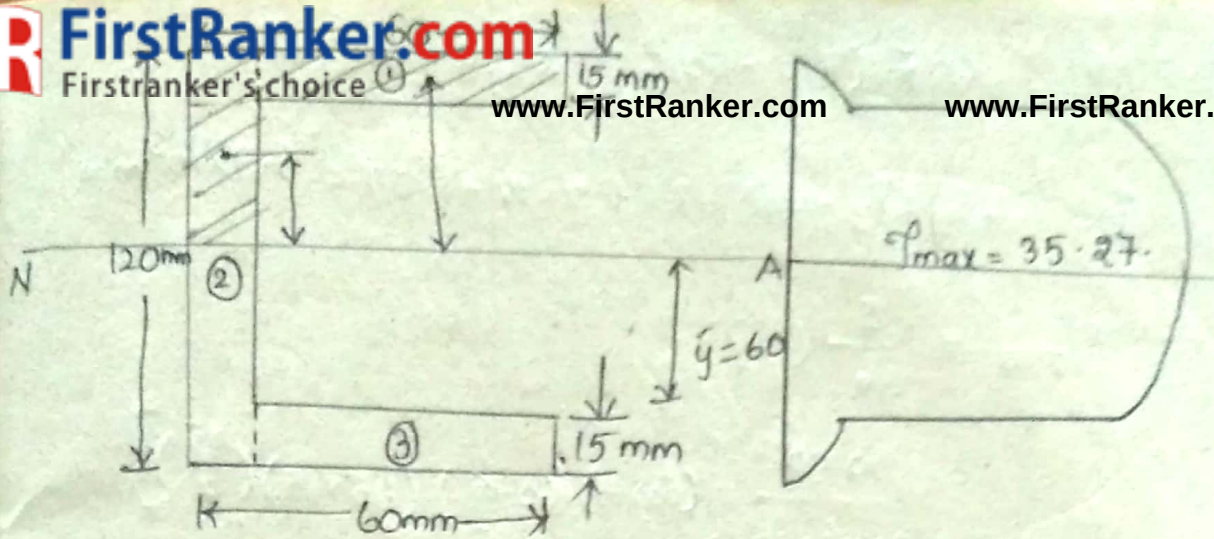
$$A\bar{y} = (125 \times 25) \left(175 - 129.16 + 25 - \frac{25}{2} \right) + 25 \left(175 - 129.16 \right) \left(\frac{175 - 129.16}{2} \right)$$

$$= 208.5 \times 10^3 \text{ mm}^3$$

$$\therefore \tau = \frac{6.25 \times 10^3 \times 208.5 \times 10^3}{29.55 \times 10^6 \times 25}$$

$$\tau = \boxed{1.76 \text{ N/mm}^2}$$

* A beam of channel-section 120 mm x 60 mm. as uniform thickness of 15 mm. Draw diagram showing the distribution of shear-stress for a vertical section where shear force is 50 kN. find the ratio b/w max & mean shear stress.



$$\bar{y} = \frac{A_1 y_1 + A_2 y_2 + A_3 y_3}{A_1 + A_2 + A_3}$$

$\therefore$ Area of $A_1 = (60-15) \times 15 = 675 \text{ mm}^2$

Area of $A_2 = (120) \times 15 = 1800 \text{ mm}^2$

Area of $A_3 = (60-15) \times 15 = 675 \text{ mm}^2$

$$\therefore y_1 = (120 - \frac{15}{2}) = 112.5$$

$$y_2 = \frac{120}{2} = 60$$

$$y_3 = \frac{15}{2} = 7.5$$

$$\therefore \bar{y} = \frac{675 \times 112.5 + 1800 \times 60 + 675 \times 7.5}{675 + 1800 + 675} = 60 \text{ mm}$$

$\therefore$ Moment of Inertia $I = I_1 + I_2 + I_3$

$$I_1 = \frac{bd^3}{12} + A (y_1 - \bar{y})^2$$

$$= \frac{45 \times 15^3}{12} + 675 (112.5 - 60)^2 = 1.87 \times 10^6 \text{ mm}^4$$

$$I_2 = \frac{15 \times 120^3}{12} + 1800 (60 - 60)^2 = 2.16 \times 10^6 \text{ mm}^4$$

$$I_3 = \frac{45 \times 15^3}{12} + 675(7.5 - 60)^2 = 1.87 \times 10^6 \text{ mm}^4$$

$$\therefore I = I_1 + I_2 + I_3 \Rightarrow 1.87 + 2.16 + 1.87 = 5.9 \times 10^6 \text{ mm}^4$$

$$\therefore \tau_{\max} = \frac{S}{Ib} (Ay)$$

$$= \frac{50 \times 10^3}{5.9 \times 10^6 \times 15} \left[(60 \times 15) \left(60 - \frac{15}{2}\right) (45 \times 15) \left(\frac{45}{2}\right) \right]$$

$$\tau_{\max} = 35.27 \text{ N/mm}^2$$

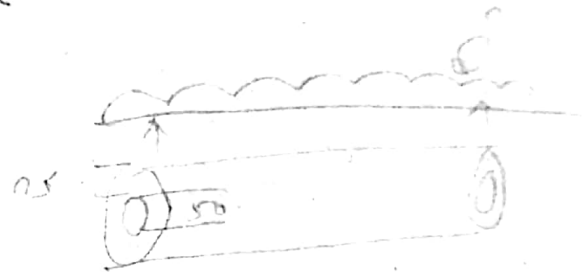
$$\therefore \tau_{\text{mean}} = \frac{S}{A} = \frac{50 \times 10^3}{3150} = 15.87 \text{ N/mm}^2$$

$$\therefore \text{Ratio b/w } \tau_{\max} \text{ \& } \tau_{\text{mean}} = \frac{35.27}{15.87} = 2.22$$

* A cast-Iron bracket subjected to bending as a c/s of I-section with unequal flanges as shown fig. If the max. compressive stress is not to exceed 20 MN/m^2 . what is the bending moment the section can take. If the section is subjected to shear force of 80 kN . Draw the shear stress distribution over the depth of the section.

5. A cast iron water main 12 m long of 500 mm inside diameter and 85 mm wall thickness runs full of water and is supported at ends. calculate the max. stress in the metal if density of cast iron is 7200 kg/m^3 and that of water is 1000 kg/m^3 .

$$\text{wt. of water} = \rho \times$$



$$\rho = 1000 \text{ kg/m}^3$$

$$\text{wt. of water} = \rho \times \frac{\pi}{4} \times d^2 \times 9.81 \Rightarrow \text{N/m}$$

$$\text{wt. of metal} = \frac{\pi}{4} d^2 - \frac{\pi}{4} (D)$$

$$= \frac{\pi}{4} D^2 - d^2 \Rightarrow \text{N/m}$$

is bent into a circular arc of radius 10m. Determine the maximum stress induced and the max. bending moment. Take $E = 2 \times 10^5 \text{ N/mm}^2$

Given data:- Steel plate $b = 120 \text{ mm}$
 thickness $t = 20 \text{ mm}$

Radius of curvature $R = 10 \text{ m} \Rightarrow 10 \times 10^3 \text{ mm}$

$E = 2 \times 10^5 \text{ N/mm}^2$

Young modulus.

$\sigma_{\max} = ?$; $M_{\max} = ?$

$$\frac{M}{I} = \frac{E}{R}$$

$$I = \frac{bd^3}{12} = \frac{120 \times 20^3}{12} = 80,000 \text{ mm}^4$$

$$\frac{M}{80,000} = \frac{2 \times 10^5}{10 \times 10^3}$$

$$M = \frac{2 \times 10^5 \times 80 \times 10^3}{10 \times 10^3}$$

$$M = 16 \times 10^5 \text{ N-mm}$$

$$= 1.6 \times 10^6 \text{ N-mm}$$

$$M = 1.6 \text{ kN-m}$$

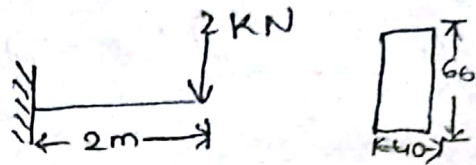
$$\frac{M}{I} = \frac{\sigma}{y}$$

$$\frac{1.6 \times 10^6}{80,000} = \frac{\sigma}{y} = \frac{\sigma}{10} = 20$$

$$\sigma = 200 \text{ N/mm}^2$$

A cantilever of length 2 mts. fails when a load of 2 kN is applied at the free end. The section of the beam is 40 mm x 60 mm. Find the stress at the failure.

Given Data:-



cantilever beam span $(l) = 2\text{m} \rightarrow 2000\text{ mm}$

$(b) = 40\text{ mm}$

$(d) = 60\text{ mm}$

$P = 2\text{ kN} \rightarrow 2 \times 10^3\text{ N}$

Max. Bending moment $M = Pl$
 $= 2 \times 10^3 \times 2000$
 $= 4 \times 10^6\text{ N-mm}$

$$\therefore \frac{M}{I} = \frac{\sigma}{y}$$

$$\frac{4 \times 10^6}{720 \times 10^3} \cdot 30 = \sigma$$

$$\sigma = 166.67\text{ N/mm}^2$$

$$I = \frac{bd^3}{12}$$

$$= \frac{40 \times 60^3}{12}$$

$$= 720 \times 10^3$$

A square beam 20 mm x 20 mm in section and 3 m long is supported at the ends. The beam fails when a point load of 400 N applied at the center of the beam, what uniformly distributed load per length will break a cantilever of the same material. 40 mm wide 60 mm deep and 3 mts long.

Given Data:

$$b = 20 \text{ mm}$$

$$d = 20 \text{ mm}$$

$$l = 2 \text{ m} \rightarrow 2000 \text{ mm}$$

$$\text{load } p = 400 \text{ N}$$

$$\text{Max. BM at point load center } M = \frac{pl}{4} = \frac{400 \times 2000}{4} = 2 \times 10^5 \text{ N-mm}$$

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$\frac{2 \times 10^5}{13.33 \times 10^3} \cdot 10 = \sigma$$

$$\sigma = 150.3 \text{ N/mm}^2$$

$$I = \frac{bd^3}{12}$$

$$= \frac{20 \times 20^3}{12} = 13.33 \times 10^3 \text{ mm}^4$$

Cantilever beam:

$$b = 40 \text{ mm}$$

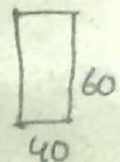
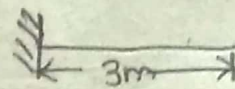
$$d = 60 \text{ mm}$$

$$l = 3 \text{ m} \rightarrow 3000 \text{ mm}$$

$$\sigma = 150 \text{ N/mm}^2$$

$$y = 30 \text{ mm}$$

$$\frac{M}{I} = \frac{\sigma}{y}$$



$$I = \frac{bd^3}{12} = 720 \times 10^3$$

$$M = \frac{150}{30} \times 720 \times 10^3 = 3.6 \times 10^5 \text{ N-mm}$$

$$\left(\because \text{point load} = wl \right)$$

$$Udl = \frac{wl^2}{2}$$

Maximum bending moment in case of cantilever beam with udl.

$$= \frac{\omega \times 3000}{2}$$

$$= 1500 \times 3000 \omega$$

$$3000 \times 1500 \omega = 36 \times 10^5$$

$$\omega = \frac{36 \times 10^5}{3000 \times 1500}$$

$$= 0.8 \text{ N/mm}$$

$$= 800 \text{ N/m}$$

A timber beam of rectangular section of length 8m is S.S beam. The beam carries a udl of 12 kN/m run over the entire length and a point load of 10 kN at 3m from the left support. If the depth of the beam is two times the width of the stress in the timber is not to exceed 8 N/mm^2 . Find the suitable dimensions of the beam.

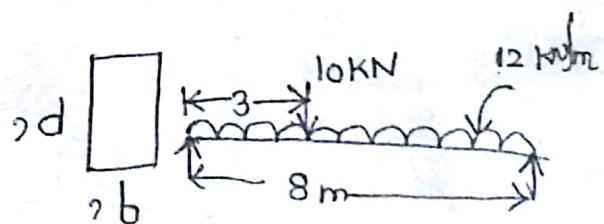
Given data:

$$d = 2b$$

$$\sigma = 8 \text{ N/mm}^2$$

$$P = 10 \text{ kN}$$

$$l = 8 \text{ m}$$



Reactions:-

$$R_A + R_B = 10 + (12 \times 8) = 106 \text{ kN}$$

$$\sum M_A = 0$$

$$\Rightarrow -R_B \times 8 + 10 \times 3 + (12 \times 8) \times 4 = 0$$

Portion AC:- (0 to 3)

$$F_x = R_A - 12x$$

$$F_{x=0} = 0$$

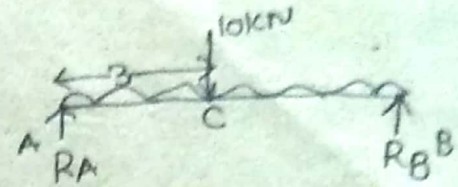
$$F_{x=3} = 54.25 - 12 \times 3 = 18.25 \text{ kN}$$

Bending moment:-

$$M_x = 54.25 - 12x \left(\frac{x}{2}\right)$$

$$M_{x=0} = 0$$

$$M_{x=3} = 108.7 \text{ kN/m}$$



Portion BC:- (0 to 5 m)

$$F_x = -R_B + 12x$$

$$F_{x=0} = -51.75$$

$$F_{x=5} = 8.25 \text{ kN}$$

$$M_x = R_B x - 12x \cdot \frac{x}{2}$$

$$M_{x=0} = 0$$

$$M_{x=5} = 51.75 + 12 \times 5 \cdot \frac{5}{2} = 108.75$$

To find the point of zero shear force equate

Effect equation in the portion BC to zero

$$F_x = -R_B + 12x$$

$$12x = R_B$$

$$x = \frac{51.75}{12} \Rightarrow 4.31$$

$$B_M = R_B x - 12x^2$$

$$M_x = 3 = 51.75 \times 4.31 - 12 (4.31)^2$$

$$= 111.5 \text{ kN-m}$$

$$\therefore \frac{M}{I} = \frac{\sigma}{y}$$

$$\frac{3 \times 111.5 \times 10^6}{2 \times b^4} = \frac{8}{b}$$

$$\frac{3 \times 111.5 \times 10^6}{2 \times 8} = \frac{b^4}{b} b^3$$

$$b = 275.5 \text{ mm}$$

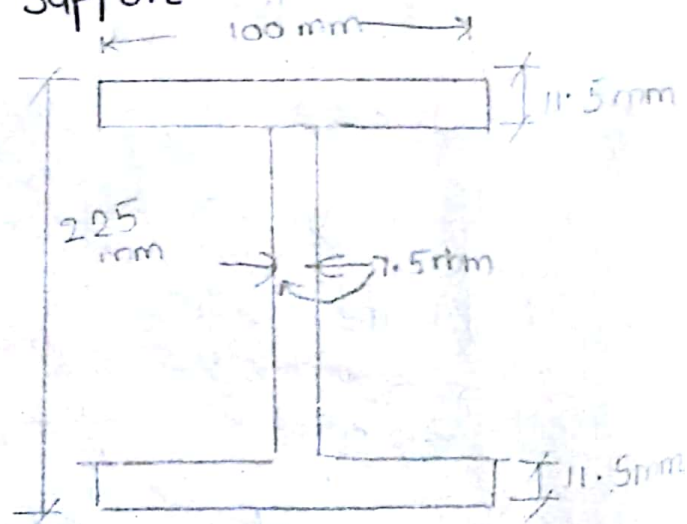
$$\therefore d = 2b$$

$$= 2 \times 275.5$$

$$= 550.9 \text{ mm.}$$

24/8/16

* An I-section shown in figure is simply supported over a span of 12m. If the max. permissible bending stress is 80 N/mm^2 what concentrated load can be carried at a distance 4m from right support.



$\sigma_{\max} = 80 \text{ N/mm}^2$

Neutral axis depth $\bar{y} = \frac{225}{2} = 112.5 \text{ mm}$.

Moment of Inertia $= \frac{Bd^3}{12} - \frac{bd^3}{12}$ ($\because$ Symmetrical).

$= \frac{100 \times 225^3}{12} - \frac{(100-75)(225)^3}{12}$

$= 94.92 \times 10^6 -$

$= 31.38 \times 10^6 \text{ mm}^4$

Let w be the concentrated load at distance a mts at right support then

max. bending moment (m) $= \frac{wab}{2}$ ($\because w = kN$)

$= \frac{w \times 8 \times 4}{1 \times 2 \times 3}$

$= \frac{8w}{3} \text{ KN-m}$

Using the relation $\frac{M}{I} = \frac{\sigma}{y}$

$\therefore \frac{\frac{8w}{3} \times 10^6}{31.38 \times 10^6} = \frac{80 \text{ N/mm}^2}{112.5 \text{ mm}}$

$\frac{8w}{3} \times 10^6 = \frac{80}{112.5} \times 31.38 \times 10^6$

$\therefore w = 8.36 \text{ KN}$

as shown figure. The beam of I-section as the dimensions of I-section carries a UDL 40 kN/m run on a span of 10 m . calculate the max. stress produced due to bending.

Given data:-

$$\text{Span}(l) = 10 \text{ m} \Rightarrow 10 \times 10^3 \text{ mm}$$

$$\text{UDL } (w) = 40 \text{ kN/m}$$

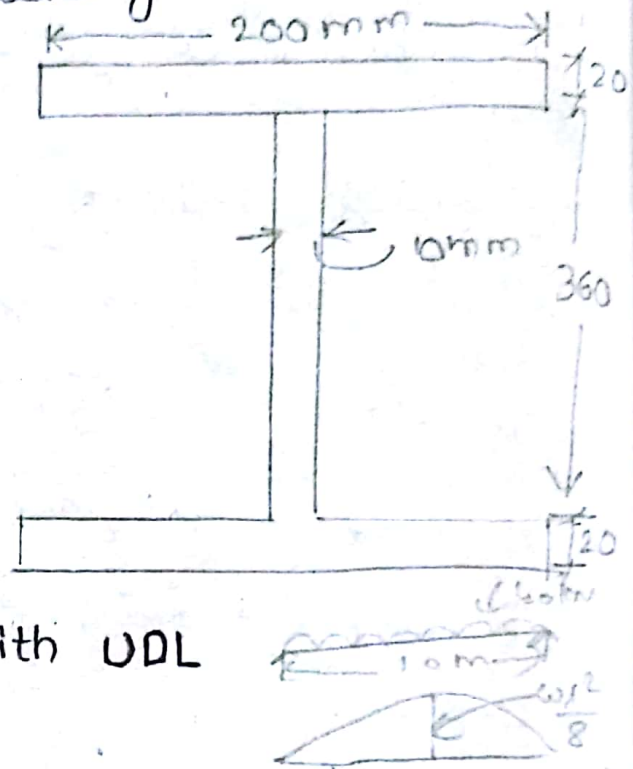
$$\text{Max. stress } (\sigma) = ?$$

Max. bending moment in case of simply supported beam with UDL

$$\begin{aligned} M &= \frac{wl^2}{8} \\ &= \frac{40 \times (10^2)}{8} \\ &= 500 \text{ kN-m} \\ &\Rightarrow 500 \times 10^6 \text{ N-mm} \end{aligned}$$

Neutral axis depth $\bar{y} = \frac{200}{2} = 200 \text{ mm}$

$$\begin{aligned} \text{MOI } I &= \frac{bd^3}{12} - \frac{bd^3}{12} \\ &= \frac{200 \times 400^3}{12} - \frac{190 \times 360^3}{12} \\ &= 327.94 \times 10^6 \text{ mm}^4 \end{aligned}$$



$$\frac{500 \times 10^6}{327.94 \times 10^6} \times 200 \text{ mm} = 304.9 \text{ N/mm}^2$$

* A cast-iron bracket subjected to bending as the cross-section of I-form with unequal flanges the dimensions of the section are shown in figure. Find the position of the neutral axis and moment of inertia of the section about the neutral axis. If the max. bending moment on the section is 40 Mega N/mm. Determine the max. bending stress. What is the nature of the stress.

Given data:-

Max. bending moment (m) = 40 MN-mm
= $40 \times 10^6 \text{ N-mm}$

Neutral axis depth:

Let $\bar{y}$ be the neutral axis distance from bottom fibre.

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3}$$

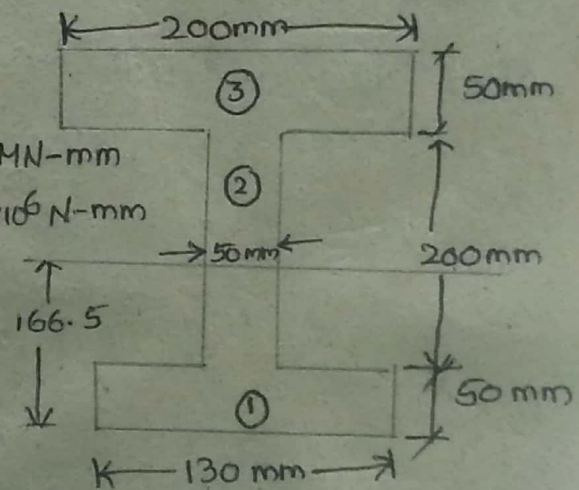
Here: $a_1 = 130 \times 50 = 6500 \text{ mm}^2$

$a_2 = 200 \times 50 = 10,000 \text{ mm}^2$

$a_3 = 200 \times 50 = 10,000 \text{ mm}^2$

y_1 = distance of neutral axis of bottom flange from the bottom.

$$= \frac{50}{2} = 25 \text{ mm}$$



$$y_3 = 50 + 200 + \frac{50}{2} = 275 \text{ mm}$$

$$= \frac{6500 \times 25 + 10,000 \times 150 + 10,000 \times 275}{6500 + 10,000 + 10,000}$$

$$\bar{y} = 166.50 \text{ mm (Bottom)}$$

$$\bar{y} = 133.5 \text{ mm (Top)}$$

Moment of Inertia:

$$\therefore \bar{I} = I_1 + I_2 + I_3$$

Here I_1 = Moment of inertia of bottom flange about the neutral axis.

$$= \frac{bd^3}{12} + (A_1)(y_1 - \bar{y})^2$$

$$= \frac{130 \times 50^3}{12} (6500) (25 - 166.5)^2$$

$$= 131.49 \times 10^6 \text{ mm}^4$$

I_2 = Moment of inertia of web flange above the neutral axis

$$= \frac{bd^3}{12} + (A_2)(y_2 - \bar{y})^2$$

$$= \frac{50 \times 200^3}{12} (10,000) (150 - 166.5)^2$$

$$= 36.05 \times 10^6 \text{ mm}^4$$

I_3 = Moment of inertia of top flange above the neutral axis

$$\frac{200 \times 50^3}{12} + (A_3) (y_3 - \bar{y})^2$$

$$= 119.8 \times 10^6 \text{ mm}^4$$

$$I = 131.49 \times 10^6 + 36.05 \times 10^6 + 119.8 \times 10^6$$
$$= 287.34 \times 10^6 \text{ mm}^4$$

Maximum bending stress:-

$$\frac{M}{I} = \frac{\sigma}{y}$$

Maximum tensile stress:- (σ_t)

$$\frac{M}{I} = \frac{\sigma_t}{y}$$

$$\frac{40 \times 10^6}{287.34 \times 10^6} = \frac{\sigma_t}{133.5}$$

$$\sigma_t = 18.58 \text{ N/mm}^2$$

Maximum compressive stress σ_c

$$\frac{40 \times 10^6}{287.34 \times 10^6} = \frac{\sigma_c}{166.5}$$

$$\sigma_c = 23.178 \text{ N/mm}^2$$

$$\therefore \text{Max. stress} = 23.18 \text{ N/mm}^2$$

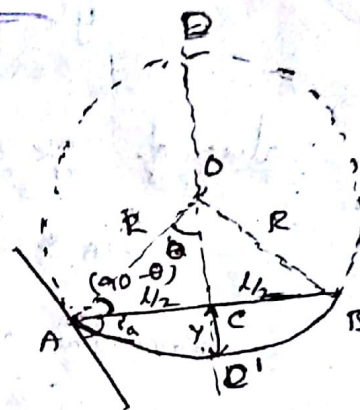
Nature of stress = Compression.

* If a beam carries UDL or a point load, the beam is deflected from its original position.

* Radius of curvature is given by $\frac{M}{I} = \frac{E}{R}$

Member bending into a circular Arc

A member AB of span l subjected to a uniform BM M so that the member is bent to a circular shape. Let R be the radius of the bent form of the member.



Let the deflection at the center of the span be $CO' = y$ (for a practical beam the deflection y is a small quantity)

~~But $DC \cdot CE = AC \cdot ED$~~

Hence $\tan \theta = 0$, Hence θ becomes the slope at

the slope is $\frac{dy}{dx} = \tan \theta = 0$

Now $AC = BC = \frac{l}{2}$

Also from the geometry of a circle, we know that

$$AC \times BC = DC \times CE$$

$$\frac{l}{2} \times \frac{l}{2} = (2R - y) \times y$$

$$(\because DC = DE - CE = 2R - y)$$

$$\frac{l^2}{4} = 2Ry - y^2$$

For practical beam, the deflection y is a small quantity. ~~Square~~ Square of small quantity is negligible

$$\frac{l^2}{4} = 2Ry$$

From bending equation,

$$\frac{M}{I} = \frac{E}{R}$$

$$R = \frac{E \times I}{M}$$

$$y = \frac{L^2}{8 \times \frac{EI}{M}}$$

$$y = \frac{ML^2}{8EI}$$

value of slope θ

$$\sin \theta = \frac{AC}{AO} = \frac{\frac{L}{2}}{R} = \frac{L}{2R}$$

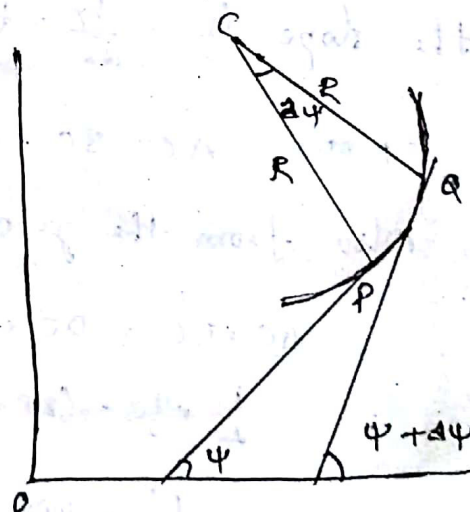
Since the angle θ is very small, hence $\sin \theta = \theta$

$$\theta = \frac{L}{2R}$$

$$= \frac{L}{2 \times \frac{EI}{M}} = \theta = \frac{ML}{2EI}$$

Equation among slope, deflection & Radius of Curvature

Consider an elemental length $PA = ds$ of a curve. let the tangents at P and Q make angles ψ and $\psi + d\psi$ with the x-axis. let the



normals at P and Q meet at C. Then C called the centre of curvature of the curve at any point b/w P and Q on the curve. The distance $CP = CQ = R$ is called the radius of curvature

obviously $ds = R d\psi$

$$R = \frac{ds}{d\psi}$$

But if x and y be the coordinates of P , then
 $\tan \psi = \frac{dy}{dx}$, $\sin \psi = \frac{dy}{ds}$, $\cos \psi = \frac{dx}{ds}$ — (1)

$$R = \frac{ds}{d\psi} = \frac{\left(\frac{ds}{dx}\right)}{\left(\frac{d\psi}{dx}\right)} = \frac{\left(\frac{1}{\cos \psi}\right)}{\left(\frac{d\psi}{dx}\right)}$$

$$R = \frac{\sec \psi}{\left(\frac{d\psi}{dx}\right)} \quad \text{--- (2)}$$

$\tan \psi = \frac{dy}{dx} \rightarrow$ differentiating w.r.t. x

$$\sec^2 \psi \cdot \frac{d\psi}{dx} = \frac{d^2y}{dx^2}$$

$$\frac{d\psi}{dx} = \frac{\frac{d^2y}{dx^2}}{\sec^2 \psi} \quad \text{--- (3)}$$

substituting (3) in (2)

$$\begin{aligned} R &= \frac{\sec \psi}{\frac{d^2y}{dx^2} \cdot \sec^2 \psi} = \frac{\sec \psi}{\left(\frac{d^2y}{dx^2} \cdot \sec^2 \psi\right)} \\ &= \frac{\sec \psi \cdot \sec^2 \psi}{\frac{d^2y}{dx^2}} = \frac{\sec^3 \psi}{\left(\frac{d^2y}{dx^2}\right)} \end{aligned}$$

Reciprocating on both sides

$$\begin{aligned} \frac{1}{R} &= \frac{\frac{d^2y}{dx^2}}{\sec^3 \psi} = \frac{\frac{d^2y}{dx^2}}{(\sec^2 \psi)^{3/2}} \\ &= \frac{\frac{d^2y}{dx^2}}{(1 + \tan^2 \psi)^{3/2}} \end{aligned}$$

For a practical exam, the slope $\tan \phi$ at any point is a small quantity, hence $\tan \phi$ can be neglected

$$\frac{1}{R} = \frac{d^2y}{dx^2} \quad \text{--- (4)}$$

from Bending equation $\frac{M}{I} = \frac{E}{R}$

$$\frac{1}{R} = \frac{M}{EI} \quad \text{--- (5)}$$

$$\text{deflection} = y$$

$$\text{slope} = \frac{dy}{dx}$$

Equating (4) and (5)

$$\frac{M}{EI} = \frac{d^2y}{dx^2} \Rightarrow \boxed{M = EI \frac{d^2y}{dx^2}} \rightarrow \text{bending moment}$$

Differentiating above equation

$$\frac{dM}{dx} = EI \frac{d^3y}{dx^3}$$

but $\frac{dM}{dx} = F$ shear force

$$\boxed{F = EI \frac{d^3y}{dx^3}} \text{--- shear force}$$

Differentiating above equation

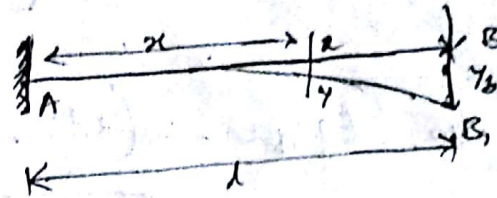
$$\frac{dF}{dx} = EI \frac{d^4y}{dx^4}$$

But $\frac{dF}{dx} = w$ the rate of loading

$$\boxed{w = EI \frac{d^4y}{dx^4}} \rightarrow \text{Rate of loading}$$

free end

consider any section
X of the cantilever
distant x from the
fixed end A.



The bending moment at the section is given by

$$EI \frac{d^2y}{dx^2} = -W(l-x)$$

by integrating above equation

$$EI \frac{dy}{dx} = -W \left(lx - \frac{x^2}{2} \right) + C_1$$

where C_1 is a constant of integration

we have at $x=0$, $\frac{dy}{dx} = 0$, $C_1 = 0$

$$EI \frac{dy}{dx} = -W \left(lx - \frac{x^2}{2} \right) \quad \text{(i) slope equation}$$

Integrating again

$$EI y = -W \left(\frac{lx^2}{2} - \frac{x^3}{6} \right) + C_2$$

where C_2 is a constant of integration

At A the deflection being zero, we have

At $x=0$, $y=0$, $C_2 = 0$

$$EI y = -W \left(\frac{lx^2}{2} - \frac{x^3}{6} \right) \quad \text{(ii) deflection equation}$$

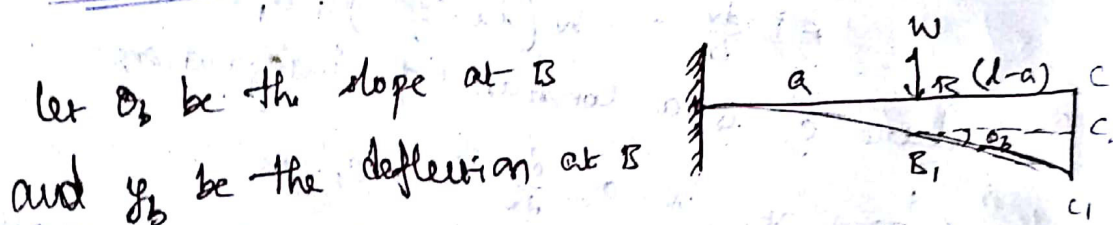
The slope and deflection at the free end can
be determined by putting $x=l$ in above equations
let the slope and deflection at B be θ_b and y_b

$$\text{we have } EI \theta_b = -W \left(l \cdot l - \frac{l^2}{2} \right)$$

$$\text{Also } EI \theta_b = -W \left(\frac{1 \cdot l^2}{2} - \frac{l^3}{6} \right) = -\frac{Wl^3}{3}$$

$$\boxed{\theta_b = -\frac{Wl^3}{3EI}} \quad (-ve \text{ downward deflection})$$

Cantilever of length carrying a concentrated
load W at a distance from the fixed end



let θ_b be the slope at B
and y_b be the deflection at B

$$\text{we have } \theta_b = \frac{W a^2}{2EI} \quad \text{and } y_b = \frac{W a^3}{3EI}$$

The beam will bend only b/w A and B; but from B to C it will remain straight since the B.M b/w B and C is zero.

let the end C deflect to C_1 and B deflect to B_1

$$\text{we have } BB_1 = y_b = \frac{W a^3}{3EI}$$

let B_1C_1 be perpendicular to CC_1

$$CC_1 = BB_1 = y_b = \frac{W a^3}{3EI}$$

$$\text{But } C_2C_1 = (l-a)\theta_b$$

$$\boxed{y_c = CC_1 + C_1C_2 = \frac{W a^3}{3EI} + (l-a) \frac{W a^2}{2EI}}$$

run over the whole length carrying a UDL per unit

The B.M at any section A at distance x from the fixed end

$$EI \frac{d^2y}{dx^2} = -\frac{W}{2}(l-x)^2$$

By integrating

$$EI \frac{dy}{dx} = +\frac{W}{6}(l-x)^3 + C_1$$

At A the fixed end the slope being 0, we have

At $x=0, \frac{dy}{dx} = 0 \therefore \boxed{C_1 = -\frac{Wl^3}{6}}$

$$EI \frac{dy}{dx} = \frac{W}{6}(l-x)^3 - \frac{Wl^3}{6} \quad \text{--- (i) slope equation}$$

Integrating again, we get

$$EI \cdot y = -\frac{W}{24}(l-x)^4 + \frac{Wl^3}{6}x + C_2 \quad \text{--- (ii) deflection equation}$$

Since at A deflection is zero

At $x=0, y=0$

$$0 = -\frac{Wl^4}{24} + \frac{Wl^3}{6} \cdot 0 + C_2 \therefore \boxed{C_2 = \frac{Wl^4}{24}}$$

from equations (i) and (ii) the slope and deflection at any section can be determined

To find the slope θ_b at B putting $x=l$ in slope equation

$$EI \theta_b = -\frac{Wl^3}{6} \therefore \boxed{\theta_b = -\frac{Wl^3}{6EI}}$$

$$EI \frac{y}{2} = -\frac{wl^4}{84} + \frac{wl^4}{24} = -\frac{wl^4}{8}$$

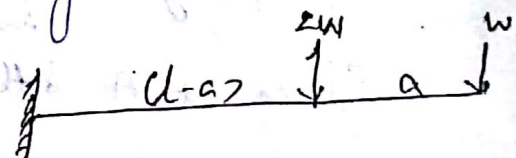
$$\boxed{y_b = -\frac{wl^4}{8EI}} \quad \therefore \text{downward deflection}$$

Ex: A horizontal cantilever of uniform section of length l carries two point loads, w at the free end and $2w$ at a distance of a from the free end. Find the maximum deflection due to this loading.

If the cantilever is a steel tube of circular section 100mm external dia and 6mm thick and $l = 1.5m$ and $a = 0.6m$ determine the value of w so that the max bending stress is $140N/mm^2$ and calculate the max deflection for the loading. $E = 2 \times 10^5 N/mm^2$

Sol

deflection at the free



end due to load $w = \frac{wl^3}{3EI}$

Deflection at free end due to the load $2w$ alone

$$= \frac{2w(l-a)^3}{3EI} + \frac{2w(l-a)^2}{2EI} \cdot a$$

Total deflection at free end is

$$= \frac{wl^3}{3EI} + \frac{2w(l-a)^3}{3EI} + \frac{2w(l-a)^2}{EI} \cdot a$$

$$= \frac{w}{3EI} [l^3 - 2(l-a)^3 + 3(l-a)^2 a]$$

$$= \frac{W}{3EI} [l^3 + 2l^3 - 6l^2a + 6la^2 - 2a^3 + 3l^2a - 6la^2 + 3a^3]$$

$$= \frac{W}{3EI} [3l^3 - 3l^2a + a^3]$$

$$\delta = \frac{W}{3EI} [3l^2(l-a) + a^3]$$

$$\text{Max BM} = M = wl + 2w(l-a)$$

$$= w(3l-2a) = w(3 \times 1.5 - 2 \times 0.6)$$

$$= 3.3 \text{ W NM}$$

$$\therefore f = \frac{M}{I} y = 140 \text{ N/mm}^2$$

$$\frac{3.3W \times 1000}{1.965 \times 10^6} \times 50 = 140$$

$$W = 1667 \text{ N}$$

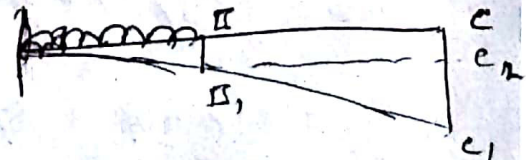
$$\text{Max. deflection} = \delta = \frac{W}{3EI} [3l^2(l-a) + a^3]$$

$$= \frac{1667}{3 \times 1.965 \times 10^6 \times 2 \times 10^5} [3 \times 2.25 \times 0.9 + 0.216] \times 1000$$

$$= 8.894 \text{ mm}$$

Cantilever of length l carrying a UDL of w per unit run for a distance from fixed end.

$$\theta_b = \frac{wa^3}{6EI}$$



$$\text{Deflection at } C = \frac{wa^4}{8EI} + (l-a) \frac{wa^3}{6EI}$$

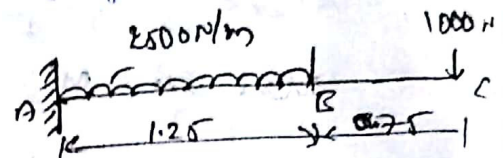
The portion of the cantilever from C to B will remain straight having a slope of $\frac{wa^3}{6EI}$

A cantilever of length 2m carries a UDL of 2500N per m for a length of 1.25m from the fixed end and a point load of 1000N at the free end. If the section is rectangular 120mm wide and 200mm deep, find the deflection at the free end. $E = 10000 \text{ N/mm}^2$

sd

$$I = \frac{bd^3}{12} = \frac{120 \times 200^3}{12}$$

$$= 1.3824 \times 10^8 \text{ mm}^4$$



Deflection at C due to uniformly distributed load on AB

$$= \frac{Wa^4}{8EI} + \frac{Wa^3}{6EI} (l-a)$$

$$= \frac{2500 \times 1.25^4}{8EI} + \frac{2500 \times 1.25^3 \times 0.75}{6EI}$$

$$= \frac{1373}{EI}$$

Deflection at C due to point load at C = $\frac{Wl^3}{3EI}$

$$= \frac{1000 \times 2^3}{3EI} = \frac{2667}{EI}$$

∴ Net deflection at C

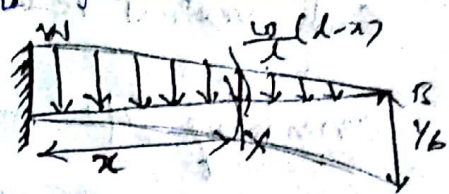
$$\frac{1373}{EI} + \frac{2667}{EI} = \frac{4040}{EI}$$

$$= \frac{4040 (1000)^3}{10^4 \times 1.3824 \times 10^8} = 2.922 \text{ mm}$$

Cantilever of length l carrying a distributed load whose intensity varies uniformly from zero at free end to w per unit run at the fixed end

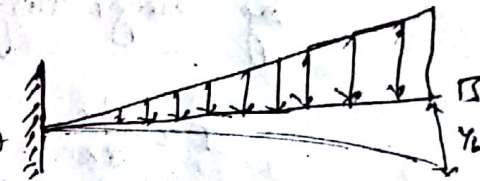
$$\text{Deflection at B} = \frac{wl^4}{30EI}$$

$$\text{slope at B } \theta_B = -\frac{wl^3}{24EI}$$

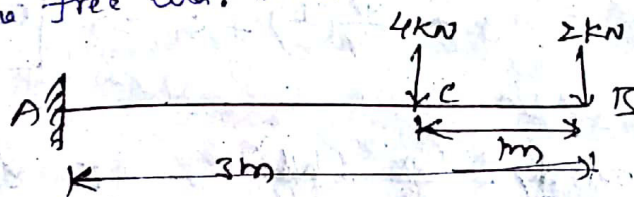


Load zero at fixed end and w per unit at free end

$$\text{Deflection at B } y_B = \frac{11}{120} \cdot \frac{wl^4}{EI}$$



Ex: A cantilever of length 3m carries two point loads of 2kN at the free end and 4kN at a distance of 1m from the free end. Find the deflection at free end



Let y_1 = deflection at the free end due to 2kN load

y_2 = deflection at the free end due to 4kN load alone

$$y_1 = \frac{wL^3}{3EI} = \frac{2000 \times 3600^3}{3 \times 2 \times 10^5 \times 10^8} = 0.9 \text{ mm}$$

$$y_2 = \frac{wa^2}{3EI} + \frac{wa^2}{2EI} (L-a)$$

$$= \frac{4000 \times 2000^3}{3 \times 2 \times 10^5 \times 10^8} + \frac{4000 \times 2000^3}{2 \times 2 \times 10^5 \times 10^8} (3000 - 2000)$$

$$= 0.54 + 0.40 = 0.94 \text{ mm}$$

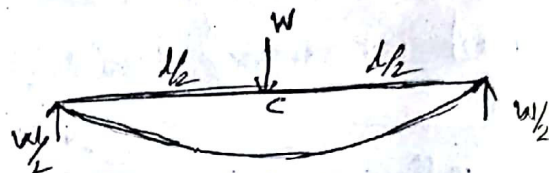
Total deflection at free end = $y_1 + y_2$

$$= 0.9 + 0.94$$

$$= 1.84 \text{ mm}$$

(i) SSB of span l carrying a point load

Each vertical reaction = $\frac{w}{2}$



Considering left half AC

The BM at any section in AC distance x from A

$$EI \frac{d^2y}{dx^2} = \frac{w}{2} x$$

by integrating, we get

$$EI \cdot \frac{dy}{dx} = \frac{wx^2}{4} + C_1$$

Since max deflection occurs at mid span C, the slope at C is zero $x = \frac{l}{2}$

$$\frac{dy}{dx} = 0$$

$$0 = \frac{w}{4} \left(\frac{l}{2}\right)^2 + C_1 \Rightarrow \boxed{C_1 = -\frac{wl^2}{16}}$$

$$EI \cdot \frac{dy}{dx} = \frac{wx^2}{4} - \frac{wl^2}{16} \quad \text{--- (i) slope equation}$$

To find the slope at A, put $x=0$

$$EI \theta_A = -\frac{wl^2}{16} \quad \therefore \boxed{\theta_A = -\frac{wl^2}{16EI}}$$

By integrating slope equation, we get

$$EI \cdot y = \frac{wx^3}{12} - \frac{wl^2}{16} x + C_2$$

Since at A the deflection is zero

we have at $x=0$

$$y=0$$

$$C_2 = 0$$

To find the deflection at c

put $x = \frac{l}{2}$

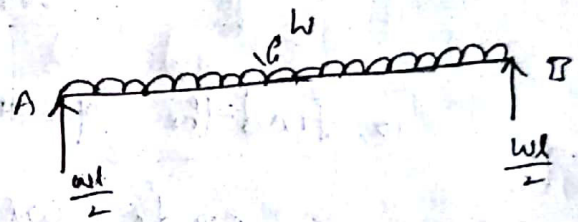
$$EI y_c = \frac{wl}{12} \left(\frac{l}{2}\right)^3 - \frac{wl^2}{16} \cdot \frac{l}{2}$$

$$y_c = -\frac{wl^3}{48EI}$$

downward deflection

(ii) SSB of span l carrying a UDL of w per unit run over the whole span

Each vertical reaction = $\frac{wl}{2}$



B.M at any section distant x from the end A

$$EI \frac{d^2y}{dx^2} = \frac{wl}{2} x - \frac{wx^2}{2}$$

By integrating $EI \frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} + C_1$

The loading being symmetrical, max deflection will occur at mid span and hence slope at mid span equals to zero

i.e., at $x = \frac{l}{2}$

$$\frac{dy}{dx} = 0$$

$$0 = \frac{wl}{4} \left(\frac{l}{2}\right)^2 - \frac{w}{6} \left(\frac{l}{2}\right)^3 + C_1$$

$$C_1 = -\frac{wl^3}{24}$$

$$EI \frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} - \frac{wl^3}{24} \quad \text{--- slope equation}$$

Integrating again $EI y = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3}{24} x + C_2$

$$\therefore C_2 = 0$$

$$EIY = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3}{24}x \quad \text{--- Deflection equation}$$

To find the max deflection which occurs at mid span section C. putting $x = \frac{l}{2}$

$$EIY_c = \frac{wl}{12} \left(\frac{l}{2}\right)^3 - \frac{w}{24} \left(\frac{l}{2}\right)^4 - \frac{wl^3}{24} \cdot \frac{l}{2}$$

$$\boxed{Y_c = -\frac{5}{384} \cdot \frac{wl^4}{EI}} \rightarrow \text{downward deflection}$$

To find the slope at A, put $x=0$ in the slope equation

$$EI \frac{dy}{dx} = \frac{wl}{4} (0) - \frac{4w(0)^3}{6} - \frac{wl^3}{24}$$

$$\boxed{\theta_a = -\frac{wl^3}{24EI}} \rightarrow \text{downward deflection}$$

Ex: A timber beam 100mm wide 250mm deep is simply supported beam of 4m. Find the UDL that can be applied on the beam over the whole span so that the deflection at the centre may not exceed 6mm. $E = 1.12 \times 10^4 \text{ N/mm}^2$

sol: For the beam, $M_oI = \frac{bd^3}{12} = \frac{100 \times 250^3}{12} = 1.3021 \times 10^8 \text{ mm}^4$

$$EI = 1.12 \times 10^4 \times 1.3021 \times 10^8 = 1.4583 \times 10^{12} \text{ Nmm}^2$$

$$\text{Deflection at the Centre } \delta = \frac{5}{384} \cdot \frac{wl^4}{EI}$$

$$w \times \frac{384EI\delta}{5l^4} = \frac{384 \times 1.4583 \times 10^{12} \times 6}{5 \times 4000^4}$$

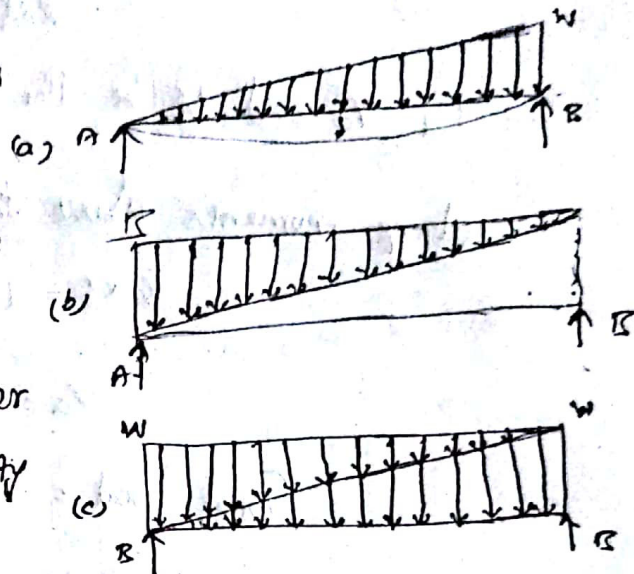
$$= 2.625 \text{ N/mm}$$

$$= 2.625 \times 1000 \text{ N/m} = 2625 \text{ N/m}$$

$$= 2.625 \text{ kN/m}$$

A SSB of span l carries a triangular load whose intensity varies from zero at one end to w at other end. Find the deflection at the centre.

When the beam carries both the above triangular loads, the deflection at the centre will be 2δ . But the two triangular loads together constitute a UDL of intensity w .



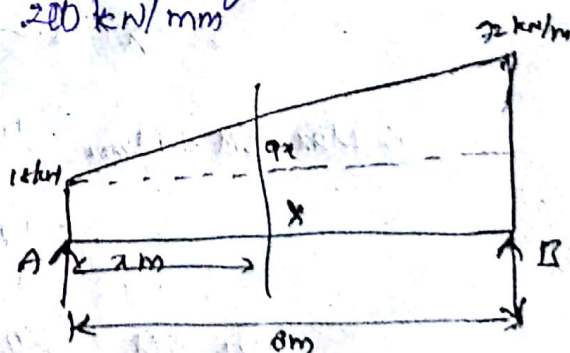
$$\therefore 2\delta = \frac{5}{384} \cdot \frac{w l^4}{EI}$$

$$\therefore \delta = \frac{5}{768} \cdot \frac{w l^4}{EI}$$

Ex A horizontal beam of symmetrical section SS at the ends, carries a load whose intensity varies uniformly from 18 kN/m at one end to 72 kN/m at the other. Find the central deflection if the span is 6 m , the section is 450 mm deep and the maximum bending stress is 90 N/mm^2 . Take $E = 200 \text{ kN/mm}^2$.

sol Average load intensity

$$w = \frac{18+72}{2} = 45 \text{ kN/m}$$



$$\text{Central deflection } \delta = \frac{5}{384} \cdot \frac{w l^4}{EI}$$

$$EI \cdot \delta = \frac{5}{384} w l^4 = \frac{5}{384} w l \cdot l^3$$

$$EI \delta = \frac{5}{384} (45 \times 6) 6^3 \times 10^9 \text{ kNmm}^3$$

$$EI \delta = 7.59375 \times 10^{11} \text{ kNmm}^3 \quad \text{--- (1)}$$

$$= \frac{90 \text{ kN}}{2 \times 5} = \frac{27}{5} \text{ N/mm} = \frac{27}{5} \times 10^{-3} \text{ kN/mm} \quad \text{--- (2)}$$

Let V_a and V_b be the reactions at the supports

Taking moments about the end B,

$$V_a \times 6 = 18 \times 6 \times \frac{6}{2} + \frac{1}{2} \times 6 \times 54 \times \frac{6}{3}$$

$$V_a = 108 \text{ kN}$$

$$\text{Total Load} = 45 \times 6 = 270 \text{ kN}$$

$$V_b = 270 - 108 = 162 \text{ kN}$$

At any section distant x from A, the shear force

$$s = 108 - (18x + \frac{1}{2} \times 9x)$$

$$= 108 - 18x - \frac{9}{2}x^2$$

For the section of zero shear

$$108 - 18x - \frac{9}{2}x^2 = 0 \Rightarrow x^2 + 4x - 24 = 0$$

$$\therefore x = 3.292 \text{ m}$$

$$\therefore \text{Max. BM} = M_{\text{max}} = 108x - 9x^2 - \frac{3}{2}x^3$$

$$= 108 \times 3.292 - 9 \times 3.292^2 - \frac{3}{2} \times 3.292^3$$

$$\text{max BM} = 204.487 \text{ kNm} = 204.487 \times 10^3 \text{ kNmm}$$

from equation (2) $\frac{27}{5} \times 10^{-3} = 204.487 \times 10^{-3}$

$$I = 5.1122 \times 10^6 \text{ mm}^4$$

from equation (1) $E I \delta = 7.55375 \times 10^{11}$

$$\delta = \frac{7.55375}{E} = 7.423 \text{ mm}$$

Lecture No. 6

-Thick Cylinders-

6-1 Difference in treatment between thin and thick cylinders - basic assumptions:

The theoretical treatment of thin cylinders assumes that the hoop stress is constant across the thickness of the cylinder wall (Fig. 6.1), and also that there is no pressure gradient across the wall. Neither of these assumptions can be used for thick cylinders for which the variation of hoop and radial stresses is shown in (Fig. 6.2), their values being given by the Lamé equations: -

$$\sigma_H = A + \frac{B}{r^2} \quad \dots 6.1$$

$$\sigma_r = A - \frac{B}{r^2} \quad \dots 6.2$$

Where: -

σ_H = Hoop stress ($\frac{N}{m^2} = Pa$).

σ_r = Radial stress ($\frac{N}{m^2} = Pa$).

r = Radius (m). A and B are Constants.

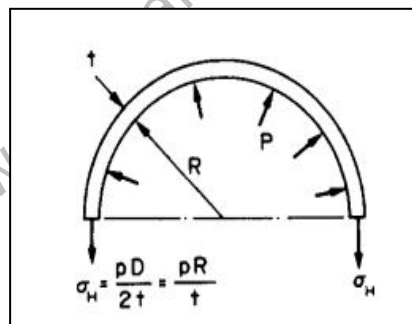


Figure 6.1: - Thin cylinder subjected to internal pressure.

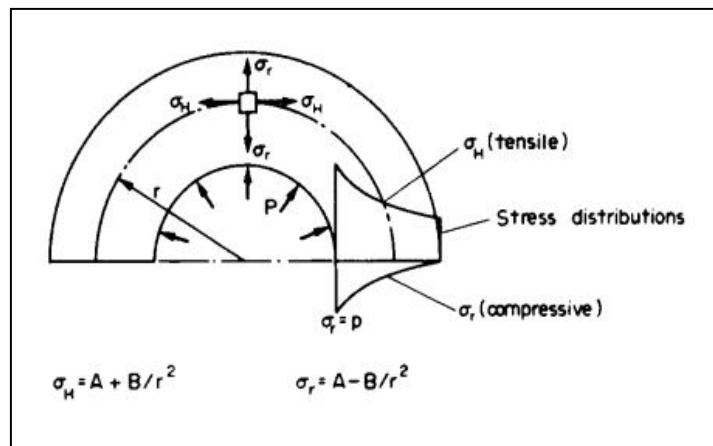


Figure 6.2: - Thick cylinder subjected to internal pressure.

6-2 Thick cylinder- internal pressure only: -

Consider now the thick cylinder shown in (Fig. 6.3) subjected to an internal pressure P , the external pressure being zero.

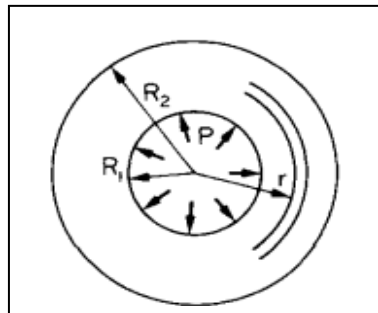


Figure 6.3: - Cylinder cross section.

The two known conditions of stress which enable the Lamé constants A and B to be determined are:

$$\text{At } r = R_1, \quad \sigma_r = -P \quad \text{and} \quad \text{at } r = R_2, \quad \sigma_r = 0$$

Note: -The internal pressure is considered as a negative radial stress since it will produce a radial compression (i.e. thinning) of the cylinder walls and the normal stress convention takes compression as negative.

Substituting the above conditions in eqn. (6.2),

$$\sigma_r = A - \frac{B}{r^2}$$

$$-P = A - \frac{B}{R_1^2} \text{ and } 0 = A - \frac{B}{R_2^2}$$

$$\text{Then } A = \frac{PR_1^2}{(R_2^2 - R_1^2)} \text{ and } B = \frac{PR_1^2 R_2^2}{(R_2^2 - R_1^2)}$$

Substituting A and B in equations 6.1 and 6.2,

$$\sigma_r = \frac{PR_1^2}{(R_2^2 - R_1^2)} \left[1 - \frac{R_2^2}{r^2} \right] \quad \dots 6.3$$

$$\sigma_H = \frac{PR_1^2}{(R_2^2 - R_1^2)} \left[1 + \frac{R_2^2}{r^2} \right] \quad \dots 6.4$$

6-3 Longitudinal stress: -

Consider now the cross-section of a thick cylinder with closed ends subjected to an internal pressure P_1 and an external pressure P_2 , (Fig. 6.4).

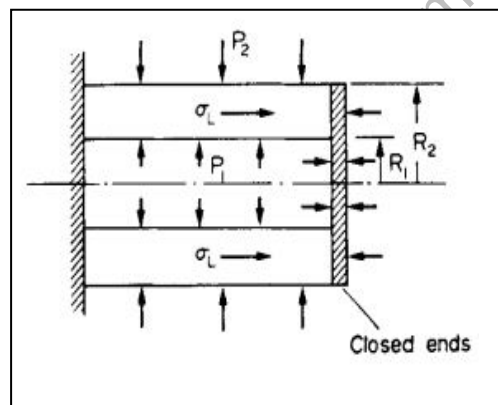


Figure 6.4: - Cylinder longitudinal section.

For horizontal equilibrium:

$$P_1^2 * \pi R_1^2 - P_2^2 * \pi R_2^2 = \sigma_L * \pi [R_2^2 - R_1^2]$$

Where σ_L is the longitudinal stress set up in the cylinder walls,

Longitudinal stress,

$$\sigma_L = \frac{P_1 R_1^2 - P_2 R_2^2}{(R_2^2 - R_1^2)} \quad \dots 6.5$$

But for $P_2 = 0$ (no external pressure),

$$\sigma_L = \frac{P_1 R_1^2}{(R_2^2 - R_1^2)} = A, \text{ constant of the Lamé equations.} \quad \dots 6.6$$

6-4 Maximum shear stress: -

It has been stated in section 6.1 that the stresses on an element at any point in the cylinder wall are principal stresses.

It follows, therefore, that the maximum shear stress at any point will be given by equation of Tresca theory as,

$$\frac{\sigma_y}{2} = \tau_{max} = \frac{\sigma_1 - \sigma_3}{2} \quad \dots 6.7$$

$$\tau_{max} = \frac{\sigma_H - \sigma_r}{2} \quad \dots 6.8$$

$$\tau_{max} = \frac{1}{2} \left[\left(A + \frac{B}{r^2} \right) - \left(A - \frac{B}{r^2} \right) \right] \quad \dots 6.9$$

$$\tau_{max} = \frac{B}{r^2} \quad \dots 6.10$$

6-5 Change of diameter: -

It has been shown that the diametral strain on a cylinder equals the hoop or circumferential strain.

Change of diameter = diametral strain x original diameter.

= circumferential strain x original diameter.

With the principal stress system of hoop, radial and longitudinal stresses, all assumed tensile, the circumferential strain is given by

$$\epsilon_H = \frac{1}{E} (\sigma_H - \nu \sigma_r - \nu \sigma_L) \quad \dots 6.11$$

$$\delta D = \frac{D}{E} (\sigma_H - \nu \sigma_r - \nu \sigma_L) \quad \dots 6.12$$

Similarly, the change of length of the cylinder is given by,

$$\delta L = \frac{L}{E} (\sigma_L - \nu \sigma_r - \nu \sigma_H) \quad \dots 6.13$$

6-6 Comparison with thin cylinder theory: -

In order to determine the limits of D/t ratio within which it is safe to use the simple thin cylinder theory, it is necessary to compare the values of stress given by both thin and thick cylinder theory for given pressures and D/t values. Since the maximum hoop stress is normally the limiting factor, it is this stress which will be considered.

Thus for various D/t ratios the stress values from the two theories may be plotted and compared; this is shown in (Fig. 6.5).

Also indicated in (Fig. 6.5) is the percentage error involved in using the thin cylinder theory.

It will be seen that the error will be held within 5 % if D/t ratios in excess of 15 are used.

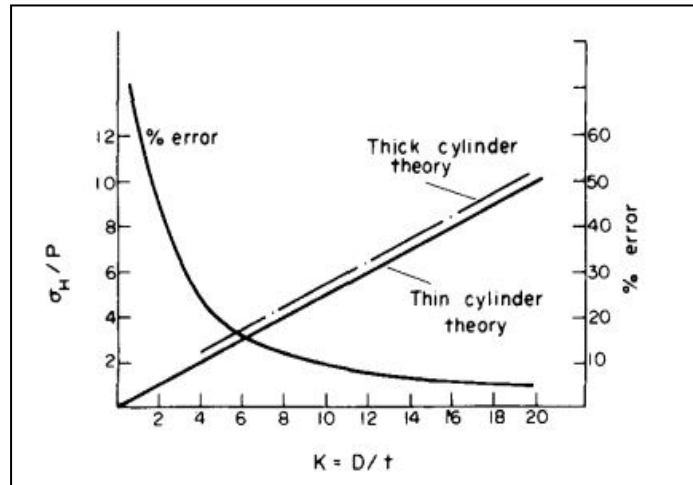


Figure 6.5: - Comparison of thin and thick cylinder theories for various diameter/thickness ratios.

6-7 Compound cylinders:-

From the sketch of the stress distributions in Figure 6.6 it is evident that there is a large variation in hoop stress across the wall of a cylinder subjected to internal pressure. The material of the cylinder is not therefore used to its best advantage. To obtain a more uniform hoop stress distribution, cylinders are often built up by shrinking one tube on to the outside of another. When the outer tube contracts on cooling the inner tube is brought into a state of compression. The outer tube will conversely be brought into a state of tension. If this compound cylinder is now subjected to internal pressure the resultant hoop stresses will be the algebraic sum of those resulting from internal pressure and those resulting from shrinkage as

drawn in Fig. 6.6; thus a much smaller total fluctuation of hoop stress is obtained. A similar effect is obtained if a cylinder is wound with wire or steel tape under tension.

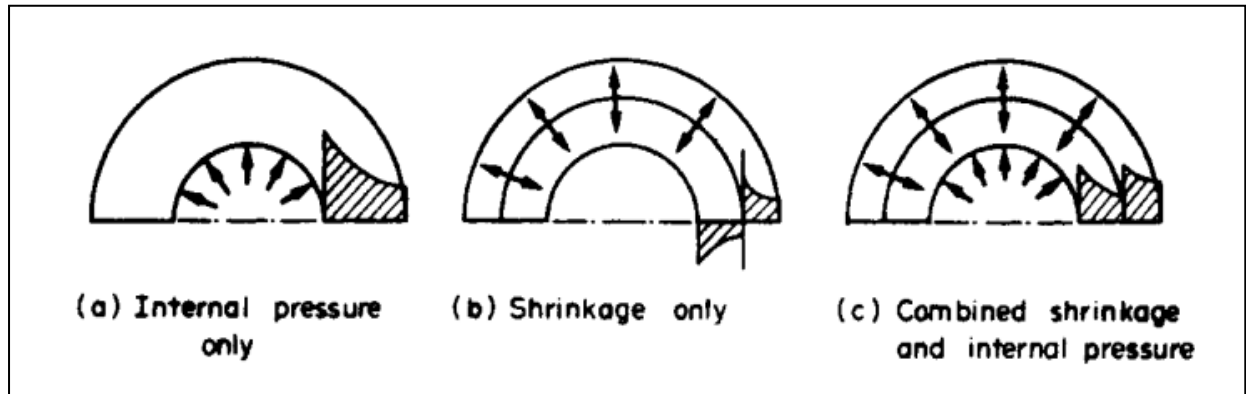


Figure 6.6: - Compound cylinders-combined internal pressure and shrinkage effects.

The method of solution for compound cylinders constructed from similar materials is to break the problem down into three separate effects:

- (a) shrinkage pressure only on the inside cylinder.
- (b) shrinkage pressure only on the outside cylinder.
- (c) internal pressure only on the complete cylinder.

For each of the resulting load conditions there are two known values of radial stress which enable the Lamé constants to be determined in each case

condition (a) shrinkage - internal cylinder:

$$\text{At } r = R_1, \quad \sigma_r = 0$$

At $r = R_c$, $\sigma_r = -p$ (compressive since it tends to reduce the wall thickness)

condition (b) shrinkage - external cylinder:

$$\text{At } r = R_2, \quad \sigma_r = 0$$

At $r = R_c$, $\sigma_r = -p$

condition (c) internal pressure - compound cylinder:

At $r = R_2$, $\sigma_r = 0$

At $r = R_1$, $\sigma_r = -P_1$

Thus for each condition the hoop and radial stresses at any radius can be evaluated and the principle of superposition applied, i.e. the various stresses are then combined algebraically to produce the stresses in the compound cylinder subjected to both shrinkage and internal pressure. In practice this means that the compound cylinder is able to withstand greater internal pressures before failure occurs or, alternatively, that a thinner compound cylinder (with the associated reduction in material cost) may be used to withstand the same internal pressure as the single thick cylinder it replaces.

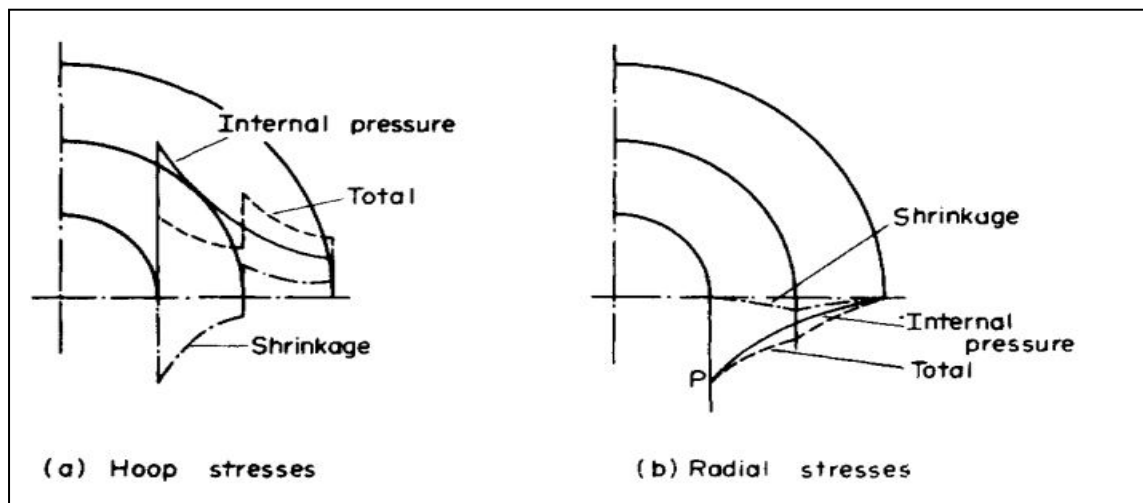


Figure 6.7: - Distribution of hoop and radial stresses through the walls of a compound cylinder.

Example 6-1: - A thick cylinder of 100 mm internal radius and 150 mm external radius is subjected to an internal pressure of 60 MN/m² and an external pressure of 30 MN/m². Determine the hoop and radial stresses at the inside and outside of the cylinder together with the longitudinal stress if the cylinder is assumed to have closed ends.

Solution: -

At $r = 0.1\text{m}$, $\sigma_r = -60\text{MPa}$.

$r = 0.15\text{ m}$, $\sigma_r = -30\text{ MPa}$.

So,

$$-60 = A - 100B \quad \dots 1$$

$$-30 = A - 44.5B \quad \dots 2$$

By solving equations 1 and 2,

$$A = -6 \text{ and } B = 0.54$$

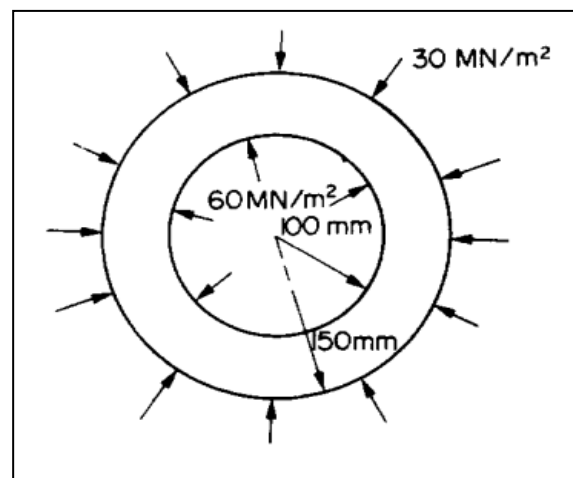
Therefore at $r = 0.1\text{m}$

$$\sigma_H = A + \frac{B}{r^2} = -6 + \frac{0.54}{(0.1)^2} = 48\text{MPa}.$$

At $r = 0.15\text{m}$,

$$\sigma_H = A + \frac{B}{r^2} = -6 + \frac{0.54}{(0.15)^2} = 18\text{MPa}$$

$$\sigma_L = \frac{P_1 R_1^2 - P_2 R_2^2}{(R_2^2 - R_1^2)} = \frac{60(0.1)^2 - 30(0.15)^2}{(0.15^2 - 0.1^2)} = -6\text{MPa i.e. compression}.$$



Example 6-2: - An external pressure of 10 MN/m^2 is applied to a thick cylinder of internal diameter 160 mm and external diameter 320 mm. If the maximum hoop stress permitted on the inside wall of the cylinder is limited to 30 MN/m^2 , what maximum internal pressure can be applied assuming the cylinder has closed ends? What will be the change in outside diameter when this pressure is applied? $E = 207 \text{ GN/m}^2$, $\nu = 0.29$.

Solution: -

$$\text{At } r=0.08\text{m, } \sigma_r=-P, \quad \frac{1}{r^2}=156$$

$$\text{At } r = 0.16 \text{ m, } \sigma_r = -10, \quad \frac{1}{r^2}=39$$

And at $r = 0.08\text{m}$, $\sigma_H = 30\text{MPa}$

$$-10 = A - 39B \quad \dots(1)$$

$$30 = A + 156B \quad \dots(2)$$

Subtracting (1) from (2), $A = -2$ and $B = 0.205$

Therefore, at $r = 0.08$, $\sigma_r = -p = A - 156B = -2 - 156 \times 0.205 = -34\text{MPa}$.

i.e. the allowable internal pressure is 34 MN/m^2 .

The change in diameter is given by

$$\delta D = \frac{D}{E} (\sigma_H - \nu \sigma_r - \nu \sigma_L) \quad \dots (3)$$

$$\text{But } \sigma_r = -10 \text{ MN/m}^2, \sigma_H = A + \frac{B}{r^2} = -2 + 39 \times 0.205 = 6 \text{ MN/m}^2$$

And finally, $\sigma_L = \frac{P_1 R_1^2 - P_2 R_2^2}{(R_2^2 - R_1^2)} = \frac{(34 \times 0.08^2 - 10 \times 0.16^2)}{(0.16^2 - 0.08^2)} = -1.98 \text{ MPa}$ i.e compressive.

Substitute σ_r , σ_H and σ_L in eqn. 3,

$$\delta D = \frac{0.32}{207 \times 10^9} [6 - 0.29(-10) - 0.29(-1.98)] 10^6 = 14.7 \mu\text{m}$$

Example 6-3: - A compound cylinder is formed by shrinking a tube of 250 mm internal diameter and 25 mm wall thickness onto another tube of 250 mm external diameter and 25 mm wall thickness, both tubes being made of the same material. The stress set up at the junction owing to shrinkage is 10 MN/m^2 . The compound tube is then subjected to an internal pressure of 80 MN/m^2 . Compare the hoop stress distribution now obtained with that of a single cylinder of 300 mm external diameter and 50 mm thickness subjected to the same internal pressure.

A solution is obtained as described before by considering the effects of shrinkage and internal pressure separately and combining the results algebraically.

Shrinkage only - outer tube,

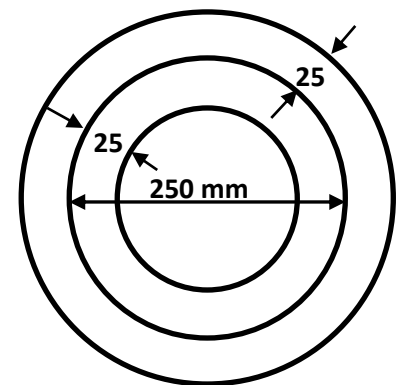
At $r = 0.15$, $\sigma_r = 0$ and at $r = 0.125$, $\sigma_r = -10 \text{ MN/m}^2$

$$0 = A - \frac{B}{(0.15^2)} = A - 44.5B$$

$$-10 = A - \frac{B}{(0.125^2)} = A - 64B$$

$$\therefore B = 0.514, \quad A = 22.85$$

hoop stress at 0.15 m radius = $A + 44.5B = 45.7 \text{ MPa}$.



hoop stress at 0.125 m radius = $A + 64B = 55.75 \text{ MPa}$.

Shrinkage only- inner tube,

At $r = 0.10$, $\sigma_r = 0$ and at $r = 0.125$, $\sigma_r = -10 \text{ MN/m}^2$

$$0 = A - \frac{B}{(0.1^2)} = A - 100B$$

$$-10 = A - \frac{B}{(0.125^2)} = A - 64B$$

$$\therefore B = -0.278, \quad A = -27.8$$

hoop stress at 0.125 m radius = $A + 64B = -45.6 \text{ MPa}$.

hoop stress at 0.10 m radius = $A + 100B = -55.6 \text{ MPa}$.

Considering internal pressure only (on complete cylinder)

At $r = 0.15$, $\sigma_r = 0$ and at $r = 0.10$, $\sigma_r = -80$

$$0 = A - \frac{B}{(0.15^2)} = A - 44.5B$$

$$-80 = A - \frac{B}{(0.1^2)} = A - 100B$$

$$\therefore B = 1.44, \quad A = 64.2$$

$$\text{At } r = 0.15 \text{ m}, \quad \sigma_H = A + 44.5B = 128.4 \text{ MN/m}^2$$

$$r = 0.125 \text{ m}, \quad \sigma_H = A + 64B = 156.4 \text{ MN/m}^2$$

$$r = 0.1 \text{ m}, \quad \sigma_H = A + 100B = 208.2 \text{ MN/m}^2$$

The resultant stresses for combined shrinkage and internal pressure are then:

$$\text{outer tube: } r = 0.15 \quad \sigma_H = 128.4 + 45.7 = 174.1 \text{ MN/m}^2.$$

$$r = 0.125 \quad \sigma_H = 156.4 + 55.75 = 212.15 \text{ MN/m}^2.$$

$$\text{inner tube: } r = 0.125 \quad \sigma_H = 156.4 - 45.6 = 110.8 \text{ MN/m}^2.$$

$$r = 0.1 \quad \sigma_H = 208.2 - 55.6 = 152.6 \text{ MN/m}^2.$$

.....END.....

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23/12

Torsion of circular shafts & springs

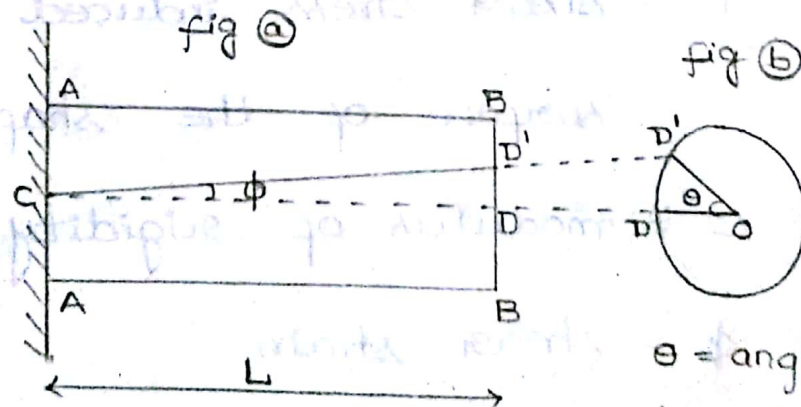
A shaft is said to be in torsion when equal and opposite torques are applied at the ends of the shaft.

The torque is equal to the product of the force applied and the radius of the shaft.

Due to the application of the torques at the two ends, the shaft is subjected to a twisting moment.

This causes the shear stresses and shear strains in the material of the shaft.

Torsion equation:



θ = angle of twist

ϕ = strain angle

* Consider a shaft fixed at A-A. and

Let CD is any line on the outer surface of the shaft.

Now let the shaft is subjected to a torque (T) at the end: $B-B$.

As a result of this torque, the shaft at the end $B-B$ will rotate clockwise and every cross-section of the shaft will be subjected to shear stresses.

The point D will shift to D' and hence line CD will be deflected to CD' as shown in figure.

Let R = radius of the shaft

L = length of the shaft

T = Torque applied at the end.

τ = Shear stress induced at the surface of the shaft.

C = modulus of rigidity

ϕ = shear strain

θ = angle of twist

$$\tan \phi = \frac{BB'}{CB}$$

$$\Rightarrow \phi = \frac{BB'}{CB} = \frac{BB'}{L}$$

Now from fig (b), $BB' = OB \cdot \theta$
 $BB' = R\theta$

$\therefore$ Shear strain at outer surface,

$$\phi = \frac{R\theta}{L}$$

$\therefore$ Modulus of rigidity, $C = \frac{\tau}{\phi}$

$$C = \frac{\text{shear stress}}{\text{shear strain}}$$

$$C = \frac{\tau}{\frac{R\theta}{L}}$$

$$\Rightarrow C = \frac{\tau L}{R\theta}$$

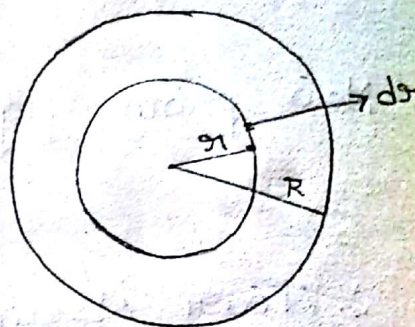
$$\Rightarrow \boxed{\frac{C\theta}{L} = \frac{\tau}{R}}$$

If q is the shear stress induced at a radius ' r ' from the center of the shaft.

$$\therefore \frac{\tau}{R} = \frac{q}{r} = \frac{C\theta}{L}$$

* Consider a shaft subjected to a torque as shown in figure.

* Consider an elementary circular ring of thickness 'dr' at a distance 'r' as shown in figure.



∴ Area of ring, $dA = 2\pi r \cdot dr$

From eqn we get

$$\frac{\tau}{R} = \frac{q}{r}$$

$$\Rightarrow q = \frac{\tau}{R} \cdot r$$

∴ Turning force on the elementary ring

$$= q \cdot dA$$

$$= \frac{\tau}{R} \cdot r \cdot dA$$

∴ Total turning moment on the circular ring

$$dT = \left(\frac{\tau}{R} \cdot r \cdot dA \right) \times r$$

$$T = F \times r$$

$$= \frac{\tau}{R} \cdot r^2 \cdot dA$$

$$T = \int_0^R \frac{\tau}{R} \cdot r^2 \cdot dA$$

$$= \frac{\tau}{R} \int_0^R r^2 \cdot dA$$

$$= \frac{\tau}{R} \cdot J$$

$$\int_0^R r^2 dA = J$$

$$\frac{T}{J} = \frac{\tau}{R}$$

$$\therefore \text{Torsion equation, } \frac{T}{J} = \frac{\tau}{R} = \frac{C\theta}{L} = \frac{\theta}{\phi}$$

Polar modulus:

Polar modulus is defined as the ratio of polar moment of inertia to the radius of the shaft.

It is also called torsional section modulus.

It is denoted by Z_p .

$$Z_p = \frac{J}{R}$$

For solid circular section

$$J = \frac{\pi D^4}{32}, \quad R = \frac{D}{2}$$

$$Z_P = \frac{J}{R}$$

$$= \frac{\pi D^4}{32} \cdot \frac{D}{2}$$

$$Z_P = \frac{\pi D^3}{16}$$

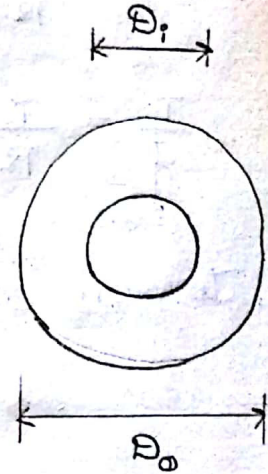
For hollow circular section

$$J = \frac{\pi D_o^4 - \pi D_i^4}{32}$$

$$R = \frac{D_o}{2}$$

$$Z_P = \frac{\frac{\pi(D_o^4 - D_i^4)}{32}}{\frac{D_o}{2}}$$

$$Z_P = \frac{\pi(D_o^4 - D_i^4)}{16 D_o}$$



Strength of a shaft & torsional rigidity

The strength of a shaft means the max. torque or max. power, the shaft can transmit.

Torsional rigidity / stiffness of the shaft is defined as the product of modulus of rigidity (C) and polar moment of inertia of the shaft.

$$\therefore \text{Torsional rigidity} = C \times J$$

torque required to produce a twist of one radian per unit length of the shaft.

- * A solid shaft of 150 mm diameter is used to transmit torque. Find the maximum torque transmitted by the shaft if the max. shear stress is 45 N/mm^2 .

Sol Given: Diameter, $D = 150 \text{ mm}$

Shear stress, $\tau = 45 \text{ N/mm}^2$

Polar modulus, $Z_p = \frac{J}{R}$

$$\begin{aligned} \text{Polar moment of inertia, } J &= \frac{\pi D^4}{32} \\ &= \frac{\pi \times 150^4}{32} \\ &= 49.7 \times 10^6 \text{ mm}^4 \end{aligned}$$

Radius of shaft, $R = \frac{D}{2} = \frac{150}{2}$

$$R = 75 \text{ mm}$$

$$\frac{T}{J} = \frac{\tau}{R}$$

$$\Rightarrow \frac{T}{49.7 \times 10^6} = \frac{45}{75}$$

$$T = 29.82 \text{ kN-m}$$

* The shearing stress of a solid shaft is not to exceed $40 \frac{\text{N}}{\text{mm}^2}$ when a torque transmitted is $20,000 \text{ N-m}$. Determine the minimum diameter of the shaft.

Sol Given: Shear stress, $\tau = 40 \text{ N/mm}^2$

Torque, $T = 20,000 \text{ N-m}$

$$= 20,000 \times 10^3 \text{ N-mm}$$

$$\frac{T}{J} = \frac{\tau}{R}$$

$$\Rightarrow \frac{20,000 \times 10^3}{\frac{\pi D^4}{32}} = \frac{40}{\frac{D}{2}}$$

$$\Rightarrow \frac{203.7 \times 10^6}{D^4} = \frac{80}{D}$$

$$\Rightarrow D^3 = \frac{203.7 \times 10^6}{80}$$

$$D^3 = 2546250$$

$$D = 136 \text{ mm}$$

$$D = 136.5 \text{ mm}$$

Power, $P = \frac{2\pi N}{60} \cdot T$ Watts

N = speed in rpm

T = torque.

* In a hollow circular shaft of outer and inner diameters of 20 cm and 10 cm respectively. The shear stress is not to exceed 40 N/mm^2 . Find the maximum torque which the shaft can carry?

Sol Given: Shear stress, $\tau = 40 \text{ N/mm}^2$

External diameter, $D_o = 20 \text{ cm}$
 $= 200 \text{ mm}$

Internal diameter, $D_i = 10 \text{ cm}$
 $= 100 \text{ mm}$

Torque, $T = ?$

$$J = \frac{\pi (D_o^4 - D_i^4)}{32}$$

$$= \frac{\pi (200^4 - 100^4)}{32}$$

$$J = 147.26 \times 10^6 \text{ mm}^4$$

$$\frac{T}{J} = \frac{\tau}{R}$$

$$\Rightarrow \frac{T}{147.26 \times 10^6} = \frac{40}{100}$$

$$\Rightarrow T = 58.9 \times 10^6 \text{ N-mm}$$

$$T = 58.9 \text{ kN-m}$$

* Find the maximum shear stress induced in a solid circular shaft of diameter 15 cm. When the shaft transmit 150 kW at 180 rpm.

Sol Given: Diameter, $D = 15 \text{ cm}$

$$= 150 \text{ mm}$$

Power transmitted, $P = 150 \text{ kW}$

$$= 150 \times 10^3 \text{ W}$$

$$N = 180 \text{ rpm.}$$

$$\text{Power, } P = \frac{2\pi N T}{60} \text{ Watts}$$

$$150 \times 10^3 = \frac{2 \times \pi \times 180}{60} \times T$$

$$J = \frac{\pi D^4}{32} = \frac{\pi \times 150^4}{32}$$

$$J = 49.7 \times 10^6 \text{ mm}^4$$

$$R = \frac{D}{2} = \frac{150}{2} = 75 \text{ mm}$$

$$\frac{T}{J} = \frac{\tau}{R}$$

$$\Rightarrow \frac{4957.74 \times 10^3}{49.7 \times 10^6} = \frac{\tau}{75}$$

$$\tau = 12 \text{ N/mm}^2$$

* Determine the diameter of solid steel shaft which will transmit 90 kW at 160 rpm. Also determine the length of the shaft if the twist must not exceed

1°. The max. shear stress is 60 N/mm²,

Modulus of rigidity is $84 \times 10^4 \text{ N/mm}^2$.

Sol Given: Power, $P = 90 \text{ kW}$
 $= 90 \times 10^3 \text{ W}$

$N = 160 \text{ rpm}$

Shear stress, $\tau = 60 \text{ N/mm}^2$

Angle of twist, $\theta = 1^\circ$

$$\theta = 0.017$$

Power transmitted, $P = \frac{2\pi N T}{60}$

$$\Rightarrow 90 \times 10^3 = \frac{2 \times \pi \times 160 \times T}{60}$$

$$\Rightarrow T = 5371.47 \text{ N-m.}$$

$$\frac{T}{J} = \frac{\tau}{R} = \frac{C\theta}{L}$$

$$\Rightarrow \frac{5371.47 \times 10^3}{\frac{\pi D^4}{32}} = \frac{60}{\frac{D}{2}} = \frac{8 \times 10^4 \times 0.017}{L}$$

$$\frac{54.71 \times 10^6}{D^4} = \frac{120}{D} = \frac{1360}{L}$$

$$\Rightarrow \frac{54.71 \times 10^6}{D^3} = 120$$

$$\Rightarrow D^3 = 46.96 \text{ mm}$$

$$\frac{120}{46.96} = \frac{1360}{L}$$

$$\Rightarrow L = 872.21 \text{ mm}$$

Two shafts of the same material and of same length are subjected to same torque. If the first shaft is of a solid circular section and the second shaft is of hollow circular section, whose internal dia. is $\frac{2}{3}$ rd of the outer dia. And the max. shear stress developed in each shaft is the same. Compare the weights of the shafts.

31 Let T = torque transmitted by shaft.

τ = max. shear stress developed in each shaft.

D = diameter of solid circular shaft.

D_o = external diameter of hollow circular shaft.

D_i = internal diameter of hollow circular shaft.

$$= \frac{2}{3} D_o$$

L = length of each shaft.

w = weight density of the material.

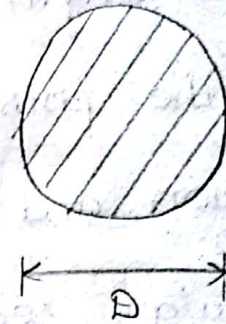
Case-1: Solid circular shaft.

$$\frac{T}{J} = \frac{\tau}{R}$$

$$\frac{T}{\frac{\pi D^4}{32}} = \frac{\tau}{\frac{D}{2}}$$

$$\frac{T}{\frac{\pi D^3}{16}} = \tau$$

$$T = \frac{\pi D^3}{16} \tau \quad \rightarrow \textcircled{1}$$



Case-2: Hollow circular shaft.

$$\frac{T}{J} = \frac{\tau}{R}$$

$$J = \frac{\pi (R_o^4 - R_i^4)}{32}$$

$$= \frac{\pi}{32} \left(R_o^4 - \left(\frac{2}{3} R_o \right)^4 \right)$$

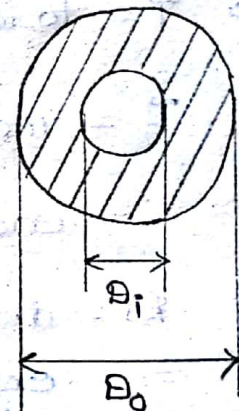
$$= \frac{\pi R_o^4}{32} \left(1 - \frac{16}{81} \right)$$

$$= \frac{\pi R_o^4}{32} \left(\frac{65}{81} \right)$$

$$R = \frac{R_o}{2}$$

$$\frac{T \times 32 \times 81}{\pi R_o^4 \times 65} = \frac{2\tau}{R_o}$$

$$\Rightarrow \frac{T \times 2592}{\pi R_o^3 \times 65} = 2\tau$$



$$T = \frac{\pi D_0^3 \tau \times 65}{1296} \quad \rightarrow (2)$$

$\therefore$ Torque transmitted by each shaft is equal

$$\frac{\pi D^3}{16} \tau = \frac{\pi D_0^3 \tau \times 65}{1296}$$

$$\frac{D^3}{16} = \frac{D_0^3 \times 65}{1296}$$

$$D^3 = \frac{D_0^3 \times 65 \times 16}{1296}$$

$$D^3 = \frac{D_0^3 \times 65}{81}$$

$$D = \sqrt[3]{\frac{D_0^3 \times 65}{81}}$$

$$D = 0.929 D_0$$

$\therefore$ Weight of solid shaft, $W_s = \rho \times V$

$$W_s = \omega \times V$$

$$= \omega \times \frac{\pi}{4} D^2 \times L$$

Weight of hollow shaft, $W_h = \omega \times V$

$$W_h = \omega \times \frac{\pi}{4} (D_0^2 - D_i^2) \times L$$

$$= \omega \times \frac{\pi}{4} \left(D_0^2 - \left(\frac{2}{3} D_0 \right)^2 \right) \times L$$

$$= \omega \times \frac{\pi}{4} D_0^2 \left(1 - \frac{4}{9} \right) \times L$$

$$\begin{aligned}
 \text{Ratio of weights} &= \frac{W_s}{W_h} \\
 &= \frac{W_s \times \frac{\pi}{4} D^2 \times L}{W_s \times \frac{\pi}{4} D_o^2 \times \frac{5}{9} \times L} \\
 &= \frac{D^2}{D_o^2 \times \frac{5}{9}} \\
 &= \frac{9 \times (0.929 D_o)^2}{D_o^2 \times 5} \\
 &= 1.55
 \end{aligned}$$

$$\frac{W_s}{W_h} = 1.55$$

$$W_s = 1.55 W_h$$

* A hollow shaft of external diameter 120 mm transmit 300 kW power at 200 rpm.

Determine the internal diameter if the max. shear stress is not to exceed 60 N/mm²

Sol Given: External diameter, $D_o = 120$ mm

Power transmitted, $P = 300$ kW

$$= 300 \times 10^3 \text{ W}$$

Speed, $N = 200$ rpm

Internal dia, $\phi_i = ?$

$$\text{Power, } P = \frac{2\pi NT}{60}$$

$$\Rightarrow 300 \times 10^3 = \frac{2\pi \times 300 \times T}{60}$$

$$T = 14323.9 \text{ N-m.}$$

$$\frac{T}{J} = \frac{\tau}{R}$$

$$\Rightarrow \frac{14323.9 \times 10^3}{\pi (\phi_o^4 - \phi_i^4) \times 32} = \frac{\tau}{\frac{\phi_o}{2}}$$

$$\Rightarrow \frac{32 \times 14323.9 \times 10^3}{\pi (120^4 - \phi_i^4)} = \frac{60 \times 2}{120}$$

$$120^4 - \phi_i^4 = 145.9 \times 10^6$$

$$\phi_i^4 = 120^4 - 145.9 \times 10^6$$

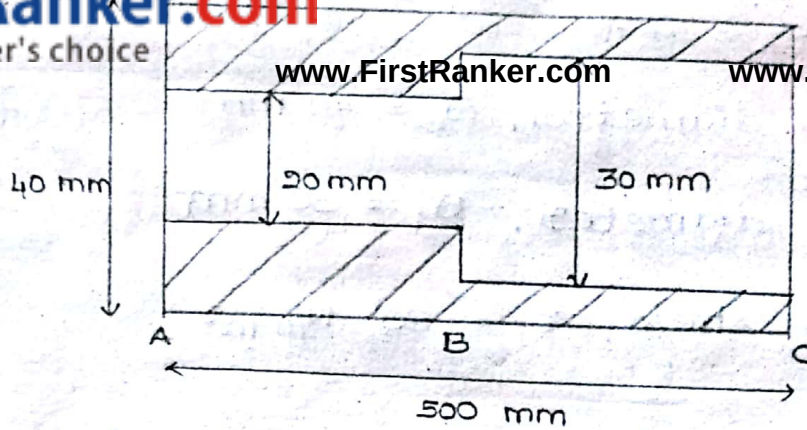
$$\phi_i^4 = 61460000.$$

$$\phi_i = 88.54 \text{ mm.}$$

- * When a shaft is made up of different lengths and of different diameters, the torque transmitted by individual section should be calculated first.
- * This strength of such a shaft is the minimum value of this torques.

A shaft ABC of 500 mm length and 40 mm external diameter is bored for a part of its length AB to a 20 mm diameter and for the remaining length BC to a 30 mm diameter bored. If the shear stress is not to exceed 80 N/mm^2 find the maximum power the shaft can transmit at a speed of 200 rpm.

If the angle of twist in the shaft length of 20 mm diameter bored is equal to ^{that in} _x the 30 mm diameter bored. Find the length of the shaft that has been bored to 20 mm and 30 mm diameter.



Given:

Shaft AB:

Outer diameter, $D_o = 40 \text{ mm}$

Inner diameter, $D_i = 20 \text{ mm}$

Shear stress, $\tau = 80 \text{ N/mm}^2$

Polar moment of inertia, $J = \frac{\pi}{32} (D_o^4 - D_i^4)$

$$= \frac{\pi}{32} (40^4 - 20^4)$$

$$= 235.61 \times 10^3 \text{ mm}^4$$

$$\text{Radius, } R = \frac{D_o}{2} = \frac{40}{2} = 20 \text{ mm}$$

$$\frac{T}{J} = \frac{\tau}{R}$$

$$\Rightarrow \frac{T}{235.61 \times 10^3} = \frac{80}{20}$$

$$\Rightarrow T = 942.44 \times 10^3 \text{ N-mm}$$

$$T = 942.44 \text{ N-m}$$

Outer diameter, $D_o = 40 \text{ mm}$

Inner diameter, $D_i = 30 \text{ mm}$

Shear stress, $\tau = 80 \text{ N/mm}^2$

$$\text{Polar moment of inertia, } J = \frac{\pi}{32} (40^4 - 30^4) \\ = 171.8 \times 10^3 \text{ mm}^4$$

$$\text{Radius, } R = \frac{D_o}{2} = \frac{40}{2} = 20 \text{ mm}$$

$$\frac{T}{J} = \frac{\tau}{R}$$

$$\Rightarrow \frac{T}{171.8 \times 10^3} = \frac{80}{20}$$

$$T = 687.2 \times 10^3 \text{ N-mm}$$

$$T = 687.2 \text{ N-m}$$

$\therefore$ Strength of a shaft $(T) = 687.2 \text{ N-m}$

$$\therefore \text{Power, } P = \frac{2\pi NT}{60}$$

$$= \frac{2 \times \pi \times 200 \times 687.2}{60}$$

$$= 14.39 \times 10^3 \text{ Watts}$$

$$= 14.39 \text{ kW}$$

$$\text{Torsion equation, } \frac{T}{J} = \frac{C\theta}{L}$$

For shaft AB, $\theta = \frac{TL_1}{CJ_1}$

For shaft BC, $\theta = \frac{TL_2}{CJ_2}$

$$\therefore \frac{TL_1}{CJ_1} = \frac{TL_2}{CJ_2}$$

$$\frac{L_1}{J_1} = \frac{L_2}{J_2}$$

$$\frac{L_1}{235.61 \times 10^3} = \frac{L_2}{171.8 \times 10^3}$$

$$L_1 = 1.371 L_2$$

from diagram, $L_1 + L_2 = 500$

$$\Rightarrow 1.371 L_2 + L_2 = 500$$

$$L_2 = 210.88 \text{ mm}$$

$$L_1 + 210.88 = 500$$

$$\Rightarrow L_1 = 289.12 \text{ mm}$$

Combined bending moment & torsion:

* When a shaft is transmitting torque it is subjected to shear stress.

* At the same time the shaft is also subjected to bending moment and

* Hence each particle in the shaft is subjected to shear stress and bending stress.

* $\therefore$ It is necessary to find the principal stresses and maximum shear stress.

* Let T = torque at the section.

D = diameter of the shaft.

M = Bending moment.

q = shear stress

σ = Bending stress at a point produced by bending moment.

* $\therefore$ Shear stress is given by.

$$\frac{q}{r} = \frac{T}{J}$$

$$\Rightarrow q = \frac{T}{J} \cdot r$$

$$= \frac{T \times 32}{\pi D^4} \times \frac{D}{2}$$

$$q = \frac{16T}{\pi D^3}$$

given by

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$\sigma = \frac{M}{I} \cdot y$$

$$\sigma = \frac{M}{I} \cdot y$$

$$= \frac{64 M}{\pi D^4} \times \frac{D}{2}$$

$$\sigma_x = \frac{32 M}{\pi D^3}$$

* Major principal stress,

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2}$$

$$= \frac{\sigma_x}{2} + \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau^2}$$

$$= \frac{32 M}{\pi D^3 \times 2} + \sqrt{\left(\frac{32 M}{\pi D^3 \times 2}\right)^2 + \left(\frac{16 T}{\pi D^3}\right)^2}$$

$$= \frac{16 M}{\pi D^3} + \frac{16}{\pi D^3} \sqrt{\left(\frac{M}{2}\right)^2 + T^2}$$

$$\sigma_1 = \frac{16}{\pi D^3} \left(M + \sqrt{M^2 + T^2} \right)$$

* Minimum principal stress,

$$\sigma_2 = \frac{16}{\pi D^3} \left(M - \sqrt{M^2 + T^2} \right)$$

$$\tau = \frac{\frac{16}{\pi D^3} (M + \sqrt{M^2 + T^2}) - \frac{16}{\pi D^3} (M - \sqrt{M^2 + T^2})}{2}$$

$$= \frac{16}{\pi D^3} \sqrt{M^2 + T^2}$$

Position of plane, $\tan 2\theta = \frac{T}{M}$
For hollow circular shaft:

$$\sigma_1 = \frac{16 D_o}{\pi (D_o^4 - D_i^4)} (M + \sqrt{M^2 + T^2})$$

$$\sigma_2 = \frac{16 D_o}{\pi (D_o^4 - D_i^4)} (M - \sqrt{M^2 + T^2})$$

$$\tau = \frac{16 D_o}{\pi (D_o^4 - D_i^4)} \sqrt{M^2 + T^2}$$

* A solid shaft of diameter 80 mm is subjected to a twisting moment of 8 MN-mm and a bending moment of 5 MN-mm at a point. Determine.

(i) Principal stresses

(ii) Position of the plane.

Given: Diameter, $D = 80 \text{ mm}$

Twisting moment, $T = 8 \text{ MN-mm}$

$(T, \text{ N-mm}) = 8 \times 10^6 \text{ N-mm}$

Bending moment, $M = 5 \text{ MN-mm}$

① Principal stresses

Major principal stress, $\sigma_1 = \frac{16}{\pi D^3} (M + \sqrt{M^2 + T^2})$

$$\sigma_1 = \frac{16}{\pi \times 80^3} \left((5 \times 10^6) + \sqrt{(5 \times 10^6)^2 + (8 \times 10^6)^2} \right)$$

$$= 9.94 \times 10^{-6} (14.43 \times 10^6)$$

$$= 143.43 \text{ N/mm}^2$$

Minor principal stress, $\sigma_2 = \frac{16}{\pi D^3} (M - \sqrt{M^2 + T^2})$

$$\sigma_2 = \frac{16}{\pi \times 80^3} \left((5 \times 10^6) - \sqrt{(5 \times 10^6)^2 + (8 \times 10^6)^2} \right)$$

$$= 9.94 \times 10^{-6} (-4.43 \times 10^6)$$

$$= -44.03 \text{ N/mm}^2 \text{ (tensile)}$$

② Position of the plane

$$\tan 2\theta = \frac{T}{M}$$

$$= \frac{8 \times 10^6}{5 \times 10^6}$$

$$= 1.6$$

$$2\theta = 57.99$$

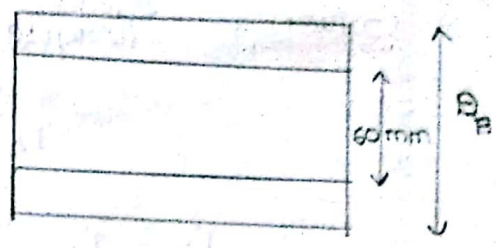
$$\theta = 28.99$$

- * A shaft made up of two or more different materials and behaving as a single shaft is known as composite shaft.
- * But the angle of twist in each shaft will be equal.
- * The total torque transmitted by the shaft is the sum of the torques transmitted by each individual shaft.
- * A composite shaft consist of steel rod 60mm diameter surrounded by a closely fitted tube of brass. Find the outside diameter of the tube so that when a torque of $1000 \text{ N}\cdot\text{m}$ is applied to the composite shaft, it will be shared equally by the two materials. Take C for steel $= 8.4 \times 10^4 \text{ N/mm}^2$ and C for brass $= 4.2 \times 10^4 \text{ N/mm}^2$.
Find also the max. shear stress in each

a length of 4m.

Sol For steel shaft:

Diameter, $D_s = 60 \text{ mm}$



Polar moment of inertia, $J_s = \frac{\pi D_s^4}{32} = \frac{\pi \times 60^4}{32} \text{ mm}^4$

$$J_s = \frac{\pi \times 60^4}{32} \text{ mm}^4$$

Let τ_s be the shear stress developed in steel shaft.

Let T_s be the torque transmitted by steel shaft.

Let θ_s be the angle of twist for steel shaft

For Brass shaft:

Outer diameter, $D_o = ?$

Inner diameter, $D_i = 60 \text{ mm}$

Polar moment of inertia, $J_B = \frac{\pi (D_o^4 - D_i^4)}{32}$
 $= \frac{\pi}{32} (D_o^4 - 60^4) \text{ mm}^4$

Let τ_b = shear stress developed in brass shaft.

T_L = torque transmitted by brass shaft.

Given Torque, $T = 1000 \text{ N-m}$

$$T_A + T_B = 1000$$

$$T_A = T_B$$

$$\therefore T_A = 500 \text{ N-m}$$

$$T_B = 500 \text{ N-m}$$

Torque equation: $\frac{T}{J} = \frac{C\theta}{L}$

$$T = \frac{C\theta J}{L}$$

$$\text{For steel, } T_A = \frac{C_A \theta_A J_A}{L_A}$$

$$\text{For Brass, } T_B = \frac{C_B \theta_B J_B}{L_B}$$

$$T_A = T_B \Rightarrow \frac{C_A \theta_A J_A}{L_A} = \frac{C_B \theta_B J_B}{L_B}$$

$$\Rightarrow C_A J_A = C_B J_B \quad \theta_A = \theta_B$$

$$L_A = L_B$$

$$\frac{C_A}{C_B} = \frac{J_B}{J_A}$$

$$\Rightarrow \frac{8.4 \times 10^4}{4.2 \times 10^4} = \frac{\frac{\pi}{32} (80^4 - 60^4)}{\frac{\pi}{32} \times 60^4}$$

$$D_o^4 = (25.92 \times 10^6) + 60^4$$

$$= 38.88 \times 10^6$$

$$D_o = 78.96 \text{ mm}$$

$$D_o \approx 79 \text{ mm}$$

Torsion equation: $\frac{T}{J} = \frac{\tau}{R}$

$$\Rightarrow \tau = \frac{TR}{J}$$

For steel, $\tau_s = \frac{T_s R_s}{J_s}$

$$= \frac{500 \times 10^3 \times \frac{60}{2}}{\frac{\pi}{32} \times 60^4}$$

$$\tau_s = 11.78 \text{ N/mm}^2$$

For Brass, $\tau_b = \frac{T_b R_b}{J_b}$

$$= \frac{500 \times 10^3 \times \frac{79}{2}}{\frac{\pi}{32} (79^4 - 60^4)}$$

$$= \frac{19.75 \times 10^6}{2.55 \times 10^6}$$

$$\tau_b = 7.74 \text{ N/mm}^2$$

Torsion equation: $\frac{T}{J} = \frac{C\theta}{L} \Rightarrow \theta = \frac{TL}{CJ}$

For Δ

$$\theta = \frac{500 \times 10^3 \times 4 \times 10^3}{8.14 \times 10^4 \times \frac{\pi}{32} \times 60^4}$$

$$= \frac{2000 \times 10^6}{0.1 \times 10^{12}}$$

$$= 2000 \times 0.0187 \text{ radians}$$

$$\approx 0.02 \text{ radians}$$

Columns & Struts

Column or strut is defined as a member of structure which is subjected to axial compressive load.

* If the member of the structure is vertical and both of its ends are fixed while subjected to compressive load, the member is known as column.

* If the member of the structure is not vertical is known as strut.

Failure of a column:

The failure of a column takes place due to any one of the following stresses.

- ① Direct compressive stress
- ② Buckling stress
- ③ Combined direct compressive and buckling stress.

②

① Failure of a short column:

The compressive stress induced is given by

$$\sigma = \frac{P}{A}$$

* If the compressive load on the short column is gradually increased, a stage will reach when the column will be on the point of failure by crushing.

* The stress induced in the column corresponding to this load is known as crushing stress and the load is called crushing load.

② Failure of a long column:

* A long column is known as long column if the length of the column in comparison to its lateral dimensions is very large. Such columns do not fail by crushing alone but also by bending the load at which the column just buckles is known as buckling load (or) crippling load.

The buckling load is less than the crushing load for a long column.

Let: l = length of a long column.

P = load at which the column has just buckled.

A = cross-sectional area of column.

e = max. bending of the column at the center.

σ_0 = stress due to direct load: $= \frac{P}{A}$

σ_b = stress due to bending at the center of column. $= \frac{M}{Z} = \frac{Pe}{Z}$

Max. stress $= \sigma_0 + \sigma_b$

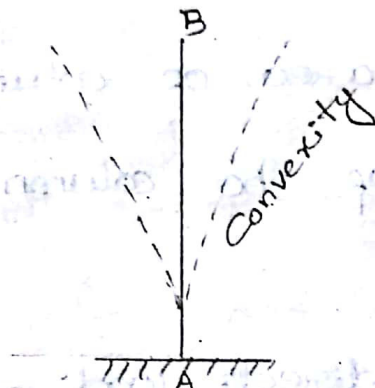
Min. stress $= \sigma_0 - \sigma_b$

Assumptions made in Euler's column theory:

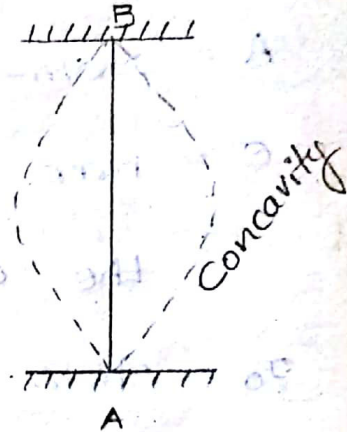
- ① Column is initially perfect straight.
- ② Load is applied axially.
- ③ Cross-section of the column is uniform.
- ④ The length of the column is very large as compared to other dimensions.
- ⑤ The direct stress is small as compared

- ④ ⑤ The self-weight of the column is negligible.

Sign conventions:



① positive.



② Negative.

End conditions:

- * Both ends are fixed.
- * One end is fixed, other end free.
- * Both ends are hinged.
- * One end is fixed, other end hinged.

1/3

Expression for crippling load when both the ends of the column are hinged:

- * The load at which the column just buckles is called 'crippling load'.

* Consider a column AB of length 'l'

area. hinged at both ends.

A and B.

* Let P be the crippling load at which the column has just buckled.



* Due to the crippling load, the column will deflect into a curved shape as shown in figure.

* Consider any section at a distance ' x ' from B.

* Let y = deflection at the section.

* The moment due to crippling load.

$$M_x = -Py$$

$$\therefore EI \frac{d^2y}{dx^2} = M_x$$

$$\Rightarrow EI \frac{d^2y}{dx^2} = -Py$$

$$EI \frac{d^2y}{dx^2} + Py = 0$$

Dividing by ' EI ' on both sides

$$\frac{d^2y}{dx^2} + \frac{P}{EI} \cdot y = 0$$

$$\frac{d^2y}{dx^2} = -\theta^2$$

$$\theta^2 y + \frac{P}{EI} \cdot y = 0$$

$$\theta^2 + \frac{P}{EI} = 0$$

$$m^2 + \frac{P}{EI} = 0$$

$$m^2 = -\frac{P}{EI}$$

$$m = \pm i \sqrt{\frac{P}{EI}}$$

$$m = \alpha \pm i\beta \Rightarrow y_c = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$$

$$y_c = y = e^{0 \cdot x} \left(C_1 \cos \sqrt{\frac{P}{EI}} x + C_2 \sin \sqrt{\frac{P}{EI}} x \right)$$

$$y = C_1 \cos x \cdot \sqrt{\frac{P}{EI}} + C_2 \sin x \cdot \sqrt{\frac{P}{EI}}$$

$$\text{At } x=0, y=0$$

$$\Rightarrow 0 = C_1 \cos 0 + C_2 \sin 0$$

$$0 = C_1 + 0$$

$$\boxed{C_1 = 0}$$

$$\text{At } x=l, y=0$$

$$\Rightarrow 0 = C_1 \cos 0 + C_2 \sin l \sqrt{\frac{P}{EI}}$$

$$C_2 \sin l \sqrt{\frac{P}{EI}} = 0$$

$$\sin l \sqrt{\frac{P}{EI}} = 0$$

$$\sin l \sqrt{\frac{P}{EI}} = \sin \pi$$

$$\frac{P}{EI} = \frac{\pi^2}{l^2}$$

$$P = \frac{\pi^2 EI}{l^2}$$

Expression for crippling load when one end of the column is fixed and other end is free:

Let y = deflection at any distance ' x ' from A.

$\therefore$ Bending moment at the section, $M_x = P(a-y)$

$$EI \frac{d^2 y}{dx^2} = M_x$$

$$\Rightarrow EI \frac{d^2 y}{dx^2} = P(a-y)$$

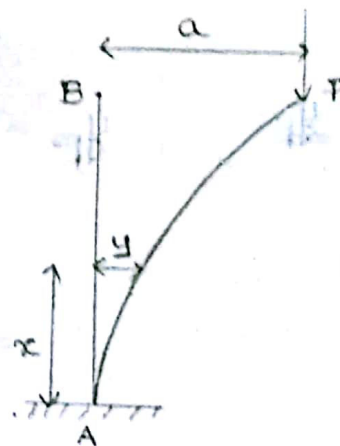
$$EI \frac{d^2 y}{dx^2} - P(a-y) = 0$$

$$EI \frac{d^2 y}{dx^2} + Py = Pa$$

$$\frac{d^2 y}{dx^2} + \frac{P}{EI} y = \frac{Pa}{EI}$$

$$D^2 y + \frac{P}{EI} y = \frac{Pa}{EI}$$

$$y \left(D^2 + \frac{P}{EI} \right) = \frac{Pa}{EI}$$



yc

$$y \left(m^2 + \frac{P}{EI} \right) = 0$$

$$m^2 + \frac{P}{EI} = 0$$

$$m^2 = -\frac{P}{EI}$$

$$m = \pm i \sqrt{\frac{P}{EI}}$$

$$y_c = C_1 \cos x \sqrt{\frac{P}{EI}} + C_2 \sin x \sqrt{\frac{P}{EI}}$$

yp

$$y_p = \frac{Pa}{EI \cdot \left(\frac{d^2}{dx^2} + \frac{P}{EI} \right)}$$

$$= \frac{\frac{Pa}{EI}}{\frac{d^2}{dx^2} + \frac{P}{EI}}$$

$$= \frac{a}{\left(1 + \frac{EI}{P} \frac{d^2}{dx^2} \right)}$$

$$= a \left(1 + \frac{EI}{P} \frac{d^2}{dx^2} \right)^{-1}$$

$$= a \left(1 - \frac{EI}{P} \frac{d^2}{dx^2} \right)$$

$$= a - \frac{EI}{P} \frac{d^2 a}{dx^2}$$

$$y_p = a$$

$$\therefore y = y_c + y_p$$

$$y = C_1 \cos x \sqrt{\frac{P}{EI}} + C_2 \sin x \sqrt{\frac{P}{EI}} + a$$

(9)

$$\Rightarrow 0 = c_1 \cos 0 + c_2 \sin 0 + a$$

$$0 = c_1 + a$$

$$\boxed{c_1 = -a}$$

(ii) At $x=0, \frac{dy}{dx}=0$

$$\frac{dy}{dx} = c_1 \left(-\sin x \sqrt{\frac{P}{EI}} \right) \sqrt{\frac{P}{EI}} + c_2 \left(\cos x \sqrt{\frac{P}{EI}} \right) \sqrt{\frac{P}{EI}} + 0$$

$$\Rightarrow 0 = c_1 \sin 0 + c_2 \cos 0 \sqrt{\frac{P}{EI}}$$

$$c_2 \sqrt{\frac{P}{EI}} = 0$$

$$\therefore \boxed{c_2 = 0}$$

$$y = -a \cos x \sqrt{\frac{P}{EI}} + a$$

At $x=l, y=a$

$$a = -a \cos l \sqrt{\frac{P}{EI}} + a$$

$$-a \cos l \sqrt{\frac{P}{EI}} = 0$$

$$\cos l \sqrt{\frac{P}{EI}} = 0$$

$$\cos l \sqrt{\frac{P}{EI}} = \cos \frac{\pi}{2}$$

$$l \sqrt{\frac{P}{EI}} = \frac{\pi}{2}$$

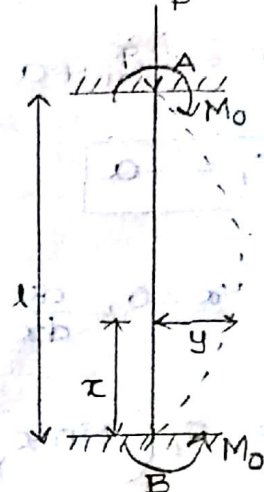
$$\frac{P}{EI} = \frac{\pi^2}{4l^2} \Rightarrow$$

$$\boxed{P = \frac{\pi^2 EI}{4l^2}}$$

10) ends of the column are fixed:

Let y = deflection at a section x from B as shown in fig.

M_0 = fixed end moment at A and B.



$\therefore$ Bending moment, $M = M_0 - Py$

$$EI \frac{d^2 y}{dx^2} = M_x$$

$$EI \frac{d^2 y}{dx^2} = M_0 - Py$$

$$EI \frac{d^2 y}{dx^2} + Py = M_0$$

$$\frac{d^2 y}{dx^2} + \frac{P}{EI} \cdot y = \frac{M_0}{EI}$$

$$y = y_c + y_p$$

y_c

$$\frac{d^2 y}{dx^2} = 0 \Rightarrow 0 + \frac{P}{EI} \cdot y = 0$$

$$y \left(0 + \frac{P}{EI} \right) = 0$$

$$m^2 + \frac{P}{EI} = 0$$

$$m^2 = - \frac{P}{EI}$$

$$m = \pm i \sqrt{\frac{P}{EI}}$$

y_p

$$y_p = \frac{\frac{M_0}{EI}}{\left(D^2 + \frac{P}{EI}\right)}$$

$$= \frac{\frac{M_0}{EI}}{\frac{P}{EI} \left(1 + \frac{D^2}{\frac{P}{EI}}\right)}$$

$$= \frac{\frac{P}{EI} \cdot \frac{M_0}{P}}{\frac{P}{EI} \left(1 + \frac{EI}{P} D^2\right)}$$

$$= \frac{M_0}{P} \left(1 + \frac{EI}{P} D^2\right)^{-1}$$

$$= \frac{M_0}{P} \left(1 - \frac{EI}{P} D^2\right)$$

$$= \frac{M_0}{P} - \frac{EI}{P} D^2 \left(\frac{M_0}{P}\right)$$

$$y_p = \frac{M_0}{P} - \frac{EI}{P} (0) \quad D^2 = \frac{d^2}{dx^2}$$

$$y_p = \frac{M_0}{P}$$

$$\therefore y = y_c + y_p$$

$$y = c_1 \cos x \sqrt{\frac{P}{EI}} + c_2 \sin x \sqrt{\frac{P}{EI}} + \frac{M_0}{P}$$

$$\text{At } x=0, y=0$$

$$\Rightarrow 0 = c_1 \cos 0 + c_2 \sin 0 + \frac{M_0}{P}$$

$$0 = c_1 + \frac{M_0}{P}$$

$$\Rightarrow c_1 = -\frac{M_0}{P}$$

(12)

$$\Rightarrow \frac{dy}{dx} = c_1 \left(-\sin x \sqrt{\frac{P}{EI}} \right) \sqrt{\frac{P}{EI}} + c_2 \left(\cos x \sqrt{\frac{P}{EI}} \right) \sqrt{\frac{P}{EI}} + 0$$

$$\Rightarrow 0 = c_1 (-\sin 0) + c_2 \cos 0 \sqrt{\frac{P}{EI}}$$

$$0 = 0 + c_2 \cdot \sqrt{\frac{P}{EI}}$$

$$c_2 \sqrt{\frac{P}{EI}} = 0$$

$$\boxed{c_2 = 0}$$

At $x=0$

$$y = -\frac{M_0}{P} \cos x \sqrt{\frac{P}{EI}} + \frac{M_0}{P}$$

At $x=1$, $y=0$

$$\Rightarrow 0 = -\frac{M_0}{P} \cos 1 \sqrt{\frac{P}{EI}} + \frac{M_0}{P}$$

$$\Rightarrow \frac{M_0}{P} \cos 1 \sqrt{\frac{P}{EI}} = \frac{M_0}{P}$$

$$\cos 1 \sqrt{\frac{P}{EI}} = 1$$

$$\cos 1 \sqrt{\frac{P}{EI}} = \cos 0$$

$$1 \sqrt{\frac{P}{EI}} = 0$$

$$\frac{P}{EI} = \frac{4\pi^2}{12}$$

$$\boxed{P = \frac{4\pi^2 EI}{12}}$$

3) is fixed and other end is hinged

Let y = deflection at a section
 x from B (as shown in
fig.

H = horizontal reaction at A



$\therefore$ Bending moment, $M = H(1-x) - Py$

$$EI \frac{d^2y}{dx^2} = M(x) \quad ||$$

$$\Rightarrow EI \frac{d^2y}{dx^2} = H(1-x) - Py$$

$$EI \frac{d^2y}{dx^2} + Py = H(1-x)$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI} \cdot y = \frac{H}{EI} (1-x)$$

$$y = y_c + y_p$$

y_c

$$\frac{d^2}{dx^2} = 0$$

$$0^2 y + \frac{P}{EI} \cdot y = 0$$

$$y \left(0^2 + \frac{P}{EI} \right) = 0$$

$$y \left(m^2 + \frac{P}{EI} \right) = 0$$

$$m^2 + \frac{P}{EI} = 0$$

$$m^2 = -\frac{P}{EI}$$

$$m = \pm i \sqrt{\frac{P}{EI}}$$

1)

$$= \frac{\frac{H}{EI} \left(\frac{H}{P} \right) (1-x)}{\frac{H}{EI} \left(1 + \frac{B^2}{\frac{P}{EI}} \right)}$$

$$= \frac{\frac{H}{P} (1-x)}{\left(1 + \frac{EI}{P} B^2 \right)}$$

$$= \frac{H}{P} (1-x) \left(1 + \frac{EI}{P} B^2 \right)^{-1}$$

$$= \frac{H}{P} (1-x) \left(1 - \frac{EI}{P} B^2 \right)$$

$$= \frac{H}{P} (1-x) - \frac{EI}{P} B^2 \frac{H}{P} (1-x)$$

$$y_p = \frac{H}{P} (1-x)$$

$$\frac{d^2}{dx^2} \frac{H}{P} (1-x)$$

$$\frac{H}{P} \frac{d^2}{dx^2} (1-x)$$

$$y = y_c + y_p$$

$$\Rightarrow y = c_1 \cosh x \sqrt{\frac{P}{EI}} + c_2 \sinh x \sqrt{\frac{P}{EI}} + \frac{H}{P} \frac{d}{dx} (0-1)$$

$$\frac{H}{P} (0-0)$$

$$0 = \left(\frac{H}{EI} + \frac{H}{P} (1-x) \right)$$

$$\text{At } x=0, y=0$$

$$\Rightarrow 0 = c_1 \cosh 0 + c_2 \sinh 0 + \frac{H(1-0)}{P}$$

$$c_1 = -\frac{H}{P}$$

$$\frac{dy}{dx} = C_1 \left(-\sin x \sqrt{\frac{P}{EI}} \right) \left(\sqrt{\frac{P}{EI}} \right) + C_2 \left(\cos x \sqrt{\frac{P}{EI}} \right) \left(\sqrt{\frac{P}{EI}} \right)$$

$$0 = C_1 \sin 0 + C_2 \cos 0 \sqrt{\frac{P}{EI}} + \frac{H(1-\phi)}{P}$$

$$C_2 \sqrt{\frac{P}{EI}} - \frac{H}{P} = 0$$

$$C_2 = \frac{H}{P} \cdot \sqrt{\frac{EI}{P}}$$

$$\text{At } x=L, y=0$$

$$0 = -\frac{HL}{P} \cos L \sqrt{\frac{P}{EI}} + \frac{H}{P} \sqrt{\frac{EI}{P}} \sin L \sqrt{\frac{P}{EI}}$$

$$\frac{H}{P} \sqrt{\frac{EI}{P}} \sin L \sqrt{\frac{P}{EI}} = \frac{HL}{P} \cos L \sqrt{\frac{P}{EI}}$$

$$\tan L \sqrt{\frac{P}{EI}} = \frac{HL}{P} \cdot \frac{P}{H} \sqrt{\frac{P}{EI}}$$

$$\tan L \sqrt{\frac{P}{EI}} = L \sqrt{\frac{P}{EI}}$$

$$L \sqrt{\frac{P}{EI}} = 4.5$$

$$\tan x = x$$

$$x = 4.5$$

$$\frac{P}{EI} = \left(\frac{4.5}{L} \right)^2$$

$$P = \frac{20.25 EI}{L^2}$$

$$P = \frac{2\pi^2 EI}{L^2}$$

Both ends are hinged - $\frac{\pi^2 EI}{l^2}$

One end is fixed other is free - $\frac{\pi^2 EI}{4l^2}$

Both ends are fixed - $\frac{4\pi^2 EI}{l^2}$

One end is fixed other is hinged - $\frac{2\pi^2 EI}{l^2}$

Effective length of a column:

* The effective length of a given column with given conditions is the length of an equivalent column of the same material and cross section with hinged ends and having the value of crippling load to that of the given column.

* Let l_e = effective length of column

l = length of column.

P = crippling load.

* Then the crippling load for any type of end conditions is

S.NO	End conditions	Coupling load		Relationship between effective length & actual length
		Actual length	Effective length	
1.	Both ends are hinged	$\frac{\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{l_e^2}$	$l_e = l$
2.	One end fixed & other end free	$\frac{\pi^2 EI}{4l^2}$	$\frac{\pi^2 EI}{l_e^2}$	$l_e = 2l$
3.	Both ends are fixed	$\frac{4\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{l_e^2}$	$l_e = \frac{l}{2}$
4.	One end fixed & other end hinged	$\frac{9\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{l_e^2}$	$l_e = \frac{l}{\sqrt{2}}$

Crippling stress:

* Moment of inertia, $I = AK^2$

* Crippling load, $P = \frac{\pi^2 EI}{l_e^2}$

$$= \frac{\pi^2 E AK^2}{l_e^2}$$

$$= \frac{\pi^2 EA}{\left(\frac{l_e}{K}\right)^2}$$

$$\sigma_c = \frac{P}{A} = \frac{\pi^2 E A (I_e/k)^2}{A}$$

$$\sigma_c = \frac{\pi^2 E}{(I_e/k)^2}$$

$$\sigma_c = \frac{\pi^2 E}{\lambda^2}$$

$$\lambda = \frac{l_e}{k}$$

Slenderness ratio:

It is the ratio of length of a column to the least radius of gyration of column is known as 'Slenderness ratio'.

$$\therefore \text{Slenderness ratio} = l/k$$

* A solid round bar 3m long 5 cm diameter is used as a strut with both ends hinged. Determine crippling load.

$$\text{Take } E = 2 \times 10^5 \text{ N/mm}^2$$

Sol Given: Length, $l = 3 \text{ m}$

$$= 3000 \text{ mm}$$

$$\text{Diameter, } d = 5 \text{ cm}$$

$$= 50 \text{ mm}$$

$$E = 2 \times 10^5 \text{ N/mm}^2$$

$$\text{Moment of Inertia, } I = \frac{\pi d^4}{64}$$

$$= \frac{\pi \times 50^4}{64}$$

$$I = 30.67 \times 10^4 \text{ mm}^4$$

$$\therefore \text{Crippling load, } P = \frac{\pi^2 EI}{l^2}$$

$$= \frac{\pi^2 \times 2 \times 10^5 \times 30.67 \times 10^4}{3000^2}$$

$$P = 67.26 \times 10^3 \text{ N}$$

$$P = 67.26 \text{ kN}$$

* A column of timber section 15 cm x 20 cm is 6 m long both ends fixed. Take

$E = 17.5 \text{ kN/mm}^2$. Determine.

(i) Crippling load.

(ii) Safe load if $\text{FOS} = 3$

Sol Given: Length, $l = 6 \text{ m}$
 $= 6000 \text{ mm}$

Width, $b = 15 \text{ cm}$

$= 150 \text{ mm}$

Depth, $d = 20 \text{ cm}$

$= 200 \text{ mm}$

$E = 17.5 \text{ kN/mm}^2 = 17.5 \times 10^3 \text{ N/mm}^2$

$$I_{xx} = 100 \times 10^6 \text{ mm}^4$$

$$I_{yy} = \frac{b^3 d}{12} = \frac{150^3 \times 200}{12}$$

$$I_{yy} = 56.25 \times 10^6 \text{ mm}^4$$

① Crumpling load, $P = \frac{4\pi^2 EI}{l^2}$

$$P = \frac{0.4 \times \pi^2 \times 17.5 \times 10^3 \times 56.25 \times 10^6}{6000^2}$$

$$P = 1079.48 \times 10^3 \text{ N}$$

$$P = 1079.48 \text{ kN}$$

② Safe load = $\frac{\text{Working or crumpling load}}{\text{factor of safety}}$

$$= \frac{1079.48}{3}$$

$$= 359.82 \text{ kN}$$

* A hollow mild steel section 6m long 4 cm internal dia and 5mm thickness is used as strut with both ends hinged. Find the crumpling load and

Safe load. Take $FOS = 3$, $E = 2 \times 10^5 \text{ N/mm}^2$

Internal dia, $D_1 = 40 \text{ mm}$
 $= 40 \text{ mm}$

Thickness, $t = 5 \text{ mm}$

External dia, $D_0 = 50 \text{ mm}$

$E = 2 \times 10^5 \text{ N/mm}^2$

Moment of inertia, $I = \frac{\pi D_0^4}{64} - \frac{\pi D_1^4}{64}$
 $= \frac{\pi \times 50^4}{64} - \frac{\pi \times 40^4}{64}$

$I = 18.13 \times 10^4 \text{ mm}^4$

Crippling load, $P = \frac{\pi^2 EI}{L^2}$
 $= \frac{\pi^2 \times 2 \times 10^5 \times 18.13 \times 10^4}{6000^2}$

$P = 9940.8 \text{ N}$

Safe load = $\frac{\text{Crippling load}}{\text{FOS}}$

$= \frac{9940.8}{3}$

$= 3313.6 \text{ N}$

A solid round bar 4 m long and 5 cm in dia was found to extend 4.6 mm under a tensile load of 50 kN. This bar is used as a strut with both ends hinged. Determine the buckling load for bar and also the factor of safety. Take FOS = 4.

Sol Given: Length of bar, $l = 4 \text{ m} = 4000 \text{ mm}$

Dia of bar, $d = 5 \text{ cm} = 50 \text{ mm}$

Extension of bar, $\delta l = 4.6 \text{ mm}$

Load, $W = 50 \text{ kN} = 50 \times 10^3 \text{ N}$

Factor of safety = 4

Moment of inertia of circular bar

$$I = \frac{\pi d^4}{64} = \frac{\pi \times 50^4}{64} = 306.79 \times 10^3 \text{ mm}^4$$

Young's modulus, $E = \frac{\sigma}{\epsilon} = \frac{W/A}{\delta l/l}$

$$E = \frac{W}{A} \times \frac{l}{\delta l}$$

$$= \frac{50 \times 10^3 \times 4000}{\frac{\pi}{4} \times 50^2 \times 4.6}$$

$$E = 22.14 \times 10^3 \text{ N/mm}^2$$

$$E = 22.14 \times 10^3 \text{ N/mm}^2$$

Buckling load, $P = \frac{\pi^2 EI}{l^2}$

$$= \frac{\pi^2 \times 22.14 \times 10^3 \times 306.79 \times 10^3}{4000^2}$$

$$P = 4.189 \text{ kN}$$

$$\text{Safe load} = \frac{\text{Buckling load}}{\text{FOS}}$$

$$= \frac{4189.8}{4}$$

$$\text{Safe load} = 1047.45 \text{ N}$$

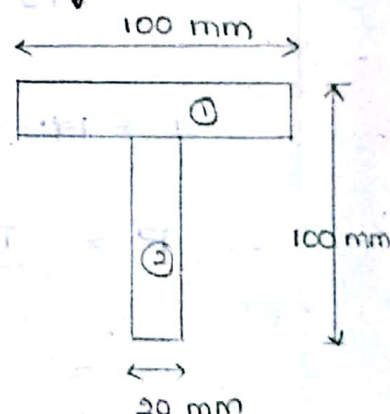
* Determine the crippling load for a T-section of dimensions $10\text{cm} \times 10\text{cm} \times 2\text{cm}$ and a length of 5m when it is used as a structure with both of its ends are hinged.

Take $E = 2 \times 10^5 \text{ N/mm}^2$.

Given:

Length, $L = 5\text{m} = 5000 \text{ mm}$

$$E = 2 \times 10^5 \text{ N/mm}^2$$



Let $\bar{y}$ = centroid of T-section from bottom.

$$\bar{y} = \frac{A_1 y_1 + A_2 y_2}{A_1 + A_2}$$

$$= \frac{(100 \times 20) \left(80 + \frac{20}{2}\right) + (20 \times 80) \left(\frac{80}{2}\right)}{(100 \times 20) + (20 \times 80)}$$

$$\bar{y} = 67.77 \text{ mm}$$

$$I_{xx} = I_{xG} + A (y_1 - \bar{y})^2$$

$$I_{xx} = \frac{100 \times 20^3}{12} + (100 \times 20) (90 - 67.77)^2$$

$$+ \frac{20 \times 80^3}{12} + (20 \times 80) (40 - 67.77)^2$$

$$= 920000 + 988345.8 + 1233876.64$$

$$= 314.92 \times 10^4 \text{ mm}^4$$

Moment of inertia about y-y axis

$$I_{yy} = \frac{20 \times 100^3}{12} + \frac{80 \times 20^3}{12}$$

$$= 172 \times 10^4 \text{ mm}^4$$

$$I = 172 \times 10^4 \text{ mm}^4$$

$$\therefore P = \frac{\pi^2 EI}{L^2}$$

$$= \frac{\pi^2 \times 2 \times 10^5 \times 172 \times 10^4}{5000^2}$$

$$P = 135.8 \times 10^3 \text{ N}$$

$$P = 135.8 \text{ kN}$$

* Using Euler's formulae calculate critical stress for a series of struts having a slenderness ratio of 40, 80, 120, 160 and 200 under the following conditions-

Take $E = 2.05 \times 10^5 \text{ N/mm}^2$.

Sol ① Both ends hinged.

$$l_e = l$$

$$\text{Critical stress} = \frac{\pi^2 E}{(l_e/k)^2}$$

$$\text{Slenderness, } l/k = 40$$

$$\text{Critical stress} = \frac{\pi^2 \times 2.05 \times 10^5}{40^2}$$

$$= 1264.54 \text{ N}$$

$$l/k = 80 \Rightarrow \text{critical stress} = \frac{\pi^2 \times 2.05 \times 10^5}{80^2}$$

$$= 316.13 \text{ N}$$

$$l/k = 120 \Rightarrow \text{C.S.} = \frac{\pi^2 \times 2.05 \times 10^5}{120^2}$$

$$= 140.5 \text{ N}$$

$$l/k = 160 \Rightarrow \text{C.S.} = \frac{\pi^2 \times 2.05 \times 10^5}{160^2}$$

$$= 79 \text{ N}$$

$$l/k = 200 \Rightarrow \text{C.S.} = \frac{\pi^2 \times 2.05 \times 10^5}{200^2}$$

$$= 50.58 \text{ N}$$

② Both ends fixed

$$l_e = l/2$$

$$\text{Critical stress} = \frac{\pi^2 E}{(l_e/k)^2} = \frac{\pi^2 E}{(l/2k)^2} = \frac{\pi^2 E}{\frac{1}{4} (l/k)^2}$$

$$= \frac{4\pi^2 E}{(l/k)^2}$$

$$\frac{1}{k} = 40 \Rightarrow C.A = \frac{4 \times \pi^2 \times 2.05 \times 10^5}{40^2}$$

$$= 5058.17 \text{ N}$$

$$\frac{1}{k} = 80 \Rightarrow C.A = \frac{4 \times \pi^2 \times 2.05 \times 10^5}{80^2}$$

$$= 1264.54 \text{ N}$$

$$\frac{1}{k} = 120 \Rightarrow C.A = \frac{4 \times \pi^2 \times 2.05 \times 10^5}{120^2}$$

$$= 562 \text{ N}$$

$$\frac{1}{k} = 160 \Rightarrow C.A = \frac{4 \times \pi^2 \times 2.05 \times 10^5}{160^2}$$

$$= 316.35 \text{ N}$$

$$\frac{1}{k} = 200 \Rightarrow C.A = \frac{4 \times \pi^2 \times 2.05 \times 10^5}{200^2}$$

$$= 202.32 \text{ N}$$

Rankine's formulae:

The empirical formula given by Rankine

$$\frac{1}{P} = \frac{1}{P_c} + \frac{1}{P_e}$$

$$P_c = \sigma_c A$$

P_c = crushing load.

P = crippling load by Rankine's formula.

σ_c = ultimate crushing stress

A = cross sectional area

$$\frac{1}{P} = \frac{P_E + P_C}{P_C P_E}$$

$$P = \frac{P_C P_E}{P_E + P_C} = \frac{P_C P_E}{P_E \left(1 + \frac{P_C}{P_E}\right)} = \frac{P_C}{1 + \frac{P_C}{P_E}}$$

$$P = \frac{\sigma_C \cdot A}{1 + \left(\frac{\sigma_C \cdot A}{\frac{\pi^2 E I}{l_e^2}} \right)}$$

$$= \frac{\sigma_C \cdot A}{1 + \frac{\sigma_C \cdot A \cdot l_e^2}{\pi^2 E I}}$$

$$I = A K^2$$

$$= \frac{\sigma_C \cdot A}{1 + \frac{\sigma_C l_e^2}{\pi^2 E K^2}} = \frac{\sigma_C \cdot A}{1 + \frac{\sigma_C}{\pi^2 E} \left(\frac{l_e}{K}\right)^2}$$

$$P = \frac{\sigma_C \cdot A}{1 + a \left(\frac{l_e}{K}\right)^2}$$

$$\frac{\sigma_C}{\pi^2 E} = a$$

Above eqn gives crippling load by Rankine's formulae.

The values of σ_C and A for different column materials are given below.

	σ_C	A
① Wrought iron	250 N/mm ²	1/9000
② Cast iron	550 N/mm ²	1/1600
③ Mild steel	320 N/mm ²	1/7500
④ Timber	50 N/mm ²	1/450

The external and internal dia of a hollow cast iron column are 5 cm & 4 cm. If the length of the column is 3 m and both of its ends are fixed. Determine the crippling load by using Rankine's formula.

Sol External dia, $\Theta = 5 \text{ cm} = 50 \text{ mm}$

Internal dia, $d = 4 \text{ cm} = 40 \text{ mm}$

Length of column, $l = 3 \text{ m} = 3000 \text{ mm}$

For cast iron, $\sigma_c = 550 \text{ N/mm}^2$

$a = 1/1600$

Area, $A = \frac{\pi}{4} (\Theta^2 - d^2)$

$= \frac{\pi}{4} (50^2 - 40^2)$

$A = 706.85 \text{ mm}^2$

Effective length, $l_e = l/2 = \frac{3000}{2} = 1500 \text{ mm}$

$I = AK^2$

$\frac{\pi (\Theta^4 - d^4)}{64} = 706.85 \times K^2$

$\frac{\pi (50^4 - 40^4)}{64} = 706.85 \times K^2$

$181.13 \times 10^3 = 706.85 \times K^2$

$K^2 = \frac{181.13 \times 10^3}{706.85} = 256.24$

$K = \sqrt{256.24} = 16$

$$P = \frac{\sigma_c A}{\left(1 + \frac{1}{1600} \left(\frac{1500}{16}\right)^2\right)}$$

$$= \frac{550 \times 706.85}{1 + 5.49}$$

$$P = 59.9 \times 10^3 \text{ N}$$

$$P = 59.9 \text{ kN}$$

* A 1.5 m long column as a circular cross-section of 5 cm diameter one end of the column is fix and other one is free.

Taking FOS as 3. Calculate the safe load using ① Rankine's formulae $\sigma_c = 560 \text{ N/mm}^2$

$$a = 1/1600.$$

② Euler's formulae, $E = 1.2 \times 10^5 \text{ N/mm}^2$

So) Given: Length of column, $l = 1.5 \text{ m} = 1500 \text{ mm}$

Diameter, $d = 5 \text{ cm} = 50 \text{ mm}$.

Factor of safety, $FOS = 3$.

①

Effective length, $l_e = 2l = 2 \times 1500 = 3000 \text{ mm}$.

$$\text{Area, } A = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times 50^2 = 1963.49 \text{ mm}^2$$

$$I = AK^2$$

$$\frac{\pi d^4}{64} = 1963.49 \times K^2$$

$$K = 12.5$$

$$\text{Crippling load, } P = \frac{\sigma_c A}{1 + a \cdot \left(\frac{le}{K}\right)^2}$$

$$= \frac{560 \times 1963.49}{1 + \frac{1}{1600} \left(\frac{3000}{12.5}\right)^2}$$

$$P = \frac{1099554.4}{1 + 36}$$

$$P = 29717.68 \text{ N}$$

$$\text{Safe load} = \frac{\text{Buckling load}}{\text{FOS}}$$

$$= \frac{29717.68}{3}$$

$$= 9905.89 \text{ N}$$

② Euler's formulae

$$\text{Crippling load, } P = \frac{\pi^2 EI}{4l^2}$$

$$= \frac{\pi^2 \times 1.2 \times 10^5 \times 30.6 \times 10^4}{4(1500)^2}$$

$$P = 40267.9 \text{ N}$$

$$\text{Safe load} = \frac{\text{Crippling load}}{\text{FOS}}$$

$$= \frac{40267.9}{3}$$

$$= 13422.63 \text{ N}$$