UNIT-1

Introduction

Definitions

Signal: A function of one (or) more independent variable which contain some information is called signal.

Ex:- Electric Voltage (or) Current, such as radio signals, TV signal.

System: A system is set of elements (or) functional block that are connected together and produce an o/p in response to an i/p signal.

Ex:- A audio amplifier, attenuator, TV set, transmitter, receiver etc.

Classification of signals:-

The signals can be classified into two parts depending upon independent variable (time)

a) Continous Time (CT) signal.

b) Discrete Time (DT) signal.

Both the CT and DT signals can be classified into following parts.

a) periodic & non-periodic signals

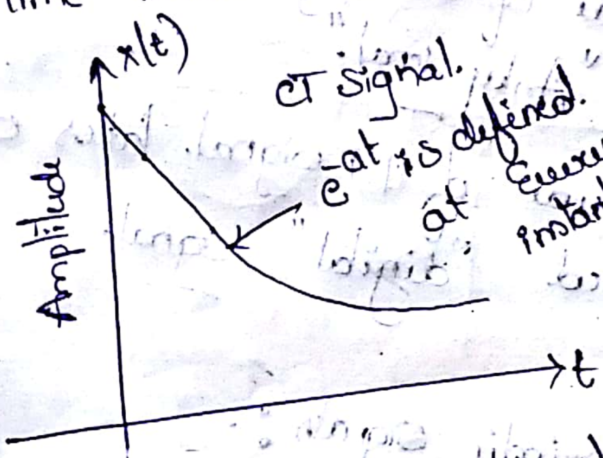
b) Even & Odd signals

c) Energy & power signals.

d) Deterministic & random signals.

CT & DT signals :- 1) A "CT" signal is defined continously w.r.t time.

2) A "DT" signal is defined only at specific & regular time instant.



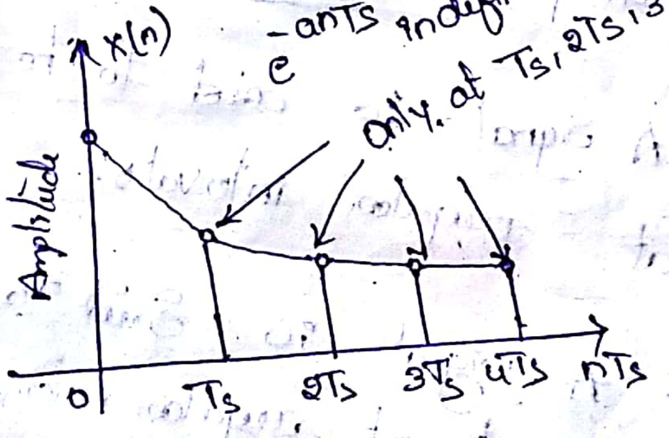


Fig: CT & DT signals.

Continous fun of time : $x(t) = e^{-at}$

~~Continuous~~ Descrete fun of time : $x(n) = e^{-anTs}$

Significance:-

1) Anlog Circuit process CT signal. Such ckt are op-amp, filters, amplifier etc.

2) Digital ckt process DT signal. Such circuit are micro processors, Counters, flip-flops etc.

Anlog & Digital system:-

⇒ when amplitude of CT signal varies continously, it is called "Anlog signal."

⇒ when Amplitude of DT signal takes only finite values it is called "digital" signal.

a)

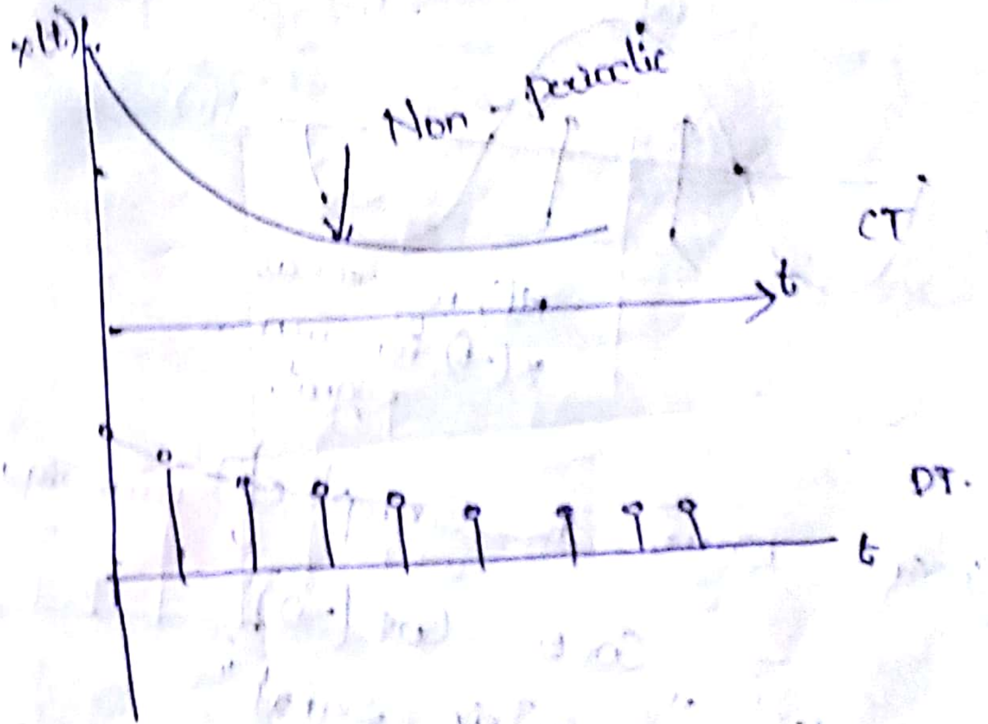
Periodic & Non-periodic signals:-

Periodic: A signal is said to be periodic if it repeats at regular intervals.

Non-periodic: A signal is said to be non-periodic if it not repeat at regular intervals.

Ex: CT for Periodic

DT for Non-periodic



Condition for periodicity of CT signal.

The CT signal repeat after certain period T_0 i.e

$$x(t) = x(t+T_0) \quad \& \quad x(n) = x_1(t) + x_2(t)$$

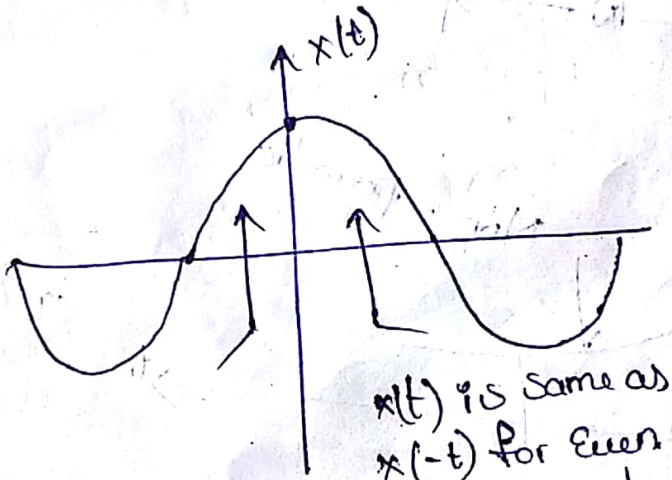
Condition for periodicity of DT signal.

Consider DT cosine wave $x(n) = \cos(2\pi f_0 n)$

Even & Odd Signals :-

Even Signal :- A signal is said to be Even signal if inversion of time does not change the amplitude.

Condition for signal to be Even.	$\begin{cases} x(t) = x(-t) \\ x(n) = x(-n) \end{cases}$



x(t) is same as x(-t) for Even signal.

Cosine wave is Example of Even signal

$$\cos\theta = \cos(-\theta)$$

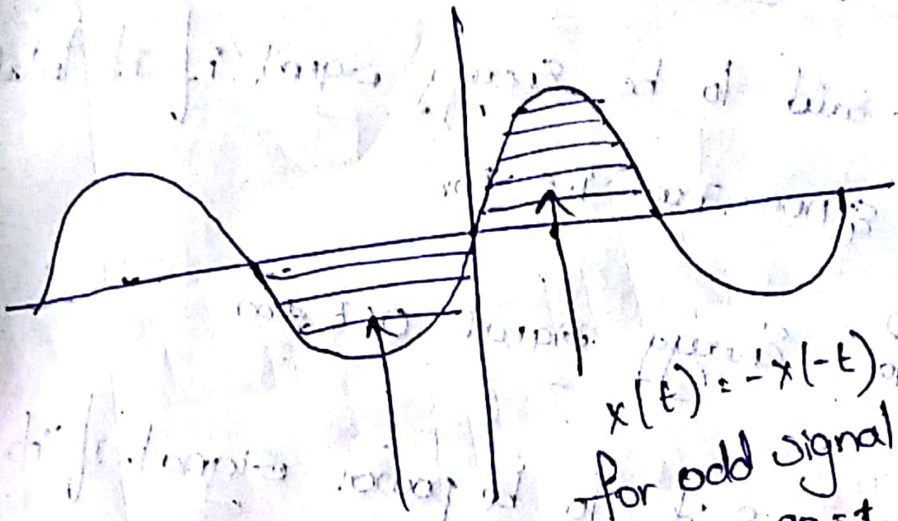
* also called "Symmetric signal"

Odd Signal :-

A signal is said to be odd signal if inversion of time axis also inverts Amplitude of the signal.

Condition for signal to be odd. $\begin{cases} x(t) = -x(-t) \\ x(n) = -x(-n) \end{cases}$

* It is also called anti-symmetric signal.



$x(t) = -x(-t)$

for odd signal

$\sin 90 = 1$

$\sin(-90) = -1$

Sine wave is odd.

⇒ Sine wave is Example of odd signal

Sign ⇒ Even & odd symmetry of the signal have specific harmonic (or) freq content.

⇒ Even & odd symmetry property is used in filter design.

$$x(t) = x_e(t) + x_o(t)$$

Even ↑ odd ↑

Continous time signal

Even part $= x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right]$

odd part $x_o(t) = \frac{1}{2}\left[x(t) - x(-t)\right]$

and $E = \sum_{n=-\infty}^{\infty} |x(n)|^2$ for DT signal.

Power of CT & DT Signal :-

$$\boxed{\begin{array}{l} \text{Power } P = \lim\limits_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt \\ \& \quad P = \lim\limits_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 \end{array}}$$

for CT Signal.

for DT Signal.

Deterministic and Random Signal.

→ A Deterministic signal can be completely represented by Mathematical Equation any time

Ex $x(t) = \cos \omega t$

$x(n) = \cos 2\pi f n$

→ A signal which can't be represented by any Mathematical Eqn is called random signal.

Here we are taking
→ Variance
→ Co-Variance.

Problems

Determine whether the following DT signal are Periodic (or) not ? if periodic determine fundamental Period.

i) $\cos(0.01\pi n) = x(n)$

ii) $\cos(3\pi n)$

iii) $\sin(3n)$

iv) $\cos\frac{2\pi n}{5} + \cos\frac{2\pi n}{7}$

v) $\cos(n/8)\cos n\pi/8$

vi) $\sin(\pi + 0.2n)$

vii) $e^{(3\pi/4)n}$

i) $x(n) = \cos(0.01\pi n)$

Compare with $x(n) = \cos 2\pi f n$

$2\pi f n = 0.01\pi n$

$f = \frac{0.01}{2} = \frac{1}{200} = k/N$

Here f is expressed as ratio of two integers with $k=1$ & $N=200$

Periodic, $N = 200$

ii) $x(n) = \cos(3\pi n)$

Compare with $x(n) = \cos 2\pi f n$.

$\cancel{\cos} 2\pi f \cancel{n} = \cancel{\cos}(3\pi \cancel{n})$

$f = \frac{3\cancel{\pi}}{2\cancel{\pi}} = 3/2$

$f = k/N$, $N = 2$

Hence

iii) $x(n)$: Cos 3n

Compare with $x(n) = \cos 2\pi f n$.

$$\cos 2\pi f n = \sin 3n$$

$$f = \frac{\sin 3}{\cos 2\pi} = K/N$$

Which is not ratio of two integers.

The signal is non-periodic

iv) $x(n) = \cos\frac{2\pi n}{5} + \cos\frac{2\pi n}{7}$

$x(n) = \cos 2\pi f_1 n + \cos 2\pi f_2 n$

$$\cancel{2\pi} f_1 \cancel{n} = \frac{\cancel{2\pi n}}{5}$$

$\boxed{f = 1/5}$; $N_1 = 5$

$$\cancel{2\pi} f_2 \cancel{n} = \frac{\cancel{2\pi n}}{7}$$

$\boxed{f_2 = 1/7}$ $N_2 = 7$

$\frac{N_1}{N_2} = 5/7$ is the ratio of two integers.

the sequence is periodic. The periodic of $x(n)$ is least Common Multiple of N_1 & N_2. Here least Common Multiple of $N_1 = 5$ and $N_2 = 7$.

Therefore this sequence is $\boxed{\text{Periodic with } N = 35}$

v) $x(n) = \cos(n/8) \cos n\pi/8$

Here $2\pi f_1 n = n/8 \Rightarrow f_1 = \frac{1}{16\pi}$ which is not rational

$2\pi f_2 n = n\pi/8 \Rightarrow f_1 = 1/16$ which is rational.

Thus $\cos(n/8)$ is non-periodic and $\cos(n\pi/8)$ is Periodic. $x(n)$ is non-periodic since it is the product of periodic & non-periodic signal.

vi) $x(n) = \sin(\pi + 0.2n)$

Compare with $x(n) = \sin(2\pi f n + \theta)$

$\theta = \pi$ i.e phase shift &

$2\pi f n = 0.2n$

$f = \frac{0.2}{2\pi} = \frac{1}{10\pi}$ which is not rational.

Hence this signal is non-periodic.

vii) $x(n) = e^{j\pi/4 n}$

$\cos \pi/4 n + j\sin \pi/4 n$

Compare with $x(n) = \cos 2\pi f n + j \sin 2\pi f n$

Here $2\pi f n = \pi/4 n \Rightarrow f = 1/8 = k/N$

which is rational

Hence this signal is

Periodic with $\boxed{N = 8}$

1) Determine whether the following signals are Energy signals (or) power signals and calculate Energy (or) power.

a) $x(n) = \left(\frac{1}{2}\right)^n u(n)$ c) $x(t) = \mathrm{rect}\left(t/T_0\right)$

b) $x(t) = \cos^2 \omega_0 t$ d) $x(t) = \mathrm{rect}\left(t/T_0\right)\cos\omega_0 t$

We have follow the given steps:-

① Observe the signal carefully. If it is periodic & infinite duration then it can be power signal. Hence calculate its power directly.

② If the signal is periodic but of finite duration, then it can be Energy signal. Hence calculate its Energy directly.

③ If the signal is not periodic, then it can be Energy signal. Hence calculate it Energy directly..

i) $x(n) = \left(\frac{1}{2}\right)^n u(n)$.

This signal is not periodic. Hence as per step 3.

Calculate its Energy directly

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} \left[\left(\frac{1}{2}\right)^n\right]^2 = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n$$

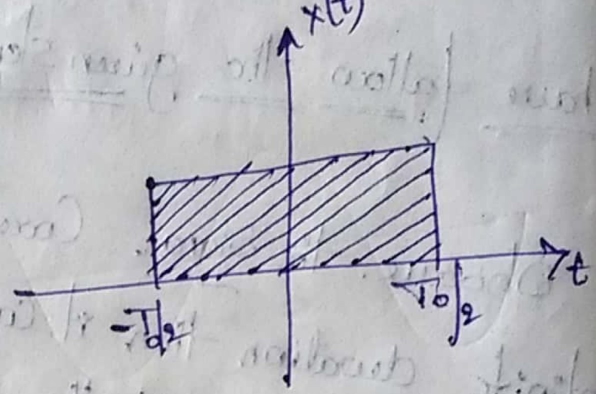
Here case $\sum_{n=0} a^n = \frac{1}{1-a}$ for $|a|<1$. The above Eqn will be.

$$E = \frac{1}{1-1/4} = 4/3$$

Since, Energy is finite & non-zero, it is Energy signal with $E = 4/3$

ii) $x(t) = \text{rect}\left(t/T_0\right)$

The $\text{rect}\left(t/T_0\right)$



$$\text{rect}\left(t/T_0\right) = \begin{cases} 1 & \text{for } -T_0/2 \le t \le T_0/2 \\ 0 & \text{Else where} \end{cases}$$

Hence it can be Energy signal as it non-periodic. Hence, Calculate Energy directly Per signal as per step 3

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-T_0/2}^{T_0/2} (1)^2 dt$$

$$\left. t \right]_{-T_0/2}^{T_0/2} = T_0 .$$

The Energy is finite and non-Zero. it is Energy Signal with $E = T_0$

(ii) $x(t) = \cos^2\omega_0 t$

This is Squared Cosine wave, hence it is Periodic. Therefore this can be periodic signal. As per step 1, Calculate power of this signal directly

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt$$

The given signal $x(t) = \cos^2\omega_0 t$ has some period T_0 & it is real signal.

$$P = \lim_{T_0\to\infty} \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} [\cos^2\omega_0 t]^2 dt$$

Hence $[\cos^2\omega_0 t]^2 = \cos^4\omega_0 t$. it can be Expanded by standard trigonometric relation.

$$P = \lim_{T_0\to\infty} \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \frac{1}{8}[3+4\cos 2\omega_0 t + \cos 4\omega_0 t] dt$$

$$\lim_{T_0\to\infty} \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} 3/8 \, dt + \lim_{T_0\to\infty} \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} 4\cos 2\omega_0 t \, dt + \underbrace{\lim_{T_0\to\infty} \int_{-T_0/2}^{T_0/2} \cos 4\omega_0 t \, dt}$$

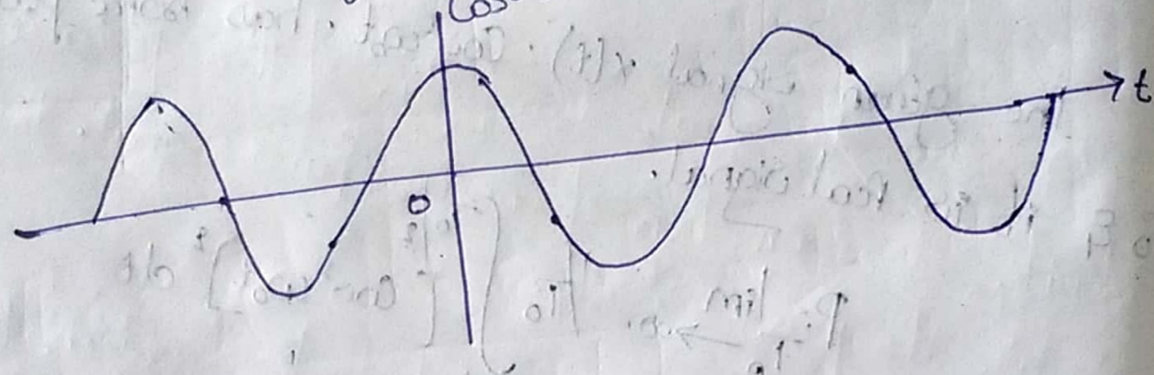
This term will be also zero Since it is integration of cosine wave over "full cycle".

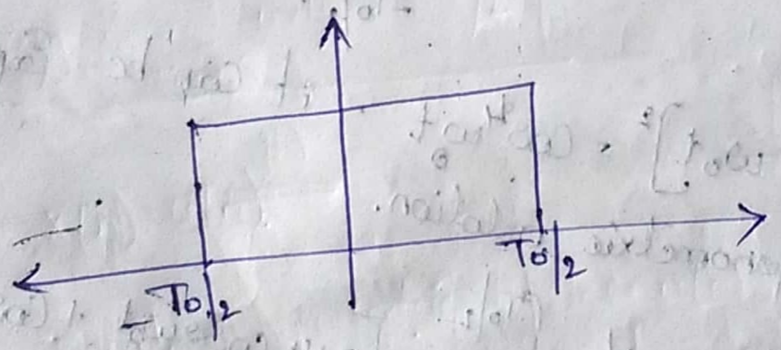
$$\lim_{T_0 \to \infty} \frac{1}{T_0} \cdot \frac{3}{8} \left[t \right]_{-T_0/2}^{T_0/2} + 0 + 0$$

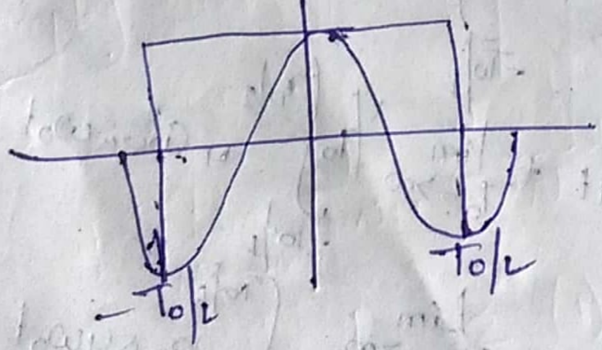
$$\lim_{T_0 \to \infty} \frac{1}{T_0} \cdot \frac{3}{8} T_0 = 3/8$$

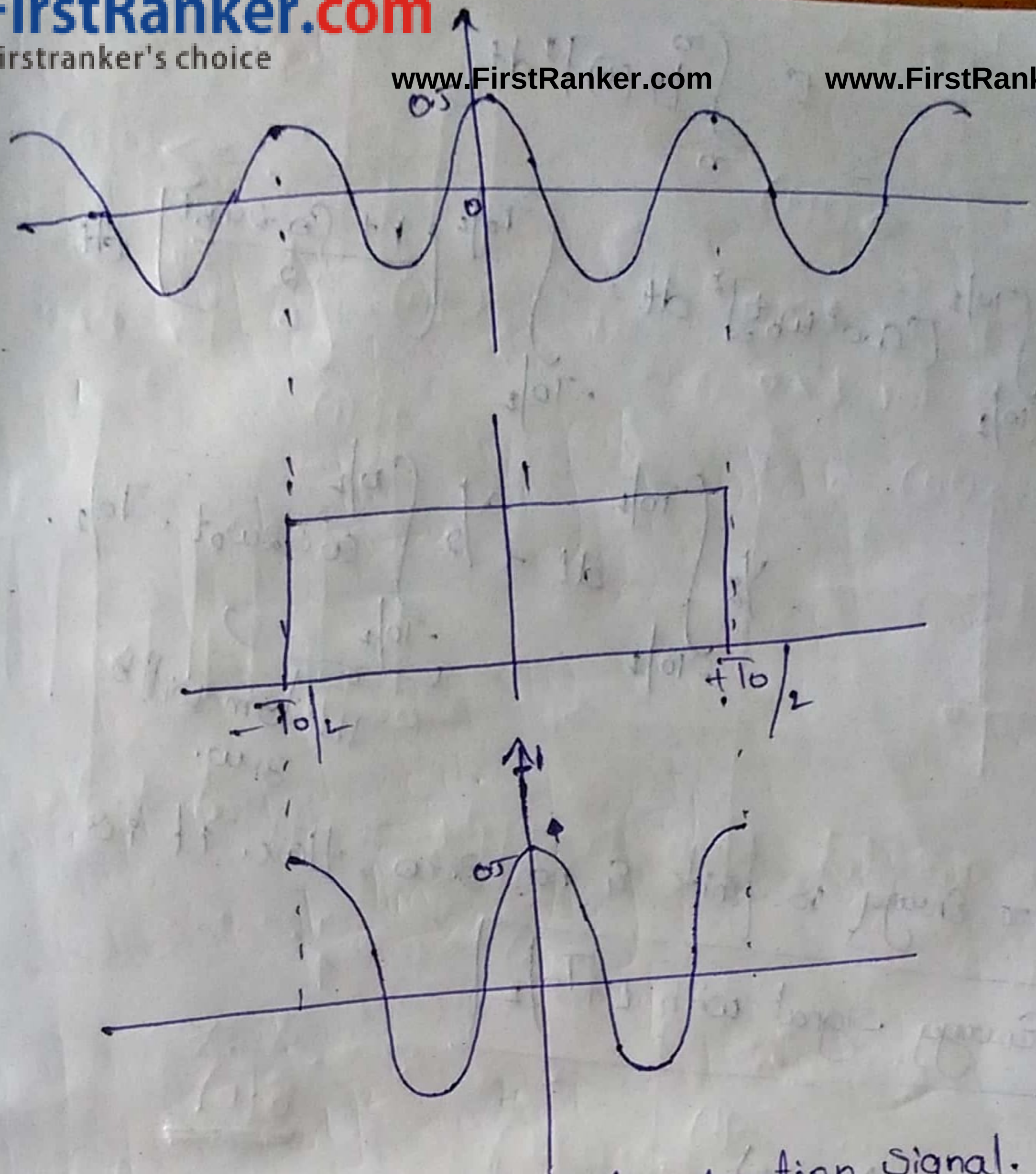
IV) $x(t) = \text{rect}(t/T_0) \cos \omega_0 t$

The given function is the product of cosine wave & rect function.









→ $\cos\omega_0 t$ is periodic & infinite duration signal.

Basically it is power signal.

→ $\cos\omega_0 t$ is Multiplied with the rectangular pulse.

→ Hence the resultant signal is Cosine wave of duration $-T_0/2 \le t \le T_0/2$

it is assumed that there are Multiple No. of cycle of cosine wave in $-T_0/2 \le t \le T_0/2$

The final signal is periodic but finite duration. Hence it is Can be Energy Signal.

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-T_0/2}^{T_0/2} [\cos^2 \omega_0 t]^2 dt = \int_{-T_0/2}^{T_0/2} \left[\frac{1+\cos 2\omega_0 t}{2}\right] dt$$

$$= \frac{1}{2}\int_{-T_0/2}^{T_0/2} dt + \frac{1}{2}\underbrace{\int_{-T_0/2}^{T_0/2} \cos 2\omega_0 t}_{0} = T_0/2.$$

This term will be zero.

Here Energy is finite & non-zero, Hence it is

$$\boxed{\text{Energy signal with } E = T_0/2}$$

v) $x(n) = u(n)$.

This signal is periodic (since $u(n)$) repeat after every sample) and of infinite duration. Hence it may be power signal.

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} (1)^2$$

Here $\sum_{n=0}^{N} (1)^2$ mean $1+1+1+1 \ldots$ for $n=0$ to N

In other words

1+1+1+1+ ...... (N+1) times: (N+1) Therefore above Equ

will be

$$P = \lim_{N\to\infty} \frac{1}{2N+1}(N+1)$$

$$= \lim_{N\to\infty} \frac{N+1}{2N+1} = \lim_{N\to\infty} \frac{1+1/N}{2+1/N} = 1/2$$

The power is finite & non-zero. hence unit step function is power signal with $= P = 1/2$

Elementry Signal:

→ standard signal are used for the analysis of System

→ These standard signal are.

a) unit step function.

b) unit impulse function.

c) unit ramp function.

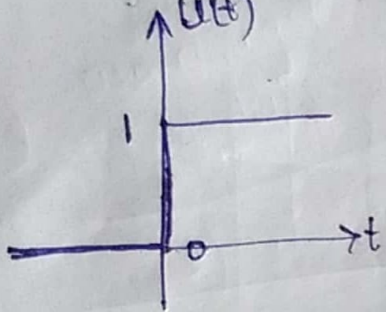
d) Complex Exponential function.

e) Sinusoidal function.

1) Unit Step :-

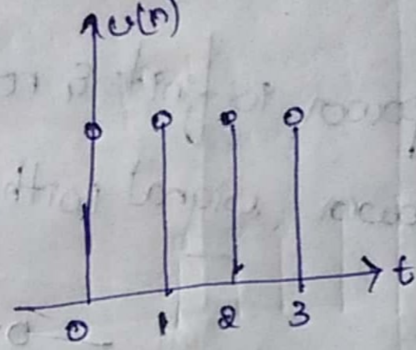
CT $\Rightarrow u(t)$

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$



DT $\Rightarrow u(n)$

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$



$\Rightarrow$ it is generated when DC supply is applied to the circuit

$$u(n) = \{0, 0, \underset{\uparrow}{1}, 1, 1, 1, \ldots\}$$

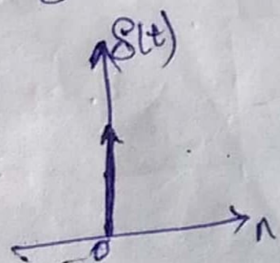
2) Unit impulse :-

δ(t)

Area under unit impulse approaches 1 as its width approaches zero. Thus it has zero value every where except $t = 0$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \quad \& \quad t \to 0$$

$$\delta(t) = 0 \quad \text{for } t \neq 0$$

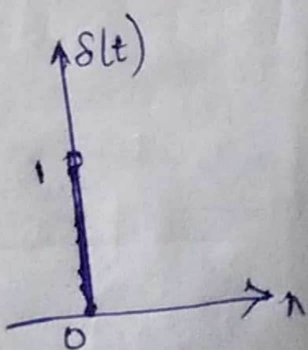


Unit Step

δ(n)

Amplitude of unit sample is '1' at $n = 0$ & it has zero value at all other value of n.

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$



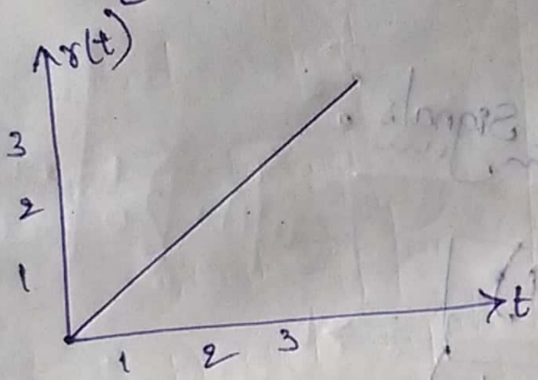
$\delta(n) = \{\ldots, 0, 1, 0, 0, 0\}$

3) Unit Ramp:-

CT

it is linearly growing fun for positive value of independent variable

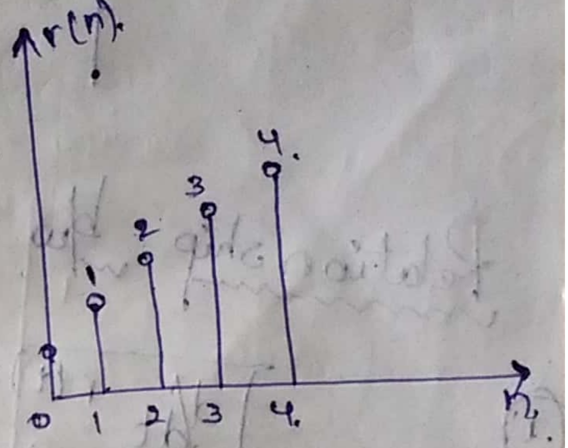
$$r(t) = \begin{cases} t & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$



DT

The amplitude of every sample increase linearly with its number for positive value of "n".

$$r(n) = \begin{cases} n & \text{for } n \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$



→ The ramp fun. indicate linear relationship.

→ It indicate constant current charging of the capacitor.

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Complex Exponential & Sinusoidal signals:-

CT

1) it is exponentially growing (or) decaying signal

$x(t) = be^{at}$

b & a are real.

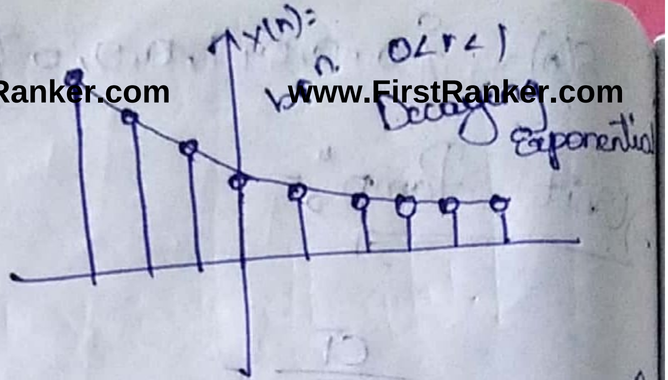
DT

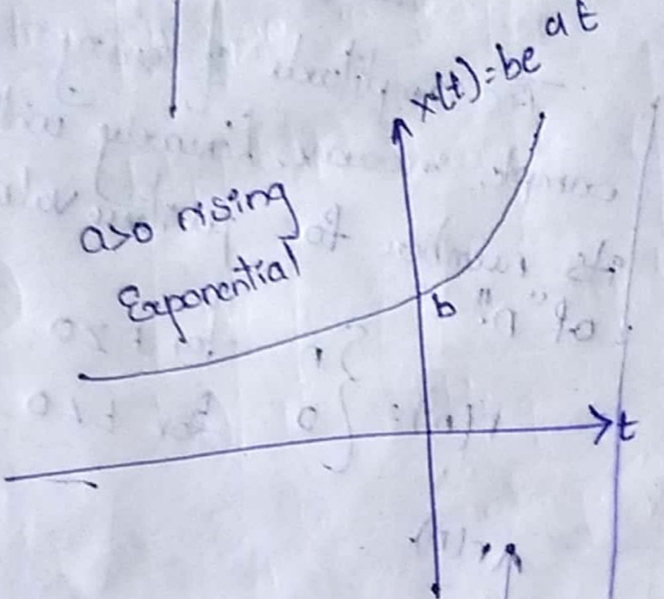
$x(n) = br^n$

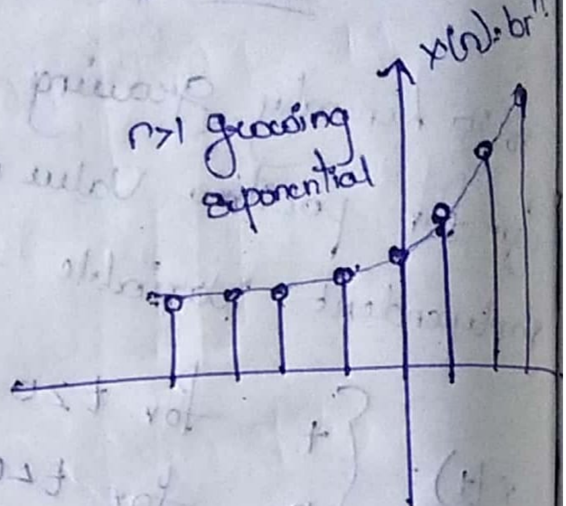
if $r = e^{\alpha}$ then

$x(n) = be^{\alpha n}$

Here b & α are real







Relationship b/w the signals :-

(1) $$\boxed{\frac{d}{dt} r(t) = u(t)}$$

$$dr(t) = dt\, u(t)$$

$$\int dr(t) = \int u(t)\, dt$$

$$\boxed{r(t) = \int u(t)\, dt}$$

Relation b/w unit step & unit ramp

Signal

② $\frac{d}{dt} u(t) = \delta(t)$

(or) $u(t) = \int \delta(t)\,dt$

Ex: find The derivative of the following Signal.

1) $x(t) = u(t) - u(t-a)$, $a>0$

2) $x(t) = t\left[u(t) - u(t-a)\right]$, $a>0$

3) $x(t) = \operatorname{sgn}(t) = \begin{cases} 1 & t>0 \\ -1 & t<0 \end{cases}$

1) $\frac{d}{dt} x(t) = \frac{d}{dt}\left[u(t) - u(t-a)\right]$

$= \frac{d}{dt} u(t) - \frac{d}{dt} u(t-a)$

$= \delta(t) - \delta(t-a)$

2) $\frac{d}{dt} x(t) = \frac{d}{dt}\left[t\left[u(t) - u(t-a)\right]\right]$

$y(t) = u(t) - u(t-a)$

$\frac{d}{dt} x(t) = \frac{d}{dt}\left[t\, y(t)\right]$

$\frac{d}{dt} t\, y(t) = t \frac{d}{dt} y(t) + y(t) \frac{d}{dt} t$

$= t\left[\delta(t) - \delta(t-a)\right] + y(t) \cdot 1 =$

$t\left[\delta(t) - \delta(t-a)\right] + u(t) - u(t-a)$

Operations on Signals :-

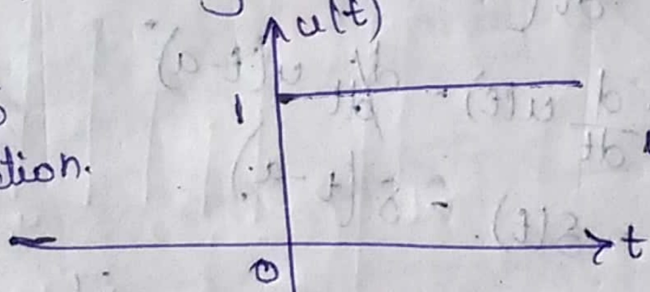
Transformation in independent Variable of signal.

Independent Variable 't (or) n' Can be Manipulated by

1) Delay / Advancing
2) Time folding,
3) Time Scaling.

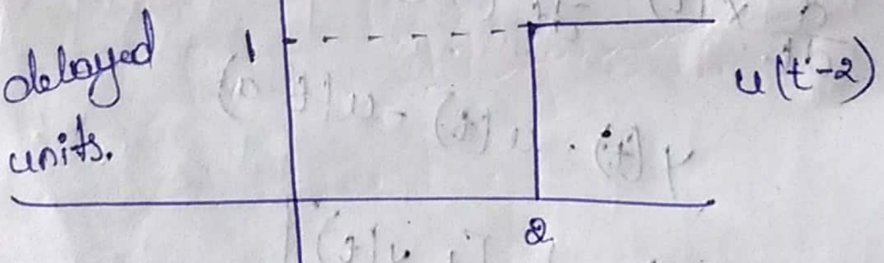
1) Delay / Advancing

unit step function.

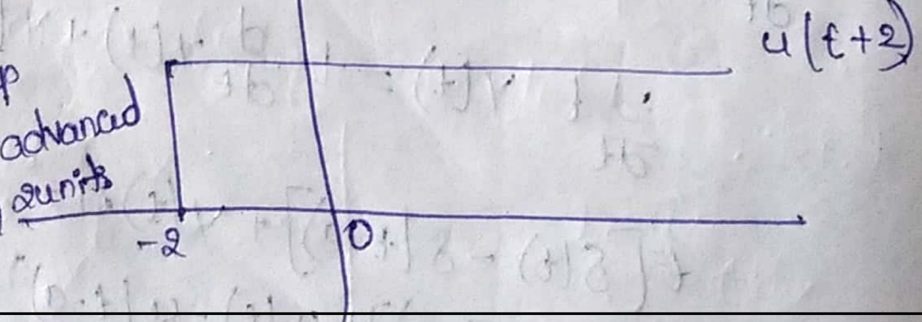


$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$

unit step function delayed by 2 units.



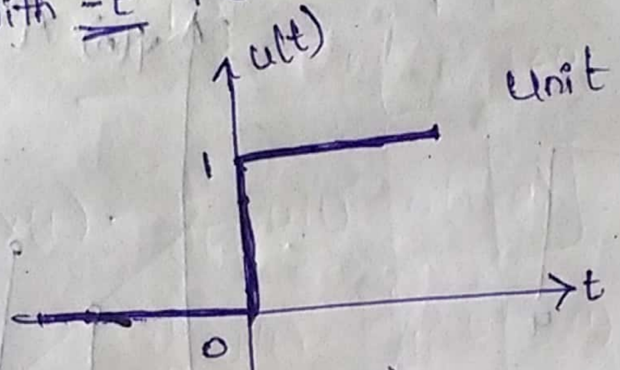
unit step function advanced by 2 units



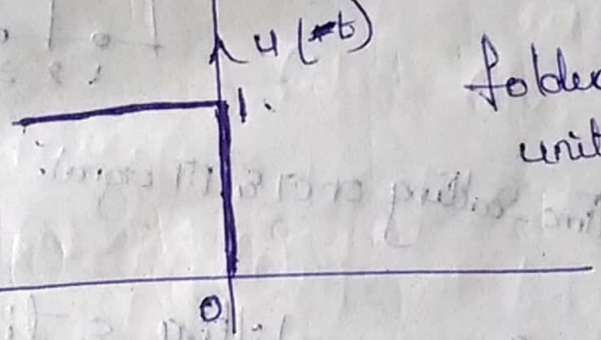
1) when function is advanced it is shifted left.
2) when function is delay it is shifted Right.

2) Time folding :-

The time folding operation is used in Convolution. Consider the Continous time Signal. $x(t)$. Then it's time folded signal is obtained by replacing "t" with "-t" i.e



unit step Signal.



folded unit step Signal.

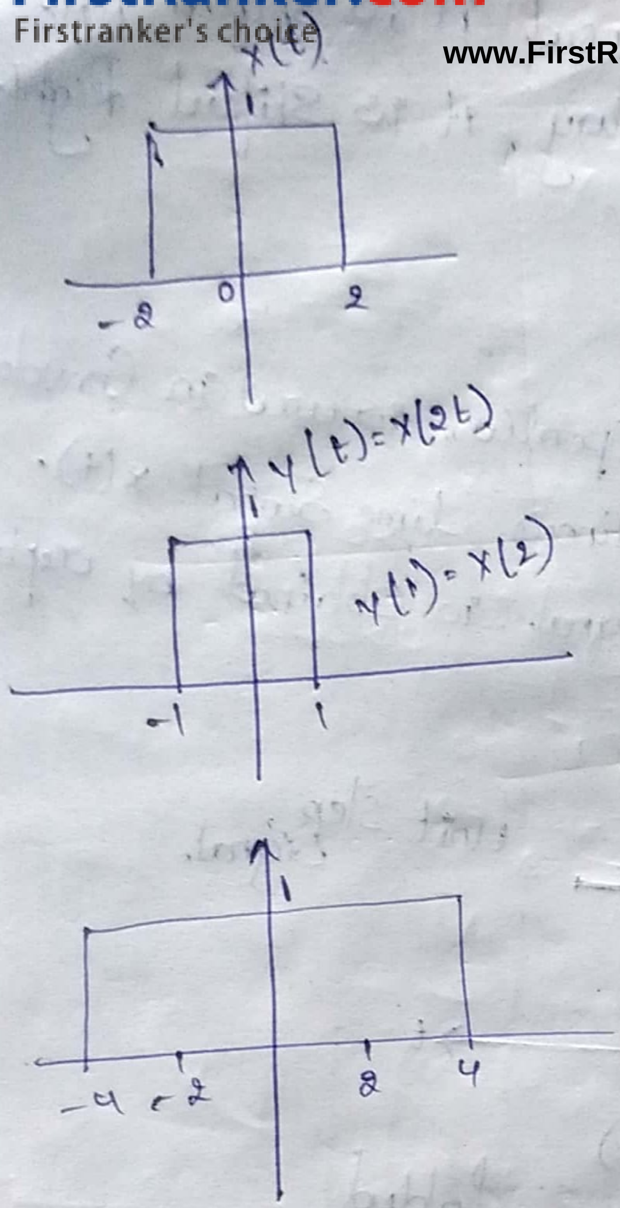
Time Scalling :

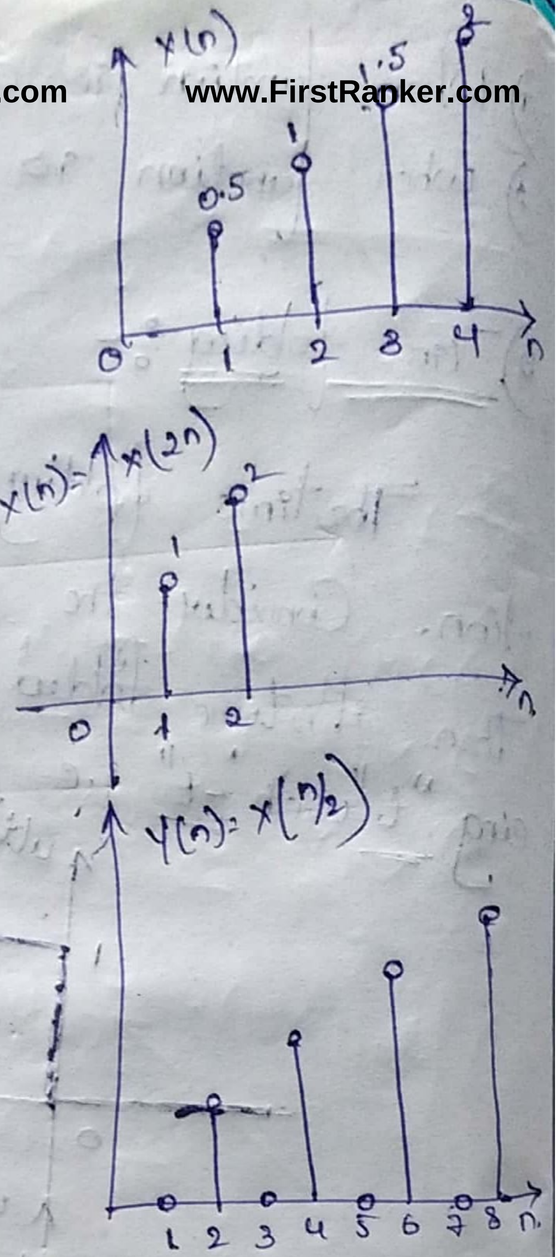
Two types of time Scalling

1) Time Compression : The time axis is Compressed.

$y(t) = x(2t)$

2) Time Expansion : The time axis is Expanded.





Time Scalling on CT & DT Signal.

Precedence Rule for Time shifting & time Scaling

Rules

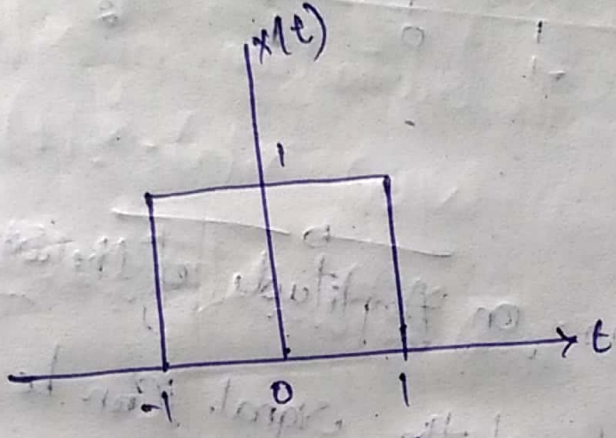
⇒ first do the shifting operation.

⇒ then do the time scalling operation.

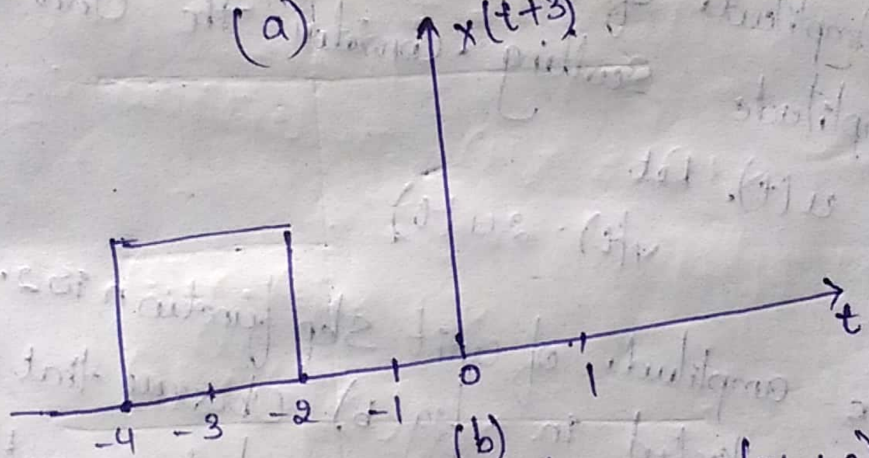
Example :-

⇒ Let us Consider $y(t) = x(-2t+3)$, The $x(t)$ is rectangular pulse of amplitude duration $-1 \le t \le 1$.

Step 1 :- Shift $x(t)$ to Left by 3, to get $x(t+3)$.

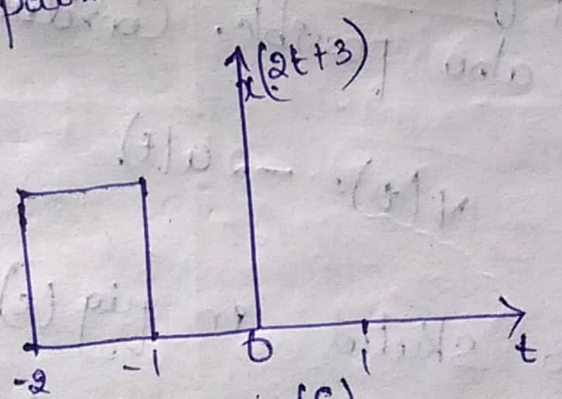


(a)



(b)

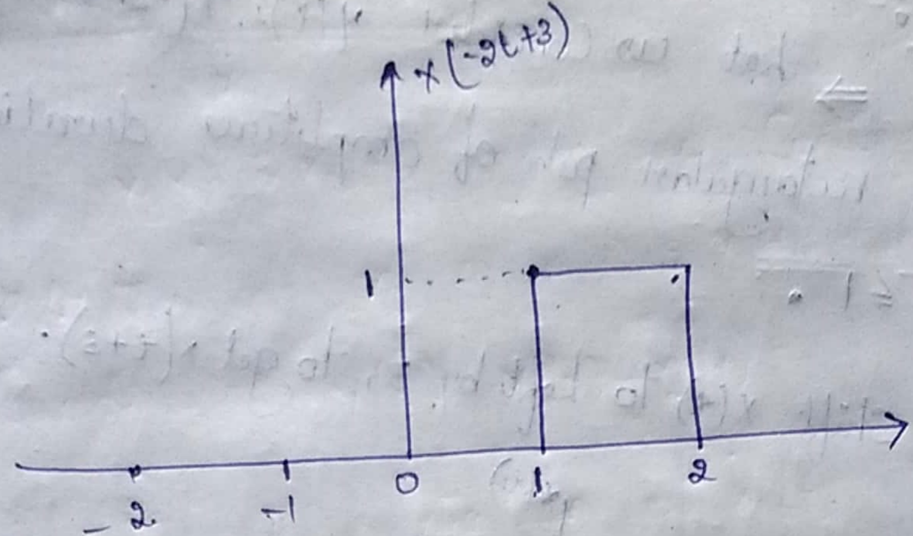
Step 2: Compress $x(t+3)$ by '2' to get $x(2t+3)$.



(c)

Step ③ The $x(2t+3)$ of fig (c) is folded in time to

Sol: $x(-2t+3)$ This is show fig given below.



Tranformation on Amplitude of the Signals

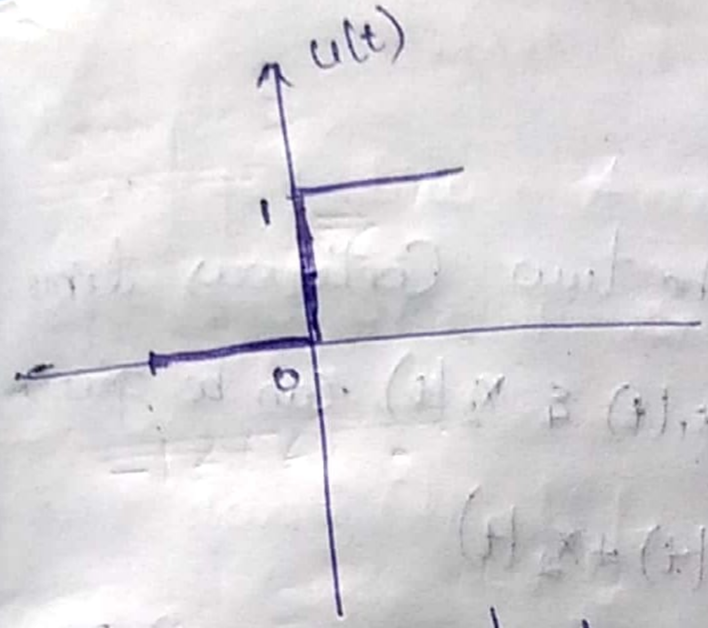
The Amplitude of the Signal can be changed with amplitude scalling : Consider the unit step function $u(t)$. Let

$$y(t) = 2u(t)$$

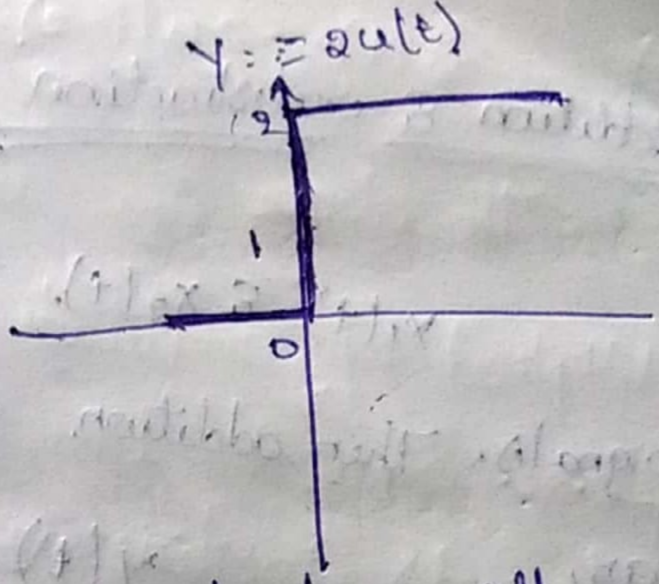
Here amplitude of unit step function is 2. This function is skected in fig(b) Observe that the amplitude of step function '2'. Similarly negative amplitudes are also possible. Consider

$$y(t) = -2u(t)$$

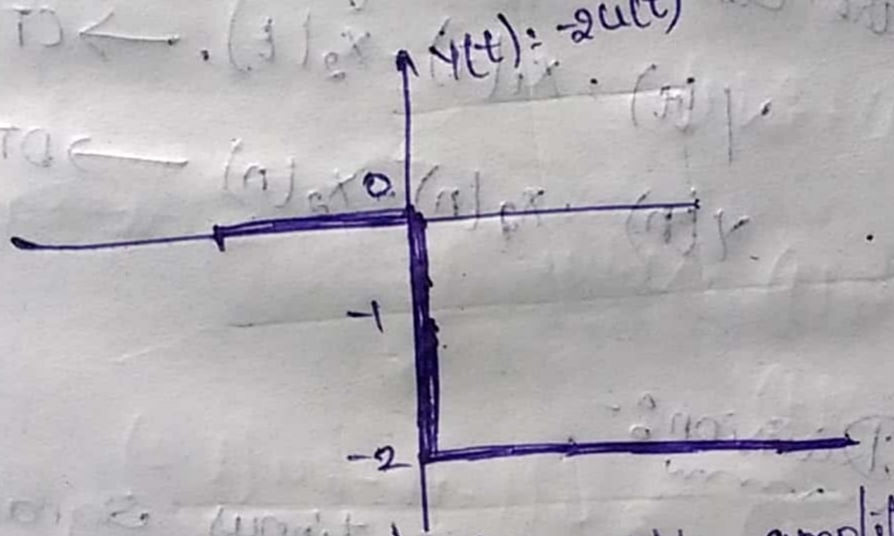
This fun is skected in fig (c) observe that the step function has -ve amplitude i.e "-2"



Unit step function



Step fun with amplitude '2' (Positive)



Step function with amplitude "2" (Negative)

Amplitude Scaling can also be performed on discrete time signal. Consider the unit step sequence u(n) Let $y(n) = 2u(n)$.

Addition & Substraction

$x_1(t)$ & $x_2(t)$ be the two Continous time signals. Then addition of $x_1(t)$ & $x_2(t)$ can be given as,

$$y(t) = x_1(t) + x_2(t)$$

Similarly, the substraction of $x_1(t)$ & $x_2(t)$ is given as.

$$y(t) = x_1(t) - x_2(t) \longrightarrow CT$$

$$y(n) = x_1(n) - x_2(n) \longrightarrow DT$$

Multiplication & Division:-

Let $x_1(t)$ & $x_2(t)$ are Continous signal then their Multiplication given as:

$$y(t) = x_1(t) \cdot x_2(t)$$

$$y(n) = x_1(n) \cdot x_2(n)$$

$$y(t) = \frac{x_1(t)}{x_2(t)}$$

$$y(n) = \frac{x_1(n)}{x_2(n)}.$$

Differentiation & Integration:-

Let $x(t)$ be the Continous time Signal. Then its differentiation. w.r.to given as.

$$y(t) = \frac{d}{dt}x(t).$$

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

Let the current i(t) is following through an inductor the voltage across it will be

$$V(t) = L \frac{d}{dt} i(t).$$

Here y(t) is integration of x(t), integration is used to represent voltage across the capacitor "C"

$$V(t) = \frac{1}{C} \int_{-\infty}^{t} i(\tau)\, d\tau.$$

Problem:-

Draw the waveform represented by following step function.

a) $f_1(t) = 2u(t-1)$
b) $f_2(t) = -2u(t-2)$
c) $f(t) = f_1(t) + f_2(t)$
d) $f(t) = f_1(t) - f_2(t)$

1) $f_1(t) = 2u(t-1)$

The above Eqn. Represents a unit step function multiplied by amplitude of 2. There is a time shift of 1sec. This time shift will be towards positive value of t.

2) $f_2(t) = -2u(t-2)$

The above Equ represents a unit step fun Multiplied by amplitude of -2. There is time shift of 2sec. Since the time shift is subtracted it will be towards positive value. of fig (b) shows the generation of $f_2(t)$ of above Equ.

3) $f(t) = f_1(t) + f_2(t)$

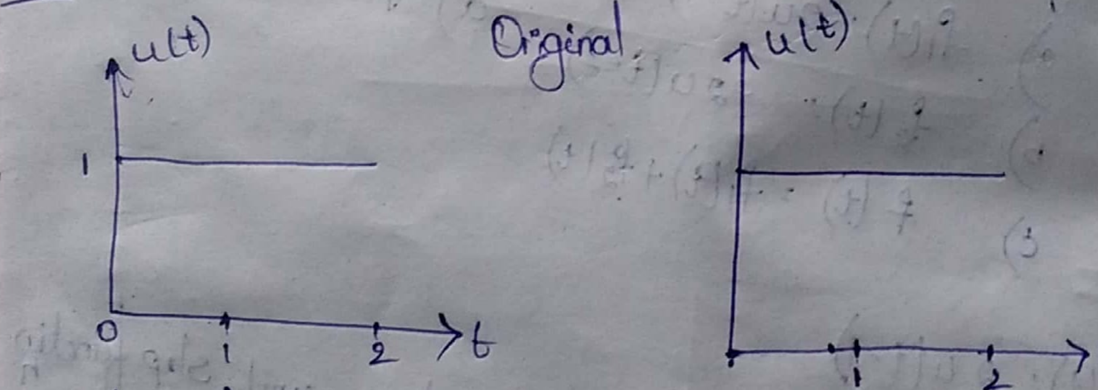
$f_1(t)$ & $f_2(t)$ values in the above Equation

we are getting $f(t)$

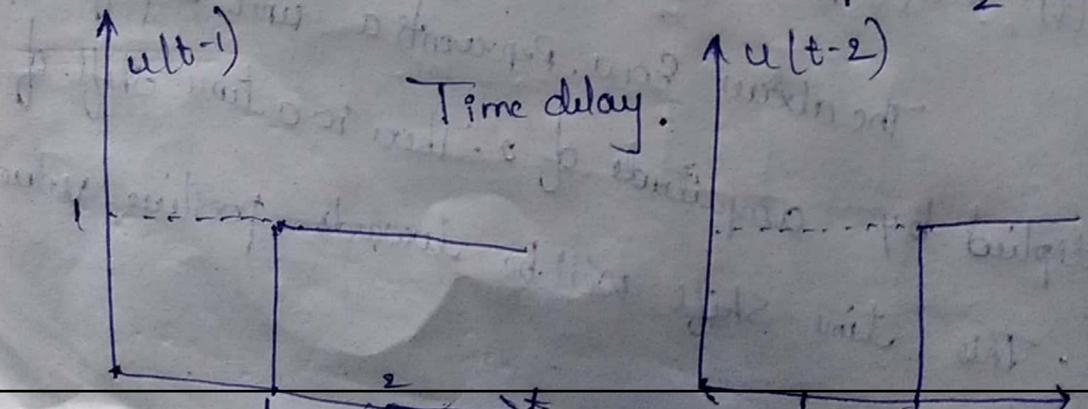
$$f(t) = 2u(t-1) - 2u(t-2)$$

IV) $f(t) = f_1(t) - f_2(t)$.

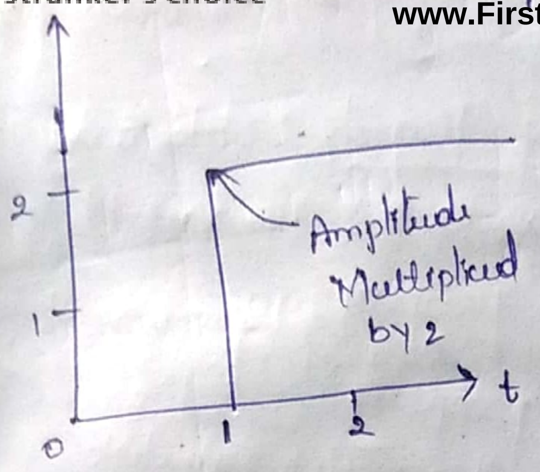
Original

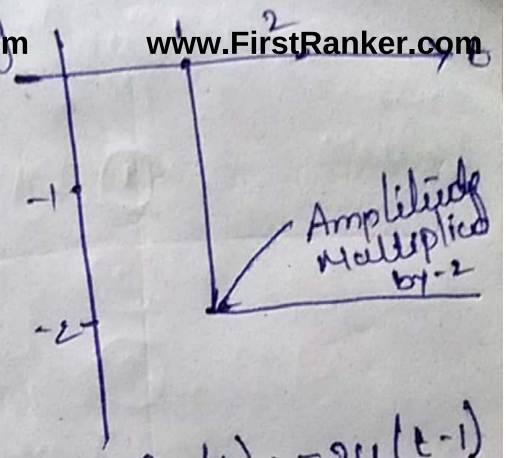


Time delay.



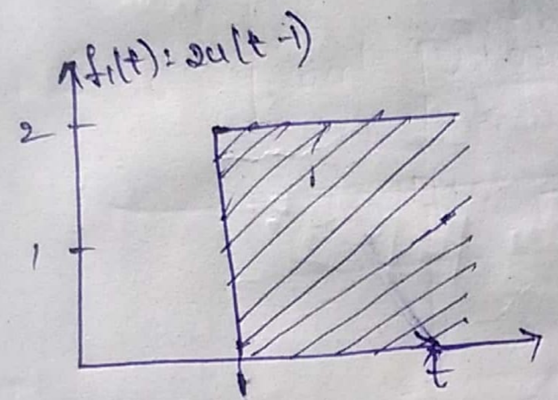
Amplitude Scaling

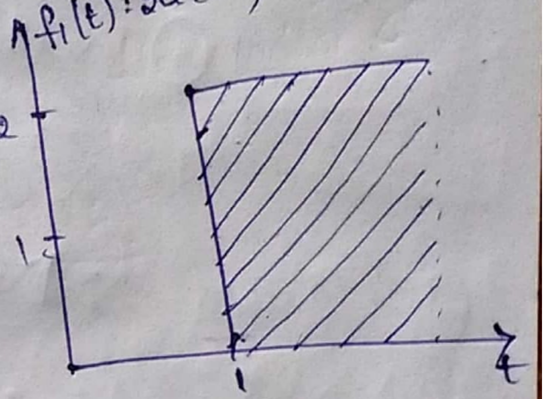


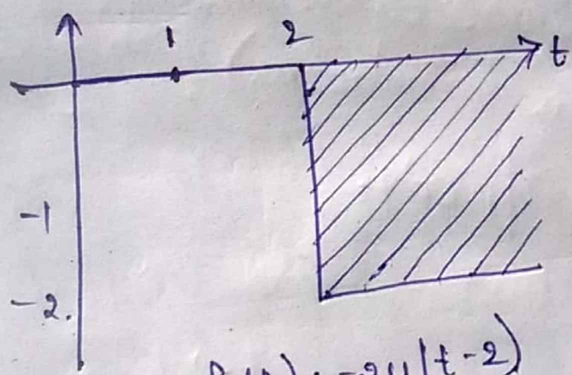


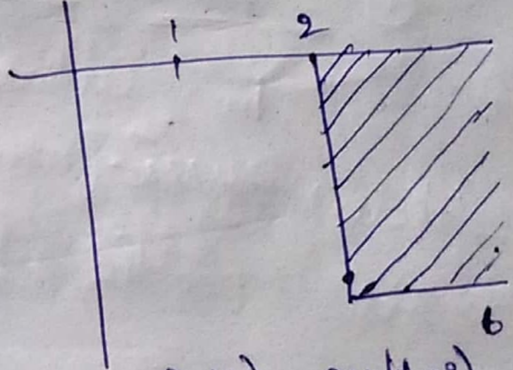
$f_2(t) = -2u(t-1)$

$f_1(t) = 2u(t-1)$

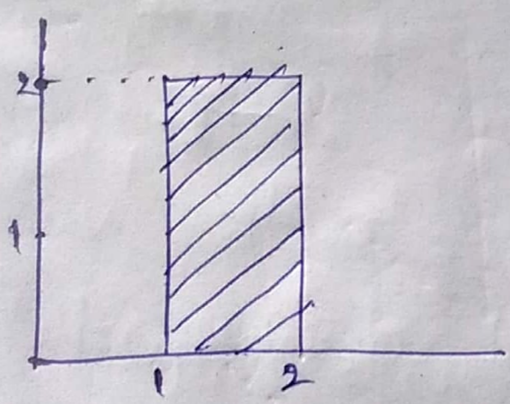


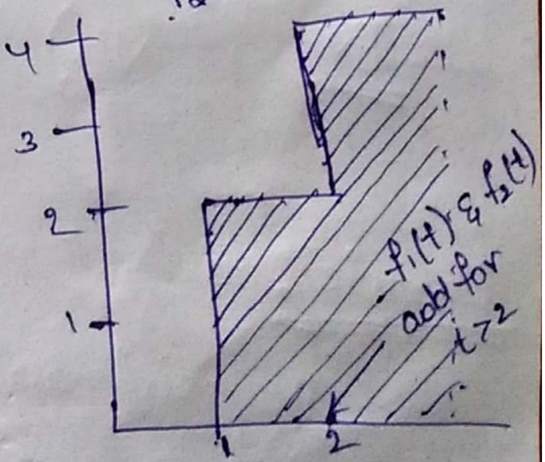






$f_2(t) = -2u(t-2)$ $f_2(t) = -2u(t-2)$





$f(t) = 2u(t-1) - (-2u(t-2))$

$f_1(t) + f_2(t) = f(t)$

$= 2u(t-1) + 2u(t-2))$

System & their properties

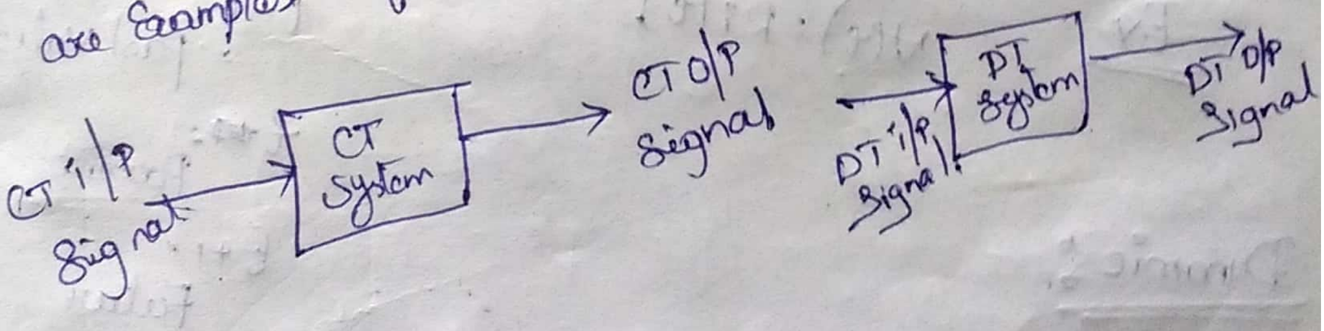
→ A System is a set of Elements (or) functional blocks that are Connected together & produce an o/p in response to an i/p signal.

Classification

two types of systems
1. Continous time System.
2. Discrete time System.

CT: It handle Continous time signals. Anlog filters, amplifiers, attenuators, anlog transmitter & receiver etc.

DT: it handle discrete time Signal. Computers, Printers, Microprocessor, Memories, Shift registers etc. are Examples of discrete time system.



CT & DT Can be further classified based on properties

1. Dynamicity Property : static & dynamic
2. Shift invariance : Time Variant & Time In Variant
3. Linearity propery : Linear & non linear
4. causality proporty : Causal & non-Causal
5. stability property : stable & unstable System
6. Invertibility property : Inversible & non-inversible

Dynamicity property :-

1. Static System :-

The Continous time system is said to be Static (or) ~~Dynamic~~ (memory less. instantaneous) if its o/p depends upon the present i/p only.

Ex $V(t) = R i(t)$

$t-1$: past value
t : Present
$t+1$: Futur Valu

Dynamic :

The Continous time system is said to be Dynamic if its o/p values depend upon the present i/p & past values.

(Or)

In dynamic system the n^{th} o/p sample value depend upon n^{th} i/p sample & just previous i.e $(n-1)^{th}$ i/p samples. This system need to be store the previous sample value.

$$y(n) = x(n) + x(n-1)$$

② Time Invariant & Time Variant System :-

Time invariant :- A Continous time system is time invariant if the time shift in the i/p signal result in corresponding time shift in the o/p. Ex Nigh & day with time

$$f(x(t-t_1)) = y(t-t_1)$$

Time Variant : A Continous system is time Varient if the time shift in the i/p signal result so there is no time shift in the o/p then it is said to be Time Variant system.

$$f(x(n-k)] = y(t).$$

Ex Temprature in a day. ~~time~~ temperature is Varie with the time.

Causal & Non - Causal System :

Causal :- The system is said to be Causal if its o/p at any time depends upon present & past i/p's only.

Ex $y(n) = x(n) \cdot x(n-1)$.

Non Causal :

The system is said to be Non Causal if o/p at any time depends up on present, past, future i/p values.

$$y(n) = x(n) + x(n-1) + x(n+1)$$

Linear & Non - Linear System :-

Linear : A system is said to be Linear if it satisfy the super position principle.

Super position principle - Sum of ~~produc~~ two i/p is Equal to the sum of the two individual i/p.

$$f[a_1 x_1(t) + a_2 x_2(t)] = a_1 y_1(t) + a_2 y_2(t)$$

Non-linear:- A system is said to be Non-linear if it don't satisfy the supper position principer.

$y_1(t) = f(x_1(t))$ $y_2(t) = f(x_2(t))$

$f[a_1 x_1(t) + a_2 x_2(t)] = a_1 y_1(t) + a_2 y_2(t)$

Stable & unstable System:

When every bounded i/p produces bound o/p then the system is "stable."

If the system produce unbounded o/p for bounded i/p then it is "unstable."

Problem

Determine whether the following continous time system are stable (or) not?

1) $y(t) = t\,x(t)$ 2) $y(t) = x(t) \sin 100\pi t$

i) $y(t) = t\,x(t)$

⇒ Here Let $x(t)$ be bounded And, $t \to \infty$, $y(t) \to \infty$

$x(t)$ is Multiplied by 't'.

2) $y(t) = x(t) \sin 100\pi t$

Let $x(t)$ is bounded. Here $x(t)$ is Multiplied by $\sin 100\pi t$. We know that value of the sine function

... Hence o/p $y(t)$ is bounded as long as $x(t)$

is bounded. Hence this System is Stable.

Ex
Determine whether the following discrete time system are stable (or) not?
i) $y(n) = x(n) + x(n-1) + x(n-1)$ ii) $y(n) = r^n x(n)$ $r>1$

Problems Determine whether the following Continuous time system are Causal (or) non-Causal.
1) $y(t) = x(t)\cos(t+1)$ 2) $y(t) = x(2t)$ 3) $y(t) = x(-t)$
4) $\frac{dy(t)}{dt} + 10y(t) + 5 = x(t)$ 5) $y(t) = \int_{-\infty}^{t} x(t)\,dt$

1) $y(t) = x(t)\cos(t+1)$

Here observe that $y(t)$ depends upon present i/p $x(t)$. A Cosine function can be Calculated at $t+1$. Hence this is Causal system.

2) $y(t) = x(2t)$

Here, if $t=2$ then,

$y(2) = x(2\cdot 2)$
$= x(4)$

Thus o/p at $t=2$ depends on future i/p $x(4)$. Hence this is non-Causal system.

iii) $y(t) = x(-t)$

Here if $t=-2$ then $y(-2) = x(-(-2))$.

Thus o/p depends upon future i/p. Hence this is non-Causal system.

iv) $\frac{dy(t)}{dt} + 10y(t) + 5 = x(t)$

Here observe that o/p $y(t)$ depends present i/p. Hence this is Causal system.

v) $y(t) = \int_{-\infty}^{t} x(t)\,dt$

Here o/p depends upon present & past i/p. Hence this is a Causal system.

Check whether the following Continuous time system as Linear (or) non-linear.

1) $y(t) = t\,x(t)$ 2) $y(t) = x^2(t)$

$y_1(t) = f[x_1(t)] = t x_1(t)$, $y_2(t) = f[x_2(t)] = t x_2(t)$

Hence Linear Combination of o/p become.

$y_3(t) = a_1 y_1(t) + a_2 y_2(t)$

$= a_1 t x_1(t) + a_2 t x_2(t)$

Now response to the Linear Combination of i/p becomes.

$$y_3'(t) = f[a_1x_1(t) + a_2x_2(t)]$$

$$= t[a_1x_1(t) + a_2x_2(t)]$$

$$= a_1tx_1(t) + a_2tx_2(t)$$

On Comparing above Eqs $y_3(t) = y_3'(t)$

Hence this is a Linear system.

ii) $y(t) = x^2(t)$

The o/p of the system to two i/p $x_1(t)$ & $x_2(t)$

$$f[x_1(t)] = x_1^2(t)$$

$$f(x_2(t)) = x_2^2(t)$$

Hence Linear Combination of these o/p becomes

$$y_3(t) = a_1y_1(t) + a_2y_2(t)$$

$$= a_1x_1^2(t) + a_2x_2^2(t)$$

Now let us find the response of the System to Combination of i/p

$$y_3'(t) = f[a_1x_1(t) + a_2x_2(t)]$$

$$= [a_1x_1(t) + a_2x_2(t)]^2$$

$a_2^2 x_2^2(t) + 2a_1a_2x_1(t)x_2(t)$.

—o—

check whether the following Continuous time system are ~~time~~ time invariant (or) time variant.

1) $y(t) = \sin x(t)$ 2) $y(t) = t\,x(t)$ 3) $y(t) = x(t)\cos 200\pi t$.

Anlogy b/w Vectors & Signals:-

Signals Can be represented in terms of Orthogonal function. These Orthogonal fun. Satisfy specific properties.

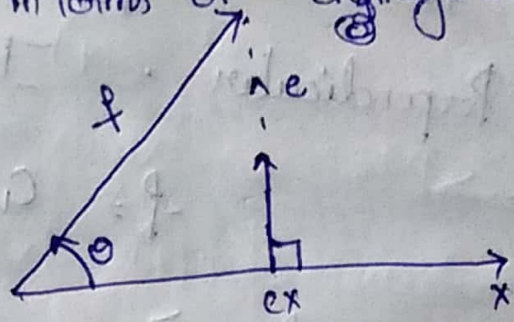


fig-1.

Orthogonality Concept in Vector:-

All the Signals are basically Vectors. A Vector Can be represented in terms of its Co-ordinate. System. For Example Consider the Vector f. as shown fig 1. There is another Vector x. The dot product of 'f' and 'x'

$$f \cdot x = |f|\,|x| \cos\theta$$

Here 'θ' is angle b/w f & x.

In the above fig 'cx' is the Component of Vector 'f' along 'x'. In Other words 'cx' is the Projection of 'f' on 'x'. Here 'f' can be Expressed as Vector addition as:

$$f = cx + e$$

Here 'e' is an Error Vector. Note 'e' is minimum only if it is perpendicular.

Two other possibilities where e is not Perpendicular. In this Case observe that

$$f = c_1 x + e_1 = c_2 x_2 + e_2$$

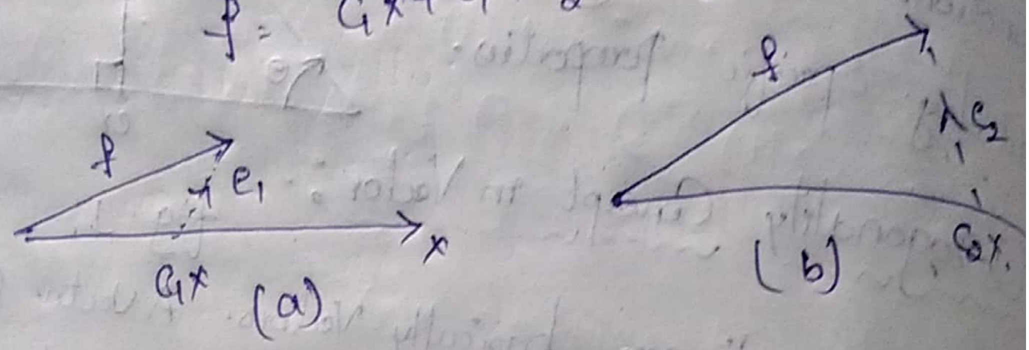


fig 2. c_1 & c_2 are greater than c.

But e_1 & e_2 are greater than e. Here 'e' is minimum Only when it is ⊥ to 'x'. The Component of f along x is cx. it is also given as $|f| \cos\theta$.

$$c|x| = f \cos\theta$$

Multiplying both side by $|x|$

$$c|x|^2 = |f| \, |x| \cos\theta$$

R.H.S of above Equ. represents the dot product of Vector f & x. Hence

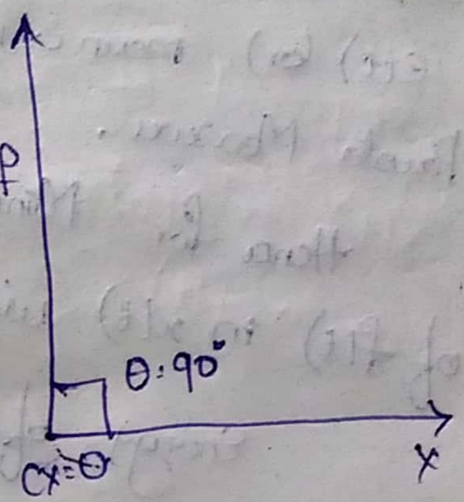
$$c|x|^2 = f \cdot x$$

$$c = 1/|x|^2 \, f \cdot x$$

$$x \cdot x = |x|^2 \, ; \quad c = \frac{f \cdot x}{x \cdot x}.$$

... will be Cancelled. Since f.x & x.x are vector products.

fig 2(a) & 2(b) observe that $C_1 x$ will be zero when 'f' is ⊥r to 'x'. In other words, will not have component along 'x' then 'f' and 'x' are ⊥r to each other.



Hence the dot product f.x will zero i.e

$$f.x = |f||x| \cos\theta$$

$$|f||x| \cos 90° = 0.$$

The vector 'f' and 'x' are said to be orthogonal if their dot product is zero. In other word, vectors are mutually perpendicular are orthogonal. if they are mutually perpendicular.

<u>Orthogonality in signals:-</u>

Now let us apply the orthogonality concept of vectors to real signals. Let us consider signal f(t) to be represented in terms of x(t) over an interval t_1 & t_2

$f(t) \approx c\,x(t) + e(t) \rightarrow (a)$

$e(t) = f(t) - c\,x(t) \rightarrow (b)$

→ Minimum value of e(t) will give best approximation of f(t) in x(t).

→ Minimum value of ε(t), minimum Energy of e(t) (or) mean square value of e(t) series approximate measure.

Hence for Minimum Energy of e(t), representation of f(t) in x(t) will better.

Energy of e(t) will be

$$E_e = \int_{t_1}^{t_2} e^2(t)\,dt$$

And Mean square value of e(t) will be given

$$\overline{e^2(t)} = \frac{1}{t_2-t_1}\int_{t_1}^{t_2} e^2(t)\,dt$$

$$\overline{e^2(t)} = \frac{E_e}{t_2-t_1}$$

Here E_e is Energy of e(t) over the interval of t_1 to t_2. And $\overline{e^2(t)}$ is Mean square value e(t).

from Equ ⑤

$$E_c = \int_{t_1}^{t_2} [f(t) - c\,x(t)]^2 dt$$

[Here the value of $e(t)^2$ over the interval t_1 to t_2

And $\overline{e^2(t)}$ is Mean Square value $e(t)$]×

~~Here~~ the value of 'c' should be selected such that E_c will be Minimum. This can obtained by differentiating E_c w.r.to c & Equating it to zero i.e

for Minimum E_c, $\frac{dE_c}{dc} = 0$

$$\text{i.e } \frac{d}{dc}\left[\int_{t_1}^{t_2} [f(t) - c\,x(t)]^2\right] dt = 0$$

$$= \frac{d}{dc}\int_{t_1}^{t_2} f^2(t)\,dt - \frac{d}{dc}\,2c\,f(t)\cdot x(t)\,dt + \frac{d}{dc}\int_{t_1}^{t_2} c^2 x^2(t)\,dt = 0$$

= first term is independent of 'c' hence it will be zero.

$$-2\int_{t_1}^{t_2} f(t)\cdot x(t)dt + 2c\int_{t_1}^{t_2} x^2(t)\,dt = 0$$

$$= 2\left[-2\int_{t_1}^{t_2} f(t)\cdot x(t)\,dt + c\int_{t_1}^{t_2} x^2(t)\,dt = 0\right.$$

$$\int_{t_1}^{t_2} f(t)\, x(t)\, dt + c\int_{t_1}^{t_2} x^2(t)\,dt = 0$$

$$c\int_{t_1}^{t_2} x^2(t)\,dt = \int_{t_1}^{t_2} f(t)\, x(t)\, dt$$

Component of $x(t)$ Contained in $f(t)$:

$$\boxed{c = \frac{\int_{t_1}^{t_2} f(t)\, x(t)\, dt}{\int_{t_1}^{t_2} x^2(t)\,dt}}$$

Here are clearly observe that above Eqn is Similar to the system Equation.

The Denominator of the above Eqn represent Energy of $x(t)$. it Can't be zero. Hence numerator must be zero to make 'c' zero. If 'c' is zero there will be no Component of $f(t)$ along $x(t)$. then $f(t)$ and $x(t)$ are said to be Orthogonal Over an interval $[t_1\ t_2]$ i.e

Orthogonality $\int_{t_1}^{t_2} f(t)\, x(t)\, dt = 0$

Similarly if $f(t)$ and $x(t)$ are Complex Signals, then they are Orthogonal. Over an interval $[t_1, t_2]$

for $\int_{t_1}^{t_2} f(t)\, x(t)\, dt = 0$

if $x(t)$ & $f(t)$ are Orthogonal signal then they are Orthogonal Over an terval $[t_1, t_2]$ if

$f(t)\, x^*(t)\, dt = 0$ (or) $\int_{t_1}^{t_2} f^*(t)\, x(t)\, dt = 0.$

$x^*(t)$ is Complex conjugate of $x(t)$.

Problem Show that the following signal are Orthogonal Over an interval $[0, 1]$

$f(t) = 1$, $x(t) = \sqrt{3}\,(1-2t)$

Sol We know that the signals are Orthogonal

if $\int_{t_1}^{t_2} f(t)\, x(t) = 0$

$\int_{t_1}^{t_2} f(t)\, x(t)\, dt = \int_0^1 1 \left[\sqrt{3}\,(1-2t)\right] dt$

$= \int_0^1 \sqrt{3}\, dt - \int_0^1 2\sqrt{3}\, t\, dt$

$$\sqrt{3}\int_0^1 dt - 2\sqrt{3}\int_0^1 t\,dt$$

$$\sqrt{3}\cdot[t]_0^1 - 2\sqrt{3}\left[\frac{t^2}{2}\right]_0^1 = 0$$

Thus the two given signal are Orthogonal over interval $[0,1]$.

2) A rectangular function is defined as.

$$f(t) = \begin{cases} A & \text{for } 0 \le t \le \pi/2 \\ -A & \text{for } \pi/2 \le t \le 3\pi/2 \\ A & \text{for } 3\pi/2 \le t \le 2\pi \end{cases}$$

Approximate above fun by $A\cos t$ b/w the interval $(0, 2\pi)$ such that Mean Squre Error is Minimum.

Sol

$f(t) = c\,x(t)$

$x(t) = A\cos t$

Here

$$c = \frac{\int_{t_1}^{t_2} f(t)x(t)\,dt}{\int_{t_1}^{t_2} x^2(t)\,dt} = \frac{\int_0^{\pi/2} A\cdot A\cos t\,dt + \int_{\pi/2}^{3\pi/2}(-A)A\cos t\,dt + \int_{3\pi/2}^{\pi/2} A\cdot A\cos t\,dt}{\int_0^{2\pi}(A\cos t)^2\,dt}$$

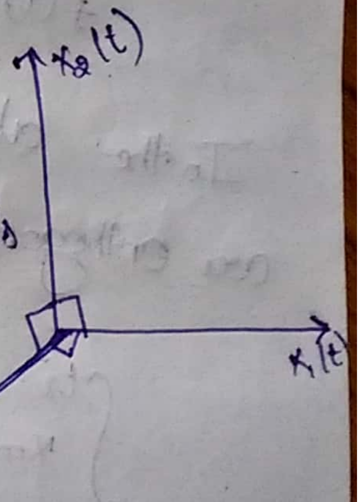
$$= \frac{A^2\left[\sin t\right]_0^{\pi/2} - A^2\left[\sin t\right]_{\pi/2}^{3\pi/2} + A^2\left[\sin t\right]_{3\pi/2}^{2\pi}}{A^2\int_0^{2\pi} \frac{1+\cos 2t}{2}\, dt}$$

$$= \frac{A^2\left[\sin \pi/2 - \sin 0\right] - A^2\left[\sin \frac{3\pi}{2} - \sin \pi/2\right] + A^2\left[\sin 2\pi - \sin 3\pi/2\right]}{\frac{A^2}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi}}$$

$$= 4/\pi$$

Thus $f(t) = 4/\pi \, A \cos t$ is the required Approximation.

Orthogonal Signal Space :-



Let $x_1(t)$, $x_2(t)$ & $x_3(t)$ be Orthogonal to Each other. This Mean these three signals will be mutually ⊥r to Each other.

it forms a three dimensional signal Space. Such signal Space. This signal Space is used to represent any signal lying in that Space.

If there are ... Mutually Orthogonal Signal space. Any signal f(t) can be represented in this dimensional signal space.

Signal Approximation using Orthogonal functions

Let us consider the set of signal which are mutually orthogonal over an interval $[t_1\ t_2]$. These signals can represents any signal f(t) as,

$$f(t) \simeq c_1 x_1(t) + c_2 x_2(t) + \dots + c_N x_N(t)$$

$$f(t) = \sum_{n=1}^{N} c_n x_n(t)$$

In the above Eqn any two signals $x_m(t)$ & $x_n(t)$ are orthogonal over an interval $[t_1\ t_2]$ i.e

$$\int_{t_1}^{t_2} x_m(t) x_n(t)\, dt = \begin{cases} 0 & \text{for } m \neq n \\ E_n & \text{for } m = n \end{cases}$$

In the above Eqn Observe that any two different signals are orthogonal, when m=n it is the same signal.

$$\int_{t_1}^{t_2} x_n(t) x_n(t)\, dt = \int_{t_1}^{t_2} x_n^2(t)\, dt = E_n$$

... Energy of the signals i.e E_n.

Error $e(t)$ in the approximation of Equ is given as $e(t) = f(t) - \sum_{n=1}^{N} C_n x_n(t)$

Hence Error Energy will be

$$E_e = \int_{t_1}^{t_2} e^2(t)\,dt = \int_{t_1}^{t_2} \left[f(t) - \sum_{n=1}^{N} C_n x_n(t)\right]^2 dt.$$

Here E_e is the fun. of $C_1, C_2, C_3 \ldots C_N$

Hence E_e will be Minimized w.r.to C_i if

$$\frac{\partial E_e}{\partial C_j} = 0$$

$$\frac{\partial}{\partial C_j}\left\{\int_{t_1}^{t_2}\left[f(t) - \sum_{n=1}^{N} C_n x_n(t)\right]^2 dt\right\} = 0 \longrightarrow (1)$$

Above Eq. will be Executed for $i = 1, 2, 3 \ldots N$

$$\frac{\partial}{\partial C_j}\left[\int_{t_1}^{t_2} f^2(t)\,dt - \int_{t_1}^{t_2}\sum_{n=1}^{N} 2C_n f(t)\, x_n(t)\,dt + \int_{t_1}^{t_2}\sum_{n=1}^{N} C_n^2\, x_n^2(t)\right\} = 0.$$

Above Equ is Executed for $i = 1, 2, 3 \ldots N$

Here observe that first integration term is independent of C_i. Hence its derivative will be zero

The derivative of second and third integration terms will be non zero only when $n=i$ these terms will be constant & their derivatives are zero.

$$\frac{\partial}{\partial c_i}\left\{-\int_{t_1}^{t_2} 2c_i f(t)\,x_i(t) + \int_{t_1}^{t_2} c_i^2 x_i^2(t)\,dt\right\}=0$$

$$-2\int_{t_1}^{t_2} f(t)\,x_i(t)\,dt + 2c_i\int_{t_1}^{t_2} x_i^2(t)\,dt = 0 \qquad i=1,2\ldots N$$

$$c_i = \frac{\int_{t_1}^{t_2} f(t)\,x_i(t)\,dt}{\int_{t_1}^{t_2} x_i^2(t)\,dt}$$

We know that $\int_{t_1}^{t_2} x_i^2(t)\,dt = E_i$ i.e Energy.

Hence above Eqn becomes.

$$\boxed{c_i = \frac{1}{E_i}\int_{t_1}^{t_2} f(t)\,x_i(t)\,dt.} \longrightarrow (a)$$

 Mean Square Error :-

Now let us Consider the Mean Square Error in signal approximation using Orthogonal functions.

The Error Energy is given by Equ.

$$E_e = \int_{t_1}^{t_2} \left[f(t) - \sum_{n=1}^{N} C_n x_n(t) \right]^2 dt.$$

$$= \int_{t_1}^{t_2} f^2(t)\, dt - 2\int_{t_1}^{t_2} \sum_{n=1}^{N} C_n f(t)\, x_n(t)\, dt + \int_{t_1}^{t_2} \sum_{n=1}^{N} C_n^2 x_n^2(t)\, dt$$

last integration term is Energy of $x(n)$ i.e E_n.

And with the help of Equ ⓐ we can write middle term of above Equation as.

$$\int_{t_1}^{t_2} f(t)\, x_n(t)\, dt = C_n E_n$$

$$E_e = \int_{t_1}^{t_2} f^2(t)\, dt - 2\sum_{n=1}^{N} C_n C_n E_n + \sum_{n=1}^{N} C_n^2 E_n.$$

$$= \int_{t_1}^{t_2} f^2(t)\, dt - 2\sum_{n=1}^{N} C_n^2 E_n + \sum_{n=1}^{N} C_n^2 E_n.$$

$$= \int_{t_1}^{t_2} f^2(t)\,dt - \sum_{n=1}^{N} C_n^2 E_n$$

The Mean sauer Error & Error Energy are related as.

$$\overline{e^2(t)} = \frac{E_e}{t_2-t_1} = \frac{1}{t_2-t_1}\left[\int_{t_1}^{t_2} f^2(t)\,dt - \sum_{n=1}^{N} C_n^2 E_n\right]$$

In the above Equ. $C_n^2 E_n$ is always positive Hence Error Energy E_e Can be reduced if number of terms. N used for representation are increased

ideally. $E_e \to 0$. & $N \to \infty$ under this Condition, the Orthogonal signal set is said to be Complex.

Closed (or) Complete set of Orthogonal function

The Mean square Error approaches zero as number of terms $C_n^2 E_n$ are Made infinite

$$0 = \frac{1}{t_2-t_1}\left[\int_{t_1}^{t_2} f^2(t)\,dt - \sum_{n=1}^{\infty} C_n^2 E_n\right] \text{ with}$$

$$\overline{e^2(t)} = 0 \quad \text{as } N = \infty$$

$$\int_{t_1} f^2(t)\,dt = \sum_{n=1}^{\infty} C_n^2 E_n$$

With N approaching infinity Equ can be written as.

$$f(t) = \sum_{n=1}^{\infty} C_n x_n(t)$$

Here $x_1(t), x_2(t), \ldots x_n(t)$ is a set of Mutually Orthogonal function. it is Said to be Complete (or) Closed Set if there Exists no function $p(t)$ for which

$$\int_{t_1}^{t_2} p(t)\, x_n(t)\, dt = 0 \quad \text{for } n = 1, 2 \ldots$$

if $p(t)$ Exists & above integral is zero. then Obviously. $p(t)$ must be a member of set $\{x_n(t)\}$

For the set of mutually Orthogonal signals. $x_n(t)$ over an interval (t_1, t_2).

$$\int_{t_1}^{t_2} x_m(t)\, x_n(t)\, dt = \begin{cases} 0 & \text{if } m \neq n \\ E_n & \text{if } m = n \end{cases}$$

For this Complete set, the function $f(t)$ is Expressed as.

$f(t) = C_1 x_1(t) + C_2 x_2(t) + \cdots$

$$C_i = \frac{\int_{t_1}^{t_2} f(t)\, x_i(t)\, dt}{\int_{t_1}^{t_2} x_i^2(t)\, dt} = \frac{1}{E_i} \int_{t_1}^{t_2} f(t)\, x_i(t)\, dt$$

The set of $x_n(t)$ is Called Orthogonal basis functions.

Orthogonality in Complex functions:

Consider that the set of signals $x_1(t), x_2(t), x_3(t) \ldots$ are Complex then they are Mutually Orthogonal if

$$\int_{t_1}^{t_2} x_m(t)\, x_n^*(t)\, dt = \int_{t_1}^{t_2} x_m^*(t)\, x_n(t)\, dt = \begin{cases} 0 & \text{for} \\ E_n & \text{for} \end{cases}$$

Then $f(t)$ Can be Expressed as,

$$\boxed{f(t) = \sum_{n=1}^{\infty} C_n x_n(t)}$$

where C_n is given in the Similar fashion of above Eqn.

$$C_n = \frac{1}{E_n}\int_{t_1}^{t_2} f(t)\, x_n^*(t)\,dt$$

Where E_n is given for Complex signals are.

$$E_n = \int_{t_1}^{t_2} x_n(t) \cdot x_n^*(t)\,dt$$

---o---

Trigonometric Fourier Series:-

We know that any fun. $f(t)$ Can be Expressed as

$$f(t) = \sum_{n=1}^{\infty} C_n x_n(t).$$

Here $x_n(t)$ represent Orthogonal signal set. This Equ is They are also Called basic function.

Called generalized Fourier Series.

$m \neq n$

$m = n$

We have seen that the set.

$$\{1, \cos\omega_0 t, \cos 2\omega_0 t \dots \cos n\omega_0 t, \dots \sin\omega_0 t, \sin 2\omega_0 t \dots \sin n\omega_0 t \dots\}$$

is Orthogonal Over the period T_0. Here ω_0 is Called fundamental frequency. and $n\omega_0$ is Called n^{th} harmonic. There is DC Component of $\cos n\omega_0 t$ at $n=0$ i.e

UNIT-2

→ Trigonometric fourier Series.

As we know that $\sin n\omega_0 t$ & $\cos n\omega_0 t$ both are orthogonal over the given interval, Now we choose a composite set of function consisting of a set $\cos n\omega_0 t$ & $\sin n\omega_0 t$ for $(n=0,1,2,\dots)$ & forms a complete orthogonal set.

∴ for $n=0$, $\sin n\omega_0 t = 0$ & for $n=1$ $\cos n\omega_0 t = 1$

& the set of orthogonal fn are given as $1, \cos\omega_0 t, \cos 2\omega_0 t \dots \cos n\omega_0 t \dots, \sin\omega_0 t, \sin 2\omega_0 t \dots \sin n\omega_0 t \dots$

Now any fn $f(t)$ can be represented in terms of these functions over any interval

$$(0,T) \text{ (or) } (t_0, t_0+T) \text{ (or) } \left(t_0, t_0+\frac{2\pi}{\omega_0}\right)$$

$$\Rightarrow f(t) = a_0 + a_1\cos\omega_0 t + \dots a_n\cos n\omega_0 t + \dots + b_1\sin\omega_0 t + b_2\sin 2\omega_0 t \dots + b_n \sin n\omega_0 t$$

$$\left(t_0, t_0+\frac{2\pi}{\omega_0}\right)$$

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos n\omega_0 t + b_n \sin n\omega_0 t\right) \quad \text{——— (1)}$$

$$(t_0 < t_0 < t_0+T)$$

eq (1) is the say trigonometric fourier series representation of $f(t)$ over the interval (t_0, t_0+T)

where $a_0, a_1 \dots a_n, b_1, b_2 \dots b_n$ are the components of $f(t)$ along the mutually orthogonal set (or) the constant values, & are given by

As we have,

$$C_{12} = \frac{\int_{t_1}^{t_2} f_1(t) f_2(t)\, dt}{\int_{t_1}^{t_2} f_2^2(t)\, dt}$$

llly

$$a_n = \frac{\int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t\, dt}{\int_{t_0}^{t_0+T} \cos^2 n\omega_0 t\, dt}$$

$$\int_{t_0} \cos^2 n\omega_0 t\, dt = \int_{t_0} \left[\frac{1+\cos 2n\omega_0 t}{2}\right] dt$$

$$= \frac{1}{2}\int_{t_0}^{t_0+T} \left[1+\cos 2n\omega_0 t\right] dt$$

$$= \frac{1}{2}\left[t - \frac{\sin 2n\omega_0 t}{2n\omega_0}\right]_{t_0}^{t_0+T}$$

$$= \frac{1}{2}\left[t_0+T-t_0+\frac{\sin 2n\omega_0(t_0+T)}{2n\omega_0} - \frac{\sin 2n\omega_0 t_0}{2n\omega_0}\right]$$

$$= \frac{1}{2}\left[T+\frac{\sin(2n\omega_0 t_0+2n\omega_0 \frac{2\pi}{\omega_0})}{2n\omega_0} - \frac{\sin 2n\omega_0 t}{2n\omega_0}\right]$$

$$= \frac{1}{2}\left[T+\frac{1}{2n\omega_0}\left\{\sin(2n\omega_0 t+4n\pi) - \sin(2n\omega_0 t_0)\right\}\right]$$

$$= \frac{1}{2}\left[T+\frac{1}{2n\omega_0}\left\{\sin(2n\omega_0 t_0) - \sin(2n\omega_0 t_0)\right\}\right]$$

$$= \frac{1}{2}[T+0] = T/2$$

$$a_n = \frac{\int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t\, dt}{\int_{t_0}^{t_0+T}\cos^2 n\omega_0 t\, dt} = \frac{\int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t\, dt}{T/2}$$

$$\boxed{a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t\, dt}$$

let $n=0$,

$$a_0 = \frac{\int_{t_0}^{t_0+T} f(t)\cos(0)\, dt}{\int_{t_0}^{t_0+T}\cos^2(0)\, dt}$$

$$a_0 = \frac{\int_{t_0}^{t_0+T} f(t)\, dt}{\int_{t_0}^{t_0+T}(1)\, dt}$$

$$a_0 = \frac{\int_{t_0}^{t_0+T} f(t)\, dt}{[t]_{t_0}^{t_0+T}} = \int_{t_0}^{t_0+T} f(t)\, dt$$

$$\boxed{a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,dt}$$

& $$b_n = \frac{\int_{t_0}^{t_0+T} f(t)\sin n\omega_0 t\,dt}{\int_{t_0}^{t_0+T} \sin^2\omega_0 t\,dt}$$

$$\boxed{b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin n\omega_0 t\,dt}$$

The constant term a_0 is the average value of $f(t)$ over the interval (t_0, t_0+T), & thus a_0 is the dc component of $f(t)$ over this interval.

→ Alternate form of the trigonometric series:-

we have
$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n\cos n\omega_0 t + b_n\sin n\omega_0 t)$$

$$a_n\cos n\omega_0 t + b_n\sin n\omega_0 t = A_n\cos[n\omega_0 t + \phi_n]$$

where $A_n = \sqrt{a_n^2 + b_n^2}$ &

$$\phi_n = -\tan^{-1}\left(\frac{b_n}{a_n}\right)$$

$$\boxed{f(t) = a_0 + \sum_{n=1}^{\infty} A_n\cos(n\omega_0 t + \phi_n)}$$

the coefficients A_n are called spectral amplitudes & ϕ_n is the spectral phase.

Ily $F_n = \dfrac{\int_{t_0}^{t_0+T} f(t)\,(e^{jn\omega_0 t})\,dt}{\int_{t_0}^{t_0+T} (e^{jn\omega_0 t})(e^{jn\omega_0 t})\,dt}$

$$\boxed{F_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\, e^{-jn\omega_0 t}\, dt}$$

Thus any fn may be expressed as a discrete sum of exponential functions $\{e^{jn\omega_0 t}\}$, $(n=0,\pm1,\pm2 \ldots)$ over an interval $t_0 < t < t_0+T$.

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$\& \quad F_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\, e^{-jn\omega_0 t}\, dt$$

These two eq are referred as Fourier series pair

→ Relation b/w the trignometric & the exponential fourier series:

Now consider an exponential fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \quad (t_0 < t < t_0+T)$$

$$= F_0 + F_1 e^{j\omega_0 t} + F_2 e^{j2\omega_0 t} \ldots F_n e^{jn\omega_0 t} \ldots$$
$$F_{-1} e^{-j\omega_0 t} + F_{-2} e^{-j2\omega_0 t} \ldots F_{-n} e^{-jn\omega_0 t} \ldots \quad \text{—— (A)}$$

where $F_n = \dfrac{1}{T}\displaystyle\int_{t_0}^{t_0+T} f(t)\, e^{-jn\omega_0 t}\, dt$ —— (1)

Ily $F_{-n} = \dfrac{1}{T}\displaystyle\int_{t_0}^{t_0+T} f(t)\, e^{jn\omega_0 t}\, dt$ —— (2)

from (1) & (2) F_n & F_{-n} are complex conjugates

i.e $F_n = F_{-n}^{*}$

Now let $F_n = \alpha_n + j\beta_n$ —— (3)

$F_{-n} = \alpha_n - j\beta_n$ —— (4)

adding these two eq we get & subtracting

$$\alpha_n = \frac{1}{2}(F_n + F_{-n}) \quad ③$$

$$\beta_n = \frac{1}{2j}(F_n - F_{-n})$$

(or) $$2\alpha_n = F_n + F_{-n} \quad ③$$

$$-2\beta_n = +j(F_n - F_{-n})$$

sub ③ & ④ in eq Ⓐ

$$f(t) = F_0 + (\alpha_1 + j\beta_1)e^{j\omega_0 t} + (\alpha_2 + j\beta_2)e^{j2\omega_0 t} \ldots + (\alpha_n + j\beta_n)e^{jn\omega_0 t} + \ldots (\alpha_1 - j\beta_1)e^{-j\omega_0 t} + (\alpha_2 - j\beta_2)e^{-j2\omega_0 t} \ldots (\alpha_n - j\beta_n)e^{-jn\omega_0 t} + \ldots$$

$$f(t) = F_0 + \left[(\alpha_1 e^{j\omega_0 t} + \alpha_2 e^{j2\omega_0 t} \ldots \alpha_{+1} e^{-j\omega_0 t} + \alpha_{+2} e^{-j2\omega_0 t} \ldots \alpha_n e^{jn\omega_0 t} + \alpha_{+n} e^{-jn\omega_0 t}) + j(\beta_1 e^{j\omega_0 t} + \beta_2 e^{j2\omega_0 t} + \ldots \beta_n e^{jn\omega_0 t} + \beta_{+1} e^{-j\omega_0 t} + \beta_{+2} e^{-j2\omega_0 t} \ldots - \beta_{+n} e^{-jn\omega_0 t} \ldots)\right]$$

$$f(t) = F_0 + \sum_{n=-\infty}^{\infty} \left[\alpha_n (e^{jn\omega_0 t} + e^{-jn\omega_0 t}) + j\beta_n (e^{jn\omega_0 t} - e^{-jn\omega_0 t})\right]$$

$$f(t) = F_0 + \sum_{n=-\infty}^{\infty} \left[2\alpha_n \left(\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}\right) + j^2 2\beta_n \left(\frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}\right)\right]$$

$$f(t) = F_0 + \sum_{n=-\infty}^{\infty} \left[2\alpha_n \cos n\omega_0 t - 2\beta_n \sin n\omega_0 t\right]$$

Now compare this with the standard trigonometric eq

$$f(t) = a_0 + \sum_{n=-\infty}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$\Rightarrow F_0 = a_0 \quad \Big| \quad 2\alpha_n = a_n \quad \Big| \quad -2\beta_n = b_n$$

$a_0 = F_0$

$a_n = 2\alpha_n = F_n + F_{-n}$

$b_n = -2\beta_n = j(F_n - F_{-n})$

This is the representation of trigonometric in terms of exponential

Ily we can " exponential in terms of trigonometric as is

$a_n = F_n + F_{-n}$ $\quad b_n = j(F_n - F_{-n})$

$$\Rightarrow \frac{b_n}{j} = F_n - F_{-n}$$

$a_n = F_n + F_{-n}$ — (5) $\Rightarrow -jb_n = F_n - F_{-n}$ — (6)

Adding & subtracting (5) & (6) we get

$$F_n = \frac{1}{2}[a_n - jb_n] \text{ \&}$$

$$F_{-n} = \frac{1}{2}[a_n + jb_n]$$

→ Representation of a periodic fn by the fourier series over the entire interval $(-\infty < t < \infty)$:

Upto know we represent a given fn $f(t)$ by the F S over a finite interval (t_0, t_0+T) & outside this interval, the fn & its corresponding F S are need not be equal. This equality b/w $f(t)$ & its series holds over the interval (t_0, t_0+T), Now we want that this equality holds over the entire interval $(-\infty < t < \infty)$

Now we consider some function $f(t)$ & its exponential F.S representation over an interval (t_0, t_0+T)

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \quad (t_0 < t < t_0+T) \text{ — (1)}$$

$$\text{where } \omega_0 = \frac{2\pi}{T}$$

The two sides of the equation need not be equal outside this interval.

Let the right-hand side of (1) be $\phi(t)$

Thus $f(t) = \phi(t) \quad (t_0 < t < t_0+T)$.

adding these two eq & subtracting

$$\alpha_n = \frac{1}{2}(F_n + F_{-n}) \quad (3)$$

$$\beta_n = \frac{1}{2j}(F_n - F_{-n})$$

(or) $2\alpha_n = F_n + F_{-n} \quad (3)$

$$-2\beta_n = +j(F_n - F_{-n})$$

sub (3) & (4) in eq (A)

$$f(t) = F_0 + (\alpha_1 + j\beta_1)e^{j\omega_0 t} + (\alpha_2 + j\beta_2)e^{j2\omega_0 t} \ldots + (\alpha_n + j\beta_n)e^{jn\omega_0 t} + \ldots (\alpha_1 - j\beta_1)e^{-j\omega_0 t} + (\alpha_2 - j\beta_2)e^{-j2\omega_0 t} \ldots (\alpha_n - j\beta_n)e^{-jn\omega_0 t} + \ldots$$

$$f(t) = F_0 + \left[(\alpha_1 e^{j\omega_0 t} + \alpha_2 e^{j2\omega_0 t} \ldots \alpha_{+1} e^{-j\omega_0 t} + \alpha_{+2} e^{-j2\omega_0 t} \ldots \alpha_n e^{jn\omega_0 t} + \alpha_{+n} e^{-jn\omega_0 t}) + j(\beta_1 e^{j\omega_0 t} + \beta_2 e^{j2\omega_0 t} + \ldots \beta_n e^{jn\omega_0 t} + \beta_{+1} e^{-j\omega_0 t} + \beta_{+2} e^{-j2\omega_0 t} \ldots - \beta_{+n} e^{-jn\omega_0 t} \ldots)\right]$$

$$f(t) = F_0 + \sum_{n=-\infty}^{\infty} \left[\alpha_n (e^{jn\omega_0 t} + e^{-jn\omega_0 t}) + j\beta_n (e^{jn\omega_0 t} - e^{-jn\omega_0 t})\right]$$

$$f(t) = F_0 + \sum_{n=-\infty}^{\infty} \left[2\alpha_n \left(\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}\right) + j^2 2\beta_n \left(\frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}\right)\right]$$

$$f(t) = F_0 + \sum_{n=-\infty}^{\infty} \left[2\alpha_n \cos n\omega_0 t - 2\beta_n \sin n\omega_0 t\right]$$

Now compare this with the standard trignometric eq

$$f(t) = a_0 + \sum_{n=-\infty}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$\Rightarrow F_0 = a_0 \quad \Big| \quad 2\alpha_n = a_n \quad \Big| \quad -2\beta_n = b_n$$

where $\phi(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$ (for all values of t)

Now consider the fn '$\phi(t+T)$',

$$\phi(t+T) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0(t+T)}$$

$$= \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} e^{jn\omega_0 T}$$

$$\phi(t+T) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} e^{j2\pi n}$$

$$\phi(t+T) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} = \phi(t)$$

$$\phi(t+T) = \phi(t)$$

i.e, the fn $\phi(t)$ repeats itself after every T seconds, such fn is called a periodic fn.

ie the exponential (or Trignometric) F S ~~depend~~ repeats then -lves every T seconds. Thus if $f(t)$ be a periodic fn of period T, then it can be represented by an exponential (or) trignometric) F.S over the entire interval $(-\infty < t < \infty)$.

$\therefore$ A periodic fn $f(t)$ with period T can be rep by a F.S over the entire interval $(-\infty < t < \infty)$,

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \quad (-\infty < t < \infty)$$

where $\omega_0 = \frac{2\pi}{T}$

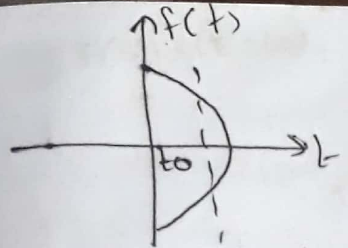
& $F_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t) e^{-jn\omega_0 t} dt$

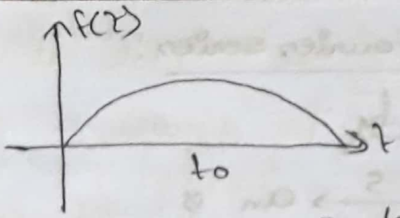
→ Fourier series - Dirichlet's conditions:

The sufficient conditions under which a s/g $f(t)$ can be represented in terms of its fourier series must satisfy are called dirichlet's conditions. They are

(i) The fn $f(t)$ is a single-valued fn of the variable 't' in the interval (t, t_n)

ie the fn $f(t)$ must have single value at any instant of time

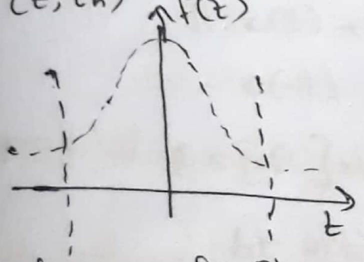


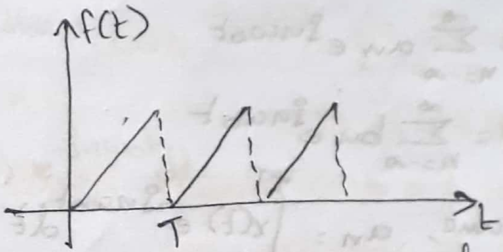


At to, it has 2 values do it, is not a single valued fn

it is a single valued fn

ii) The fn f(t) has a finite number of discontinuities in the interval (t, t_n)



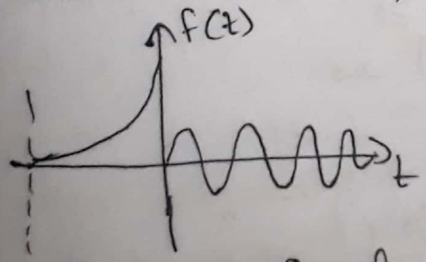


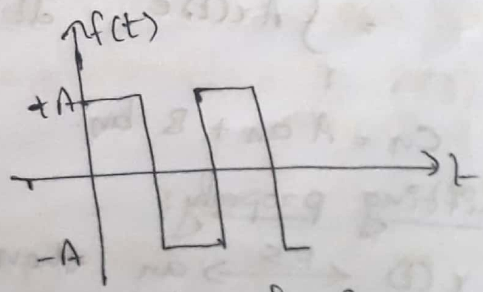
it has no finite number of discontinuities as it is not possible to find the value of f(t) at such a number of discontinuities

It has finite number of distinuities & the value of f(t) at the distincontinuities can be calculated

$$f(t=T) = \frac{f(T^-) + f(T^+)}{2}$$

iii) The fn f(t) has a finite number of minima & maxima in the interval (t_1, t_2)



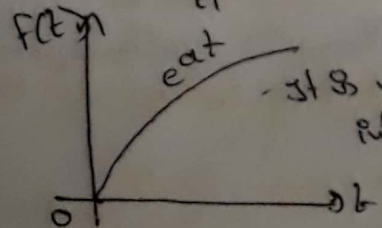


It has not fixed number of manima & maxima

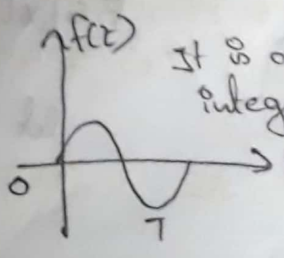
It has fixed number of minima & maxima(1)

iv) The fn f(t) is absolutely integrable

$$\text{ie } \int_{t_1}^{t_2} |f(t)|\, dt < \infty$$



It is not absolutely integrable



It is absolutely integrable

where $\phi(t) = \sum_{n=-\infty} F_n e^{jn\omega_0 t}$ (for all values of t)

Now consider the fn '$\phi(t+T)$',

$$\phi(t+T) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0(t+T)}$$

$$= \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \cdot e^{jn\omega_0 T}$$

$$\phi(t+T) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} e^{j2\pi n}$$

$$\phi(t+T) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} = \phi(t)$$

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∴ A periodic fn $f(t)$ with period T can be rep by a F.S over the entire interval $(-\infty < t < \infty)$,

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \quad (-\infty < t < \infty)$$

where $\omega_0 = \frac{2\pi}{T}$

$$\& \quad F_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\, e^{-jn\omega_0 t}\, dt$$

→ Fourier Series - Dirichlet's conditions:

The sufficient conditions under which a s/g $f(t)$ can be represented in terms of its fourier series must satisfy are called dirichlet's conditions. They are

(i) The fn $f(t)$ is a single-valued fn of the variable 't' in the interval (t, t_n)

i.e the fn $f(t)$ must have single value at any instant of time

→ Properties of Fourier series:

1) linearity property:

If $x(t) \xleftrightarrow{F.S} a_n$ &

$y(t) \xleftrightarrow{F.S} b_n$ then

$$A x(t) + B y(t) \xleftrightarrow{FS} A a_n + B b_n$$

Proof: we have

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}$$

& $$y(t) = \sum_{n=-\infty}^{\infty} b_n e^{jn\omega_0 t}$$

Also we have, $$a_n = \int_T x(t) e^{-jn\omega_0 t} dt$$

& $$b_n = \int_T y(t) e^{-jn\omega_0 t} dt$$

let $c(t) = A x(t) + B y(t)$ then,

$$c_n = \int_T [A x(t) + B y(t)] e^{-jn\omega_0 t} dt$$

$$= \int_T A x(t) e^{-jn\omega_0 t} dt + \int_T B y(t) e^{-jn\omega_0 t} dt$$

$$c_n = A a_n + B b_n$$

2) time shifting property:

If $x(t) \xleftrightarrow{FS} a_n$ then

$$x(t-t_0) \xleftrightarrow{FS} e^{-jn\omega_0 t_0} a_n$$

proof: we have

$$Fs[x(t)] = a_n = \int_T x(t) e^{-jn\omega_0 t} dt$$

$$FS[x(t-t_0)] = \int_T x(t-t_0) e^{-jn\omega_0 t} dt$$

$$F(S)[x(t-t_0)] = \int_T x(p)\, e^{-jn\omega_0(p+t_0)}\,dp$$

$$= \int_T x(p)\, e^{-jn\omega_0 p} \cdot e^{-jn\omega_0 t_0}\,dp$$

$$= e^{-jn\omega_0 t_0} \int_T x(p)\, e^{-jn\omega_0 p}\,dp$$

$$= e^{-jn\omega_0 t_0}\, a_n$$

3) time-Reversal property:

If $x(t) \xleftrightarrow{FS} a_n$ then

$x(-t) \xleftrightarrow{FS} a_{-n}$

proof: $FS[x(t)] = a_n = \int_T x(t)\, e^{-jn\omega_0 t}\,dt$

let $y(t) = x(-t)$

$$F.S[y(t)] = \int_T y(t)\, e^{-jn\omega_0 t}\,dt$$

$$= \int_T x(-t)\, e^{-jn\omega_0 t}\,dt$$

let $p = -t \Rightarrow dp = -dt$

$$= \int_{t_0}^{t_0+T} x(-t)\, e^{-jn\omega_0 t}\,dt$$

we have

$p = -t \qquad dp = -dt$

$$= \int_{-t_0}^{-(t_0+T)} x(p)\, e^{jpn\omega_0}\,(-dp)$$

$$= \int_{t_0}^{t_0+T} x(p)\, e^{jn\omega_0 p}\,dp$$

$= a_{-n}$ is the FS coefficients of the time reversal of a s/g are time reversal of the F.S coefficients of the corresponding signal.

4) Time scaling:

$x(t) \xleftrightarrow{FS} a_n$

then $x(at) \xleftrightarrow{FS} a_n$ but the fundamental fq is $a\omega_0$

$FS[x(t)] =$

$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}$, then

$x(at) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 at}$

where 'a' is the scaling factor.

5) frequency shifting:

If $x(t) \xleftrightarrow{FS} a_n$ then

$x(t) e^{jm\omega_0 t} \xleftrightarrow{FS} a_{n-m}$

proof: $FS[x(t)] = a_n = \int_{\langle T \rangle} x(t) e^{-jn\omega_0 t} dt$

$FS[x(t) e^{jm\omega_0 t}] = \int_{\langle T \rangle} x(t) e^{jn\omega_0 t} \cdot e^{-jn\omega_0 t} dt$

$= \int_{T} x(t) e^{-j(n-m)\omega_0 t} dt$

$= a_{n-m}$

6) Conjugation:

If $x(t) \xleftrightarrow{FS} a_n$ then

$x(t) \xleftrightarrow{FS} a_{-n}$

$FS[x(t)] = a_n = \int_{\langle T \rangle} x(t) e^{-jn\omega_0 t} dt$

$FS[x(t)] = \int_{\langle T \rangle} x^*(t) (e^{-jn\omega_0 t})^* dt$

$FS[x(t)] = \int_{\langle T \rangle} x^*(t) e^{jn\omega_0 t} dt$

$= a^*_{-n}$

→ Symmetry conditions:

1) If a periodic fn is symmetrical about the vertical axis, the corresponding fourier series contains only cosine terms.

2) If a periodic fn is antisymmetrical about the vertical axis, the F.S contains sine terms only.

To prove this,

consider a fn $f_e(t)$, it is said to be an even fn of it, if

$$f_e(t) = f_e(-t)$$

& $f_o(t)$ is said to be an odd fn of 'f' if

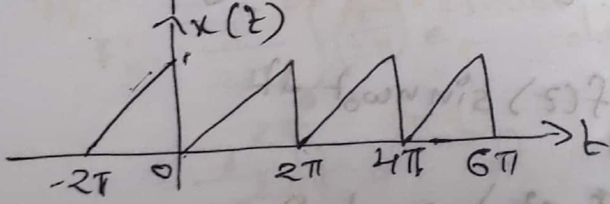
$$f_o(t) = -f_o(-t)$$

properties of even & odd fn:

1) product of an even & an odd fn is an odd fn
2) " even & even fn "
3) " odd & even fn is odd fn
4) " odd & odd " even fn

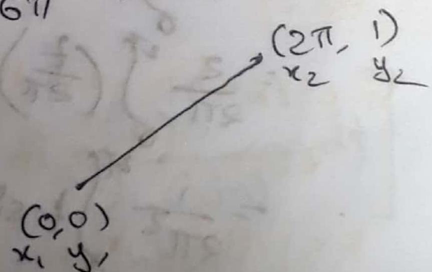
problems

→ Find the cosine representation FS for the signal shown in fig



time period, $T = 2\pi$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$



$$y - y_1 = m(x - x_1)$$

$$x(t) - 0 = \frac{1-0}{2\pi - 0}(t - 0)$$

$$x(t) = \frac{1}{2\pi}(t)$$

$$x(t) = \frac{t}{2\pi} \text{ for } 0 \le t < 2\pi \quad \text{for } 0 \le t < T$$

Multiply t with amplitude & divide with period

$$\frac{(1)t}{2\pi} = \frac{t}{2\pi} \text{ for a given interval}$$

we have the trigonometric F.S.,

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right)$$

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{t}{2\pi}\,dt = \frac{1}{2}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\,dt$$

$$= \frac{2}{2\pi}\int_0^{2\pi}\left(\frac{t}{2\pi}\right)\cos(nt)\,dt$$

$$= \frac{1}{2\pi^2}\left[t\int \cos nt\,dt - \int (1)\frac{\sin nt}{n}\,dt\right]_0^{2\pi}$$

$$= \frac{1}{2\pi^2}\left[\frac{t\sin nt}{n} + \frac{\cos nt}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi^2}\left[0 + \frac{1}{n^2} - 0 - \frac{1}{n^2}\right] = 0$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t\,dt$$

$$= \frac{2}{2\pi}\int_0^{2\pi}\left(\frac{t}{2\pi}\right)\sin nt\,dt$$

$$= \frac{1}{2\pi^2}\int_0^{2\pi} t\sin nt\,dt$$

$$b_n = \frac{1}{2\pi^2}\left[t\int \sin nt\,dt - \int (1)\frac{-\cos nt}{n}\,dt\right]_0^{2\pi}$$

$$= \frac{1}{2\pi^2}\left[\frac{-t\cos nt}{n} + \frac{\sin nt}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi^2}\left[\frac{(-2\pi)\cos 2\pi n}{n} + 0 + 0 + 0\right]$$

$$\frac{1}{2\pi^2}\left[\frac{(-2\pi)\cos 2\pi n}{n} + 0 + 0 + 0\right]$$

$$= -\frac{1}{\pi n}\cos 2\pi n = \frac{-1}{n\pi}(1) = \frac{-1}{n\pi}$$

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right]$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty}\left[0 + \frac{-1}{n\pi}\sin nt\right]$$

$$= \frac{1}{2} - \sum_{n=1}^{\infty}\frac{1}{n\pi}\sin(nt)$$

$$\therefore f(t) = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin nt}{n}$$

→ find out above for exponential F.S.

$$f(t) = \frac{t}{2\pi} \text{ for } 0 \le t \le 2\pi$$

we have, $f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$

$$F_n = \frac{1}{T}\int_0^T f(t)\, e^{-jn\omega_0 t}\, dt$$

$$F_n = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{t}{2\pi}\right) e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{4\pi^2}\int_0^{2\pi} t \cdot e^{-jnt}\, dt$$

$$= \frac{1}{4\pi^2}\left[t\int e^{-jnt}\, dt - \int \frac{(1)e^{-jnt}}{(-jn)}\, dt\right]$$

$$= \frac{1}{4\pi^2}\left[\frac{+te^{-jnt}}{-jn} - \frac{e^{-jnt}}{j^2 n^2}\right]$$

$$= \frac{1}{4\pi^2}\left[\frac{te^{-jnt}}{-jn} + \frac{e^{-jnt}}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi^2}\left[\frac{2\pi e^{-j2\pi n}}{-jn} + \frac{e^{-j2\pi n}}{n^2} - 0 - \frac{1}{n^2}\right]$$

$$= \frac{1}{(4\pi)^2}\left[\frac{2\pi(1)}{-jn} + \frac{1}{n^2} - \frac{1}{n^2}\right]$$

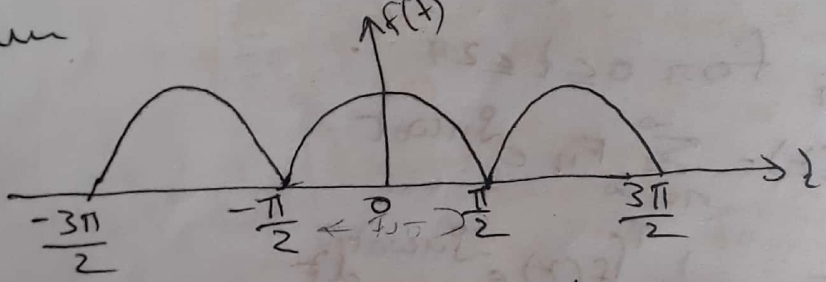
$$= \frac{1}{-j2\pi n} = \frac{j}{2\pi n}$$

$$\Rightarrow f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} \frac{j}{2\pi n} e^{jnt}$$

$$f(t) = \frac{j}{2\pi} \sum_{n=-\infty}^{\infty} \left(\frac{e^{jnt}}{n}\right)$$

→ Det the trignometric FS & exponential FS of a fully rectified cosine fn shown in fig. & draw the complex spectrum



let $t_0 = -\frac{\pi}{2}$ | $f(t) = \cos\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

then $t_0 + T = \frac{\pi}{2}$

$$T = \frac{\pi}{2} - t_0 = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{\pi} = 2 \text{ rad/sec}$$

$$f(t) = a_0 + \sum_{n=-\infty}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,dt$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} (\cos t)\,dt$$

$$= \frac{1}{\pi}\left[+\sin t\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{\pi}\left[+\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)\right] = \frac{1}{\pi}(1+1)$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t \, dt$$

$$= \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \cos t \cos 2nt \, dt$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \left[\cos(2n+1)t + \cos(2n-1)t\right] dt$$

$$= \frac{1}{\pi}\left[\frac{\sin(2n+1)t}{(2n+1)} + \frac{\sin(2n-1)t}{(2n-1)}\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{\pi}\left[\frac{\sin(2n+1)\frac{\pi}{2}}{(2n+1)} - \frac{\sin(2n+1)(-\frac{\pi}{2})}{(2n+1)} + \frac{\sin(2n-1)\frac{\pi}{2}}{(2n-1)} - \frac{\sin(2n-1)(-\frac{\pi}{2})}{(2n-1)}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\sin(2n+1)\frac{\pi}{2}}{(2n+1)} + \frac{2\sin(2n-1)\frac{\pi}{2}}{(2n-1)}\right]$$

$$a_n = \frac{2}{\pi}\left[\frac{(-1)^n}{(2n+1)} + \frac{(-1)^{n+1}}{2n-1}\right]$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin n\omega_0 t \, dt$$

$$= \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \cos t \sin 2nt \, dt$$

$$b_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \cos t \sin 2t \, dt$$

$$= \frac{1}{\pi}\left[\int_{-\pi/2}^{\pi/2}\left[\sin(2n+1)t + \sin(2n-1)t\right\} dt\right]$$

$$= \frac{1}{\pi}\left[\frac{-\cos(2n+1)t}{2n+1} - \frac{\cos(2n-1)t}{2n-1}\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{\pi}\left[\frac{-\cos(2n+1)\frac{\pi}{2}}{2n+1} + \frac{\cos(2n+1)(\frac{-\pi}{2})}{(2n+1)} - \frac{\cos(2n-1)\frac{\pi}{2}}{(2n-1)} + \frac{\cos(2n-1)(-\frac{\pi}{2})}{2n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{-\cos(2n+1)\frac{\pi}{2}}{(2n+1)} + \frac{\cos(2n+1)\frac{\pi}{2}}{2n+1} - \frac{\cos(2n-1)\frac{\pi}{2}}{(2n-1)} + \frac{\cos(2n-1)\frac{\pi}{2}}{(2n-1}\right]$$

$$= \frac{0}{2}$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t]$$

$$f(t) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \left[\frac{2}{\pi}\left(\frac{(-1)^n}{2n+1} + \frac{(-1)^{n+1}}{2n-1} \right) \cos 2nt + 0 \right]$$

ex exponential F.S :

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$F_n = \frac{1}{T} \int_{t_0}^{t_0+T} f(t)\, e^{-jn\omega_0 t}\, dt$$

$$F_n = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos t \cdot e^{-j2nt}\, dt$$

$$= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \left(\frac{e^{jt} + e^{-jt}}{2} \right) e^{-j2nt}\, dt$$

$$= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \left[e^{jt} \cdot e^{-j2nt} + e^{-jt} \cdot e^{-j2nt} \right] dt$$

$$= \frac{1}{2\pi} \left[\int_{-\pi/2}^{\pi/2} e^{-j(+2n-1)t}\, dt + \int_{-\pi/2}^{\pi/2} e^{-j(2n+1)t}\, dt \right]$$

$$= \frac{1}{2\pi} \left[\frac{e^{-j(2n-1)t}}{-j(2n-1)} + \frac{e^{-j(2n+1)t}}{-j(2n+1)} \right]_{-\pi/2}^{\pi/2}$$

$$\frac{1}{\ } \left[\frac{e^{-j(2n-1)t}}{\ } + e^{-j(2n+1)t} \right]^{\pi/2}$$

$$\frac{1}{2\pi}\left[\frac{e^{j(2n-1)\frac{\pi}{2}} - e^{-j(2n-1)\frac{\pi}{2}}}{+j(2n-1)} + \frac{e^{j(2n+1)\frac{\pi}{2}} - e^{-j(2n+1)\frac{\pi}{2}}}{j(2n+1)}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(2n-1)\frac{\pi}{2}}{(2n-1)} + \frac{\sin(2n+1)\frac{\pi}{2}}{(2n+1)}\right]$$

$$F_n = \frac{1}{\pi}\left[\frac{(-1)^{n+1}}{2n-1} + \frac{(-1)^n}{2n+1}\right]$$

$$F_0 = \frac{1}{\pi}\left[\frac{(-1)^1}{-1} + \frac{1}{1}\right] = \frac{1}{\pi}[1+1] = \frac{2}{\pi}$$

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{1}{\pi}\left[\frac{(-1)^{n+1}}{(2n-1)} + \frac{(-1)^n}{2n+1}\right] e^{j2nt}$$

$$F_n = \frac{1}{\pi}\left[\frac{(-1)^n}{2n+1} + \frac{(-1)^{n+1}}{2n-1}\right]$$

$$F_1 = \frac{1}{\pi}\left[\frac{(-1)^1}{3} + \frac{(-1)^2}{2-1}\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{3} + 1\right] = \frac{1}{\pi}\left[1 - \frac{1}{3}\right] = \frac{1}{\pi}\left[\frac{2}{3}\right] = \frac{2}{3\pi}$$

$$F_2 = \frac{1}{\pi}\left[\frac{(-1)^2}{4+1} + \frac{(-1)^{2+1}}{4-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{5} - \frac{1}{3}\right] = \frac{1}{\pi}\left[\frac{3-5}{15}\right] = \frac{-2}{15\pi}$$

Thus the F-S is

$$f(t) = \frac{2}{\pi} + \frac{2}{3\pi} e^{j2t} - \frac{2}{15\pi} e^{j4t} - \cdots$$

$$+ \frac{2}{3\pi} e^{-j2t} - \frac{2}{15\pi} e^{-j4t} + \cdots$$

$$= \frac{1}{\pi}\left[\frac{-\cos(2n+1)\frac{\pi}{2}}{(2n+1)} + \frac{\cos(2n+1)\frac{\pi}{2}}{2n+1} - \frac{\cos(2n-1)\frac{\pi}{2}}{(2n-1)} + \frac{\cos(2n-1)\frac{\pi}{2}}{(2n-1)}\right]$$

$$= 0$$

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right]$$

$$f(t) = \frac{2}{\pi} + \sum_{n=1}^{\infty}\left[\frac{2}{\pi}\left(\frac{(-1)^n}{2n+1} + \frac{(-1)^{n+1}}{2n-1}\right)\cos 2nt + 0\right]$$

for exponential F.S:

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$F_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\, e^{-jn\omega_0 t}\, dt$$

$$F_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos t \cdot e^{-j2nt}\, dt$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\left(\frac{e^{jt}+e^{-jt}}{2}\right)e^{-j2nt}\, dt$$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\left[e^{jt}\cdot e^{-j2nt} + e^{-jt}\cdot e^{-j2nt}\right] dt$$

$$= \frac{1}{2\pi}\left[\int_{-\pi/2}^{\pi/2} e^{-j(+2n-1)t}\, dt + \int_{-\pi/2}^{\pi/2} e^{-j(2n+1)t}\, dt\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j(2n-1)t}}{-j(2n-1)} + \frac{e^{-j(2n+1)t}}{-j(2n+1)}\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j(2n-1)t}}{-j(2n-1)} + \frac{e^{-j(2n+1)t}}{-j(2n+1}\right]_{-\pi/2}^{\pi/2}$$

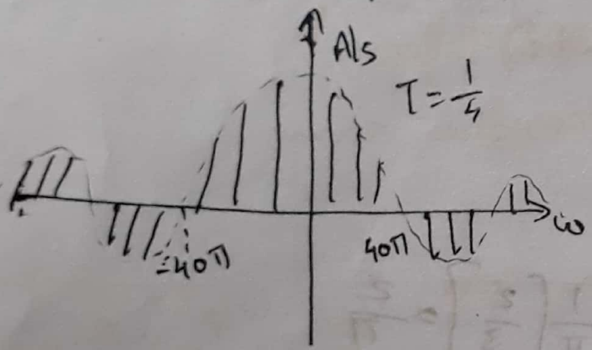
$$= \frac{1}{2\pi}\left[\frac{e^{-j(2n-1)\frac{\pi}{2}}}{-j(2n-1)} - \frac{e^{-j(2n-1)\left(-\frac{\pi}{2}\right)}}{-j(2n-1)} + \frac{e^{-j(2n+1)\pi/2}}{-j(2n+1)} - \dots\right.$$

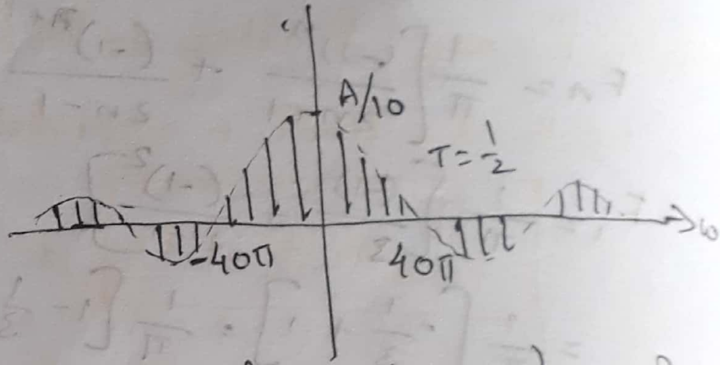
Fourier transforms:

Deriving fourier transform from fourier series (or) representation of an arbitary fn over the entire interval $(-\infty,\infty)$:

As we know that any non periodic signal can be represented in terms of its sum of exp fns over any finite interval $(t_0<t<t_0+T)$ & any periodic signal can be represented in $(-\infty,\infty)$.

Now we want to represent an arbitary fn (non periodic) a sum of exponential fn over the entire interval $(-\infty<t<\infty)$





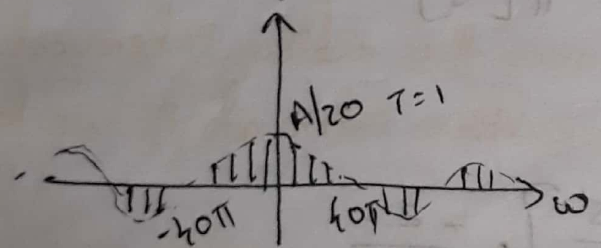


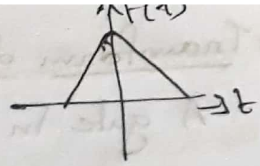
fig shows the spectrum of a periodic gate fn for some specific values of T.

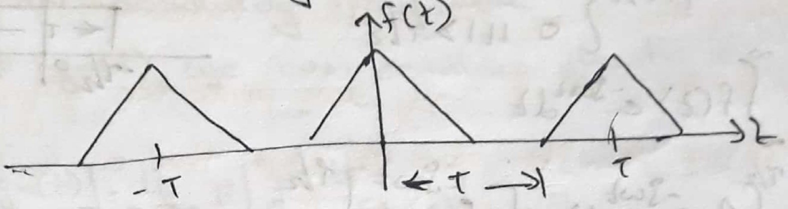
If we can observe the spectrum, then as the period 'T' is made larger, the fundamental fq becomes smaller. The fq spectrum becomes denser. But the amplitudes of the fq components becomes smaller.

The shape of the spectrum remains unaltered.

Now, consider an arbitary fn $f(t)$, we want to represent this function as a sum of exponential fns over the entire interval $(-\infty<t<\infty)$

This can be achieved by constructing a new periodic fn $f_T(t)$ of period T, where the fn $f(t)$ repeats itself for every T seconds.





Now this fn $f_T(t)$ is a periodic fn & it can be represented with exponential FS over the entire interval $(-\infty, \infty)$

In the limit, if T becomes ∞, then the pulses in the periodic fn repeat after an ∞ (infinite) Interval.

i.e in the limit $T \to \infty$ $f_T(t)$ & $f(t)$ are same

$$\boxed{\lim_{T \to \infty} f_T(t) = f(t)}$$

Thus the FS representing $f_T(t)$ over the entire interval will also represent $f(t)$ over the entire interval if we take $T \to \infty$ in this series

The expone FS for $f_T(t)$ can be represented by,

$$f_T(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

where $\omega_0 = \frac{2\pi}{T}$

$$\therefore F_n = \frac{1}{T} \int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_0 t} dt$$

→Fourier Transform of signum fn:

$$\text{sgn}(t) = 1, \ t > 0$$
$$= -1, \ t < 0$$

This is not absolutely integrable

so, Instead of sgn(t), we can consider the fn $e^{-a|t|}\text{sgn}(t)$ as the limit $a \to 0$

$$F[\text{sgn}(t)] = \lim_{a\to 0} F\left[e^{-a|t|}\text{sgn}(t)\right]$$

$$= \lim_{a\to 0} \int_{-\infty}^{\infty} e^{-a|t|}\text{sgn}(t)\, dt\, e^{-j\omega t}$$

$$= \lim_{a\to 0}\left[\int_0^{\infty} e^{-at} e^{-j\omega t} dt - \int_{-\infty}^{0} e^{at}\cdot e^{-j\omega t} dt\right]$$

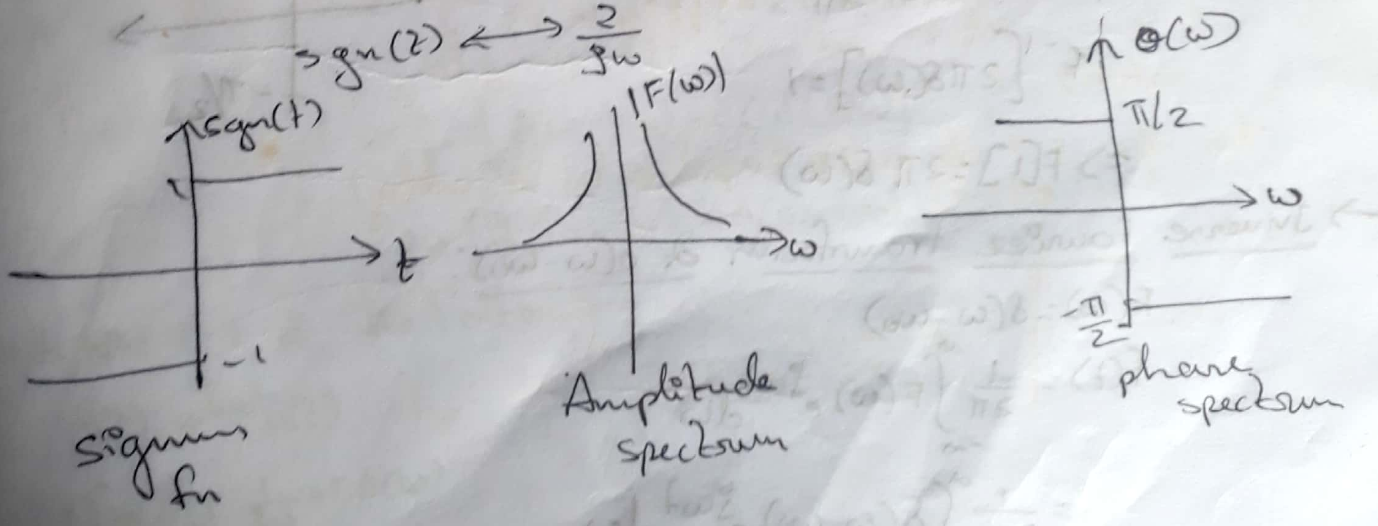
$$= \lim_{a\to 0}\left[\int_0^{\infty} e^{-(a+j\omega)t} dt - \int_{-\infty}^{0} e^{(a-j\omega)t} dt\right]$$

$$= \lim_{a\to 0}\left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\Bigg|_0^{\infty} - \frac{e^{(a-j\omega)t}}{a-j\omega}\Bigg|_{-\infty}^{0}\right]$$

$$= \lim_{a\to 0}\left[+\frac{1}{a+j\omega} - \frac{1}{a-j\omega}\right] = \lim_{a\to 0}\left[\frac{a-j\omega-a-j\omega}{a^2+\omega^2}\right]$$

$$= \lim_{a\to 0}\left[\frac{-2j\omega}{a^2+\omega^2}\right] = \frac{-2j\omega}{\omega^2} = \frac{-2j}{\omega} = \frac{2}{j\omega}$$

$$\text{sgn}(t) \longleftrightarrow \frac{2}{j\omega}$$



→ Fourier transform of step fn:

we have,

$$u(t) = \frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}(t)$$

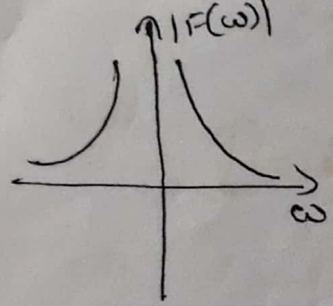
$$\mathrm{sgn}(t) = 2u(t) - 1$$

$$F(\omega) = F[u(t)] = F\left[\frac{1}{2}\right] + F\left[\frac{1}{2}\,\mathrm{sgn}\,t\right]$$

$$= \frac{1}{2}F[1] + \frac{1}{2}F[\mathrm{sgn}\,t]$$

$$= \frac{1}{2}2\pi\,\delta(\omega) + \frac{1}{2}\frac{2}{j\omega}$$

$$= \pi\,\delta(\omega) + \frac{1}{j\omega}$$



→ Inverse F.T of $\delta(\omega)$:

$$F(\omega) = \delta(\omega)$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\,e^{j\omega t}\,d\omega$$

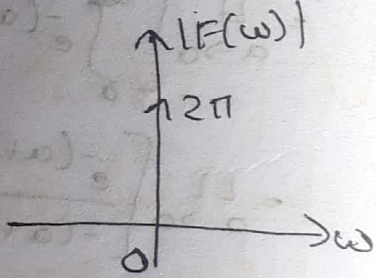
$$= \frac{1}{2\pi}(1) = \frac{1}{2\pi}$$

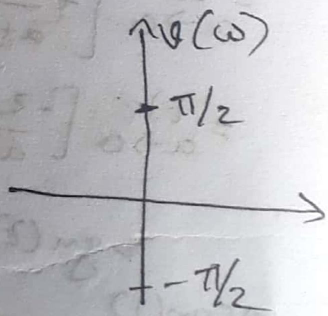
$$F^{-1}[\delta(\omega)] = f(t) = \frac{1}{2\pi}$$

$$F^{-1}[\delta(\omega)] = \frac{1}{2\pi}$$

$$F^{-1}[2\pi\,\delta(\omega)] = 1$$

$$\Rightarrow F[1] = 2\pi\,\delta(\omega)$$





→ Inverse Fourier transform of $\delta(\omega - \omega_0)$:

$$F(\omega) = \delta(\omega - \omega_0)$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0)\,e^{j\omega t}\,d\omega$$

$$F^{-1}[\delta(\omega - \omega_0)] = \frac{1}{2\pi}e^{j\omega_0 t}$$

$$F^{-1}\left[2\pi\delta(\omega-\omega_0)\right] = e^{j\omega_0 t}$$

$$F\left[e^{j\omega_0 t}\right] = 2\pi\delta(\omega-\omega_0)$$

→ F.T of cosine signal:

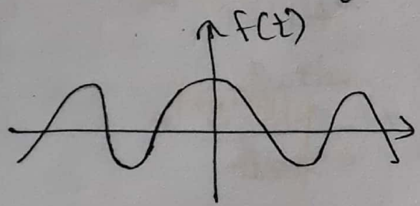
$$f(t) = \cos\omega_0 t = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$$

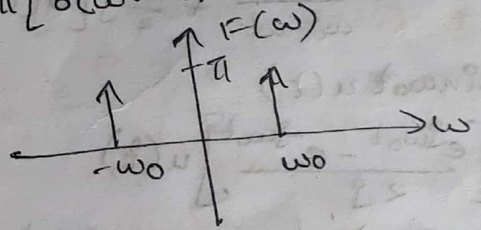
$$F[f(t)] = F(\omega) = F\left[\frac{1}{2}\left\{e^{j\omega_0 t} + e^{-j\omega_0 t}\right\}\right]$$

$$= \frac{1}{2}\left[F\left[e^{j\omega_0 t}\right] + F\left[e^{-j\omega_0 t}\right]\right]$$

$$= \frac{1}{2}\left[2\pi\delta(\omega+\omega_0) + 2\pi\,\delta(\omega-\omega_0)\right]$$

$$F[\cos\omega_0 t] = \pi\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]$$





→ F.T of sinusoidal signal:

$$f(t) = \sin\omega_0 t$$

$$= \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]$$

$$F\{f(t)\} = \frac{1}{2j}\left[F\left[e^{j\omega_0 t}\right] - F\left[e^{-j\omega_0 t}\right]\right]$$

$$= \frac{1}{2j}\left[2\pi\,\delta(\omega-\omega_0) - 2\pi\delta(\omega+\omega_0)\right]$$

$$= \frac{\pi}{j}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$

$$= j\pi\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]$$

→ Find the F.T of the following

$$f(t) = e^{j\omega_0 t}u(t)$$

$$F[u(t)] = \frac{1}{j\omega} + \pi\delta(\omega)$$

$$F\left[e^{j\omega_0 t}u(t)\right] = \frac{1}{j(\omega-\omega_0)} + \pi\,\delta(\omega-\omega_0)$$

→ $f(t) = \sin\omega_0 t\, u(t)$

$$F[\sin\omega_0 t\, u(t)] = F\left[\left\{\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right\} u(t)\right]$$

$$= \frac{1}{2j}\left[F[e^{j\omega_0 t}u(t)] - F[e^{-j\omega_0 t}u(t)]\right]$$

$$= \frac{1}{2j}\left[\left\{\frac{1}{j(\omega-\omega_0)} + \pi\delta(\omega-\omega_0)\right\} - \left\{\frac{1}{j(\omega+\omega_0)} + \pi\delta(\omega+\omega_0)\right\}\right]$$

$$= \frac{1}{2j}\left[\frac{1}{j}\left\{\frac{\omega+\omega_0-\omega+\omega_0}{(\omega-\omega_0)(\omega+\omega_0)}\right\} + \pi\left\{\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right\}\right]$$

$$= \frac{1}{2j}\left[\frac{1}{j}\left\{\frac{2\omega_0}{\omega^2-\omega_0^2}\right\} + \pi\,\delta(\omega-\omega_0) - \pi\,\delta(\omega+\omega_0)\right]$$

$$= \frac{\omega_0}{\omega_0^2-\omega^2} + \frac{\pi}{2}j\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]$$

→ $f(t) = A\sin\omega_0 t\, u(t)$

$$= A\left[\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right]u(t)$$

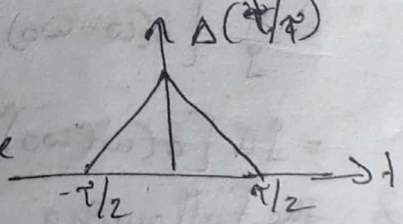
$$= \frac{A}{2j}e^{j\omega_0 t}u(t) - \frac{A}{2j}e^{-j\omega_0 t}u(t)$$

$$F[f(t)] = F\left[\frac{A}{2j}e^{j\omega_0 t}u(t) - \frac{A}{2j}e^{-j\omega_0 t}u(t)\right]$$

$$= \frac{A}{2j}\left[\pi\delta(\omega-\omega_0) + \frac{1}{j(\omega-\omega_0)}\right] - \frac{A}{2j}\left[\pi\delta(\omega+\omega_0) + \frac{1}{j(\omega-\omega_0)}\right]$$

$$= \frac{\pi A}{2j}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right] + \frac{A}{2}\left[\frac{2\omega_0}{\omega^2-\omega_0^2}\right]$$

→ F.T of triangular pulse:



$$\Delta_a(t) = \begin{cases} 1-|t| \,;\, |t| < a \\ 0 \,;\, \text{otherwise} \end{cases}$$

$$\Delta\left(\frac{t}{\tau}\right) = \begin{cases} 1 - \frac{2|t|}{\tau} \,;\, |t| < \tau/2 \\ 0 \,;\, \text{otherwise} \end{cases}$$

$$F[\omega] = F\left[\Delta\left(\frac{t}{\tau}\right)\right]$$

$$= \int_{-\tau/2}^{\tau/2}\left[1 - \frac{2|t|}{\tau}\right]e^{-j\omega t}\,dt$$

$$\int_{-\tau/2}^{0}\left(1+\frac{2t}{\tau}\right)e^{-j\omega t}\,dt+\int_{0}^{\tau/2}\left(1-\frac{2t}{\tau}\right)e^{-j\omega t}\,dt$$

$$=\int_{-\tau/2}^{0}\left(1+\frac{2t}{\tau}\right)e^{-j\omega t}\,dt+\int_{0}^{\tau/2}\left(1-\frac{2t}{\tau}\right)e^{-j\omega t}\,dt$$

$$=\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-\tau/2}^{0}+\frac{2}{\tau}\left[\frac{-te^{-j\omega t}}{j\omega}-\frac{e^{-j\omega t}}{(j\omega)^2}\right]_{-\tau/2}^{0}+\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{0}^{\tau/2}$$

$$-\frac{2}{\tau}\left[\frac{-te^{-j\omega t}}{j\omega}-\frac{e^{-j\omega t}}{(j\omega)^2}\right]_{0}^{\tau/2}$$

$$=-\cancel{\frac{1}{j\omega}}+\frac{e^{-j\omega(-\tau/2)}}{j\omega}+\frac{2}{\tau}\left[0-\frac{1}{(j\omega)^2}-\frac{\frac{\tau}{2}e^{+j\omega\tau/2}}{j\omega}+\frac{e^{j\omega\tau/2}}{(j\omega)^2}\right]$$

$$+\frac{e^{-j\omega\tau/2}}{-j\omega}+\cancel{\frac{1}{j\omega}}-\frac{2}{\tau}\left[\frac{-\frac{\tau}{2}e^{-j\omega\tau/2}}{j\omega}-\frac{e^{-j\omega\tau/2}}{(j\omega)^2}+0+\frac{1}{(j\omega)^2}\right]$$

$$=\cancel{\frac{e^{j\omega\tau/2}}{j\omega}}-\cancel{\frac{e^{j\omega\tau/2}}{j\omega}}-\frac{2}{\tau(j\omega)^2}-\cancel{\frac{e^{j\omega\tau/2}}{j\omega}}+\frac{2}{\tau}\frac{e^{j\omega\tau/2}}{(j\omega)^2}+\cancel{\frac{e^{-j\omega\tau/2}}{j\omega}}$$

$$+\frac{2}{\tau}\frac{e^{-j\omega\tau/2}}{(j\omega)^2}-\frac{2}{\tau}\frac{1}{(j\omega)^2}$$

$$=\frac{-4}{\tau(j\omega)^2}+\frac{2}{\tau}\frac{e^{j\omega\tau/2}}{(j\omega)^2}+\frac{2}{\tau}\frac{e^{-j\omega\tau/2}}{(j\omega)^2}$$

$$=\frac{2}{\tau}\left[\frac{e^{j\omega\tau/2}}{(j\omega)^2}+\frac{e^{-j\omega\tau/2}}{(j\omega)^2}-\frac{2}{(j\omega)^2}\right]$$

$$=\frac{2}{\tau}\left[\left\{\frac{e^{j\omega\tau/4}}{j\omega}\right\}^2+\left\{\frac{e^{-j\omega\tau/4}}{j\omega}\right\}^2-2\left\{\frac{e^{j\omega\tau/4}}{j\omega}\right\}\left\{\frac{e^{-j\omega\tau}}{j\omega}\right\}\right]$$

$$=\frac{2}{\tau}\left[\left\{\frac{e^{j\omega\tau/4}}{j\omega}-\frac{e^{-j\omega\tau/4}}{j\omega}\right\}^2\right]$$

$$=4\,\frac{2}{\tau}\left[\frac{e^{j\omega\tau/4}-e^{-j\omega\tau/4}}{2j\omega}\right]^2$$

$$\frac{8}{\tau}\left[\frac{\sin(\omega\tau/4)}{\omega(\tau/4)}\right]^2 \times \left(\frac{\tau}{4}\right)^2$$

$$= \frac{8}{\tau}\cdot\frac{\tau^2}{16}\left[Sa(\omega\tau/4)\right]^2$$

$$= \cancel{\frac{8}{\tau}}\ \cancel{\frac{8}{\cdot}}\ \frac{\tau}{2}\, Sa^2\left(\frac{\omega\tau}{4}\right)$$

$$= \frac{\tau}{2}\, \mathrm{sinc}^2\left(\frac{\omega\tau}{4}\right)$$

$$F\left[\Delta\left(\frac{t}{\tau}\right)\right] = \frac{\tau}{2}\, \mathrm{sinc}^2\left(\frac{\omega\tau}{4}\right)$$

→ F.T & Impulse train:

we have the exponential F.S of unit impulse train is

$$f(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$$

T_0 is the spacing b/w the

$$F\{f(t)\} = F\left[\sum_{n=-\infty}^{\infty} \delta(t - nT)\right]$$

$$= F\left[\frac{1}{T_0}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}\right]$$

$$= \frac{1}{T_0}\sum_{n=-\infty}^{\infty} F\left[e^{jn\omega_0 t}\right]$$

$$= \frac{1}{T_0}\sum_{n=-\infty}^{\infty}\left[2\pi\,\delta(\omega - n\omega_0)\right]$$

$$= \frac{2\pi}{T_0}\sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_0)$$

$$F(\omega) = \omega_0\sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_0)$$

→ F.T of aperiodic function:

Generally FT is applicable for a periodic fn, & F.T of a periodic fn does not exist,

∴ it fails to satisfy the condition of absolutely integrability.

But the transform does exist in the limit, lly for $\cos\omega_0 t$ & $\sin\omega_0 t$.

i.e we can assume the periodic fn exists only in the finite interval $(-\tau/2, \tau/2)$ & in the limit let $\tau \to \infty$ with period τ as we can express a periodic fn $f(t)$

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

taking F.T on both sides

$$F[f(t)] = F\left[\sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}\right]$$

$$= \sum_{n=-\infty}^{\infty} F_n F\left[e^{jn\omega_0 t}\right]$$

$$= \sum_{n=-\infty}^{\infty} F_n \, 2\pi\delta(\omega - n\omega_0)$$

$$\therefore F[f(t)] = 2\pi \sum_{n=-\infty}^{\infty} F_n \delta(\omega - n\omega_0)$$

∴ The F.T of a periodic s/g consists of impulses located at the harmonic fq of the signal & the strength of each impulse is same as 2π times the value of the corresponding coefficient in the exponential F.S

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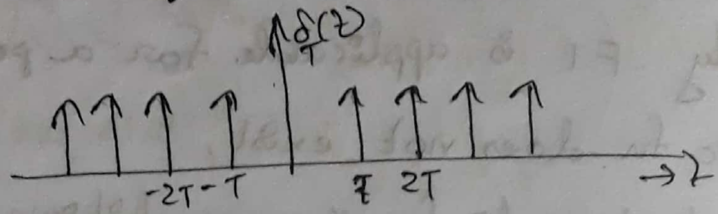
→ Find the F.T of sequence of equidistant impulses.



Now we consider a sequence of equidistant impulses of unit strength & seperated by T sec, & let it be $\delta_T(t)$

$$\delta_T(t) = \delta(t) + \delta(t-T) + \delta(t+2T) \cdots$$
$$+ \delta(t+T) + \delta(t+2T) \cdots$$

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

This is a periodic s/g with period T & then we can find its F.S

The F.S of $\delta_T(t)$ is

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

where $F_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta_T(t) e^{-jn\omega_0 t} dt$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jn\omega_0 t} dt$$

$$F_n = \frac{1}{T}(1) = \frac{1}{T}$$

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{jn\omega_0 t}$$

$$\boxed{\delta_T(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega$$

$$E = \int_{-\infty}^{\infty} f(t)f(t)\,dt$$

$$= \int_{-\infty}^{\infty} f(t)\,\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega\,dt$$

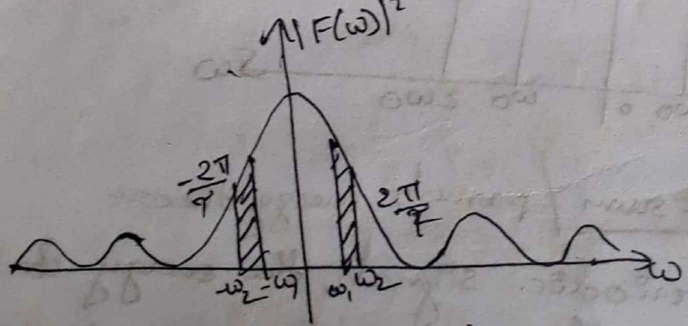
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,d\omega \int_{-\infty}^{\infty} f(t)e^{j\omega t}\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,F(-\omega)\,d\omega$$

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega$$

$$\int_{-\infty}^{\infty} f^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega$$

$\therefore$ The energy of a signal is given by $\frac{1}{2\pi}$ times the area under the curve $|F(\omega)|^2$



The energy contained in the freq components with in a band of fq (ω_1, ω_2) is $\frac{1}{2\pi}$ times the area of $|F(\omega)|^2$ under the band (ω_1, ω_2)

There is also a band of (-ve $(-\omega_1, \omega_2)$ fq which also has ~~already~~ exactly the same amount of energy as that in (ω_1, ω_2)

thus the energy contained in the fz band (ω_1, ω_2) is given by

$$\Delta E = 2 \cdot \frac{1}{2\pi} \int_{\omega_1}^{\omega_2} |F(\omega)|^2 d\omega$$

$$= \frac{1}{\pi} \int_{\omega_1}^{\omega_2} |F(\omega)|^2 d\omega$$

$\frac{1}{\pi} |F(\omega)|^2 \rightarrow$ represents the energy per unit band width

which represents the energy density denoted as $S(\omega)$

$$S(\omega) = \frac{1}{\pi} |F(\omega)|^2$$

$\therefore$ The energy ΔE associated with components of fz lieing in the interval (ω_1, ω_2) is

$$\Delta E = \int_{\omega_1}^{\omega_2} S(\omega) d\omega \text{ is}$$

$$E = \int_0^{\infty} S(\omega) d\omega$$

$\rightarrow$ Find FT of $e^{-2t} u(t-1)$

$$F[e^{-at} u(t)] = \frac{1}{a + j\omega}$$

$$F[e^{-2t} u(t)] = \frac{1}{2 + j\omega}$$

$$F[e^{-2(t-1)} u(t-1)] = \frac{1}{2 + j\omega} e^{-j\omega(1)}$$

$$e^2 F[e^{-2t} u(t-1)] = \frac{e^{-j\omega}}{2 + j\omega}$$

$$F[e^{-2t} u(t-1)] = \frac{e^{-(2 + j\omega)}}{2 + j\omega}$$

→ Find $te^{-3t}u(t)$

$$F[e^{-3t}u(t)] = \frac{1}{3+j\omega}$$

$$f(t) \longleftrightarrow F(\omega)$$

$$-jtf(t) \longleftrightarrow \frac{dF}{d\omega}$$

$$tf(t) \longleftrightarrow j\frac{dF}{d\omega}$$

$$F[te^{-3t}u(t)] = j\frac{d}{d\omega}\left(\frac{1}{3+j\omega}\right)$$

$$= \frac{-j}{(3+j\omega)^2}(j) = \frac{1}{(3+j\omega)^2}$$

→ $f(t) = e^{-0.5t}$

$$f(at) = \frac{1}{|a|}F\left(\frac{\omega}{a}\right)$$

$$F[e^{-0.5t}] \longleftrightarrow \frac{1}{|0.5|}F\left(\frac{\omega}{0.5}\right)$$

→ F.T $\cos\omega_0 t$

$$F[\cos\omega_0 t] = \lim_{T\to\infty}\int_{-T/2}^{T/2}\cos\omega_0 t\, e^{-j\omega t}\,dt$$

$$= \lim_{T\to\infty}\int_{-T/2}^{T/2}\left\{\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}\right\}e^{-j\omega t}\,dt$$

$$= \frac{1}{2}\lim_{T\to\infty}\int_{-T/2}^{T/2}\left\{e^{-j(\omega-\omega_0)t} + e^{-j(\omega+\omega_0)t}\right\}dt$$

$$= \frac{1}{2}\lim_{T\to\infty}\left[\frac{e^{-j(\omega-\omega_0)t}}{-j(\omega-\omega_0)} + \frac{e^{-j(\omega+\omega_0)t}}{-j(\omega+\omega_0)}\right]_{-T/2}^{T/2}$$

$$= \frac{1}{2}\lim_{T\to\infty}\left[\frac{e^{-j(\omega-\omega_0)T/2}}{-j(\omega-\omega_0)} + \frac{e^{-j(\omega+\omega_0)T/2}}{-j(\omega+\omega_0)} + \frac{e^{+j(\omega-\omega_0)T/2}}{j(\omega-\omega_0)} + \frac{e^{j(\omega+\omega_0)T/2}}{\ldots}\right]$$

$$\frac{2}{\tau} \lim_{T\to\infty} \left[\frac{\sin(\omega-\omega_0)\tau/2}{\omega-\omega_0} + \frac{\sin(\omega+\omega_0)\tau/2}{\omega+\omega_0} \right]$$

$$= \pi \lim_{T\to\infty} \left[\frac{2}{\tau\pi} Sa(\omega-\omega_0)\tau/2 + \frac{2}{2\pi} \{ Sa(\omega+\omega_0)\tau/2 \} \right]$$

$$= \pi \lim_{T\to\infty} \left[\frac{k}{\pi} Sa(\omega-\omega_0)\tau/2 + \frac{k}{\pi} Sa(\omega+\omega_0)\tau/2 \right]$$

$$= \pi \left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0) \right]$$

$\rightarrow f(t) = t e^{-at} u(t)$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} t e^{-at} u(t) e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} t e^{-(a+j\omega)t} dt$$

$$= t \int e^{-(a+j\omega)t} dt - \int (1) \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} dt$$

$$= \left[t \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} + \frac{e^{-(a-j\omega)t}}{-(a+j\omega)^2} \right]_0^{\infty}$$

$$F(\omega) = \frac{1}{(a+j\omega)^2}$$

F.T properties

1) Linearity property:

If $a x(t) + \cancel{b y(t)} \xleftrightarrow{F.T} a X(\omega) + b Y(\omega)$

$f(t) = a x(t) + b y(t)$

$F(\omega) = \int_{-\infty}^{\infty} [a x(t) + b y(t)] e^{-j\omega t} dt$

$= \left[a \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\right] + b \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt$

$= a X(\omega) + b Y(\omega)$

2) Time shift property:

$x(t \pm t_0) = e^{\pm j n \omega_0 t} \cancel{dt}\, x(j\omega)$

$f(t - t_0) = \int_{-\infty}^{\infty} f(t - t_0) e^{-j n \omega_0 t} dt.$

put $t - t_0 = p$

$t = p + t_0$

$f(P) = \int_{-\infty}^{\infty} f(p) e^{-j n \omega_0 (p + t_0)} dt$

$= \int_{-\infty}^{\infty} f(p) e^{-j n \omega_0 p} \cancel{e^{j n \omega_0 t}} dt$

$= F(\omega) e^{-j n \omega_0 t}$

Fq shifting: $F^{-1}[X(\omega - \omega_0)] = x(t) e^{\pm j\omega_0 t}$

$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$

$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega - \omega_0) e^{j\omega t} d\omega$

$\omega - \omega_0 = P$

$\frac{1}{2\pi} \int F(P) e^{j(P + \omega_0)} d\omega \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} F(P) e^{jPt} e^{j\omega_0 t}$

$f(t) e^{\pm j\omega t} d\omega$

Time reversal:

$$x(-t) \xleftrightarrow{F.T} x(-\omega)$$

$$F(f(-t)) = \int_{-\infty}^{\infty} f(-t)\, e^{+j\omega t}\, dt$$

$$= F(-\omega)$$

Time scaling:

$$F[x(at)] = \frac{1}{a}\, x\left(\frac{\omega}{a}\right)$$

$$x(at) = \int_{-\infty}^{\infty} f(at) \cdot e^{-j\omega t}\, dt$$

$at = p$
$t = p/a$

$$\int_{-\infty}^{\infty} f(p)\, e^{-j\omega(p/a)}\, dt$$

$$= \int_{-\infty}^{\infty} f(p) \cdot e^{-\frac{1}{a}[j\omega p]}$$

$$= \frac{1}{a}\, x\left(\frac{\omega}{a}\right)$$

Convolution:

$$f(t) = h(t) * x(t) \xleftrightarrow{F.T} F(\omega) = X(\omega) \cdot Y(\omega)$$

$$h(t) * x(t) = \int_{-\infty}^{\infty} h(t) * x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t} * x(t)\, e^{-j\omega t}\, dt$$

$$= H(\omega)\, X(\omega)$$

$\cos \longleftrightarrow$

→ Convolution:

$$f(t) = x(t) * y(t) \xleftrightarrow{FT} F(\omega) = X(\omega)\cdot Y(\omega)$$

A convolution operation is transformed to modulation in frequency domain.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} [x(t) * y(t)]\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau) y(t-\tau)\, d\tau\right] e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} y(t-\tau) e^{-j\omega t}\, dt\right] d\tau.$$

put $t - \tau = \alpha$, then $t = \tau + \alpha$
$dt = d\alpha$

$$F(\omega) = \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} y(\alpha)\, e^{-j\omega(\tau+\alpha)}\, d\alpha\right] d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} y(\alpha) e^{-j\omega\tau} \cdot e^{-j\omega\alpha}\, d\alpha\right] d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau \int_{-\infty}^{\infty} y(\alpha)\, e^{-j\omega\alpha}\, d\alpha$$

$$F(\omega) = X(\omega)\cdot Y(\omega)$$

→ Frequency differentiation:

If $x(t) \xleftrightarrow{F.T} X(\omega)$, then

$$-jt\, x(t) \xleftrightarrow{F.T} \frac{d}{d\omega} X(\omega)$$

Differentiating the fq spectrum is equivalent to multiplying the time domain signal by complex number $-jt$:

proof: $x(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

$$\frac{d}{d\omega} x(\omega) = \int_{-\infty}^{\infty} x(t) \frac{d}{d\omega}\left[e^{-j\omega t}\right] dt$$

$$= \int_{-\infty}^{\infty} x(t)(-jt) e^{-j\omega t} dt$$

$$= -jt \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\frac{d}{d\omega} x(\omega) = -jt \, x(\omega)$$

→ Time Differentiation:

If $x(t) \xleftrightarrow{F.T} x(\omega)$, then

$$\frac{d}{dt} x(t) \xleftrightarrow{F.T} j\omega \, x(\omega)$$

Differentiation in time domain corresponds to multiplying by $j\omega$ in fq domain. It accentuates high frequency components of the signal.

proof: $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\omega) e^{j\omega t} d\omega$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\omega) \left[\frac{d}{dt} e^{j\omega t}\right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\omega) j\omega e^{j\omega t} d\omega$$

$$\int_{-\infty}^{\infty} [j\omega \, x(\omega)] e^{j\omega t} d\omega$$

$$\frac{d\,x(t)}{dt} \longleftrightarrow j\omega X(\omega)$$

→ Parseval's Theorem or Rayleigh's theorem:

If $x(t) \xleftrightarrow{F.T} X(\omega)$, then

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Energy of the signal can be obtained by interchanging its energy spectrum.

Proof:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t) \cdot x^*(t)\, dt \quad \text{— (1)}$$

Inverse F.T states that

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Taking conjugate of both sides

$$x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t} d\omega$$

substitute $x^*(t)$ in eq (1)

$$E = \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t}\right] d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega) \int_{-\infty}^{\infty} x(t) e^{-j\omega t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega) \cdot X(\omega)\, d\omega$$

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

$\because \omega = 2\pi f,\ d\omega = 2\pi\, df$

$$E = \frac{1}{\cancel{2\pi}}\int_{-\infty}^{\infty} |X(\omega)|^2 \cancel{2\pi}\, df = \int_{-\infty}^{\infty} |X(f)|^2 df$$

→ Introduction to Hilbert transform:

Hilbert transform of a signal $x(t)$ is defined as the transform in which phase angle of all components of the signal shifted by $\pm 90°$.

Hilbert transform of $x(t)$ is represented with $\bar{x}(t)$, & it is given by

$$\bar{x}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(k)}{t-k} \, dk$$

The inverse Hilbert transform is given by

$$\hat{x}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\bar{x}(k)}{(t-k)} \, dk$$

$x(t), \bar{x}(t)$ is called Hilbert transform pair

Properties of Hilbert Transform

A signal $x(t)$ and its Hilbert transform $\bar{x}(t)$ are have

1) The same amplitude spectrum
2) The same autocorrelation function
3) The energy spectral density is same as $\bar{x}(t)$
4) $x(t)$ & $\bar{x}(t)$ are orthogonal
5) The Hilbert transform of $\bar{x}(t)$ is $-x(t)$
6) If fourier transform exist then Hilbert transform also exists for energy & power signals.

UNIT-3

Sampling Theorem

Sampling Theorem:

A continous time signal can be completely represented in its samples and recovered back if the sampling frequency is twice of the highest frequency content of the signal. i.e

$$f_s \geq 2W$$

where f_s is the sampling frequency
W is the higher fq content.

→ proof of sampling theorem are two parts:
1) Representation of $x(t)$ in terms of its samples
2) Reconstruction of $x(t)$ from its samples.

→ Reconstruction of signal from its samples

step1: Take Inverse Fourier transform of $x(f)$ which is in terms of $x_\delta(f)$.

2: show that $x(t)$ is obtained back with the help of interpolation function.

step1: Relation between $x(f)$ & $x_\delta(f)$

let us assume $f_s = 2W$, then as per below diagram

$x_\delta(f) = f_s x(f)$ for $-W \leq f \leq W$

$x(f) = \frac{1}{f} x_\delta(f)$ —— (1) $f_s = 2W$

DTFT $X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

$$x(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n} \quad \text{—— (2)}$$

In above equation 'f' is the fq of DT signal. If we replace $x(f)$ by $x_\delta(f)$, then f becomes frequency of CT signal. i.e.

$$x_\delta(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi \frac{f}{f_s} n}$$

In above equation 'f' is frequency of CT signal. And $\frac{f}{f_s} = f_q$ of DT signal.

$\therefore x(n) = x(nT_s)$

$$x_\delta(f) = \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f n T_s} \quad \text{—— (3)}$$

$\because \frac{1}{f_s} = T_s$

substitute above equation in eq (1)

$$X(f) = \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f n T_s}$$

Inverse Fourier transform of above equation gives $x(t)$ i.e.,

$$x(t) = IFT\left\{ \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f n T_s} \right\}$$

$$x(t) = \int_{-\infty}^{\infty} \left[\frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f n T_s} \right] e^{j2\pi f t} df$$

Here the integration can be taken from $-\omega \le f \le \omega$. since $x(f) = \frac{1}{f_s} x_\delta(f)$ for $-\omega \le f \le$

$$\therefore x(t) = \int_{-\omega}^{\omega} \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f n T_s} \cdot e^{j2\pi f t} df$$

Interchanging the order of summation & integration.

$$x(t) = \sum_{n=\infty}^{\infty} x(nT_s)\frac{1}{f_s}\int_{-\omega}^{\omega} e^{j2\pi f(t-nT_s)}\, df$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\cdot\frac{1}{f_s}\cdot\left[\frac{e^{j2\pi f(t-nT_s)}}{j2\pi(t-nT_s)}\right]_{-\omega}^{\omega}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\cdot\frac{1}{f_s}\left[\frac{e^{j2\pi\omega(t-nT_s)} - e^{-j2\pi\omega(t-nT_s)}}{j2\pi(t-nT_s)}\right]$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\cdot\frac{1}{f_s}\,\frac{\sin 2\pi W(t-nT_s)}{\pi(t-nT_s)}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\,\frac{\sin\pi(2Wt - 2W_nT_s)}{\pi(f_st - f_snT_s)}$$

Here $f_s = 2W$, hence $T_s = \frac{1}{f_s} = \frac{1}{2W}$

simplifying above equation,

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\frac{\sin\pi(2Wt-n)}{\pi(2Wt-n)}$$

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\mathrm{sinc}(2Wt-n)$$

$$\because \frac{\sin\pi\theta}{\pi\theta} = \mathrm{sinc}\,\theta$$

step 2: Let us interpret the above equation.

Expanding we get,

$$x(t) = \cdots + x(-2T_s)\,\mathrm{sinc}(2Wt+2) + x(-T_s)\,\mathrm{sinc}(2Wt+1) + x(0)\,\mathrm{sin}(2Wt) + x(T_s)\,\mathrm{sin}(2Wt-1) + \cdots$$

→ Graphical and analytical proof for Band limited signal:

1) A band limited signal of finite energy, which has no fq components higher than W Hertz, is completely described by specifying the values of the signal at instants of time seperated by $\frac{1}{2W}$ seconds and

2) A band limited signal of finite energy, which has no frequency components higher than W Hertz, may be completely recovered from the knowledge of its samples taken at the rate of 2W samples per second

The first part of above statement tells about sampling of the signal and 2nd part tells about reconstruction of the signal. Above statement can be combined & stated alternately as follows:

see the first page.

part I: Representation of $x(t)$ in its samples $x(nT_s)$

step 1: Define $x_\delta(t)$

2: Fourier transform of $x_\delta(t)$ i.e $X_\delta(f)$

3: Relation between $X(f)$ & $X_\delta(f)$

4: Relation between $x(t)$ & $x(nT_s)$

step 1: Define $x_\delta(t)$

step 2: Fourier transform of $x_\delta(t)$ i.e $X_\delta(f)$.

step 3 Relation between $X(f)$ & $X_\delta(f)$

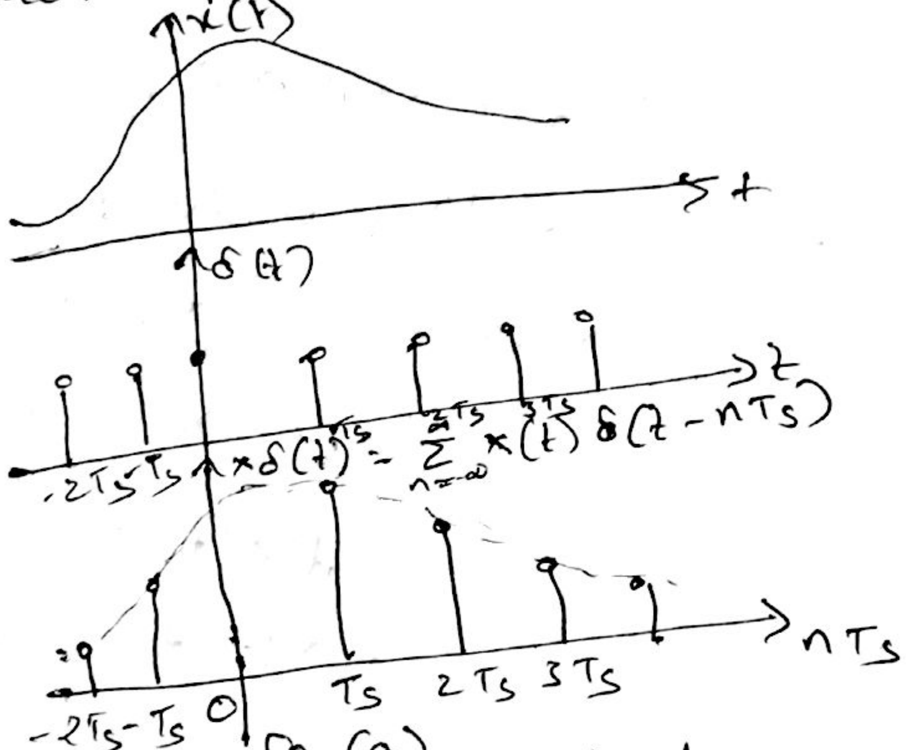
4 Relation between $x(nT_s)$ & $x(t)$

step 1) Define $x_\delta(t)$

The sampled signal $x_\delta(t)$ is given as

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(t)\,\delta(t - nT_s) \quad \text{--- (1)}$$

Here observe that $x_\delta(t)$ is the product of x_δ and impulse train $\delta(t)$ as shown in fig(a)



fig(a)

In equation (1), $\delta(t - nT_s)$ indicates the samples placed at $\pm T_s, \pm 2T_s, \pm 3T_s$ and so on.

step 2: F T of $x_\delta(t)$ i.e $X_\delta(f)$

Taking F.T of eq (1)

$$X_\delta(f) = F\,T\left\{ \sum_{n=-\infty}^{\infty} x(t)\,\delta(t - nT_s) \right\}$$

$$= F\cdot T\,\{\text{product of } x(t) \text{ \& impulse train}\}$$

We know that F.T of product of time domain becomes convolution in frequency domain i.e,

$$X_\delta(f) = F.T\{x(t) * F.T\{\delta(t-nT_s)\}\} \quad -②$$

By definitions, $x(t) \xleftrightarrow{F.T} X(f)$ &

$$\delta(t-nT_s) \xleftrightarrow{F.T} f_s \sum_{n=-\infty}^{\infty} \delta(f-nf_s)$$

∴ eq ② becomes

$$X_\delta(f) = X(f) * f_s \sum_{n=-\infty}^{\infty} \delta(f-nf_s)$$

∵ convolution is linear,

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f) * \delta(f-nf_s)$$

$$= f_s \sum_{n=-\infty}^{\infty} X(f-nf_s)$$

$$= \cdots f_s X(f-2f_s) + f_s X(f-f_s) + f_s X(f) + f_s X(f-f_s) + f_s X(f-2f_s)$$

→ Sampling Techniques:

Here, We have different types of sampling the signals.

Ideal sampling (or) Instantaneous sampling (or) Impulse sampling:

Ideal sampling is same as instantaneous sampling. In fig(1) shows the

Switching sampler :

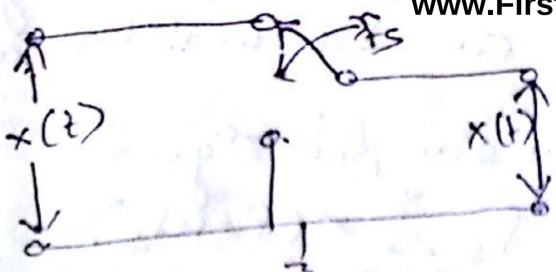


fig (a)

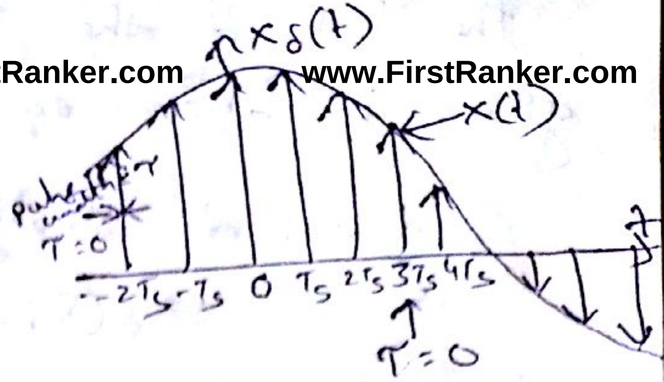


fig (b)

If closing time 't' of the switch approaches zero the output $x_\delta(t)$ gives only instantaneous value. The waveform shown in fig (b). Since the width of the pulse approaches zero, the instantaneous sampling gives train of impulses in $x_\delta(t)$. The area of each impulse in the sampled version is equal to instantaneous value of input signal $x(t)$.

We know that the train of impulses can be represented mathematically as,

$$S_\delta(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \quad \text{——— (1)}$$

This is called sampling function and its waveform is shown in fig (a). The sampled signal $x_\delta(t)$ is given by multiplication of $x(t)$ and $s_\delta(t)$.

$$\therefore x_\delta(t) = x(t)\, s_\delta(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\, \delta(t - nT_s) \quad \text{— (2)}$$

The F.T of the ideally sampled signal given by above eq. can be written as

spectrum of ideally sampled signal

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

→ Natural Sampling (or) chopper sampling

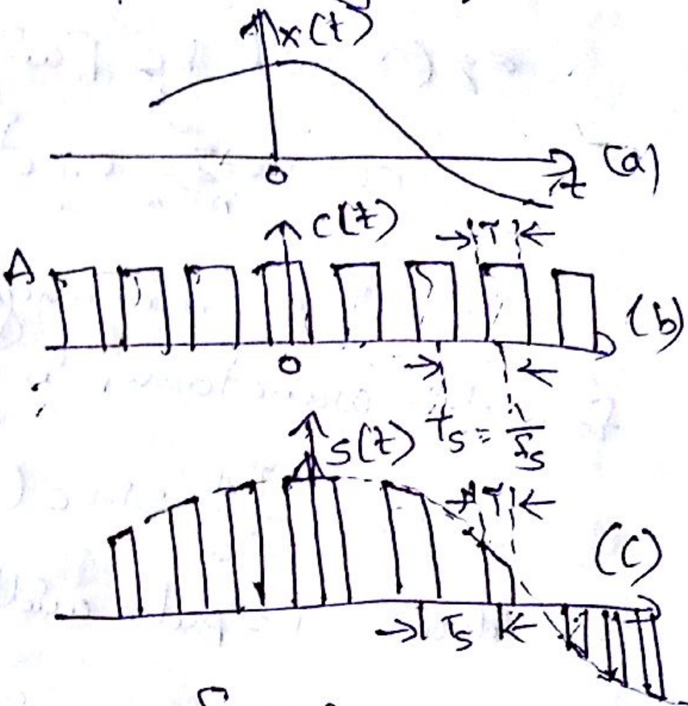
In instantaneous sampling, we have seen that the samples whose width τ approach zero. Because of this impracticable method the power in the instantaneously sampled pulse is negligible hence it is not suitable for transmission. Therefore the possible methods like natural sampling & flat top sampling are used.

In natural sampling, the pulse has a finite width τ. The waveform of the sampled signal appears to be chopped off from the original signal waveform.

Let us consider an analog continuous time signal $x(t)$ to be sampled at the rate of f_s Hz and f_s is the higher than Nyquist rate such that sampling theorem is satisfied. A sampled signal $s(t)$ is obtained by multiplication of the sampling function & signal $x(t)$.

...this method, c(t) is a train of periodic pulses of width τ and frequency equal to f_s Hz.

fig 1



fig(2)

Fig (1) shows a functional diagram of natural sampler. When c(t) goes high, a switch 's' is closed. Therefore,

$s(t) = x(t)$ when $c(t) = A$ (amplitude of c(t))

$s(t) = 0$ when $c(t) = 0$

Signal s(t) can also be defined mathematically as $s(t) = c(t)\, x(t)$ —— ①

Here c(t) is the periodic train of pulses of width τ & f_q f_s.

Exponential F.S for periodic waveform is given as

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi nt/T_0} \quad \text{—— ②}$$

For the periodic pulse train of c(t) we have $T_0 = T_s = \frac{1}{f_s}$ = period of c(t)

$f_0 = f_s = \frac{1}{T_0} = \frac{1}{T_s}$: Frequency of $c(t)$

eq ② will be [with $x(t) = c(t)$]

$$c(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi f_s n t}$$ putting $\frac{1}{T_0} = f_s$

$c(t)$ is a rectangular pulse train. c_n for this waveform is given as

$$\therefore c_n = \frac{TA}{T_0} \operatorname{sinc}(f_n T)$$

Here T = pulse width = τ

f_n = Harmonic fz

$f_n = n f_s$ or $f_n = \frac{n}{T_0} = n f_0$

$$\therefore c_n = \frac{\tau A}{T_s} \operatorname{sinc}(f_n \tau) \quad \text{——— ④}$$

substitute c_n value in eq ③

$$c(t) = \sum_{n=-\infty}^{\infty} \frac{\tau A}{T_s} \operatorname{sinc}(f_n \tau) e^{j2\pi f_s n t}$$

on putting the value of $c(t)$ in

$$s(t) = c(t)\, x(t)$$

$$s(t) = \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} \operatorname{sinc}(f_n \tau) e^{j2\pi f_s n t} \cdot x(t)$$

This equation represents naturally sampled signals.

F.T of $s(t)$.

$S(f) = F.T\{s(t)\}$

$$= \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} \operatorname{sinc}(f_n \tau)\, F.T\{e^{j2\pi f_n t} \cdot x(t)\}$$

for shifting property of F.T, that

$$e^{j2\pi f_s n t}\, x(t) \longleftrightarrow x(f - f_s n)$$

$$S(f) = \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} \operatorname{sinc}(f_n \tau)\, x(f - f_s n)$$

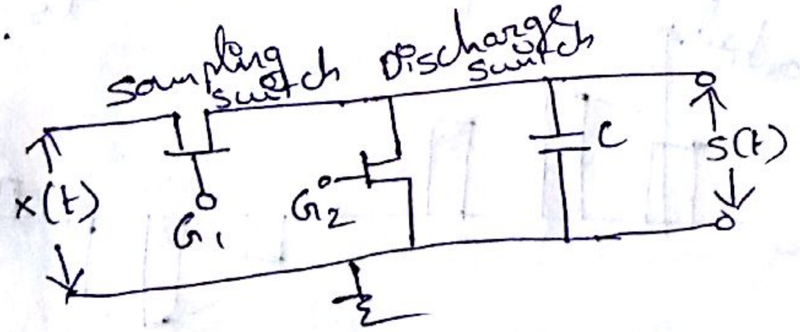
we know that $f_n = n f_s$

spectrum of naturally sampled signal

$$S(f) = \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} \operatorname{sinc}(n f_s \tau)\, x(f - n f_s)$$

→ Flat top sampling (or) Rectangular pulse sampling:

Natural sampling is little complex, but it is very easy to get flat top samples. The top of the samples remains constant and equal to instantaneous value of base band signal $x(t)$ at the start of the sampling. The duration of each sample is τ and sampling rate is equal to $f_s = \frac{1}{T_s}$.

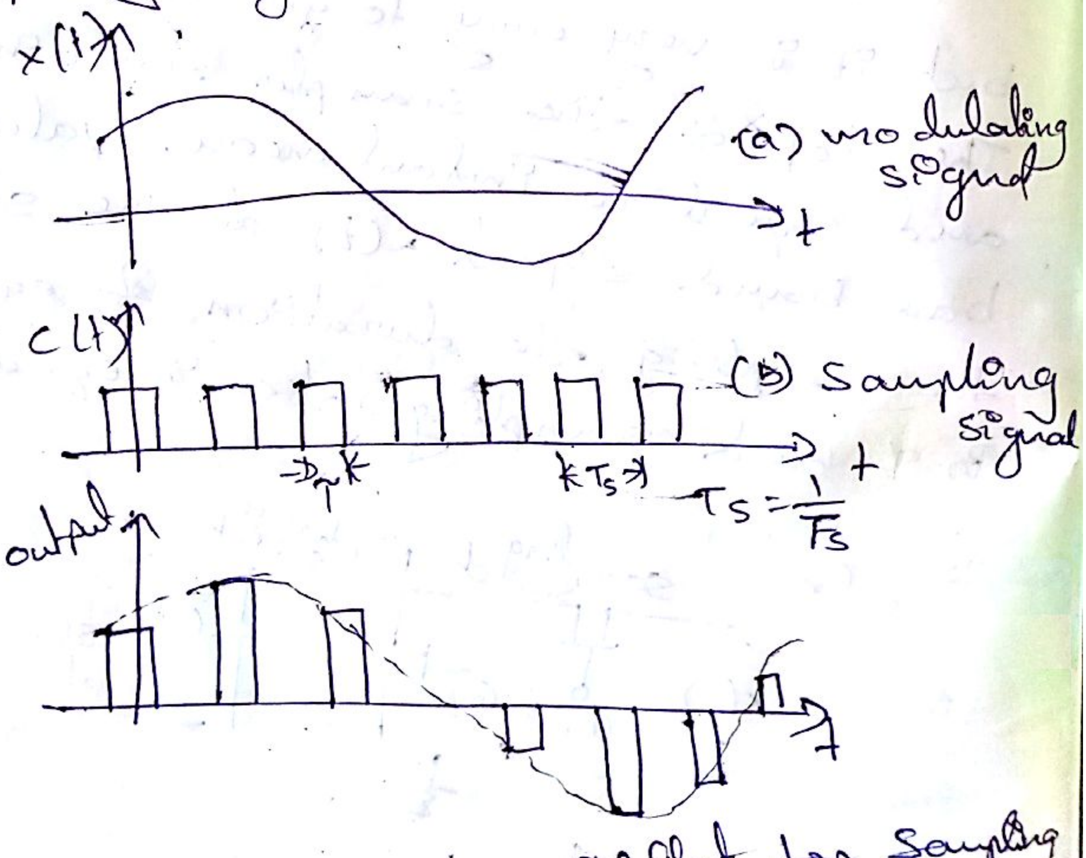


In fig (i), the sample and hold circuit is

used for generating flat top samples and waveforms shown in fig (2)

The switch S_1 closes at each sampling instant to sample the modulating signal. The capacitor 'C' holds the sampled voltage for period τ at the end of which switch S_2 is closed in order to discharge the capacitor.

Thus the signal generated as a result of sample & hold process is the flat top sampled signal. The spectrum of the generated flat top sampling signal along with the modulating signal and the sampling signal is shown below fig (2)



(c) flat top sampling signal

[Flat top sampling is mostly used in digital trans]

Flat top sampling $s(t)$ can be mathematically considered as convolution of the sampled signal and pulse signal $h(t)$.

$$\therefore s(t) = x_\delta(t) * h(t) \quad \text{--- (1)}$$

$$x(t) * \delta(t) = x(t) \quad \text{--- (2)}$$

Convolution of $x_\delta(t)$ & $h(t)$, we get a pulse whose duration is equal to $h(t)$ only but amplitude is defined by $x_\delta(t)$.

$x_\delta(t)$ is given as

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\ \delta(t - nT_s) \quad \text{--- (3)}$$

From eq (1)

$$s(t) = x_\delta(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x_\delta(u)\ h(t-u)\ du$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(nT_s)\ \delta(u - nT_s)\ h(t-u)\ du$$

from eq (3)

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \int_{-\infty}^{\infty} \delta(u - nT_s)\ h(t-u)\ du \quad \text{--- (4)}$$

From the shifting property of delta function we know that,

$$\int_{-\infty}^{\infty} f(t)\ \delta(t - t_0) = f(t_0) \quad \text{--- (5)}$$

Using this equation we can write eq ⑥

as

$$s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\, h(t - nT_s) \longrightarrow ⑥$$

$$s(t) = x_\delta(t) * h(t)$$

By taking F·T of both sides

$$S(f) = X_\delta(f)\, H(f) \longrightarrow ⑦$$

Convolution in time domain is converted to multiplication in fq domain.

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \longrightarrow ⑧$$

eq ⑦ becomes

spectrum of flat top sampled signal

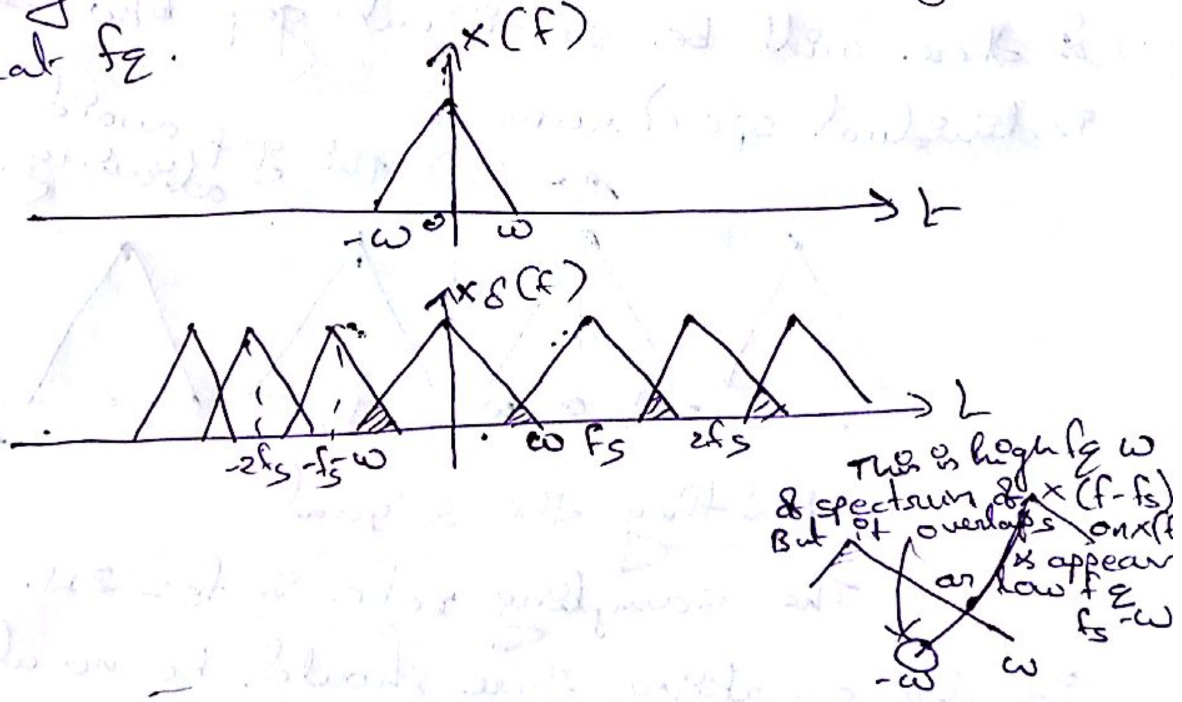
$$S(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)\, H(f)$$

→ Effects of undersampling (Aliasing):

When considering the reconstruction of a signal, you should already be familiar with the idea of nyquist rate. This concept allows us to find the sampling rate that will provide for perfect reconstruction of our signal. If we sample at too low of a rate (below the Nyquist rate), then problem will arise that will make perfect reconstruction impossible. This problem is known as aliasing.

Aliasing occurs when there is an overlap in the shifted, periodic copies of our original signal F.T. i.e, spectrum.

In fq domain, that part of the signal will overlap with the periodic signals next to it. In this overlap the values of the fq will be added together and the shape of the signals spectrum will be unwantingly altered. This overlapping, or aliasing, makes it impossible to correctly determine the correct strength of that fq.



Aliasing: When the high fq interferes with low fq & appears as low fq, then the phenomenon is called aliasing.

Effects of aliasing:

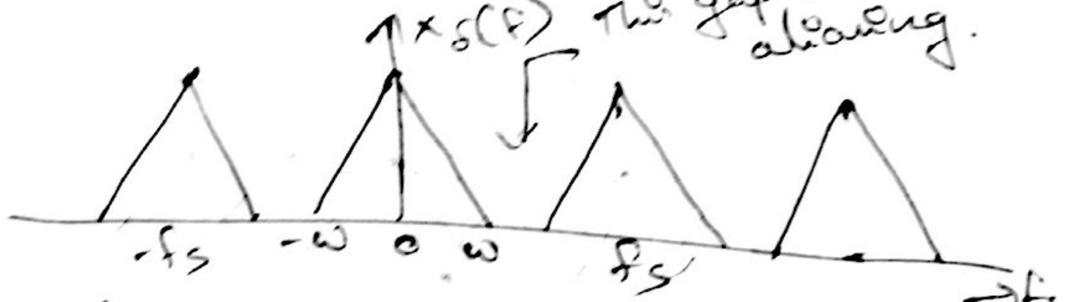
1) Since high & low fq interfere with each other, distortion is generated.
2) The data is lost and it cannot be recovered.

Different ways to avoid aliasing:

Aliasing can be avoided by two methods
1) sampling rate $f_s \geq 2W$
2) strictly band limit the signal to 'W'

1) Sampling rate $f_s \geq 2W$

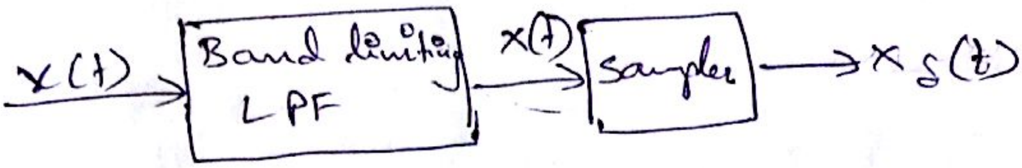
When the sampling rate is made higher than 2W, then the spectrum will not overlap & there will be sufficient gap b/w the individual spectrum.



2) Band limiting the signal:

The sampling rate is $f_s = 2W$. The ideally speaking there should be no aliasing. But there can be few components higher than 2W. These components create aliasing. Hence a LPF is used before sampling the signals. Thus the o/p of ~~Band limited~~ LPF ~~is strictly band limited & there are no fq~~

... higher than W. Then there is no aliasing.



→ Nyquist Rate & Nyquist Interval

Nyquist Rate:

When the sampling rate becomes exactly equal to 2W samples/sec, for a given bandwidth of W Hz, then it is called Nyquist rate.

Nyquist rate = 2W HZ.

Nyquist interval: It is the time interval between any two adjacent samples when sampling rate is Nyquist rate.

Nyquist ~~rate~~ Interval $= \frac{1}{2W}$ seconds.

UNIT – IV

SIGNAL TRANSMISSION THROUGH LINEAR SYSTEMS

UNIT – IV

SIGNAL TRANSMISSION THROUGH LINEAR SYSTEMS

Linear Systems:

A system is said to be linear when it satisfies superposition and homogenate principles. Consider two systems with inputs as $x_1(t)$, $x_2(t)$, and outputs as $y_1(t)$, $y_2(t)$ respectively. Then, according to the superposition and homogenate principles,

$$T\,[a_1\,x_1(t) + a_2\,x_2(t)] = a_1\,T[x_1(t)] + a_2\,T[x_2(t)]$$

$$\therefore\ T\,[a_1\,x_1(t) + a_2\,x_2(t)] = a_1\,y_1(t) + a_2\,y_2(t)$$

From the above expression, is clear that response of overall system is equal to response of individual system.

Example:

$y(t) = 2x(t)$

Solution:

$y_1\,(t) = T[x_1(t)] = 2x_1(t)$

$y_2\,(t) = T[x_2(t)] = 2x_2(t)$

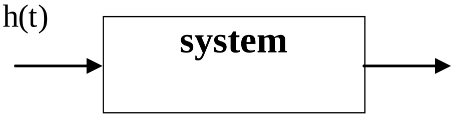
$T\,[a_1\,x_1(t) + a_2\,x_2(t)] = 2[\,a_1\,x_1(t) + a_2\,x_2(t)]$

Which is equal to $a_1y_1(t) + a_2\,y_2(t)$. Hence the system is said to be linear.

Impulse Response:

The impulse response of a system is its response to the input δ(t) when the system is initially at rest. The impulse response is usually denoted h(t). In other words, if the input to an initially at rest system is δ(t) then the output is named h(t).

δ(t) h(t)



Liner Time variant (LTV) and Liner Time Invariant (LTI) Systems

If a system is both liner and time variant, then it is called liner time variant (LTV) system.

If a system is both liner and time Invariant then that system is called liner time invariant (LTI) system.

Response of a continuous-time LTI system and the convolution integral

(i) Impulse Response:

The ***impulse response*** h(t) of a continuous-time LTI system (represented by **T)** is defined to be the response of the system when the input is δ(t), that is,

$$h(t)= T\{ \delta(t)\} \text{ ---------- (1)}$$

(ii) Response to an Arbitrary Input:

The input ***x(*** t) can be expressed as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \quad \text{--------(2)}$$

Since the system is linear, the response ***y(*** t of the system to an arbitrary input x(t) can be expressed as

$$y(t) = \mathbf{T}\{x(t)\} = \mathbf{T}\left\{\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right\}$$

$$= \int_{-\infty}^{\infty} x(\tau)\mathbf{T}\{\delta(t-\tau)\}\,d\tau \quad \text{--------(3)}$$

Since the system is time-invariant, we have

$$h(t-\tau) = \mathbf{T}\{\delta(t-\tau)\} \quad \text{--------(4)}$$

Substituting **Eq.** (4) into **Eq.** (3), we obtain

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \quad \text{-------(5)}$$

Equation *(5)* indicates that a continuous-time LTI system is completely characterized by its impulse response *h(t)*.

(iii) Convolution Integral:

Equation (5) defines the convolution of two continuous-time signals x (t) and h(t) denoted by

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \quad \text{-------(6)}$$

Equation (6) is commonly called the convolution integral. Thus, we have the fundamental result that the output of any continuous-time LTI system is the convolution of the input x (t) with the impulse response h(t) of the system. The following figure illustrates the definition of the impulse response h(t) and the relationship of Eq. (6).

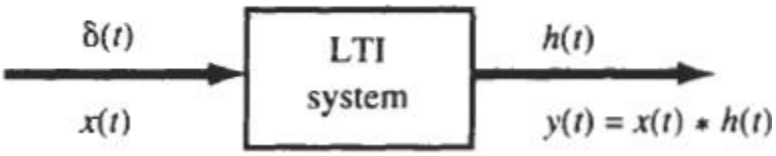


Fig. : Continuous-time LTl system.

(iv) Properties of the Convolution Integral:

The convolution integral has the following properties.

1. *Commutative:*

$$x(t) * h(t) = h(t) * x(t)$$

2. *Associative:*

$$\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\}$$

3. *Distributive:*

$$x(t) * \{h_1(t)\} + h_2(t)\} = x(t) * h_1(t) + x(t) * h_2(t)$$

(v) Step Response:

The step response s(t) of a continuous-time LTI system (represented by **T)** is defined to be the response of the system when the input is u(t); that is,

S(t)= T{u(t)}

In many applications, the step response s(t) is also a useful characterization of the system. The step response s(t) can be easily determined by,

$$s(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)\,d\tau = \int_{-\infty}^{t} h(\tau)\,d\tau$$

Thus, the step response s(t) can be obtained by integrating the impulse response h(t). Differentiating the above equation with respect to t, we get

$$h(t) = s'(t) = \frac{ds(t)}{dt}$$

Thus, the impulse response h(t) can be determined by differentiating the step response s(t).

Distortion less transmission through a system:

Transmission is said to be distortion-less if the input and output have identical wave shapes. i.e., in distortion-less transmission, the input x(t) and output y(t) satisfy the condition:

$$y(t) = Kx(t - t_d)$$

Where t_d = delay time and

k = constant.

Take Fourier transform on both sides

$$FT[y(t)] = FT[Kx(t - t_d)]$$

$$= K\, FT[x(t - t_d)]$$

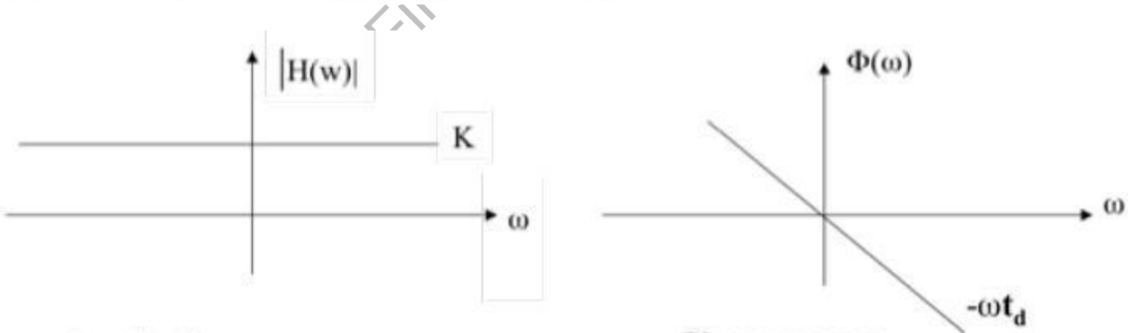
According to time shifting property,

$$Y(w) = KX(w)e^{-j\omega t_d}$$

Thus, distortion less transmission of a signal x(t) through a system with impulse response h(t) is achieved when

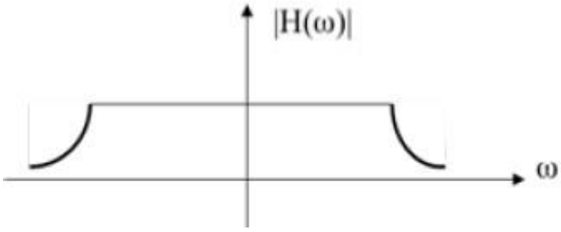
$|$**H(ω)|=K and** (amplitude response)

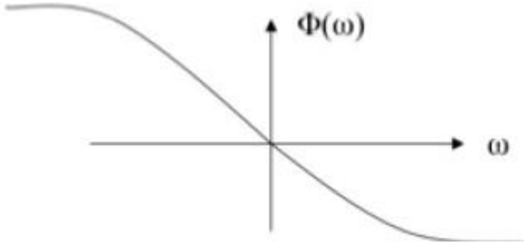
$$\Phi(\omega) = -\omega t_d = -2\pi f t_d \quad phase response$$



Amplitude response **Phase response**

A physical transmission system may have amplitude and phase responses as shown below:





FILTERING

One of the most basic operations in any signal processing system is filtering. Filtering is the process by which the relative amplitudes of the frequency components in a signal are changed or perhaps some frequency components are suppressed. As we saw in the preceding section, for continuous-time LTI systems, the spectrum of the output is that of the input multiplied by the frequency response of the system. Therefore, an LTI system acts as a filter on the input signal. Here the word "filter" is used to denote a system that exhibits some sort of frequency-selective behavior.

A. Ideal Frequency-Selective Filters:

An ideal frequency-selective filter is one that exactly passes signals at one set of frequencies and completely rejects the rest. The band of frequencies passed by the filter is referred to as the pass band, and the band of frequencies rejected by the filter is called the stop band.

The most common types of ideal frequency-selective filters are the following.

1. Ideal Low-Pass Filter:

An ideal low-pass filter (LPF) is specified by

$$|H(\omega)| = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

The frequency w_c is called the cutoff frequency.

2. Ideal High-Pass Filter:

An ideal high-pass filter (HPF) is specified by

$$|H(\omega)| = \begin{cases} 0 & |\omega| < \omega_c \\ 1 & |\omega| > \omega_c \end{cases}$$

3. Ideal Bandpass Filter:

An ideal bandpass filter (BPF) is specified by

$$|H(\omega)| = \begin{cases} 1 & \omega_1 < |\omega| < \omega_2 \\ 0 & \text{otherwise} \end{cases}$$

4. Ideal Bandstop Filter:

An ideal bandstop filter (BSF) is specified by

$$|H(\omega)| = \begin{cases} 0 & \omega_1 < |\omega| < \omega_2 \\ 1 & \text{otherwise} \end{cases}$$

The following figures shows the magnitude responses of ideal filters

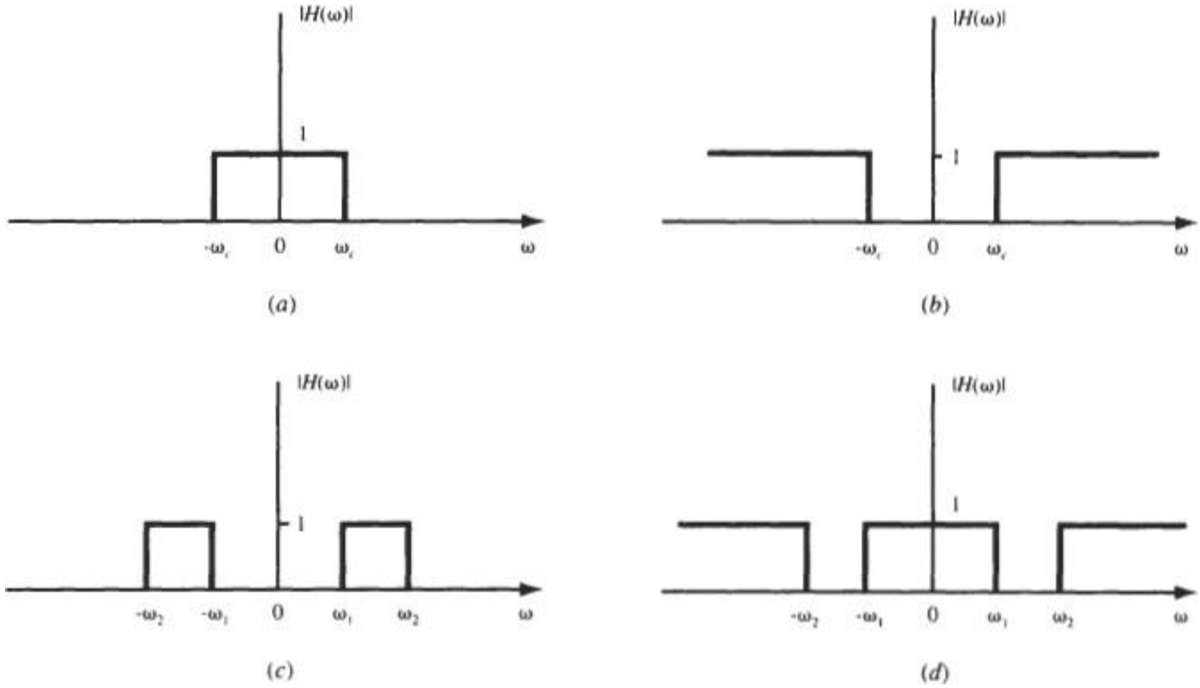


Fig: Magnitude responses of ideal filters (a) Ideal Low-Pass Filter (b)Ideal High-Pass Filter

© Ideal Bandpass Filter (d) Ideal Bandstop Filter

UNIT – V

LAPLACE TRANSFORMS

UNIT – V

LAPLACE TRANSFORMS

THE LAPLACE TRANSFORM:

we know that for a continuous-time LTI system with impulse response h(t), the output **y(t)** of the system to the complex exponential input of the form e^{st} is

$$y(t) = T\{e^{st}\} = H(s)e^{st}$$

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt$$

A. Definition:

The function H(s) is referred to as the Laplace transform of h(t). For a general continuous-time signal x(t), the Laplace transform X(s) is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

The variable s is generally complex-valued and is expressed as

$$s = \sigma + j\omega$$

Relation between Laplace and Fourier transforms:

Laplace transform of x(t)

$$X(S) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

Substitute s= σ + jω in above equation.

$$\rightarrow X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}\,dt$$

$$= \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}\,dt$$

$$\therefore X(S) = F.T[x(t)e^{-\sigma t}]$$

$$X(S) = X(\omega) \qquad for\ s = j\omega$$

Inverse Laplace Transform:

We know that

$$X(S) = F.T[x(t)e^{-\sigma t}]$$

$$\rightarrow x(t)e^{-\sigma t} = F.T^{-1}[X(S)] = F.T^{-1}[X(\sigma + j\omega)]$$

$$= \frac{1}{2}\pi \int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t} d\omega$$

$$x(t) = e^{\sigma t} \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t} d\omega$$

Here, $\sigma + j\omega = s$

$$jd\omega = ds \rightarrow d\omega = ds/j$$

$$\therefore x(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} X(s)e^{st} ds \dots ..$$

Conditions for Existence of Laplace Transform:

Dirichlet's conditions are used to define the existence of Laplace transform. i.e.

- The function f has finite number of maxima and minima.
- There must be finite number of discontinuities in the signal f ,in the given interval of time.
- It must be absolutely integrable in the given interval of time. i.e.

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$$

Initial and Final Value Theorems

If the Laplace transform of an unknown function x(t) is known, then it is possible to determine the initial and the final values of that unknown signal i.e. x(t) at $t=0^+$ and $t=\infty$.

Initial Value Theorem

Statement: If x(t) and its 1st derivative is Laplace transformable, then the initial value of x(t) is given by

$$x(0^+) = \lim_{s\to\infty} SX(S)$$

Final Value Theorem

Statement: If x(t) and its 1st derivative is Laplace transformable, then the final value of x(t) is given by

$$x(\infty) = \lim_{s\to\infty} SX(S)$$

Properties of Laplace transform:

The properties of Laplace transform are:

Linearity Property

If $x(t) \overset{\text{L.T}}{\longleftrightarrow} X(s)$

& $y(t) \overset{\text{L.T}}{\longleftrightarrow} Y(s)$

Then linearity property states that

$$ax(t) + by(t) \overset{\text{L.T}}{\longleftrightarrow} aX(s) + bY(s)$$

Time Shifting Property

If $x(t) \overset{\text{L.T}}{\longleftrightarrow} X(s)$

Then time shifting property states that

$$x(t - t_0) \overset{\text{L.T}}{\longleftrightarrow} e^{-st_0} X(s)$$

Frequency Shifting Property

If $x(t) \overset{\text{L.T}}{\longleftrightarrow} X(s)$

Then frequency shifting property states that

$$e^{s_0 t} . x(t) \overset{\text{L.T}}{\longleftrightarrow} X(s - s_0)$$

Time Reversal Property

If $x(t) \overset{\text{L.T}}{\longleftrightarrow} X(s)$

Then time reversal property states that

$$x(-t) \overset{\text{L.T}}{\longleftrightarrow} X(-s)$$

Time Scaling Property

If $x(t) \overset{\text{L.T}}{\longleftrightarrow} X(s)$

Then time scaling property states that

$$x(at) \overset{\text{L.T}}{\longleftrightarrow} \frac{1}{|a|} X(\frac{s}{a})$$

Differentiation and Integration Properties

If $x(t) \overset{\text{L.T}}{\longleftrightarrow} X(s)$

Then differentiation property states that

$$\frac{dx(t)}{dt} \overset{\text{L.T}}{\longleftrightarrow} s . X(s)$$

$$\frac{d^n x(t)}{dt^n} \overset{\text{L.T}}{\longleftrightarrow} (s)^n . X(s)$$

The integration property states that

$$\int x(t)dt \stackrel{\mathrm{L.T}}{\longleftrightarrow} \frac{1}{s}X(s)$$

$$\iiint \ldots \int x(t)dt \stackrel{\mathrm{L.T}}{\longleftrightarrow} \frac{1}{s^n}X(s)$$

Multiplication and Convolution Properties

If $x(t) \stackrel{\mathrm{L.T}}{\longleftrightarrow} X(s)$

and $y(t) \stackrel{\mathrm{L.T}}{\longleftrightarrow} Y(s)$

Then multiplication property states that

$$x(t).\,y(t) \stackrel{\mathrm{L.T}}{\longleftrightarrow} \frac{1}{2\pi j}X(s) * Y(s)$$

The convolution property states that

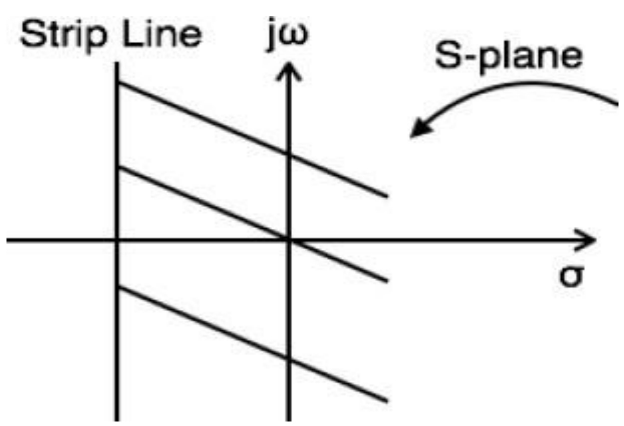
$$x(t) * y(t) \stackrel{\mathrm{L.T}}{\longleftrightarrow} X(s).\,Y(s)$$

<u>**Region of convergence**</u>.

The range variation of ζ for which the Laplace transform converges is called region of convergence.

Properties of ROC of Laplace Transform

- ROC contains strip lines parallel to jω axis in s-plane.



- If x(t) is absolutely integral and it is of finite duration, then ROC is entire s-plane.
- If x(t) is a right sided sequence then ROC : Re{s} > ζ_o.
- If x(t) is a left sided sequence then ROC : Re{s} < ζ_o.
- If x(t) is a two sided sequence then ROC is the combination of two regions.

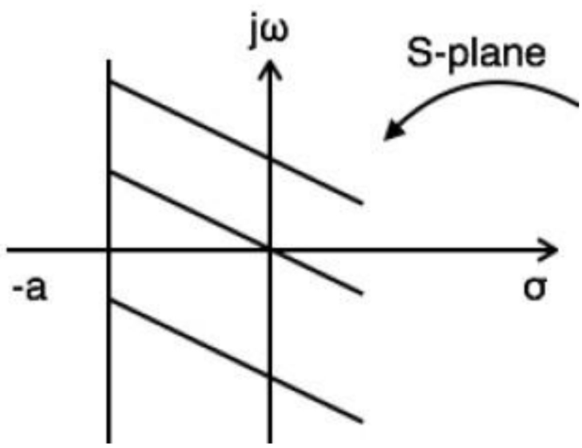
ROC can be explained by making use of examples given below:

Example 1: Find the Laplace transform and ROC of x(t)=e− at u(t) x(t)=e^{-at}u(t)

$$L.T[x(t)] = L.T[e -^{at} u(t)] = \frac{1}{S+a}$$

$$Re > -a$$

$$ROC : Res >> -a$$

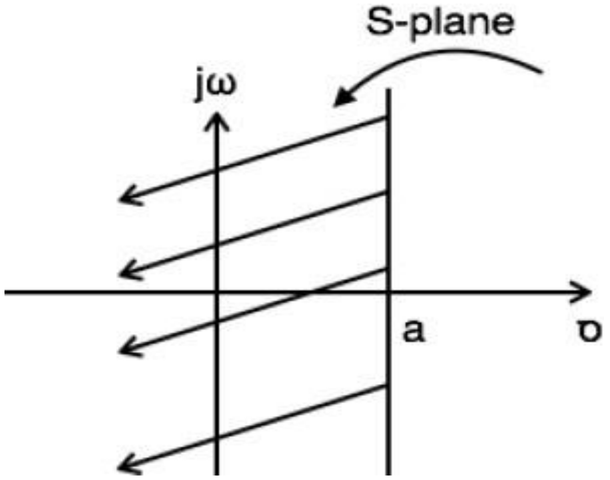


Example 2: Find the Laplace transform and ROC of x(t)=e at u(−t) x(t)=e^{at}u(−t)

$$L.T[x(t)] = L.T[e^{at} u(t)] = \frac{1}{S-a}$$

$$Res < a$$

$$ROC : Res < a$$

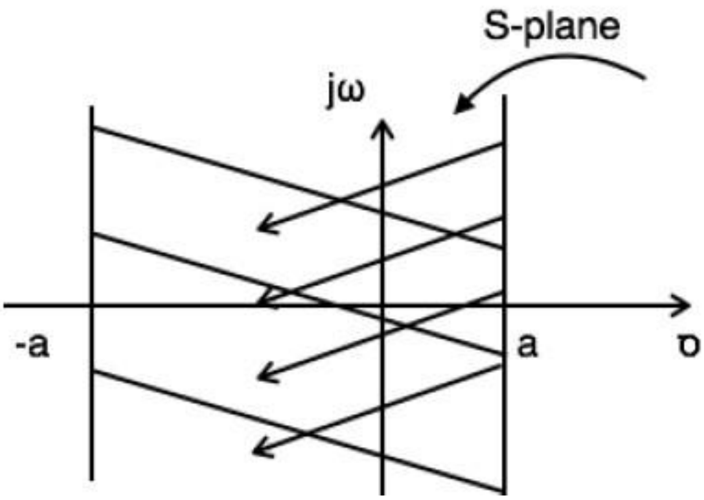


Example 3: Find the Laplace transform and ROC of x(t)=e −at u(t)+e at u(−t)
$x(t)=e^{-at}u(t)+e^{at}u(-t)$

$$L.T[x(t)] = L.T[e^{-at}u(t) + e^{at}u(-t)] = \frac{1}{S+a} + \frac{1}{S-a}$$

For $\frac{1}{S+a} Re\{s\} > -a$

For $\frac{1}{S-a} Re\{s\} < a$

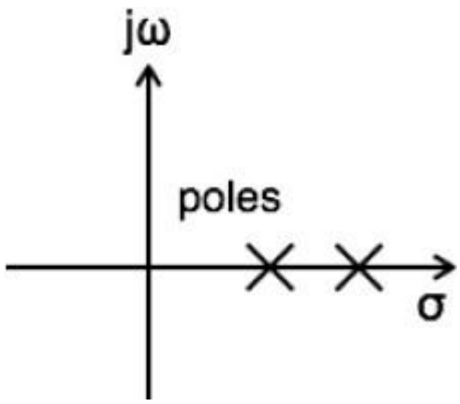


Referring to the above diagram, combination region lies from –a to a. Hence,

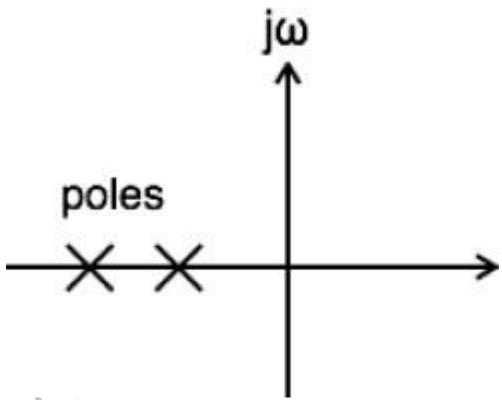
ROC: −a<Res<a

Causality and Stability

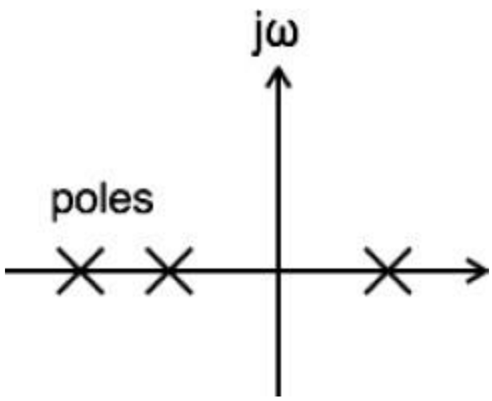
- For a system to be causal, all poles of its transfer function must be right half of s-plane.



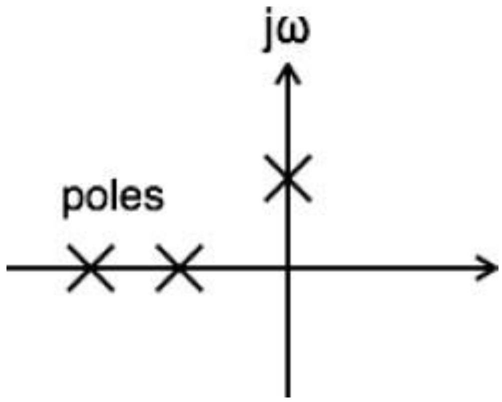
- A system is said to be stable when all poles of its transfer function lay on the left half of s-plane.



- A system is said to be unstable when at least one pole of its transfer function is shifted to the right half of s-plane.



- A system is said to be marginally stable when at least one pole of its transfer function lies on the jω axis of s-plane



LAPLACE TRANSFORMS OF SOME COMMON SIGNALS

A. Unit Impulse Function δ(*t*):

$$\mathscr{L}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-st}\,dt = 1 \qquad \text{all } s$$

B. Unit Step Function *u(t*):

$$\mathscr{L}[u(t)] = \int_{-\infty}^{\infty} u(t)e^{-st}\,dt = \int_{0^+}^{\infty} e^{-st}\,dt$$

$$= -\frac{1}{s}e^{-st}\Big|_{0^+}^{\infty} = \frac{1}{s} \qquad \mathrm{Re}(s) > 0$$

$$\text{where } 0^+ = \lim_{\varepsilon \to 0}(0 + \varepsilon).$$

Some Laplace Transforms Pairs:

$x(t)$	$X(s)$	ROC
$\delta(t)$	1	All s
$u(t)$	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$
$-u(-t)$	$\frac{1}{s}$	$\mathrm{Re}(s) < 0$
$tu(t)$	$\frac{1}{s^2}$	$\mathrm{Re}(s) > 0$
$t^k u(t)$	$\frac{k!}{s^{k+1}}$	$\mathrm{Re}(s) > 0$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) < -\mathrm{Re}(a)$
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$
$-te^{-at}u(-t)$	$\frac{1}{(s+a)^2}$	$\mathrm{Re}(s) < -\mathrm{Re}(a)$
$\cos \omega_0 t u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathrm{Re}(s) > 0$
$\sin \omega_0 t u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{Re}(s) > 0$
$e^{-at}\cos \omega_0 t u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$
$e^{-at}\sin \omega_0 t u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$

UNIT-6

Z-Transform

Z-Transform

Analysis of continuous time LTI systems can be done using z-transforms. It is a powerful mathematical tool to convert differential equations into algebraic equations.

The bilateral (two sided) z-transform of a discrete time signal x(n) is given as

$$Z.T[x(n)] = X(Z) = \Sigma_{n=-\infty}^{\infty} x(n) z^{-n}$$

The unilateral (one sided) z-transform of a discrete time signal x(n) is given as

$$Z.T[x(n)] = X(Z) = \Sigma_{n=0}^{\infty} x(n) z^{-n}$$

Z-transform may exist for some signals for which Discrete Time Fourier Transform (DTFT) does not exist.

Concept of Z-Transform and Inverse Z-Transform

Z-transform of a discrete time signal x(n) can be represented with X(Z), and it is defined as

$$X(Z) = \Sigma_{n=-\infty}^{\infty} x(n) z^{-n} \ldots\ldots (1)$$

If $Z = re^{j\omega}$ then equation 1 becomes

$$X(re^{j\omega}) = \Sigma_{n=-\infty}^{\infty} x(n) [re^{j\omega}]^{-n}$$

$$= \Sigma_{n=-\infty}^{\infty} x(n) [r^{-n}] e^{-j\omega n}$$

$$X(re^{j\omega}) = X(Z) = F.T[x(n) r^{-n}] \ldots\ldots (2)$$

The above equation represents the relation between Fourier transform and Z-transform

$$X(Z)|_{z=e^{j\omega}} = F.T[x(n)].$$

Inverse Z-transform:

$$X(re^{j\omega}) = F.T[x(n)r^{-n}]$$
$$x(n)r^{-n} = F.T^{-1}[X(re^{j\omega}]$$
$$x(n) = r^n F.T^{-1}[X(re^{j\omega})]$$
$$= r^n \frac{1}{2\pi} \int X(re^j\omega)e^{j\omega n} d\omega$$
$$= \frac{1}{2\pi} \int X(re^j\omega)[re^{j\omega}]^n d\omega \ldots\ldots (3)$$

Substitute $re^{j\omega} = z$.

$$dz = jre^{j\omega} d\omega = jzd\omega$$
$$d\omega = \frac{1}{j} z^{-1} dz$$

Substitute in equation 3.

$$3 \rightarrow x(n) = \frac{1}{2\pi} \int X(z) z^n \frac{1}{j} z^{-1} dz = \frac{1}{2\pi j} \int X(z) z^{n-1} dz$$

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$x(n) = \frac{1}{2\pi j} \int X(z) z^{n-1} dz$$

Z-Transform Properties:

Z-Transform has following properties:

Linearity Property:

If $x(n) \xleftrightarrow{\text{Z.T}} X(Z)$

and $y(n) \xleftrightarrow{\text{Z.T}} Y(Z)$

Then linearity property states that

$$a\,x(n) + b\,y(n) \xleftrightarrow{\text{Z.T}} a\,X(Z) + b\,Y(Z)$$

Time Shifting Property:

If $x(n) \xleftrightarrow{\text{Z.T}} X(Z)$

Then Time shifting property states that

$$x(n-m) \xleftrightarrow{\text{Z.T}} z^{-m}X(Z)$$

Multiplication by Exponential Sequence Property

If $x(n) \xleftrightarrow{\text{Z.T}} X(Z)$

Then multiplication by an exponential sequence property states that

$$a^n\,.\,x(n) \xleftrightarrow{\text{Z.T}} X(Z/a)$$

Time Reversal Property

If $x(n) \xleftrightarrow{\text{Z.T}} X(Z)$

Then time reversal property states that

$$x(-n) \xleftrightarrow{\text{Z.T}} X(1/Z)$$

Differentiation in Z-Domain OR Multiplication by n Property

If $x(n) \stackrel{\text{Z.T}}{\longleftrightarrow} X(Z)$

Then multiplication by n or differentiation in z-domain property states that

$n^k x(n) \stackrel{\text{Z.T}}{\longleftrightarrow} [-1]^k z^k \frac{d^k X(Z)}{dZ^K}$

Convolution Property

If $x(n) \stackrel{\text{Z.T}}{\longleftrightarrow} X(Z)$

and $y(n) \stackrel{\text{Z.T}}{\longleftrightarrow} Y(Z)$

Then convolution property states that

$x(n) * y(n) \stackrel{\text{Z.T}}{\longleftrightarrow} X(Z).Y(Z)$

Correlation Property

If $x(n) \stackrel{\text{Z.T}}{\longleftrightarrow} X(Z)$

and $y(n) \stackrel{\text{Z.T}}{\longleftrightarrow} Y(Z)$

Then correlation property states that

$x(n) \otimes y(n) \stackrel{\text{Z.T}}{\longleftrightarrow} X(Z).Y(Z^{-1})$

Initial Value and Final Value Theorems

Initial value and final value theorems of z-transform are defined for causal signal.

Initial Value Theorem

For a causal signal x(n), the initial value theorem states that

$$x(0) = \lim_{z\to\infty} X(z)$$

This is used to find the initial value of the signal without taking inverse z-transform

Final Value Theorem

For a causal signal x(n), the final value theorem states that

$$x(\infty) = \lim_{z\to 1}[z-1]X(z)$$

This is used to find the final value of the signal without taking inverse z-transform

Region of Convergence (ROC) of Z-Transform

The range of variation of z for which z-transform converges is called region of convergence of z-transform.

Properties of ROC of Z-Transforms

- ROC of z-transform is indicated with circle in z-plane.
- ROC does not contain any poles.
- If x(n) is a finite duration causal sequence or right sided sequence, then the ROC is entire z-plane except at z = 0.
- If x(n) is a finite duration anti-causal sequence or left sided sequence, then the ROC is entire z-plane except at z = ∞.
- If x(n) is a infinite duration causal sequence, ROC is exterior of the circle with radius a. i.e. |z| > a.
- If x(n) is a infinite duration anti-causal sequence, ROC is interior of the circle with radius a. i.e. |z| < a.
- If x(n) is a finite duration two sided sequence, then the ROC is entire z-plane except at z = 0 & z = ∞.

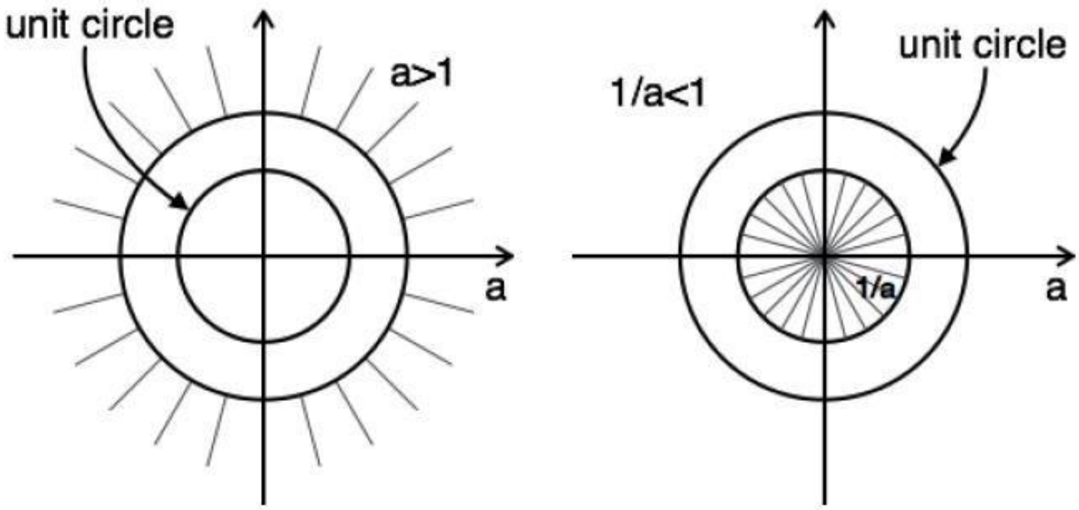
The concept of ROC can be explained by the following example:

Example 1: Find z-transform and ROC of $a^{n}u[n]+a^{-n}u[-n-1]$ $a^nu[n]+a^{-n}u[-n-1]$

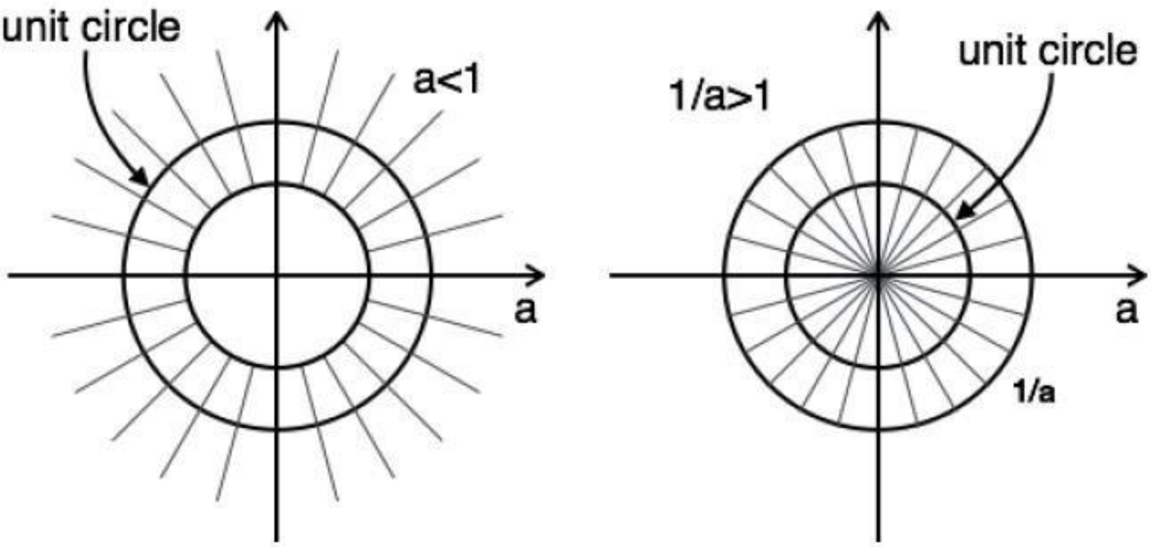
$$Z.T[a^n u[n]] + Z.T[a^{-n}u[-n-1]] = \frac{Z}{Z-a} + \frac{Z}{Z\frac{-1}{a}}$$

$$ROC: |z| > a \qquad ROC: |z| < \frac{1}{a}$$

The plot of ROC has two conditions as a > 1 and a < 1, as we do not know a.



In this case, there is no combination ROC.



Here, the combination of ROC is from $a<|z|<1/a$

Hence for this problem, z-transform is possible when $a < 1$.

Causality and Stability

Causality condition for discrete time LTI systems is as follows:

A discrete time LTI system is causal when

- ROC is outside the outermost pole.
- In The transfer function H[Z], the order of numerator cannot be grater than the order of denominator.

Stability Condition for Discrete Time LTI Systems

A discrete time LTI system is stable when

- its system function H[Z] include unit circle |z|=1.
- all poles of the transfer function lay inside the unit circle |z|=1.

Z-Transform of Basic Signals

$x[n]$	$X(z)$	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}, \frac{z}{z-1}$	$\|z\|>1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}, \frac{z}{z-1}$	$\|z\|<1$
$\delta[n-m]$	z^{-m}	All z except 0 if $(m>0)$ or ∞ if $(m<0)$
$a^n u[n]$	$\frac{1}{1-az^{-1}}, \frac{z}{z-a}$	$\|z\|>\|a\|$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}, \frac{z}{z-a}$	$\|z<\|a\|$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}, \frac{az}{(z-a)^2}$	$\|z\|>\|a\|$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}, \frac{az}{(z-a)^2}$	$\|z\|<\|a\|$
$(n+1)a^n u[n]$	$\frac{1}{(1-az^{-1})^2}, \left[\frac{z}{z-a}\right]^2$	$\|z\|>\|a\|$
$(\cos\Omega_0 n)u[n]$	$\frac{z^2-(\cos\Omega_0)z}{z^2-(2\cos\Omega_0)z+1}$	$\|z\|>1$
$(\sin\Omega_0 n)u[n]$	$\frac{(\sin\Omega_0)z}{z^2-(2\cos\Omega_0)z+1}$	$\|z\|>1$
$(r^n\cos\Omega_0 n)u[n]$	$\frac{z^2-(r\cos\Omega_0)z}{z^2-(2r\cos\Omega_0)z+r^2}$	$\|z\|>r$
$(r^n\sin\Omega_0 n)u[n]$	$\frac{(r\sin\Omega_0)z}{z^2-(2r\cos\Omega_0)z+r^2}$	$\|z\|>r$
$\begin{cases} a^n & 0\le n\le N-1 \\ 0 & \text{otherwise}\end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\|>0$

Some Properties of the Z- Transform:

Property	Sequence	Transform	ROC
	$x[n]$	$X(z)$	R
	$x_1[n]$	$X_1(z)$	R_1
	$x_2[n]$	$X_2(z)$	R_2
Linearity	$a_1x_1[n]+a_2x_2[n]$	$a_1X_1(z)+a_2X_2(z)$	$R' \supset R_1 \cap R_2$
Time shifting	$x[n-n_0]$	$z^{-n_0}X(z)$	$R' \supset R \cap \{0<\|z\|<\infty\}$
Multiplication by z_0^n	$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$R' = \|z_0\|R$
Multiplication by $e^{j\Omega_0 n}$	$e^{j\Omega_0 n}x[n]$	$X(e^{-j\Omega_0}z)$	$R' = R$
Time reversal	$x[-n]$	$X\left(\frac{1}{z}\right)$	$R' = \frac{1}{R}$
Multiplication by n	$nx[n]$	$-z\frac{dX(z)}{dz}$	$R' = R$
Accumulation	$\sum_{k=-\infty}^{n} x[n]$	$\frac{1}{1-z^{-1}}X(z)$	$R' \supset R \cap \{\|z\|>1\}$
Convolution	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	$R' \supset R_1 \cap R_2$

Inverse Z transform:

Three different methods are:

1. Partial fraction method
2. Power series method
3. Long division method

Partial fraction method:

- In case of LTI systems, commonly encountered form of z-transform is

$$X(z) = \frac{B(z)}{A(z)}$$

$$X(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}}$$

Usually $M < N$

- If $M > N$ then use long division method and express $X(z)$ in the form

$$X(z) = \sum_{k=0}^{M-N} f_k z^{-k} + \frac{\tilde{B}(z)}{A(z)}$$

where $\bar{B}(z)$ now has the order one less than the denominator polynomial and use partial fraction method to find z-transform

- The inverse z-transform of the terms in the summation are obtained from the transform pair and time shift property

$$1 \xleftrightarrow{z} \delta[n]$$

$$z^{-n_o} \xleftrightarrow{z} \delta[n-n_o]$$

- If $X(z)$ is expressed as ratio of polynomials in z instead of z^{-1} then convert into the polynomial of z^{-1}
- Convert the denominator into product of first-order terms

$$X(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 \prod_{k=1}^{N}(1 - d_k z^{-1})}$$

where d_k are the poles of $X(z)$

For distinct poles

- For all distinct poles, the $X(z)$ can be written as

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{(1 - d_k z^{-1})}$$

- Depending on ROC, the inverse z-transform associated with each term is then determined by using the appropriate transform pair
- We get

$$A_k(d_k)^n u[n] \xleftrightarrow{z} \frac{A_k}{1 - d_k z^{-1}},$$

$$\text{with ROC} \quad z > d_k \qquad \text{OR}$$
$$-A_k(d_k)^n u[-n-1] \xleftrightarrow{z} \frac{A_k}{1 - d_k z^{-1}},$$
$$\text{with ROC} \quad z < d_k$$

- For each term the relationship between the ROC associated with $X(z)$ and each pole determines whether the right-sided or left sided inverse transform is selected

For Repeated poles

- If pole d_i is repeated r times, then there are r terms in the partial-fraction expansion associated with that pole

$$\frac{A_{i_1}}{1-d_i z^{-1}}, \frac{A_{i_2}}{(1-d_i z^{-1})^2}, \ldots, \frac{A_{i_r}}{(1-d_i z^{-1})^r}$$

- Here also, the ROC of $X(z)$ determines whether the right or left sided inverse transform is chosen.

$$A\frac{(n+1)\ldots(n+m-1)}{(m-1)!}(d_i)^n u[n] \overset{z}{\longleftrightarrow} \frac{A}{(1-d_i z^{-1})^m}, \qquad \text{with ROC}|z| > d_i$$

- If the ROC is of the form $|z| < d_i$, the left-sided inverse z-transform is chosen, ie.

$$-A\frac{(n+1)\ldots(n+m-1)}{(m-1)!}(d_i)^n u[-n-1] \overset{z}{\longleftrightarrow} \frac{A}{(1-d_i z^{-1})^m}, \quad \text{with ROC}|z| < d_i$$

Deciding ROC

- The ROC of $X(z)$ is the intersection of the ROCs associated with the individual terms in the partial fraction expansion.
- In order to chose the correct inverse z-transform, we must infer the ROC of each term from the ROC of $X(z)$.
- By comparing the location of each pole with the ROC of $X(z)$.
- Chose the right sided inverse transform: if the ROC of $X(z)$ has the radius greater than that of the pole associated with the given term
- Chose the left sided inverse transform: if the ROC of $X(z)$ has the radius less than that of the pole associated with the given term

Partial fraction method

- It can be applied to complex valued poles
- Generally the expansion coefficients are complex valued

- If the coefficients in $X(z)$ are real valued, then the expansion coefficients corresponding to complex conjugate poles will be complex conjugate of each other

- Here we use information other than ROC to get unique inverse transform
- We can use causality, stability and existence of DTFT
- If the signal is known to be causal then right sided inverse transform is chosen
- If the signal is stable, then t is absolutely summable and has DTFT
- Stability is equivalent to existence of DTFT, the ROC includes the unit circle in the z-plane, ie. $|z| = 1$
- The inverse z-transform is determined by comparing the poles and the unit circle
- If the pole is inside the unit circle then the right-sided inverse z-transform is chosen
- If the pole is outside the unit circle then the left-sided inverse z-transform is chosen

Power series expansion method

- Express $X(z)$ as a power series in z^{-1} or z as given in z-transform equation
- The values of the signal $x[n]$ are then given by coefficient associated with z^{-n}
- Main disadvantage: limited to one sided signals

- Signals with ROCs of the form $|z| > a$ or $|z| < a$
- If the ROC is $|z| > a$, then express $X(z)$ as a power series in z^{-1} and we get right sided signal
- If the ROC is $|z| < a$, then express $X(z)$ as a power series in z and we get left sided signal

Long division method:

- Find the z-transform of

$$X(z) = \frac{2 + z^{-1}}{1 - \frac{1}{2}z^{-1}}, \text{with ROC} \quad |z| > \frac{1}{2}$$

- Solution is: use long division method to write $X(z)$ as a power series in z^{-1}, since ROC indicates that $x[n]$ is right sided sequence
- We get

$$X(z) = 2 + 2z^{-1} + z^{-2} + \frac{1}{2}z^{-3} + \ldots$$

- Compare with z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- We get

$$x[n] = 2\delta[n] + 2\delta[n-1] + \delta[n-2] + \frac{1}{2}\delta[n-3] + \ldots$$

- If we change the ROC to $|z| < \frac{1}{2}$, then expand $X(z)$ as a power series in z using long division method
- We get

$$X(z) = -2 - 8z - 16z^2 - 32z^3 + \ldots$$

- We can write $x[n]$ as

$$x[n] = -2\delta[n] - 8\delta[n+1] - 16\delta[n+2] - 32\delta[n+3] + \dots$$

- Find the z-transform of

$$X(z) = e^{z^2}, \text{with ROC all } z \text{ except } |z| = \infty$$

- Solution is: use power series expansion for e^a and is given by

$$e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!}$$

- We can write $X(z)$ as

$$X(z) = \sum_{k=0}^{\infty} \frac{(z^2)^k}{k!}$$

$$X(z) = \sum_{k=0}^{\infty} \frac{z^{2k}}{k!}$$

- We can write $x[n]$ as

$$x[n] = \begin{cases} 0 & n > 0 \text{ or } n \text{ is odd} \\ \frac{1}{(\frac{-n}{2})!}, & \text{otherwise} \end{cases}$$

Example: A finite sequence *x [n]* is defined as

$$x[n] = \{5, 3, -\underset{\uparrow}{2}, 0, 4, -3\}$$

Find X(z) and its ROC.

Sol: We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-2}^{3} x[n]z^{-n}$$

$$= x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3}$$
$$= 5z^2 + 3z - 2 + 4z^{-2} - 3z^{-3}$$

For z not equal to zero or infinity, each term in $X(z)$ will be finite and consequently $X(z)$ will converge. Note that $X(z)$ includes both positive powers of z and negative powers of z. Thus, from the result we conclude that the ROC of $X(z)$ is $\mathbf{0} < |z| < \infty$.

Example: Consider the sequence

$$x[n] = \begin{cases} a^n & 0 \le n \le N-1,\ a > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find $X(z)$ **and plot the poles and zeros of X(z) .**

Sol:

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n = \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$

From the above equation we see that there is a pole of $(N-1)^{th}$ order at $z = 0$ and a pole at $z = a$. Since x[n] is a finite sequence and is zero for $n < 0$, the ROC is $|z| > 0$. The N roots of the numerator polynomial are at

$$z_k = ae^{j(2\pi k/N)} \qquad k = 0, 1, \ldots, N-1$$

The root at k = **0** cancels the pole at $z = a$. The remaining zeros of $X(z)$ are at

$$z_k = ae^{j(2\pi k/N)} \qquad k = 1, \ldots, N-1$$

The pole-zero plot is shown in the following figure with N=8