Dynamics of Machinery

Prepared by

Peetala Bhaskar Rao
Asst. Prof.
Mechanical Engineering

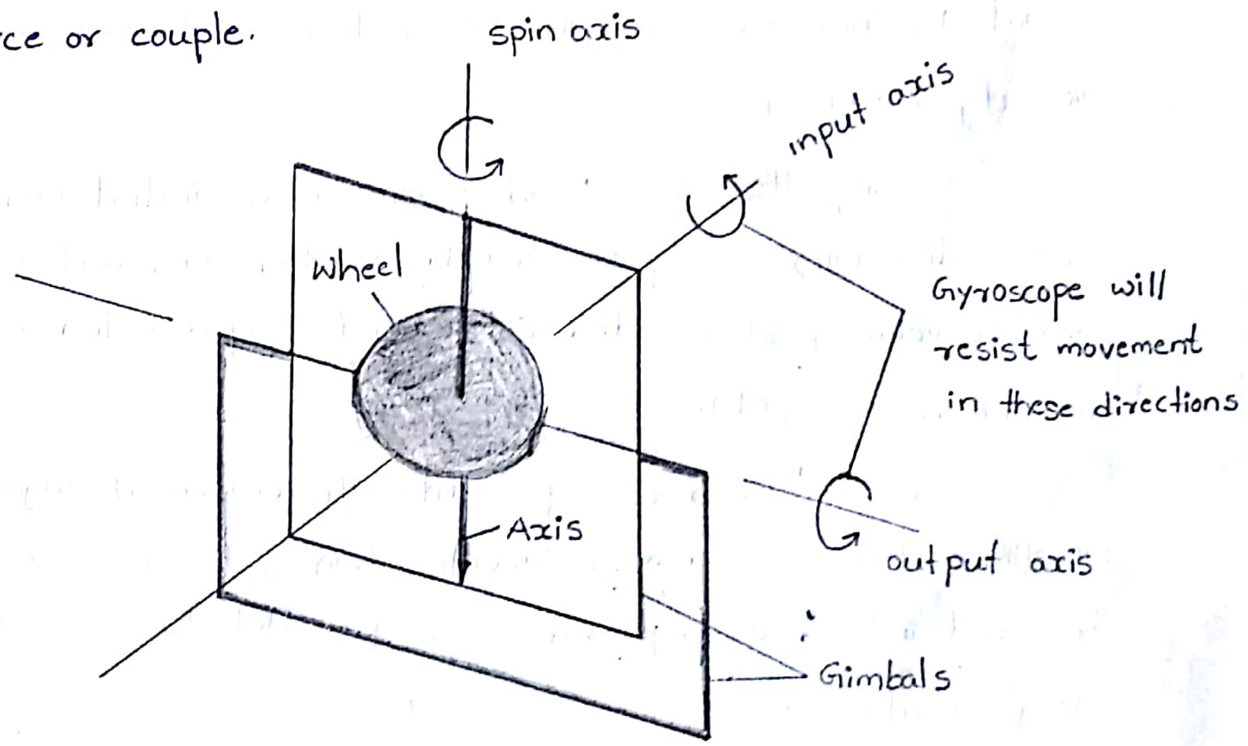
DOM

Precession

When a body moves along a curved path with a uniform linear velocity, a force in the direction of centripetal acceleration (known as centripetal force) has to be applied externally over the body, so that it moves along the required ~~path~~ curved path. This external force applied is known as active force.

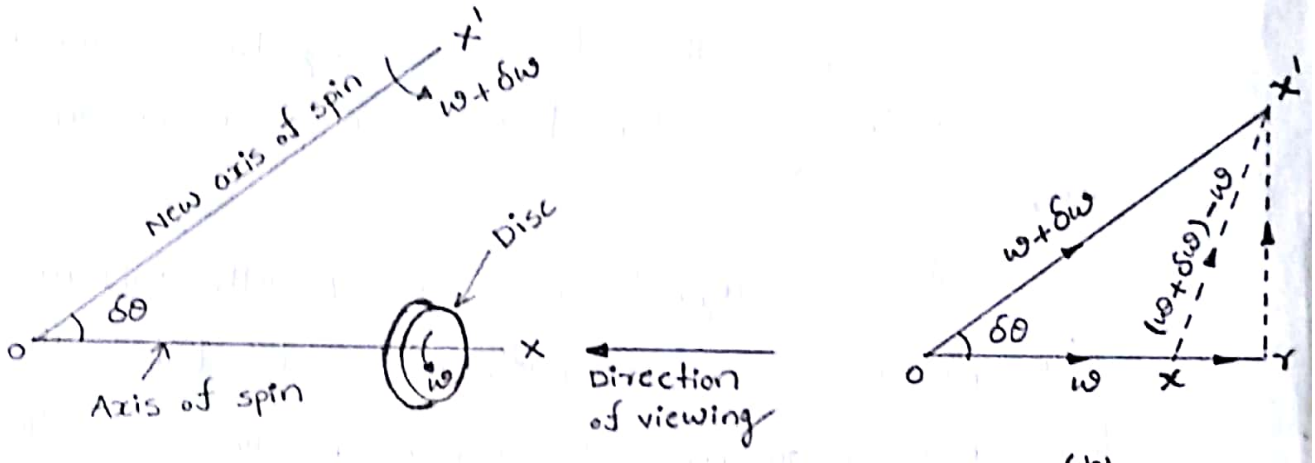
When a body, itself, is moving with uniform linear velocity along a circular path, it is subjected to the centrifugal force radially outwards. This centrifugal force is called reactive force. The action of the reactive or centrifugal force is to tilt or move the body along radially outward direction.

Whenever the effect of any force or couple over a moving or rotating body is to be considered, it should be with respect to the reactive force or couple and not with respect to active force or couple.



Precessional Angular Motion

The angular acceleration is the rate of change of angular velocity with respect to time. It is a vector quantity and may be represented by drawing a vector diagram with the help of right hand screw rule.



(a) (b)

Consider a disc, as shown in Fig (a), revolving or spinning about the axis OX (known as axis of spin) in anticlockwise when seen from the front, with an angular velocity ω in a plane at angles to the paper.

After a short interval of time δt, let the disc be spinning about the new axis of spin OX' (at an angle δθ) with an angular velocity (ω+δω).

Using the right hand screw rule, initial angular velocity of the disc (ω) is represented by vector ox; and the final angular velocity of the disc (ω+δω) is represented by vector ox' as shown in Fig (b).

The vector xx' represents the change of angular velocity in time δt i.e the angular acceleration of the disc. This may be resolved into two components, one parallel to ox & the other perpendicular to ox.

component of angular acceleration in the direction of ox

$$\alpha_t = \frac{xr}{\delta t} = \frac{or - ox}{\delta t} = \frac{ox'\cos\delta\theta - ox}{\delta t}$$

$$= \frac{(\omega + \delta\omega)\cos\delta\theta - \omega}{\delta t} = \frac{\omega\cos\delta\theta + \delta\omega\cos\delta\theta - \omega}{\delta t}$$

since $\delta\theta$ is very small, therefore substituting $\cos\delta\theta = 1$

$$\alpha_t = \frac{\omega + \delta\omega - \omega}{\delta t} = \frac{\delta\omega}{\delta t}$$

In the limit, when $\delta t \to 0$

$$\alpha_t = \underset{\delta t \to 0}{Lt} \left(\frac{\delta\omega}{\delta t}\right) = \frac{d\omega}{dt}$$

component of angular acceleration in the direction perpendicular to ox

$$\alpha_c = \frac{rx'}{\delta t} = \frac{ox'\sin\delta\theta}{\delta t} = \frac{(\omega + \delta\omega)\sin\delta\theta}{\delta t} = \frac{\omega\sin\delta\theta + \delta\omega\sin\delta\theta}{\delta t}$$

since $\delta\theta$ is very small, therefore substituting $\sin\delta\theta = \delta\theta$

$$\alpha_c = \frac{\omega\delta\theta + \delta\omega\delta\theta}{\delta t} = \frac{\omega\delta\theta}{\delta t}$$

(Neglecting $\delta\omega\delta\theta$, being very small)

In the limit when $\delta t \to 0$

$$\alpha_c = \underset{\delta t \to 0}{Lt} \frac{\omega\delta\theta}{\delta t} = \omega\frac{d\theta}{dt} = \omega . \omega_p \qquad \left(\because \omega_p = \frac{d\theta}{dt}\right)$$

$\therefore$ Total angular acceleration of the disc

$= \text{vector } xx' = \text{vector sum of } \alpha_t \text{ and } \alpha_c$

$$= \frac{d\omega}{dt} + \omega\frac{d\theta}{dt} = \frac{d\omega}{dt} + \omega.\omega_p$$

where $\frac{d\theta}{dt}$ is the angular velocity of the axis of

~~the~~ spin about a certain axis, which is perpendicular to the plane in which the axis of spin is going to rotate. This angular velocity of the axis of spin (i.e $\frac{d\theta}{dt}$) is known as angular velocity of precession and is denoted by ω_p. The axis, about which the axis of spin is to turn, is known as axis of precession. The angular motion of the axis of spin about the axis of precession is known as precessional angular motion.

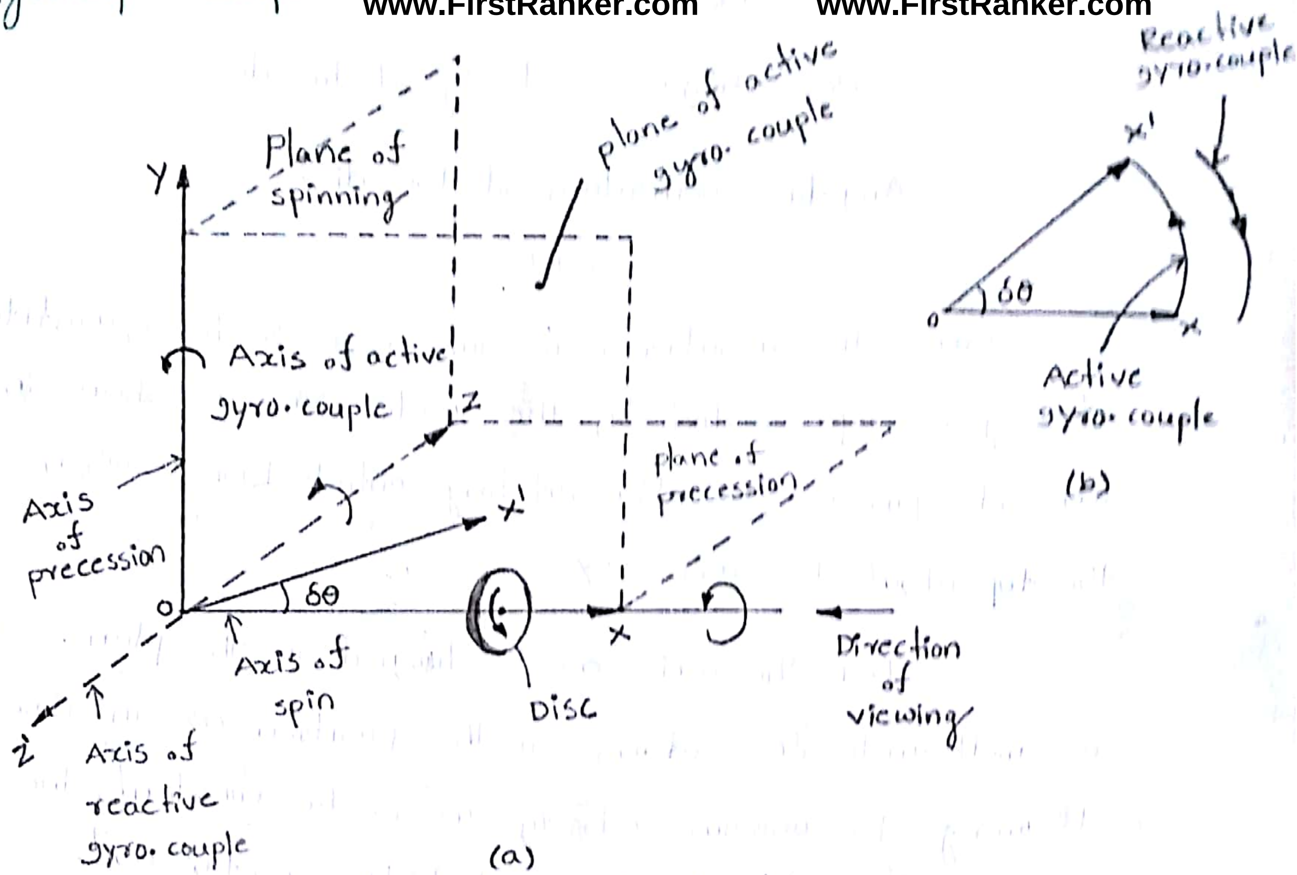
The axis of precession is perpendicular to the plane in which the axis of spin is going to rotate.

If the angular velocity of the disc remains constant at all positions of the axis of spin, then $\frac{d\theta}{dt}$ is zero, and thus α_c is zero.

If the angular velocity of the disc changes the direction but remains constant in magnitude, then angular acceleration of the disc is given by

$$\alpha_c = \omega \frac{d\theta}{dt} = \omega . \omega_p$$

The angular acceleration α_c is known as gyroscopic acceleration.



(a)

Fig: Gyroscopic Couple

Consider a disc spinning with an angular velocity ω rad/s about the axis of spin OX, in anti-clockwise direction when seen from the front, as shown in Fig (a). Since the plane in which the disc is rotating is parallel to the plane YOZ, therefore it is called plane of spinning. The plane XOZ is a horizontal plane and the axis of spin rotates in a plane parallel to the horizontal plane about an axis OY.

In other words, the axis of spin is said to be rotating or processing about an axis OY. In other words, the axis of spin is said to be rotating or processing about an axis OY (which is perpendicular to both the axes OX and OZ) at an angular velocity ω_p rad/s. This horizontal plane XOZ is called plane of precession and OY is the axis of precession.

Let I = Mass moment of inertia of the disc about ox,

ω = Angular velocity of the disc

$\therefore$ Angular momentum of the disc

$$= I\omega$$

Since the angular momentum is a vector quantity, therefore it may be represented by the vector ox, as shown in Fig.(b). The axis of spin ox is also rotating anticlockwise when seen from the top about the axis oy.

Let the axis ox is turned in the plane xoz through a small angle $\delta\theta$ radians to the position ox' in time δt seconds. Assuming the angular velocity ω to be constant, the angular momentum will now be represented by vector ox'.

$\therefore$ change in angular momentum

$$= \vec{ox'} - \vec{ox} = \vec{xx'} = \vec{ox}\,\delta\theta$$

$$= I.\omega.\delta\theta$$

and rate of change of angular momentum

$$= I\omega \times \frac{\delta\theta}{\delta t}$$

Since the rate of change of angular momentum will result by the application of a couple to the disc, therefore the couple applied to the disc causing precession,

$$\left(\because \frac{d\theta}{dt} = \omega_p\right)$$

$$C = \underset{\delta t \to 0}{Lt}\, I.\omega \times \frac{\delta\theta}{\delta t} = I\omega.\frac{d\theta}{dt} = I.\omega.\omega_p$$

where ω_p = Angular velocity of precession of the axis of spin or the speed of rotation of the axis of spin about the axis of precession oy.

In S.I units, the units of C is N-m when I is in kg-m^2

u (P) A uniform disc of diameter 300 mm and of mass 5 kg is mounted on one end of an arm of length 600mm. The other end of the arm is free to rotate in a universal bearing. If the disc rotates about the arm with a speed of 300 rpm. clockwise, looking from the front, with what speed will it precess about the vertical axis

Sol:- $d = 300\text{ mm}$ (or) $r = 150\text{mm} = 0.15\text{ m}$

$m = 5\text{ kg}$

$l = 600\text{ mm} = 0.6\text{ m}$

$N = 300\text{ rpm}$ (or) $\omega = \frac{2\pi N}{60} = \frac{2\pi \times 300}{60} = 31.42\text{ rad/sec}$

we know that the mass moment of inertia of the disc, about an axis through its centre of gravity & perpendicular to the plane of disc,

$$I = \frac{1}{2} m r^2 = \frac{1}{2} \times 5 \times (0.15)^2 = 0.056\text{ kg-m}^2$$

and couple due to mass of disc,

$$C = mgl = 5 \times 9.81 \times 0.6 = 29.43\text{ N-m}$$

Let ω_p = speed of precession

we know that couple, $C = I \omega \omega_p$

$$29.43 = 0.056 \times 31.42 \times \omega_p$$

$$\omega_p = 16.7\text{ rad/s}$$

Effect of the Gyroscopic couple on an Aeroplane

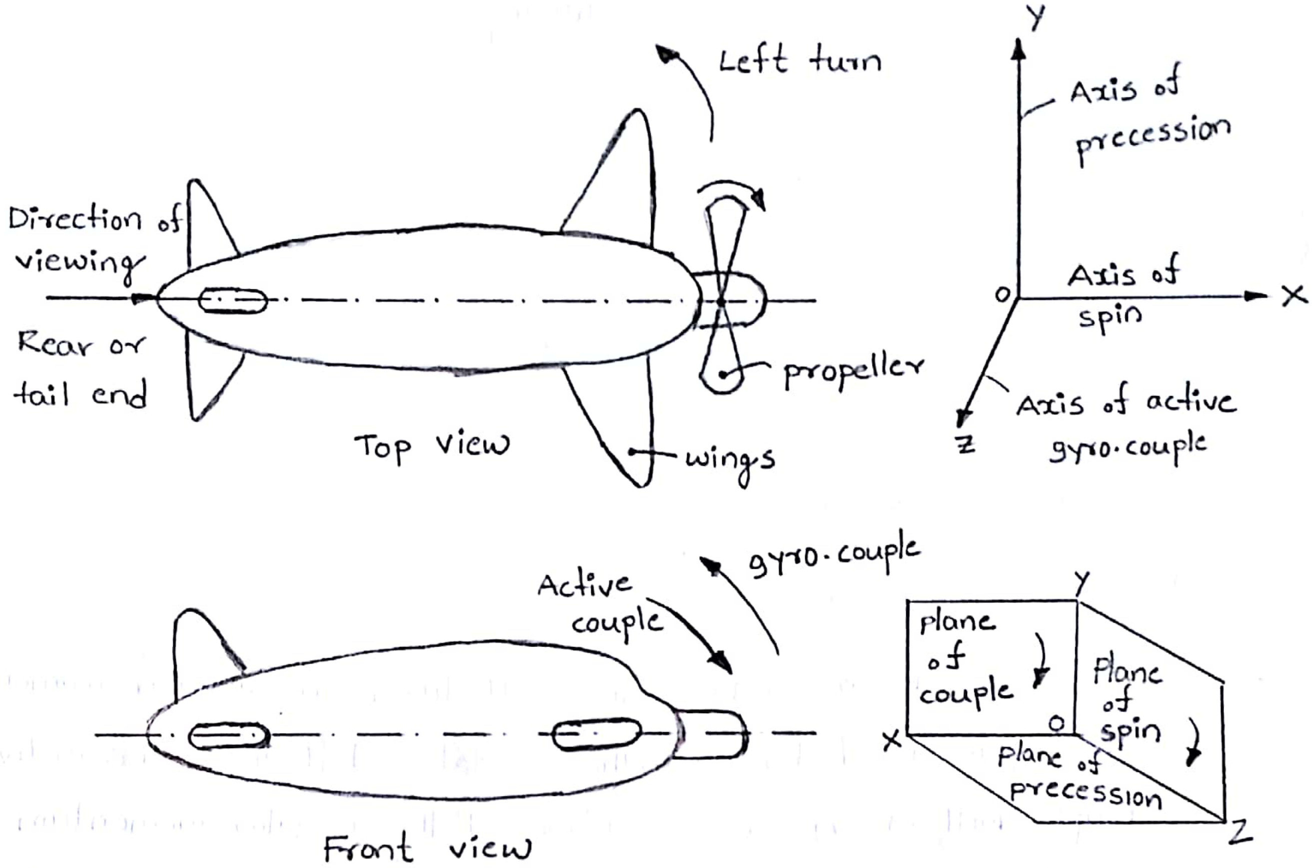


Fig: Aeroplane taking a left turn

The top and front view of an aeroplane are shown in Fig. Let engine or propeller rotates in the clockwise direction, when seen from the rear or tail end and the aeroplane takes a turn to the left.

Let

ω – Angular velocity of the engine in rad/s

m – Mass of the engine and the propeller in kg

k – Radius of gyration in meters

I – Mass moment of inertia of the engine and the propeller in $kg\text{-}m^2 = mk^2$

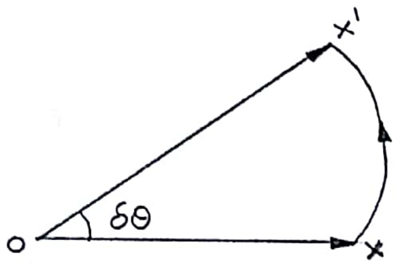
V – Linear velocity of the aeroplane in m/s

R – Radius of curvature in meters, and

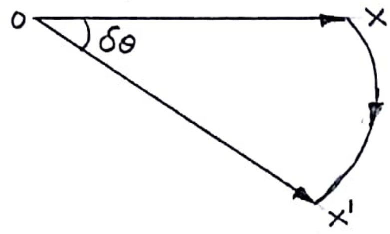
ω_p – Angular velocity of precession $= \frac{V}{R}$ rad/s

Gyroscopic couple acting on the aeroplane

$C = I \cdot \omega \cdot \omega_p$



a) Aeroplane taking left turn



b) Aeroplane taking right turn

Fig: Effect of gyroscopic couple on an aeroplane

Before taking the left turn, the angular momentum vector is represented by ox. When it takes left turn, the active gyroscopic couple will change the direction of the angular momentum vector from ox to ox' as shown in Fig (a). The vector xx', in the limit, represents the change of angular momentum or the active gyroscopic couple and is perpendicular to ox. Thus the plane of active gyroscopic couple xoy will be perpendicular to xx', i.e vertical in this case, as shown in Fig (b).

By applying right hand screw rule to vector xx', we find that the direction of active gyroscopic couple is clockwise as shown in the front view of Fig. In other words, for left hand turning, the active gyroscopic couple on the aeroplane in the axis oz will be clockwise as shown in Fig. The reactive gyroscopic couple (equal in magnitude of active gyroscopic couple) will act in the opposite direction (i.e in the anticlockwise direction) and the effect of this couple is, therefore, to raise the nose and dip the tail of the aeroplane.

An aeroplane makes a complete half circle of 50 meters radius, towards left, when flying at 200 kmph. The rotary engine and the propeller of the plane has a mass of 400 kg and a radius of gyration of 0.3 m. The engine rotates at 2400 rpm clockwise when viewed from the rear. Find the gyroscopic couple on the aircraft and state its effect on it.

$R = 50$ m , $v = 200$ kmph $= 55.6$ m/s , $m = 400$ kg

$k = 0.3$m , $N = 2400$ rpm (or) $\omega = \frac{2\pi N}{60} = \frac{2\pi \times 2400}{60}$

$\omega = 251.4$ rad/s

we know that mass moment of inertia of the engine & the propeller,

$$I = mk^2 = 400 \times (0.3)^2 = 36 \text{ kg-m}^2$$

and angular velocity of precession,

$$\omega_p = \frac{v}{R} = \frac{55.6}{50} = 1.11 \text{ rad/s}$$

we know that gyroscopic couple acting on the aircraft

$$C = I\omega\omega_p = 36 \times 251.4 \times 1.11 = 10046 \text{ N-m}$$

When the aeroplane turns towards left, the effect of the gyroscopic couple is to lift the nose upwards & tail downwards.

Effect of Gyroscopic couple on a Naval ship during steering

Steering is the turning of a complete ship in a curve towards left or right, while it moves forward. Consider the ship taking a left turn, and rotor rotates in the clockwise direction when viewed from the stern, as shown in Fig. The effect of gyroscopic couple on a naval ship during steering taking left or right turn may be obtained in the similar way as for an aeroplane.

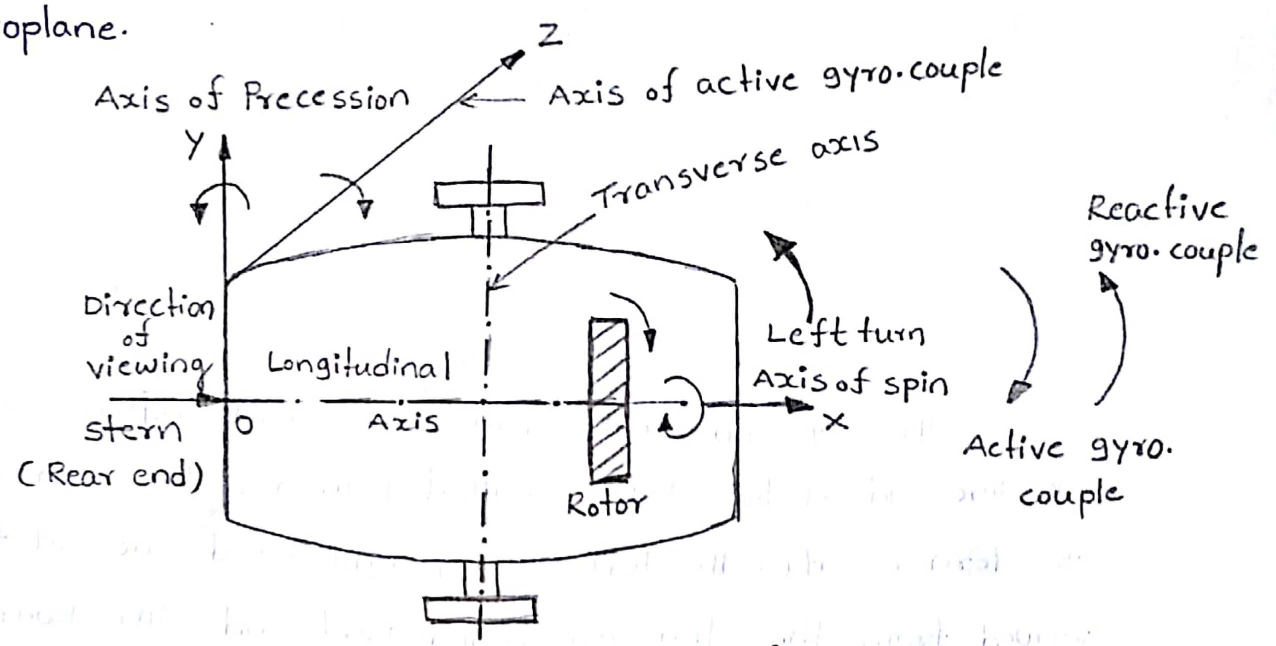
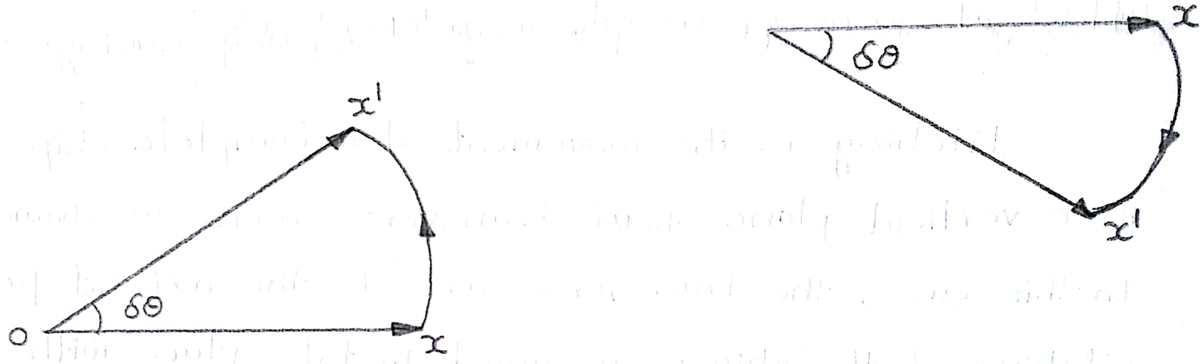


Fig: Naval ship taking a left turn

When the rotor of the ship rotates in the clockwise direction when viewed from the stern, it will have its angular momentum vector in the direction ox as shown in Fig.(a). As the ship steers to the left, the active gyroscopic couple will change the angular momentum vector from ox to ox'. The vector xx' now represents the active gyroscopic couple and is perpendicular to ox. Thus the plane of active gyroscopic couple is perpendicular to xx' and its direction in the axis oz for left hand turn is clockwise as shown in Fig. The reactive gyroscopic couple of the same magnitude will act in the opposite direction (i.e. in anticlockwise direction). The effect of this reactive gyroscopic couple is to raise the bow & lower the stern.

10

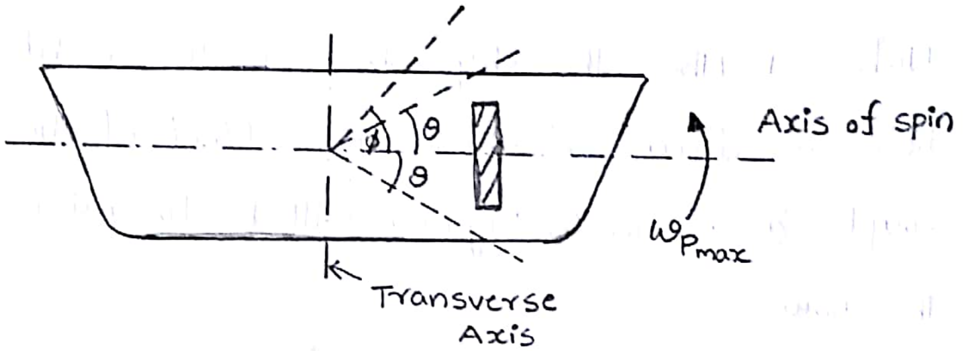


a) Steering to the left b) Steering to the right

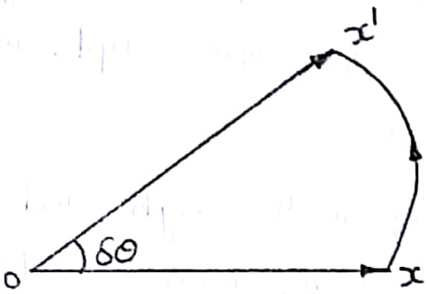
Fig: Effect of gyroscopic couple on a naval ship during steering

Effect of gyroscopic couple on a Naval ship during Pitching

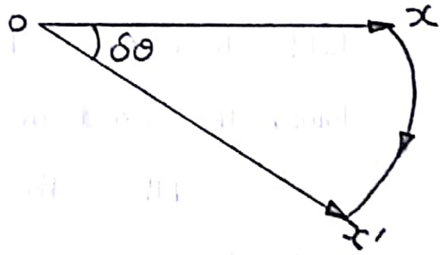
Pitching is the movement of a complete ship up and down in a vertical plane about transverse axis, as shown in Fig (a). In this case, the transverse axis is the axis of precession. The pitching of the ship is assumed to take place with simple harmonic motion i.e. the motion of the axis of spin about transverse axis is simple harmonic.



a) Pitching of a naval ship



b) Pitching upward



c) Pitching downward

Fig: Effect of gyroscopic couple on a naval ship during pitching

$\therefore$ Angular displacement of the axis of spin from mean position after time t seconds,

$$\theta = \phi \sin \omega_1 \cdot t$$

where ϕ – Amplitude of swing i.e. maximum angle turned from the mean position in radians

ω_1 – Angular velocity of S.H.M

$$\omega_1 = \frac{2\pi}{\text{Time period of S.H.M in seconds}} = \frac{2\pi}{t_p} \text{ rad/s}$$

Angular velocity of precession,

$$\omega_p = \frac{d\theta}{dt} = \frac{d}{dt}(\phi \sin\omega_1 \cdot t) = \phi\omega_1 \cos\omega_1 \cdot t$$

The angular velocity of precession will be maximum, if $\cos\omega_1 t = 1$

$\therefore$ Maximum angular velocity of precession

$$\omega_{p_{max}} = \phi\omega_1 = \phi \frac{2\pi}{t_p}$$

Let I = Moment of inertia of the rotor in $kg\text{-}m^2$

ω = Angular velocity of the rotor in rad/s

$\therefore$ Maximum gyroscopic couple,

$$C_{max} = I \cdot \omega \cdot \omega_{p_{max}}$$

When the pitching is upward, the effect of the reactive gyroscopic couple, as shown in Fig (b), will try to move the ship toward star-board. On the other hand, if the pitching is downward, the effect of the reactive gyroscopic couple, as shown in Fig (c), is to turn the ship towards port side.

A ship propelled by a turbine rotor which has a mass of 5 tonnes and a speed of 2100 rpm. The rotor has a radius of gyration of 0.5m and rotates in a clockwise direction when viewed from the stern. Find the gyroscopic effects in the following conditions:

1. The ship sails at a speed of 30 km/h and steers to the left in a curve having 60m radius.
2. The ship pitches 6° above and 6° below the horizontal position. The bow is descending with its maximum velocity. The motion due to pitching is simple harmonic and the periodic time is 20 seconds.
3. The ship rolls and at a certain instant it has an angular velocity of 0.03 rad/s clockwise when viewed from stern.

Determine also the maximum angular acceleration during pitching. Explain how the direction of motion due to gyroscopic effect is determined in each case.

Sol:- $m = 5 \text{ tonnes} = 5000 \text{ kg}$, $N = 2100 \text{ rpm}$ $k = 0.5 \text{ m}$

$$\omega = \frac{2\pi N}{60} = \frac{2\pi * 2100}{60}$$

$$\omega = 220 \text{ rad/s}$$

1. When the ship steers to the left

$V = 30 \text{ km/h} = 8.33 \text{ m/s}$, $R = 60 \text{ m}$

we know that angular velocity of precession

$$\omega_p = \frac{V}{R} = \frac{8.33}{60} = 0.14 \text{ rad/s}$$

and mass moment of inertia of the rotor

$$I = mk^2 = 5000 (0.5)^2 = 1250 \text{ kg-m}^2$$

$\therefore$ Gyroscopic couple, $C = I\omega\omega_p = 1250 * 220 * 0.14$

$$C = 38500 \text{ N-m}$$

When the rotor rotates in a clockwise direction when viewed from the stern and the ship steers to the left, the effect of reactive gyroscopic couple is to raise the bow and lower the stern.

2. When the ship pitches with the bow descending

$$\phi = 6^\circ = \frac{6 \times \pi}{180} = 0.105 \text{ rad}$$

we know that angular velocity of simple harmonic motion

$$\omega_1 = \frac{2\pi}{t_p} = \frac{2\pi}{20} = 0.3142 \text{ rad/s}$$

and maximum angular velocity of precession

$$\omega_{P_{max}} = \phi\,\omega_1 = 0.105 \times 0.3142 = 0.033 \text{ rad/s}$$

$\therefore$ Maximum gyroscopic couple

$$C_{max} = I\,\omega\,\omega_{P_{max}} = 1250 \times 220 \times 0.033 = 9075 \text{ N-m}$$

Since the ship is pitching with the bow descending, therefore the effect of this maximum gyroscopic couple is to turn the ship towards port side.

3. When the ship rolls

Since the ship rolls at an angular velocity of 0.03 rad/s, therefore angular velocity of precession when the ship rolls,

$$\omega_P = 0.03 \text{ rad/s}$$

$\therefore$ Gyroscopic couple, $C = I\,\omega\,\omega_P = 1250 \times 220 \times 0.03$

$$C = 8250 \text{ N-m}$$

In case of rolling of a ship, the axis of precession is always parallel to the axis of spin for all positions, therefore there is no effect of gyroscopic couple.

Maximum angular acceleration during pitching

$$\alpha_{max} = \phi\,(\omega_1)^2 = 0.105\,(0.3142)^2 = 0.01 \text{ rad/s}^2$$

Effect of Gyroscopic couple on a Four Wheel Drive Moving in a Curved Path

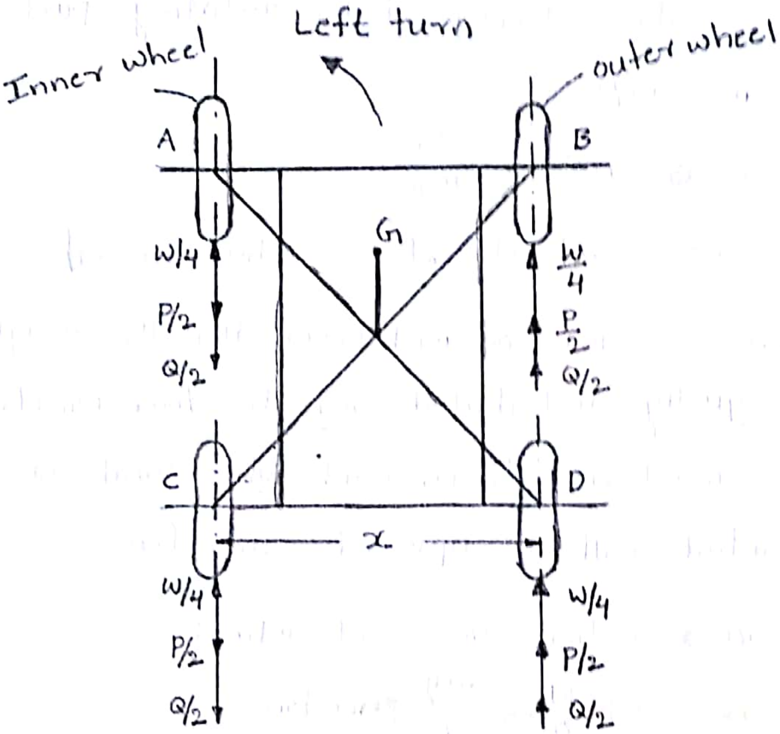


Fig: Four wheel drive moving in a curved path

Consider the four wheels A, B, C and D of an automobile locomotive taking a turn towards left as shown in Fig. The wheels A and C are inner wheels, whereas B and D are outer wheels. The centre of gravity (C.G) of the vehicle lies vertically above the road surface.

Let
- m – Mass of the vehicle in kg
- W – Weight of the vehicle in Newtons $= mg$
- r_W – Radius of the wheels in meters
- R – Radius of curvature in meters $(R > r_W)$
- h – Distance of centre of gravity, vertically above the road surface in meters
- x – width of track in meters
- I_W – Mass moment of inertia of one of the wheels in kg-m²
- ω_W – Angular velocity of the wheels or velocity of spin in rad/s

I_E – Mass moment of inertia of the rotating parts of the engine in kg m²

ω_E – Angular velocity of the rotating parts of the engine in rad/s

G – Gear ratio $= \frac{\omega_E}{\omega_W}$

v – Linear velocity of the vehicle in m/s $= \omega_W r_W$

A little consideration will show, that the weight of the vehicle (W) will be equally distributed over the four wheels which will act downwards. The reaction between each wheel and the road surface of the same magnitude will act upwards. Therefore

Road reaction over each wheel

$$= \frac{W}{4} = \frac{mg}{4} \text{ newtons}$$

Let us now consider the effect of the gyroscopic couple and centrifugal couple on the vehicle.

1. Effect of the gyroscopic couple

Since the vehicle takes a turn towards left due to the precession and other rotating parts, therefore a gyroscopic couple will act.

We know that, velocity of precession,

$$\omega_P = \frac{v}{R}$$

∴ Gyroscopic couple due to 4 wheels,

$$C_W = 4 I_W \omega_W \omega_P$$

and gyroscopic couple due to the rotating parts of the engine

$$C_E = I_E \omega_E \omega_P = I_E G \cdot \omega_W \cdot \omega_P \qquad \left(\because G = \frac{\omega_E}{\omega_W}\right)$$

∴ Net gyroscopic couple,

$$C = C_W \pm C_E = 4 I_W \omega_W \omega_P \pm I_E G \cdot \omega_W \cdot \omega_P$$

$$C = \omega_W \cdot \omega_P (4 I_W \pm G I_E)$$

The positive sign is used when the wheels and rotating parts of the engine rotate in the same direction. If the rotating parts of the engine revolves in opposite direction, then negative sign is used.

Due to the gyroscopic couple, vertical reaction on the road surface will be produced. The reaction will be vertically upwards on the outer wheels and vertically downwards on the inner wheels.

Let the magnitude of this reaction at the two outer or inner wheels be P newtons. Then

$$P * x = C \quad \text{or} \quad P = C/x$$

∴ Vertical reaction at each of the outer or inner wheels

$$P/2 = C/2x$$

Note: We have discussed above that when rotating parts of the engine rotate in opposite directions, then −ve sign is used, i.e. net gyroscopic couple,

$$C = C_W - C_E$$

when $C_E > C_W$, then C will be −ve. Thus the reaction will be vertically downwards on the outer wheels and vertically upwards on the inner wheels.

2. Effect of the centrifugal couple

Since the vehicle moves along a curved path, therefore centrifugal force will act outwardly at the centre of gravity of the vehicle. The effect of this centrifugal force is also to overturn the vehicle.

We know that centrifugal force,

$$F_c = \frac{mv^2}{R}$$

The couple tending to overturn the vehicle or overturning couple,

$$C_o = F_c \times h = \frac{mv^2}{R} \times h$$

This overturning couple is balanced by vertical reactions, which are vertically upwards on the outer wheels & vertically downwards on the inner wheels. Let the magnitude of this reaction at the two outer or inner wheels be Q. Then

$$Q \times x = C_o \quad \text{or} \quad Q = \frac{C_o}{x} = \frac{mv^2 h}{R \cdot x}$$

∴ Vertical reaction at each of the outer or inner wheels

$$\frac{Q}{2} = \frac{mv^2 h}{2Rx}$$

∴ Total vertical reaction at each of the outer wheel,

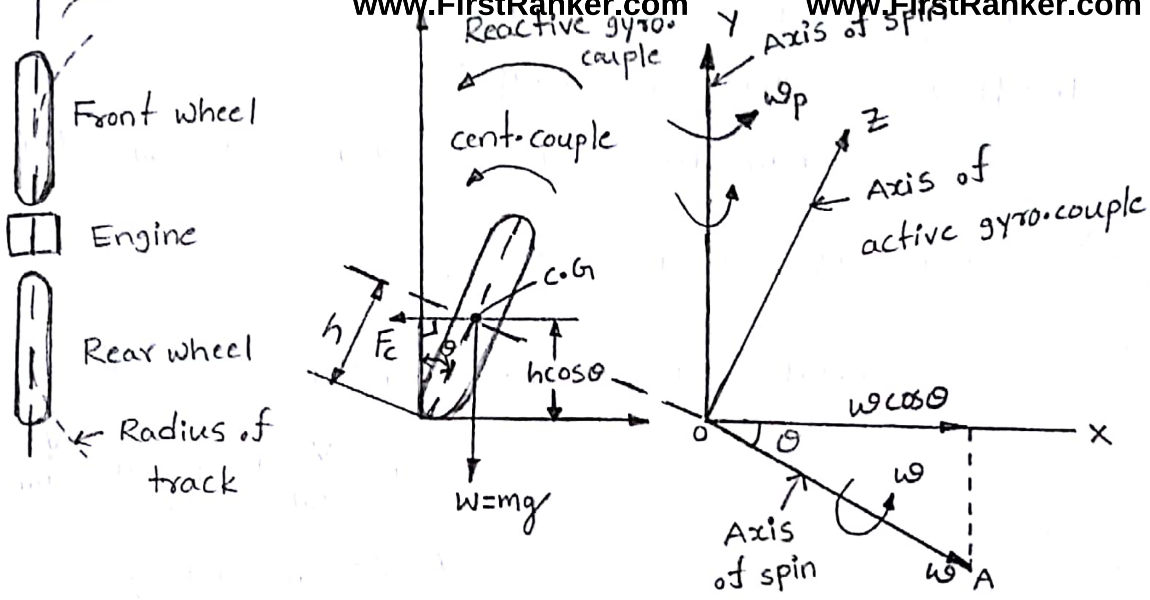
$$P_o = \frac{W}{4} + \frac{P}{2} + \frac{Q}{2}$$

and total vertical reaction at each of the inner wheel

$$P_I = \frac{W}{4} - \frac{P}{2} - \frac{Q}{2}$$

A little consideration will show that when the vehicle is running at high speeds, P_I may be zero or even negative. This will cause the inner wheels to leave the ground thus tending to overturn the automobile. In order to have the contact between the inner wheels and the ground, the sum of $\frac{P}{2}$ & $\frac{Q}{2}$ must be less than $\frac{W}{4}$.

Stability of a two wheel vehicle taking a Turn



(a) (b) (c)

Fig: stability of a two wheel vehicle taking a turn

Consider a two wheel vehicle (say a scooter or motor cycle) taking a right turn as shown in Fig (a).

Let m – Mass of the vehicle & its rider in kg

W – Weight of the vehicle & its rider in newtons $= mg$

h – Height of the centre of gravity of the vehicle and rider

r_w – Radius of the wheels

R – Radius of track or curvature

I_W – Mass moment of inertia of each wheel

I_E – Mass moment of inertia of the rotating parts of the engine

ω_W – Angular velocity of the wheels

ω_E – Angular velocity of the engine

G = Gear ratio $= \dfrac{\omega_E}{\omega_W}$

v = Linear velocity of the vehicle $= \omega_W \times r_w$

θ = Angle of heel. It is the inclination of the vehicle to the vertical for equilibrium.

Let us now consider the effect of the gyroscopic couple and centrifugal couple on the vehicle.

1. Effect of gyroscopic couple

we know that $V = r_w * \omega_w$ (or) $\omega_w = V/r_w$

and $\omega_E = G\omega_W = G * \frac{V}{r_w}$

$\therefore$ Total $(I * \omega) = 2 I_w \omega_w \pm I_E \omega_E$

$$= 2 I_w * \frac{V}{r_w} \pm I_E * G * \frac{V}{r_w}$$

$$= \frac{V}{r_w}(2 I_w \pm G I_E)$$

and velocity of precession, $\omega_p = V/R$

A little consideration will show that when the wheels move over the curved path, the vehicle is always inclined at an angle θ with the vertical plane as shown in Fig(b). This angle is known as angle of heel.

In other words, the axis of spin is inclined to the horizontal at an angle θ, as shown in Fig(c). Thus the angular momentum vector $I\omega$ due to spin is represented by OA inclined to OX at an angle θ. But the precession axis is vertical. Therefore the spin vector is resolved along OX.

$\therefore$ Gyroscopic couple

$$C_1 = I\omega\cos\theta * \omega_p = \frac{V}{r_w}(2 I_w \pm G I_E)\cos\theta * \frac{V}{R}$$

$$= \frac{V^2}{R \cdot r_w}(2 I_w \pm G I_E)\cos\theta$$

... the same direction as

2. Effect of centrifugal couple

we know that centrifugal force

$$F_c = \frac{mv^2}{R}$$

This force acts horizontally through the centre of gravity along the outward direction.

$\therefore$ Centrifugal couple, $C_2 = F_c \times h\cos\theta$

$$= \left(\frac{mv^2}{R}\right) h\cos\theta$$

since the centrifugal couple has a tendency to overturn the vehicle, therefore

Total overturning couple,

C_o = Gyroscopic couple + Centrifugal couple

$$= \frac{v^2}{R \cdot r_w}\left(2I_w + GI_E\right)\cos\theta + \frac{mv^2}{R} \times h\cos\theta$$

$$= \frac{v^2}{R}\left[\frac{2I_w + GI_E}{r_w} + mh\right]\cos\theta$$

we know that balancing couple $= mgh\sin\theta$

The balancing couple acts in clockwise direction when seen from the front of the vehicle. Therefore for stability, the overturning couple must be equal to the balancing couple, i.e.

$$\frac{v^2}{R}\left(\frac{2I_w + GI_E}{r_w} + m \cdot h\right)\cos\theta = mgh\sin\theta$$

From this expression, the value of the angle of heel (θ) may be determined, so that the vehicle does not skid.

Friction

When a body slides over another, the motion is resisted by a force called the force of friction. The force arises from the fact that the surfaces, though planed and made smooth, have ridges and depressions that interlock and the relative movement is resisted. Thus, the force of friction on a body is parallel to the sliding surfaces and acts in a direction opposite to that of the sliding body.

There are phenomena, where it is necessary to reduce the force of friction whereas in some cases it must be increased. In case of lathe slides, journal bearings, etc., where the power transmitted is reduced due to friction, it has to be decreased by the use of lubricated surfaces. In processes where the power is transmitted through friction, attempts are made to increase it to transmit more power. Examples are friction clutches and belt drives. Even the tightness of a nut and bolt is dependent mainly on the force of friction. Had there been no friction between the nut and the surface on which it is tightened, the nut would loosen off at the moment the spanner is removed after tightening the surface on which it is tightened, the nut would loosen off at the moment the spanner is removed after tightening.

Kinds of Friction

Usually, three kinds of friction, depending upon the conditions of surfaces are considered.

1. Dry Friction

Dry friction is said to occur when there is relative motion between two completely unlubricated surfaces. It is further divided into two types :

a) Solid Friction When the two surfaces have a sliding motion relative to each other.

b) Rolling Friction Friction due to rolling of one surface over another. (e.g., ball and roller bearings)

2. Skin or Greasy Friction

When the two surfaces in contact have a minute thin layer of lubricant between them, it is known as skin or greasy friction. Higher spots on the surface break through the lubricant and come in contact with the other surface.

Skin friction is also termed as boundary friction.

3. Flim Friction

when the two surfaces in contact are completely separated by a Lubricant, friction will occur due to the shearing of different layers of the lubricant. This is known as film friction or viscous friction.

Laws of Friction

Experiments have shown that the force of friction

→ is directly proportional to normal reaction between the two surfaces

→ opposes the motion between the surfaces

→ depends upon the materials of the two surfaces

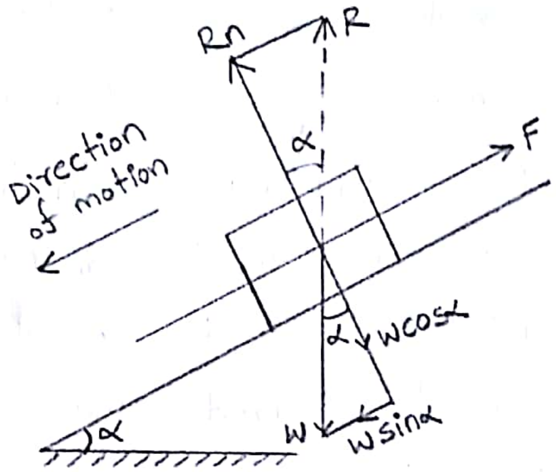
→ is independent of the area of contact

→ is independent of the velocity of sliding

The last of these laws is not true in the strict sense as it has been found that the friction force decreases slightly with the increase in velocity.

Inclined Plane

1. Body at Rest



When a body is at rest on an inclined plane making an angle α with the horizontal, the forces acting on the body are

W, weight of body in downward direction

Rn, normal reaction

F, force resisting the motion of body

From equilibrium conditions,

$F = w\sin\alpha$ and $R_n = w\cos\alpha$

If the angle of inclination of the plane is increased, the body will just slide down the plane of its own when

$$w\sin\alpha = F = \mu R_n = \mu w\cos\alpha$$

$$\tan\alpha = \mu = \tan\phi$$

$$\alpha = \phi$$

This maximum value of the angle of inclination of the plane with the horizontal when the body starts sliding on its own is known as the angle of repose

Let W – Weight of the body

α – Angle of inclination of the plane to the horizontal

ϕ – Limiting angle of friction for the contact surfaces

P – Effort applied in a given direction in order to cause the body to slide with uniform velocity parallel to the plane, considering friction

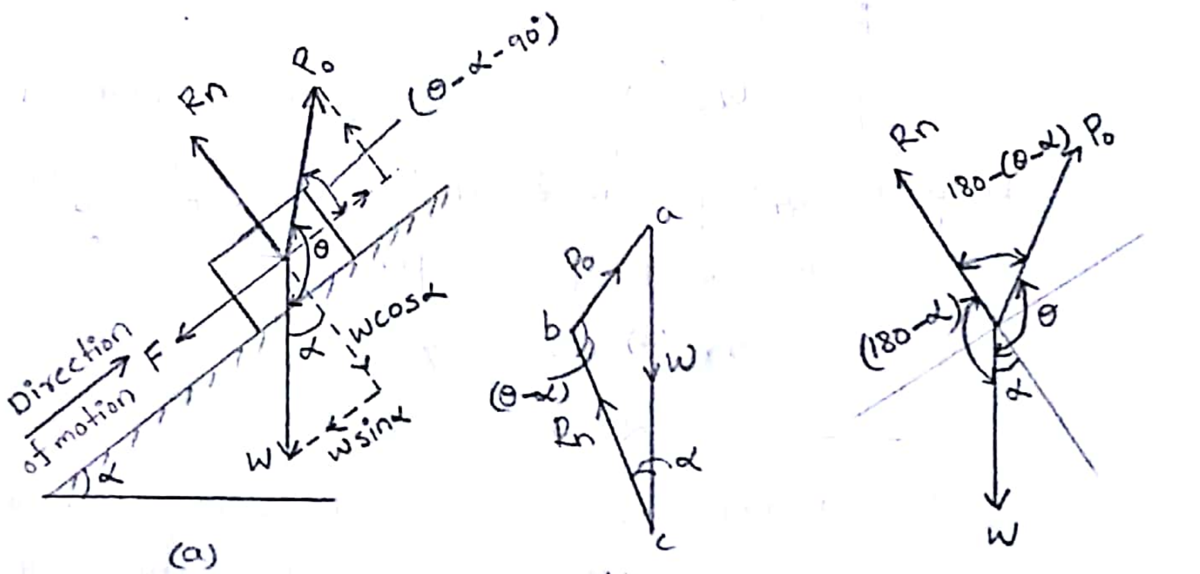
P_0 – Effort required to move the body up the plane neglecting friction

θ – Angle which the line of action of P makes with the weight of the body W

μ – Coefficient of friction between the surfaces of the plane and the body

R_n – Normal reaction

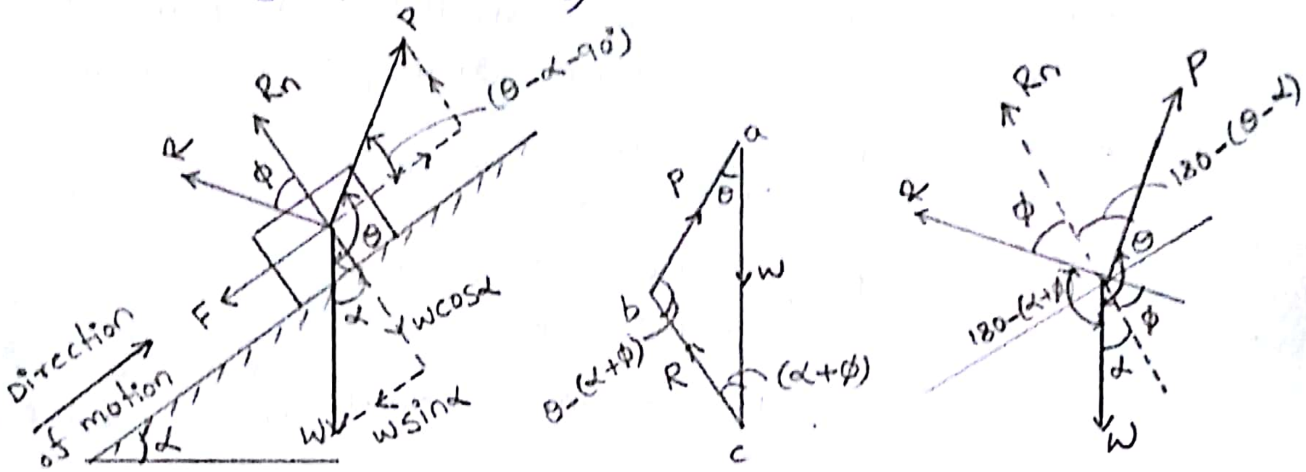
R – Resultant reaction



(a) (b) (c)

When the friction is neglected, the body is in equilibrium under the action of the three forces, i.e. P_0, W & R_n as shown in Fig (a). The triangle of forces is shown in Fig (b). Now applying sine rule for these three concurrent forces

$$\frac{P_0}{\sin\alpha} = \frac{W}{\sin(\theta-\alpha)} \quad (or) \quad P_0 = \frac{W\sin\alpha}{\sin(\theta-\alpha)} \longrightarrow (1)$$



(a) (b) (c)

When friction is taken into account, a frictional force $F = \mu R_n$ acts in the direction opposite to the motion of the body, as shown in Fig (a). The resultant reaction R between the plane and the body is inclined at an angle ϕ with the normal reaction R_n. The triangle of forces is shown in Fig (b). Now applying the sine rule,

$$\frac{P}{\sin(\alpha+\phi)} = \frac{W}{\sin[\theta-(\alpha+\phi)]} \quad (or) \quad P = \frac{W\sin(\alpha+\phi)}{\sin[\theta-(\alpha+\phi)]} \longrightarrow (2)$$

1. The effort P_0 (or P) may also be obtained by applying Lami's theorem

$$\frac{P_0}{\sin(180-\alpha)} = \frac{W}{\sin[180^\circ-(\theta-\alpha)]} \quad (or) \quad \frac{P_0}{\sin\alpha} = \frac{W}{\sin(\theta-\alpha)}$$

2. The effort P_0 (or P) may also be obtained by resolving the forces along the plane and perpendicular to the plane and then applying $\Sigma H = 0$ and $\Sigma V = 0$

Notes:

1. When the effort applied is horizontal, then $\theta = 90^\circ$. In that case, the equations (1) & (2) may be written as

$$P_0 = \frac{W\sin\alpha}{\sin(90-\alpha)} = \frac{W\sin\alpha}{\cos\alpha} = W\tan\alpha$$

$$P = \frac{W\sin(\alpha+\phi)}{\sin[90-(\alpha+\phi)]} = \frac{W\sin(\alpha+\phi)}{\cos(\alpha+\phi)} = W\tan(\alpha+\phi)$$

2. When the effort applied is parallel to the plane, then $\theta = 90^\circ + \alpha$. In that case, the equations ① & ② may be written as

$$P_0 = \frac{W\sin\alpha}{\sin[90+\alpha-\alpha]} = W\sin\alpha$$

$$P = \frac{W\sin(\alpha+\phi)}{\sin[(90+\alpha)-(\alpha+\phi)]} = \frac{W\sin(\alpha+\phi)}{\cos\phi}$$

$$= \frac{W(\sin\alpha\cos\phi + \cos\alpha\sin\phi)}{\cos\phi} = W(\sin\alpha + \cos\alpha\cdot\tan\phi)$$

$$P = W(\sin\alpha + \mu\cos\alpha)$$

Efficiency of Inclined plane

The ratio of the effort required neglecting friction (i.e P_0) to the effort required considering friction (i.e P) is known as efficiency of the inclined plane. Mathematically, efficiency of the inclined plane,

$$\eta = \frac{P_0}{P}$$

For the motion of the body up the plane

$$\text{Efficiency, } \eta = \frac{P_0}{P} = \frac{W\sin\alpha}{\sin(\theta-\alpha)} \times \frac{\sin[\theta-(\alpha+\phi)]}{W\sin(\alpha+\phi)}$$

$$\eta = \frac{\sin\alpha}{\sin\theta\cdot\cos\alpha - \cos\theta\cdot\sin\alpha} \times \frac{\sin\theta\cos(\alpha+\phi) - \cos\theta\cdot\sin(\alpha+\phi)}{\sin(\alpha+\phi)}$$

Multiplying the numerator & denominator by $\sin(\alpha+\phi)\sin\theta$,

$$\eta = \frac{\cot(\alpha+\phi) - \cot\theta}{\cot\alpha - \cot\theta}$$

Notes:

1. When effort is applied horizontally, then $\theta = 90^\circ$

$$\eta = \frac{\tan\alpha}{\tan(\alpha+\phi)}$$

2. When effort is applied parallel to the plane, then $\theta = 90^\circ + \alpha$

$$\eta = \frac{\cot(\alpha+\phi) - \cot(90^\circ+\alpha)}{\cot\alpha - \cot(90^\circ+\alpha)} = \frac{\cot(\alpha+\phi) + \tan\alpha}{\cot\alpha + \tan\alpha} = \frac{\sin\alpha\cdot\cos\phi}{\sin(\alpha+\phi)}$$

Screw Friction

The screws, bolts, studs, nuts etc. are widely used in various machines and structures for temporary fastenings. These fastenings have screw threads, which are made by cutting a continuous helical groove on a cylindrical surface. If the threads are cut on the outer surface of a solid rod, these are known as external threads. But if the threads are cut on the internal surface of a hollow rod, these are known as internal threads. The screw threads are mainly of two types i.e. v-threads & square threads.

The v-threads are stronger and offer more frictional resistance to motion than square threads. Moreover, the v-threads have an advantage of preventing the nut from slackening. In general, the v-threads are used for the purpose of tightening pieces together e.g. bots & nuts etc. But the square threads are used in screw jacks, vice screws etc.

The following terms are important for the study of screw:

Helix : It is the curve traced by a particle, while describing a circular path at a uniform speed and advancing in the axial direction at a uniform rate. In other words, it is the curve traced by a particle while moving along a screw thread.

Pitch : It is the distance from a point of a screw to a corresponding point on the next thread, measured parallel to the axis of the screw.

Lead : It is the distance, a screw thread advances axially in one turn.

Depth of thread : It is the distance between the top & bottom surfaces of a thread (also known as crest & root of a thread)

Single-threaded screw : If the lead of a screw is equal to its pitch, it is known as single-threaded screw.

Multi threaded screw: If more than one thread is cut in one lead distance of a screw, it is known as multi-threaded screw e.g. in a double threaded screw, two threads are cut in one lead length. In such cases, all the threads run independently along the length of the rod. Mathematically,

$$\text{Lead} = \text{Pitch} \times \text{Number of threads}$$

Helix Angle: It is the slope or inclination of the thread with the horizontal. Mathematically,

$$\tan\alpha = \frac{\text{Lead of screw}}{\text{circumference of screw}}$$

$$= \frac{p}{\pi d} \text{ ------ (In single-threaded screw)}$$

$$= \frac{n.p}{\pi d} \text{ ------ (In multi-threaded screw)}$$

where α - Helix angle

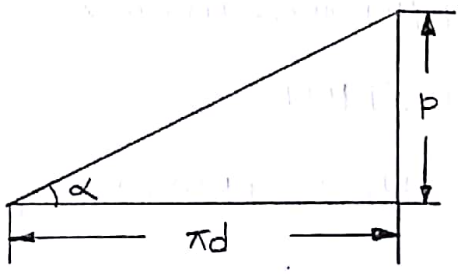
p - pitch of the screw

d - Mean diameter of the screw

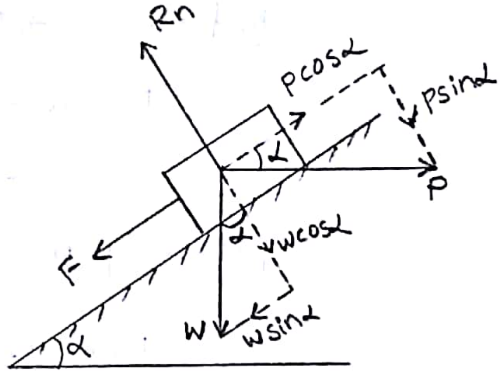
n - Number of threads in one lead

Torque Required to lift the load by a screw jack

If one complete turn of a screw thread by imagined to be unwound, from the body of the screw and developed, it will form an inclined plane as shown in Fig(a)



(a) Development of a screw



(b) Forces acting on the screw

Let p – Pitch of the screw

d – Mean diameter of the screw

α – Helix angle

P – Effort applied at the circumference of the screw to lift the load

W – Load to be lifted

μ – Coefficient of friction, between the screw & nut $= \tan\phi$ where ϕ is the friction angle

From the geometry of the Fig(a), we find that

$$\tan\alpha = \frac{p}{\pi d}$$

Since the principle on which a screw jack works is similar to that of an inclined plane, therefore the force applied on the lever of a screw jack may be considered to be horizontal as shown in Fig(b).

Since the load is being lifted, therefore the force of friction (F) will act downwards. All the forces acting on the screw are shown in Fig(b).

Resolving the forces along the plane,

$$P\cos\alpha = W\sin\alpha + F = W\sin\alpha + \mu R_n \longrightarrow ①$$

and resolving the forces perpendicular to the plane

$$R_n = P\sin\alpha + W\cos\alpha \longrightarrow ②$$

substituting this value of R_n in equation ①

$$P\cos\alpha = W\sin\alpha + \mu(P\sin\alpha + W\cos\alpha)$$

$$= W\sin\alpha + \mu \cdot P\sin\alpha + \mu W\cos\alpha$$

$$P\cos\alpha - \mu P\sin\alpha = W\sin\alpha + \mu \cdot W\cos\alpha$$

$$P(\cos\alpha - \mu\sin\alpha) = W(\sin\alpha + \mu\cos\alpha)$$

$$\therefore \quad P = W \times \frac{\sin\alpha + \mu\cos\alpha}{\cos\alpha - \mu\sin\alpha}$$

substituting the value of $\mu = \tan\phi$ in the above equation,

$$P = W \times \frac{\sin\alpha + \tan\phi\cos\alpha}{\cos\alpha - \tan\phi\sin\alpha}$$

Multiplying the numerator & denominator by $\cos\phi$

$$P = W \times \frac{\sin\alpha\cos\phi + \sin\phi\cos\alpha}{\cos\alpha\cos\phi - \sin\alpha\sin\phi} = W \times \frac{\sin(\alpha+\phi)}{\cos(\alpha+\phi)}$$

$$P = W\tan(\alpha+\phi)$$

$\therefore$ Torque required to overcome friction between the screw & Nut,

$$T_1 = P \times \frac{d}{2} = W\tan(\alpha+\phi)\frac{d}{2}$$

when the axial load is taken ~~by~~ up by a thrust collar or a flat surface, as shown in Fig (b), so that the load does not rotate with the screw, then the torque required to overcome friction at the collar,

$$T_2 = \mu_1 \cdot W\left(\frac{R_1+R_2}{2}\right) = \mu_1 W R$$

where R_1 & R_2 – outside & inside radii of the collar

R – Mean radius of the collar

μ_1 – coefficient of friction for the collar

Total torque required to overcome friction (i.e to rotate the screw),

$$T = T_1 + T_2 = P \times \frac{d}{2} + \mu_1 W R$$

If an effort P_1 is applied at the end of a lever of arm length l, then the total torque required to overcome friction must be equal to the torque applied at the end of the lever, i.e.

$$\therefore T = P \times \frac{d}{2} = P_1 \times l$$

Notes:

1. When the nominal diameter (d_o) and the core diameter (d_c) of the screw thread is given, then the mean diameter of the screw,

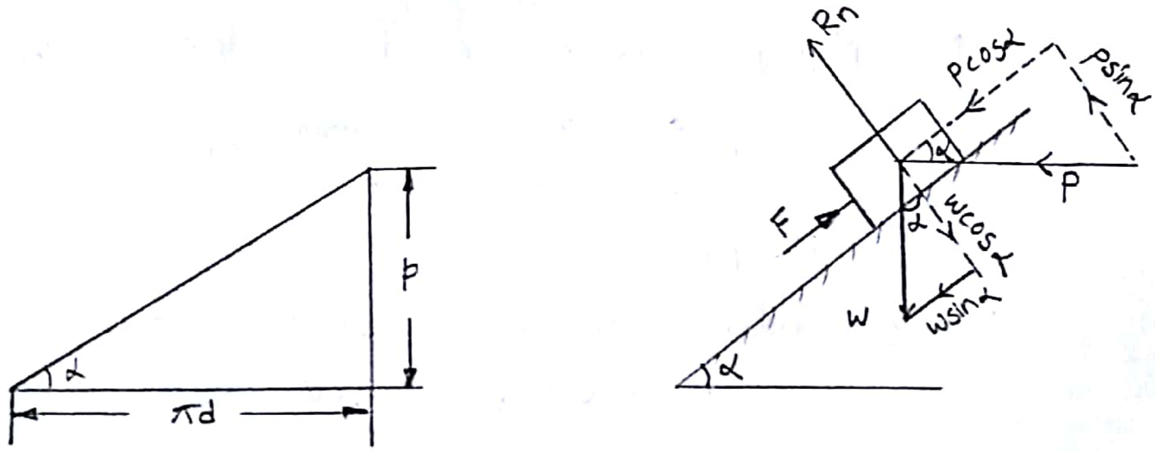
$$d = \frac{d_o + d_c}{2} = d_o - \frac{p}{2} = d_c + \frac{p}{2}$$

2. Since the mechanical advantage is the ratio of load lifted (W) to the effort applied (P_1) at the end of the lever, therefore mechanical advantage,

$$M.A = \frac{W}{P_1} = \frac{W \times 2l}{P \times d} \qquad \because P_1 = \frac{Pd}{2l}$$

$$M.A = \frac{W \times 2l}{W \tan(\alpha+\phi) d} = \frac{2l}{d \cdot \tan(\alpha+\phi)}$$

Torque Required to Lower the load by a Screw Jack



(a) (b)

Let
p – Pitch of the screw
d – Mean diameter of the screw
α – Helix angle
W – weight to be lowered

P- Effort applied at the circumference of the screw to lower the load

μ- coefficient of friction between the screw & nut = $\tan\phi$

From the geometry of the figure, we find that

$$\tan\alpha = \frac{p}{\pi d}$$

Since the load is being lowered, therefore the force of friction will act upwards. All the forces acting on the screw are shown in Fig(b).

Resolving the forces along the plane,

$$P\cos\alpha = F - W\sin\alpha = \mu R_n - W\sin\alpha \longrightarrow ①$$

and resolving the forces perpendicular to the plane,

$$R_n = W\cos\alpha - P\sin\alpha \longrightarrow ②$$

substituting this value of R_n in equation ①

$$P\cos\alpha = \mu(W\cos\alpha - P\sin\alpha) - W\sin\alpha$$

$$= \mu.W\cos\alpha - \mu P\sin\alpha - W\sin\alpha$$

$$P\cos\alpha + \mu.P\sin\alpha = \mu.W\cos\alpha - W\sin\alpha$$

$$P(\cos\alpha + \mu\sin\alpha) = W(\mu\cos\alpha - \sin\alpha)$$

$$P = W \times \frac{\mu\cos\alpha - \sin\alpha}{\cos\alpha + \mu\sin\alpha}$$

substituting the value of $\mu = \tan\phi$ in the above equation,

$$P = W \times \frac{\tan\phi\cos\alpha - \sin\alpha}{\cos\alpha + \tan\phi\sin\alpha}$$

Multiplying the numerator & denominator by $\cos\phi$

$$P = W \times \frac{\sin\phi\cos\alpha - \sin\alpha\cos\phi}{\cos\alpha\cos\phi + \sin\phi\sin\alpha} = W \times \frac{\sin(\phi-\alpha)}{\cos(\phi-\alpha)}$$

$$P = W\tan(\phi - \alpha)$$

$\therefore$ Torque required to overcome friction between the screw & nut

$$T = P \times \frac{d}{2} = W\tan(\phi-\alpha) \times \frac{d}{2}$$

Note:

when $\alpha > \phi$, then $P = \tan(\alpha - \phi)$

Over hauling and self Locking screws

The effort required at the circumference of the screw to lower the load is

$$P = W \tan(\phi - \alpha)$$

and the torque required to lower the load

$$T = P \times \frac{d}{2} = W \tan(\phi - \alpha) \times \frac{d}{2}$$

In the above expression, if $\phi < \alpha$, then torque required to lower the load will be negative. In other words, the load will start moving downward without the application of any torque. such a condition is known as over hauling screws. If however, $\phi > \alpha$, the torque required to lower the load will positive, indicating that an effort is applied to lower the load. such a screw is known as self locking screw. In other words, a screw will be self locking if the friction angle is greater than helix angle or coefficient of friction is greater than tangent of helix angle i.e μ or $\tan\phi > \tan\alpha$.

Efficiency of self Locking screws

we know that efficiency of the screw

$$\eta = \frac{\tan\alpha}{\tan(\alpha + \phi)}$$

and for self locking screws, $\phi \geq \alpha$ or $\alpha \leq \phi$

$\therefore$ Efficiency of self locking screws,

$$\eta \leq \frac{\tan\phi}{\tan(\phi + \phi)} \leq \frac{\tan\phi}{\tan 2\phi} \leq \frac{\tan\phi\,(1 - \tan^2\phi)}{2\tan\phi}$$

$$\leq \frac{1}{2} - \frac{\tan^2\phi}{2} \qquad \tan 2\phi = \frac{2\tan\phi}{1 - \tan^2\phi}$$

From this expression we see that efficiency of self locking screws is less than $\frac{1}{2}$ or 50%. If the efficiency is more than 50%, then the screw is said to be overhauling.

... a square threaded screw jack is 50mm. The pitch of the thread is 10mm. The coefficient of friction is 0.15. What force must be applied at the end of a 0.7 m long lever, which is perpendicular to the longitudinal axis of the screw to raise a load of 20kN and to lower it?

Sol:- $d = 50\,mm = 0.05\,m$, $p = 10\,mm$, $\mu = \tan\phi = 0.15$, $l = 0.7\,m$

$W = 20\,kN = 20\times10^3\,N$

We know that, $\tan\alpha = \frac{p}{\pi d} = \frac{10}{\pi\times 50} = 0.0637$

Let P_1 = Force required at the end of the lever

Force required to raise the load

We know that force required at the circumference of screw

$$P = W\tan(\alpha+\phi) = W\left[\frac{\tan\alpha+\tan\phi}{1-\tan\alpha\tan\phi}\right]$$

$$P = 20\times10^3\left[\frac{0.0637+0.15}{1-0.0637\times0.15}\right] = 4314\,N$$

Now the force required at the end of the lever may be found out by the relation,

$$P_1\times l = P\times\frac{d}{2}$$

$$P_1 = \frac{P\times d}{2l} = \frac{4314\times0.05}{2\times0.7} = 154\,N$$

Force required to lower the load

We know that the force required at the circumference of screw

$$P = W\tan(\phi-\alpha) = W\left[\frac{\tan\phi-\tan\alpha}{1+\tan\phi\tan\alpha}\right]$$

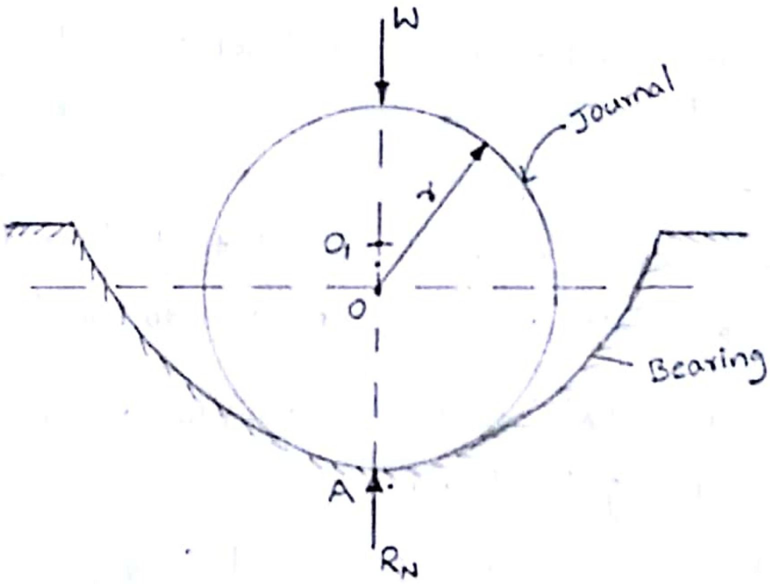
$$P = 20\times10^3\left[\frac{0.15-0.0637}{1+0.15\times0.0637}\right] = 1710\,N$$

Now the force required at the end of the lever may be found out by the relation

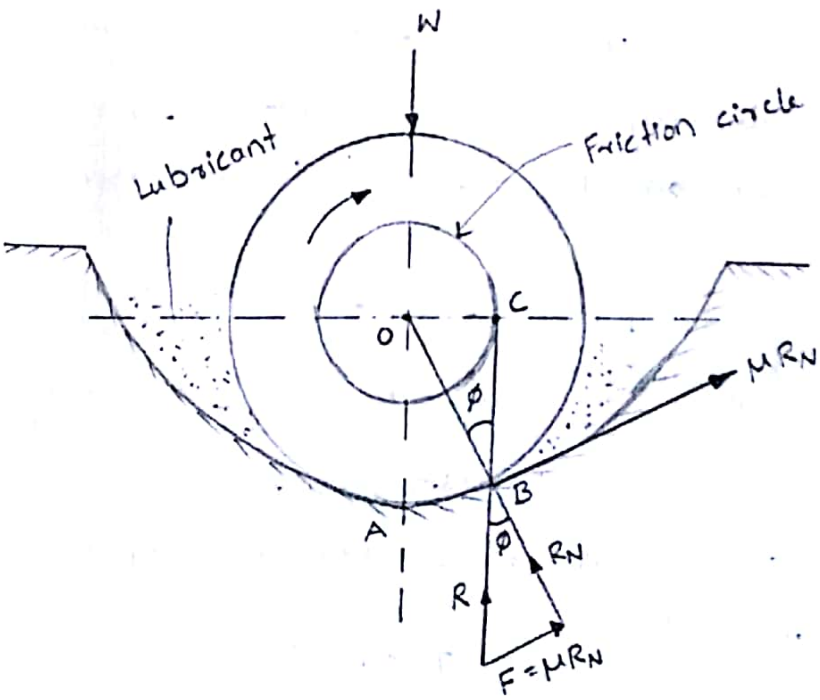
$$P_1\times l = P\times\frac{d}{2} \quad (or) \quad P_1 = \frac{P\times d}{2l} = \frac{1710\times0.05}{2\times0.7} = 61\,N$$

Friction in Journal Bearing – Friction Circle

A journal bearing forms a turning pair as shown in Fig. The fixed pair is called a bearing and that portion of the inner element (i.e shaft) which fits in the bearing is called a journal. The journal is slightly less in diameter than the bearing, in order to permit the free movement of the journal in a bearing.



(a)



(b)

Fig: Friction in journal bearing

When the bearing is not lubricated (or the journal is stationary), then there is a line contact between the two elements as shown in Fig(a).

The load W on the journal and normal reaction R_N (equal to W) of the bearing acts through the centre. The reaction R_N acts vertically upwards at point A. This point A is known as seat or point of pressure.

Now consider a shaft rotating inside a bearing in clockwise direction as shown in Fig (b). The lubricant between the journal and bearing forms a thin layer which gives rise to a greasy friction. Therefore, the reaction R does not act vertically upward, but acts at another point of pressure B. This is due to the fact that when shaft rotates, a frictional force $F = \mu R_N$ acts at the circumference of the shaft which has a tendency to rotate the shaft in opposite direction of motion and this shifts the point A to point B.

In order that the rotation may be maintained, there must be a couple rotating the shaft.

Let ϕ – Angle between R & R_N

μ – Coefficient of friction between the journal & bearing

T – Frictional torque in N-m

r – Radius of the shaft in metres.

For uniform motion, the resultant force acting on the shaft must be zero and the resultant turning moment on the shaft must be zero.

$$R = W \quad \text{and} \quad T = W * OC = W * OB\sin\phi = W \cdot r\sin\phi$$

since ϕ is very small, therefore substituting $\sin\phi = \tan\phi$

$$\therefore \quad T = W \cdot r \cdot \tan\phi = \mu W r$$

If the shaft rotates with angular velocity ω rad/s, then power wasted in friction,

$$P = T\omega = \frac{2\pi N T}{60} \text{ watts}$$

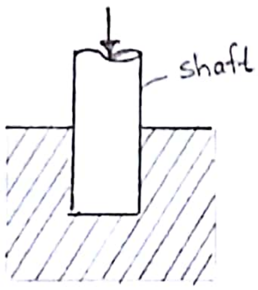
→ If a circle is drawn with centre O & radius $OC = r\sin\phi$, then this circle is called friction circle of a bearing.

Friction of Pivot and Collar Bearing

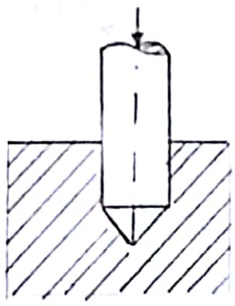
The rotating shafts are frequently subjected to axial thrust. The bearing surfaces such as pivot and collar bearings are used to take this axial thrust of the rotating shaft. The propeller shafts of ships, the shafts of steam turbines, and vertical machine shafts are examples of shafts which carry an axial thrust.

The bearing surfaces placed at the end of a shaft to take the axial thrust are known as pivots. The pivot may have a flat surface or conical surface as shown in Fig (a) and (b). When the cone is truncated, it is then known as truncated or trapezoidal pivot as shown in Fig (c).

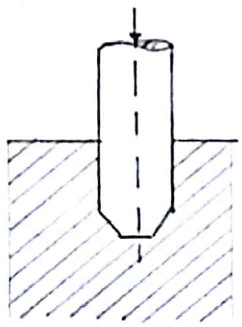
The collar may have flat bearing surface or conical bearing surface, but the flat surface is most commonly used. There may be a single collar, as shown in Fig (d) or several collars along the length of a shaft, as shown in Fig (e) in order to reduce the intensity of pressure.



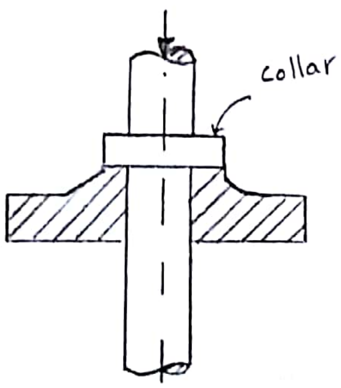
a) Flat pivot



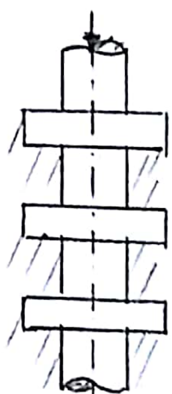
b) Conical pivot



c) Truncated pivot



d) Single flat collar



e) Multiple flat collar

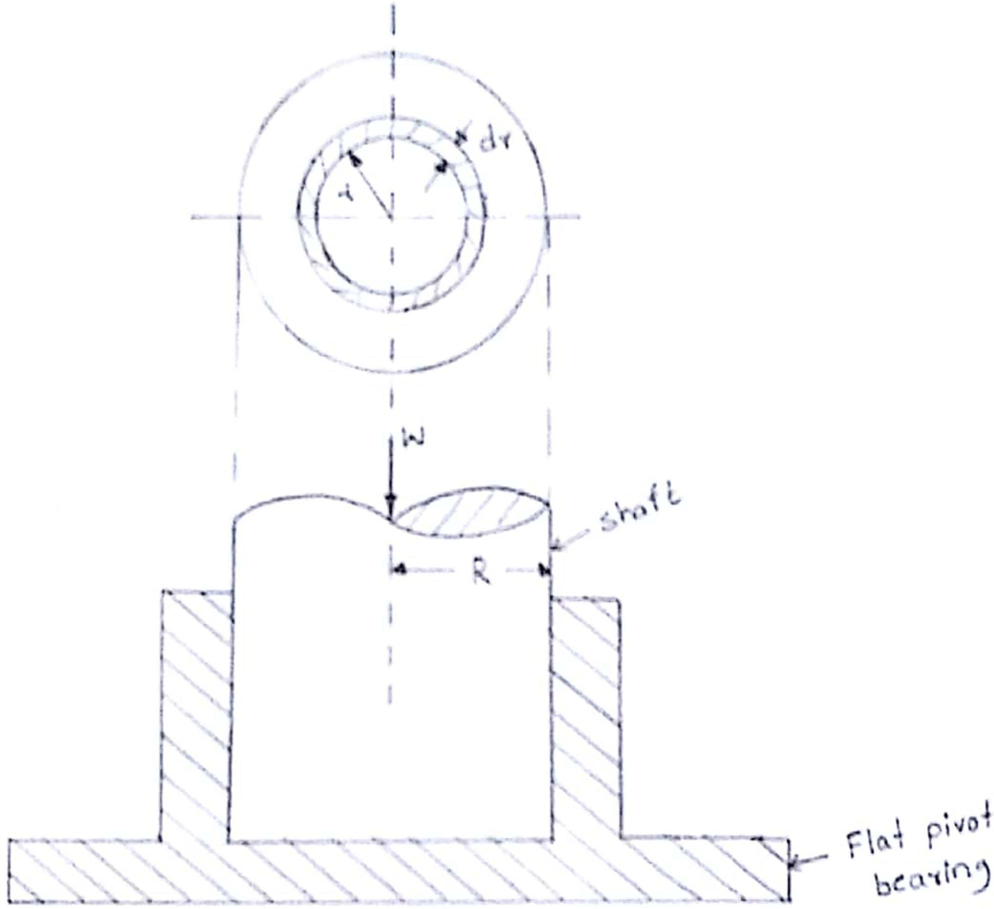
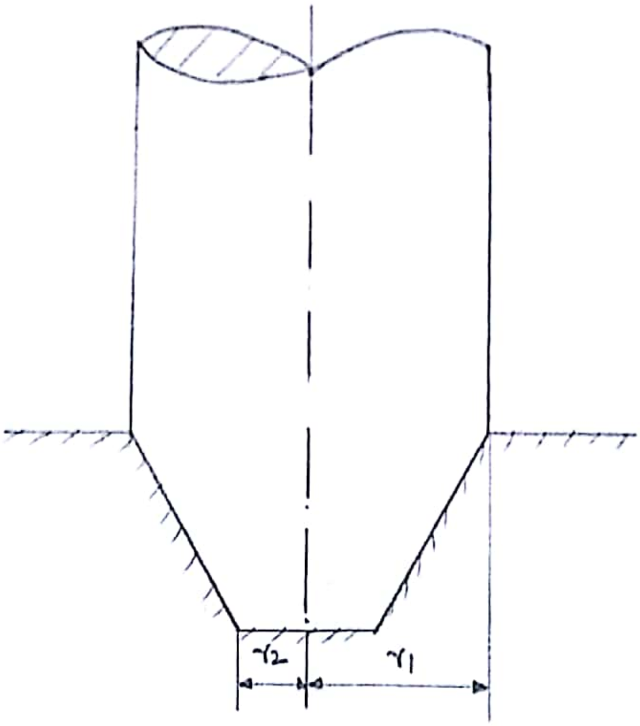
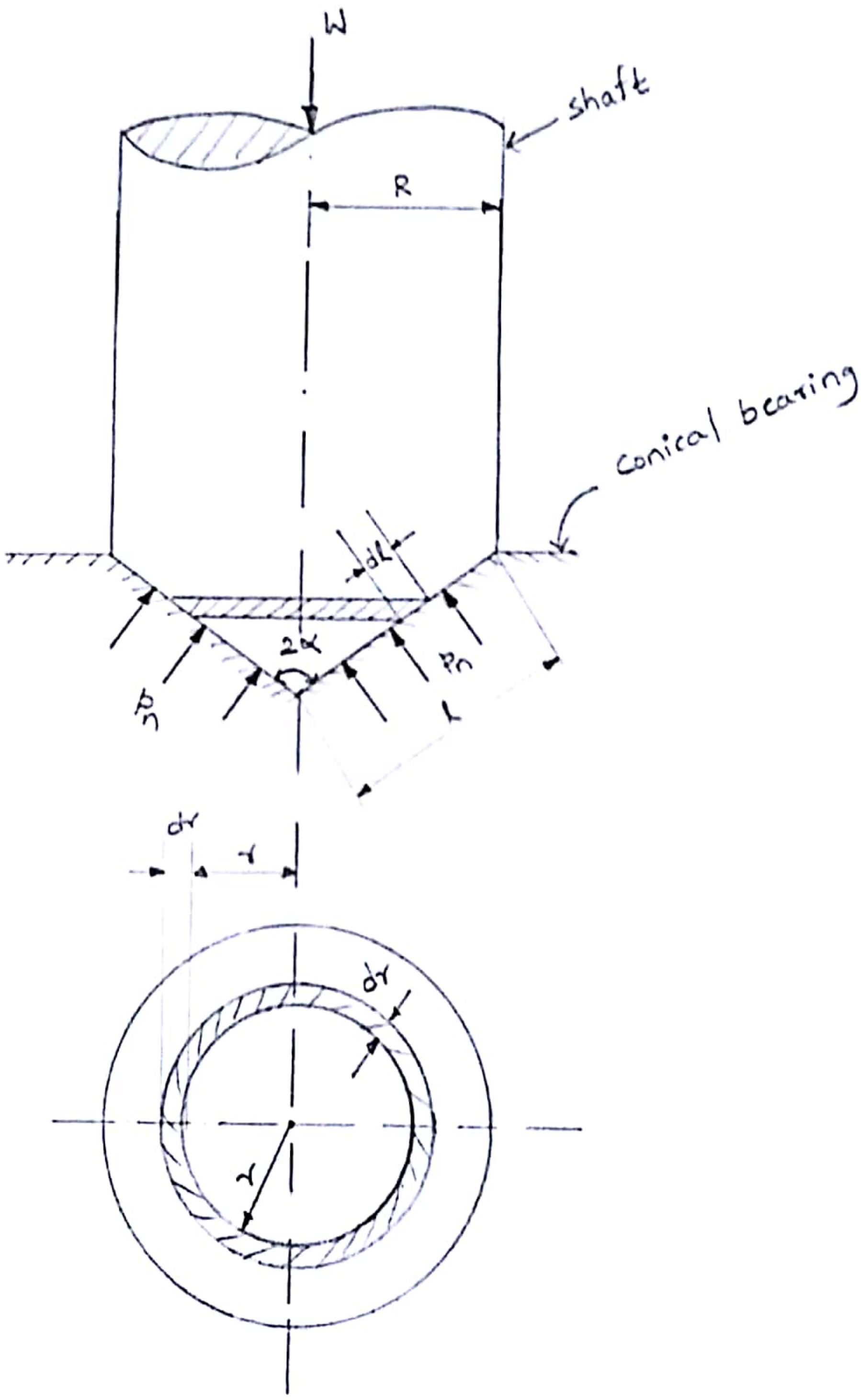


Fig: Foot step bearing

Trapezoidal pivot bearing



Conical Pivot Bearing



Flat Collar Bearing

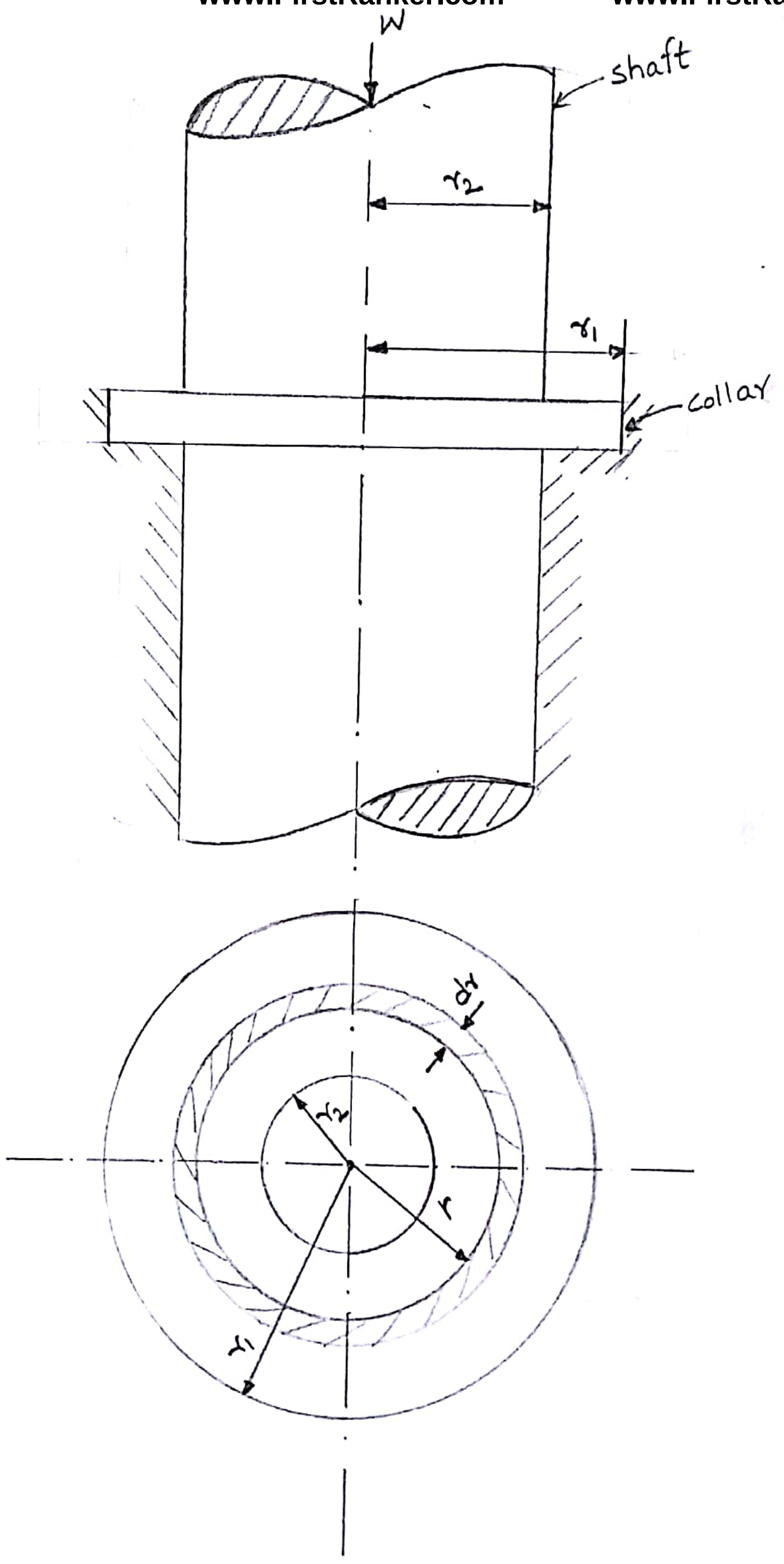


Fig: Single collar bearing

Flat Pivot Bearing

When a vertical shaft rotates in a flat pivot bearing (known as foot step bearing), as shown in Fig, the sliding friction will be along the surface of contact between the shaft & the bearing.

Let W = Load transmitted over the bearing surface

R = Radius of bearing surface

p = Intensity of pressure per unit area of bearing surface between rubbing surfaces, and

μ = Coefficient of friction

We will consider the following two cases:

1. When there is a uniform pressure; and
2. When there is a uniform wear.

Considering uniform pressure

When the pressure is uniformly distributed over the bearing area, then

$$p = \frac{W}{\pi R^2}$$

Consider a ring of radius r and thickness dr of the bearing area.

$\therefore$ Area of bearing surface, $A = 2\pi r.dr$

Load transmitted to the ring,

$$\delta W = p * A = p * 2\pi r \cdot dr \longrightarrow ①$$

Frictional resistance to sliding on the ring acting tangentially at radius r,

$$F_r = \mu \cdot \delta W = \mu * p * 2\pi r \cdot dr$$

$$F_r = 2\pi \mu p r dr$$

$\therefore$ Frictional torque on the ring,

$$T_r = F_r * r = 2\pi\mu p r dr * r = 2\pi\mu p r^2 dr \longrightarrow ②$$

Integrating this equation within the limits from 0 to R for the total frictional torque on the pivot bearing.

$\therefore$ Total frictional torque, $T = \int_0^R 2\pi\mu p r^2 dr$

$$T = 2\pi\mu p \int_0^R r^2 dr$$

$$= 2\pi\mu p \left[\frac{r^3}{3}\right]_0^R = 2\pi\mu p * \frac{R^3}{3} = \frac{2}{3} * \pi\mu p R^3$$

$$\left(\because p = \frac{W}{\pi R^2}\right)$$

$$T = \frac{2}{3} * \pi\mu * \frac{W}{\pi R^2} * R^3 = \frac{2}{3}\mu W R$$

When the shaft rotates at ω rad/s, then power lost in friction,

$$P = T\omega = T * \frac{2\pi N}{60}$$

where N = Speed of shaft in rpm

Considering uniform wear

The rate of wear depends upon the intensity of pressure (p) and the velocity of rubbing surfaces (v). It is assumed that the rate of wear is proportional to the product of intensity of pressure and the velocity of rubbing surfaces (i.e $p.v$). Since the velocity of rubbing surfaces increases with the distance (i.e. radius r) from the axis of the bearing, therefore for uniform wear

$$p \cdot r = c \text{ (a constant)}$$

$$p = \frac{c}{r}$$

and the load transmitted to the ring,

$$\delta W = p \times 2\pi r \cdot dr \longrightarrow \text{(from equation ①)}$$

$$= \frac{C}{r} \times 2\pi r \cdot dr = 2\pi C \cdot dr$$

∴ Total load transmitted to the ~~ring~~ bearing

$$W = \int_0^R 2\pi C dr = 2\pi C [r]_0^R = 2\pi C R$$

$$C = \frac{W}{2\pi R}$$

we know that frictional torque acting on the ring,

$$T_r = 2\pi\mu p r^2 dr = 2\pi\mu \times \frac{C}{r} \times r^2 dr \quad (\because p = \frac{C}{r})$$

$$T_r = 2\pi\mu C \cdot r \cdot dr \longrightarrow ③$$

∴ Total frictional torque on the bearing,

$$T = \int_0^R 2\pi\mu C r dr = 2\pi\mu C \left[\frac{r^2}{2}\right]_0^R$$

$$= 2\pi\mu C \times \frac{R^2}{2} = \pi\mu C R^2 \quad (\because C = \frac{W}{2\pi R})$$

$$T = \pi\mu \times \frac{W}{2\pi R} \times R^2 = \frac{1}{2} \times \mu W R$$

Conical Pivot Bearing

The conical pivot bearing supporting a shaft carrying a load W is shown in Fig.

Let. p_n - Intensity of pressure normal to the cone,

α - Semi angle of the cone,

μ - coefficient of friction between the shaft & the bearing

R - Radius of the shaft

consider a small ring of radius r and thickness dr. Let dl is the length of ring along the cone, such that

$$dl = dr \cdot \operatorname{cosec}\alpha$$

Area of the ring, $(\because dl = dr \operatorname{cosec}\alpha)$

$$A = 2\pi r \cdot dl = 2\pi r \cdot dr \operatorname{cosec}\alpha$$

considering uniform pressure

We know that normal load acting on the ring,

$$\delta W_n = \text{Normal pressure} * \text{Area}$$

$$= p_n * 2\pi r \cdot dr \operatorname{cosec}\alpha$$

and vertical load acting on the ring,

$$\delta W = \text{Vertical component of } \delta W_n = \delta W_n \sin\alpha$$

$$= p_n * 2\pi r \cdot dr \operatorname{cosec}\alpha \cdot \sin\alpha = p_n * 2\pi r \cdot dr$$

$\therefore$ Total vertical load transmitted to the bearing,

$$W = \int_0^R p_n * 2\pi r \cdot dr = 2\pi p_n \left[\frac{r^2}{2}\right]_0^R = 2\pi p_n * \frac{R^2}{2} = \pi R^2 * p_n$$

$$p_n = \frac{W}{\pi R^2}$$

We know that frictional force on the ring acting tangentially at radius r,

$$F_r = \mu \cdot \delta W_n = \mu \cdot p_n \cdot 2\pi r \cdot dr \operatorname{cosec}\alpha = 2\pi\mu \cdot p_n \operatorname{cosec}\alpha \cdot r\,dr$$

and frictional torque acting on the ring,

$$T_r = P_r \times r$$

$$= 2\pi\mu \cdot p_n \operatorname{cosec}\alpha \cdot r\,dr \times r = 2\pi\mu \cdot p_n \operatorname{cosec}\alpha \cdot r^2\,dr$$

Integrating the expression within the limits from 0 to R for the total frictional torque on the conical pivot bearing.

$\therefore$ Total frictional torque,

$$T = \int_0^R 2\pi\mu\, p_n \operatorname{cosec}\alpha \cdot r^2\,dr = 2\pi\mu \cdot p_n \operatorname{cosec}\alpha \left[\frac{r^3}{3}\right]_0^R$$

$$T = 2\pi\mu \cdot p_n \cdot \operatorname{cosec}\alpha \times \frac{R^3}{3} = \frac{2\pi R^3}{3} \times \mu \cdot p_n \cdot \operatorname{cosec}\alpha \longrightarrow (1)$$

substituting the value of p_n in equation (1)

$$T = \frac{2\pi R^3}{3} \times \mu \times \frac{W}{\pi R^2} \times \operatorname{cosec}\alpha = \frac{2}{3} \times \mu W R \operatorname{cosec}\alpha$$

Note: If slant length (l) of the cone is known, then

$$T = \frac{2}{3}\mu W l \qquad (\because\ l = R\operatorname{cosec}\alpha)$$

Considering uniform wear

In Fig, let p_r be the normal intensity of pressure at a distance r from the central axis. We know that, in case of uniform wear, the intensity of pressure varies inversely with the distance.

$$\therefore\ p_r \cdot r = C \quad (\text{a constant}) \quad \text{or} \quad p_r = \frac{C}{r}$$

and the load transmitted to the ring,

$$\delta W = p_r \times 2\pi r \cdot dr = \frac{C}{r} \times 2\pi r \cdot dr = 2\pi C \cdot dr$$

$\therefore$ Total load transmitted to the bearing,

$$W = \int_0^R 2\pi C \cdot dr = 2\pi C\,[r]_0^R = 2\pi C \cdot R \quad (\text{or}) \quad C = \frac{W}{2\pi R}$$

we know that frictional torque acting on the ring,

$$T_r = 2\pi\mu \cdot p_r \operatorname{cosec}\alpha \cdot r^2 \cdot dr = 2\pi\mu \times \frac{C}{r} \times \operatorname{cosec}\alpha \cdot r^2 \cdot dr$$

$$= 2\pi\mu \cdot C \cdot \operatorname{cosec}\alpha \cdot r \cdot dr$$

$\therefore$ Total frictional torque acting on the bearing,

$$T = \int_0^R 2\pi\mu \cdot C \cdot \operatorname{cosec}\alpha \cdot r \cdot dr = 2\pi\mu C \cdot \operatorname{cosec}\alpha \left[\frac{r^2}{2}\right]_0^R$$

$$T = 2\pi\mu \cdot c \cdot \operatorname{cosec}\alpha * \frac{R^2}{2} = \pi\mu \cdot C \cdot \operatorname{cosec}\alpha \cdot R^2$$

substituting the value of C, we have

$$T = \pi\mu * \frac{W}{2\pi R} * \operatorname{cosec}\alpha \cdot R^2 = \frac{1}{2} * \mu W R \cdot \operatorname{cosec}\alpha$$

$$T = \frac{1}{2} * \mu W \cdot l$$

Trapezoidal or truncated conical Pivot Bearing

If the pivot bearing is not conical, but a frustrum of a cone with r_1 & r_2, the external and internal radius as shown in Fig, then

Area of the bearing surface,

$$A = \pi[(r_1)^2 - (r_2)^2]$$

$\therefore$ Intensity of uniform pressure,

$$p_n = \frac{W}{A} = \frac{W}{\pi[(r_1)^2 - (r_2)^2]} \longrightarrow ①$$

Considering uniform pressure

The total torque acting on the bearing is obtained by integrating the value of T_r within the limits r_1 & r_2

$\therefore$ Total torque acting on the bearing,

$$T = \int_{r_2}^{r_1} 2\pi\mu \cdot p_n \operatorname{cosec}\alpha \cdot r^2 \cdot dr = 2\pi\mu \cdot p_n \operatorname{cosec}\alpha \left[\frac{r^3}{3}\right]_{r_2}^{r_1}$$

$$= 2\pi\mu \cdot p_n \cdot \operatorname{cosec}\alpha \left[\frac{r_1^3 - r_2^3}{3}\right]$$

substituting the value of p_n from equation ①

$$T = 2\pi\mu * \frac{W}{\pi[r_1^2 - r_2^2]} * \operatorname{cosec}\alpha \left[\frac{r_1^3 - r_2^3}{3}\right]$$

$$T = \frac{2}{3} * \mu . W . \operatorname{cosec}\alpha \left[\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right]$$

Considering uniform wear

The load transmitted to the ring,

$$\delta W = 2\pi C . dr$$

∴ Total load transmitted to the ring,

$$W = \int_{r_2}^{r_1} 2\pi C\, dr = 2\pi C \left[r\right]_{r_2}^{r_1} = 2\pi C (r_1 - r_2)$$

$$C = \frac{W}{2\pi(r_1 - r_2)} \longrightarrow ②$$

we know that the torque acting on the ring, considering uniform wear, is

$$T_r = 2\pi\mu . C . \operatorname{cosec}\alpha . r . dr$$

∴ Total torque acting on the bearing,

$$T = \int_{r_2}^{r_1} 2\pi\mu . C . \operatorname{cosec}\alpha . r . dr = 2\pi\mu . C . \operatorname{cosec}\alpha \left[\frac{r^2}{2}\right]_{r_2}^{r_1}$$

$$T = \pi\mu C . \operatorname{cosec}\alpha \left[r_1^2 - r_2^2\right]$$

substituting the value of C from equation ②

$$T = \pi\mu * \frac{W}{2\pi(r_1 - r_2)} * \operatorname{cosec}\alpha . \left[r_1^2 - r_2^2\right]$$

$$T = \frac{1}{2} * \mu W (r_1 + r_2) \operatorname{cosec}\alpha = \mu W R \operatorname{cosec}\alpha$$

where R = Mean radius of the bearing $= \frac{r_1 + r_2}{2}$

A shaft has a number of a collars integral with it. The external diameter of the collars is 400 mm and the shaft diameter is 250mm If the intensity of pressure is 0.35 N/mm² (uniform) and the coefficient of friction is 0.05, estimate: 1. power absorbed when the shaft runs at 105 rpm. carrying a load of 150kN; and 2. number of collars required.

Sol:-

$d_1 = 400$ mm (or) $r_1 = 200$ mm, $p = 0.35$ N/mm², $\mu = 0.05$

$d_2 = 250$ mm (or) $r_2 = 125$ mm, $W = 150$kN $= 150 \times 10^3$ N

$N = 105$ rpm (or) $\omega = \frac{2\pi N}{60} = \frac{2\pi \times 105}{60} = 11$ rad/s

1. Power absorbed

We know that for uniform pressure, total frictional torque transmitted,

$$T = \frac{2}{3}\mu W\left[\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right] = \frac{2}{3} \times 0.05 \times 150 \times 10^3\left[\frac{200^3 - 125^3}{200^2 - 125^2}\right]$$

$$T = 5000 \times 248 = 1240 \times 10^3 \text{ N-mm}$$

$$T = 1240 \text{ N-m}$$

$\therefore$ Power absorbed, $P = T\omega$

$$= 1240 \times 11 = 13640 \text{ W} = 13.64 \text{ kW}$$

2. Number of collars required

Let n = Number of collars required

We know that the intensity of uniform pressure (p)

$$p = \frac{W}{n \cdot \pi[r_1^2 - r_2^2]} = \frac{150 \times 10^3}{n \cdot \pi[200^2 - 125^2]} = \frac{1.96}{n}$$

$$n = \frac{1.96}{0.35} = 5.6 \approx 6$$

The thrust of a propeller shaft in a marine engine is taken up by a number of collars integral with the shaft which is 300 mm in diameter. The thrust on the shaft is 200kN and the speed is 75 rpm. Taking μ constant and equal to 0.05 and assuming intensity of pressure as uniform and equal to 0.3 N/mm², find the external diameter of the collars and the number of collars required, if the power lost in friction is not to exceed 16 kW.

Sol:-

$d_2 = 300$ mm or $r_2 = 150$ mm $= 0.15$ m

$W = 200$ kN $= 200 \times 10^3$ N, $\mu = 0.05$, $p = 0.3$ N/mm²

$P = 16$ kW $= 16 \times 10^3$ W, $N = 75$ rpm (or) $\omega = \frac{2\pi N}{60} = \frac{2\pi \times 75}{60} = 7.86$ rad/s

Let T = Total frictional torque transmitted in N-m

we know that power lost in friction, $P = T \times \omega$

$$16 \times 10^3 = T \times 7.86$$

$$T = 2036 \text{ N-m}$$

External diameter of the collar

Let d_1 = External diameter of the collar in meters $= 2r_1$

we know that for uniform pressure, total frictional torque

$$T = \frac{2}{3}\mu W\left[\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right] = \frac{2}{3}\mu W\left[\frac{r_1^2 + r_2^2 + r_1 r_2}{r_1 + r_2}\right]$$

$$2036 = \frac{2}{3} \times 0.05 \times 200 \times 10^3\left[\frac{r_1^2 + (0.15)^2 + r_1 \times 0.15}{r_1 + 0.15}\right]$$

$$2036 \times 3\,(r_1 + 0.15) = 20 \times 10^3\left[r_1^2 + 0.15 r_1 + 0.0225\right]$$

Dividing throughout by 20×10^3

$$0.305\,(r_1 + 0.15) = r_1^2 + 0.15 r_1 + 0.0225$$

$$r_1^2 - 0.155 r_1 - 0.0233 = 0$$

Solving this as a quadratic equation

$$r_1 = \frac{0.155 \pm \sqrt{(0.155)^2 + 4 \times 0.0233}}{2} = \frac{0.155 \pm 0.342}{2}$$

$r_1 = 0.2485$ m $= 248.5$ mm

$\therefore d_1 = 2r_1 = 2 \times 248.5 = 497$ mm

Number of collars

Let n = Number of collars

we know that intensity of pressure (p),

$$p = \frac{W}{n\pi\left[r_1^2 - r_2^2\right]}$$

$$0.3 = \frac{200 \times 10^3}{n\pi\left[(248.5)^2 - (150)^2\right]} = \frac{1.62}{n}$$

$$n = \frac{1.62}{0.3} = 5.4 \text{ (or) } 6$$

Friction clutches

A friction clutch has its principal application in the transmission of power of shafts and machines which must be started and stopped frequently. Its application is also found in cases in which power is to be delivered to machines partially or fully loaded. The force of friction is used to start the driven shaft from rest and gradually brings it up to the proper speed without excessive slipping of the friction surfaces. In automobiles, friction clutch is used to connect the engine to the driven shaft. In operating such a clutch, care should be taken so that the friction surfaces engage easily and gradually brings the driven shaft up to proper speed. The proper alignment of the bearing must be maintained and it should be located as close to the clutch as possible. It may be noted that

1. The contact surfaces should develop a frictional force that may pick up and hold the load with reasonably low pressure between the contact surfaces.

2. The heat of friction should be rapidly dissipated and tendency to grab should be at a minimum.

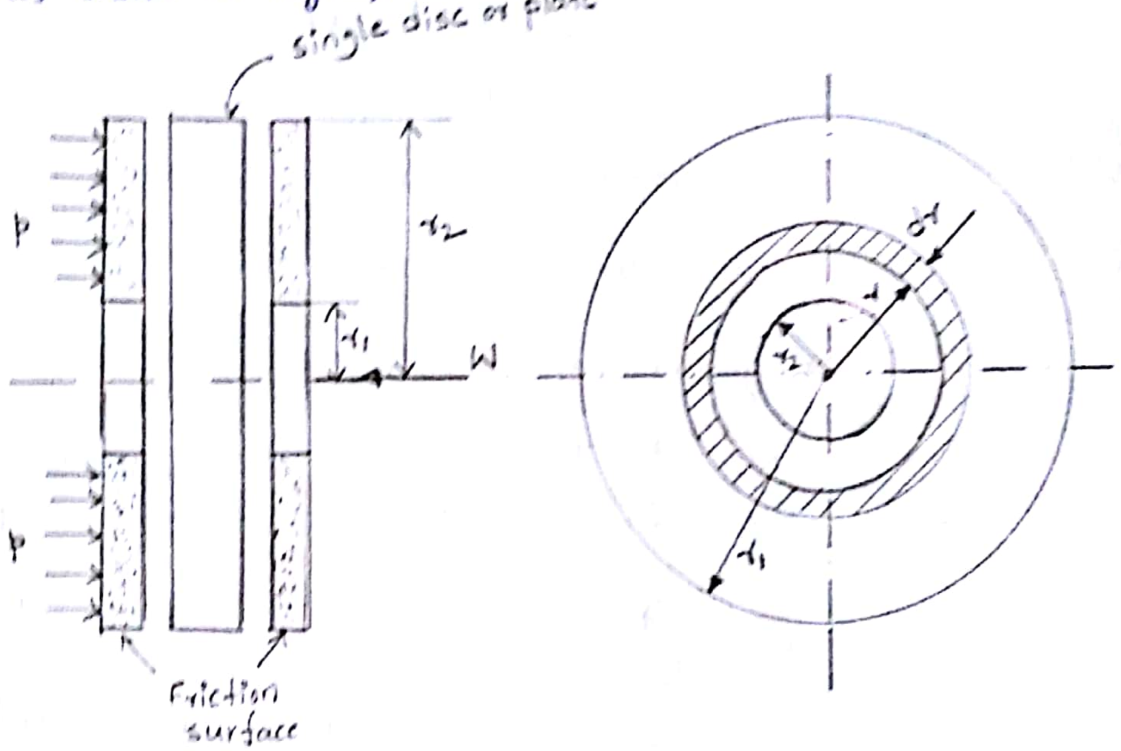
3. The surfaces should be backed by a material stiff enough to ensure a reasonably uniform distribution of pressure.

The friction clutches of the following types

1. Disc or plate clutches (single disc or multiple disc clutch)
2. Cone clutches
3. Centrifugal clutches

The disc and cone clutches are based on the same theory as the pivot & collar bearings.

Now consider two friction surfaces, maintained in contact by an axial thrust W as shown in Fig(a)



Let T = Torque transmitted by the clutch

p = Intensity of axial pressure with which the contact surfaces are held together

r_1 & r_2 = External & Internal radii of friction faces

μ = co-efficient of friction

Consider an elementary ring of radius r and thickness dr as shown in Fig(b).

We know that area of contact surface or friction surface

$= 2\pi r \cdot dr$

$\therefore$ Normal or axial force on the ring

δW = Pressure × Area

$\delta W = p \times 2\pi r \cdot dr$

and the frictional force on the ring acting tangentially at radius r,

$F_r = \mu \cdot \delta W = \mu \times p \times 2\pi r \cdot dr$

Frictional torque acting on the ring

$$T_r = F_r \times r = \mu \times p \times 2\pi r \cdot dr \times r$$

$$T_r = 2\pi \mu \cdot p \cdot r^2 \, dr$$

We shall now consider the following two cases:

1. when there is a uniform pressure
2. when there is a uniform wear

Considering uniform pressure

When the pressure is uniformly distributed over the entire area of the friction face, then the intensity of pressure,

$$p = \frac{W}{\pi[r_1^2 - r_2^2]} \longrightarrow (1)$$

where W = Axial thrust with which the contact or friction surfaces are held together

The frictional torque on the elementary ring of radius r and thickness dr is

$$T_r = 2\pi \mu p r^2 \, dr$$

Integrating this equation with the limits r_2 to r_1 for the total frictional torque.

∴ Total frictional torque acting on the friction surface or on the clutch

$$T = \int_{r_2}^{r_1} 2\pi \mu p r^2 \, dr = 2\pi \mu p \left[\frac{r^3}{3}\right]_{r_1}^{r_2} = \frac{2}{3}\pi \mu p [r_1^3 - r_2^3]$$

substituting the value of p from equation (1)

$$T = \frac{2}{3}\pi \mu \times \frac{W}{\pi[r_1^2 - r_2^2]} \times [r_1^3 - r_2^3] = \frac{2}{3}\mu W\left[\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right] = \frac{2}{3}\mu W R$$

$$R = \text{Mean radius of friction surface} = \frac{2}{3}\left[\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right]$$

Considering uniform wear

Let p be the normal intensity of pressure at a distance r from the axis of the clutch. Since the intensity of pressure varies inversely with the distance, therefore

$$p.r = c \quad \text{or} \quad p = c/r$$

and the normal force on the ring,

$$\delta W = p \times 2\pi r\, dr = \frac{c}{r} \times 2\pi r\, dr = 2\pi c\, dr$$

$\therefore$ Total force acting on the friction surface

$$W = \int_{r_2}^{r_1} 2\pi c\, dr = 2\pi c\,[r]_{r_2}^{r_1} = 2\pi c\,[r_1 - r_2]$$

$$c = \frac{W}{2\pi(r_1 - r_2)}$$

we know that the frictional torque acting on the ring

$$T_r = 2\pi \mu p r^2 dr = 2\pi\mu \times \frac{c}{r} \times r^2 dr = 2\pi\mu c r\, dr$$

$\therefore$ Total frictional torque on the friction surface,

$$T = \int_{r_2}^{r_1} 2\pi\mu c r\, dr = 2\pi\mu c \left[\frac{r^2}{2}\right]_{r_2}^{r_1} = 2\pi\mu c \left[\frac{r_1^2 - r_2^2}{2}\right]$$

$$= \pi\mu c\,[r_1^2 - r_2^2] = \pi\mu \times \frac{W}{2\pi(r_1 - r_2)}\,[r_1^2 - r_2^2]$$

$$T = \frac{1}{2}\mu W (r_1 + r_2) = \mu W R$$

where R = Mean radius of the friction surface $= \frac{r_1 + r_2}{2}$

F

Consider a pair of friction surface as shown in fig(a). Since the area of contact of a pair of friction surface is a frustrum of a cone, therefore the torque transmitted by the cone clutch may be determined in the similar manner as discussed for conical pivot bearings.

Let p_n – Intensity of pressure with which the conical friction surfaces are held together (i.e normal pressure between contact surfaces),

r_1 & r_2 – Outer & Inner radius of friction surfaces

R – Mean radius of the friction surface $= \frac{r_1 + r_2}{2}$

α – semi angle of the cone (also called face angle of the cone) or the angle of the friction surface with the axis of the clutch

μ – Coefficient of friction between contact surfaces

b – width of the contact surfaces (also known as face width or clutch face)

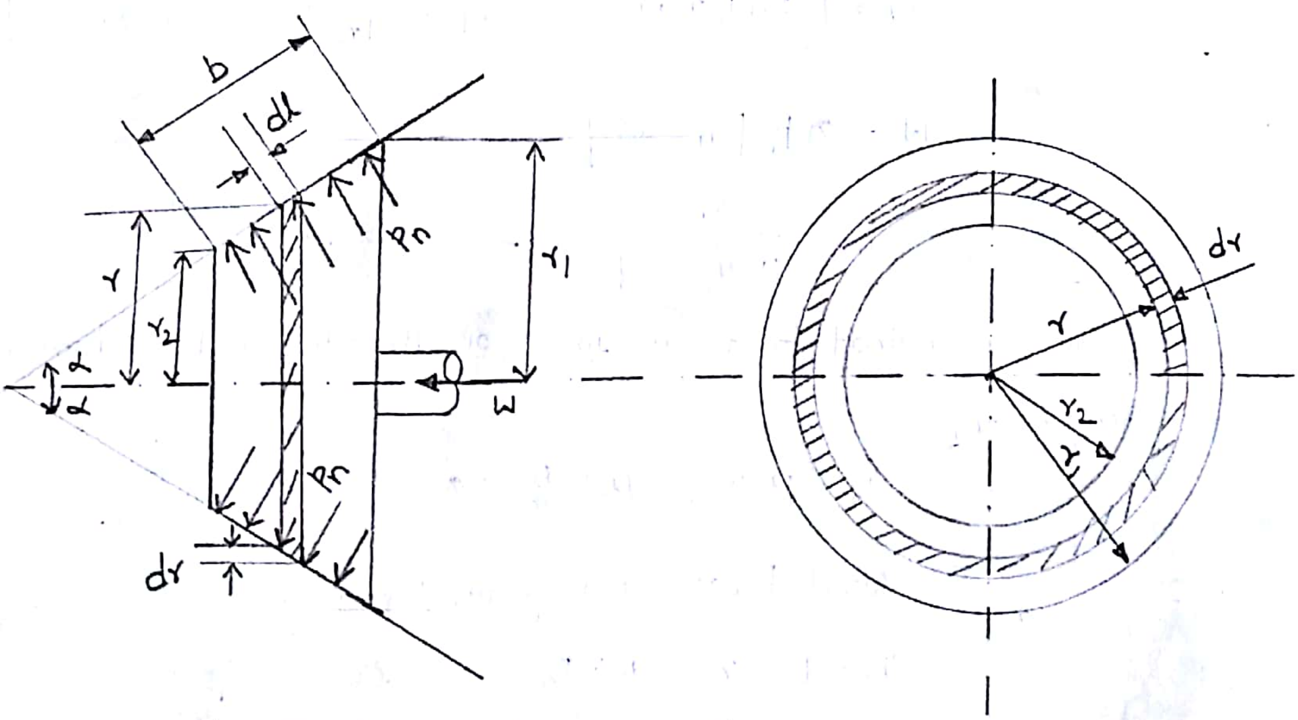


Fig: Friction surfaces as a frustrum of a cone

Consider a small ring of radius r and thickness dr, as shown in Fig(b). Let dl is length of ring of the friction surface, such that

$$dl = dr \cdot \operatorname{cosec}\alpha$$

$\therefore$ Area of the ring, $A = 2\pi r \cdot dl = 2\pi r\, dr \operatorname{cosec}\alpha$

we shall consider the following two cases:

1. When there is a uniform pressure
2. when there is a uniform wear

Considering uniform pressure

We know that normal load acting on the ring

δw_n = Normal pressure $*$ Area of ring

$$\delta w_n = p_n * 2\pi r\, dr \operatorname{cosec}\alpha$$

and the axial load acting on the ring,

δw = Horizontal component of δw_n (i.e. in the direction of W)

$$\delta w = \delta w_n \sin\alpha = p_n * 2\pi r\, dr \operatorname{cosec}\alpha \cdot \sin\alpha = 2\pi p_n r\, dr$$

$\therefore$ Total axial load transmitted to the clutch or the axial spring force required,

$$W = \int_{r_2}^{r_1} 2\pi p_n r\, dr = 2\pi p_n \left[\frac{r^2}{2}\right]_{r_2}^{r_1} = 2\pi p_n \left[\frac{r_1^2 - r_2^2}{2}\right]$$

$$W = \pi p_n \left[r_1^2 - r_2^2\right]$$

$$\therefore p_n = \frac{W}{\pi\left[r_1^2 - r_2^2\right]} \longrightarrow (1)$$

we know that frictional force on the ring acting tangentially at radius r,

$$F_r = \mu \delta w_n = \mu * p_n * 2\pi r\, dr \operatorname{cosec}\alpha$$

$\therefore$ Frictional torque acting on the ring,

$$T_r = F_r * r = \mu * p_n * 2\pi r\, dr \operatorname{cosec}\alpha * r$$

$$T_r = 2\pi \mu p_n \operatorname{cosec}\alpha \cdot r^2\, dr$$

Integrating this expression within the limits from r_2 to r_1 for the total frictional torque on the clutch.

$\therefore$ Total frictional torque,

$$T = \int_{r_2}^{r_1} 2\pi\mu p_n \operatorname{cosec}\alpha \cdot r^2 dr = 2\pi\mu p_n \operatorname{cosec}\alpha \left[\frac{r^3}{3}\right]_{r_2}^{r_1}$$

$$T = 2\pi\mu p_n \operatorname{cosec}\alpha \left[\frac{r_1^3 - r_2^3}{3}\right]$$

substituting the value of p_n from equation ①, we get

$$T = 2\pi\mu * \frac{W}{\pi[r_1^2 - r_2^2]} * \operatorname{cosec}\alpha \left[\frac{r_1^3 - r_2^3}{3}\right]$$

$$T = \frac{2}{3}\mu W \operatorname{cosec}\alpha \left[\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right] \longrightarrow ②$$

Considering uniform wear

Let p_r be the normal intensity of pressure at a distance r from the axis of the clutch. We know that, in case of uniform wear, the intensity of pressure varies inversely with the distance.

$$p_r \times r = c \quad (\text{or}) \quad p_r = \frac{c}{r}$$

we know that the normal load acting on the ring,

$$\delta W_n = \text{Normal pressure} * \text{Area of ring}$$

$$= p_r * 2\pi r dr \operatorname{cosec}\alpha$$

and axial load acting on the ring

$$\delta W = \delta W_n \sin\alpha = p_r * 2\pi r dr \operatorname{cosec}\alpha * \sin\alpha = p_r * 2\pi r dr$$

$$\delta W = \frac{c}{r} 2\pi r dr = 2\pi c\, dr$$

$\therefore$ Total axial load transmitted to the clutch,

$$W = \int_{r_2}^{r_1} 2\pi c\, dr = 2\pi c [r]_{r_2}^{r_1} = 2\pi c[r_1 - r_2]$$

$$c = \frac{W}{2\pi[r_1 - r_2]} \longrightarrow ③$$

we know that frictional force acting on the ring,

$$F_r = \mu \delta W_n = \mu * p_r * 2\pi r dr \operatorname{cosec}\alpha$$

and frictional torque acting on the ring,

$$T_r = F_r \times r = \mu \times P_r \times 2\pi r\,dr\,\mathrm{cosec}\alpha \times r$$

$$= \mu \times \frac{C}{r} \times 2\pi r^2\,dr\,\mathrm{cosec}\alpha$$

$$T_r = 2\pi\mu C.\,\mathrm{cosec}\alpha.\,r\,dr$$

∴ Total frictional torque acting on the clutch

$$T = \int_{r_2}^{r_1} 2\pi\mu C.\,\mathrm{cosec}\alpha.\,r\,dr = 2\pi\mu C.\,\mathrm{cosec}\alpha\left[\frac{r^2}{2}\right]_{r_2}^{r_1}$$

$$T = 2\pi\mu C.\,\mathrm{cosec}\alpha\left[\frac{r_1^2 - r_2^2}{2}\right]$$

substituting the value of C from equation ③, we have

$$T = 2\pi\mu \times \frac{W}{2\pi[r_1 - r_2]} \times \mathrm{cosec}\alpha\left[\frac{r_1^2 - r_2^2}{2}\right]$$

$$T = \mu W.\,\mathrm{cosec}\alpha\left[\frac{r_1 + r_2}{2}\right] = \mu W R\,\mathrm{cosec}\alpha \longrightarrow ④$$

where $R = \frac{r_1 + r_2}{2}$ = Mean radius of friction surface

since the normal force acting on the friction surface $W_n = \frac{W}{\sin\alpha}$

therefore the equation ④ may be written as

$$T = \mu W_n R \longrightarrow ⑤$$

In order to determine the mass and size of shoes, the following procedure is adopted :

1. Mass of the shoes

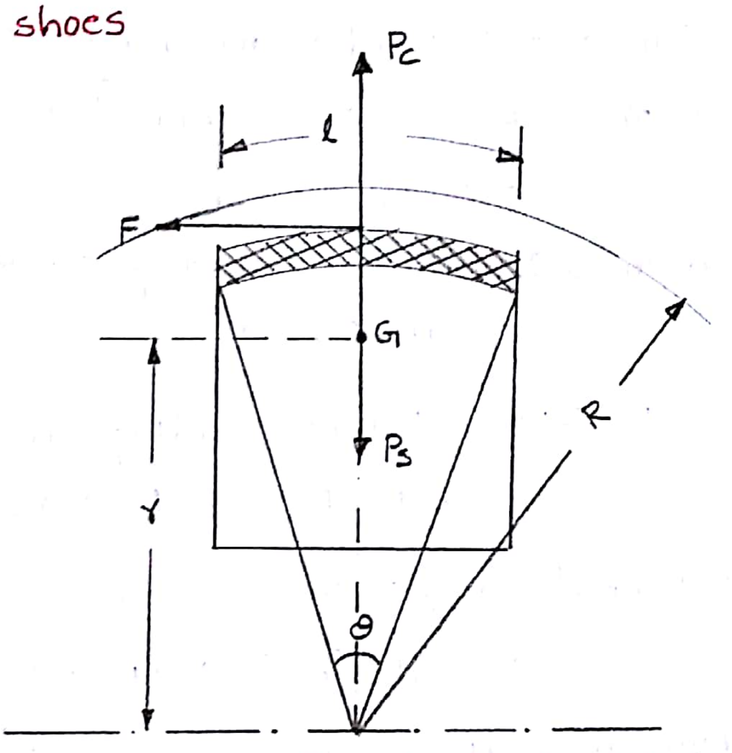


Fig: Forces on a shoe of centrifugal clutch

Consider one shoe of a centrifugal clutch as shown in Fig.

Let m - Mass of each shoe

n - Number of shoes

r - Distance of centre of gravity of the shoe from the centre of the spider

R - Inside radius of the pulley rim

N - Running speed of the pulley in rpm

ω - Angular running speed of the pulley in rad/s

ω_1 - Angular speed at which the engagement begins to take place

μ - Coefficient of friction between the shoe & rim

we know that the centrifugal force acting on each shoe at the running speed,

$$P_c = m \cdot \omega^2 \cdot r$$

and the inward force on each shoe exerted by the spring at the speed at which engagement begins to take place,

$$P_s = m(\omega_1)^2 r$$

∴ The net outward radial force (i.e. centrifugal force) with the shoe presses against the rim at the running speed

$$= P_c - P_s$$

and the frictional force acting tangentially on each shoe,

$$F = \mu (P_c - P_s)$$

∴ Frictional torque acting on each shoe,

$$= F * R = \mu (P_c - P_s) * R$$

and total torque transmitted,

$$T = \mu (P_c - P_s) R * n = n F R$$

From this expression, the mass of the shoes (m) may be evaluated.

2. Size of the shoes

Let l – contact length of the shoes

b – width of the shoes

R – contact radius of the shoes. It is same as the inside radius of the rim of the pulley

θ – Angle subtended by the shoes at the centre of the spider in radians

p – Intensity of pressure exerted on the shoe. In order to ensure reasonable life, the intensity of pressure may be taken as 0.1 N/mm²

we know that

$$\theta = \frac{l}{R} \text{ rad} \quad (\text{or}) \quad l = \theta . R$$

∴ Area of contact of the shoe,

$$A = l . b$$

and the force with which the shoe presses against the rim

$= A * p = l \cdot b \cdot p$

Since the force with which the shoe presses against the rim at the running speed is $(P_c - P_s)$,

$$l \cdot b \cdot p = P_c - P_s$$

From this expression, the width of shoe (b) may be obtained.

Ⓟ A single plate clutch, with both sides effective, has outer & inner diameters 300mm & 200mm respectively. The maximum intensity of pressure at any point in the contact surface is not to exceed 0.1 N/mm². If the coefficient of friction is 0.3, determine the power transmitted by a clutch at a speed 2500 rpm.

Sol:-

$d_1 = 300\,mm$ (or) $r_1 = 150\,mm$, $p_{max} = 0.1\ N/mm^2$

$d_2 = 200\,mm$ (or) $r_2 = 100\,mm$

$\mu = 0.3$, $N = 2500$ rpm (or) $\omega = \frac{2\pi N}{60} = \frac{2\pi * 2500}{60} = 261.8$ rad/s

Since the intensity of pressure (p) is maximum at the inner radius (r_2), therefore for uniform wear,

$p.r_2 = C$ (or) $C = 0.1 * 100 = 10$ N/mm

we know that the axial thrust

$W = 2\pi C(r_1 - r_2)$ (or) $C = \frac{W}{2\pi(r_1 - r_2)} = 3142\,N$

and mean radius of the friction surfaces for uniform wear

$R = \frac{r_1 + r_2}{2} = \frac{150+100}{2} = 125\,mm = 0.125\,m$

we know that torque transmitted,

$T = n.\mu.W R = 2 * 0.3 * 3142 * 0.125$

$T = 235.65$ N-m ($\because n = 2$, for both sides of plate surface)

$\therefore$ Power transmitted by a clutch,

$P = T.\omega = 235.65 * 261.8 = 61693\,W = 61.693\,kW$

A conical friction clutch is used to transmit 90 kW at 1500 rpm. The semi-cone angle is 20° and the coefficient of friction is 0.2. If the mean diameter of the bearing surface is 375mm and the intensity of normal pressure is not to exceed 0.25 N/mm², find the dimensions of the conical bearing surface and the axial load required.

Sol:- $P = 90\text{ kW} = 90 \times 10^3\text{ W}$, $N = 1500\text{ rpm}$ (or) $\omega = \frac{2\pi N}{60} = 156\text{ rad/s}$

$\alpha = 20°$, $\mu = 0.2$, $D = 375\text{ mm}$ (or) $R = 187.5\text{ mm}$, $p_n = 0.25\text{ N/mm}^2$

Dimensions of the conical bearing surface

Let r_1 & r_2 – External & internal radii of the bearing surface

b – width of the bearing surface in mm

T – Torque transmitted

We know that power transmitted

$$P = T\omega$$

$$90 \times 10^3 = T \times 156$$

$$T = 577\text{ N-m} = 577 \times 10^3\text{ N-mm}$$

and the torque transmitted

$$T = 2\pi \mu p_n R^2 b$$

$$577 \times 10^3 = 2\pi \times 0.2 \times 0.25 \times 187.5^2 \times b$$

$$b = 52.2\text{ mm}$$

We know that, $r_1 + r_2 = 2R = 2 \times 187.5 = 375\text{ mm} \rightarrow (1)$

$r_1 - r_2 = b\sin\alpha = 52.2 \sin 20° = 18\text{ mm} \rightarrow (2)$

From equations (1) & (2)

$r_1 = 196.5\text{ mm}$, $r_2 = 178.5\text{ mm}$

Axial load required

Since in case of friction clutch, uniform wear is considered and the intensity of pressure is maximum at the minimum contact surface radius (r_2), therefore

$p_n \cdot r_2 = C$ (or) $C = 0.25 \times 178.5 = 44.6\text{ N/mm}$

we know that the axial load required

$$W = 2\pi C (r_1 - r_2) \qquad C = \frac{W}{2\pi (r_1 - r_2)}$$

$$W = 2\pi \times 44.6\,(196.5 - 178.5) = 5045\ N$$

(P) An engine developing 45kw at 1000 rpm is fitted with a cone clutch built inside the flywheel. The cone has a face angle of 12.5° and a maximum mean diameter of 500mm. The coefficient of friction is 0.2. The normal pressure on the clutch face is not to exceed 0.1 N/mm². Determine: 1. the axial spring force necessary to engage to clutch, & 2. the face width required

Sol:-

$P = 45kw = 45 \times 10^3\ W$, $N = 1000$ rpm (or) $\omega = \frac{2\pi N}{60} = 104.7$ rad/s

$\alpha = 12.5°$, $D = 500mm$ (or) $R = 250mm = 0.25m$, $\mu = 0.2$

$P_n = 0.1\ N/mm^2$

1. Axial spring force necessary to engage the clutch

we know that power developed by the clutch

$$P = T \cdot \omega$$
$$45 \times 10^3 = T \times 104.7$$
$$T = 430\ N\text{-}m$$

we also know that the torque developed by the clutch

$$T = \mu W_n R$$
$$430 = 0.2 \times W_n \times 0.25$$
$$W_n = 8600\ N$$

and axial spring force necessary to engage the clutch

$$W_e = W_n(\sin\alpha + \mu\cos\alpha)$$
$$= 8600\,(\sin 12.5° + 0.2\cos 12.5) = 3540\ N$$

2. Face width required

Let b - Face width required

we know that normal load acting on the friction surface

$$W_n = P_n \times 2\pi R b$$
$$8600 = P_n \times 2\pi \times 250 \times b = 0.1 \times 2\pi \times 250 \times b$$
$$b = 54.7\ mm$$

Brakes

A brake is a device by means of which artificial frictional resistance is applied to a moving machine member, in order to retard or stop the motion of a machine.

In the process of performing this function, the brake absorbs either kinetic energy of the moving member or potential energy given up by objects being lowered by hoists, elevators etc. The energy absorbed by brakes is dissipated in the form of heat. This heat is dissipated in the surrounding air (or water which is circulated through the passages in the brake drum) so that excessive heating of the brake lining does not take place.

The capacity of a brake depends upon the following factors:

1. The unit pressure between the braking surfaces
2. The coefficient of friction between the braking surfaces
3. The peripheral velocity of the brake drum
4. The projected area of the friction surfaces, and
5. The ability of the brake to dissipate heat equivalent to the energy being absorbed.

The major functional difference between a clutch and a brake is that a clutch is used to keep the driving and driven member moving together, whereas brakes are used to stop a moving member or to control its speed.

Types of Brakes

The brakes, according to the means used for transforming the energy by the braking elements, are classified as:

1. Hydraulic brakes e.g. pumps or hydrodynamic brake & fluid agitator,
2. Electric brakes e.g. generators and eddy current brakes, and
3. Mechanical brakes

Single Block or Shoe Brake

A single block or shoe brake is shown in Fig. It consists of a block or shoe which is pressed against the rim of a revolving brake wheel drum. The block is made of a softer material than the rim of the wheel. This type of a brake is commonly used on railway trains and tram cars. The friction between the block and the wheel causes a tangential braking force to act on the wheel, which retard the rotation of the wheel. The block is pressed against the wheel by a force applied to one end of a lever to which the block is rigidly fixed as shown in Fig. The other end of the lever is pivoted on a fixed fulcrum O.

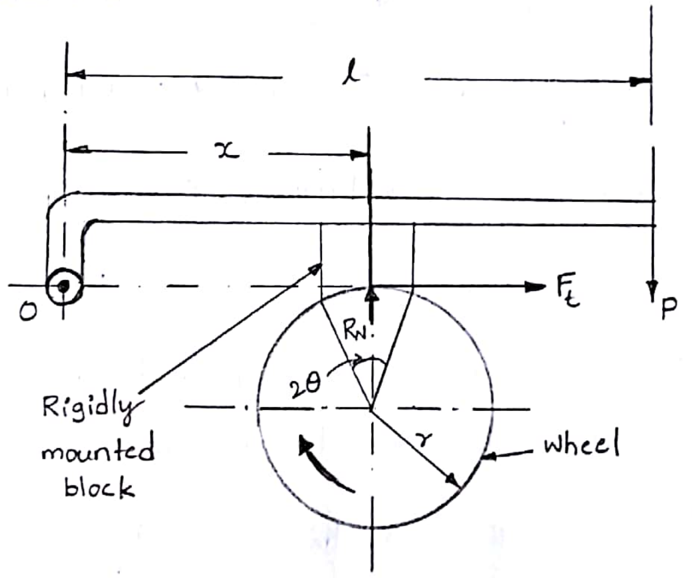
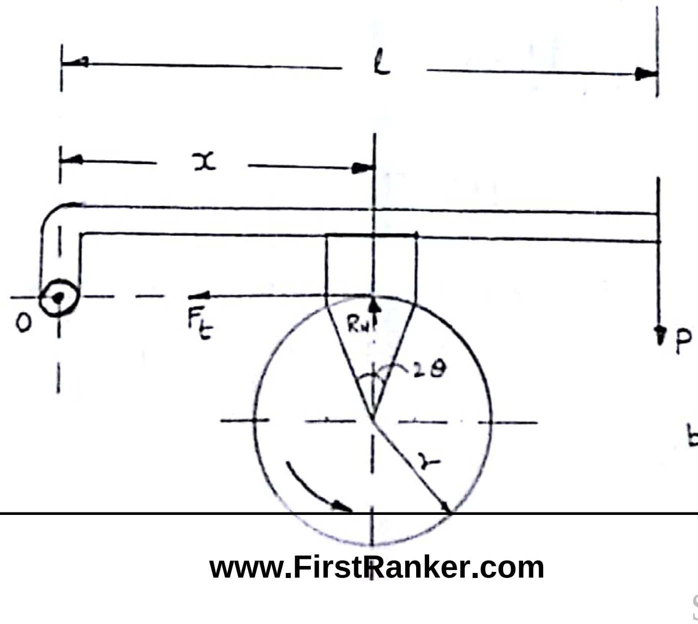


Fig: Line of action of tangential force passes through the fulcrum of the lever.

a) clockwise rotation of brake wheel



b) Anticlockwise rotation of brake wheel

Let P - Force applied at the end of the lever

R_N - Normal force pressing the brake block on the wheel

r - Radius of the wheel

2θ - Angle of contact surface of the block

F_t - Tangential braking force or the frictional force acting at the contact surface of the block & the wheel.

μ - Coefficient of friction

If the angle of contact is less than 60°, then it may be assumed that the normal pressure between the block and the wheel is uniform. In such cases, tangential braking force on the wheel,

$$F_t = \mu R_N$$

and the braking torque, $T_B = F_t * r = \mu \cdot R_N \cdot r$

Let us now consider the following three cases:

Case 1:

When the line of action of tangential braking force (F_t) passes through the fulcrum O of the lever, and the brake wheel rotates clockwise as shown in Fig (a), then for equilibrium, taking moments about the fulcrum O, we have

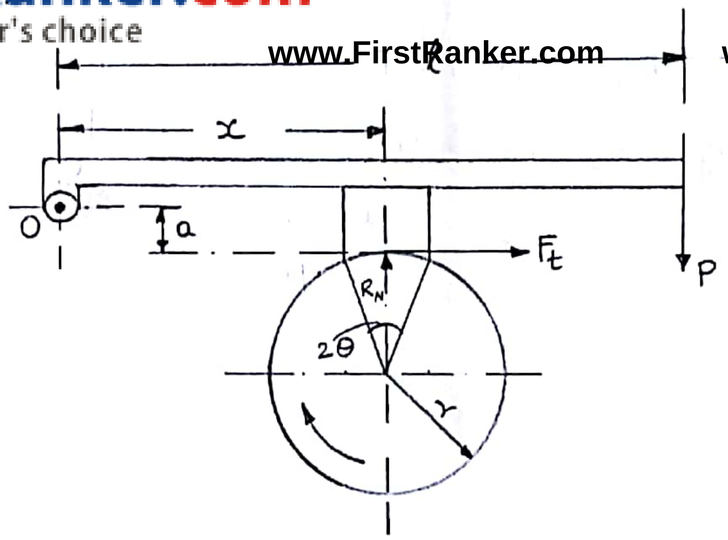
$$R_N * x - P * l = 0$$

$$R_N = \frac{P * l}{x}$$

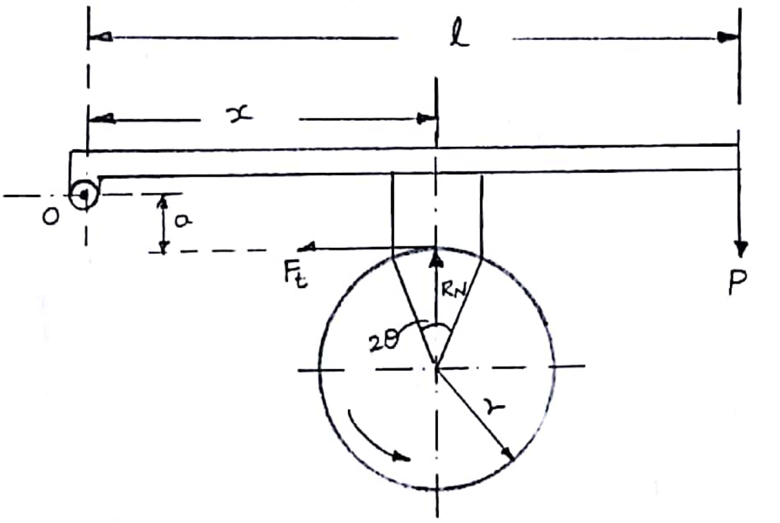
$\therefore$ Braking torque, $T_B = \mu \cdot R_N \cdot r = \mu * \frac{P \cdot l}{x} * r = \frac{\mu \cdot P \cdot l \cdot r}{x}$

It may be noted that when the brake wheel rotates anticlock as shown in Fig (b), then the braking torque is same, i.e

$$T_B = \mu \cdot R_N \cdot r = \frac{\mu \cdot P \cdot l \cdot r}{x}$$



a) clockwise rotation of brake wheel



b) Anticlockwise rotation of brake wheel

Fig: Line of action F_t Passes below the fulcrum

When the line of action of the tangential braking force (F_t) passes through a distance 'a' below the fulcrum O, and the brake wheel rotates clockwise as shown in Fig (a), then for equilibrium, taking moments about the fulcrum O,

$$R_N * x + F_t * a - P * l$$

$$R_N * x + \mu R_N * a = P.l \quad (or) \quad R_N = \frac{P.l}{x + \mu a}$$

and braking torque, $T_B = \mu R_N r$

$$T_B = \frac{\mu \cdot P \cdot l \cdot r}{x + \mu a}$$

When the brake wheel rotate anticlockwise, as shown in Fig (b), then for equilibrium,

$$R_N * x - P * l - F_t * a = 0$$

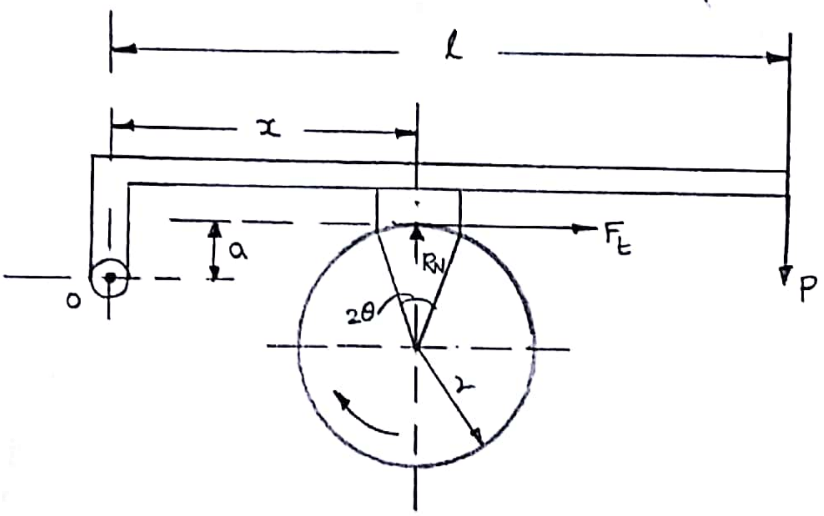
$R_N \times x - P \cdot l - \mu R_N \times a = 0$

$R_N(x - \mu a) = P \cdot l$ (or) $R_N = \dfrac{P \cdot l}{x - \mu a}$

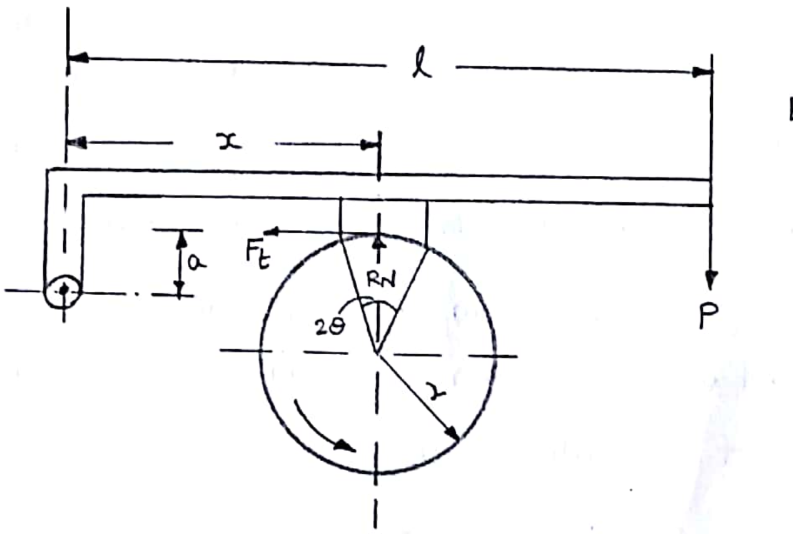
and braking torque, $T_B = \mu R_N r$

$$T_B = \frac{\mu \cdot P \cdot l \cdot r}{x - \mu a}$$

case 3:



a) clockwise rotation of brake wheel



b) Anticlockwise rotation of brake wheel

Fig: Line of action of F_t passes above the fulcrum

When the line of action of the tangential braking force (F_t) passes through a distance 'a' above the fulcrum O, and the brake wheel rotates clockwise as shown in Fig (a), then for equilibrium, taking moments about the fulcrum O, we have

$$R_N \times x - P \times l - F_t \times a = 0$$

$$R_N \times x - P \times l - \mu R_N \times a = 0$$

$R_N(x - \mu a) = P.l$ (or) $R_N = \dfrac{P.l}{x-\mu a}$

and braking torque, $T_B = \mu \cdot R_N \cdot r$

$$T_B = \frac{\mu P.l \cdot r}{x-\mu a}$$

When the brake wheel rotates anticlockwise as shown in Fig(b), then for equilibrium, taking moments about the fulcrum O, we have

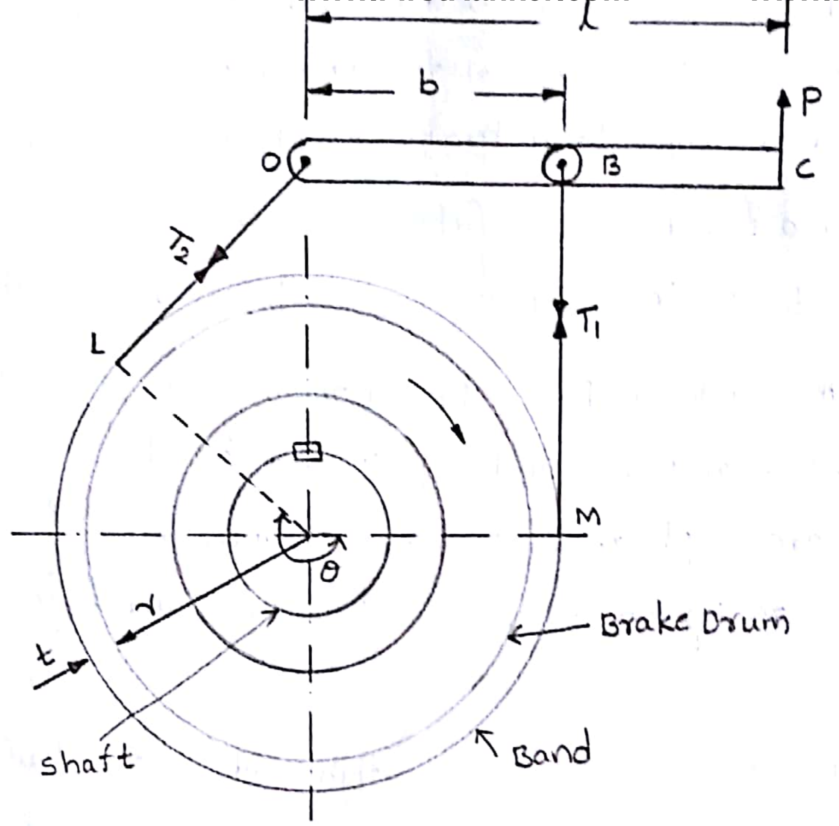
$$R_N * x + F_t * a - P.l = 0$$

$$R_N * x + \mu R_N * a - P.l = 0 \quad (\text{or}) \quad R_N = \frac{P.l}{x+\mu a}$$

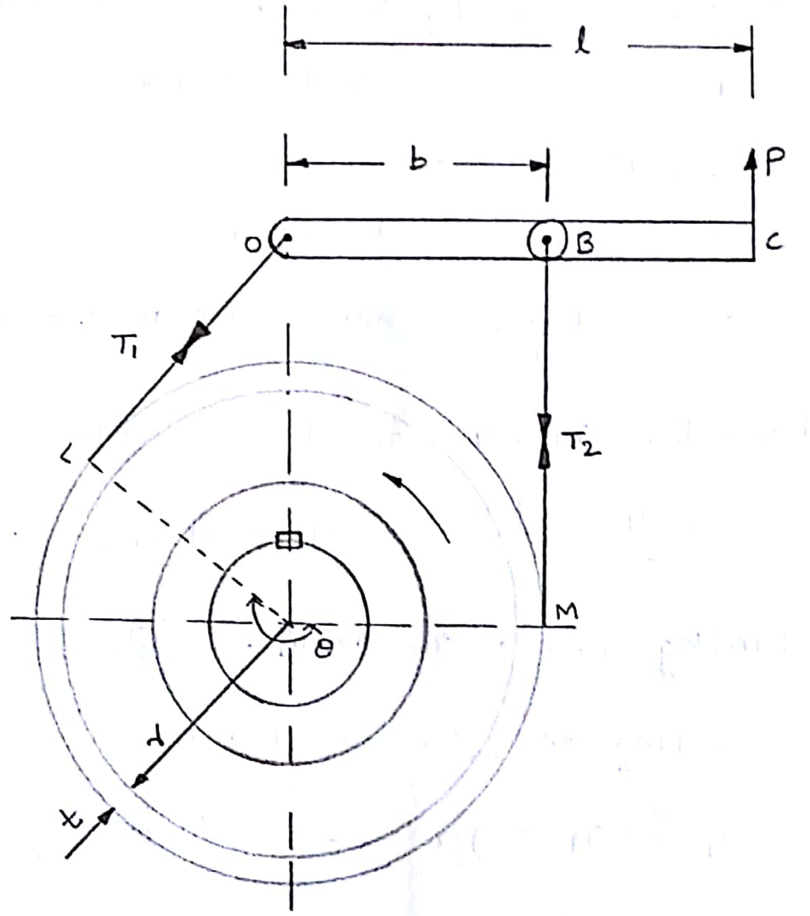
braking torque, $T_B = \mu \cdot R_N \cdot r$

$$T_B = \frac{\mu \cdot P.l \cdot r}{x+\mu a}$$

Simple Band Brake



a) Clockwise rotation of drum



b) Anticlockwise rotation of drum

... band brake consists of a flexible band of leather, one or more ropes, or a steel lined with friction material, which embraces a part of the circumference of the drum. A band brake, as shown in Fig. is called a simple band brake in which one end of the band is attached to a fixed pin or fulcrum of the lever while the other end is attached to the lever at a distance b from the fulcrum.

When a force P is applied to the lever at C, the lever turns about the fulcrum pin O and tightens the band on the drum and hence the brakes are applied. The friction between the band and the drum provides the braking force. The force P on the lever at C may be determined as,

Let T_1 – Tension in the tight side of the band

T_2 – Tension in the slack side of the band

θ – Angle of lap (or embrace) of the band on the drum

μ – Coefficient of friction between the band & the drum

r – Radius of the drum

t – Thickness of the band

r_e – Effective radius of the drum $= r + \frac{t}{2}$

We know that limiting ratio of the tension is given by

$$\frac{T_1}{T_2} = e^{\mu\theta} \quad \text{(or)} \quad 2.3\log\left(\frac{T_1}{T_2}\right) = \mu\theta$$

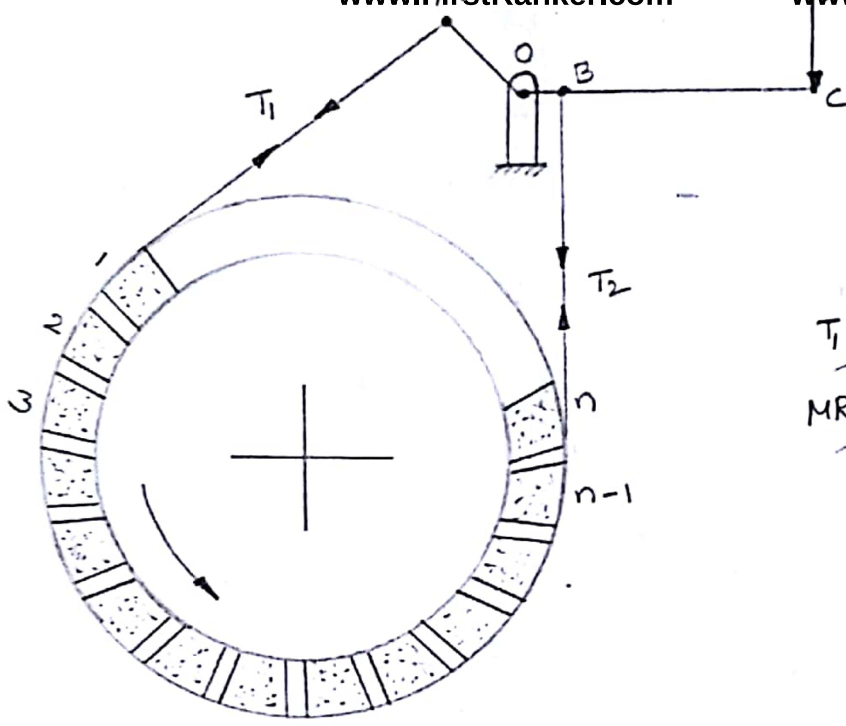
and braking force on the drum $= T_1 - T_2$

$\therefore$ Braking torque on the drum

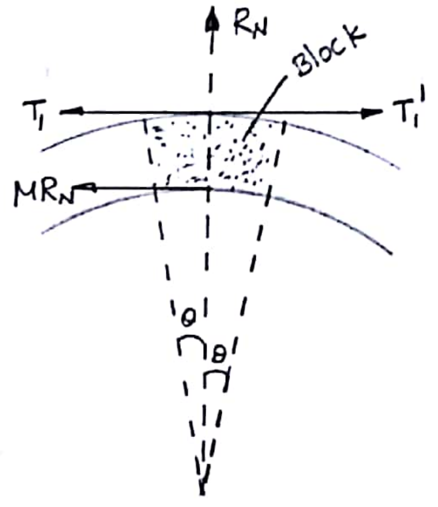
$$T_B = (T_1 - T_2)\, r \quad \text{—— (Neglecting thickness of band)}$$

$$= (T_1 - T_2)\, r_e \quad \text{—— (considering thickness of band)}$$

Band and Block Brake



(a)



(b)

The band brake may be lined with blocks of wood or other material, as shown in Fig. The friction between the blocks and the drum provides braking action. Let there are 'n' number of blocks, each subtending an angle 2θ at the centre and the drum rotates in anticlockwise direction.

Let T_1 – Tension in the tight side

T_2 – Tension in the slack side

μ – Coefficient of friction between the blocks & drum

T_1' – Tension in the band between the first & second block

T_2', T_3' etc. – Tension in the band between the second & third block, between the third & fourth block etc.

Consider one of the blocks (say first block) as shown in Fig (b). This is in equilibrium under the action of the following forces:

1. Tension in the tight side (T_1)
2. Tension in the slack side (T_1') or tension in the band between the first & second block
3. Normal reaction of the drum on the block (R_N)
4. The force of friction (μR_N)

... forces radially

$$(T_1 + T_1')\sin\theta = R_N \longrightarrow ①$$

Resolving the forces tangentially

$$(T_1 - T_1')\cos\theta = \mu R_N \longrightarrow ②$$

Dividing equation ② by ①, we have

$$\frac{(T_1 - T_1')\cos\theta}{(T_1 + T_1')\sin\theta} = \frac{\mu R_N}{R_N}$$

$$(T_1 - T_1') = \mu \tan\theta\,(T_1 + T_1')$$

$$\frac{T_1}{T_1'} = \frac{1+\mu\tan\theta}{1-\mu\tan\theta}$$

Similarly, it can be proved for each of the blocks that

$$\frac{T_1'}{T_2'} = \frac{T_2'}{T_3'} = \frac{T_3'}{T_4'} = \cdots\cdots = \frac{T_{n-1}}{T_2} = \frac{1+\mu\tan\theta}{1-\mu\tan\theta}$$

$$\therefore \quad \frac{T_1}{T_2} = \frac{T_1}{T_1'} \times \frac{T_1'}{T_2'} \times \frac{T_2'}{T_3'} \times \cdots\cdots \times \frac{T_{n-1}}{T_2} = \left(\frac{1+\mu\tan\theta}{1-\mu\tan\theta}\right)^n$$

Braking torque on the drum of effective radius r_e

$$T_B = (T_1 - T_2)\, r_e$$

$$T_B = (T_1 - T_2)\, r \quad ---- \text{(Neglecting thickness of band)}$$

Note: For the first block, the tension in the tight side is T_1 and in the slack side is T_1' and for the second block, the tension in the tight side is T_1' and in the slack side is T_2'. Similarly for the third block, the tension in the tight side is T_{n-1} and in the slack side is T_2.

A band and block brake, having 14 blocks each of which subtends an angle of 15° at the centre, is applied to a drum of 1m effective diameter. The drum and flywheel mounted on the same shaft has a mass of 2000 kg and a combined radius of gyration of 500 mm. The two ends of the band are attached to pins on opposite sides of the brake lever at distances of 30 mm and 120 mm from the fulcrum. If a force of 200 N is applied at a distance of 750 mm from the fulcrum, find: 1. maximum braking torque, 2. angular retardation of the drum, and 3. time taken by the system to come to rest from the rated speed of 360 rpm. The coefficient of friction between blocks and drum may be taken as 0.25.

Sol:-

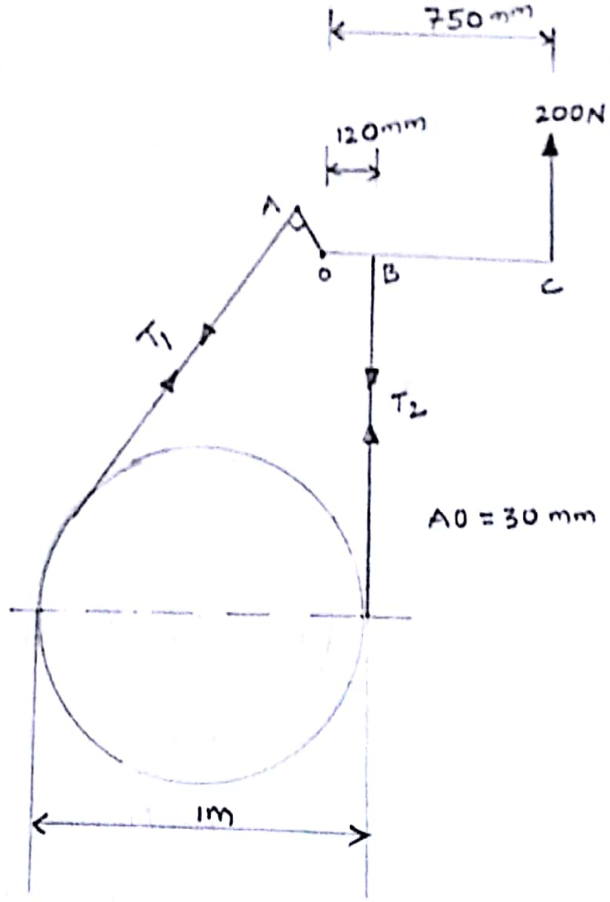
$n = 14$, $2\theta = 15°$ (or) $\theta = 7.5°$

$d = 1m$ (or) $r = 0.5m$, $m = 2000\,kg$

$k = 500mm = 0.5m$, $P = 200\,N$

$N = 360$ rpm , $l = 750$ mm

$\mu = 0.25$



1. Maximum braking torque

The braking torque will be maximum when OB > OA and the drum rotates anticlockwise as shown in Fig.

The force P must act upwards and the end of the band attached to A is tight under tension T_1 and the end of the band attached to B is slack under tension T_2.

Taking moments about O

$$200 \times 750 + T_1 \times 30 - T_2 \times 120 = 0$$

$$12T_2 - 3T_1 = 15000 \longrightarrow (1)$$

we know that

$$\frac{T_1}{T_2} = \left(\frac{1 + \mu \tan\theta}{1 - \mu \tan\theta}\right)^n$$

$$\frac{T_1}{T_2} = \left(\frac{1+0.25 \tan 7.5^\circ}{1-0.25 \tan 7.5^\circ}\right)^{14}$$

$$= \left(\frac{1+0.25*0.1317}{1-0.25*0.1317}\right)^{14} = (1.068)^{14}$$

$$\frac{T_1}{T_2} = 2.512 \longrightarrow (2)$$

From (1) & (2)

$T_1 = 8440$ N, & $T_2 = 3360$ N

we know that maximum braking torque,

$$T_B = (T_1 - T_2) r = (8440 - 3360) 0.5 = 2540 \text{ N-m}$$

2. Angular retardation of the drum

Let α = Angular retardation of the drum

we know that braking torque

$$T_B = I\alpha = mk^2\alpha$$

$$2540 = 2000 (0.5)^2 \alpha$$

$$\alpha = 5.08 \text{ rad/s}^2$$

3. Time taken by the system to come to rest

Let t = Required time

Since the system is to come to rest from the rated speed of 360 rpm.

Initial angular speed, $\omega_1 = \frac{2\pi * 360}{60} = 37.7$ rad/s

and final angular speed, $\omega_2 = 0$

we know that $\omega_2 = \omega_1 - \alpha t$ (-ve sign due to retardation)

$$t = \frac{\omega_1}{\alpha} = \frac{37.7}{5.08}$$

$$t = 7.42 \text{ sec}$$

Dynamometer

A dynamometer is a brake but in addition it has a device to measure the frictional resistance. Knowing the frictional resistance, we may obtain the torque transmitted and hence the power of the engine.

Types of Dynamometers

Following are the two types of dynamometers, used for measuring the brake power of an engine.

1. Absorption dynamometers, and
2. Transmission dynamometers

In the absorption dynamometers, the entire energy or power produced by the engine is absorbed by the friction resistances of the brake and is transformed into heat, during the process of measurement. But in the transmission dynamometers, the energy is not wasted in friction but is used for doing work. The energy or power produced by the engine is transmitted through the dynamometer to some other machines where the power developed is suitably measured.

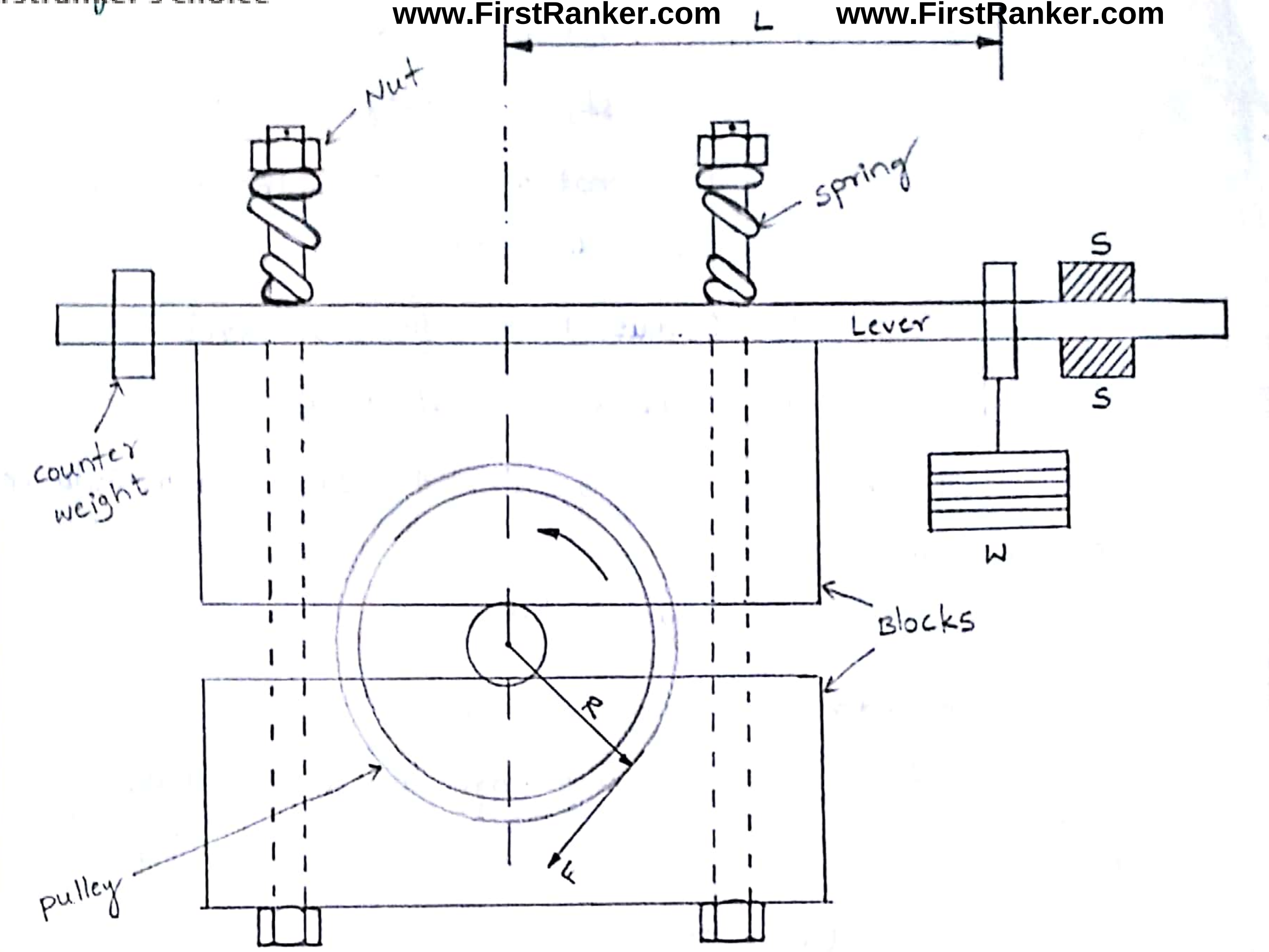
Absorption Dynamometers

1. Prony Brake dynamometer
2. Rope Brake dynamometer

Transmission Dynamometers

1. Epicyclic-train dynamometer
2. Belt transmission dynamometer
3. Torsion dynamometer

Prony Brake Dynamometer



A simplest form of an absorption dynamometer is a prony brake dynamometer, as shown in Fig. It consists of two wooden blocks placed around a pulley fixed to the shaft of an engine whose power is required to be measured. The blocks are clamped by means of two bolts and nuts, as shown in Fig. A helical spring is provided between the nut and the upper block to adjust the pressure on the pulley to control its speed. The upper block has a long lever attached to it and carries a weight W at its outer end. A counter weight is placed at the other end of the lever which balances the brake when unloaded. Two stops S, S are provided to limit the motion of the lever.

When the brake is to be put in operation, the long end of the lever is loaded with suitable weights W and the nuts are tightened until the engine shaft runs at a constant speed and the lever is in horizontal position. Under these conditions, the moment due to the weight W must balance the moment of the frictional resistance between the blocks & pulley.

Let W – Weight at the outer end of the lever in newtons

L – Horizontal distance of the weight W from the centre of ~~gravity~~ the pulley in metres,

F – Frictional resistance between the blocks and the pulley in newtons,

R – Radius of the pulley in metres

N – speed of the shaft in rpm

We know that the moment of the frictional resistance or torque on the shaft,

$$T = W \cdot L = F \cdot R \quad N\text{-}m$$

Workdone in one revolution

$$= \text{Torque} \times \text{Angle turned in radians}$$

$$= T \times 2\pi \quad N\text{-}m$$

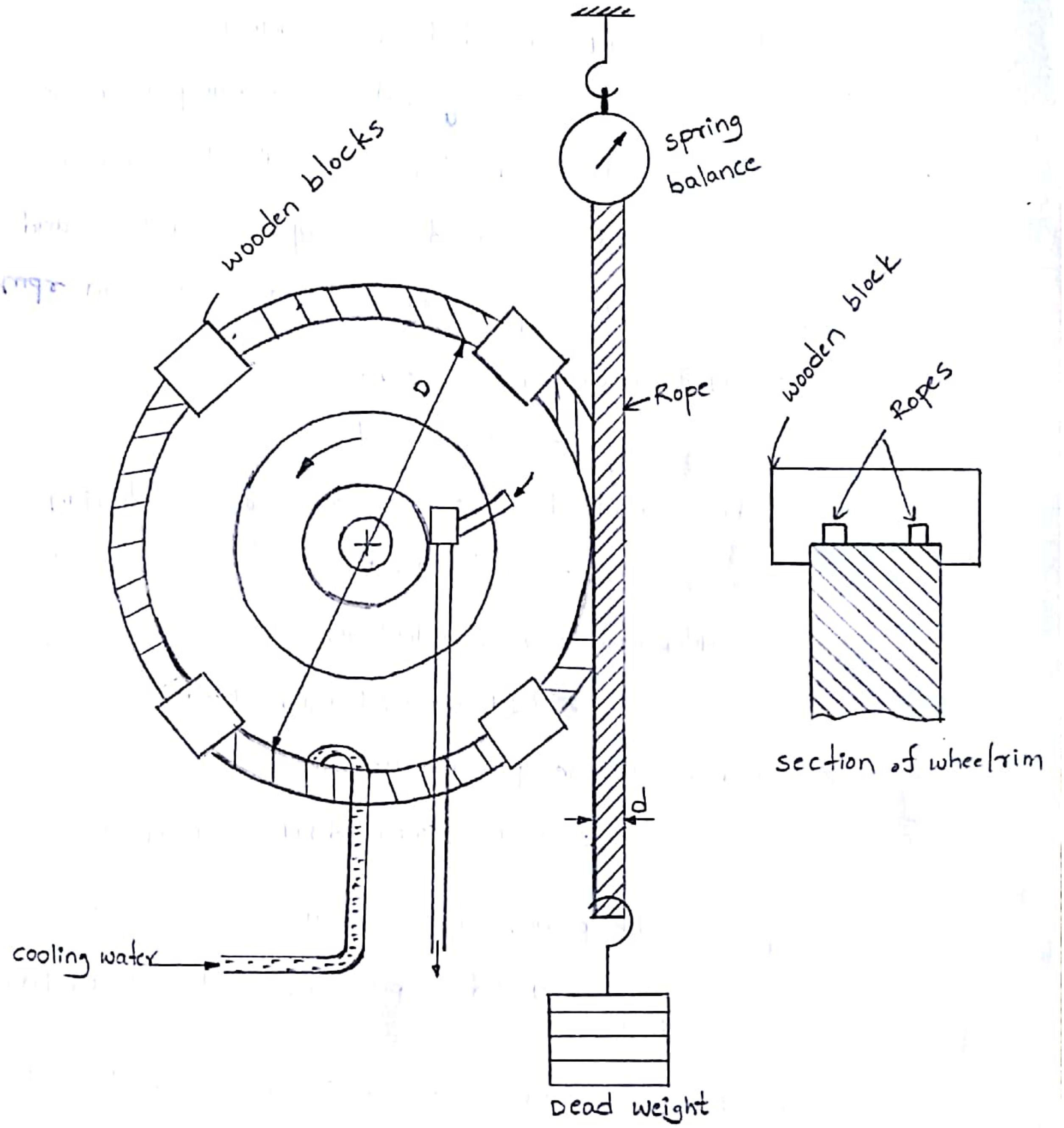
$\therefore$ workdone per minute

$$= T \times 2\pi N \quad N\text{-}m$$

We know that brake power of the engine

$$B.P = \frac{\text{workdone per minute}}{60} = \frac{T \times 2\pi N}{60} = \frac{W \cdot L \times 2\pi N}{60} \text{ watts}$$

Rope Brake Dynamometer



It is another form of absorption type dynamometer which is most commonly used for measuring the brake power of the engine. It consists of one, two or more ropes wound around the flywheel or rim of a pulley fixed rigidly to the shaft of an engine. The upper end of the ropes is attached to a spring balance while the lower end of the ropes is kept in position by applying a dead weight as shown in Fig. In order to prevent the slipping of the rope over the flywheel, wooden blocks are placed at intervals around the circumference of the flywheel.

In the operation of the brake, the engine is made to run at a constant speed. The frictional torque, due to the rope, must be equal to the torque being transmitted by the engine.

Let W – Dead load in newtons,

S – spring balance reading in newtons,

D – Diameter of the wheel in metres

d – Diameter of rope in metres, and

N – speed of the engine shaft in rpm

∴ Net load on the brake

$$= (W-S) \ N$$

we know that distance moved in one revolution

$$= \pi(D+d) \ m$$

∴ Workdone per revolution

$$= (W-S)\pi(D+d) \quad N\text{-}m$$

and workdone per minute

$$= (W-S)\pi(D+d)N \quad N\text{-}m$$

∴ Brake power of the engine,

$$B.P = \frac{\text{Workdone per min}}{60} = \frac{(W-S)\pi(D+d)N}{60} \text{ watts}$$

If the diameter of the rope (d) is neglected, then brake power of the engine,

$$B.P = \frac{(W-S)\pi DN}{60} \text{ watts}$$

Dynamic Force Analysis

Dynamic forces are associated with accelerating masses. As all machines have some accelerating parts, dynamic forces are always present when the machines operate. In situations where dynamic forces are dominant or comparable with magnitudes of external forces and operating speeds are high, dynamic analysis has to be carried out.

For example, in case of rotors which rotate at speeds more than 80000 rpm, even the slightest eccentricity of the centre of mass from the axis of rotation produces very high dynamic forces. This may lead to vibrations, wear, noise or even machine failure.

D-Alembert's Principle

Consider a rigid body acted upon by a system of forces. The system may be reduced to a single resultant force acting on the body whose magnitude is given by the product of the mass of the body and the linear acceleration of the centre of mass of the body.

According to Newton's second law of motion,

$$F = ma \longrightarrow (1)$$

where F – Resultant force acting on the body

m – Mass of the body

a – Linear acceleration of the centre of mass of the body

The equation (1) may also be written as:

$$F - ma = 0 \longrightarrow (2)$$

A little consideration will show, that if the quantity $-ma$ be treated as a force, equal, opposite and with the same line of action as the resultant force F, and include this force with the system of forces of which F is the resultant, then the complete system of forces will be in equilibrium. This principle is known as D-Alembert's principle. The equal & opposite force $-ma$ is known as reversed effective force or the inertia force (F_i).

The equation ② may be written as

$$F + F_i = 0$$

Thus, D-Alembert's principle states that the resultant force acting on a body together with the reversed effective force (or inertia force), are in equilibrium.

Inertia force & Inertia Torque

The inertia force is an imaginary force, which when acts upon a rigid body, brings it in an equilibrium position. It is numerically equal to the accelerating force in magnitude, but opposite in direction.

Inertia force = − Accelerating force = $-m \cdot a$

where m − Mass of the body

a − Linear acceleration of the centre of gravity of the body.

Similarly, the inertia torque is an imaginary torque, which when applied upon the rigid body, brings it in equilibrium position. It is equal to the accelerating couple in magnitude but opposite in direction.

Dynamic Analysis of Slider-Crank Mechanisms

www.FirstRanker.com

(Analytical Method for velocity & Acceleration of the Piston)

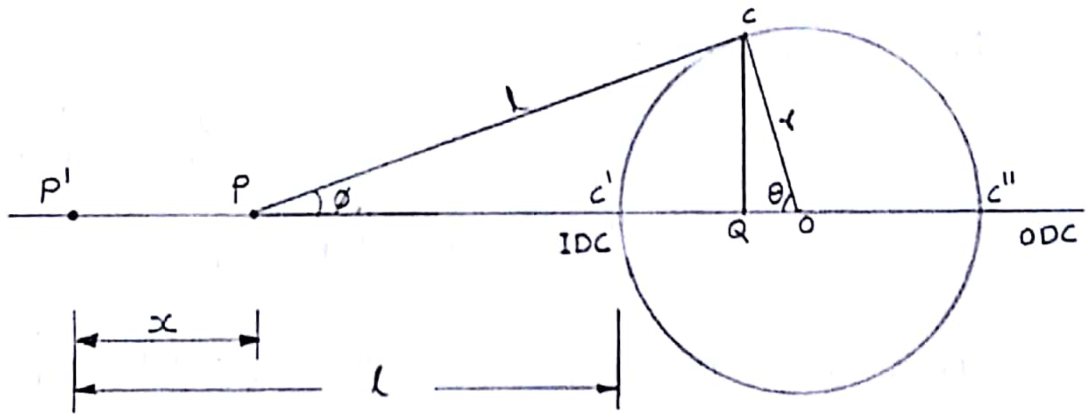


Fig: Motion of a crank and connecting rod of a reciprocating steam engine

Consider the motion of a crank and connecting rod of a reciprocating steam engine as shown in Fig. Let OC be the crank & PC be the connecting rod. Let the crank rotates with angular velocity of ω rad/s and the crank turns through an angle θ from the inner dead centre (IDC). Let x be the displacement of a reciprocating body P from IDC after time t seconds, during which the crank has turned through an angle θ.

Let l - Length of connecting rod between the centres

r - Radius of crank or crank pin circle

ϕ - Inclination of connecting rod to the line of stroke PO

n - Ratio of length of connecting rod to the radius of crank $= l/r$

Velocity of the piston

From the geometry of Fig

$$x = P'P = OP' - OP = (P'C' + C'O) - (PQ + QO)$$

$= (l+r) - (l\cos\phi + r\cos\theta)$ $\quad \because PQ = l\cos\phi$

$\quad QO = r\cos\theta$

$$= r(1-\cos\theta) + l(1-\cos\phi)$$

$$= r\left[(1-\cos\theta) + \frac{l}{r}(1-\cos\phi)\right]$$

$\sin^2\phi + \cos^2\phi = 1$

$\cos^2\phi = 1 - \sin^2\phi$

$$x = r\left[(1-\cos\theta) + n(1-\cos\phi)\right] \longrightarrow ①$$

From triangles CPQ and CQO,

$$CQ = l\sin\phi = r\sin\theta \quad (\text{or}) \quad \frac{l}{r} = \frac{\sin\theta}{\sin\phi}$$

$$\therefore \quad n = \frac{\sin\theta}{\sin\phi} \quad (\text{or}) \quad \sin\phi = \frac{1}{n}\sin\theta \longrightarrow ②$$

we know that, $\cos\phi = (1-\sin^2\phi)^{\frac{1}{2}} = \left(1 - \frac{\sin^2\theta}{n^2}\right)^{\frac{1}{2}}$

Expanding the above expression by binomial theorem,

$$\cos\phi = 1 - \frac{1}{2} * \frac{\sin^2\theta}{n^2} + \text{----}$$

(Neglecting higher terms)

$$1 - \cos\phi = \frac{\sin^2\theta}{2n^2} \longrightarrow ③$$

substituting the value of $(1-\cos\phi)$ in equation ①

$$x = r\left[(1-\cos\theta) + n * \frac{\sin^2\theta}{2n^2}\right]$$

$$x = r\left[(1-\cos\theta) + \frac{\sin^2\theta}{2n}\right] \longrightarrow ④$$

Differentiating equation ④ with respect to θ,

$$\frac{dx}{d\theta} = r\left[\sin\theta + \frac{1}{2n} * 2\sin\theta\cdot\cos\theta\right] = r\left[\sin\theta + \frac{\sin 2\theta}{2n}\right] \longrightarrow ⑤$$

$(\because 2\sin\theta\cos\theta = \sin 2\theta)$

$\therefore$ Velocity of P with respect to O (or) Velocity of the piston P,

$$V_{PO} = V_P = \frac{dx}{dt} = \frac{dx}{d\theta} * \frac{d\theta}{dt} = \frac{dx}{d\theta} * \omega$$

substituting the value of $dx/d\theta$ from equation ⑤,

$$V_{PO} = V_P = \omega . r\left[\sin\theta + \frac{\sin 2\theta}{2n}\right] \longrightarrow ⑥$$

Acceleration of the piston

Since the acceleration is the rate of change of velocity, therefore acceleration of the piston P,

$$a_P = \frac{dv_P}{dt} = \frac{dv_P}{d\theta} * \frac{d\theta}{dt} = \frac{dv_P}{d\theta} * \omega$$

Differentiating equation ⑥ with respect to θ,

$$\frac{dv_P}{d\theta} = \omega . r \left[\cos\theta + \frac{\cos 2\theta * 2}{2n}\right] = r.\omega\left[\cos\theta + \frac{\cos 2\theta}{n}\right]$$

substituting the value of $\frac{dv_P}{d\theta}$ in the above equation, we have

$$a_P = \omega . r\left[\cos\theta + \frac{\cos 2\theta}{n}\right] * \omega$$

$$a_P = r.\omega^2\left[\cos\theta + \frac{\cos 2\theta}{n}\right] \longrightarrow ⑦$$

Notes :

1. When crank is at the inner dead centre, then $\theta = 0°$

$$a_P = r.\omega^2\left[\cos 0° + \frac{\cos 0°}{n}\right] = r\omega^2\left[1 + \frac{1}{n}\right]$$

2. When the crank is at the outer dead centre, then $\theta = 180°$

$$a_P = r\omega^2\left[\cos 180° + \frac{\cos 2*180}{n}\right] = r\omega^2\left[-1 + \frac{1}{n}\right]$$

As the direction of motion is reversed at the outer dead centre therefore changing the sign of the above expression

$$a_P = r\omega^2\left[1 - \frac{1}{n}\right]$$

Angular velocity and acceleration of the connecting rod

Consider the motion of a connecting rod and a crank as shown in Fig.

From the geometry of the figure,

$$CQ = l\sin\phi = r\sin\theta \qquad \left(\because n = \frac{l}{r}\right)$$

$$\therefore \sin\phi = \frac{r}{l}\sin\theta = \frac{\sin\theta}{n}$$

Differentiating both sides with respect to time t,

$$\cos\phi * \frac{d\phi}{dt} = \frac{\cos\theta}{n} * \frac{d\theta}{dt} = \frac{\cos\theta}{n} * \omega \qquad \left(\because \frac{d\theta}{dt} = \omega\right)$$

Since the angular velocity of the connecting rod PC is same as the angular velocity of point P with respect to C and is equal to $\frac{d\phi}{dt}$, therefore angular velocity of the connecting rod

$$\omega_{PC} = \frac{d\phi}{dt} = \frac{\cos\theta}{n} * \frac{\omega}{\cos\phi} = \frac{\omega}{n} * \frac{\cos\theta}{\cos\phi}$$

we know that, $\cos\phi = (1-\sin^2\phi)^{\frac{1}{2}} = \left(1 - \frac{\sin^2\theta}{n^2}\right)^{\frac{1}{2}}$

$$\therefore \omega_{PC} = \frac{\omega}{n} * \frac{\cos\theta}{\left(1 - \frac{\sin^2\theta}{n^2}\right)^{\frac{1}{2}}} = \frac{\omega}{n} * \frac{\cos\theta}{\frac{1}{n}(n^2-\sin^2\theta)^{1/2}}$$

$$\omega_{PC} = \frac{\omega\cos\theta}{(n^2-\sin^2\theta)^{1/2}} \longrightarrow (1)$$

Angular acceleration of the connecting rod PC,

α_{PC} = Angular acceleration of P with respect to C

$$\alpha_{PC} = \frac{d(\omega_{PC})}{dt}$$

we know that

$$\frac{d(\omega_{PC})}{dt} = \frac{d(\omega_{PC})}{d\theta} * \frac{d\theta}{dt} = \frac{d(\omega_{PC})}{d\theta} * \omega \longrightarrow (2)$$

Now differentiating equation (1), we get

$$\frac{d(\omega_{PC})}{d\theta} = \frac{d}{d\theta}\left[\frac{\omega\cos\theta}{(n^2-\sin^2\theta)^{1/2}}\right]$$

$$= \omega\left[\frac{[(n^2-\sin^2\theta)^{1/2}(-\sin\theta)] - [(\cos\theta) * \frac{1}{2}(n^2-\sin^2\theta)^{-\frac{1}{2}} * -2\sin\theta\cos\theta]}{n^2-\sin^2\theta}\right]$$

$$= \omega\left[\frac{(n^2-\sin^2\theta)^{1/2}(-\sin\theta) + (n^2-\sin^2\theta)^{-1/2}\sin\theta\cos^2\theta}{n^2-\sin^2\theta}\right]$$

$$= \left[\frac{(n^2-\sin^2\theta)^{1/2} - (n^2-\sin^2\theta)^{-1/2}\cos^2\theta}{n^2-\sin^2\theta}\right] * -\omega\sin\theta$$

$$\frac{d(\omega_{PC})}{d\theta} = \left[\frac{(n^2-\sin^2\theta) - \cos^2\theta}{(n^2-\sin^2\theta)^{3/2}}\right] * -\omega\sin\theta \quad \left(\because \text{Dividing \& multiplying by } (n^2-\sin^2\theta)^{1/2}\right)$$

$$\frac{d(\omega_{PC})}{d\theta} = \frac{-\omega\sin\theta}{(n^2-\sin^2\theta)^{3/2}}\left[n^2-(\sin^2\theta+\cos^2\theta)\right]$$

$$= \frac{-\omega\sin\theta\,(n^2-1)}{(n^2-\sin^2\theta)^{3/2}}$$

$$\therefore \alpha_{PC} = \frac{d(\omega_{PC})}{d\theta}\times\omega = \frac{-\omega^2\sin\theta\,(n^2-1)}{(n^2-\sin^2\theta)^{3/2}} \longrightarrow (3)$$

The –ve sign shows that the sense of the acceleration of the connecting rod is such that it tends to reduce the angle ϕ.

Notes:

1. Since $\sin^2\theta$ is small as compared to n^2, therefore it may be neglected. Thus equations (1) & (3) are reduced to

$$\omega_{PC} = \frac{\omega\cos\theta}{n}, \quad \alpha_{PC} = \frac{-\omega^2\sin\theta\,(n^2-1)}{n^3}$$

2. Also in equation (3), unity is small as compared to n^2, hence the term unity may be neglected.

$$\alpha_{PC} = \frac{-\omega^2\sin\theta}{n}$$

Engine Force Analysis

(Forces on the Reciprocating parts of an engine, neglecting the weight of the connecting rod)

The various forces acting on the reciprocating parts of a horizontal engine are shown in Fig. The expressions for these forces, neglecting the weight of the connecting rod.

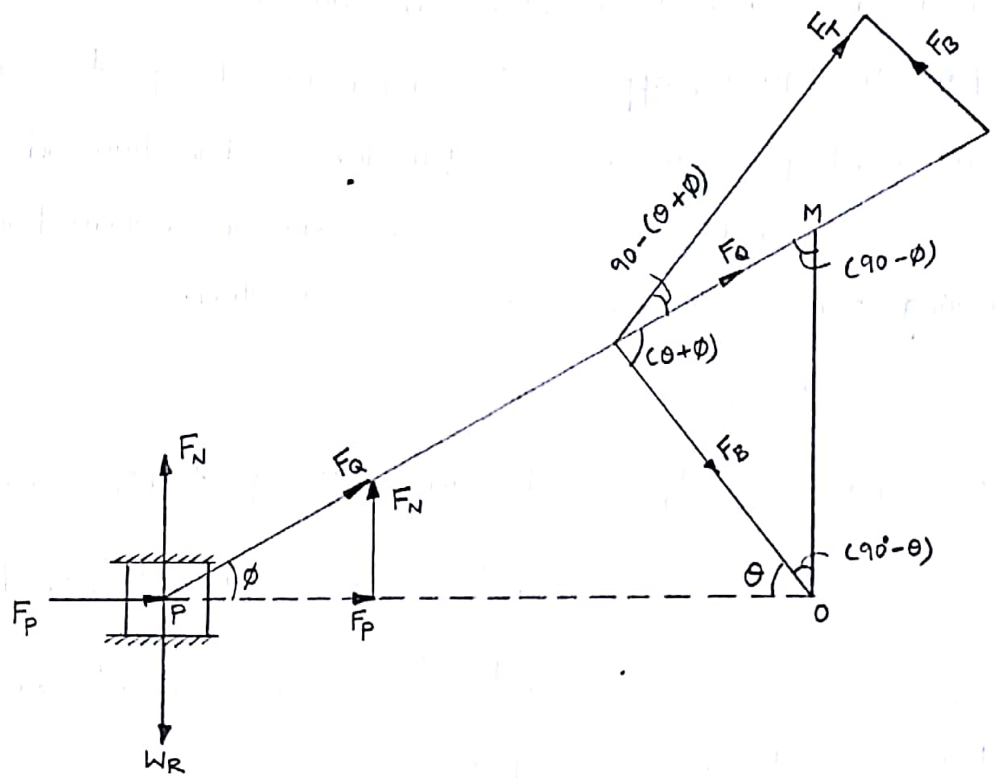


Fig: Forces on the reciprocating parts of an engine

1. Piston effort

It is the net force acting on the piston or crosshead pin, along the line of stroke. It is denoted by F_P.

Let m_R – Mass of the reciprocating parts, e.g. piston, crosshead pin or gudgeon pin etc., in kg

W_R – weight of the reciprocating parts in newtons

$W_R = m_R \cdot g$

We know that acceleration of the reciprocating parts,

$$a_R = a_P = r\omega^2 \left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

∴ Accelerating force or inertia force of the reciprocating parts,

$$F_I = m_R \cdot a_R = m_R \cdot \omega^2 \cdot r \left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

It may be noted that in a horizontal engine, the reciprocating parts are accelerated from rest, during the first half of the stroke (i.e when the piston moves from inner dead centre to outer dead centre). It is, then, retarded during the latter half of the stroke (i.e. when the piston moves from outer dead centre to inner dead centre). The inertia force due to the acceleration of the reciprocating parts, opposes the force on the piston due to the difference of pressure in the cylinder on the two sides of the piston. On the other hand, the inertia force due to retardation of the reciprocating parts, helps the force on the piston.

Therefore,

Piston effort, F_P = Net load on the piston $\mp$ Inertia force

$= F_L \mp F_I$ --- (Neglecting frictional resistance)

$= F_L \mp F_I - R_F$ --- (Considering frictional resistance)

Where R_F – Frictional resistance

The –ve sign is used when the piston is accelerated, and +ve sign is used when the piston is retarded.

In a double acting reciprocating steam engine, net load on the piston,

$$F_L = p_1 A_1 - p_2 A_2 = p_1 A_1 - p_2 (A_1 - a)$$

where p_1, A_1 – Pressure & cross-sectional area on the back end side of the piston,

p_2, A_2 – Pressure & cross-sectional area on the crank end side of the piston

a – cross-sectional area of the piston rod

Notes:

1. If p is the net pressure of steam or gas on the piston and D is diameter of the piston, then

Net load on the piston, F_L = Pressure * Area = $p * \frac{\pi}{4} D^2$

2. In case of a vertical engine, the weight of the reciprocating parts assists the piston effort during the downward stroke (i.e when the piston moves from top dead centre to bottom dead centre) and opposes during the upward stroke of the piston (i.e when the piston moves from bottom dead centre to top dead centre).

$\therefore$ Piston effort, $F_P = F_L \mp F_I \pm W_R - R_F$.

2. Force acting along the connecting rod

It is denoted by F_Q in Fig. From the geometry of the figure, we find that

$$F_Q = \frac{F_P}{\cos\phi}$$

we know that, $\cos\phi = \sqrt{1 - \frac{\sin^2\theta}{n^2}}$

$$F_Q = \frac{F_P}{\sqrt{1 - \frac{\sin^2\theta}{n^2}}}$$

3. Thrust on the sides of the cylinder walls or normal reaction on the guide bars

It is denoted by F_N in Fig. From the geometry of the figure, we find that ($\because F_Q = \frac{F_P}{\cos\phi}$)

$$F_N = F_Q \sin\phi = \frac{F_P}{\cos\phi} * \sin\phi = F_P \tan\phi$$

4. Crank-pin effort and thrust on crank shaft bearings

The force acting on the connecting rod F_Q may be resolved into two components, one perpendicular to the crank and the other along the crank. The component of F_Q perpendicular to crank is known as crank-pin effort and it is denoted by F_T in Fig.

The component of F_Q along the crank produces a thrust on the crank shaft bearings and it is denoted by F_B in Fig.

Resolving F_Q perpendicular to the crank,

$$F_T = F_Q \sin(\theta+\phi) = \frac{F_P}{\cos\phi} * \sin(\theta+\phi)$$

and resolving F_Q along the crank,

$$F_B = F_Q \cos(\theta+\phi) = \frac{F_P}{\cos\phi} * \cos(\theta+\phi)$$

5. Crank effort or turning moment or torque on the crank shaft

The product of the crank-pin effort (F_T) and the crank pin radius (r) is known as crank effort or turning moment or torque on the crank shaft.

$$\text{Crank effort, } T = F_T * r = \frac{F_P \sin(\theta+\phi)}{\cos\phi} * r$$

$$= \frac{F_P(\sin\theta.\cos\phi + \cos\theta.\sin\phi)}{\cos\phi} * r$$

$$= F_P\left(\sin\theta + \cos\theta * \frac{\sin\phi}{\cos\phi}\right) * r$$

$$T = F_P(\sin\theta + \cos\theta.\tan\phi) * r \longrightarrow (1)$$

we know that $l\sin\phi = r\sin\theta$

$$\sin\phi = \frac{r}{l}\sin\theta = \frac{\sin\theta}{n} \qquad \left(\because n = \frac{l}{r}\right)$$

$$\cos\phi = \sqrt{1-\sin^2\phi}$$

$$\cos\phi = \sqrt{1 - \frac{\sin^2\theta}{n^2}} = \frac{1}{n}\sqrt{n^2 - \sin^2\theta}$$

$$\therefore \tan\phi = \frac{\sin\phi}{\cos\phi}$$

$$\tan\phi = \frac{\sin\theta}{n} * \frac{n}{\sqrt{n^2-\sin^2\theta}} = \frac{\sin\theta}{\sqrt{n^2-\sin^2\theta}}$$

Substituting the value of $\tan\phi$ in equation (1), we have crank effort,

$$T = F_P\left(\sin\theta + \frac{\cos\theta\sin\theta}{\sqrt{n^2-\sin^2\theta}}\right) * r = F_P * r\left(\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2-\sin^2\theta}}\right)$$

Crank effort, $T = F_p \times r \left(\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2 - \sin^2\theta}} \right)$

Note: since $\sin^2\theta$ is very small as compared to n^2 therefore neglecting $\sin^2\theta$, we have

$$\text{Crank effort, } T = F_p \times r \left(\sin\theta + \frac{\sin 2\theta}{n} \right)$$

Turning Moment Diagram

The turning moment diagram (also known as crank-effort diagram) is the graphical representation of the turning moment or crank-effort for various positions of the crank. It is plotted on cartesian co-ordinates, in which the turning moment is taken as the ordinate and crank angle as abscissa.

Turning Moment Diagram for a single cylinder Double acting steam Engine

A turning moment diagram for a single cylinder double acting steam engine is shown in Fig. The vertical ordinate represents the turning moment and the horizontal ordinate represents the crank angle.

We know that the turning moment on the crankshaft

$$T = F_p \times r\left(\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2 - \sin^2\theta}}\right)$$

where

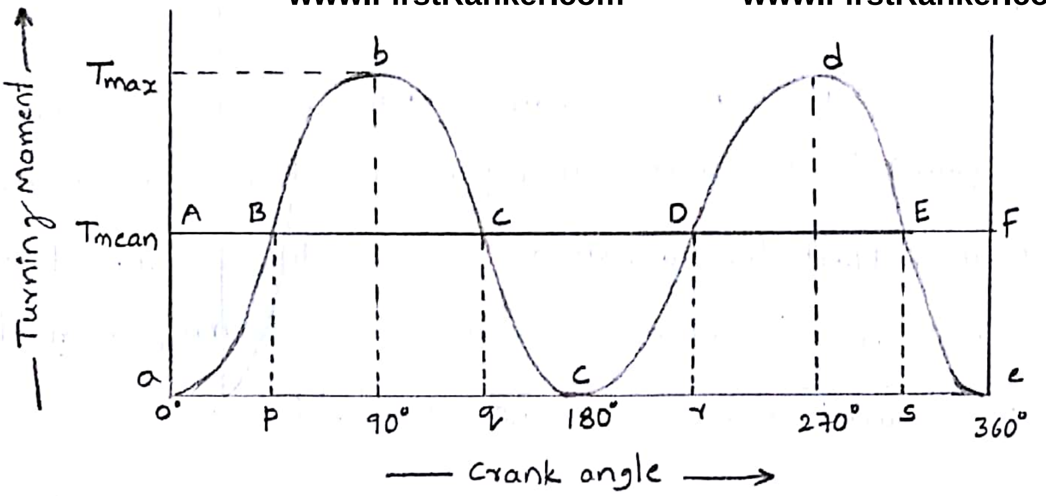
F_p – Piston effort

r – Radius of crank

n – Ratio of the connecting rod length & radius of crank

θ – Angle turned by the crank from inner dead centre

From the above expression, we see that the turning moment (T) is zero, when the crank angle (θ) is zero. It is maximum when the crank angle is 90° and it is again zero when crank angle is 180°.



This is shown by the curve abc in Fig. and it represents the turning moment diagram for outstroke. The curve cde is the turning moment diagram for instroke and is somewhat similar to the curve abc.

Since the workdone is the product of the turning moment and the angle turned, therefore the area of the turning moment diagram represents the work done per revolution. In actual practice, the engine is assumed to work against the mean resisting torque, as shown by a horizontal line AF. The height of the ordinate aA represents the mean height of the turning moment diagram. Since it is assumed that the work done by the turning moment per revolution is equal to the work done against the mean resisting torque, therefore the area of the rectangle aAFe is proportional to the work done against the mean resisting torque.

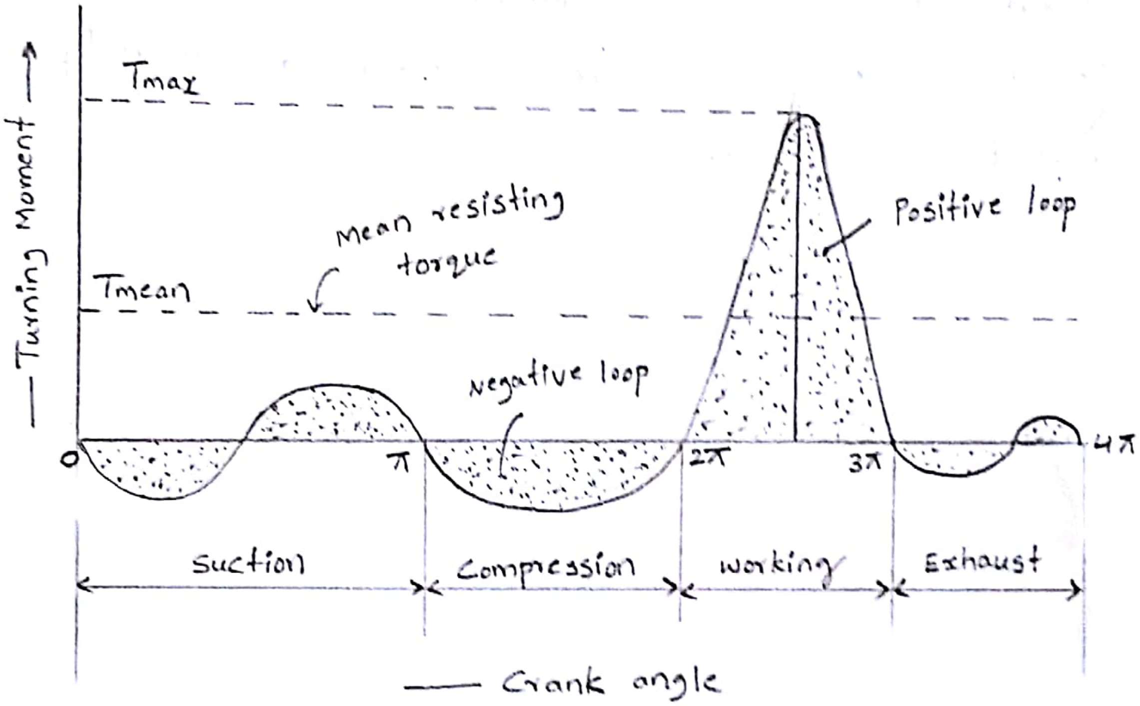
Notes:

1. When the turning moment is positive (i.e when the engine torque is more than the mean resisting torque) as shown between points B and C (or D and E) in Fig, the crankshaft accelerates and the work is done by the steam.

2. When the turning moment is negative (i.e when the engine torque is less than the mean resisting torque) as shown between points C and D in Fig, the crankshaft retards and the work is done on the steam.

Turning Moment Diagram for a Four stroke Cycle Internal Combustion Engine

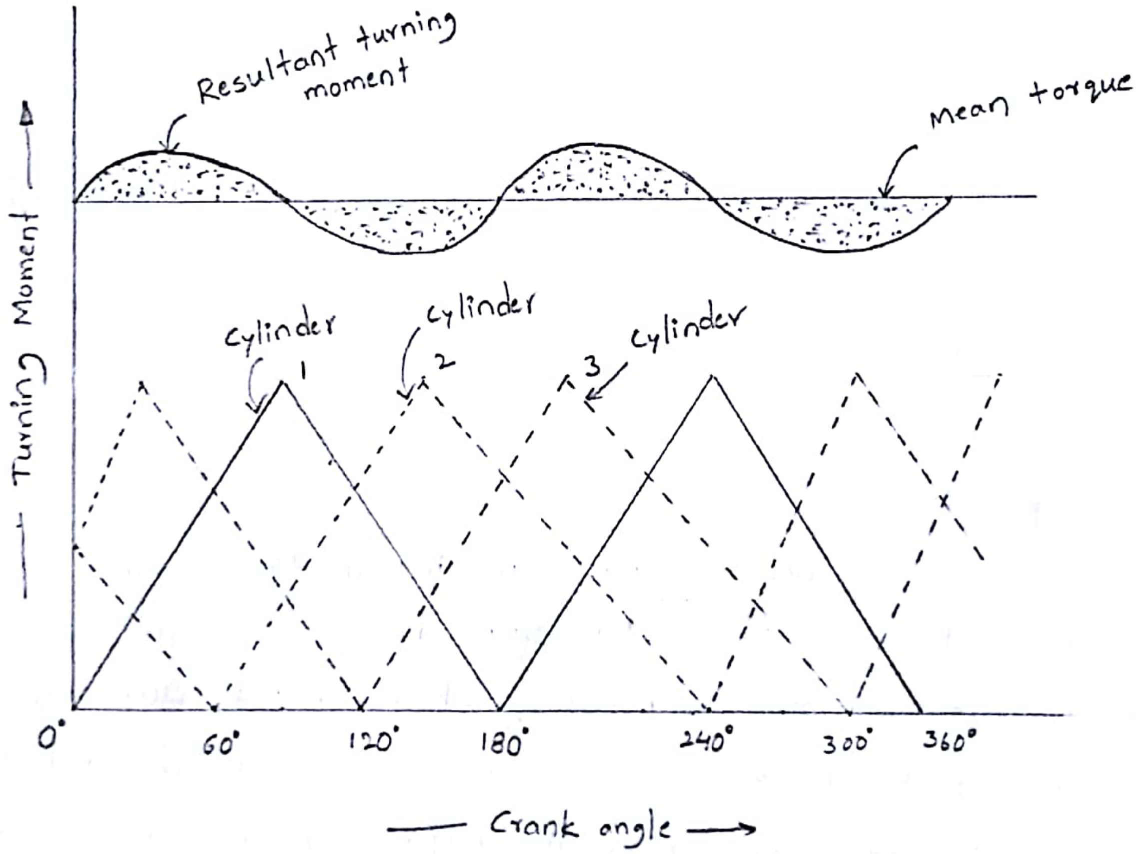
A turning moment diagram for a four stroke cycle cycle internal combustion engine is shown in Fig. we know that in a four stroke cycle internal combustion engine, there is one working stroke after the crank has turned through two revolutions, i.e. 720° (or 4π radians).



Since the pressure inside the engine cylinder is less than the atmospheric pressure during the suction stroke, therefore a negative loop is formed as shown in Fig. During the compression stroke, the work done on the gases, therefore a higher negative loop is obtained. During the expansion or working stroke, the fuel burns and the gases expand, therefore a large positive loop is obtained. In this stroke, the work is done by the gases. During exhaust stroke, the work is done on the gases, therefore a negative loop is obtained. It may be noted that the effect of the inertia forces on the piston is taken into account in Fig.

Turning Moment Diagram for a Multi-Cylinder Engine

The separate turning moment diagram for a compound steam engine having three cylinders and the resultant turning moment diagram is shown in Fig. The resultant turning moment diagram is the sum of the turning moment diagrams for the three cylinders. It may be noted that the first cylinder is the high pressure cylinder, second cylinder is the intermediate cylinder and the third cylinder is the low pressure cylinder. The crank, in case of three cylinders, are usually placed at 120° to each other.



Fluctuation of Energy

The fluctuation of energy may be determined by the turning moment diagram for one complete cycle of operation. Consider the turning moment diagram for a single cylinder double acting steam engine as shown in Fig.

We see that the mean resisting torque line AF cuts the turning moment diagram at points B, C, D and E. When the crank moves from a to p, the work done by the engine is equal to the area aBp, whereas the energy required is represented by the area aABp. In other words, the engine has done less work (equal to the area aAB) than the requirement. This amount of energy is taken from the flywheel and hence the speed of the flywheel decreases. Now the crank moves from p to q, the work done by the engine is equal to the area pBbCq, whereas the requirement of energy is represented by the area pBCq. Therefore the engine has done more work than the requirement. This excess work (equal to the area BbC) is stored in the flywheel and hence the speed of the flywheel increases while the crank moves from p to q.

Similarly, when the crank moves from q to r, more work is taken from the engine than is developed. This loss of work is represented by the area CcD. To supply this loss, the flywheel gives up some of its energy and thus the speed decreases while the crank moves from q to r. As the crank moves from r to s, excess energy is again developed given by the area DdE and the speed again increases. As the piston moves from s to e, again there is a loss of work and the speed decreases. The variations of energy above and below the mean resisting torque line are called fluctuations of energy. The areas BbC, CcD, DdE, etc. represent fluctuations of energy.

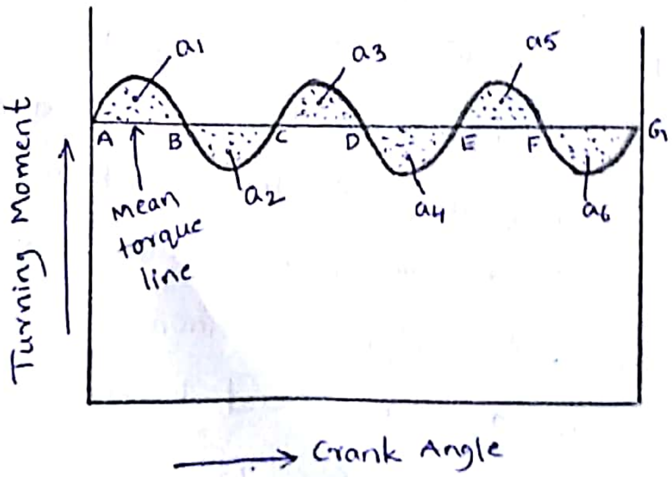
A little consideration will show that the engine has a maximum speed either at q or at s. This is due to the fact that the flywheel absorbs energy while the crank moves from p to q and from r to s. On the other hand, the engine has a minimum speed either at p or at r. The reason is that the flywheel gives out some of its energy when the crank moves from a to p and q to r. The difference between the maximum and the minimum energies is known as maximum fluctuation of energy.

 www.FirstRanker.com

Determination of Maximum Fluctuation of Energy

A turning moment diagram for a multi-cylinder engine is shown by a wavy curve in Fig. The horizontal line AG represents the mean torque line.

Let a_1, a_3, a_5 be the areas above the mean torque line and a_2, a_4 and a_6 be the areas below the mean torque line. These areas represent some quantity of energy which is either added or subtracted from the energy of the moving parts of the engine.



Let the energy in the flywheel at A = E, then from Fig, we have

Energy at B $= E + a_1$

Energy at C $= E + a_1 - a_2$

Energy at D $= E + a_1 - a_2 + a_3$

Energy at E $= E + a_1 - a_2 + a_3 - a_4$

Energy at F $= E + a_1 - a_2 + a_3 - a_4 + a_5$

Energy at G $= E + a_1 - a_2 + a_3 - a_4 + a_5 - a_6$

= Energy at A (i.e. cycle repeats after G)

Let us now suppose that the greatest of these energies is at B and least at E.

Therefore,

Maximum energy in flywheel

$= E + a_1$

Minimum energy in the flywheel

$= E + a_1 - a_2 + a_3 - a_4$

∴ Maximum fluctuation of energy,

ΔE = Maximum energy − Minimum energy

$= (E + a_1) - (E + a_1 - a_2 + a_3 - a_4)$

$$\Delta E = a_2 - a_3 + a_4$$

Coefficient of Fluctuation of Energy (C_E)

It may be defined as the ratio of the maximum fluctuation of energy to the work done per cycle.

$$C_E = \frac{\text{Maximum fluctuation of energy}}{\text{Work done per cycle}}$$

Work done per cycle $= T_{mean} * \theta$

where T_{mean} – Mean torque

θ – Angle turned (in radians), in one revolution

$= 2\pi$, in case of steam engine and two stroke internal combustion engines

$= 4\pi$, in case of four stroke internal combustion engines

The mean torque (T_{mean}) is N-m

$$T_{mean} = \frac{P * 60}{2\pi N} = \frac{P}{\omega}$$

where P – Power transmitted in watts

N – speed in rpm

ω – Angular speed in rad/s $= \frac{2\pi N}{60}$

The work done per cycle may also be obtained

$$\text{work done per cycle} = \frac{P * 60}{n}$$

where n – Number of working strokes per minute

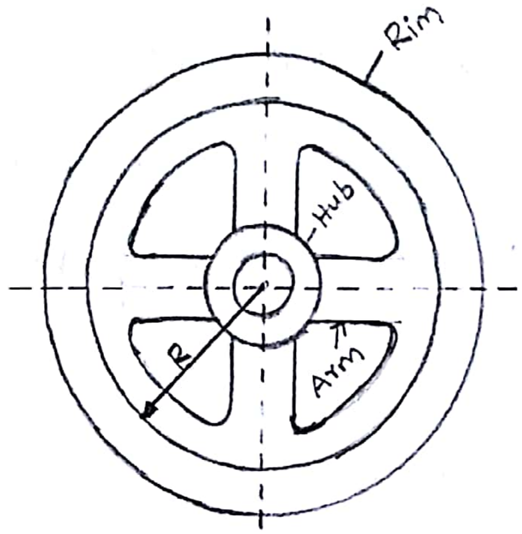
$= N$, in case of steam engines and two stroke internal combustion

$= \frac{N}{2}$, in case of four stroke internal combustion engines

C_E values

1. Single cylinder, double acting steam engine – 0.21
2. Cross-compound steam engine – 0.096
3. Single cylinder, single acting, 4S gas engine – 1.93
4. Four cylinders, single acting, 4S gas engine – 0.066
5. Six cylinders, single acting, 4S gas engine – 0.031

Energy Stored in a Flywheel



A flywheel is shown in Fig. that when a flywheel absorbs energy, its speed increases and when it gives up energy, its speed decreases.

Let m – Mass of the flywheel in kg

k – Radius of gyration of the flywheel in meters

I – Mass moment of inertia of the flywheel about its axis of rotation in kg-m² $= mk^2$

N_1 & N_2 – Max. & Min. speeds during the cycle in rad/s

ω_1 & ω_2 – Max. & Min. angular speeds during the cycle in rad/s

N = Mean speed during the cycle in rpm $= \frac{N_1 + N_2}{2}$

ω = Mean angular speed during the cycle in rad/s $= \frac{\omega_1 + \omega_2}{2}$

C_s = coefficient of fluctuation of speed $= \frac{N_1 - N_2}{N}$

(or) $\frac{\omega_1 - \omega_2}{\omega}$

Kinetic energy of the flywheel,

$$E = \frac{1}{2} * I \cdot \omega^2 = \frac{1}{2} * m \cdot k^2 \cdot \omega^2 \quad \text{in N-m or joules}$$

As the speed of the flywheel changes from ω_1 to ω_2, the maximum fluctuation of energy.

$$\Delta E = \text{Maximum K.E} - \text{Minimum K.E}$$

$$= \frac{1}{2} * I\omega_1^2 - \frac{1}{2} * I\omega_2^2 = \frac{1}{2} * I\left[\omega_1^2 - \omega_2^2\right]$$

$$= \frac{1}{2} * I\,(\omega_1 + \omega_2)(\omega_1 - \omega_2) = I\omega(\omega_1 - \omega_2)$$

$$\Delta E = I\omega(\omega_1 - \omega_2) \longrightarrow ① \qquad \left(\because \omega = \frac{\omega_1 + \omega_2}{2}\right)$$

$$= I\cdot\omega^2\left(\frac{\omega_1 - \omega_2}{\omega}\right) \qquad (\because \text{Multiplying \& dividing by } \omega)$$

$$\Delta E = I\omega^2 C_s = mk^2\omega^2 C_s \rightarrow ② \qquad (\because I = mk^2)$$

$$\Delta E = 2.E.C_s \longrightarrow ③ \qquad \left(\because E = \frac{1}{2} I\omega^2\right)$$

The radius of gyration (k) may be taken equal to the mean radius of the rim (R), because the thickness of rim is very small as compared to the diameter of rim. Therefore, substituting $k=R$ in equation ②

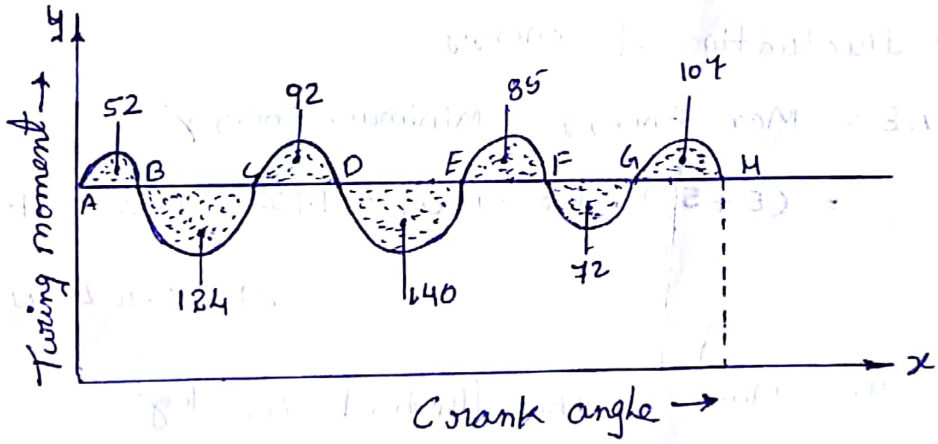
$$\Delta E = m.R^2.\omega^2.C_s = m.v^2.C_s$$

where v = Mean linear velocity (i.e at the mean radius) in m/s $= R\cdot\omega$.

(P) The turning moment diagram for a multicylinder engine has been drawn to a scale 1mm = 600 N-m vertically and 1mm = 3° horizontally. The intercepted areas between the output torque curve and the mean resistance line, taken in order from one end, as follows: +52, −124, +92, −140, +85, −72 & + 107 mm^2 when the engine is running at a speed of 600 rpm. If the total fluctuation of speed is not to exceed ± 1.5% of the mean, find the necessary mass of the flywheel of radius 0.5m.

Sol:- N = 600 rpm (or) $\omega = \frac{2\pi \times 600}{60} = 62.84$ rad/s

R = 0.5m



Since the total fluctuation of speed is not to exceed ± 1.5% of the mean speed,

$$\omega_1 - \omega_2 = 3\%\ \omega = 0.03\omega$$

coefficient of fluctuation of speed

$$C_s = \frac{\omega_1 - \omega_2}{\omega} = 0.03$$

Since the turning moment scale is 1mm = 600 N-m & crank angle scale is 1mm = 3° = $\frac{3 \times \pi}{180} = \frac{\pi}{60}$ rad

1 mm² on turning moment diagram

$$= 600 \times \frac{\pi}{60} = 31.42 \text{ N-m}$$

Let the total energy at A = E

Energy at B = E + 52

Energy at C = E + 52 − 124 = E − 72

Energy at D = E − 72 + 92 = E + 20

Energy at E = E + 20 − 140 = E − 120

Energy at F = E − 120 + 85 = E − 35

Energy at G = E − 35 − 72 = E − 107

Energy at H = E − 107 + 107 = E = Energy at A.

Max. fluctuation of energy

ΔE = Max. Energy − Minimum Energy

$= (E+52) - (E-120) = 172 = 172 \times 31.42$

$\Delta E = 5404$ N-m

Let m = Mass of the flywheel in kg

Max. fluctuation of Energy $(\Delta E) = m R^2 \omega^2 C_s$

$5404 = m (0.5)^2 (62.84)^2 0.03$

$m = 183$ kg

Governors

The function of a governor is to regulate the mean speed of an engine, when there are variations in the load e.g. when the load on an engine increases, its speed decreases, therefore it becomes necessary to increase the supply of working fluid. On the other hand, when the load on the engine decreases, its speed increases and thus less working fluid is required. The governor automatically controls the supply of working fluid to the engine with the varying load conditions and keeps the mean speed within certain limits.

When the load increases, the configuration of the governor changes and a valve is moved to increase the supply of the working fluid; conversely, when the load decreases, the engine speed increases and the governor decreases the supply of working fluid.

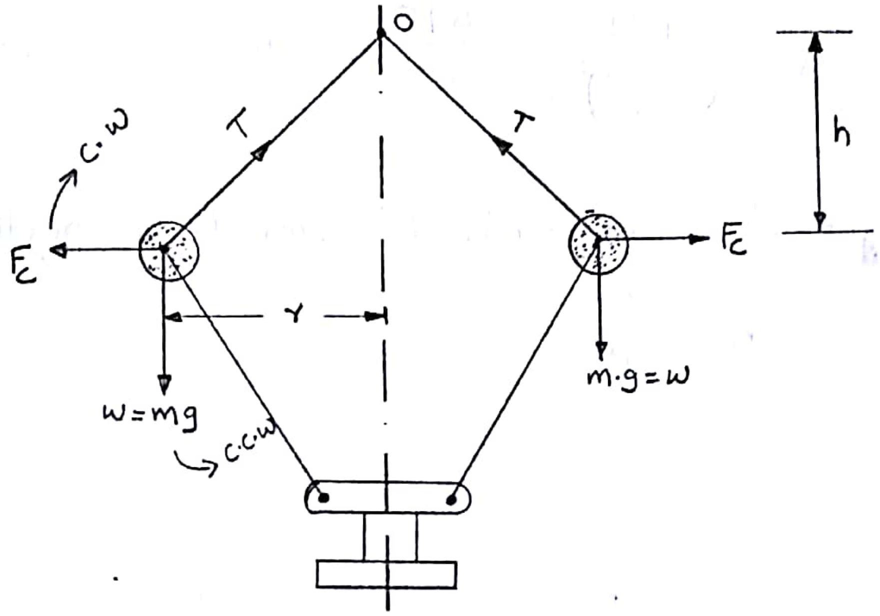
Note: The function of a flywheel in an engine is entirely different from that of a governor. It controls the speed variation caused by the fluctuations of the engine turning moment during each cycle of operation. It does not control the speed variations caused by a varying load. The varying demand for power is met by the governer regulating the supply of working fluid.

Types of Governors

The governors may, broadly, be classified as

1. Centrifugal Governors
2. Inertia Governors

Watt Governor (or) Simple Conical Governor



Let

- m – Mass of the ball in kg
- w – Weight of the ball in newtons = m.g
- T – Tension in the arm in newtons
- ω – Angular velocity of the arm and ball about the spindle axis in rad/s
- r – Radius of the path of rotation of the ball i.e., horizontal distance from the centre of the ball to the spindle axis in metres.
- F_c – Centrifugal force acting on the ball in newtons

$$F_c = m \cdot r \cdot \omega^2$$

- h – Height of the governer in meters

It is assumed that the weight of the arms, links and the sleeve are negligible as compared to the weight of the balls. Now, the ball is in equilibrium under the action of F_c, T & w.

Taking moments about point O.

$$\Sigma M_o = 0$$

$$F_c * h = m \cdot g * r$$

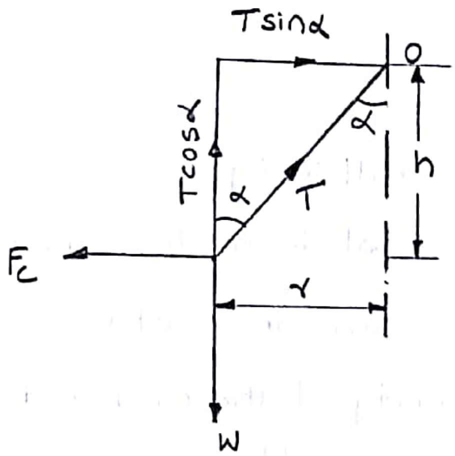
$$\cancel{m \cdot r \cdot \omega^2 * h = m \cdot g \cdot r} \qquad h = \frac{g}{\omega^2} \longrightarrow ①$$

$g = 9.81\ m/s^2$, $\omega = \frac{2\pi N}{60}$

$$\therefore \quad h = \frac{9.81}{\left(\frac{2\pi N}{60}\right)^2} = \frac{895}{N^2} \text{ metres} \longrightarrow ②$$

The height of a governor h, is inversely proportional to N^2.

$$h \propto \frac{1}{N^2}$$



$$\tan\alpha = \frac{r}{h}$$

The equilibrium of the mass

$\Sigma H = 0$

$T\sin\alpha - F_c = 0$

$T\sin\alpha = F_c$

$T\sin\alpha = m \cdot r \cdot \omega^2 \longrightarrow ①$

$\Sigma V = 0$

$T\cos\alpha - w = 0$

$T\cos\alpha = w$

$T\cos\alpha = m \cdot g \longrightarrow ②$

$$\frac{T\sin\alpha}{T\cos\alpha} = \frac{m \cdot r \cdot \omega^2}{m \cdot g}$$

$$\tan\alpha = \frac{r \cdot \omega^2}{g}$$

$$\frac{r}{h} = \frac{r\omega^2}{g}$$

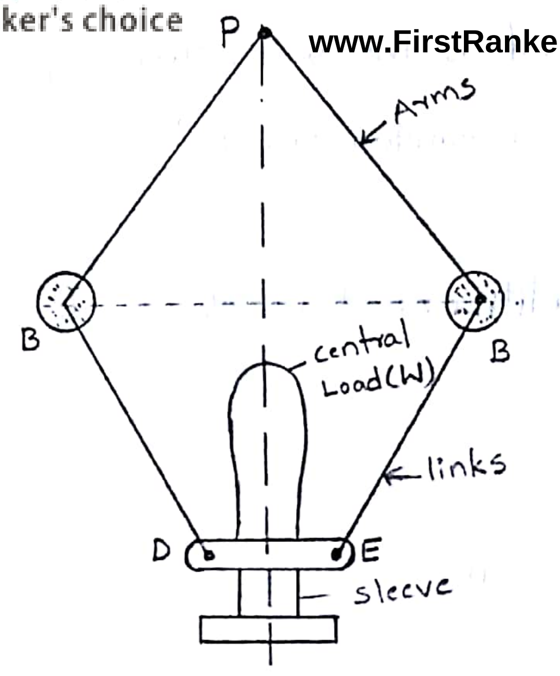
where $g = 9.81\ m/s^2$

$\omega = \frac{2\pi N}{60}$ rad/s

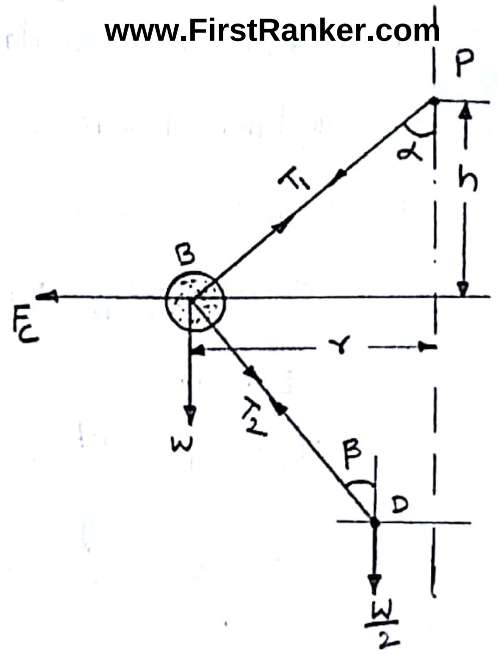
$$h = \frac{g}{\omega^2}$$

$$\therefore h = \frac{895}{N^2} \text{ metres}$$

Porter Governor



(a)



(b)

The Porter governor is a modification of Watt's governor, with central load attached to the sleeve as shown in Fig (a).

Consider the forces acting on one-half of the governor as shown in Fig (b).

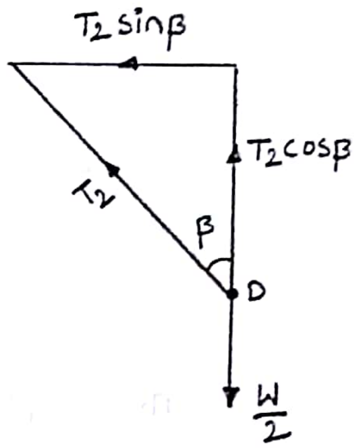
Let

- m – Mass of each ball in kg
- w – weight of each ball in newtons = m.g
- M – Mass of the central load in kg
- W – weight of the central load in newtons = M.g
- r – Radius of rotation in metres,
- h – Height of governor in metres
- N – Speed of the balls in rpm
- ω – Angular speed of the balls in rad/s = $\frac{2\pi N}{60}$
- F_c – Centrifugal force acting on the ball in newtons
  $F_c = m \cdot r \cdot \omega^2$
- T_1 & T_2 – Force in the arm & in the link in newtons
- α & β – Angle of inclination of the arm & of the link to the vertical

Determining the relation between the height of the governor (h) and the angular speed of the balls (ω) by using method of resolution of forces and Instantaneous method.

Method of resolution of forces

considering the equilibrium of the forces acting at D,

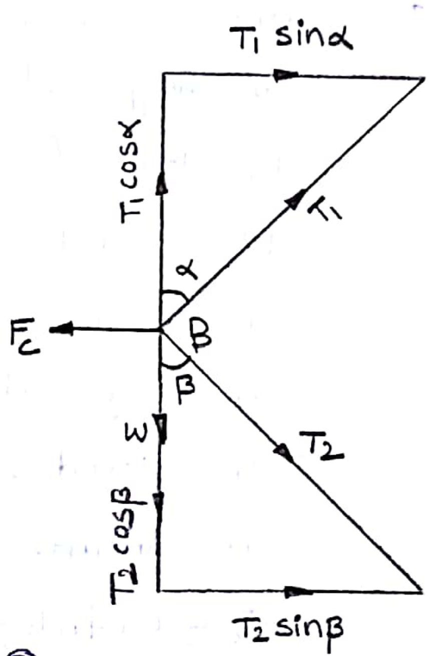


$\Sigma V = 0$

$$T_2 \cos\beta = \frac{W}{2} = \frac{M \cdot g}{2}$$

$$T_2 = \frac{M \cdot g}{2\cos\beta} \longrightarrow ①$$

Again, considering the equilibrium of the forces acting on B. The point B is in equilibrium under the action of forces w, F_c, T_1 & T_2.



$\Sigma V = 0$

$$T_1 \cos\alpha - T_2 \cos\beta - w = 0$$

$$T_1 \cos\alpha = T_2 \cos\beta + w$$

$$T_1 \cos\alpha = \frac{M \cdot g}{2} + m \cdot g \longrightarrow ②$$

$\Sigma H = 0$

$$T_1 \sin\alpha + T_2 \sin\beta - F_c = 0$$

$$T_1 \sin\alpha + \frac{M \cdot g}{2\cos\beta} * \sin\beta - F_c = 0$$

$$T_1 \sin\alpha = F_c - \frac{M \cdot g}{2} * \tan\beta \longrightarrow ③$$

Dividing equation ③ by equation ②

$$\frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{F_c - \frac{M \cdot g}{2} * \tan\beta}{\frac{M \cdot g}{2} + m \cdot g}$$

$$\left(\frac{M\cdot g}{2}+m\cdot g\right)\operatorname{Tan}\alpha = F_c - \frac{M\cdot g}{2} * \operatorname{Tan}\beta$$

$$\frac{M\cdot g}{2}+m\cdot g = \frac{F_c}{\operatorname{Tan}\alpha} - \frac{M\cdot g}{2} * \frac{\operatorname{Tan}\beta}{\operatorname{Tan}\alpha}$$

Substituting $\frac{\operatorname{Tan}\beta}{\operatorname{Tan}\alpha} = q$, $\operatorname{Tan}\alpha = \frac{r}{h}$

$$\frac{M\cdot g}{2}+mg = m\cdot r\cdot \omega^2 * \frac{h}{r} - \frac{M\cdot g}{2} * q$$

$$m\cdot\omega^2\cdot h = m\cdot g + \frac{M\cdot g}{2}(1+q)$$

$$h = \left[m\cdot g + \frac{M\cdot g}{2}(1+q)\right]\frac{1}{m\cdot\omega^2} = \frac{m+\frac{M}{2}(1+q)}{m} * \frac{g}{\omega^2} \longrightarrow (4)$$

$$\omega^2 = \left[m\cdot g + \frac{M\cdot g}{2}(1+q)\right]\frac{1}{m\cdot h} = \frac{m+\frac{M}{2}(1+q)}{m} * \frac{g}{h}$$

$$N^2 = \frac{m+\frac{M}{2}(1+q)}{m} * \frac{g}{h}\left(\frac{60}{2\pi}\right)^2 \qquad \because g = 9.81\ m/s^2$$

$$N^2 = \frac{m+\frac{M}{2}(1+q)}{m} * \frac{895}{h} \longrightarrow (5)$$

Notes: 1. When the length of arms are equal to the length of links and the points P and D lie on the same vertical line, then

$$\operatorname{Tan}\alpha = \operatorname{Tan}\beta \quad (or) \quad q = \frac{\operatorname{Tan}\beta}{\operatorname{Tan}\alpha} = 1$$

Therefore, the equation (5) becomes

$$N^2 = \frac{(m+M)}{m} * \frac{895}{h} \longrightarrow (6)$$

2. When the loaded sleeve moves up and down the spindle, the frictional force acts on it in a direction opposite to that of the motion of sleeve.

If F = Frictional force acting on the sleeve in newtons, then equations (5) and (6) may be written as

$$N^2 = \frac{m\cdot g + \left(\frac{M\cdot g \pm F}{2}\right)(1+q)}{m\cdot g} * \frac{895}{h} \longrightarrow (7)$$

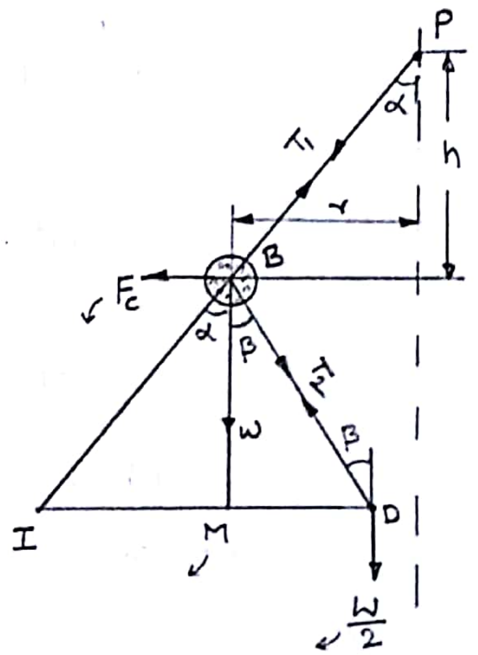
$$N^2 = \frac{m \cdot g + (M \cdot g \pm F)}{m \cdot g} \times \frac{895}{h} \quad (\because q = 1) \longrightarrow (8)$$

The +ve sign is used when the sleeve moves upwards or the governor speed increases and −ve sign is used when the sleeve moves downwards or the governor speed decreases.

3. On comparing the equation (6) with equation (2) of watt's governor we find that the mass of the central load (M) increases the height of governor in the ratio $\frac{m+M}{m}$.

Instantaneous centre method

Taking moments about the point I,



$$F_c \times BM = w \times IM + \frac{W}{2} \times ID$$

$$= m \cdot g \times IM + \frac{M \cdot g}{2} \times ID$$

$$\therefore F_c = m \cdot g \times \frac{IM}{BM} + \frac{M \cdot g}{2} \times \frac{ID}{BM}$$

$$= m \cdot g \times \frac{IM}{BM} + \frac{M \cdot g}{2}\left(\frac{IM + MD}{BM}\right)$$

$$F_c = m \cdot g \tan\alpha + \frac{M \cdot g}{2}(\tan\alpha + \tan\beta)$$

Dividing throughout by $\tan\alpha$

$$\frac{F_c}{\tan\alpha} = m \cdot g + \frac{M \cdot g}{2}\left(1 + \frac{\tan\beta}{\tan\alpha}\right) = m \cdot g + \frac{M \cdot g}{2}(1+q)$$

$\because \tan\alpha = \frac{IM}{BM}$

$\tan\beta = \frac{MD}{BM}$

$q = \frac{\tan\beta}{\tan\alpha}$

we know that $F_c = m \cdot r \cdot \omega^2$ & $\tan\alpha = \frac{r}{h}$

$$\therefore m \cdot r \cdot \omega^2 \times \frac{h}{r} = m \cdot g + \frac{M \cdot g}{2}(1+q)$$

$$h = \frac{m \cdot g + \frac{M \cdot g}{2}(1+q)}{m} \times \frac{1}{\omega^2} = \frac{m + \frac{M}{2}(1+q)}{m} \times \frac{g}{\omega^2}$$

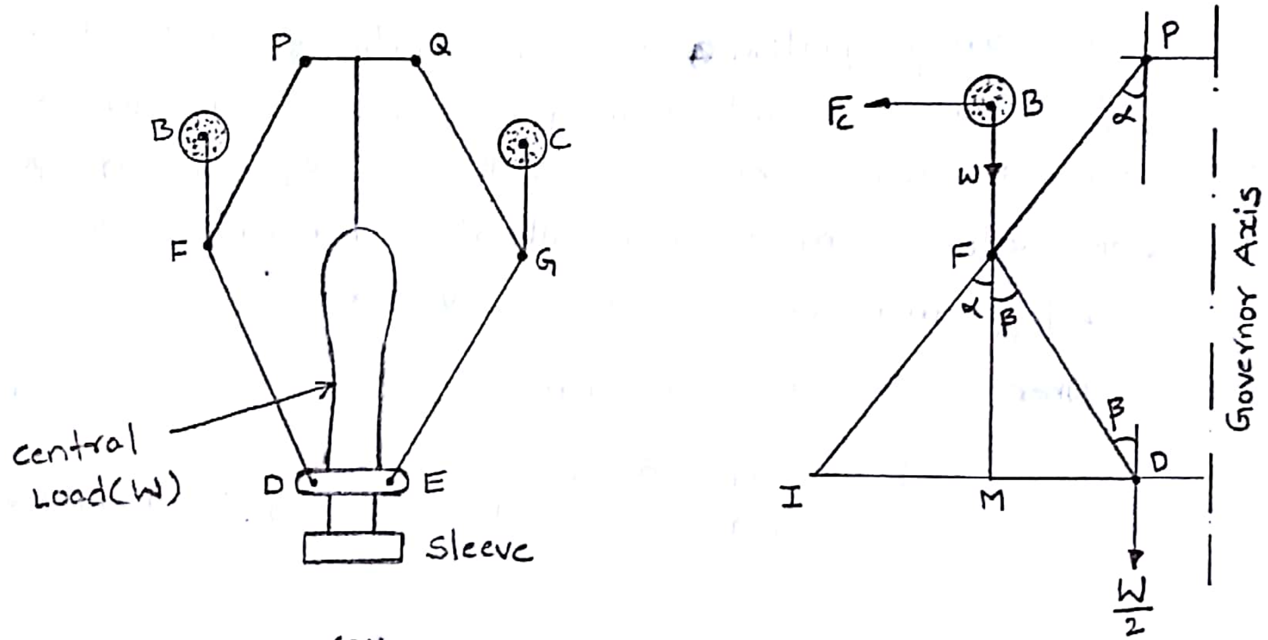
when $\tan\alpha = \tan\beta$ or $q = 1$, then

$$h = \frac{m+M}{m} \times \frac{g}{\omega^2}$$

Proell Governor

The Proell governor has the balls fixed at B and C to the extension of the links DF and EG, as shown in Fig (a). The arms FP and GQ are pivoted at P and Q respectively.

Consider the equilibrium of the forces on one-half of the governor as shown in Fig (b). The instantaneous centre (I) lies on the intersection of the line PF produced and the line from D drawn perpendicular to the spindle axis. The perpendicular BM is drawn on ID.



(a) (b)

Taking moments about I, using the same notations as in Porter governor,

$$F_c * BM = W * IM + \frac{W}{2} * ID = m.g * IM + \frac{M.g}{2} * ID \longrightarrow ①$$

$$F_c = m.g * \frac{IM}{BM} + \frac{M.g}{2}\left(\frac{IM+MD}{BM}\right) \qquad (\because ID = IM + MD)$$

Multiplying and dividing by FM, we have

$$F_c = \frac{FM}{BM}\left[m.g * \frac{IM}{FM} + \frac{M.g}{2}\left(\frac{IM}{FM}+\frac{MD}{FM}\right)\right]$$

$$= \frac{FM}{BM}\left[m.g * \tan\alpha + \frac{M.g}{2}(\tan\alpha + \tan\beta)\right]$$

$$F_c = \frac{FM}{BM} * \tan\alpha\left[m.g + \frac{M.g}{2}\left(1+\frac{\tan\beta}{\tan\alpha}\right)\right]$$

We know that $F_c = m.r.\omega^2$, $\tan\alpha = \frac{r}{h}$ and $q = \frac{\tan\beta}{\tan\alpha}$

$$m \cdot r \cdot \omega^2 = \frac{FM}{BM} * \frac{r}{h}\left[m \cdot g + \frac{M \cdot g}{2}(1+q)\right]$$

$$\omega^2 = \frac{FM}{BM}\left[\frac{m + \frac{M}{2}(1+q)}{m}\right]\frac{g}{h} \longrightarrow (2)$$

Substituting $\omega = 2\pi N/60$, and $g = 9.81\ m/s^2$

$$N^2 = \frac{FM}{BM}\left[\frac{m + \frac{M}{2}(1+q)}{m}\right]\frac{895}{h} \longrightarrow (3)$$

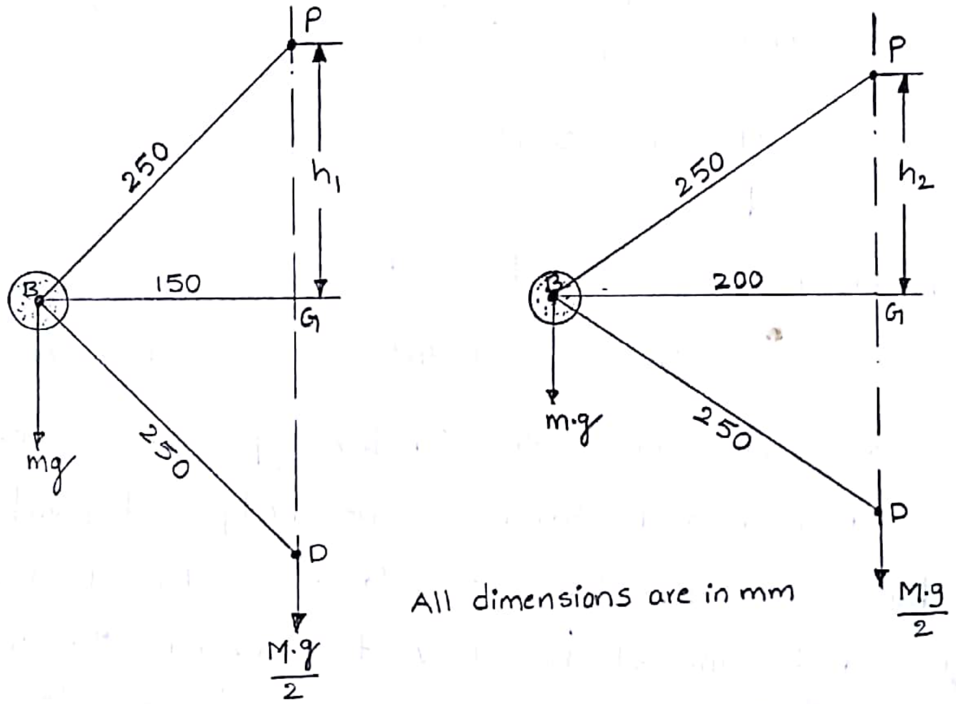
Notes: 1. The equation (1) may be applied to any given configuration of the governor.

2. Comparing equation (4) with the equation (6) of the Porter governor, we see that the equilibrium speed reduces for the given values of m, M and h. Hence in order to have the same equilibrium speed for the given values of m, M and h, balls of smaller masses are used in the Proell governor than in the Porter governor.

3. When $\alpha = \beta$, then $q = 1$. Therefore equation (3) may be written as

$$N^2 = \frac{FM}{BM}\left(\frac{m+M}{m}\right)\frac{895}{h} \longrightarrow (4) \quad (h \text{ in meters})$$

A Porter governor has equal arms each 250 mm long and pivoted on the axis of rotation. Each ball has a mass of 5 kg and the mass of the central load on the sleeve is 15 kg. The radius of rotation of the ball is 150 mm when the governor begins to lift and 200 mm when the governor is at maximum speed. Find the minimum & maximum speeds and range of speed of the governor.



a) Minimum Position

b) Maximum Position

The minimum and maximum position of the governor are shown in Fig (a) and (b).

$$BP = BD = 250\,mm = 0.25\,m, \quad r_1 = 150\,mm = 0.15\,m$$

$$r_2 = 200\,mm = 0.2\,m, \quad m = 5\,kg, \quad M = 15\,kg.$$

Minimum speed when $r_1 = BG = 0.15\,m$

Let N_1 = Minimum speed

From Fig (a), we find that height of the governor

$$h_1 = PG = \sqrt{(PB)^2 - (BG)^2} = \sqrt{(0.25)^2 - (0.15)^2} = 0.2\,m$$

We know that

$$(N_1)^2 = \frac{m+M}{m} * \frac{895}{h_1} = \frac{5+15}{5} * \frac{895}{0.2} = 17900$$

$$N_1 = 133.8 \text{ rpm}$$

Maximum speed when $r_2 = BG = 0.2\,m$

Let N_2 = Maximum speed

From Fig (b), we find that height of the governor,

$$h_2 = PG = \sqrt{(PB)^2 - (BG)^2} = \sqrt{(0.25)^2 - (0.2)^2} = 0.15\,m$$

we know that

$$(N_2)^2 = \frac{m+M}{m} * \frac{895}{h_2} = \frac{5+15}{5} \times \frac{895}{0.15} = 23867$$

$$N_2 = 154.5 \text{ rpm}$$

Range of speed

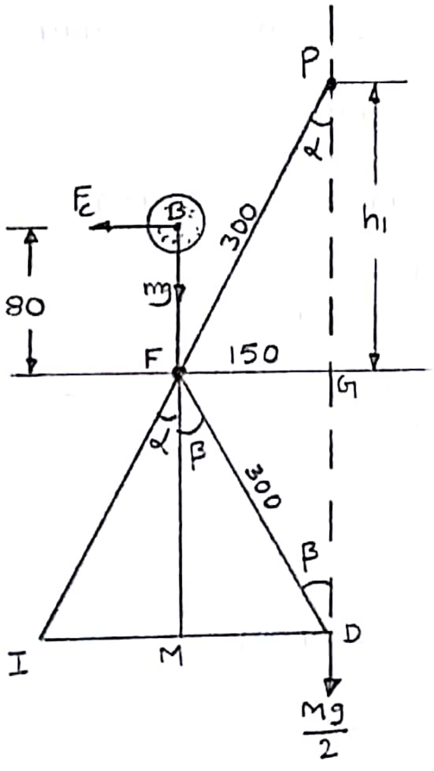
We know that range of speed

$$= N_2 - N_1 = 154.5 - 133.8 = 20.7 \text{ rpm}$$

A Proell governor has equal arms of length 300 mm. The upper and lower ends of the arms are pivoted on the axis of the governor. The extension arms of the lower links are each 80mm long and parallel to the axis when the radii of rotation of the balls are 150mm and 200mm. The mass of each ball is 10kg and the mass of the central load is 100 kg. Determine the range of speed of the governor.

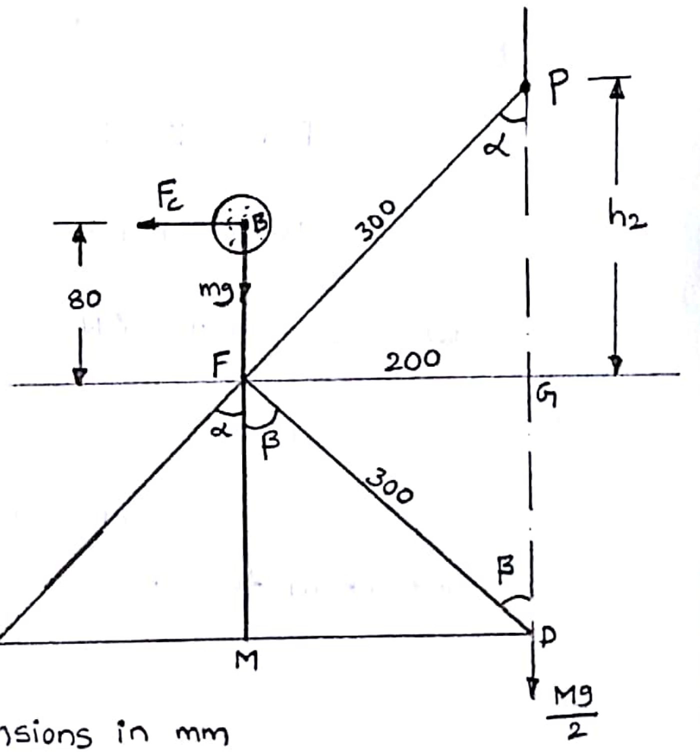
$PF = DF = 300\text{ mm}$, $BF = 80\text{ mm}$, $r_1 = 150\text{ mm}$, $r_2 = 200\text{ mm}$

$m = 10\text{ kg}$, $M = 100\text{ kg}$.



a) Minimum Position

All dimensions in mm



b) Maximum position

First of all, let us find the minimum & maximum speed of the governor. The minimum & maximum position of the governor is shown in Fig.

Let N_1 – Minimum speed when radius of rotation, $r_1 = FG = 150\text{mm}$

N_2 – Maximum speed when radius of rotation, $r_2 = FG = 200\text{ mm}$

From Fig (a), we find that height of the governor,

$$h_1 = PG = \sqrt{(PF)^2 - (FG)^2} = \sqrt{(300)^2 - (150)^2} = 260\text{ mm} = 0.26\text{ m}$$

$= 260 \text{ mm} = 0{\cdot}26 \text{ m}$

$\therefore \quad BM = BF + \ldots = 340 \text{ mm}$

we know that $\quad (N_1)^2 = \dfrac{FM}{BM}\left(\dfrac{m+M}{m}\right)\dfrac{895}{h_1}$ $\qquad \therefore \alpha = \beta$ or $q = 1$

$$= \frac{0{\cdot}26}{0{\cdot}34}\left(\frac{10+100}{10}\right)\frac{895}{0{\cdot}26}$$

$$(N_1)^2 = 28956$$

$$N_1 = 170 \text{ rpm}$$

Now from Fig (b), we find that height of the governor

$$h_2 = PG = \sqrt{(PF)^2 - (FG)^2} = \sqrt{(300)^2 - (200)^2}$$

$$h_2 = 224 \text{ mm} = 0{\cdot}224 \text{ m}$$

$$FM = GD = PG = 224 \text{ mm} = 0{\cdot}224 \text{ m}$$

$$\therefore \quad BM = BF + FM = 80 + 224 = 304 \text{ mm} = 0{\cdot}304 \text{ m}$$

we know that $\quad (N_2)^2 = \dfrac{FM}{BM}\left(\dfrac{m+M}{m}\right)\dfrac{895}{h_2}$ $\qquad \therefore \alpha = \beta$ or $q = 1$

$$= \frac{0{\cdot}224}{0{\cdot}304}\left(\frac{10+100}{10}\right)\frac{895}{0{\cdot}224} = 32385$$

$$N_2 = 180 \text{ rpm}$$

we know that range of speed

$$= N_2 - N_1 = 180 - 170 = 10 \text{ rpm}$$

Sensitiveness of a Governor

A governor is said to be sensitive when it readily responds to a small change of speed. The movement of the sleeve for a fractional change of speed is the measure of sensitivity.

As a governor is used to limit the change of speed of the engine between minimum to full-load conditions, the sensitiveness of a governor is also defined as the ratio of the difference between the maximum and the minimum speeds (range of speed) to the mean equilibrium speed. Thus,

$$\text{Sensitiveness} = \frac{\text{range of speed}}{\text{mean speed}}$$

$$= \frac{N_2 - N_1}{N} = \frac{2(N_2 - N_1)}{N_1 + N_2}$$

where

N – Mean speed

N_1 – Minimum speed corresponding to full load conditions

N_2 – Maximum speed corresponding to no-load conditions

Stability

A governor is said to be stable if it brings the speed of the engine to the required value and there is not much hunting. The ball masses occupy a definite position for each speed of the engine within the working range.

Obviously, the stability & the sensitivity are two opposite characteristics.

Hunting

Sensitiveness of a governor is a desirable quality. However, if a governor is too sensitive, it may fluctuate continuously, because when the load on the engine falls, the sleeve rises rapidly to a maximum position. This shuts off the fuel supply to the extent to affect a sudden fall in the speed. As the speed falls to below the mean value, the sleeve again moves rapidly and falls to a minimum position to increase the fuel supply. The speed subsequently rises and becomes more than the average with the result that the sleeve again rises to reduce the fuel supply. This process continues and is known as hunting.

Isochronism

A governor with a range of speed zero is known as an isochronous governor. This means that for all positions of the sleeve or the balls, the governor has the same equilibrium speed. Any change of speed results in moving the balls and the sleeve to their extreme positions. However, an isochronous governor is not practical due to friction at the sleeve.

In case of a Porter governor running at speeds N_1 & N_2 rpm.

$$(N_1)^2 = \frac{m + \frac{M}{2}(1+q)}{m} * \frac{895}{h_1} \longrightarrow ①$$

$$(N_2)^2 = \frac{m + \frac{M}{2}(1+q)}{m} * \frac{895}{h_2} \longrightarrow ②$$

For isochronism, range of speed should be zero i.e. $N_2 - N_1 = 0$ (or) $N_1 = N_2$. Therefore from equations ① & ②, $h_1 = h_2$, which is impossible in case of a Porter governor. Hence a Porter governor cannot be isochronous.

In case of a Hartnell governor running speeds N_1 & N_2 rpm

$$M \cdot g + S_1 = 2 F_{c1} * \frac{x}{y} = 2 * m \left(\frac{2\pi N_1}{60}\right)^2 r_1 * \frac{x}{y} \longrightarrow ③$$

$$M \cdot g + S_2 = 2 F_{c2} * \frac{x}{y} = 2 * m \left(\frac{2\pi N_2}{60}\right)^2 r_2 * \frac{x}{y} \longrightarrow ④$$

For isochronism, $N_2 = N_1$. Therefore from equations ③ & ④

$$\frac{M \cdot g + S_1}{M \cdot g + S_2} = \frac{r_1}{r_2}$$

Note: The isochronous governor is not of practical use because the sleeve will move to one of its extreme positions immediately the speed deviates from the isochronous speed.

Balancing UNIT-5

The high speed of engines and other machines are having rotating & reciprocating parts should be completely balanced as far as possible. If these parts are not properly balanced the dynamic forces are set up. These forces not only increase the loads on bearings and stresses in the various members, but also produce unpleasant and even dangerous vibrations.

The balancing of unbalanced forces caused by rotating masses, in order to minimise pressure on the main bearings when an engine is running.

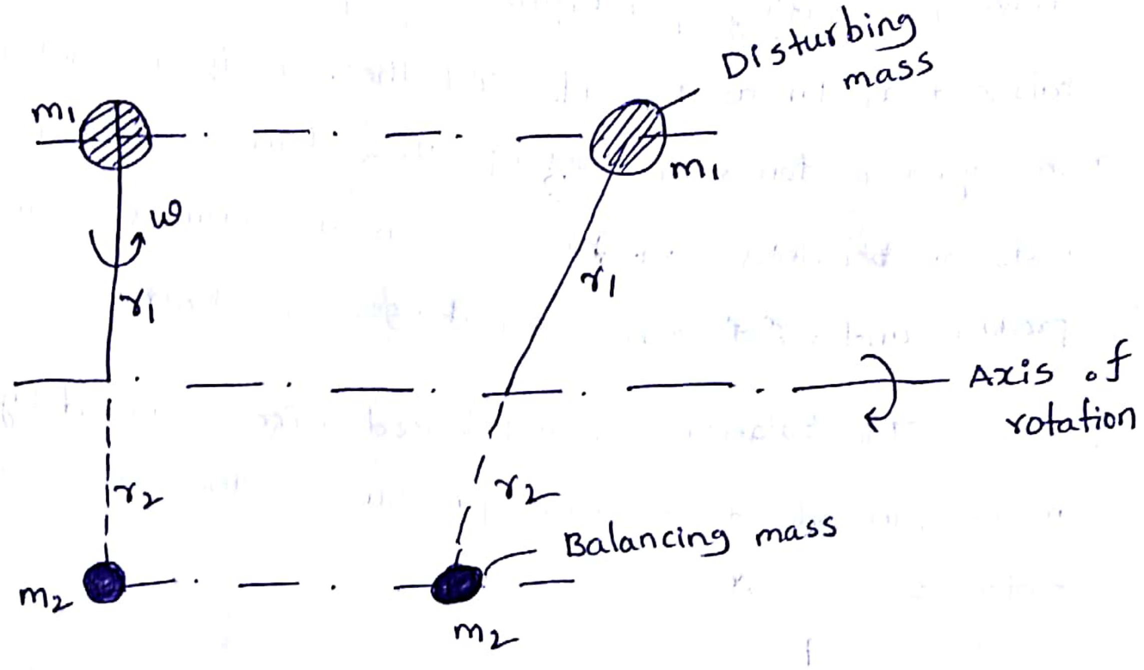
Balancing of Rotating Masses

Whenever a certain mass is attached to a rotating shaft, it exerts some centrifugal force, whose effect is to bend the shaft and to produce vibrations in it.

In order to prevent the effect of centrifugal force, another mass is attached to the opposite side of the shaft, at such a position so as to balance the effect of the centrifugal force of the first mass. This is done in such a way that the centrifugal force of both the masses are made to be equal and opposite.

The process of providing the second mass in order to counteract the effect of the centrifugal force of the first mass, is called balancing of rotating masses.

i) Balancing of a single rotating mass by a single mass rotating in the same plane



Centrifugal force exerted by the mass m_1 on the shaft

$$F_{c_1} = m_1 \cdot r_1 \cdot \omega^2 \longrightarrow (1)$$

r_2 = Radius of rotation of the balancing mass m_2 (i.e distance b/w the axis of rotation of the shaft & the centre of gravity of mass m_2

Centrifugal force due to m_2

$$F_{c_2} = m_2 \cdot r_2 \cdot \omega^2 \longrightarrow (2)$$

Equating (1) & (2)

$$m_1 \cdot r_1 \omega^2 = m_2 \cdot r_2 \cdot \omega^2$$

$$m_1 r_1 = m_2 r_2$$

The centrifugal forces are proportional to the product of the mass and radius of rotation of respective masses, because ω^2 is same for each mass)

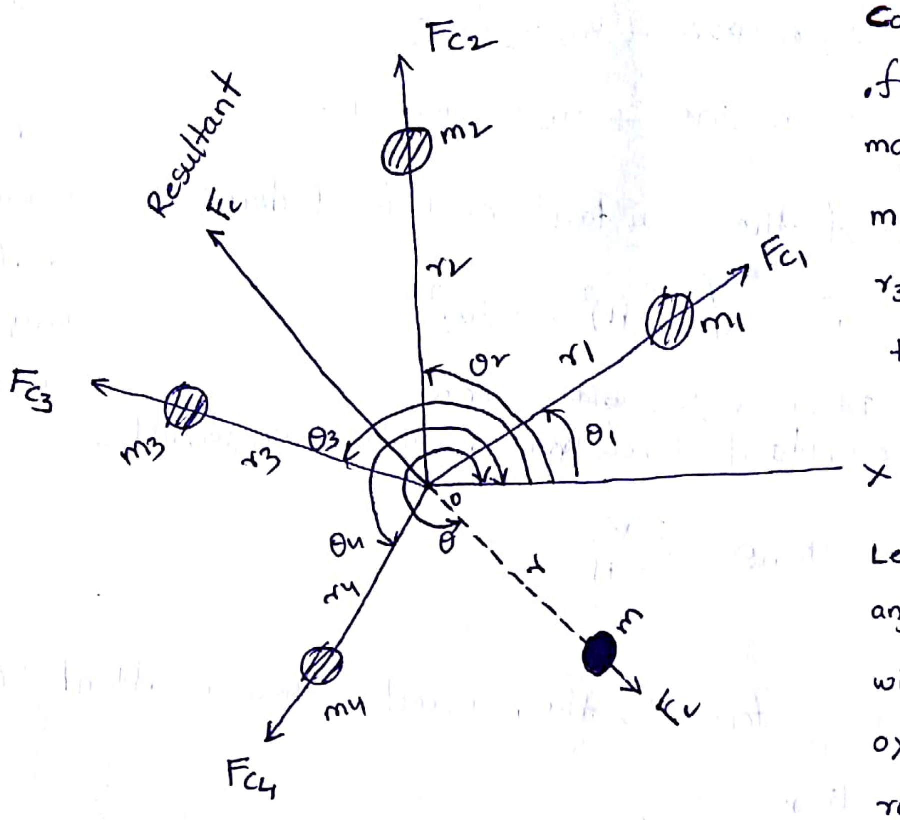
Consider a disturbing mass m_1 attached to a shaft rotating at ω rad/s as shown in Fig. Let r_1 be the radius of rotation of the mass m_1 (i.e distance b/w the axis of rotation of the shaft and the centre of gravity of the mass m_1).

$$F_{c_1} = m_1 r_1 \omega^2$$

This centrifugal force acts radially outwards and thus produces bending moment on the shaft. In order to counteract the effect of this force, a balancing mass (m_2) may be attached in the same plane of rotation as that of disturbing mass (m_1) such that centrifugal forces due to two masses are equal

Balancing of Several Masses rotating in the same plane

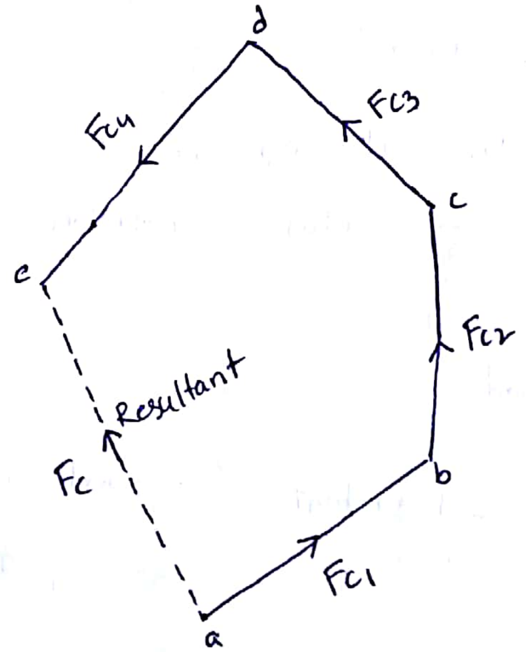
www.FirstRanker.com



(a) space diagram

Consider any number of masses (say four) of magnitude m_1, m_2, m_3 & m_4 at distances of r_1, r_2, r_3 & r_4 from the axis of the rotating shaft.

Let $\theta_1, \theta_2, \theta_3, \theta_4$ be the angles of these masses with the horizontal line ox. Let these masses rotate about an axis through O and $\perp^{lar}$ to the plane of paper, with a constant velocity of ω rad/s.



b) Vector diagram

The magnitude & position of the balancing mass may be found out analytically or graphically.

Analytical method

1) First of all, find out the centrifugal force (or the product of the mass & its radius of rotation) exerted by each mass on the rotating shaft.

2) Resolving the centrifugal forces horizontally & vertically

$$\Sigma H = m_1 r_1 \cos\theta_1 + m_2 r_2 \cos\theta_2 + \cdots$$

$$\Sigma V = m_1 r_1 \sin\theta_1 + m_2 r_2 \sin\theta_2 + \cdots$$

3) Magnitude of the resultant centrifugal force

$$F_c = \sqrt{(\Sigma H)^2 + (\Sigma V)^2}$$

It is Angle, ~~with~~ which

4) θ = resultant force makes with horizontal

$$\tan\theta = \frac{\Sigma V}{\Sigma H}$$

5) The balancing force is then equal to the resultant force, but in opposite direction.

6) Magnitude of the balancing mass

$$F_c = m \cdot r$$

m – balancing mass

r – radius of rotation

Graphical method

$m r \omega^2$ – Resultant centrifugal force

$m r$ = Resultant of $m_1 r_1$, $m_2 r_2$ ----.

Four masses m_1, m_2, m_3 & m_4 are 200kg, 300kg, 240 kg & 260 kg respectively. The corresponding radii of rotation are 0.2 m, 0.15 m, 0.25 m & 0.3 m respectively and the angles between successive masses are 45°, 75° & 135°. Find the position & magnitude of the balance mass required, if its radius of rotation is 0.2m.

$m_1 = 200\text{ kg}$, $m_2 = 300\text{ kg}$, $m_3 = 240\text{ kg}$, $m_4 = 260\text{ kg}$

$r_1 = 0.2\text{m}$, $r_2 = 0.15\text{m}$, $r_3 = 0.25\text{m}$, $r_4 = 0.3\text{ m}$

$\theta_1 = 0°$, $\theta_2 = 45°$, $\theta_3 = 45° + 75° = 120°$, $\theta_4 = 45° + 75° + 135°$
$\theta_4 = 255°$

$r = 0.2\text{m}.$

Let m = Balancing mass
θ = The angle which the balancing mass makes with m_1

Since the magnitude of centrifugal forces are proportional to the product of each mass & its radius, $(\because F_c = m r)$

$m_1 r_1 = 200 * 0.2 = 40\text{ kg-m}$

$m_2 r_2 = 300 * 0.15 = 45\text{ kg-m}$

$m_3 r_3 = 240 * 0.25 = 60\text{ kg-m}$

$m_4 r_4 = 260 * 0.3 = 78\text{ kg-m}$

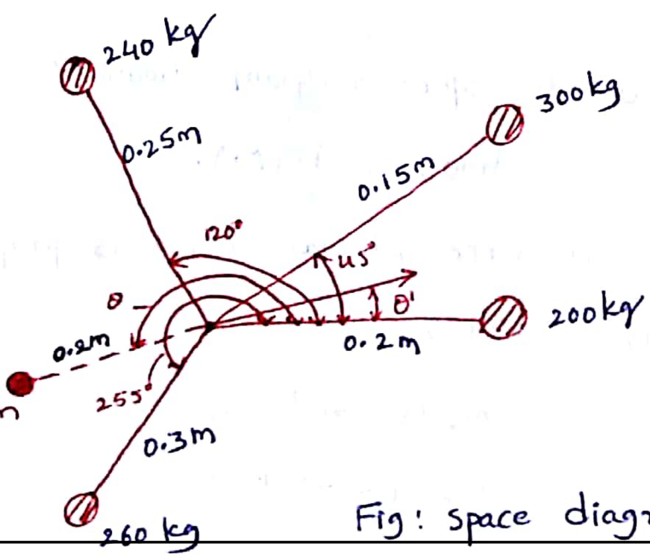


Fig: space diagram

Analytical Method

Resolving m_1r_1, m_2r_2, m_3r_3 & m_4r_4 horizontally & vertically

$$\Sigma H = m_1r_1\cos\theta_1 + m_2r_2\cos\theta_2 + m_3r_3\cos\theta_3 + m_4r_4\cos\theta_4$$

$$= 40\cos 0 + 45\cos 45 + 60\cos 120^\circ + 78\cos 255^\circ = 21.6 \text{ kg-m}$$

$$\Sigma V = m_1r_1\sin\theta_1 + m_2r_2\sin\theta_2 + m_3r_3\sin\theta_3 + m_4r_4\sin\theta_4$$

$$= 40\sin 0 + 45\sin 45 + 60\sin 120 + 78\sin 255 = 8.5 \text{ kg-m}$$

$\therefore$ Resultant $= R = \sqrt{(\Sigma H)^2 + (\Sigma V)^2} = 23.2$ kg-m

$$F_c = mr = R$$

$$23.2 = mr = m * 0.2$$

$$m = 116 \text{ kg}$$

$$\tan\theta' = \frac{\Sigma V}{\Sigma H} = 0.3935$$

$$\theta' = 21.48^\circ$$

since θ' is the angle of the resultant R from the horizontal mass of 200 kg, therefore the angle of the balancing mass from the horizontal mass of 200 kg

$$\theta = 180 + 21.48 = 201.48^\circ$$

Graphical method

1) First of all, draw the space diagram showing the positions of all the given masses as shown in Fig (a).

2) since the centrifugal force of each mass is proportional to the product of the mass & radius

$m_1r_1 = 40$ kg-m $m_3r_3 = 60$ kg-m

$m_2r_2 = 45$ kg-m $m_4r_4 = 78$ kg-m

3) Now draw the vector diagram with the above values, to some suitable scale, as shown in Fig (b). The closing side of the polygon ae represents the resultant force. By measurement we find that ae = 23 kg-m.

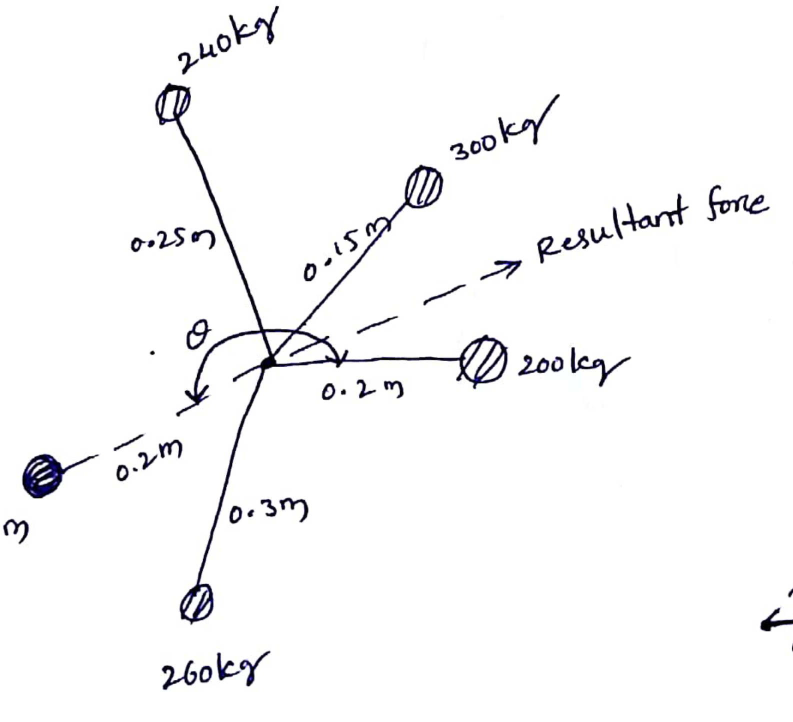
4) The balancing force is equal to the resultant force, but opposite in direction as shown in fig (a). Since the balancing force is proportional to m.r

$$m \times 0.2 = \text{vector } ea = 23 \text{ kg-m}$$

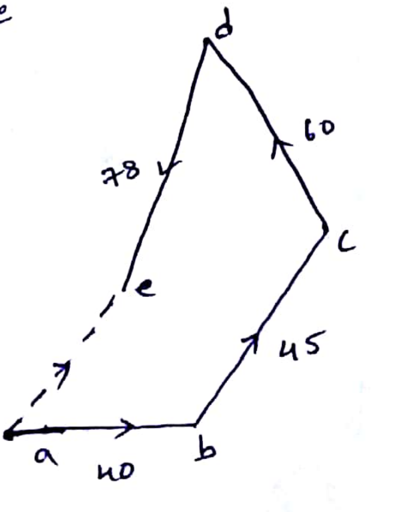
$$m = 115 \text{ kg}$$

By measurement we also find that the angle of inclination of the balancing mass (m) from the horizontal mass of 200 kg

$$\theta = 201^\circ$$



(a) space diagram



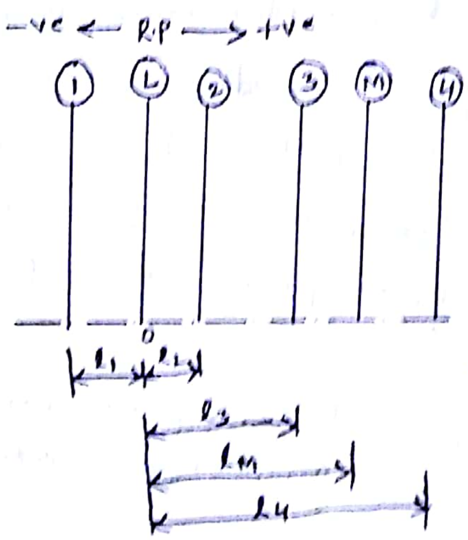
www.FirstRanker.com

Balancing of Several Masses Rotating in Different Planes

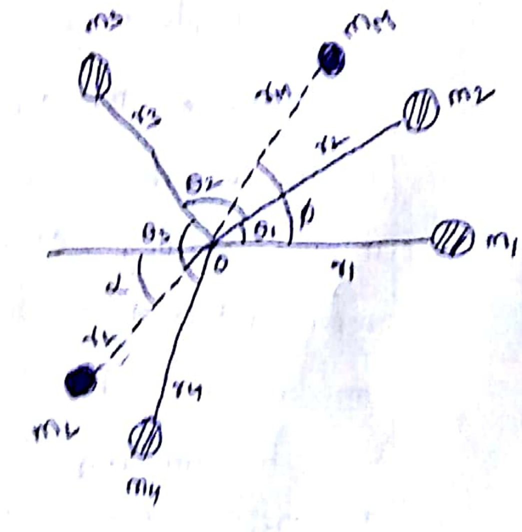
When several masses revolve in different planes, they may be transferred to a reference plane (R.P), which may be defined as the plane passing through a point on the axis of rotation and perpendicular to it. The effect of transferring a revolving mass (in one plane) to a reference plane is to cause a force of magnitude equal to the centrifugal force of the revolving mass to act in the reference plane, together with a couple of magnitude equal to the product of the force and the distance between the plane of rotation and the reference plane. In order to have a complete balance of the several revolving masses in different planes, the following two conditions must be satisfied.

1. The forces in the reference plane must balance, i.e. the resultant force must be zero.
2. The couples about the reference plane must balance, i.e. the resultant couple must be zero.

Let us now consider four masses m_1, m_2, m_3 & m_4 revolving in planes 1, 2, 3 and 4 respectively as shown in Fig (a). The relative angular positions of these masses are shown in the end view Fig (b).



a) Position of planes of the masses



b) Angular position of the masses

The magnitude of the balancing masses m_L and m_M in planes L & M may be obtained as

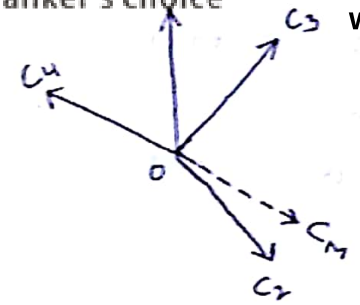
1. Take one of the planes, say L as the reference plane (R.P). The distances of all the other planes to the left of the reference plane may be regarded as negative, and those to the right as positive.

2. Tabulate the data as shown in Table. The planes are tabulated in the same order in which they occur, regarding from left to right.

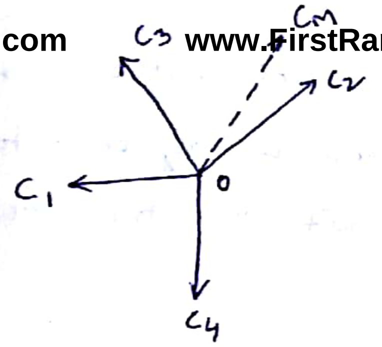
Plane (1)	Mass (m) (2)	Radius (r) (3)	Cent. force $\div \omega^2$ (m.r) (4)	Distance from plane L (l) (5)	Couple $\div \omega^2$ (m.r.l) (6)
1	m_1	r_1	$m_1 r_1$	$-l_1$	$-m_1 r_1 l_1$
L (R.P)	m_L	r_L	$m_L r_L$	0	0
2	m_2	r_2	$m_2 r_2$	l_2	$m_2 r_2 l_2$
3	m_3	r_3	$m_3 r_3$	l_3	$m_3 r_3 l_3$
M	m_M	r_M	$m_M r_M$	l_M	$m_M r_M l_M$
4	m_4	r_4	$m_4 r_4$	l_4	$m_4 r_4 l_4$

3. A couple may be represented by a vector drawn perpendicular to the plane of the couple; The couple C_1 introduced by transferring m_1 to the reference plane through O is proportional to $m_1 r_1 l_1$ and acts in a plane through om_1 and perpendicular to the paper. The vector representing this couple is drawn in the plane of the paper and $\perp^{lar}$ to om_1 as shown by oc_1 in Fig (c).

Similarly, the vectors oc_2, oc_3 & oc_4 are drawn perpendicular to om_2, om_3 & om_4 respectively and in the plane of the paper.



c) Couple vector

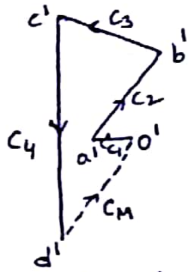


d) Couple vectors turned counter clockwise through a right angle.

4. The couple vectors as discussed above, are turned counter clockwise through a right angle for convenience of drawing as shown in Fig(d). Now the vectors We see that their relative positions remains unaffected. Now the vectors om_2, om_3 & oc_2, oc_3 & oc_4 are parallel and in the same direction as om_2, om_3 & om_4, while the vector oc_1 is parallel to om_1 but in opposite direction.

Hence the couple vectors are drawn radially outwards for the masses on one side of the reference plane and radially inward for the masses on the other side of the reference plane.

5. Now draw the couple polygon as shown in Fig (e). The vector $d'o'$ represents the balanced couple. Since the balanced couple C_M is proportional to $m_M r_M l_M$, therefore



e) couple polygon

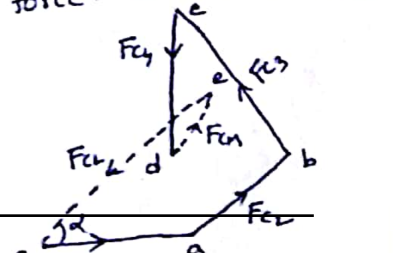
$$C_M = m_M \cdot r_M \cdot l_M = \text{Vector } d'o'$$

$$m_M = \frac{\text{vector } d'o'}{r_M \cdot l_M}$$

From this expression, the value of the balancing mass m_M in the plane M may be obtained, and the angle of inclination ϕ of this mass m_M in the plane M may be obtained.

6. Now draw the force polygon as shown in Fig (f). The vector eo (in the direction from e to o) represents the balanced force. Since the balanced force is proportional to $m_L \cdot r_L$, therefore

$$m_L \cdot r_L = \text{vector } eo$$



(f) Force Polygon

$$m_L = \frac{\text{vector } eo}{r_L}$$

From this expression, the value of the balancing mass m_L in the plane L may be obtained and the angle of inclination α of this mass with the horizontal may be measured from Fig (b).

Balancing of Reciprocating Masses

The various forces acting on the reciprocating parts of an engine. The resultant of all the forces acting on the body of the engine due to inertia forces only is known as unbalanced force or shaking force. Thus if the resultant of all the forces due to inertia effects is zero, then there will be no unbalanced force, but even then an unbalanced couple or shaking couple will be present.

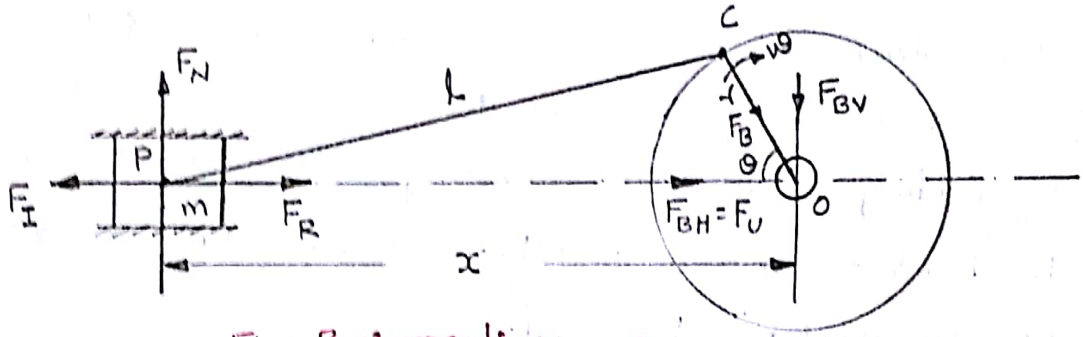


Fig: Reciprocating engine mechanism

Consider a horizontal reciprocating engine mechanism as shown in Fig.

Let F_R – Force required to accelerate the reciprocating parts,

F_I – Inertia force due to reciprocating parts,

F_N – Force on the sides of the cylinder walls (or) normal force acting on the cross-head guides

F_B – Force acting on the crankshaft bearing or main bearing

Since F_R and F_I are equal in magnitude but opposite in direction, therefore they balance each other. The horizontal component of F_B (ie F_{BH}) acting along the line of reciprocation is also equal and opposite to F_I. This force $F_{BH} = F_U$ is an unbalanced force or shaking force and required to be properly balanced.

The force on the sides of the cylinder walls (F_N) and the vertical component of F_B (i.e. F_{BV}) are equal and opposite and thus form a shaking couple of magnitude $F_N \times x$ or $F_{BV} \times x$.

From above we see that the effect of the reciprocating parts is to produce a shaking force and a shaking couple. Since the shaking force and a shaking couple vary in magnitude and direction during the engine cycle, therefore they cause very objectionable vibrations.

Thus the purpose of balancing the reciprocating masses is to eliminate the shaking force and a shaking couple. In most of the mechanisms, we can reduce the shaking force and a shaking couple by adding appropriate balancing mass, but it is usually not practical to eliminate them completely. In other words, the reciprocating masses are only partially balanced.

Note: The masses rotating with the crankshaft are normally balanced and they do not transmit any unbalanced or shaking force on the body of the engine.

Primary & Secondary Unbalanced Forces of Reciprocating Masses

Consider a reciprocating engine mechanism as shown in Fig.

Let m = Mass of the reciprocating parts

l = Length of the connecting rod PC

r = Radius of crank OC

θ = Angle of inclination of the crank with the line of stroke PO

ω = Angular speed of the crank

n = Ratio of length of the connecting rod to the crank radius = l/r

The acceleration of the reciprocating parts is approximately given by the expression,

$$a_R = \omega^2 \cdot r\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

∴ Inertia force due to reciprocating parts or force required to accelerate the reciprocating parts,

$$F_I = F_R = \text{Mass} * \text{Acceleration} = m \cdot \omega^2 \cdot r\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

The horizontal component of the force exerted on the crank shaft bearing (i.e F_{BH}) is equal and opposite to inertia force (F_I). This force is an unbalanced one and is denoted by F_U.

∴ Unbalanced force,

$$F_U = m \cdot \omega^2 \cdot r\left(\cos\theta + \frac{\cos 2\theta}{n}\right) = m \cdot \omega^2 \cdot r\cos\theta + m \cdot \omega^2 \cdot r * \frac{\cos 2\theta}{n}$$

$$F_U = F_P + F_S$$

The expression $(m \cdot \omega^2 \cdot r\cos\theta)$ is known as primary unbalanced force and $\left(m \cdot \omega^2 \cdot r * \frac{\cos 2\theta}{n}\right)$ is called secondary unbalanced force.

∴ Primary unbalanced force, $F_P = m \cdot \omega^2 \cdot r\cos\theta$

and secondary unbalanced force, $F_S = m \cdot \omega^2 \cdot r * \dfrac{\cos 2\theta}{n}$

Partial balancing of Locomotives

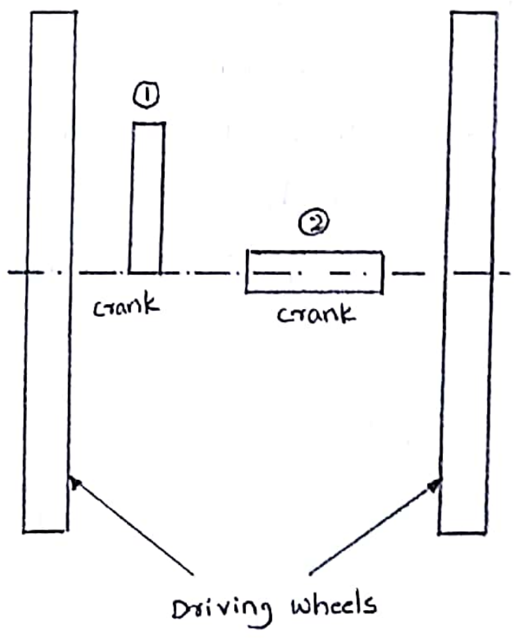
The locomotives, usually, have two cylinders with cranks placed at right angles to each other in order to have uniformity in turning moment diagram. The two cylinder locomotives may be classified as:

1. Inside cylinder locomotives 2. outside cylinder locomotives

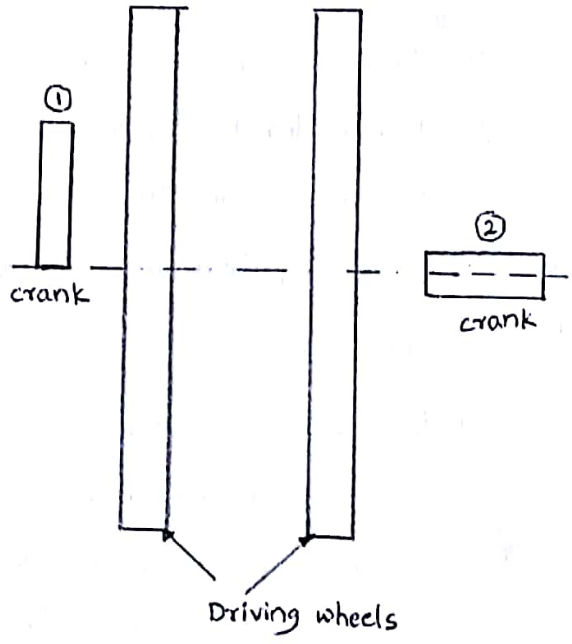
In the inside cylinder locomotives, the two cylinders are placed in between the planes of two driving wheels as shown in Fig (a); whereas in the outside cylinder locomotives, the two cylinders are placed outside the driving wheels, one on each side of the driving wheel, as shown in Fig (b). The locomotives may be

a) single or uncoupled locomotives b) Coupled locomotives

A single or uncoupled locomotive is one, in which the effort is transmitted to one pair of the wheels only; whereas in coupled locomotives, the driving wheels are connected to the leading & trailing wheel by an outside coupling rod.



a) Inside cylinder locomotives



b) outside cylinder locomotives

Effect of Partial Balancing of Reciprocating parts of Two cylinder Locomotives

The reciprocating parts are only partially balanced. Due to this partial balancing of the reciprocating parts, there is an unbalanced primary force along the line of stroke and also an unbalanced primary force perpendicular to the line of stroke. The effect of an unbalanced primary force along the line of stroke is to produce;

1. Variation in tractive force along the line of stroke
2. Swaying couple

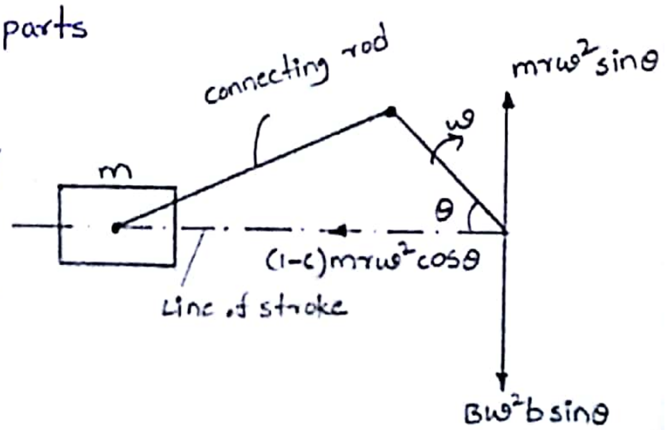
The effect of an unbalanced primary force perpendicular to the line of stroke is to produce variation in pressure on the rails, which results in hammering action on the rails. The maximum magnitude of the unbalanced force along the perpendicular to the line of stroke is known as a hammer blow.

Variation of Tractive Force

The resultant unbalanced force due to the two cylinders, along the line of stroke, is known as tractive force. Let the crank for the first cylinder be inclined at an angle θ with the line of stroke, as shown in Fig. Since the crank for the second cylinder is at right angle to the first crank, therefore the angle of inclination for the second crank will be $(90°+\theta)$.

Let m – Mass of the reciprocating parts per cylinder

c – Fraction of the reciprocating parts to be balanced



We know that unbalanced force along the line of stroke for cylinder 1

$$= (1-c)\, m r \omega^2 \cos\theta$$

Similarly, unbalanced force along the line of stroke for cylinder 2

$$= (1-c)\, m r \omega^2 \cos(90+\theta)$$

∴ As per definition, the tractive force,

$$F_T = \text{Resultant unbalanced force along the line of stroke}$$

$$= (1-c)\, m r \omega^2 \cos\theta + (1-c)\, m r \omega^2 \cos(90°+\theta)$$

$$F_T = (1-c)\, m r \omega^2 (\cos\theta - \sin\theta)$$

The tractive force is maximum (or) minimum when $(\cos\theta - \sin\theta)$ is maximum or minimum. For $(\cos\theta - \sin\theta)$ to be maximum or minimum,

$$\frac{d}{d\theta}(\cos\theta - \sin\theta) = 0 \quad \text{(or)} \quad -\sin\theta - \cos\theta = 0$$

$$-\sin\theta = \cos\theta$$

$$\therefore \tan\theta = -1 \quad \text{(or)} \quad \theta = 135° \text{ (or) } 315°$$

Thus, the tractive force is maximum or minimum when $\theta = 135°$ or $315°$

∴ Maximum & minimum value of the tractive force (or) variation in tractive force

$$= \pm (1-c)\, m r \omega^2 (\cos 135° - \sin 135°)$$

$$= \pm \sqrt{2}\, (1-c)\, m r \omega^2$$

Swaying Couple

The unbalanced forces along the line of stroke for the two cylinders constitute a couple about the centre line YY between the cylinders as shown in Fig.

This couple has swaying effect about a vertical axis, and tends to sway the engine alternately in clockwise & anticlockwise directions. Hence the couple is known as swaying couple.

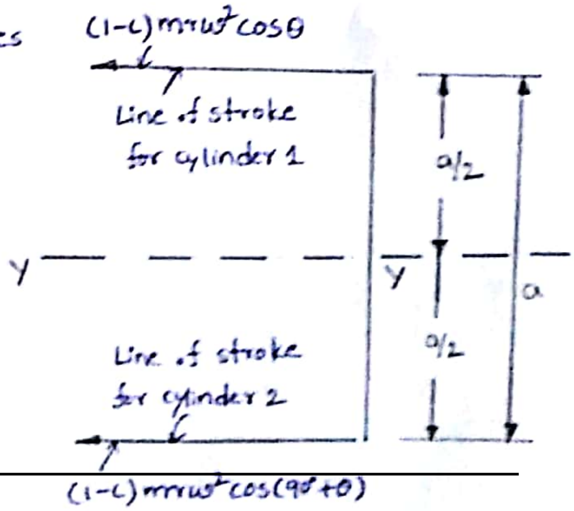
Let a - Distance between the centre lines of the two cylinders

∴ Swaying couple

$$= (1-c)\, m r \omega^2 \cos\theta \times \frac{a}{2}$$

$$- (1-c)\, m r \omega^2 (90°+\theta) \times \frac{a}{2}$$

$$= (1-c)\, m r \omega^2 \times \frac{a}{2} (\cos\theta + \sin\theta)$$



The swaying couple is maximum or minimum when $(\cos\theta + \sin\theta)$ is maximum or minimum. For $(\cos\theta + \sin\theta)$ to be maximum or minimum.

$$\frac{d}{d\theta}(\cos\theta + \sin\theta) = 0 \quad (\text{or}) \quad -\sin\theta + \cos\theta = 0 \quad (\text{or}) \quad -\sin\theta = -\cos\theta$$

$$\tan\theta = 1 \quad (\text{or}) \quad \theta = 45^\circ \ (\text{or})\ 225^\circ$$

Thus, the swaying couple is maximum or minimum when $\theta = 45^\circ$ (or) 225°

$\therefore$ Maximum & minimum value of the swaying couple

$$= \pm (1-c)\, m r \omega^2 \times \frac{a}{2}(\cos 45^\circ + \sin 45^\circ)$$

$$= \pm \frac{a}{\sqrt{2}}(1-c)\, m r \omega^2$$

Hammer Blow

The maximum magnitude of the unbalanced force along the perpendicular to the line of stroke is known as hammer blow.

The unbalanced force along the perpendicular to the line of stroke due to the balancing mass B, at a radius b, in order to balance reciprocating parts only is $B\omega^2 b \sin\theta$. This force will be maximum when $\sin\theta$ is unity, i.e when $\theta = 90^\circ$ (or) 270°

$$\therefore \quad \text{Hammer blow} = B\omega^2 b$$

Let m – Mass of reciprocating parts per cylinder

l – Length of connecting rod

r – Radius of crank

n – Ratio of length of connecting rod to crank radius $= l/r$

θ – Inclination of crank to the vertical at any instant

ω – Angular velocity of crank

Considering primary forces

Total component of primary force along the vertical line

$$F_{PV} = 2m \cdot \omega^2 \cdot r\cos^2\alpha \cdot \cos\theta$$

Total component of primary force along the horizontal line

$$F_{PH} = 2m \cdot \omega^2 \cdot r\sin^2\alpha \cdot \sin\theta$$

$\therefore$ Resultant primary force

$$F_P = \sqrt{(F_{PV})^2 + (F_{PH})^2}$$

Considering secondary forces

Total component of secondary force along vertical & horizontal line

$$F_{SV} = \frac{2m}{n} * \omega^2 \cdot r\cos\alpha \cdot \cos 2\alpha \cos 2\theta$$

$$F_{SH} = \frac{2m}{n} * \omega^2 \cdot r\sin\alpha \cdot \sin 2\alpha \sin 2\theta$$

$\therefore$ Resultant secondary force

$$F_S = \sqrt{(F_{SV})^2 + (F_{SH})^2}$$

VIBRATIONS

When elastic bodies such as a spring, a beam and a shaft are displaced from the equilibrium position by the application of external forces, and then released, they execute a **vibratory motion.** This is due to the reason that, when a body is displaced, the internal forces in the form of elastic or strain energy are present in the body. At release, these forces bring the body to its original position. When the body reaches the equilibrium position, the whole of the elastic or strain energy is converted into kinetic energy due to which the body continues to move in the opposite direction. The whole of the kinetic energy is again converted into strain energy due to which the body again returns to the equilibrium position. In this way, the vibratory motion is repeated indefinitely.

TERMS USED IN VIBRATORY MOTION

The following terms are commonly used in connection with the vibratory motions:

1. Period of vibration or time period. It is the time interval after which the motion is repeated itself. The period of vibration is usually expressed in seconds.

2. Cycle. It is the motion completed during one-time period.

3. Frequency. It is the number of cycles described in one second. In S.I. units, the frequency is expressed in hertz (briefly written as Hz) which is equal to one cycle per second.

TYPES OF VIBRATORY MOTION

The following types of vibratory motion are important from the subject point of view:

1. Free or natural vibrations. When no external force acts on the body, after giving it an initial displacement, then the body is said to be under **free or natural vibrations.** The frequency of the free vibrations is called **free or natural frequency.**

2. Forced vibrations. When the body vibrates under the influence of external force, then the body is said to be under **forced vibrations.** The external force applied to the body is a periodic disturbing force created by unbalance. The vibrations have the same frequency as the applied force.

Note: When the frequency of the external force is same as that of the natural vibrations, resonance takes place.

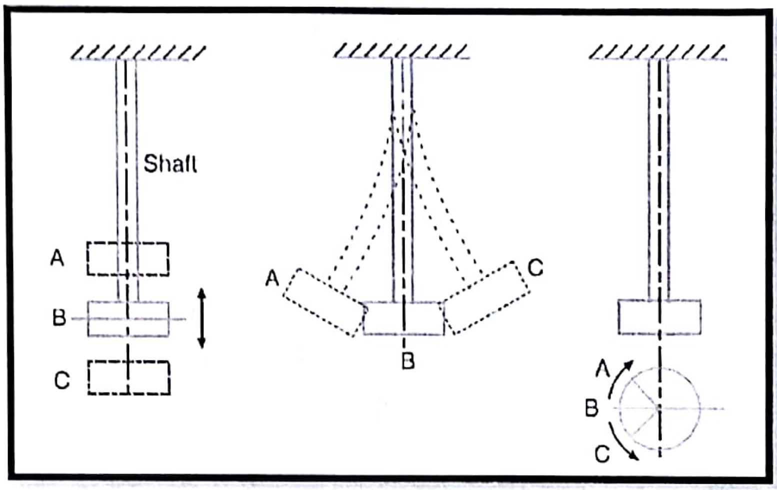
3. Damped vibrations. When there is a reduction in amplitude over every cycle of vibration, the motion is said to be **damped vibration.** This is due to the fact that a certain amount of energy possessed by the vibrating system is always dissipated in overcoming frictional resistances to the motion.

TYPES OF FREE VIBRATIONS

The following three types of free vibrations are important from the subject point of view:

1. Longitudinal vibrations, **2.** Transverse vibrations, and **3.** Torsional vibrations.

Consider a weightless constraint (spring or shaft) whose one end is fixed and the other end carrying a heavy disc, as shown in Fig. This system may execute one of the three above mentioned types of vibrations.



B = Mean position; A and C = Extreme positions.

(a) Longitudinal vibrations. (b) Transverse vibrations. (c) Torsional vibrations.

Fig: Types of free vibrations.

1. Longitudinal vibrations. When the particles of the shaft or disc moves parallel to the axis of the shaft, as shown in Fig(a), then the vibrations are known as **longitudinal vibrations.**

In this case, the shaft is elongated and shortened alternately and thus the tensile and compressive stresses are induced alternately in the shaft.

2. Transverse vibrations. When the particles of the shaft or disc move approximately perpendicular to the axis of the shaft, as shown in Fig(b), then the vibrations are known as **transverse vibrations.** In this case, the shaft is straight and bent alternately and bending stresses are induced in the shaft.

3. Torsional vibrations. When the particles of the shaft or disc move in a circle about the axis of the shaft, as shown in Fig(c), then the vibrations are known as **torsional vibrations.**

In this case, the shaft is twisted and untwisted alternately and the torsional shear stresses are induced in the shaft.

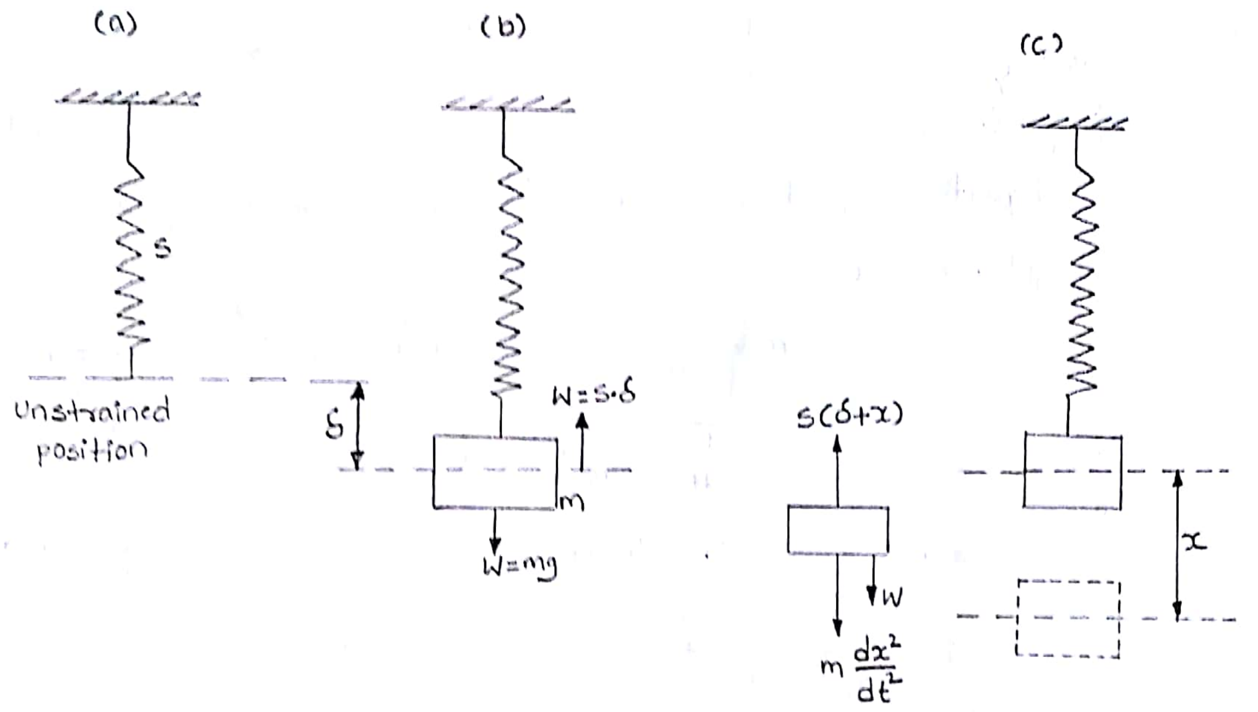
Natural Vibrations

Natural Frequency of Free Longitudinal Vibrations

The natural frequency of the free longitudinal vibrations may be determined by the following three methods:

1. Equilibrium Method

Consider a constraint (i.e spring) of negligible mass in an unstrained position, as shown in Fig (a).



Let

s – stiffness of the constraint. It is the force required to produce unit displacement in the direction of vibration, N/m

m – Mass of the body suspended from the constraint in kg

W – Weight of the body in newtons $= mg$

δ – static deflection of the spring in metres due to weight

x – Displacement given to the body by the external force, m

In the equilibrium position, as shown in Fig (b), the gravitation pull $W = mg$, is balanced by a force of spring, such that $W = s.\delta$

Since the mass is now displaced from its equilibrium position by a distance x, as shown in Fig(c), and is therefore after time t,

$$\text{Restoring force} = W - s(\delta + x) = W - s\cdot\delta - s\cdot x$$

$$= s\cdot\delta - s\cdot\delta - s\cdot x = -s\cdot x \longrightarrow ①$$

$(\because W = s\delta)$

(Taking upward force as negative)

$$\text{Accelerating force} = \text{Mass} \times \text{Acceleration}$$

$$= m \times \frac{d^2x}{dt^2} \longrightarrow ②$$

$v = \delta x/\delta t$ $a = \delta^2 x/\delta t^2$

(Taking downward force as positive)

Equating equations ① and ②, the equation of motion of the body of mass m after time t is

$$m \times \frac{d^2x}{dt^2} = -s\cdot x \quad (\text{or}) \quad m \times \frac{d^2x}{dt^2} + s\cdot x = 0$$

$$\therefore \frac{d^2x}{dt^2} + \frac{s}{m} \times x = 0 \longrightarrow ③$$

We know that the fundamental equation of simple harmonic motion

$$\frac{d^2x}{dt^2} + \omega^2 \cdot x = 0 \longrightarrow ④$$

comparing equations ③ & ④, we have

$$\omega = \sqrt{\frac{s}{m}}$$

$(\because mg = s\cdot\delta)$

$$\therefore \text{Time period, } t_p = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{s}}$$

$$\text{and natural frequency, } f_n = \frac{1}{t_p} = \frac{1}{2\pi}\sqrt{\frac{s}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$$

Taking the value of g as $9.81\ m/s^2$ and δ in metres,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{9.81}{\delta}} = \frac{0.4985}{\sqrt{\delta}}\ \text{Hz}$$

Note: The value of static deflection δ may be found out from the given conditions of the problem. For longitudinal vibrations,

$$\frac{\text{stress}}{\text{strain}} = E \quad (\text{or}) \quad \frac{W}{A} \times \frac{l}{\delta} = E \quad (\text{or}) \quad \delta = \frac{W\cdot l}{E\cdot A}$$

2. Energy Method

We know that the kinetic energy is due to the motion of the body and the potential energy is with respect to a certain datum position which is equal to the amount of work required to move the body from the datum position.

In the case of vibrations, the datum position is the mean or equilibrium position at which the potential energy of the body or the system is zero.

In the free vibrations, no energy is transferred to the system or from the system. Therefore the summation of kinetic energy & potential energy must be a constant quantity which is same at all the times.

$$\therefore \frac{d}{dt}(K.E + P.E) = 0$$

We know that kinetic energy, $K.E = \frac{1}{2} * m \left(\frac{dx}{dt}\right)^2$

and potential energy, $P.E = \left(\frac{0 + s.x}{2}\right) * x = \frac{1}{2} s.x^2$

($\because$ P.E = Mean force * Displacement)

$$\therefore \frac{d}{dt}\left[\frac{1}{2} * m \left(\frac{dx}{dt}\right)^2 + \frac{1}{2} * s.x^2\right] = 0$$

$$\frac{1}{2} * m * 2 * \frac{dx}{dt} * \frac{d^2x}{dt^2} + \frac{1}{2} * s * 2x * \frac{dx}{dt} = 0$$

$$m * \frac{d^2x}{dt^2} + s.x = 0 \quad (\text{or}) \quad \frac{d^2x}{dt^2} + \frac{s}{m} * x = 0$$

The time period & the natural frequency may be obtained as discussed in the previous method.

3. Rayleigh's Method

In this method, the maximum kinetic energy at the mean position is equal to the maximum potential energy (or strain energy) at the extreme position. Assuming the motion executed by the vibration to be simple harmonic, then

$$x = X \sin \omega t \longrightarrow ①$$

where X – Maximum displacement from mean position to extreme position

x – Displacement of the body from the mean position after time t seconds

Now differentiating equation ①, we have

$$\frac{dx}{dt} = \omega * X \cos\omega t$$

since at mean position, $t = 0$, therefore maximum velocity at the mean position,

$$v = \frac{dx}{dt} = \omega \cdot X$$

∴ Maximum kinetic energy at mean position

$$= \frac{1}{2} \times m v^2 = \frac{1}{2} * m \cdot \omega^2 X^2 \longrightarrow ②$$

and maximum potential energy at the extreme position

$$= \left(\frac{0 + s \cdot X}{2}\right) X = \frac{1}{2} * s \cdot X^2 \longrightarrow ③$$

Equating equations ② & ③

$$\frac{1}{2} * m \cdot \omega^2 \cdot X^2 = \frac{1}{2} * s \cdot X^2$$

$$\omega^2 = \frac{s}{m}, \text{ and } \omega = \sqrt{\frac{s}{m}}$$

∴ Time period, $t_p = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{s}}$

and natural frequency, $f_n = \frac{1}{t_p} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{s}{m}}$

Note: In all the above expressions, ω is known as natural circular frequency and is denoted by ω_n.

Natural Frequency of Free Transverse Vibrations For a shaft subjected to a Number of Point Loads

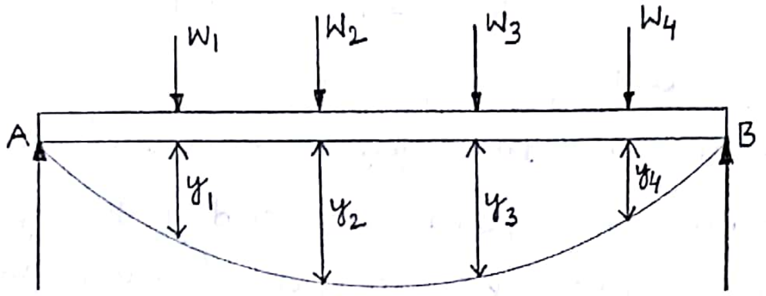


Fig: shaft carrying a number of point loads

Consider a shaft AB of negligible mass loaded with point loads W_1, W_2, W_3 and W_4 etc. in newtons, as shown in Fig. Let m_1, m_2, m_3 & m_4 etc. be the corresponding masses in kg. The natural frequency of such a shaft may be found out by the following two methods :

1- Energy (or Rayleigh's) method

Let y_1, y_2, y_3, y_4 etc. be total deflection under loads W_1, W_2, W_3 and W_4 etc. as shown in Fig.

We know that maximum potential energy

$$= \frac{1}{2} * m_1 \cdot g \cdot y_1 + \frac{1}{2} * m_2 \cdot g \cdot y_2 + \frac{1}{2} * m_3 \cdot g \cdot y_3 + \frac{1}{2} * m_4 \cdot g \cdot y_4 + ----$$

$$= \frac{1}{2} \Sigma m \cdot g \cdot y$$

and maximum kinetic energy

$$= \frac{1}{2} * m_1 (\omega \cdot y_1)^2 + \frac{1}{2} * m_2 (\omega \cdot y_2)^2 + \frac{1}{2} * m_3 (\omega \cdot y_3)^2 + \frac{1}{2} * m_4 (\omega \cdot y_4)^2 + ---$$

$$= \frac{1}{2} * \omega^2 \left[m_1 (y_1)^2 + m_2 (y_2)^2 + m_3 (y_3)^2 + m_4 (y_4)^2 + --- \right]$$

$$= \frac{1}{2} * \omega^2 \Sigma m \cdot y^2$$

(ω = circular frequency of vibration)

Equating the maximum kinetic energy to the maximum potential energy,

$$\frac{1}{2} * \omega^2 \Sigma m y^2 = \frac{1}{2} * \Sigma m \cdot g \cdot y$$

$$\omega^2 = \frac{\Sigma m \cdot g \cdot y}{\Sigma m \cdot y^2} = \frac{g \cdot \Sigma m \cdot y}{\Sigma m \cdot y^2}$$

$$\omega = \sqrt{\frac{g \Sigma m \cdot y}{\Sigma m \cdot y^2}}$$

Natural frequency of transverse vibration

$$f_n = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g\,\Sigma m\cdot y}{\Sigma m\cdot y^2}}$$

2. Dunkerley's Method

The natural frequency of transverse vibration for a shaft carrying a number of point loads and uniformly distributed load is obtained from Dunkerley's empirical formula. According to this

$$\frac{1}{(f_n)^2} = \frac{1}{(f_{n1})^2} + \frac{1}{(f_{n2})^2} + \frac{1}{(f_{n3})^2} + \cdots + \frac{1}{(f_{ns})^2}$$

Where f_n – Natural frequency of transverse vibration of the shaft carrying point loads and uniformly distributed load

f_{n1}, f_{n2}, f_{n3}, etc. – Natural frequency of transverse vibration of each point load

f_{ns} – Natural frequency of transverse vibration of the uniformly distributed load (or due to the mass of the shaft).

Now, consider a shaft AB loaded as shown in Fig.

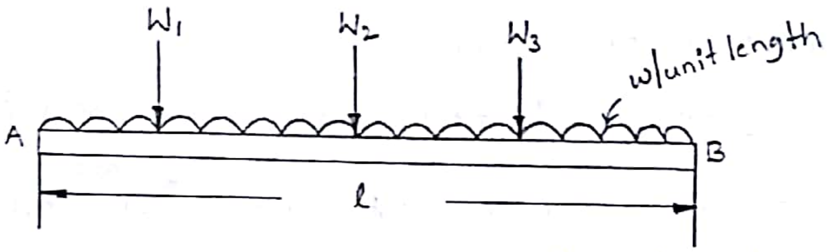


Fig: shaft carrying a number of point loads & a uniformly distributed load

Let $\delta_1, \delta_2, \delta_3$, etc – static deflection due to the load W_1, W_2, W_3 etc. when considered separately.

δ_s – static deflection due to the uniformly distributed load or due to the mass of the shaft.

We know that natural frequency of transverse vibration due to load W_1, W_2 & W_3.

$$f_{n1} = \frac{0.4985}{\sqrt{\delta_1}}\ \text{Hz}, \quad f_{n2} = \frac{0.4985}{\sqrt{\delta_2}}\ \text{Hz}, \quad f_{n3} = \frac{0.4985}{\sqrt{\delta_3}}\ \text{Hz}$$

Also natural frequency of transverse vibration due to uniformly

distributed load or weight of the shaft,

$$f_{ns} = \frac{0.5615}{\sqrt{\delta_s}} \text{ Hz}$$

Therefore, according to Dunkerley's empirical formula, the natural frequency of the whole system,

$$\frac{1}{(f_n)^2} = \frac{1}{(f_{n1})^2} + \frac{1}{(f_{n2})^2} + \frac{1}{(f_{n3})^2} + \text{------} + \frac{1}{(f_{ns})^2}$$

$$= \frac{\delta_1}{(0.4985)^2} + \frac{\delta_2}{(0.4985)^2} + \frac{\delta_3}{(0.4985)^2} + \text{---} + \frac{\delta_s}{(0.5615)^2}$$

$$\frac{1}{(f_n)^2} = \frac{1}{(0.4985)^2}\left[\delta_1 + \delta_2 + \delta_3 + \frac{\delta_s}{1.27}\right]$$

$$f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \delta_3 + \text{---} + \frac{\delta_s}{1.27}}} \text{ Hz}$$

Note: 1. When there is no uniformly distributed load or mass of the shaft is negligible, then $\delta_s = 0$

$$f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \delta_3 + \text{---}}} \text{ Hz}$$

2. The value of $\delta_1, \delta_2, \delta_3$ etc. for a simply supported shaft may be obtained from the relation

$$\delta = \frac{W a^2 b^2}{3EIl}$$

where δ – static deflection due to load W

a and b – Distances of the load from the ends

E – Young's modulus for the matrial of the shaft

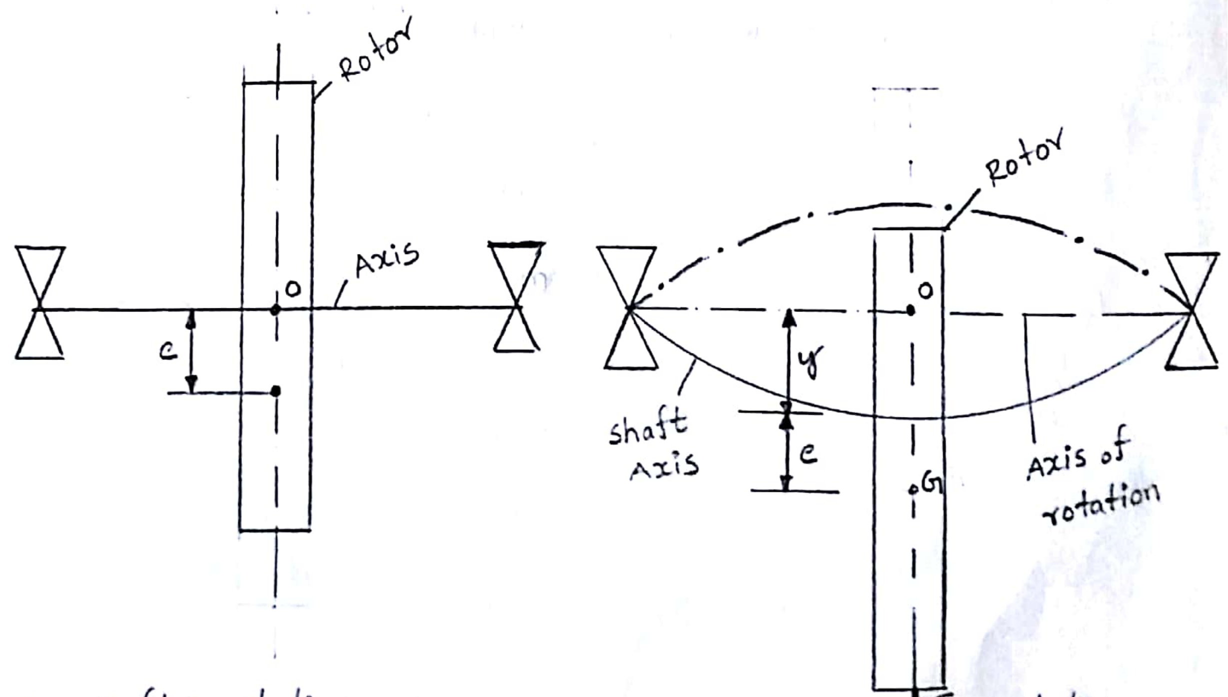
I – Moment of inertia of the shaft

l – Total length of the shaft.

Critical or Whirling Speed of a shaft

In actual practice, a rotating shaft carries different mountings and accessories in the form of gears, pulleys, etc. When the gears or pulleys are put on the shaft, the centre of gravity of the pulley or gear does not coincide with the centre line of the bearings or with the axis of the shaft, when the shaft is stationary. This means that the centre of gravity of the pulley or gear is at a certain distance from the axis of rotation and due to this, the shaft is subjected to centrifugal force. This force will bent the shaft which will further increase the distance of centre of gravity of the pulley or gear from the axis of rotation. This correspondingly increases the value of centrifugal force, which further increases the distance of centre of gravity from the axis of rotation. This effect is cumulative and ultimately the shaft fails. The bending of shaft not only depends upon the value of eccentricity (distance between centre of gravity of the pulley and the axis of rotation) but also depends upon the speed at which the shaft rotates.

The speed at which the shaft runs so that the additional deflection of the shaft from the axis of rotation becomes infinite, is known as critical or whirling speed.



a) When shaft is stationary b) When shaft is rotating

Consider a shaft of negligible mass carrying a rotor, as shown in Fig(a). The point O is on the shaft axis and G is the centre of gravity of the rotor. When the shaft is stationary, the centre line of the bearing and the axis of the shaft coincides. Fig(b) shows the shaft when rotating about the axis of rotation at a uniform speed of ω rad/s.

Let
- m – Mass of the rotor
- e – Initial distance of centre of gravity of the rotor from the centre line of the bearing or shaft axis, when the shaft is stationary
- y – Additional deflection of centre of gravity of the rotor when the shaft starts rotating at ω rad/s
- s – stiffness of the shaft i.e the load required per unit deflection of the shaft

Since the shaft is rotating at ω rad/s, therefore centrifugal force acting radially outwards through G causing the shaft to deflect is given by

$$F_c = m \cdot \omega^2 (y+e)$$

The shaft behaves like a spring. Therefore the force resisting the deflection y,

$$= s \cdot y$$

For the equilibrium position,

$$m \cdot \omega^2 (y+e) = s \cdot y$$

$$m \cdot \omega^2 \cdot y + m \cdot \omega^2 \cdot e = s \cdot y$$

$$y(s - \omega^2 m) = m \cdot \omega^2 \cdot e$$

$$y = \frac{m \cdot \omega^2 \cdot e}{s - m \cdot \omega^2} = \frac{\omega^2 \cdot e}{\frac{s}{m} - \omega^2} \longrightarrow (1)$$

We know that circular frequency,

$$\omega_n = \sqrt{\frac{s}{m}} \quad (\text{or}) \quad y = \frac{\omega^2 \cdot e}{(\omega_n)^2 - \omega^2}$$

A little consideration will show that when $\omega > \omega_n$, the value of y will be negative and the shaft deflects in the opposite direction as shown dotted in Fig (b).

In order to have the value of y always positive, both plus and minus signs are taken.

$(\because \omega_n = \omega_c)$

$$y = \pm \frac{\omega^2 \cdot e}{(\omega_n)^2 - \omega^2} = \frac{\pm e}{\left(\frac{\omega_n}{\omega}\right)^2 - 1} = \frac{\pm e}{\left(\frac{\omega_c}{\omega}\right)^2 - 1}$$

We see from the above expression that when $\omega_n = \omega_c$, the value of y becomes infinite. Therefore ω_c is the critical or whirling speed.

$\therefore$ Critical or whirling speed, $\qquad (\because \delta = \frac{m \cdot g}{s})$

$$\omega_c = \omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{g}{\delta}} \text{ Hz}$$

If N_c is the critical or whirling speed in r.p.s., then

$$2\pi N_c = \sqrt{\frac{g}{\delta}} \quad (\text{or}) \quad N_c = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{0.4985}{\sqrt{\delta}} \text{ r.p.s}$$

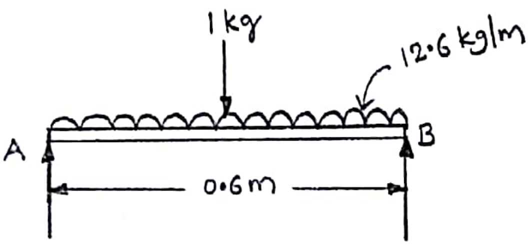
where δ – static deflection of the shaft in metres.

Hence the critical or whirling speed is the same as the natural frequency of transverse vibration but its unit will be revolutions per second.

Calculate the whirling speed of a shaft 20mm diameter and 0.6m long carrying a mass of 1 kg at its mid-point. The density of the shaft material is 40 Mg/m³, and Young's modulus is 200 GN/m². Assume the shaft to be freely supported.

$d = 20\,mm = 0.02\,m$, $l = 0.6\,m$, $m_1 = 1\,kg$, $\rho = 40\,Mg/m^3 = 40 \times 10^6\,g/m^3$

$E = 200\,GN/m^2 = 200 \times 10^9\,N/m^2$ $\quad \rho = 40 \times 10^3\,kg/m^3$



Moment of Inertia, $I = \frac{\pi}{64} d^4$

$I = \frac{\pi}{64}(0.02)^4 = 7.855 \times 10^{-9}\,m^4$

Since the density of shaft material is $40 \times 10^3\,kg/m^3$, therefore mass of the shaft per metre length,

$$m_s = \text{Area} \times \text{Length} \times \text{Density} = \frac{\pi}{4}(0.02)^2 \times 1 \times 40 \times 10^3 = 12.6\,kg/m$$

static deflection due to 1 kg of mass at the centre,

$$\delta = \frac{Wl^3}{48EI} = \frac{1 \times 9.81 \times (0.6)^3}{48 \times 200 \times 10^9 \times 7.855 \times 10^{-9}} = 28 \times 10^{-6}\,m$$

and static deflection due to mass of the shaft

$$\delta_s = \frac{5wl^4}{384EI} = \frac{5 \times 12.6 \times 9.81 \times (0.6)^4}{384 \times 200 \times 10^9 \times 7.855 \times 10^{-9}} = 0.133 \times 10^{-3}\,m$$

$\therefore$ Frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta + \frac{\delta_s}{1.27}}} = \frac{0.4985}{\sqrt{28 \times 10^{-6} + \frac{0.133 \times 10^{-3}}{1.27}}}$$

$$f_n = \frac{0.4985}{11.52 \times 10^{-3}} = 43.3\,Hz$$

Whirling speed of a shaft in r.p.s is equal to the frequency of transverse vibration in Hz,

$$N_c = 43.3\,r.p.s = 43.3 \times 60 = 2598\,r.p.m$$

Frequency of Free Damped Vibrations (Viscous Damping)

Let

m – Mass suspended from the spring

s – stiffness of the spring

x – Displacement of the mass from the mean position at time t

δ – static deflection of the spring

$\delta = \frac{m \cdot g}{s}$

c – Damping coefficient (or) the damping force per unit velocity

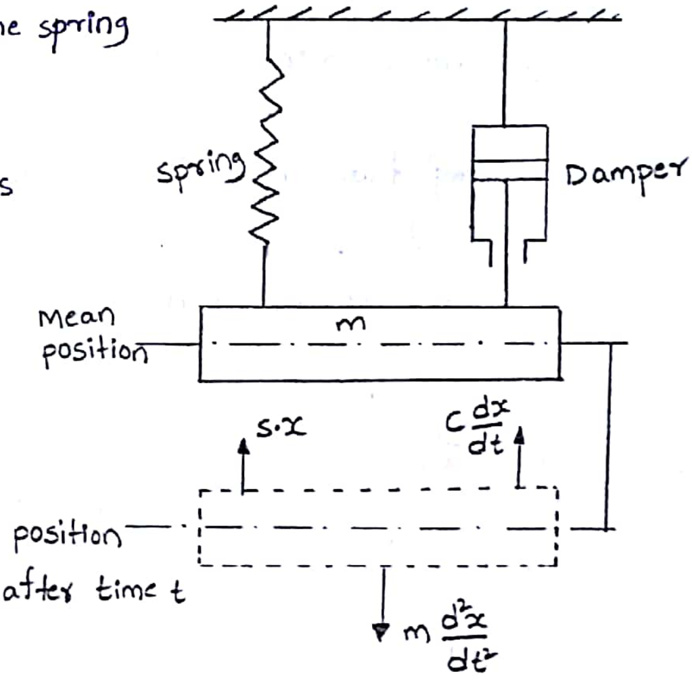


Fig: Frequency of free damped vibrations

$$a = \frac{c}{2m} \quad , \quad \omega_n = \sqrt{\frac{s}{m}}$$

circular damped vibrations – ω_d

$$\omega_d = \sqrt{\omega_n^2 - a^2} = \sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2}$$

Periodic time of vibration, $t_p = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2}} = \frac{2\pi}{\sqrt{\omega_n^2 - a^2}}$

frequency of damped vibration, $f_d = \frac{1}{t_p} = \frac{\omega_d}{2\pi}$

$$f_d = \frac{1}{2\pi}\sqrt{(\omega_n)^2 - a^2} = \frac{1}{2\pi}\sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2}$$

→ when no damper is provided in the system, then c = 0. Therefore the frequency of the undamped vibration,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{s}{m}}$$

critical damping coefficient, $c_c = 2m\sqrt{\frac{s}{m}} = 2m * \omega_n$

$$\left(\frac{c_c}{2m}\right)^2 = \frac{s}{m}$$

Damping factor (or) Damping ratio $= \frac{c}{c_c} = \frac{c}{2m\omega_n}$

 www.FirstRanker.com

Logarithmic decrement

Amplitude reduction factor, $\frac{x_1}{x_2} = e^{a \cdot t_p}$ = constant

Logarithmic decrement, $\delta = \log\left(\frac{x_1}{x_2}\right) = \log e^{a \cdot t_p}$

$$\delta = \log_e\left(\frac{x_1}{x_2}\right) = a \cdot t_p$$

$\because t_p = \frac{2\pi}{\omega_d}$

$\omega_d = \sqrt{\omega_n^2 - a^2}$

$a = \frac{c}{2m}$

$c_c = 2m\omega_n$

t_p – period of forced oscillation (or) time difference b/w two consecutive amplitudes

$$\delta = a \times \frac{2\pi}{\omega_d} = \frac{a \times 2\pi}{\sqrt{\omega_n^2 - a^2}}$$

$$\delta = \frac{\frac{c}{2m} \times 2\pi}{\omega_n\sqrt{1 - \frac{a^2}{\omega_n^2}}} = \frac{c \times 2\pi}{2m\omega_n\sqrt{1 - \left(\frac{c}{2m\omega_n}\right)^2}}$$

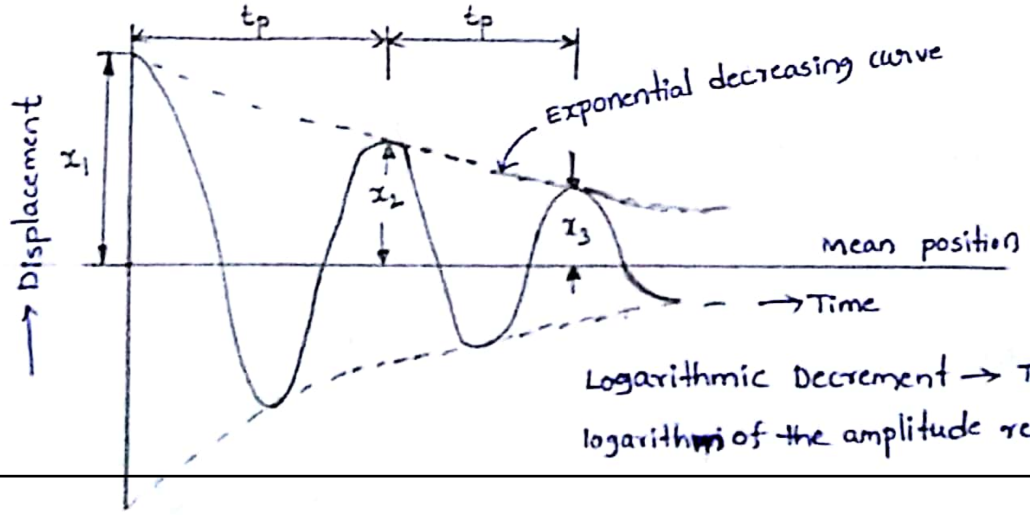
$$\delta = \frac{c \times 2\pi}{c_c\sqrt{1 - \left(\frac{c}{c_c}\right)^2}} = \frac{2\pi c}{\sqrt{(c_c)^2 - c^2}}$$

Amplitude reduction factor

$$\frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \cdots = \frac{x_n}{x_{n+1}} = e^{a \cdot t_p} = \text{constant}$$

$\therefore$ Logarithmic decrement, $\delta = \log_e\left(\frac{x_n}{x_{n+1}}\right) = a \cdot t_p = \frac{2\pi c}{\sqrt{c_c^2 - c^2}}$

x_1 & x_2 → successive values of the amplitude on the same side of the mean position

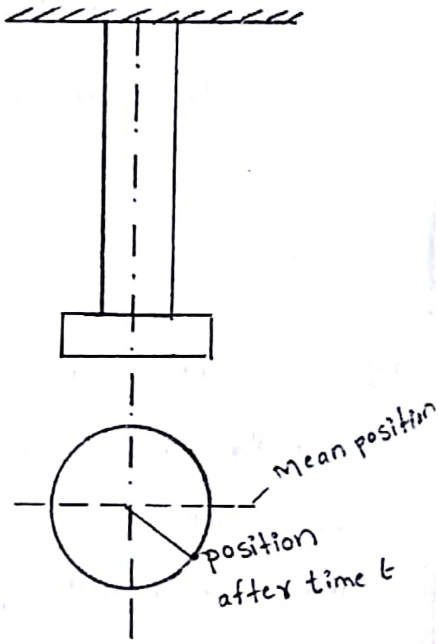


Logarithmic Decrement → The natural logarithm of the amplitude reduction factor.

When the particles of the shaft or disc move in a circle about the axis of the shaft, then the vibrations are known as torsional vibrations. In this case, the shaft is twisted and untwisted alternately & torsional shear stresses are induced in the shaft.

Natural Frequency of Free Torsional vibrations

Consider a shaft of negligible mass whose one end is fixed and the other end carrying a disc as shown in Fig.



θ – Angular displacement of the shaft from mean position after time t in radians

m – Mass of disc in kg

I – Mass moment of inertia of disc in kg-m²

$I = mk^2$

k – Radius of gyration

q – Torsional stiffness of the shaft in N-m

$\therefore$ Restoring force $= q\cdot\theta \longrightarrow ①$

Accelerating force $= I \times \frac{d^2\theta}{dt^2} \longrightarrow ②$

Equating equations ① & ②, the equation of motion is

$$I \times \frac{d^2\theta}{dt^2} = -q\theta$$

$$I \times \frac{d^2\theta}{dt^2} + q\theta = 0$$

$$\frac{d^2\theta}{dt^2} + \frac{q}{I}\theta = 0 \longrightarrow ③$$

The fundamental equation of the simple harmonic motion is

$$\frac{d^2x}{dt^2} + \omega^2 . x = 0 \longrightarrow ④ \qquad \frac{d^2\theta}{dt^2} + \omega^2 . \theta = 0$$

$$\omega = \sqrt{\frac{q}{I}}$$

Time period, $t_p = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{q}}$

Natural frequency, $f_n = \frac{1}{t_p} = \frac{1}{2\pi}\sqrt{\frac{q}{I}}$

$$\frac{T}{J} = \frac{\tau}{R} = \frac{C\theta}{l} \rightarrow \frac{T}{\theta} = \frac{C \cdot J}{l} \qquad \left(\because \frac{T}{\theta} = q\right)$$

Torsional stiffness $-q$

$$q = \frac{C \cdot J}{l}$$

C – Modulus of rigidity for the shaft material

J – Polar moment of inertia of the shaft material

$J = \frac{\pi}{32} d^4$, d – diameter of the shaft

l – Length of the shaft.

(P) A shaft of 100 mm diameter and 1 metre long has one of its end fixed and the other end carries a disc of mass 500 kg at a radius of gyration of 450 mm. The modulus of rigidity for the shaft material is 80 GN/m². Determine the frequency of torsional vibrations

$d = 100\text{mm} = 0.1\text{ m}$, $l = 1\text{ m}$, $m = 500\text{ kg}$, $k = 450\text{ mm} = 0.45\text{m}$

$C = 80\ \text{GN/m}^2 = 80 * 10^9\ \text{N/m}^2$

Polar moment of inertia of the shaft, $J = \frac{\pi}{32} d^4$

$$J = \frac{\pi}{32}(0.1)^4 = 9.82 * 10^{-6}\ \text{m}^4$$

$\therefore$ Torsional stiffness of the shaft

$$q = \frac{C \cdot J}{l} = \frac{80 * 10^9 * 9.82 * 10^{-6}}{1} = 785.6 * 10^3\ \text{N-m}$$

Mass moment of inertia of the shaft

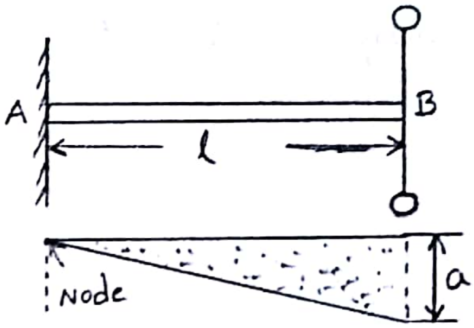
$$I = mk^2 = 500\,(0.45)^2 = 101.25\ \text{kg-m}^2$$

$\therefore$ Frequency of torsional vibrations

$$f_n = \frac{1}{2\pi}\sqrt{\frac{q}{I}} = \frac{1}{2\pi}\sqrt{\frac{785.6 * 10^3}{101.25}} = 14\ \text{HZ}$$

Torsional vibrations of a single rotor system

shaft fixed at one end and carrying a rotor at the free end as shown in Fig.



The natural frequency of torsional vibration,

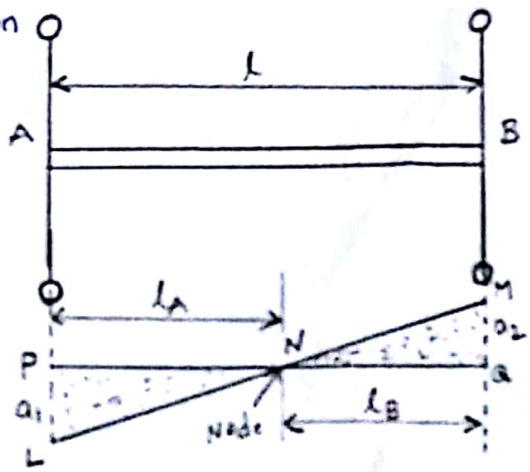
$$f_n = \frac{1}{2\pi}\sqrt{\frac{q}{I}} = \frac{1}{2\pi}\sqrt{\frac{C \cdot J}{l \cdot I}}$$

where
C – Modulus of rigidity for shaft material
J – Polar moment of inertia of shaft $= \frac{\pi}{32} d^4$
d – Diameter of shaft $\quad \because q = \frac{C \cdot J}{l}$
l – Length of shaft
m – Mass of rotor
k – radius of gyration of rotor
I – Mass moment of inertia of rotor $= mk^2$

The amplitude of vibration is zero at A and maximum at B, as shown in Fig. It may be noted that the point or the section of the shaft whose amplitude of torsional vibration is zero, is known as node. In other words at the node, the shaft remains unaffected by the vibration.

Free Torsional vibrations of a Two rotor system

Consider a two rotor system as shown in Fig. It consists of a shaft with two rotors at its ends. In this system, the torsional vibrations occur only when the two rotors A and B move in opposite directions i.e. if A moves in anticlock-wise direction then B moves in clockwise direction at the same instant and vice versa. It may be noted that the two rotors must have the same frequency.



We see from Fig. that the node lies at point N. This point can be safely assumed as a fixed end and the shaft may be considered as two separate shafts NP and NQ each fixed to one of its ends and carrying rotors at the free ends.

Let l_A – Length of part NP i.e. distance of node from rotor A.

l_B – Length of part NQ i.e. distance of node from rotor B.

I_A – Mass moment of inertia of rotor A

I_B – Mass moment of inertia of rotor B

l – Length of the shaft, d – diameter of shaft

$\therefore$ Natural frequency of torsional vibration for rotor A & rotor B

$$f_{nA} = \frac{1}{2\pi}\sqrt{\frac{C \cdot J}{l_A \cdot I_A}} \quad \& \quad f_{nB} = \frac{1}{2\pi}\sqrt{\frac{C \cdot J}{l_B \cdot I_B}}$$

Since $f_{nA} = f_{nB}$

$$\frac{1}{2\pi}\sqrt{\frac{C \cdot J}{l_A \cdot I_A}} = \frac{1}{2\pi}\sqrt{\frac{C \cdot J}{l_B \cdot I_B}}$$

$$I_A \cdot l_A = I_B \cdot l_B$$

$$\therefore \; l_A = \frac{l_B \cdot I_B}{I_A}$$

We also know that

$$l = l_A + l_B$$

The line LNM is known as elastic line for the shaft.

A ... stiffness 4 N/mm supports vertically a mass of 20 kg at the free end. The motion is resisted by the oil dashpot. It is found that the amplitude at the beginning of the fourth cycle is 0.8 times the amplitude of the previous vibration. Determine the damping force per unit velocity. Also find the ratio of the frequency of damped and undamped vibrations.

Sol:- $s = 4\ N/mm = 4000\ N/m$, $m = 20\ kg$

Damping force per unit velocity

Let c = Damping force in newtons per unit velocity i.e. in N/m/s

x_n = Amplitude at the beginning of the third cycle

x_{n+1} = Amplitude at the beginning of the fourth cycle $= 0.8\, x_n$

we know that natural circular frequency of motion

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{4000}{20}} = 14.14\ rad/s$$

and $$\log_e\left[\frac{x_n}{x_{n+1}}\right] = a * \frac{2\pi}{\sqrt{\omega_n^2 - a^2}}$$

$$\log_e\left[\frac{x_n}{0.8 x_n}\right] = a * \frac{2\pi}{\sqrt{(14.14)^2 - a^2}}$$

$$\log_e(1.25) = a * \frac{2\pi}{\sqrt{200-a^2}} \quad (or) \quad 0.223 = a * \frac{2\pi}{\sqrt{200-a^2}}$$

squaring both sides

$$0.05 = \frac{a^2 * 4\pi^2}{200 - a^2} = \frac{39.5\, a^2}{200 - a^2}$$

$$0.05(200 - a^2) = 39.5\, a^2 \quad (or) \quad 39.55\, a^2 = 10$$

$$a = 0.5$$

we know that $a = c/2m \Rightarrow c = a * 2m = 0.5 * 2 * 20 = 20\ N/m/s$

Ratio of the frequencies

Let f_{n_1} = Frequency of damped vibrations $= \frac{\omega_d}{2\pi}$

f_{n_2} = Frequency of undamped vibrations $= \frac{\omega_n}{2\pi}$

$$\therefore \frac{f_{n_1}}{f_{n_2}} = \frac{\omega_d}{2\pi} * \frac{2\pi}{\omega_n} = \frac{\omega_d}{\omega_n} = \sqrt{\frac{\omega_n^2 - a^2}{\omega_n}}$$

$$= \sqrt{\frac{(14.14)^2 - (0.5)^2}{14.14}} = 0.999 \qquad \left(\because \omega_d = \sqrt{\omega_n^2 - a^2}\right)$$

The measurements on a mechanical vibrating system show that it has a mass of 8 kg and that the springs can be combined to give an equivalent spring of stiffness 5.4 N/mm. If the vibrating system have a dashpot attached which exerts a force of 40 N when the mass has a velocity of 1 m/s, find: 1. critical damping coefficient 2. damping factor 3. logarithmic decrement, and 4. ratio of two consecutive amplitudes.

Sol:-

$m = 8\,kg$, $s = 5.4\,N/mm = 5400\,N/m$

Since the force exerted by dashpot is 40 N, and the mass has a velocity of 1 m/s, therefore

Damping coefficient (actual),

$$c = 40\ N/m/s$$

1. Critical damping coefficient

we know that critical damping coefficient

$$C_c = 2m\omega_n = 2m \times \sqrt{\frac{s}{m}} = 2 * 8 \times \sqrt{\frac{5400}{8}} = 416\ N/m/s$$

2. Damping factor

we know that damping factor

$$= \frac{c}{C_c} = \frac{40}{416} = 0.096$$

3. Logarithmic decrement

we know that logarithmic decrement

$$\delta = \frac{2\pi c}{\sqrt{C_c^2 - c^2}} = \frac{2\pi * 40}{\sqrt{(416)^2 - (40)^2}} = 0.6$$

4. Ratio of two consecutive amplitudes

Let x_n & x_{n+1} = Magnitude of two consecutive amplitudes

we know that logarithmic decrement

$$\delta = \log_e\left[\frac{x_n}{x_{n+1}}\right] \text{ (or) } \frac{x_n}{x_{n+1}} = e^{\delta} = (2.7)^{0.6} = 1.82$$