**Roll No.** **Total No. of Pages : 03**

**Total No. of Questions : 07**

**M.Sc Mathematics (2017 Batch) (Sem.–3)**

# FUNCTIONAL ANALYSIS

**Subject Code : MSM-304**

**Paper ID : [75384]**

**Time : 3 Hrs.** **Max. Marks : 80**

**INSTRUCTION TO CANDIDATES :**

1. **SECTION-A is COMPULSORY consisting of EIGHT questions carrying TWO marks each.**
2. **SECTION - B & C. have THREE questions in each section carrying SIXTEEN marks each.**
3. **Select atleast TWO questions from SECTION - B & C EACH.**

## SECTION-A

1. (a) Show that the function $\|x\| = \sqrt{(x,x)}$ defines a norm.

   (b) Prove that a metric $d$ induced by a norm on a normed space X satisfies

   (i) $d(x + a, y + a) = d(x, y)$,

   (ii) $d(\alpha x, \alpha y) = |\alpha|\, d(x, y)$,

   for all $x, y, a \in X$ and every scalar $\alpha$.

   (c) What is the zero element of the vector space $B(X, Y)$? The inverse of any element T $\in$ $B(X, Y)$?

   (d) Give an example of a Banach space that is not a Hilbert space. Justify.

   (e) Show that in an inner product space, x is orthogonal to $y$ if and only if $\|x + \alpha y\| \geq \|x\|$, for all scalars $\alpha$.

   (f) Let $H$ be a Hilbert space and let $T : H \to H$ be a bounded, linear operator. If $H$ is complex linear space and $(Tx, X)$ is real for all $x \in H$ then prove that $T$ is self-adjoint.

   (g) Let $H$ be a Hilbert space and let $U : H \to H$ be unitary. Then prove that $U$ is isometric, *i.e.*, $\|Ux\| = \|x\| \; \forall\, x \in H$ and $\|U\| = 1$provided $H \neq \{0\}$.

   (h) State Open Mapping Theorem.

## SECTION-B

2. (a) Let $\|f\| = \min\{\|f\|_\infty, 2\|f\|_1\}$ for all $f \in C[0,1]$. Prove that $\|\cdot\|$ is not a norm on $C[0,1]$.

   (b) Show that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbb{R}^n$ satisfy

$$\frac{1}{\sqrt{n}}\|x\|_1 \le \|x\|_2 \le \|x\|_1.$$

3. (a) Prove that a linear transformation $T : X \to Y$ is continuous if and only if it is bounded.

   (b) Definite integral defined by

$$f(x) = \int_a^b x(t)\, dt, \qquad x \in C[a,b],$$

   becomes a functional on that space, call it $f$. Show that $f$ is linear, bounded and has norm $\|f\| = b - a$.

4. Let $X$ be a real linear space, and $p : X \to \mathbb{R}$, is a continuous function with

$$p(x+y) \le p(x) + p(y), p(\alpha x) = \alpha\, p(x), \ \forall\ x, y \in X,\ \alpha \in \mathbb{R}, \alpha \ge 0.$$

   Let $f$ be a linear functional defined on a subspace $Z$ of $X$ such that

$$f(x) \le p(x), \forall\ x \in Z.$$

   Then prove that $f$ has a linear extension $F : Z \to X$ such that

$$F(x) \le p(x), \forall\ x \in X$$

$$F(x) = f(x), \forall\ x \in Z$$

## SECTION-C

5. (a) Show $\ell^p$ with $p \neq 2$ is not a Hilbert space.

   (b) Prove that an inner product and the corresponding norm satisfy the following Schwarz inequality

$$|\langle x, y\rangle| \leq \|x\| \, \|y\|.$$

   Further also show that the equality holds if and only if x and y are a linearly dependent.

6. (a) Let A be a linear bounded self-adjoint operator in a Hilbert space $H$. Let $u, v \in H$ and $\alpha \in \mathbb{C}$.Consider the equation $Au - \alpha u = v$. Show that for $\alpha$ nonreal (i.e. it has an imaginary part) $v$ cannot vanish unless u vanishes.

   (b) Let X be an inner product space and let ($e_k$) be an orthonormal sequence in $X$. Let $x \in H$ then prove that

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \leq \|x\|^2.$$

7. (a) Let $\{T_n\}$ be a sequence of bounded linear operators from a Banach space $X$ into a normed space $Y$ such that $\{T_n x\}$ is bounded for every $x \in X$, say,

$$\|T_n x\| \leq c_x, \; n = 1, 2, \ldots,$$

   where $c_x$ is a real number. Then prove that the sequence of the norms $\|T_n\|$ is bounded that is, there is a $c$ such that

$$\|T_n\| \leq c, \text{ for all } n.$$

   (b) Let X is the normed space of all polynomials with norm defined by

$$\|x\| = \max_i |\alpha_i| \; where \; x(t) = \alpha_0 + \alpha_1 t + \cdots + \alpha_k t^k.$$

   Use Uniform Boundedness Theorem to prove that $X$ is not complete.